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Description of bi-quadratic algebras on 3 generators with PBW basis

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Abstract

The aim of the paper is to give an explicit description of bi-quadratic algebras on 3 generators with PBW basis.

Key words: Bi-quadratic algebras, automorphism, Lie algebra, generalized Weyl algebras, diskew polynomial rings, normal element, regular element.

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Contents

1 Introduction

The aim of the paper is to give an explicit description of bi-quadratic algebras on 3 generators with PBW basis. First, we give necessary definitions and then explain the results and main ideas.

The skew bi-quadratic algebras. For a ring D and a natural number $n \geq 2$, a family $M = (m_{ij})_{i>j}$ of elements $m_{ij} \in D$ (where $1 \leq j < i \leq n$) is called a **lower triangular half-matrix** with coefficients in D. The set of all such matrices is denoted by $L_n(D)$.

Definition. Let D be a ring, $Z(D)$ be its centre, $\sigma = (\sigma_1, \ldots, \sigma_n)$ be an n-tuple of commuting endomorphisms of D, $\delta = (\delta_1, \ldots, \delta_n)$ be an n-tuple of σ-endomorphisms of D (δ_i is a σ_i-derivation of D for $i = 1, \ldots, n$), $Q = (q_{ij}) \in L_n(Z(D))$, $\mathbb{A} = (a_{ij,k})$ where $a_{ij,k} \in D$, $1 \leq j < i \leq n$ and $k = 1, \ldots, n$ and $\mathbb{B} = (b_{ij}) \in L_n(D)$. The **skew bi-quadratic algebra** (SBQA) $A = D[x_1, \ldots, x_n; \sigma, \delta, Q, \mathbb{A}, \mathbb{B}]$ is a ring generated by the ring D and elements $x_1, \ldots, x_n$ subject to the defining relations: For all $d \in D$,

$$x_i d = \sigma_i(d) x_i + \delta_i(d) \ \text{ for } \ i = 1, \ldots, n, \tag{1}$$

$$x_i x_j - q_{ij} x_j x_i = \sum_{k=1}^{n} a_{ij,k} x_k + b_{ij} \ \text{ for all } \ j < i. \tag{2}$$

Definition. In the particular case, when $\sigma_i = \mathrm{id}_D$ and $\delta_i = 0$ for $i = 1, \ldots, n$, the ring A is called the **bi-quadratic algebra** (BQA) and is denoted by $A = D[x_1, \ldots, x_n; Q, \mathbb{A}, \mathbb{B}]$.

We say that the algebra $A = D[x_1, \ldots, x_n; Q, \mathbb{A}, \mathbb{B}]$ has **PBW basis** if

$$A = \bigoplus_{\alpha \in \mathbb{N}^n} Dx^\alpha \text{ where } x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

On the set W_n of all words in the alphabet $\{x_1, \ldots, x_n\}$ (W_n is a multiplicative monoid freely generated by the elements $x_1, \ldots, x_n$) we consider the **degree-by-lexicographic ordering** where $x_1 < \cdots < x_n$. In more detail, $x_{i_1} \cdots x_{i_s} < x_{j_1} \cdots x_{j_t}$ if either $s < t$ or $s = t$, $i_1 = j_1, \ldots, i_k = j_k$ and $i_{k+1} < j_{k+1}$ for some k such that $1 \leq k < s$.

Given a bi-quadratic algebra A on $n \geq 3$ generators. For each triple $i, j, k \in \{1, \ldots, n\}$ such that $i < j < k$, there exactly two different ways to simplify the product $x_k x_j x_i$ with respect to the degree-by-lexicographic ordering:

$$x_k x_j x_i = q_{kj} q_{ki} q_{ji} x_i x_j x_k + \sum_{|\alpha| \leq 2} c_{k,j,i,\alpha} x^\alpha, \tag{3}$$

$$x_k x_j x_i = q_{kj} q_{ki} q_{ji} x_i x_j x_k + \sum_{|\alpha| \leq 2} c'_{k,j,i,\alpha} x^\alpha, \tag{4}$$

where in the first (resp. second) equality we start to simplify the product with the relation $x_k x_j = q_{kj} x_j x_k + \cdots$ (resp., $x_j x_i = q_{ji} x_i x_j + \cdots$).

Let S_n be the symmetric group of order n (the group of all bijections from a set that contains n elements to itself).

Theorem 1.1 *Let $A = D[x_1, \ldots, x_n; Q, \mathbb{A}, \mathbb{B}]$ be a bi-quadratic algebra where $n \geq 3$. Then the defining relations (2) are consistent and $A = \bigoplus_{\alpha \in \mathbb{N}^n} Dx^\alpha$ iff the following equalities hold: For all triples $i, j, k \in \{1, \ldots, n\}$ such that $i < j < k$,*

$$c_{k,j,i,\alpha} = c'_{k,j,i,\alpha}. \tag{5}$$

If (5) holds then, for all $\sigma \in S_n$, $A = \bigoplus_{\alpha \in \mathbb{N}^n} Dx_\sigma^\alpha$ where $x_\sigma^\alpha = x_{\sigma(1)}^{\alpha_1} \cdots x_{\sigma(n)}^{\alpha_n}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$.

The groups $G_n = \mathrm{Sh}_n \rtimes \mathbb{T}^n$ and $G'_n = G_n \rtimes S_n$. The following groups of linear changes of the variables $x_1, \ldots, x_n$ preserve the structure of the defining relations of the bi-quadratic algebras $A = K[x_1, \ldots, x_n; Q, \mathbb{A}, \mathbb{B}]$:

$$\begin{aligned} \mathbb{T}^n &= \{t_\lambda \,|, \lambda = (\lambda_1, \ldots, \lambda_n) \in K^{\times n}\} \simeq (K^{\times n}, \cdot), \;\; t_\lambda \leftrightarrow \lambda \text{ where } t_\lambda(x_i) = \lambda_i x_i \text{ for } i = 1, \ldots, n, \\ \mathrm{Sh}_n &= \{s_\mu \,|, \mu = (\mu_1, \ldots, \mu_n) \in K^n\} \simeq (K^n, +), \;\; s_\mu \leftrightarrow \mu \text{ where } s_\mu(x_i) = x_i + \mu_i \text{ for } i = 1, \ldots, n. \end{aligned}$$

The groups $\mathbb{T}^n$ and Sh_n are called the **algebraic n-dimensional torus** and the **shift group**, respectively. Clearly,

$$t_\lambda s_\mu t_\lambda^{-1}(x_i) = x_i + \lambda_i^{-1} \mu_i \text{ for } i = 1, \ldots, n. \tag{6}$$

So, $t_\lambda s_\mu t_\lambda^{-1} \in \mathrm{Sh}_n$, $\mathbb{T}^n \cap \mathrm{Sh}_n = \{e\}$, and the group G_n which is generated by $\mathbb{T}^n$ and Sh_n is a semi-direct product of groups $G_n = \mathrm{Sh}_n \rtimes \mathbb{T}^n$. Under the action of the group G_n the half-matrix Q does not change.

The symmetric group S_n acts on the set $\{x_1, \ldots, x_n\}$ by permuting the variables: For all $\sigma \in S_n$ and x_i, $\sigma(x_i) = x_{\sigma(i)}$. The group G'_n, which is generated by G_n and S_n, is a semi-direct product $G'_n = G_n \rtimes S_n$. The group G_n acts on the class of bi-quadratic algebras. The group G'_n acts on the class of bi-quadratic algebras $A = D[x_1, \ldots, x_n; Q, \mathbb{A}, \mathbb{B}]$ where all elements of Q are *invertible*: For $\sigma \in G'_n$,

$${}^\sigma A = D[\sigma(x_1), \ldots, \sigma(x_n); {}^\sigma Q, {}^\sigma \mathbb{A}, {}^\sigma \mathbb{B}]$$

where the triple ${}^\sigma Q, {}^\sigma \mathbb{A}, {}^\sigma \mathbb{B}$ can be explicitly written. In particular, for an element σ of S_n,

$${}^\sigma A = D[x_{\sigma(1)}, \ldots, x_{\sigma(n)}; {}^\sigma Q, {}^\sigma \mathbb{A}, {}^\sigma \mathbb{B}].$$

For each half-matrix $Q = (q_{ij})_{i>j}$ with invertible entries q_{ij}, there is its unique completion Q^c to an $n \times n$ matrix with 1 on the diagonal (i.e. $q_{ii} = 1$ for all $i = 1, \dots, n$) and $q_{ji} := q_{ij}^{-1}$ for all $i > j$.

The symmetric group S_n acts on the set of $n \times n$ matrices by simultaneously permuting their rows and columns (for $\sigma \in S_n$ and a matrix $M = (m_{ij})$, ${}^\sigma M = ({}^\sigma m_{ij})$ where ${}^\sigma m_{ij} := m_{\sigma(i)\sigma(j)}$). The matrix ${}^\sigma Q$ of the algebra ${}^\sigma A$ is the lower triangular part of the matrix ${}^\sigma(Q^c)$. Clearly, for all $\sigma \in G_n$, ${}^\sigma Q = Q$.

When we say '*up to* G_n (resp., G_n')' we mean 'up to affine change of the canonical generators $x_1, \dots, x_n$ of the bi-quadratic algebra A induced by an element of the group G_n (resp., G_n')'.

The bi-quadratic algebra of Lie type.

Definition. A bi-quadratic algebra A is called a **bi-quadratic algebra of Lie type** provided all $q_{ij} = 1$ and is denoted by $A = D[x_1, \dots, x_n; \mathbb{A}, \mathbb{B}]$.

Theorem 1.2 gives a reason for choosing the name for this class of bi-quadratic algebras.

Theorem 1.2 *Let $A = D[x_1, \dots, x_n; Q, \mathbb{A}, \mathbb{B}]$ be a bi-quadratic algebra such that D is a K-algebra over a field K and all the elements $a_{ij,k}$ and b_{ij} belong to K. Then the algebra A is a bi-quadratic algebra of Lie type iff $A \simeq D \otimes_K (U(\mathcal{G})/(Z-1))$ where $\mathcal{G}$ is a Lie algebra of dimension $n+1$ with nontrivial centre $Z(\mathcal{G})$ and $Z \in Z(\mathcal{G}) \backslash \{0\}$.*

The bi-quadratic algebras on 3 generators. Let K be a field and $A = K[x_1, x_2, x_3; Q, \mathbb{A}, \mathbb{B}]$ be a bi-quadratic algebra where $Q = (q_1, q_2, q_3) \in K^{\times 3}$,

$$\mathbb{A} = \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \lambda & \mu & \nu \end{pmatrix} \tag{7}$$

and $\mathbb{B} = (b_1, b_2, b_3)$. So, the algebra A is an algebra that is generated over the field K by the elements x_1, x_2 and x_3 subject to the defining relations:

$$x_2x_1 - q_1x_1x_2 = ax_1 + bx_2 + cx_3 + b_1, \tag{8}$$

$$x_3x_1 - q_2x_1x_3 = \alpha x_1 + \beta x_2 + \gamma x_3 + b_2, \tag{9}$$

$$x_3x_2 - q_3x_2x_3 = \lambda x_1 + \mu x_2 + \nu x_3 + b_3. \tag{10}$$

Examples of bi-quadratic algebras on 3 generators.

1. The universal enveloping algebra of any 3-dimensional Lie algebra.

2. The *3-dimensional quantum space* $\mathbb{A}^3_{q_1,q_2,q_3} := K[x_1, x_2, x_3; Q, \mathbb{A} = 0, \mathbb{B} = 0]$.

3. The algebra $U_q'(so_3)$ is generated over the field K by elements I_1, I_2 and I_3 subject to the defining relations:

$$q^{\frac{1}{2}}I_1I_2 - q^{-\frac{1}{2}}I_2I_1 = I_3, \;\; q^{\frac{1}{2}}I_2I_3 - q^{-\frac{1}{2}}I_3I_2 = I_1, \;\; q^{\frac{1}{2}}I_3I_1 - q^{-\frac{1}{2}}I_1I_3 = I_2,$$

where $q \in K \backslash 0, \pm 1\}$, [8, 7].

4. The Askey-Wilson algebras $AW(3)$ introduced by A. Zhedanov, [10]. The algebra $AW(3)$ is generated by three elements K_0, K_1 and K_2 subject to the defining relations:

$$[K_0, K_1]_w = K_2, \;\; [K_2, K_0]_w = BK_0 + C_1K_1 + D_1, \;\; [K_1, K_2]_w = BK_1 + C_0K_0 + D_0,$$

where B, C_0, C_1, D_0, $D_1 \in K$, $[L, M]_w := wLM - w^{-1}ML$ and $w \in K^\times$.

5. A construction of bi-quadratic algebras on 4 generators was introduced by James J. Zhang and Jun Zhang, [9].

Theorem 1.3 gives explicit necessary and sufficient conditions for the defining relations of the algebra A to be consistent (i.e., $A \neq 0$) and the algebra A has PBW-basis.

Theorem 1.3 *Suppose that the algebra A is generated over a field K by the elements x_1, x_2, and x_3 that satisfy the defining relations (8), (9) and (10). Then the defining relations are consistent and $A = \bigoplus_{\alpha\in\mathbb{N}^3} Kx^\alpha$ where $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$ iff the following conditions hold:*

$$(1-q_3)\alpha = (1-q_2)\mu, \tag{11}$$

$$(1-q_3)a = (1-q_1)\nu, \tag{12}$$

$$(1-q_2)b = (1-q_1)\gamma, \tag{13}$$

$$(1-q_1q_2)\lambda = 0, \tag{14}$$

$$(q_1-q_3)\beta = 0, \tag{15}$$

$$(1-q_2q_3)c = 0, \tag{16}$$

$$((1-q_3)\alpha-\mu)a + (b+q_1\gamma)\lambda - \nu\alpha + (q_1q_2-1)b_3 = 0, \tag{17}$$

$$(a-\nu)\beta + q_1\gamma\mu - q_3\alpha b + (q_1-q_3)b_2 = 0, \tag{18}$$

$$(a+(q_1-1)\nu)\gamma + b\nu - (\mu+q_3\alpha)c + (1-q_2q_3)b_1 = 0, \tag{19}$$

$$-(\mu+q_3\alpha)b_1 + (a-\nu)b_2 + (b+q_1\gamma)b_3 = 0. \tag{20}$$

Furthermore, if $A = \bigoplus_{\alpha\in\mathbb{N}^3} Kx^\alpha$ where $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$ then $A = \bigoplus_{\alpha\in\mathbb{N}^3} Kx^\alpha_\sigma$ for all $\sigma \in S_3$ where $x^\alpha_\sigma = x_{\sigma(1)}^{\alpha_1}x_{\sigma(2)}^{\alpha_2}x_{\sigma(3)}^{\alpha_3}$.

Remark. In this paper, when we say '*a bi-quadratic algebra on 3 generators*' we mean 'a bi-quadratic algebra on 3 generators *with PBW basis*' (i.e., the ones that satisfy Theorem 1.3).

4 main classes of bi-quadratic algebras A. In order to classify bi-quadratic algebra on 3 generators we have to consider the following 4 classes (in view of the S_3-action):

1. $q_1 = q_2 = q_3 = 1$,
2. $q_1 \neq 1$, $q_2 = q_3 = 1$,
3. $q_1 \neq 1$, $q_2 \neq 1$, $q_3 = 1$,
4. $q_1 \neq 1$, $q_2 \neq 1$, $q_3 \neq 1$.

Explicit descriptions of the algebras in each of the four cases are given in Section 7, Section 4, Section 5, and Section 6, respectively.

Till the end of this section,

$$A = K[x_1, x_2, x_3; Q, \mathbb{A}, \mathbb{B}]$$

is a bi-quadratic algebra on 3-generators where $Q = (q_1, q_2, q_3) \in K^{\times 3}$ and (7)–(10) hold.

1. Classification (up to isomorphism) of bi-quadratic algebras on 3 generators of Lie type, i.e. when $q_1 = q_2 = q_3 = 1$.

Theorem 1.4 *Let A be an algebra of Lie type over an algebraically closed field of characteristic zero. Then the algebra A is isomorphic to one of the following (pairwise non-isomorphic) algebras:*

1. *$P_3 = K[x_1, x_2, x_3]$, a polynomial algebra in 3 variables.*

2. *$U(sl_2(K))$, the universal enveloping algebra of the Lie algebra $sl_2(K)$.*

3. *$U(\mathcal{H}_3)$, the universal enveloping algebra of the Heisenberg Lie algebra $\mathcal{H}_3$.*

4. *$U(\mathcal{N})/(c-1) \simeq K\langle x, y, z \mid [x, y] = z,\ [x, z] = 0,\ [y, z] = 1\rangle$ and the algebra $U(\mathcal{N})/(c-1)$ is a tensor product $A_1 \otimes K[x']$ of its subalgebras, the Weyl algebra $A_1 = K\langle y, z \mid [y, z] = 1\rangle$ and the polynomial algebra $K[x']$ where $x' = x + \frac{1}{2}z^2$.*

5. *$U(\mathfrak{n}_2 \times Kz) \simeq K\langle x, y, z \mid [x, y] = y\rangle$ and z is a central element.*

6. *$U(\mathcal{M})/(c-1) \simeq K\langle x, y, z \mid [x, y] = y,\ [x, z] = 1,\ [y, z] = 0\rangle$ and the algebra $U(\mathcal{M})/(c-1)$ is a skew polynomial algebra $A_1[y; \sigma]$ where $A_1 = K\langle x, z \mid [x, z] = 1\rangle$ is the Weyl algebra and σ is an automorphism of A_1 given by the rule $\sigma(x) = x + 1$ and $\sigma(z) = z$.*

The proof of Theorem 1.4 is given in Section 7.

2. Description of bi-quadratic algebras on 3 generators when $q_1 \neq 1$, $q_2 = q_3 = 1$. An element of a ring R is called a **regular element** if it is neither a left nor right zero-divisor of R. The set of all regular elements of the ring R is denoted by $\mathcal{C}_R$. An element a of the ring R is called a **normal element** if $Ra = aR$. In Section 2, definitions and some results on the (classical) generalized Weyl algebras (GWA) and diskew polynomial rings are given.

Theorem 1.5 *Suppose that $q_1 \neq 1$, $q_2 = q_3 = 1$ and $A = K[x_1, x_2, x_3; Q, \mathbb{A}, \mathbb{B}]$.*

1. *If $\mu + \alpha \neq 0$ then (up to G_3):*

$$x_2x_1 = q_1x_1x_2,\ \ x_3x_1 = x_1(x_3 + \alpha)\ \text{ and }\ x_3x_2 = x_2(x_3 + \mu)$$

where $(\alpha, \mu) \in \{(1, \mu'), (0, 1) \mid \mu' \in K\backslash\{-1\}\}$. The elements x_1 and x_2 are regular normal elements of A. The algebra

$$A = K[x_3][x_2, x_1; \sigma, \tau, b = 0, \rho = q_1]$$

is a diskew polynomial algebra where $\tau(x_3) = x_3 - \alpha$ and $\sigma(x_3) = x_3 - \mu$. Furthermore, the algebra $A = \mathcal{D}[x_2, x_1; \sigma, \tau, a = h]$ is a generalized Weyl algebra where $\mathcal{D} = K[x_3][h; \tau\sigma]$ is a skew polynomial ring where $\tau\sigma(x_3) = x_3 - \alpha - \mu$, $\sigma(h) = q_1h$ and $\tau(h) = q_1^{-1}h$. The element h is a regular normal element of A.

   (a) *The algebra $A = K[x_3][x_1; \sigma_1][x_2; \sigma_2]$ is an iterated skew polynomial algebra where $\sigma_1(x_3) = x_3 - \alpha$, $\sigma_2(x_3) = x_3 - \mu$ and $\sigma_2(x_1) = q_1x_1$.*

   (b) *The homomorphism*

$$A(q_1, \alpha = 1, \mu = 0) \to A(q_1^{-1}, \alpha = 0, \mu = 1),\ \ x_1 \mapsto x_2,\ \ x_2 \mapsto x_1,\ \ x_3 \mapsto x_3$$

   is an isomorphism.

   (c) *In particular, the homomorphism*

$$A(q_1 = -1, \alpha = 1, \mu = 0) \to A(q_1 = -1, \alpha = 0, \mu = 1),\ \ x_1 \mapsto x_2,\ \ x_2 \mapsto x_1,\ \ x_3 \mapsto x_3$$

   is an isomorphism.

2. *If $\mu + \alpha = 0$ then (up to G_3):*

$$x_2x_1 = q_1x_1x_2 + cx_3 + b_1,\ \ x_3x_1 = x_1(x_3 + \alpha)\ \text{ and }\ x_3x_2 = x_2(x_3 - \alpha)$$

where exactly one of the following cases occurs:

   (a) *$\alpha = 0$ and $c, b_1 \in \{0, 1\}$,* *(b)* $\alpha = 1$ *and* $(c, b_1) \in \{(0,0),(1,1)\}$,

(c) $\alpha = 1$, $c = 0$, $b_1 = 1$ *or* $\alpha = 1$, $c = 1$, $b_1 \in K\backslash\{1\}$.

So, the cases (a)–(c) can be written as $(\alpha, c, b_1) \in \{0,1\}^3$ *or* $\alpha = 1$, $c = 1$, $b_1 \in K\backslash\{0,1\}$. *The algebra* $A = K[x_3][x_2, x_1; \sigma, \sigma^{-1}, b = cx_3 + b_1, \rho = q_1]$ *is a diskew polynomial algebra where* $\sigma(x_3) = x_3 + \alpha$. *Furthermore, the algebra* $A = K[x_3, h][x_2, x_1; \sigma, a = h]$ *is a classical generalized Weyl algebra where* $\sigma(x_3) = x_3 + \alpha$ *and* $\sigma(h) = q_1 h + cx_3 + b_1$.

The proof of Theorem 1.5 is given in Section 4.

3. Description of bi-quadratic algebras on 3 generators when $q_1 \neq 1$, $q_2 \neq 1$, $q_3 = 1$.

Theorem 1.6 *Suppose that* $q_1 \neq 1$, $q_2 \neq 1$, $q_3 = 1$ *and let* $A = K[x_1, x_2, x_3; Q, \mathbb{A}, \mathbb{B}]$.

1. *If* $1 - q_1 q_2 \neq 0$ *then (up to* G_3*):*

$$x_2 x_1 = q_1 x_1 x_2, \;\; x_3 x_1 = q_2 x_1 x_3 \;\text{ and }\; x_3 x_2 = x_2 x_3,$$

and so $A \simeq \mathbb{A}^3_{(q_1, q_2, 1)}$.

2. *If* $1 - q_1 q_2 = 0$ *then (up to* G_3*):*

$$x_2 x_1 = q_1 x_1 x_2, \;\; x_3 x_1 = q_1^{-1} x_1 x_3 \;\text{ and }\; x_3 x_2 = x_2 x_3 + \lambda x_1 + b_3$$

where $\lambda, b_3 \in \{0,1\}$. *The algebra* $A = K[x_1][x_3, x_2; \sigma, \sigma^{-1}, \rho = 1, b = \lambda x_1 + b_3]$ *is a diskew polynomial algebra where* $\sigma(x_1) = q_1^{-1} x_1$. *The algebra* $A = K[x_1, h][x_3, x_2; \sigma, a = h]$ *is a classical GWA where* $\sigma(x_1) = q_1^{-1} x_1$ *and* $\sigma(h) = h + b$. *If* $b_3 = 0$ *then the element* $C = h + \alpha'$ *where* $\alpha' = \lambda(1 - q_1^{-1})^{-1} x_1$ *is a central element of the algebra* A *and* $A = K[x_1, C][x_3, x_2; \sigma, a = C - \alpha']$ *is a classical GWA where* $\sigma(x_1) = q_1^{-1} x_1$ *and* $\sigma(C) = C$.

The proof of Theorem 1.5 is given in Section 5.

4. Description of bi-quadratic algebras on 3 generators when $q_1 \neq 1$, $q_2 \neq 1$, $q_3 \neq 1$. The class of algebras A is a disjoint union of subclasses that satisfy the condition that elements of the set $\{q_1 - q_3, 1 - q_1 q_2, 1 - q_2 q_3\}$ are either zero or not. Potentially, there are $2 \times 2 \times 2 = 8$ subclasses but in fact, there are only 5 since the condition that $q_1 - q_3 = 0$ implies that $1 - q_1 q_2 = 1 - q_2 q_3$; and the conditions $1 - q_1 q_2 = 0$, $1 - q_2 q_3 = 0$ imply that $q_1 - q_3 = 0$.

- Case 1: $q_1 - q_3 = 0$, $1 - q_1 q_2 = 0$.
- Case 2: $q_1 - q_3 = 0$, $1 - q_1 q_2 \neq 0$.
- Case 3: $q_1 - q_3 \neq 0$, $1 - q_1 q_2 = 0$, $1 - q_2 q_3 \neq 0$.
- Case 4: $q_1 - q_3 \neq 0$, $1 - q_1 q_2 \neq 0$, $1 - q_2 q_3 = 0$.
- Case 5: $q_1 - q_3 \neq 0$, $1 - q_1 q_2 \neq 0$, $1 - q_2 q_3 \neq 0$.

• **Case 1: $q_1 - q_3 = 0$ and $1 - q_1 q_2 = 0$, i.e., $q := q_1 = q_2^{-1} = q_3 \neq 1$ (the quantum bi-quadratic algebras).**

Definition. A bi-quadratic algebra A is called the **quantum bi-quadratic algebra** if $q := q_1 = q_2^{-1} = q_3 \neq 1$. The element q is called the **quantum parameter** of A.

In Theorem 1.7, an expression like '$\mu \in K^\times / K^{\times n}$' means '$\mu$ is unique up to $K^{\times n}$' where $K^{\times n} := \{\xi^n \,|\, \xi \in K^\times\}$ and $n \geq 2$.

Theorem 1.7 *Let A be a quantum bi-quadratic algebra over the field K with quantum parameter q_1. Then (up G_3')*

$$x_2x_1 = qx_1x_2 + cx_3 + b_1, \tag{21}$$

$$x_3x_1 = q^{-1}x_1x_3 + \beta x_2 + b_2, \tag{22}$$

$$x_3x_2 = qx_2x_3 + \lambda x_1 + b_3, \tag{23}$$

where $q \in \{q_1, q_1^{-1}\}$ and (up G_3') the parameters $c, \beta, \lambda, b_1, b_2, b_3$ belong precisely to one of the four cases below (that correspond to Cases 1–4 in the proof):

Case 1: $(c, \beta, \lambda) \in (1, K^\times/K^{\times 2}, K^\times/K^{\times 2})$ and $b_1, b_2, b_3 \in K$;

Case 2: $\lambda = 0$, $(c, \beta, b_3) \in (1, K^\times/K^{\times 4}, 1) \coprod (1, K^\times/K^{\times 2}, 0)$ and $b_1, b_2, \in K$;

Case 3: $\beta = \lambda = 0$, $(c, b_2, b_3) \in (1, 1, K^\times/K^{\times 3}) \coprod \{(1,1,0), (1,0,0)\}$ and $b_1 \in K$;

Case 4: $c = \beta = \lambda = 0$, $(b_1, b_2, b_3) \in (1, 1, K^\times/K^{\times 2}) \coprod \{(1,1,0), (1,0,0), (0,0,0)\}$.

If, in addition, $\sqrt{K} \subseteq K$ and $\sqrt[3]{K} \subseteq K$ then (up G_3') there are the following 11 types of the quantum bi-quadratic algebras:

1. *$c = \beta = \lambda = 1$ and $b_1, b_2, b_3 \in K$.*
2. *$c = \beta = 1$, $\lambda = 0$ and $b_1, b_2 \in K$, $b_3 \in \{0, 1\}$.*
3. *$c = 1$, $\beta = \lambda = 0$ and $b_1 \in K$, $(b_2, b_3) \in \{(1,1), (1,0), (0,0)\}$.*
4. *$c = \beta = \lambda = 0$ and $(b_1, b_2, b_3) \in \{(1,1,1), (1,1,0), (1,0,0), (0,0,0)\}$.*

Remark. If one applies an odd permutation of S_3 then the quantum parameter q becomes q^{-1} but for even permutations of S_3 it does not change.

- **Case 2:** $q_1 - q_3 = 0$ **and** $1 - q_1q_2 \neq 0$.

Theorem 1.8 *Suppose that all $q_i \neq 1$, $q_1 - q_3 = 0$ and $1 - q_1q_2 \neq 0$. Then (up to G_3'):*

$$x_2x_1 = q_1x_1x_2, \quad x_3x_1 = q_2x_1x_3 + \beta x_2 + b_2 \ \text{ and } \ x_3x_2 = q_1x_2x_3$$

where $\beta, b_2 \in \{0, 1\}$.

- **Case 3:** $q_1 - q_3 \neq 0$**,** $1 - q_1q_2 = 0$ **and** $1 - q_2q_3 \neq 0$.

Notice that the third condition (i.e., $1 - q_2q_3 \neq 0$) follows from the first two.

Theorem 1.9 *Suppose that all $q_i \neq 1$, $q_1 - q_3 \neq 0$ and $1 - q_1q_2 = 0$ ($\Rightarrow 1 - q_2q_3 \neq 0$). Then (up to G_3'):*

$$x_2x_1 = q_1x_1x_2, \quad x_3x_1 = q_1^{-1}x_1x_3 \ \text{ and } \ x_3x_2 = q_3x_2x_3 + \lambda x_1 + b_3$$

where $\lambda, b_3 \in \{0, 1\}$.

- **Case 4:** $q_1 - q_3 \neq 0$**,** $1 - q_1q_2 \neq 0$ **and** $1 - q_2q_3 = 0$.

Theorem 1.10 *Suppose that all $q_i \neq 1$, $q_1 - q_3 \neq 0$, $1 - q_1q_2 \neq 0$ and $1 - q_2q_3 = 0$. Then (up to G_3'):*

$$x_2x_1 = q_1x_1x_2 + cx_3 + b_1, \quad x_3x_1 = q_2x_1x_3 \ \text{ and } \ x_3x_2 = q_2^{-1}x_2x_3$$

where $c, b_1 \in \{0, 1\}$.

- **Case 5:** $q_1 - q_3 \neq 0$**,** $1 - q_1q_2 \neq 0$ **and** $1 - q_2q_3 \neq 0$.

Theorem 1.11 *Suppose that all $q_i \neq 1$, $q_1 - q_3 \neq 0$, $1 - q_1q_2 \neq 0$ and $1 - q_2q_3 \neq 0$. Then $\mathbb{A} = 0$ and $\mathbb{B} = 0$, i.e., $A \simeq \mathbb{A}_Q^3$.*

Proofs of the above results are given in Section 6.

2 Classification of bi-quadratic algebras on 2 generators

It is an easy exercise to give a classification (up to isomorphism) of bi-quadratic algebras on 2 generators (Theorem 2.1) for $D = K$.

Definition. Let K be a field, $q \in K^{\times} := K\backslash\{0\}$, and $a, b, c, \in K$. Then the algebra

$$A = K[x_1, x_2; q, a, b, c] := K\langle x_1, x_2 \,|\, x_2x_1 = qx_1x_2 + ax_1 + bx_2 + c\rangle \tag{24}$$

is a bi-quadratic algebra on 2 generators.

The algebra $A = K[x_1][x_2; \sigma, \delta]$ is a skew polynomial algebra where $\sigma(x_1) = qx_1 + b$ and $\delta(x_1) = ax_1 + c$. So, the algebra A is a Noetherian domain with PBW basis, $A = \bigoplus_{i,j\in\mathbb{N}} Kx_1^ix_2^j = \bigoplus_{i,j\in\mathbb{N}} Kx_2^ix_1^j$ and the scalars a, b, c are arbitrary.

A $\mathbb{Z}$-homomorphism $f : R \to R$ is called a **locally nilpotent map** if $R = \bigcup_{i\geq 1} \ker(f^i)$. An element r of R is called a **locally nilpotent element** if the inner derivation of R,

$$\mathrm{ad}_r : R \to R, \quad x \mapsto rx - xr$$

is a locally nilpotent map. For a K-algebra A, we denote by $\mathcal{N}(A)$ the monoid of all regular normal elements of A, $\overline{\mathcal{N}}(A) := \mathcal{N}(A)/K^{\times}$ (the factor monoid modulo its central subgroup $K^{\times}$), and

$$\widetilde{\mathcal{N}}(A) = \mathcal{N}(A)/Z(A)^{\mathrm{reg}}$$

(the factor monoid modulo the relation defined by the submonoid $Z(A)^{\mathrm{reg}} := \mathcal{N}(A) \cap Z(A)$ of regular central elements of A, i.e., $u \sim v$ if $uZ(A)^{\mathrm{reg}} \cap vZ(A)^{\mathrm{reg}} \neq \emptyset$).

Theorem 2.1 *Up to isomorphism, there are only five bi-quadratic algebras on 2 generators:*

1. *The polynomial algebra* $K[x_1, x_2]$,
2. *The Weyl algebra* $A_1 = \langle x_1, x_2 \,|\, x_2x_1 - x_1x_2 = 1\rangle$,
3. *The universal enveloping algebra of the Lie algebra* $\mathfrak{n}_2 = \langle x_1, x_2 \,|\, [x_2, x_1] = x_1\rangle$, $U(\mathfrak{n}_2) = K\langle x_1, x_2 \,|\, x_2x_1 - x_1x_2 = x_1\rangle$,
4. *The quantum plane* $\mathbb{A}_q^2 = \langle x_1, x_2 \,|\, x_2x_1 = qx_1x_2\rangle$ *where* $q \in K\backslash\{0, 1\}$, *and*
5. *The quantum Weyl algebra* $A_1(q) = \langle x_1, x_2 \,|\, x_2x_1 - qx_1x_2 = 1\rangle$ *where* $q \in K\backslash\{0, 1\}$.

Proof. Let $A = K[x_1, x_2; q, a, b, c]$ be a bi-quadratic algebra on 2 generators.

Step 1. *The algebra A is isomorphic to one of the algebras in statements 1–5*:

(i) *Suppose that* $q = 1$.
If $a = b = c = 0$ then $A \simeq K[x_1, x_2]$.
If $a = b = 0$ and $c \neq 0$ then $A \simeq A_1$.
If $(a, b) \neq (0, 0)$ then $A \simeq U(\mathfrak{n}_2)$. If, say $a \neq 0$, then

$$[a^{-1}x_2, ax_1 + bx_2 + c] = ax_1 + bx_2 + c,$$

and the result follows.

(ii) *Suppose that* $q \neq 1$. By Lemma 3.1, we can assume that $a = b = 0$, i.e., $x_2x_1 = qx_1x_2 + c$. Up to the change of the variables $x_1' = \lambda x_1$, $x_2' = x_2$ (where $\lambda \in K^{\times}$) we can assume that $c \in \{0, 1\}$. If $c = 0$ (resp., $c = 1$) then $A \simeq \mathbb{A}_q^2$ (resp., $A \simeq A_1(q)$).

Step 2. *The algebras in statements 1–5 are not isomorphic*: The polynomial algebra $K[x_1, x_2]$ is the only commutative algebra in statements 1–5. So, it remains to show that the algebras in statements 2–5 are not isomorphic.

If char $(K) = 0$ then the Weyl algebra A_1 is the only *simple* algebra in statements 2–5. If char $(K) = p > 0$ then the Weyl algebra A_1 is the only algebra in statements 2–5 such that all normal elements are central. So, it remains to show that the algebras in statements 3–5 are not isomorphic. Let $U = U(\mathfrak{n}_2)$.

(a)

$$Z(U)^{\rm reg} = \begin{cases} K^\times & \text{if char } K\text{=0,} \\ K[x_1^p, x_2^p]\backslash\{0\} & \text{if char } K\text{=p.} \end{cases}$$

$$\mathcal{N}(U) = \begin{cases} \{Z(U)^{\rm reg} x_1^i \,|\, i \geq 0\} & \text{if char } K\text{=0,} \\ \{Z(U)^{\rm reg} x_1^i \,|\, i = 0, 1, \ldots, p-1\} & \text{if char } K = p. \end{cases}$$

$$\widetilde{\mathcal{N}}(U) \simeq \begin{cases} \mathbb{N} & \text{if char } K\text{=0,} \\ \mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z} & \text{if char } K = p. \end{cases}$$

(b)

$$Z(\mathbb{A}_q^2)^{\rm reg} = \begin{cases} K^\times & \text{if } q \text{ is not a root of 1,} \\ K[x_1^m, x_2^m]\backslash\{0\} & \text{if } q \text{ is a primitive } m\text{'th a root of 1.} \end{cases}$$

$$\mathcal{N}(\mathbb{A}_q^2) = \begin{cases} \{Z(\mathbb{A}_q^2)^{\rm reg} x_1^i x_2^j \,|\, i, j \geq 0\} & \text{if } q \text{ is not a root of 1,} \\ \{Z(\mathbb{A}_q^2)^{\rm reg} x_1^i x_2^j \,|\, i, j = 0, 1, \ldots, m-1\} & \text{if } q \text{ is a primitive } m\text{'th a root of 1.} \end{cases}$$

$$\widetilde{\mathcal{N}}(\mathbb{A}_q^2) \simeq \begin{cases} \mathbb{N}\times\mathbb{N} & \text{if } q \text{ is not a root of 1,} \\ \mathbb{Z}_m \times \mathbb{Z}_m & \text{if } q \text{ is a primitive } m\text{'th a root of 1.} \end{cases}$$

(c)

$$Z(A_1(q))^{\rm reg} = \begin{cases} K^\times & \text{if } q \text{ is not a root of 1,} \\ K[x_1^m, x_2^m]\backslash\{0\} & \text{if } q \text{ is a primitive } m\text{'th a root of 1.} \end{cases}$$

$$\mathcal{N}(A_1(q))) = \begin{cases} \{Z(A_1(q))^{\rm reg} h^i \,|\, i \geq 0\} & \text{if } q \text{ is not a root of 1,} \\ \{Z(A_1(q))^{\rm reg} h^i \,|\, i = 0, 1, \ldots, m-1\} & \text{if } q \text{ is a primitive } m\text{'th a root of 1.} \end{cases}$$

$$\widetilde{\mathcal{N}}(A_1(q)) \simeq \begin{cases} \mathbb{N} & \text{if } q \text{ is not a root of 1,} \\ \mathbb{Z}_m & \text{if } q \text{ is a primitive } m\text{'th a root of 1.} \end{cases}$$

The element $x_1 \in U$ is a non-central, regular, normal, locally nilpotent element of the algebra U. There is no such an element in the algebras $\mathbb{A}_q^2$ and $A_1(q)$, see (a), (b) and (c). On the other hand, the algebras $\mathbb{A}_q^2$ and $A_1(q)$ are not isomorphic, see (b) and (c) above. $\square$

3 Bi-quadratic algebras on 3 generators

The aim of this section is to give explicit consistency conditions for 3-generated bi-quadratic algebras with PBW basis (Theorem 1.3). Many algebras from the class BQA(3) of bi-quadratic algebras on 3-generators over K are generalized Weyl algebras and diskew polynomial algebras. In this section, we collect results about these two classes of algebras that are used in the proofs of this paper.

Proof of Theorem 1.1. The theorem follows at once from the Diamond Lemma since for the degree-by-lexicographic ordering the ambiguities are $\{x_k x_j x_i \,|\, \leq i < j < k \leq n\}$ and for each of them there are exactly two different ways how to resolve them, see (3) and (4). $\square$

Proof of Theorem 1.2. $(\Rightarrow)$ The algebra A is a Lie algebra $(A, [\cdot,\cdot])$ where $[a,b] = ab - ba$. By Theorem 1.1, $A = \bigoplus_{\alpha\in\mathbb{N}^n} Dx^\alpha$, and so the vector space $\mathcal{G} := \bigoplus_{i=0}^n Kx_i$ is a Lie subalgebra of A where $x_0 = 1$. Now, by Theorem 1.1, $A \simeq D \otimes_K (U(\mathcal{G})/(Z-1))$ where $Z = x_0$.

$(\Leftarrow)$ This implication is obvious. $\square$

Proof of Theorem 1.3. In view of Theorem 1.1, we have to equate the coefficients of the only ambiguity $x_3x_2x_1$:

$$\begin{aligned}
(x_3x_2)x_1 &= (q_3x_2x_3 + \lambda x_1 + \mu x_2 + \nu x_3 + b_3)x_1\\
&= q_3x_2(q_2x_1x_3 + \alpha x_1 + \beta x_2 + \gamma x_3 + b_2)\\
&+ \lambda x_1^2 + \mu(q_1x_1x_2 + ax_1 + bx_2 + cx_3 + b_1) + \nu(q_2x_1x_3 + \alpha x_1 + \beta x_2 + \gamma x_3 + b_2) + b_3x_1\\
&= q_3q_2(q_1x_1x_2 + ax_1 + bx_2 + cx_3 + b_1)x_3 + q_3\alpha(q_1x_1x_2 + ax_1 + bx_2 + cx_3 + b_1)\\
&+ q_3\beta x_2^2 + q_3\gamma x_2x_3 + q_3b_2x_2\\
&= q_3q_2q_1x_1x_2x_3 + (q_1q_3\alpha + q_1\mu)x_1x_2 + (q_2q_3a + q_2\nu)x_1x_3 + (q_2q_3b + q_3\gamma)x_2x_3 + \lambda x_1^2\\
&+ q_3\beta x_2^2 + q_2q_3cx_3^2 + (q_3\alpha a + \mu a + b_3 + \nu\alpha)x_1 + (q_3\alpha b + q_3b_2 + \mu b + \nu\beta)x_2\\
&+ (q_3\alpha c + \mu c + \nu\gamma + q_2q_3b_1)x_3 + q_3\alpha b_1 + \mu b_1 + \nu b_2.
\end{aligned}$$

$$\begin{aligned}
x_3(x_2x_1) &= x_3(q_1x_1x_2 + ax_1 + bx_2 + cx_3 + b_1)\\
&= q_1(q_2x_1x_3 + \alpha x_1 + \beta x_2 + \gamma x_3 + b_2)x_2 + b(q_3x_2x_3 + \lambda x_1 + \mu x_2 + \nu x_3 + b_3)\\
&+ a(q_2x_1x_3 + \alpha x_1 + \beta x_2 + \gamma x_3 + b_2) + cx_3^2 + b_1x_3\\
&= q_1q_2x_1(q_3x_2x_3 + \lambda x_1 + \mu x_2 + \nu x_3 + b_3) + \alpha q_1x_1x_2 + q_1\beta x_2^2\\
&+ q_1\gamma(q_3x_2x_3 + \lambda x_1 + \mu x_2 + \nu x_3 + b_3) + q_1b_2x_2\\
&= q_1q_2q_3x_1x_2x_3 + (q_1q_2\mu + \alpha q_1)x_1x_2 + (q_1q_2\nu + aq_2)x_1x_3\\
&+ (q_1q_3\gamma + q_3b)x_2x_3 + q_1q_2\lambda x_1^2 + q_1\beta x_2^2 + cx_3^2 + (q_1q_2b_3 + q_1\gamma\lambda + b\lambda + a\alpha)x_1\\
&+ (q_1\gamma\mu + q_1b_2 + b\mu + a\beta)x_2 + (q_1\gamma\nu + b\nu + a\gamma + b_1)x_3 + q_1\gamma b_3 + bb_3 + ab_2.
\end{aligned}$$

Now, equating the coefficients of the monomials x_1x_2, x_1x_3 and x_2x_3 we obtain the equations (11), (12) and (13) that are multiplied by the nonzero scalars q_1, q_2 and q_3, respectively. So, these equations are equivalent to the equations (11), (12) and (13), respectively. Equating the coefficients of the monomials x_1^2, x_2^2 and x_3^2 we obtain the equations (14), (15) and (16), respectively. Finally, equating the coefficients of the elements x_1, x_2, x_3 and 1, we obtain the equations (17), (18), (19) and (20), respectively. $\square$

4 main classes of bi-quadratic algebras A**.** From now on let

$$A = K[x_1, x_2, x_3; Q, \mathbb{A}, \mathbb{B}]$$

be a bi-quadratic algebra on 3-generators where $Q = (q_1, q_2, q_3) \in K^{\times 3}$ and (7) – (10) hold. In order to classify bi-quadratic algebra on 3 generators we have to consider the following 4 classes (in view of the S_3-action):

1. $q_1 = q_2 = q_3 = 1$,
2. $q_1 \neq 1$, $q_2 = q_3 = 1$,
3. $q_1 \neq 1$, $q_2 \neq 1$, $q_3 = 1$,
4. $q_1 \neq 1$, $q_2 \neq 1$, $q_3 \neq 1$.

Explicit descriptions of the algebras in each of the four cases are given in Section 7, Section 4, Section 5 and Section 6, respectively. The next lemma is the starting point of analysis of the cases 2–4.

Lemma 3.1 *Suppose that $q_1 \neq 1$. Then by making the change of the variables,*

$$x_1' = x_1 - \frac{b}{1-q_1}, \ \ x_2' = x_2 - \frac{a}{1-q_1}, \ \ x_3' = x_3,$$

we can assume that $a = b = 0$, and then the equations (12) and (13) are equivalent to the equalities $\nu = 0$ and $\gamma = 0$, respectively. If, in addition, $q_3 = 1$ then (15) is equivalent to $\beta = 0$.

Proof. By making the above change of variables, the relation (8) is equal to the relation $x_2'x_1' - q_1x_1'x_2' = cx_3' + b + \frac{ab}{1-q_1}$. The rest is obvious. $\square$

Many 3-generated bi-quadratic algebras are generalized Weyl algebras and diskew polynomial rings. Below, we collected necessary results that are used in proofs.

(Classical) generalized Weyl algebras $D(\sigma, a)$ with central element a.
Definition, [1]–[4]. Let D be a ring, σ be a ring automorphism of D, a is a *central* element of D. The **(classical) generalized Weyl algebra** of rank 1 (GWA, for short) $D(\sigma, a) = D[x, y; \sigma, a]$ is a ring generated by the ring D and two elements x and y that are subject to the defining relations:

$$xd = \sigma(d)x \ \text{ and } \ yd = \sigma^{-1}(d)y \ \text{ for all } \ d \in D, \ \ yx = a \ \text{ and } \ xy = \sigma(a). \tag{25}$$

The ring D is called the *base ring* of the GWA. The automorphism σ and the element a are called the *defining automorphism* and the *defining element* of the GWA, respectively.

This is an experimental fact that many popular algebras of small Gelfand-Kirillov dimension are GWAs: the first Weyl algebra A_1 and its quantum analogue, the *quantum plane*, the *quantum sphere*, $Usl(2)$, $U_qsl(2)$, the *Heizenberg* algebra and its quantum analogues, the 2×2 quantum matrices, the *Witten's* and *Woronowicz's* deformations, Noetherian down-up algebras, etc.

The generalized Weyl algebras were introduced by myself in 1987 when I was an algebra post-graduate student at the Taras Shevchenko National University of Kyiv, the Department of Algebra and Mathematical Logic, and they were the subject of my PhD thesis "Generalized Weyl algebras and their representations" submitted at the end of 1990 (defended at the beginning of 1991).

Generalized Weyl algebras with two endomorphisms and a left normal element a. *Definition, [6, 5].* Let D be a ring, σ and τ be ring endomorphisms of D, and an element $a \in D$ be such that

$$\tau\sigma(a) = a, \ \ ad = \tau\sigma(d)a \ \text{ and } \ \sigma(a)d = \sigma\tau(d)\sigma(a) \ \text{ for all } \ d \in D. \tag{26}$$

The **generalized Weyl algebra** (GWA) of rank 1, $A = D(\sigma, \tau, a) = D[x, y; \sigma, \tau, a]$, is a ring generated by D, x and y subject to the defining relations:

$$xd = \sigma(d)x \ \text{ and } \ yd = \tau(d)y \ \text{ for all } \ d \in D, \ \ yx = a \ \text{ and } \ xy = \sigma(a). \tag{27}$$

The ring D is called the *base ring* of the GWA A. The endomorphisms σ, τ and the element a are called the *defining endomorphisms* and the *defining element* of the GWA A, respectively. By (26), the elements a and $\sigma(a)$ are left normal in D. An element d of a ring D is called *left normal* (resp., *normal*) if $dD \subseteq Dd$ (resp., $Dd = dD$). To distinguish 'old' GWAs from the 'new' ones the former are called the *classical* GWAs. Every classical GWA is a GWA as the conditions in (26) trivially hold if a is central and $\tau = \sigma^{-1}$.

Diskew polynomial rings. *Definition, [6, 5].* Let D be a ring, σ and τ be its ring endomorphisms, ρ and b be elements of D such that, for all $d \in D$,

$$\sigma\tau(d)\rho = \rho\tau\sigma(d) \ \text{ and } \ \sigma\tau(d)b = bd, \tag{28}$$

The **diskew polynomial ring** (DPR) $E := D(\sigma, \tau, b, \rho) := D[x, y; \sigma, \tau, b, \rho]$ is a ring generated by D, x and y subject to the defining relations:

$$xd = \sigma(d)x \ \text{ and } \ yd = \tau(d)y \ \text{ for all } \ d \in D, \ \ xy - \rho yx = b. \tag{29}$$

By (28), b is a left normal element of D. If $\tau\sigma$ (resp., $\sigma\tau$) is an epimorphism then ρ is a left (resp., tight) normal element of D.

Diskew polynomial rings are GWAs when ρ is a unit. If the element ρ is a unit in D then every diskew polynomial ring is a generalized Weyl algebra, Theorem 3.2, where $\omega_\rho : D \to D$, $d \mapsto \rho d \rho^{-1}$ is the inner automorphism of D that is determined by the unit ρ.

Theorem 3.2 *([6, 5]) Let $E = D[x, y; \sigma, \tau, b, \rho]$ be a diskew polynomial ring. Suppose that ρ is a unit in D. Then x and y are left regular elements of E and the ring $E = \mathcal{D}[x, y; \sigma, \tau, a = h]$ is a GWA with base ring $\mathcal{D} := D[h; \tau\sigma]$ which is a skew polynomial ring, σ and τ are ring endomorphisms of $\mathcal{D}$ that are extensions of the ring endomorphisms σ and τ of D, respectively, defined by the rule $\sigma(h) = \rho h + b$ and $\tau(h) = \tau(\rho^{-1})(h - \tau(b))$. In particular, $\tau\sigma(h) = h$ and $\sigma\tau(h) = \omega_\rho(h) = \rho\tau\sigma(\rho^{-1})h$. Furthermore, $\sigma\tau = \omega_\rho\tau\sigma$ in $\mathcal{D}$.*

The canonical left normal element C of a diskew polynomial ring. Theorem 3.3 is a criterion for an element $C = h + \alpha$ (where $\alpha \in D$) to be a left normal element in E.

Theorem 3.3 *([6, 5]) Let $E = D[x, y; \sigma, \tau, b, \rho]$ be a diskew polynomial ring such that ρ is a unit, $\mathcal{D} = D[h; \nu = \tau\sigma]$ and $C = h + \alpha$ where $h = yx$ and $\alpha \in D$. The following statements are equivalent:*

1. *The element C is left normal in E.*
2. *$\rho\alpha - \sigma(\alpha) = b$, $\nu(\alpha) = \alpha$ and $\alpha d = \nu(d)\alpha$ for all elements $d \in D$.*

   If one of the equivalent conditions holds then $[h, C] = 0$ and

   (a) *$C = \rho^{-1}(xy + \sigma(\alpha))$, $xC = \rho Cx$ and $yC = \tau(\rho^{-1})Cy$.*

   (b) *$E \simeq D[C; \nu][x, y; \sigma, \tau, a := C - \alpha]$ is a GWA where $\sigma(C) = \rho C$ and $\tau(C) = \tau(\rho^{-1})C$.*

   (c) *The element C is a left normal, left regular element of E and $E/(C) \simeq D[x, y; \sigma, \tau, -\alpha]$ is a GWA.*

   (d) *The element C is a normal element in E iff $\mathrm{im}(\nu) = D$.*

   (e) *The element C is regular iff C is right regular iff $\ker(\nu) = 0$.*

   (f) *The element C is a normal, regular element iff ν is an automorphism of D.*

A more general situation when E is a GWA is described below.

Theorem 3.4 *([6, 5]) Let $E = D[x, y; \sigma, \tau, b, \rho]$ be a diskew polynomial ring such that ρ is a unit and $\mathcal{D} = D[h; \nu = \tau\sigma]$ where $h = yx$. The following statements are equivalent:*

1. *There exists an element $C = h + \alpha \in \mathcal{D}$, where $\alpha \in D$, such that $Cd = \nu(d)C$ for all elements $d \in D$ and $\sigma(C) = \rho C$.*
2. *There is an element $\alpha \in D$ such that $\rho\alpha - \sigma(\alpha) = b$ and $\alpha d = \nu(d)\alpha$ for all elements $d \in D$.*

   If one of the equivalent conditions holds then $[h, C] = (\nu(\alpha) - \alpha)C$ and

   (a) *$C = \rho^{-1}(xy + \sigma(\alpha))$, $xC = \rho Cx$ and $yC = \tau(\rho^{-1})(C + \nu(\alpha) - \alpha)y$.*

   (b) *$E \simeq D[C; \nu][x, y; \sigma, \tau, a := C - \alpha]$ is a GWA where $\sigma(C) = \rho C$ and $\tau(C) = \tau(\rho^{-1})(C + \nu(\alpha) - \alpha)$. Furthermore, $\tau\sigma(C) = C + \nu(\alpha) - \alpha$ and $\sigma\tau(C) = \sigma\tau(\rho^{-1})(\rho C + \sigma(\nu(\alpha) - \alpha))$.*

The canonical central element C of a diskew polynomial ring (under certain conditions). The next corollary is a criterion for an element $C = h + \alpha$ (where $\alpha \in D$) to be a central element in E. It follows straightaway from Theorem 3.3.

Corollary 3.5 *([6, 5]) Let $E = D[x, y; \sigma, \tau, b, \rho]$ be a diskew polynomial ring such that ρ is a unit, $\mathcal{D} = D[h; \nu = \tau\sigma]$ and $C = h + \alpha$ where $h = yx$ and $\alpha \in D$. The following statements are equivalent.*

1. The element C is a central element of E.

2. $\rho = 1$, $\nu = 1$, $\alpha - \sigma(\alpha) = b$, and the element α belongs to the centre of D.

If one of the equivalent conditions holds then

(a) $C = xy + \sigma(\alpha)$.

(b) $E \simeq D[C][x, y; \sigma, \tau, a := C - \alpha]$ is a GWA where $\sigma(C) = C$ and $\tau(C) = C$.

(c) The element C is a regular element of E.

4 The bi-quadratic algebras with $q_1 \neq 1$, $q_2 = q_3 = 1$

The aim of this section is to prove Theorem 1.5 that gives an explicit description of bi-quadratic algebras with $q_1 \neq 1$, $q_2 = q_3 = 1$.

Proof of Theorem 1.5 . Since $q_1 \neq 1$ and $q_3 = 1$, we can assume that $a = b = \nu = \gamma = \beta = 0$, by Lemma 3.1. Since $q_1 \neq 1$ and $q_2 = 1$, we must have $1 - q_1 q_2 = 1 - q_1 \neq 0$. Then the conditions (11), (14) and (16) can be written, respectively, as follows

$$0 = 0, \quad \lambda = 0 \text{ and } 0 = 0.$$

Then the conditions (17)–(20) can be written, respectively, as follows: $b_3 = 0$, $b_2 = 0$,

$$(\mu + \alpha)c = 0 \text{ and } (\mu + \alpha)b_1 = 0. \tag{30}$$

Then the defining relations of the algebra A take the form

$$\begin{aligned} x_2x_1 &= q_1x_1x_2 + cx_3 + b_1, \\ x_3x_1 &= x_1(x_3 + \alpha), \\ x_3x_2 &= x_2(x_3 + \mu), \end{aligned}$$

where the elements c, b_1, α and μ satisfy (30).

1. *Suppose that* $\mu + \alpha \neq 0$. Then the conditions in (30) are equivalent to $c = b_1 = 0$, and the defining relations of A take the form

$$x_2x_1 = q_1x_1x_2, \quad x_3x_1 = x_1(x_3 + \alpha) \text{ and } x_3x_2 = x_2(x_3 + \mu)$$

where $\alpha, \mu \in K$. By making the change of variables $x_1' = x_1$, $x_2' = x_2$, $x_3' = \omega x_3$ (where $\omega \in K^\times$), we can assume that

$$(\alpha, \mu) \in \{(1, \mu'), (0, 1) \,|\, \mu' \in K \backslash \{-1\}\}$$

and the first part of statement 1 follows. The second part follows from Theorem 3.2.

The statements (a)–(c) are obvious.

2. *Suppose that* $\mu + \alpha = 0$. Then $\mu = -\alpha$, the conditions in (30) hold automatically, and the defining relations of the algebra A are as follows

$$x_2x_1 = q_1x_1x_2 + cx_3 + b_1, \quad x_3x_1 = x_1(x_3 + \alpha) \text{ and } x_3x_2 = x_2(x_3 - \alpha)$$

where $c, \alpha, b_1 \in K$. There are three case:

(a) $\alpha = 0$,

(b) $\alpha \neq 0$ and $c\alpha = b_1$, and

(c) $\alpha \neq 0$ and $c\alpha \neq b_1$.

The cases (a)–(c) correspond to the cases (a)–(c) of statement 2, respectively.

(a) If $\alpha = 0$, $c = 0$ and $b_1 \neq 0$ then by making the change of the variables

$$x_1' = b_1^{-1} x_1, \;\; x_2' = x_2, \;\; x_3' = x_3$$

the triple $(\alpha, c, b_1) = (0, 0, b_1)$ is transformed into the triple $(0, 0, 1)$.

If $\alpha = 0$, $c \neq 0$ and $b_1 \neq 0$ (resp., $b_1 = 0$) then by making the change of the variables

$$x_1' = b_1^{-1} x_1, \;\; x_2' = x_2, \;\; x_3' = b_1^{-1} c x_3$$

(resp., $x_1' = x_1$, $x_2' = x_2$, $x_3' = cx_3$) the triple $(\alpha, c, b_1) = (0, c, b_1)$ is transformed into the triple $(0, 1, 1)$ (resp., $(0, 1, 0)$).

(b) Suppose that $\alpha \neq 0$ and $c\alpha = b_1$. There are two cases:
(b1) $c = 0$ and $b_1 = 0$, and
(b2) $c \neq 0$ and $b_1 \neq 0$.
In the case (b1) (resp., (b2)), the substitution

$$x_1' = x_1, \;\; x_2' = x_2, \;\; x_3' = \alpha^{-1} x_3$$

(resp., $x_1' = (c\alpha)^{-1} x_1$, $x_2' = x_2$, $x_3' = \alpha^{-1} x_3$) transforms the triple (α, c, b_1) into the triple $(1, 0, 0)$ (resp., $(1, 1, 1)$).

(c) Suppose that $\alpha \neq 0$ and $c\alpha \neq b_1$. There are two cases:
(c1) $c = 0$, and
(c2) $c \neq 0$.
If $c = 0$ then $b_1 \neq 0$ and the substitution

$$x_1' = b_1^{-1} x_1, \;\; x_2' = x_2, \;\; x_3' = \alpha^{-1} x_3$$

transforms the triple $(\alpha, c = 0, b_1)$ into the triple $(1, 0, 1)$.

If $c \neq 0$ then $b_1 \neq c\alpha$ and the substitution

$$x_1' = (c\alpha)^{-1} x_1, \;\; x_2' = x_2, \;\; x_3' = \alpha^{-1} x_3$$

transforms the triple (α, c, b_1) into the triple $(1, 1, b_1)$ where $b_1 \in K \backslash \{1\}$.

So, the first part of statement 2 follows. The second part follows from Theorem 3.2. □

5 The bi-quadratic algebras with $q_1 \neq 1$, $q_2 \neq 1$, $q_3 = 1$

The aim of this section is to prove Theorem 1.6 that gives an explicit description of bi-quadratic algebras with $q_1 \neq 1$, $q_2 \neq 1$ and $q_3 = 1$.

Proof of Theorem 1.6. Since $q_1 \neq 1$ and $q_3 = 1$, we can assume that $a = b = \nu = \gamma = \beta = 0$, by Lemma 3.1. Then the equalities (11), (16), (18), (19), and (20) can be written respectively as

$$\mu = 0, \;\; c = 0, \;\; b_2 = 0, \;\; b_1 = 0 \;\; \text{and} \;\; 0 = 0.$$

Then the equalities (14) and (17) are equal respectively to the equalities

$$(1 - q_1 q_2)\lambda = 0 \;\; \text{and} \;\; (1 - q_1 q_2) b_3 = 0,$$

and the defining relations are

$$x_2 x_1 = q_1 x_1 x_2, \;\; x_3 x_1 = x_1 (q_2 x_3 + \alpha) \;\; \text{and} \;\; x_3 x_2 = x_2 x_3 + \lambda x_1 + b_3.$$

Using $x_3' = x_3 - \frac{\alpha}{1-q_2}$, we can assume that $\alpha = 0$.

1. *Suppose that* $1 - q_1q_2 \neq 0$. Then $\lambda = b_3 = 0$ and statement 1 follows.

2. *Suppose that* $1 - q_1q_2 = 0$. Then λ and b_3 are arbitrary scalars. Up to the change of the variables

$$x_1' = \lambda_1 x_1, \ \ x_2' = \lambda_2 x_2, \ \ x_3' = x_3 \ \ (\text{where } \ \lambda_i \in K^\times),$$

the elements λ and b_3 belong to the set $\{0, 1\}$ and the first part of statement 2 follows.

Clearly, the algebra A is the diskew polynomial algebra as in statement 2. If $b_3 = 0$ then the element

$$\alpha' = \lambda(1 - q_1^{-1})^{-1} x_1$$

is a solution to the equation $\alpha' - \sigma(\alpha') = b$ (where $b = \lambda x_1$). By Corollary 3.5, the element $C = h - \alpha'$ is a central element of A and the algebra A is the GWA as in statement 2. $\Box$

6 The bi-quadratic algebras with $q_1 \neq 1$, $q_2 \neq 1$, $q_3 \neq 1$

In this section we assume that $q_1 \neq 1$, $q_2 \neq 1$, $q_3 \neq 1$. So, this class of algebras is closed under the action of the symmetric group S_3 (under permutation of the canonical generators x_1, x_2, x_3). We split the class of algebras into a disjoint union of subclasses based on the condition that elements of the set $\{q_1 - q_3, 1 - q_1q_2, 1 - q_2q_3\}$ are either zero or not. Potentially, there are $2 \times 2 \times 2 = 8$ subclasses but in fact, there are only 5 since the condition that $q_1 - q_3 = 0$ implies that $1 - q_1q_2 = 1 - q_2q_3$; and the conditions $1 - q_1q_2 = 0$ and $1 - q_2q_3 = 0$ imply that $q_1 - q_3 = 0$.

- Case 1: $q_1 - q_3 = 0$, $1 - q_1q_2 = 0$.
- Case 2: $q_1 - q_3 = 0$, $1 - q_1q_2 \neq 0$.
- Case 3: $q_1 - q_3 \neq 0$, $1 - q_1q_2 = 0$, $1 - q_2q_3 \neq 0$.
- Case 4: $q_1 - q_3 \neq 0$, $1 - q_1q_2 \neq 0$, $1 - q_2q_3 = 0$.
- Case 5: $q_1 - q_3 \neq 0$, $1 - q_1q_2 \neq 0$, $1 - q_2q_3 \neq 0$.

The next lemma is often used in proofs.

Lemma 6.1 *Suppose that R is a K-algebra, $q_1, q_2 \in K$ and $q_1 \neq 1$. When we make the substitution $x = x' + \dfrac{1}{1-q_1}$ into the equalities below that hold in the algebra R (where $x, y, z, z', u, u' \in R$):*

$$yx = (q_1x+1)y + x + \frac{1}{q_1 - 1}, \ zx = \left(q_2x + \frac{1-q_2}{1-q_1}\right)z, \ xz' = z'\left(q_2x + \frac{1-q_2}{1-q_1}\right), \ ux = (q_1x+1)u,$$

and $xu' = u'(q_1x + 1)$, we obtain respectively the equalities:

$$yx' = x'(q_1y + 1), \ zx' = q_2x'z, \ x'z' = q_2z'x', \ ux' = q_1x'u \ \text{ and } \ x'u' = q_1u'x'.$$

Proof. The first equality follows from

$$yx' + \frac{y}{1-q_1} = yx = (q_1x+1)y + x + \frac{1}{q_1 - 1} = \left(q_1x' + \frac{q_1}{1-q_1} + 1\right)y + x_1' = x_1'(q_1y + 1) + \frac{y}{1-q_1}.$$

The second equality follows from

$$zx' + \frac{z}{1-q_1} = zx = \left(q_2x + \frac{1-q_2}{1-q_1}\right)z = \left(q_2x' + \frac{q_2}{1-q_1} + \frac{1-q_2}{1-q_1}\right)z = q_2x'z + \frac{z}{1-q_1}.$$

The third equality can be proven in the same manner as the second. The fourth equality follows from

$$ux' + \frac{u}{1-q_1} = ux = (q_1x+1)u = \left(q_1x' + \frac{q_1}{1-q_1} + 1\right)u = q_1x'u + \frac{u}{1-q_1}.$$

The fifth equality is proven in a similar way as the fourth. □

Since $q_1 \neq 1$, we can assume that $a = b = \nu = \gamma = 0$, by Lemma 3.1. Then the equation (11) can be written as

$$\mu = \frac{1-q_3}{1-q_2}\alpha.$$

In particular, the defining relations of the algebra A are

$$x_2x_1 = q_1x_1x_2 + cx_3 + b_1, \quad x_3x_1 = x_1(q_2x_3+\alpha) + \beta x_2 + b_2, \quad x_3x_2 = x_2\left(q_3x_3 + \frac{1-q_3}{1-q_2}\alpha\right) + \lambda x_1 + b_3.$$

If $\alpha \neq 0$ then using Lemma 6.1 we can assume that $\alpha = 0$. In more detail, first by making the substitution $x_3' = \alpha^{-1}x_3$, we can assume that $\alpha = 1$. Then applying Lemma 6.1 where $x_3 = x_3' + \frac{1}{1-q_2}$ we obtain the result.

Summarizing, we can assume that

$$a = b = \alpha = \gamma = \mu = \nu = 0, \tag{31}$$

the equalities (11)–(13) automatically hold, and the defining relations are

$$x_2x_1 = q_1x_1x_2 + cx_3 + b_1, \tag{32}$$

$$x_3x_1 = q_2x_1x_3 + \beta x_2 + b_2, \tag{33}$$

$$x_3x_2 = q_3x_2x_3 + \lambda x_1 + b_3. \tag{34}$$

6.1 Case 1: $q_1 - q_3 = 0$ and $1 - q_1q_2 = 0$, i.e., $q := q_1 = q_2^{-1} = q_3 \neq 1$ (the quantum bi-quadratic algebras)

Recall that if $q := q_1 = q_2^{-1} = q_3 \neq 1$ then the bi-quadratic algebra A is called the quantum bi-quadratic algebra and the element q is called the quantum parameter of A. In particular, all elements of the set

$$\{q_1 - q_3, 1 - q_1q_2 = 1 - q_2q_3\}$$

are equal to 0. Then *the equalities (14) –(20) hold automatically.* So, (32)–(34) are the defining relations for the quantum bi-quadratic algebra A where there is no restrictions on the defining constants apart from $q_1 = q_2^{-1} = q_3 \neq 1$. So, the defining relations can be written as in Theorem 1.7.

Classification of orbits of four actions of $\mathbb{T}^3$ on K^3. The 3-dimensional algebraic torus $\mathbb{T}^3 = K^{\times 3}$ acts in four different ways on the affine space K^3: For all $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{T}^3$ and $\xi = (\xi_1, \xi_2, \xi_3) \in K^3$,

Case 1: $\lambda \cdot \xi = (\frac{\lambda_3}{\lambda_1\lambda_2}\xi_1, \frac{\lambda_2}{\lambda_1\lambda_3}\xi_2, \frac{\lambda_1}{\lambda_2\lambda_3}\xi_3)$.

Case 2: $\lambda \cdot \xi = (\frac{\lambda_3}{\lambda_1\lambda_2}\xi_1, \frac{\lambda_2}{\lambda_1\lambda_3}\xi_2, \frac{1}{\lambda_2\lambda_3}\xi_3)$.

Case 3: $\lambda \cdot \xi = (\frac{\lambda_3}{\lambda_1\lambda_2}\xi_1, \frac{1}{\lambda_1\lambda_3}\xi_2, \frac{1}{\lambda_2\lambda_3}\xi_3)$.

Case 4: $\lambda \cdot \xi = (\frac{1}{\lambda_1\lambda_2}\xi_1, \frac{1}{\lambda_1\lambda_3}\xi_2, \frac{1}{\lambda_2\lambda_3}\xi_3)$.

The four actions and the classifications of theirs orbits (Proposition 6.2) are an essential part of the proof of Theorem 1.7. The set

$$K^3 = K^{\times 3} \coprod K^3_{\rm sing}$$

is a disjoint union of two $\mathbb{T}^3$-invariant sets where

$$K^3_{\rm sing} := K^3 \backslash K^{\times 3} = \{(\xi_1, \xi_2, \xi_3) \in K^3 \mid \xi_1\xi_2\xi_3 = 0\}.$$

For each element $\xi = (\xi_1, \xi_2, \xi_3) \in K^3$, $\mathrm{supp}(\xi) := (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{0, 1\}^3$ is called the **support** of ξ where

$$\varepsilon_i = \begin{cases} 1 & \text{if } \xi_i \neq 0, \\ 0 & \text{if } \xi_i = 0. \end{cases}$$

Notice that

$$K^3_{\rm sing} = \coprod_{i=1}^{3} K^3_{{\rm sing},i}$$

is a disjoint union of $\mathbb{T}^3$-invariant subsets $K^3_{{\rm sing},i} := \{\xi \in K^3_{\rm sing} \mid z(\xi) = i\}$ where $z(\xi)$ is the number of zero coordinates of the vector ξ. The set

$$K^3_{{\rm sing},1} = \coprod_{i=1}^{3} K^3_{{\rm sing},1}(i)$$

is a disjoint union of $\mathbb{T}^3$-invariant subsets $K^3_{{\rm sing},1}(i) := \{\xi \in K^3_{{\rm sing},1} \mid \xi_i = 0\}$. For each natural number $n \geq 1$, the image of the group homomorphism

$$\mu_n : K^\times \to K^\times, \;\; a \mapsto a^n$$

is denoted by $K^{\times n}$, and $K^\times/K^{\times n}$ is the factor group of $K^\times$ modulo $K^{\times n}$. Notice that $K^\times = K^{\times n}$ for $n = 2$ (resp., $n = 3$) iff the field K contains all quadratic (resp., cubic) roots of all elements of K.

We denote by $K^3/\mathbb{T}^3 = \{\mathbb{T}^3\xi \mid \in K^3\}$ the set of T^3-orbits in K^3. Clearly,

$$K^3/\mathbb{T}^3 = K^{\times 3}/\mathbb{T}^3 \coprod K^3_{\rm sing}/\mathbb{T}^3 = K^{\times 3}/\mathbb{T}^3 \coprod \coprod_{i=1}^{3} K^3_{{\rm sing},i}.$$

In view of the equality above, Proposition 6.2 classifies orbits of the four actions of $\mathbb{T}^3$ on K^3 (Cases 1–4).

Proposition 6.2 *1. For the action* $\lambda \cdot \xi = (\frac{\lambda_3}{\lambda_1\lambda_2}\xi_1, \frac{\lambda_2}{\lambda_1\lambda_3}\xi_2, \frac{\lambda_1}{\lambda_2\lambda_3}\xi_3)$,

(a) the map

$$K^{\times 3}/\mathbb{T}^3 \to K^\times/K^{\times 2} \times K^\times/K^{\times 2}, \;\; (\xi_1, \xi_2, \xi_3) \mapsto (\xi_1\xi_2 K^{\times 2}, \xi_1\xi_3 K^{\times 2})$$

is a bijection with the inverse $(\rho K^{\times 2}, \eta K^{\times 2}) \mapsto \mathbb{T}^3(1, \rho, \eta)$.

(b) The set $(1, K^\times/K^{\times 2}, 0) \coprod (1, 0, K^\times/K^{\times 2}) \coprod (0, 1, K^\times/K^{\times 2}) \coprod \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0)\}$ *is a set of representatives of the* $\mathbb{T}^3$*-orbits in* $K^3_{\rm sing}/\mathbb{T}^3$.

2. For the action $\lambda \cdot \xi = (\frac{\lambda_3}{\lambda_1\lambda_2}\xi_1, \frac{\lambda_2}{\lambda_1\lambda_3}\xi_2, \frac{1}{\lambda_2\lambda_3}\xi_3)$,

(a) the map

$$K^{\times 3}/\mathbb{T}^3 \to K^\times/K^{\times 4}, \;\; (\xi_1, \xi_2, \xi_3) \mapsto \frac{\xi_2}{\xi_1\xi_3^2} K^{\times 4}$$

is a bijection with the inverse $\rho K^{\times 4} \mapsto \mathbb{T}^3(1, \rho, 1)$.

(b) *The set* $(1, K^\times/K^{\times 2}, 0)\coprod\{(1,0,1),(0,1,1),(1,0,0),(0,1,0),(0,0,1),(0,0,0)\}$ *is a set of representatives of the* $\mathbb{T}^3$*-orbits in* $K^3_{\rm sing}/\mathbb{T}^3$.

3. *For the action* $\lambda\cdot\xi = (\frac{\lambda_3}{\lambda_1\lambda_2}\xi_1, \frac{1}{\lambda_1\lambda_3}\xi_2, \frac{1}{\lambda_2\lambda_3}\xi_3)$,

   (a) *the map*

$$K^{\times 3}/\mathbb{T}^3 \to K^\times/K^{\times 3},\ \ (\xi_1,\xi_2,\xi_3)\mapsto \frac{\xi_3}{\xi_1\xi_2^2}K^{\times 3}$$

   is a bijection with the inverse $\rho K^{\times 3}\mapsto \mathbb{T}^3(1,1,\rho)$.

   (b) *The set* $\{0,1\}^3\backslash\{(1,1,1)\}$ *is a set of representatives of the* $\mathbb{T}^3$*-orbits in* $K^3_{\rm sing}/\mathbb{T}^3$.

4. *For the action* $\lambda\cdot\xi = (\frac{1}{\lambda_1\lambda_2}\xi_1, \frac{1}{\lambda_1\lambda_3}\xi_2, \frac{1}{\lambda_2\lambda_3}\xi_3)$,

   (a) *the map*

$$K^{\times 3}/\mathbb{T}^3 \to K^\times/K^{\times 2},\ \ (\xi_1,\xi_2,\xi_3)\mapsto \frac{\xi_3}{\xi_2\xi_2}K^{\times 2}$$

   is a bijection with the inverse $\rho K^{\times 2}\mapsto \mathbb{T}^3(1,1,\rho)$.

   (b) *The set* $\{0,1\}^3\backslash\{(1,1,1)\}$ *is a set of representatives of the* $\mathbb{T}^3$*-orbits in* $K^3_{\rm sing}/\mathbb{T}^3$.

5. *For each of the four actions above, the map*

$$(K^{\times 3}_{{\rm sing},2}\coprod K^{\times 3}_{{\rm sing},3})/\mathbb{T}^3 \to \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\},\ \ \xi\mapsto {\rm supp}(\xi)$$

   is a bijection with the inverse $v\mapsto \mathbb{T}^3 v$.

Proof. Let $\xi=(\xi_1,\xi_2,\xi_3)\in K^{\times 3}$ and $\lambda=(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{T}^3$.

1(a) $\lambda\cdot\xi\in(1,K^\times,K^\times)$ iff $\frac{\lambda_3}{\lambda_1\lambda_2}\xi_1=1$, and in this case $\lambda\cdot\xi=(1,\xi_1\xi_2\lambda_1^{-2},\xi_1\xi_3\lambda_2^{-2})$; and the statement (a) follows.

(b) If $\xi\in K^3_{\rm sing}$ then at least one of the coordinates of the vector ξ is equal to zero. If at least two coordinates of ξ are equal to zero then the set

$$\{(1,0,0),(0,1,0),(0,0,1)(0,0,0)\}$$

is a set of representatives in this case. It remains to consider the case when exactly one coordinate of ξ is equal to zero. In view of the cyclic permutation symmetry of the action (in Case 1), it suffices to consider the case when $\xi_3=0$. Then $\lambda\cdot\xi\in(1,K^\times,0)$ iff

$$\lambda\cdot\xi=(1,\xi_1\xi_2\lambda_1^{-2},0)$$

(see the case (a)), and the statement (b) follows.

2(a) $\lambda\cdot\xi\in(1,K^\times,1)$ iff $\frac{\lambda_3}{\lambda_1\lambda_2}\xi_1=1$ and $\frac{\xi_3}{\lambda_2\lambda_3}=1$, and in this case

$$\lambda\cdot\xi=(1,\frac{\xi_2}{\xi_1\xi_3^2}\lambda_2^4,1)$$

(since $\frac{\lambda_2\xi_2}{\lambda_1\lambda_3}=\frac{\xi_2}{\xi_1\xi_3^2}\lambda_2^4$, use the two equalities); and the statement (a) follows.

(b) If $\xi\in K^3_{{\rm sing},1}(3)$ then $\lambda\cdot\xi\in(1,K^\times,0)$ iff $\lambda\cdot\xi=(1,\xi_1\xi_2\lambda_1^{-2},0)$. Therefore, the set

$$(1,K^\times/K^{\times 2},0)$$

is a set of representatives of the orbits in $K^3_{{\rm sing},1}(3)$. Clearly, $K^3_{{\rm sing},1}(1)=\mathbb{T}^3(0,1,1)$ and $K^3_{{\rm sing},1}(2)=\mathbb{T}^3(1,0,1)$. Now, the statement (b) follows from statement 5.

3(a) $\lambda\cdot\xi\in(1,1,K^\times)$ iff $\frac{\lambda_3}{\lambda_1\lambda_2}\xi_1=1$ and $\frac{\xi_2}{\lambda_1\lambda_3}=1$, and in this case

$$\lambda\cdot\xi=(1,1,\frac{\xi_3}{\xi_1\xi_2^2}\lambda_1^3)$$

(since $\frac{\xi_3}{\lambda_2\lambda_3} = \frac{\xi_3}{\xi_1\xi_2^2}\lambda_1^3$, use the two equalities); and the statement (a) follows.

(b) Straightforward.

4(a) $\lambda \cdot \xi \in (1, 1, K^\times)$ iff $\frac{\xi_1}{\lambda_1\lambda_2} = 1$ and $\frac{\xi_2}{\lambda_1\lambda_3} = 1$, and in this case

$$\lambda \cdot \xi = (1, 1, \frac{\xi_3}{\xi_1\xi_2}\lambda_1^2)$$

(since $\frac{\xi_3}{\lambda_2\lambda_3} = \frac{\xi_3}{\xi_1\xi_2}\lambda_1^2$, use the two equalities); and the statement (a) follows.

(b) Straightforward.

5. Statement 5 is obvious. $\square$

Corollary 6.3 *Suppose that* $\sqrt{K} \subseteq K$ *and* $\sqrt[3]{K} \subseteq K$. *Then for each of the four actions as in Proposition 6.2, the map* $K^3/\mathbb{T}^3 \to \{0,1\}^3$, $\mathbb{T}^3\xi \mapsto \mathrm{supp}(\xi)$ *is a bijection with the inverse* $\eta \mapsto \mathbb{T}^3\eta$.

Proof of Theorem 1.7. By writing the equalities (21)–(23) in a 'cyclicly symmetric way' ($x_2x_1 = qx_1x_2 + \cdots$, $x_1x_3 = qx_3x_1 + \cdots$ and $x_3x_2 = qx_2x_3 + \cdots$) we see that up to cyclic permutation of the canonical generators there are four cases:

Case 1: $c \neq 0$, $\beta \neq 0$, $\lambda \neq 0$.

Case 2: $c \neq 0$, $\beta \neq 0$, $\lambda = 0$.

Case 3: $c \neq 0$, $\beta = 0$, $\lambda = 0$.

Case 4: $c = 0$, $\beta = 0$, $\lambda = 0$.

By making the change of variables

$$x_1' = \lambda_1^{-1}x_1, \;\; x_2' = \lambda_2^{-1}x_2, \;\; x_3' = \lambda_3^{-1}x_3 \;\text{ where }\; \lambda_i \in K^\times, \tag{35}$$

the equations (21)–(23) can be written as

$$x_2'x_1' = qx_1'x_2' + \frac{\lambda_3}{\lambda_1\lambda_2}cx_3' + \frac{1}{\lambda_1\lambda_2}b_1, \tag{36}$$

$$x_3'x_1' = q^{-1}x_1'x_3' + \frac{\lambda_2}{\lambda_1\lambda_3}\beta x_2' + \frac{1}{\lambda_1\lambda_3}b_2, \tag{37}$$

$$x_3'x_2' = qx_2'x_3' + \frac{\lambda_1}{\lambda_2\lambda_3}\lambda x_1' + \frac{1}{\lambda_2\lambda_3}b_3. \tag{38}$$

In each of the four cases the following triples

Case 1: (c, β, λ),

Case 2: (c, β, b_3),

Case 3: (c, b_2, b_3),

Case 4: (b_1, b_2, b_3)

are transformed under the change of variables (35) as the four $\mathbb{T}^3$-actions above (see Cases 1–4 of $\mathbb{T}^3$-actions). Now, using Proposition 6.2 and using S_3-action if necessary we see that (up to G_3') the parameters $c, \beta, \lambda, b_1, b_2, b_3$ belong precisely to one of the four cases of the theorem (that correspond to Cases 1–4 above)

If, in addition, $\sqrt{K} \subseteq K$ and $\sqrt[3]{K} \subseteq K$ then the four cases are reduced to cases 1–4 at the and of the theorem. $\square$

6.2 Case 2: $q_1 - q_3 = 0$ and $1 - q_1q_2 \neq 0$

Proof of Theorem 1.8. In view of (31), the equalities (14)–(16) can be written as $\lambda = 0$, $0 = 0$, and $c = 0$, respectively. Then the equalities (17)–(19) can be written as

$$b_3 = 0, \quad 0 = 0, \quad \text{and} \quad b_1 = 0,$$

respectively. Now, the equality (20) is $0 = 0$. So, the defining relations (32)–(34) are as in the theorem. Now, using the change of variables

$$x_1' = \lambda_1 x_1, \quad x_2' = \lambda_2 x_2, \quad \text{and} \quad x_3' = x_3$$

(where $\lambda_1, \lambda_2 \in K^\times$), we can assume that $\beta, b_2 \in \{0, 1\}$. $\square$

6.3 Case 3: $q_1 - q_3 \neq 0$, $1 - q_1q_2 = 0$ and $1 - q_2q_3 \neq 0$

Notice that the third condition (i.e., $1 - q_2q_3 \neq 0$) follows from the first two.

Proof of Theorem 1.9. In view of (31), the equalities (14)–(16) can be written as $0 = 0$, $\beta = 0$ and $c = 0$, respectively. Then the equalities (17)–(19) can be written as

$$0 = 0, \quad (q_1 - q_3)b_2 = 0, \quad \text{and} \quad (1 - q_2q_3)b_1 = 0,$$

respectively, and so $b_1 = b_2 = 0$. Then the condition (20) holds. So, the defining relations (32)–(34) are as in the theorem. Now, using the change of variables

$$x_1' = \lambda_1 x_1, \quad x_2' = \lambda_2 x_2, \quad \text{and} \quad x_3' = x_3$$

(where $\lambda_1, \lambda_2 \in K^\times$), we can assume that $\lambda, b_3 \in \{0, 1\}$. $\square$

6.4 Case 4: $q_1 - q_3 \neq 0$, $1 - q_1q_2 \neq 0$ and $1 - q_2q_3 = 0$

Proof of Theorem 1.10. In view of (31), the equalities (14)–(16) can be written as $(1-q_1q_2)\lambda = 0$, $(q_1 - q_3)\beta = 0$ and $0 = 0$, respectively, and so $\lambda = \beta = 0$. Then the equalities (17)–(19) can be written as

$$(q_1q_2 - 1)b_3 = 0, \quad (q_1 - q_3)b_2 = 0, \quad \text{and} \quad 0 = 0,$$

respectively, and so $b_2 = b_3 = 0$. Then the condition (20) holds. So, the defining relations (32)–(34) are as in the theorem. Then the condition (20) holds. Now, using the change of variables

$$x_1' = \lambda_1 x_1, \quad x_2' = x_2 \quad \text{and} \quad x_3' = \lambda_3 x_3$$

(where $\lambda_1, \lambda_3 \in K^\times$), we can assume that $c, b_1 \in \{0, 1\}$. $\square$

6.5 Case 5: $q_1 - q_3 \neq 0$, $1 - q_1q_2 \neq 0$ and $1 - q_2q_3 \neq 0$

Proof of Theorem 1.11. In view of (31), the equalities (14)–(16) can be written as

$$\lambda = 0, \quad \beta = 0, \quad \text{and} \quad c = 0,$$

respectively. Then the equalities (17)–(19) can be written as $b_3 = 0$, $b_1 = 0$ and $b_2 = 0$, respectively. Then the condition (20) holds. Therefore, $\mathbb{A} = 0$ and $\mathbb{B} = 0$, i.e., $A \simeq \mathbb{A}_Q^3$. $\square$

7 The bi-quadratic algebras with $q_1 = q_2 = q_3 = 1$ (Classification of bi-quadratic algebras A of Lie type)

The aim of this section is to give a classification (up to isomorphism) of the algebras $A = [x_1, x_2, x_3; Q, \mathbb{A}, \mathbb{B}]$ of Lie type (Theorem 1.4).

The algebras A of Lie type and their classification.

Definition. We say that the algebra A is of **Lie type** if $q_1 = q_2 = q_3 = 1$.

The next theorem describes all such algebras.

Theorem 7.1 *The algebra A is of Lie type iff $A \simeq U(\mathcal{G})/(z-1)$ where $U(\mathcal{G})$ is the universal enveloping algebra of a 4-dimensional Lie algebra $\mathcal{G}$ such that z is a central nonzero element of $\mathcal{G}$.*

Proof. ($\Leftarrow$) Let $\mathcal{G} = Kx_1 \bigoplus Kx_2 \bigoplus Kx_3 \bigoplus Kz$ be a 4-dimensional Lie algebra where $\{x_1, x_2, x_3, z\}$ is a K-basis of $\mathcal{G}$. Let $U = U(\mathcal{G})$ be the universal enveloping algebra of the Lie algebra $\mathcal{G}$. Then the defining relations of the algebra U are

$$x_2x_1 - x_1x_2 = ax_1 + bx_2 + cx_3 + b_1z, \tag{39}$$

$$x_3x_1 - x_1x_3 = \alpha x_1 + \beta x_2 + \gamma x_3 + b_2z, \tag{40}$$

$$x_3x_2 - x_2x_3 = \lambda x_1 + \mu x_2 + \nu x_3 + b_3z, \tag{41}$$

$$x_iz - zx_i = 0 \text{ for } i = 1, 2, 3. \tag{42}$$

The element z belongs to the centre of the algebra U. Then the factor algebra $U/(z-1)$ is the algebra of Lie type with defining constants $a, \ldots, b_3$.

($\Rightarrow$) Let A be a bi-qudratic algebra of Lie type. It can be seen as a Lie algebra $(A, [\cdot, \cdot])$ where $[u, v] := uv - vu$. Recall that $A = \bigoplus_{\alpha \in \mathbb{N}^3} Kx^\alpha$ (Theorem 1.3). Hence, the 4-dimensional vector space

$$\mathcal{G} = Kx_1 \bigoplus Kx_2 \bigoplus Kx_3 \bigoplus Kz, \quad \text{where} \quad z = 1,$$

is a Lie subalgebra of A. Furthermore, there is an epimorphism

$$U(\mathcal{G}) \to A, \quad x_i \mapsto x_i, \quad z \mapsto 1.$$

The element z belongs to the center of the algebra $U(\mathcal{G})$. By the PBW Theorem and Theorem 1.3, $U/(z-1) \simeq A$. $\square$

Let K be an algebraically closed field of characteristic zero and $\mathcal{G}$ be a Lie algebra over K. A Lie algebra $\mathcal{G}$ is called an *abelian* Lie algebra if $[\mathcal{G}, \mathcal{G}] = 0$. The Lie algebra

$$\mathfrak{n}_2 = \langle x, y \mid [x, y] = y \rangle$$

is the only (up to isomorphism) 2-dimensional non-abelian Lie algebra: If $[x, y] = \lambda x + \mu y$ where $\lambda, \mu \in K$ and $\mu \neq 0$, then

$$[\mu^{-1}x, \lambda x + \mu y] = \lambda x + \mu y.$$

If $\mu = 0$ then $\lambda \neq 0$ and $[-\lambda^{-1}y, x] = x$.

Theorem 7.2 is a classification of simple 3-dimensional Lie algebras. The classification is known (and can be easily proven directly).

Theorem 7.2 *Let K be an algebraically closed field of characteristic zero and $\mathcal{G}$ be a 3-dimensional Lie K-algebra. Then (up to isomorphism) the Lie algebra $\mathcal{G}$ is one of the following Lie algebras:*

1. $sl_2(K) = \langle x, y, h \mid [h, x] = 2x, \ [h, y] = -2y, \ [x, y] = h \rangle$,

2. $\mathcal{H}_3 = \langle x, y, z \mid [x, y] = z,\ [x, z] = 0,\ [y, z] = 0\rangle$, *the* 3*-dimensional Heisenberg Lie algebra,*

3. $\mathfrak{n}_2 \times Kz$*, a direct product of Lie algebras (where* Kz *is an abelain* 1*-dimensional Lie algebra),*

4. *an abelian* 3*-dimensional Lie algebra.*

Theorem (7.3) is a classification of 4-dimensional Lie algebras with nontrivial centre.

Classification of 4-dimensional Lie algebras with nontrivial centre.

Lemma 7.3 *Let* $\mathcal{G}$ *be a Lie algebra with* $Z(\mathcal{G}) \neq 0$ *and* $c \in Z(\mathcal{G}) \setminus \{0\}$*. Then* Kc *is an ideal of the Lie algebra* $\mathcal{G}$ *and there is a short exact sequence of Lie algebras*

$$0 \to Kc \to \mathcal{G} \to \overline{\mathcal{G}} := \mathcal{G}/Kc \to 0.$$

We can assume that $\mathcal{G} = \overline{\mathcal{G}} \oplus Kc$*, a direct sum of vector spaces, i.e., we fix an embedding of the vector space* $\overline{\mathcal{G}}$ *into* $\mathcal{G}$*,* $\overline{\mathcal{G}} \to \mathcal{G}$*,* $v \mapsto \overline{v}$*. Then, for all* $u, v \in \mathcal{G}$*,* $[\overline{u}, \overline{v}] = \overline{[u, v]} + \lambda_{u,v}c$ *where* $\lambda_{\cdot,\cdot} : \overline{\mathcal{G}} \times \overline{\mathcal{G}} \to K, (u, v) \mapsto \lambda_{u,v}$ *is a skew-symmetric bilinear form.*

Proof. Straightforward. □

Theorem 7.4 describes all the 4-dimensional Lie algebras with nontrivial centre. We will see that all of them are non-isomorphic Lie algebras, Theorem 7.5.

Theorem 7.4 *Let* $\mathcal{G}$ *be a* 4*-dimensional Lie algebra over an algebraically closed field* K *of characteristic zero such that its centre* $Z(\mathcal{G})$ *is not equal to zero. Fix a nonzero element* c *of* $Z(\mathcal{G})$ *and let* $\overline{\mathcal{G}} = \mathcal{G}/Kc$*, a Lie factor algebra of dimension* 3*.*

1. *If* $\overline{\mathcal{G}} \simeq sl_2(K)$ *then* $\mathcal{G} \simeq sl_2(K) \times Kc$*.*

2. *If* $\overline{\mathcal{G}} \simeq \mathcal{H}_3$ *is the* 3*-dimensional Heisenberg Lie algebra,* $\mathcal{H}_3 = \langle x, y, z \mid [x, y] = z,\ [x, z] = 0,\ [y, z] = 0\rangle$ *then either* $\mathcal{G} \simeq \mathcal{H}_3 \times Kc$ *or otherwise* $\mathcal{G} \simeq \mathcal{N}$ *where* $\mathcal{N} = \langle x, y, z, c \mid [x, y] = z,\ [x, z] = 0,\ [y, z] = c\rangle$ *and* c *is a central element of* $\mathcal{N}$*.*

3. *If* $\overline{\mathcal{G}} \simeq \mathfrak{n}_2 \times Kz$ *then either* $\mathcal{G} = \mathfrak{n}_2 \times Kz \times Kc$ *or otherwise* $\mathcal{G} \simeq \mathcal{M} := \langle x, y, z, c \mid [x, y] = y,\ [x, z] = c,\ [y, z] = 0\rangle$ *and* c *is a central element of* $\mathcal{G}$*.*

4. *If* $\overline{\mathcal{G}}$ *is an abelian* 3*-dimensional Lie algebra then either* $\mathcal{G}$ *is an abelian* 4*-dimensional Lie algebra or otherwise* $\mathcal{G} \simeq \mathcal{H}_3 \times Kd$ *where* Kd *is an abelian* 1*-dimensional Lie algebra.*

Proof. 1. Fix an embedding of the vector space $sl_2(K)$ into $\mathcal{G}$, then

$$[\overline{h}, \overline{x}] = 2\overline{x} + 2\lambda_1 c,\ \ [\overline{h}, \overline{y}] = -2\overline{y} - 2\lambda_2 c,\ \text{ and }\ [\overline{x}, \overline{y}] = \overline{h} + \lambda_3 c\ \text{ for some }\ \lambda_i \in K.$$

Then the linear span of the vectors

$$x := \overline{x} + \lambda_1 c,\ \ y := \overline{y} + \lambda_2 c,\ \text{ and }\ h = h + \lambda_3 c$$

is a Lie subalgebra of $\mathcal{G}$ which is isomorphic to sl_2 since

$$[h, x] = 2x,\ \ [h, y] = -2y,\ \text{ and }\ [x, y] = h.$$

We denote this Lie subalgebra by $sl_2(K)$. Then $\mathcal{G} = sl_2(K) \times Kc$ is a direct product of Lie algebras.

2. Fix an embedding of the vector space $\mathcal{H}_3$ into $\mathcal{G}$. Then

$$[\overline{x}, \overline{y}] = \overline{z} + \lambda_3 c,\ \ [\overline{x}, \overline{z}] = \lambda_1 c,\ \text{ and }\ [\overline{y}, \overline{z}] = \lambda_2 c\ \text{ for some }\ \lambda_i \in K.$$

Replacing $\overline{z}$ by $\overline{z} + \lambda_3 c$, we can assume that $\lambda_3 = 0$.

If $\lambda_1 = \lambda_2 = 0$ then the 3-dimensional vector space $K\overline{x} \oplus K\overline{y} \oplus K\overline{z}$ is a Lie subalgebra of $\mathcal{G}$ which is isomorphic to $\mathcal{H}_3$, and $\mathcal{G} \simeq \mathcal{H}_3 \times Kc$ is a direct product of Lie algebras.

If $\lambda_2 \neq 0$ then replacing c by $\lambda_2 c$ we can assume that $\lambda_2 = 1$. Then replacing $\overline{x}$ by $\overline{x} - \lambda_1 \overline{y}$ we can assume that $\lambda_1 = 0$, i.e., $\mathcal{G} \simeq \mathcal{N}$.

If $\lambda_2 = 0$ and $\lambda_1 \neq 0$ then applying the linear transformation

$$\overline{x} \mapsto \overline{y},\ \overline{y} \mapsto -\overline{x},\ \overline{z} \mapsto \overline{z},\ c \mapsto c$$

we come to the previous case (where $\lambda_2 \neq 0$).

3. Fix an embedding of the vector space $\mathfrak{n}_2 \times Kz$ into $\mathcal{G}$. Then

$$[\overline{x}, \overline{y}] = \overline{y} + \lambda_3 c,\ [\overline{x}, \overline{z}] = \lambda_1 c,\ [\overline{y}, \overline{z}] = \lambda_2 c \ \text{ for some } \lambda_i \in K.$$

Replacing $\overline{y}$ by $\overline{y} + \lambda_3 c$ we can assume the $\lambda_3 = 0$.

If $\lambda_1 = \lambda_2 = 0$ then the 3-dimensional vector space $K\overline{x} \oplus K\overline{y} \oplus K\overline{z}$ is a Lie subalgebra of $\mathcal{G}$ which is isomorphic to $\mathfrak{n}_2 \times Kz$, and so

$$\mathcal{G} \simeq \mathfrak{n}_2 \times Kz \times Kc$$

is a direct product of Lie algebras.

If, say, $\lambda_1 \neq 0$ then replacing c by $\lambda_1 c$ we can assume the $\lambda_1 = 1$. Then replacing $\overline{y}$ by $\overline{y} - \lambda_2 \overline{x}$ we can assume these $\lambda_2 = 0$, i.e.,

$$\mathcal{G} \simeq \mathcal{M}.$$

If $\lambda_1 = 0$ and $\lambda_2 \neq 0$ then replacing c by $\lambda_2 c$ we can assume the $\lambda_2 = 1$, i.e.,

$$[\overline{x}, \overline{y}] = \overline{y},\ \ [\overline{x}, \overline{z}] = 0,\ \ [\overline{y}, \overline{z}] = c.$$

This is not a Lie bracket since $0 = [\overline{y}, [\overline{x}, \overline{z}]] = [[\overline{y}, \overline{x}], \overline{z}] + [\overline{x}, [\overline{y}, \overline{z}]] = -[\overline{y}, \overline{z}] + [\overline{x}, c] = -c$, a contradiction.

4. Fix an embedding of the 3-dimensional abelian Lie algebra $\overline{\mathcal{G}} = \langle \overline{x}_1, \overline{x}_2, \overline{x}_2 \rangle$ into $\mathcal{G}$ as vector space. We assume that the Lie algebra $\mathcal{G}$ is not abelian. To finish the proof of statement 4 we have to show that $\mathcal{G} \simeq \mathcal{H}_3 \times Kd$. Notice that

$$[\overline{x}_1, \overline{x}_2] = \lambda c,\ [\overline{x}_1, \overline{x}_3] = \mu c,\ [\overline{x}_2, \overline{x}_3] = \delta c$$

for some scalars $\lambda, \mu, \delta \in K$. Without loss of generality we may assume that $\lambda \neq 0$ since $\mathcal{G}$ is not an abelian Lie algebra. Then replacing $\overline{x}_1$ by $\lambda^{-1}\overline{x}_1$ we can assume that $\lambda = 1$. Hence the vector space

$$\langle \overline{x}_1, \overline{x}_2, c \mid [\overline{x}_1, \overline{x}_2] = c \rangle$$

is a 3-dimensional Heisenberg Lie algebra $\mathcal{H}_3$. Then replacing $\overline{x}_3$ by $\overline{x}_3 - \mu \overline{x}_2$ we can assume that $\mu = 0$, i.e.,

$$[\overline{x}_1, \overline{x}_2] = c,\ [\overline{x}_1, \overline{x}_3] = 0, \ \text{ and } [\overline{x}_2, \overline{x}_3] = \delta c.$$

If $\delta = 0$ then

$$\mathcal{G} \simeq \mathcal{H}_3 \times Kx_3$$

is a direct product of Lie algebras or $\delta \neq 0$.

If $\delta \neq 0$ then changing $\overline{x}_3$ to $\delta^{-1}\overline{x}_3$ we can assume that $\delta = 1$, i.e., $[\overline{x}_2, \overline{x}_3] = c$. Then

$$[\overline{x}_2, \overline{x}_3 + \overline{x}_1] = c - c = 0,$$

and changing $\overline{x}_3$ to $\overline{x}_3 + \overline{x}_1$ we can assume the $\delta = 0$, and we are in the situation of the previous case. $\square$

The next theorem is a classification of 4-dimensional Lie algebras with nonzero centre.

Theorem 7.5 *Let $\mathcal{G}$ be a 4-dimensional Lie algebras over an algebraically closed field of characteristic zero such that $Z(\mathcal{G}) \neq 0$. Fix a nonzero element c of $Z(\mathcal{G})$. Then $\mathcal{G}$ is isomorphic to one of the following non-isomorphic Lie algebras:*

1. *$\mathcal{G}$ is an abelian 4-dimensional Lie algebra,*

2. *$sl_2(K) \times Kc$,*

3. *$\mathcal{H}_3 \times Kc$,*

4. *$\mathcal{N} = K\langle x, y, z, c \mid [x, y] = z,\ [x, z] = 0,\ [y, z] = c\rangle$ and c is a central element of the Lie algebra $\mathcal{N}$.*

5. *$\mathfrak{n}_2 \times Kz \times Kc$.*

6. *$\mathcal{M} = \langle x, y, z, c \mid [x, y] = y,\ [x, z] = c,\ [y, z] = 0\rangle$ and c is a central element of the Lie algebra $\mathcal{M}$.*

Proof. It follows from Theorem 7.4 and the classification of 3-dimensional Lie algebras (Theorem 7.2) that the Lie algebra $\mathcal{G}$ is isomorphic to one of the algebras in statements 1–6. We denote by $\mathcal{G}_i$ the Lie algebras in the statement i. Let $n = \dim Z(\mathcal{G})$. Then

$$\begin{aligned} n = 1 &\ :\ \mathcal{G}_2,\ \mathcal{G}_4,\ \mathcal{G}_6, \\ n = 2 &\ :\ \mathcal{G}_3,\ \mathcal{G}_5, \\ n = 4 &\ :\ \mathcal{G}_1. \end{aligned}$$

So, it remains to show that the Lie algebras in the three cases above are not isomorphic. The Lie algebras $\mathcal{G}_3$ and $\mathcal{G}_4$ are nilpotent. The Lie algebras $\mathcal{G}_5$ and $\mathcal{G}_6$ are solvable but non-nilpotent. The Lie algebra $\mathcal{G}_2$ is a direct product of a simple Lie algebra and an abelian Lie algebra. Hence, all Lie algebras $\mathcal{G}_1, \ldots, \mathcal{G}_6$ are not isomorphic. □

Classification (up to isomorphism) of algebras of Lie type. Theorem 1.4 is such a classification.

Proof of Theorem 1.4. It follows from Theorem 7.1 and Theorem 7.5 that an algebra A of Lie type is isomorphic to one of the algebras in statements 1-6.

Let $\mathcal{A}_i$ be the algebra in the statement i for $i = 1, \ldots, 6$. It remains to prove that the algebras

$$\mathcal{A}_1, \ldots, \mathcal{A}_6$$

are not isomorphic. Since the algebra $\mathcal{A}_1$ is the only commutative algebra, we have to show that the algebras

$$\mathcal{A}_2, \ldots, \mathcal{A}_6$$

are not isomorphic. In Proposition 7.6, we collect isomorphism-invariant properties of the algebras $\mathcal{A}_2, \ldots, \mathcal{A}_6$ that show that they are not isomorphic. By Proposition 7.6, the algebra $\mathcal{A}_6$ is the only algebra in the set $\mathcal{A}_2, \ldots, \mathcal{A}_6$ that has trivial centre. Hence, it remains to show that the algebra

$$\mathcal{A}_2, \ldots, \mathcal{A}_5$$

are not isomorphic. By Proposition 7.6, the algebra $\mathcal{A}_2$ is the only algebra in the set $\mathcal{A}_2, \ldots, \mathcal{A}_5$ that has simple 2-dimensional module. So, it remains to show that the algebras

$$\mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$$

are not isomorphic. By Proposition 7.6, the centre of each of the algebras $\mathcal{A} = \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$ is a polynomial algebra $K[t]$ in a single a variable. The algebra

$$\mathcal{A} = \mathcal{A}_4$$

is the only algebra in the set $\mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$ that has the property that all factor algebras $\mathcal{A}/(t - \lambda)$, where $\lambda \in K$, are simple (since the Weyl algebra A_1 is a simple algebra). So, it remains to show that

$$\mathcal{A}_3 \not\simeq \mathcal{A}_5.$$

This follows from statements $2(c)$ and $4(c)$ of Proposition 7.6. □

Proposition 7.6 *Let K be an algebraically closed field of characteristic zero.*

1. *Let $\mathcal{A}_2 = U(sl_2(K))$. Then*
   (a) *$Z(\mathcal{A}_2) = K[C]$ where C is the Casimir element.*
   (b) *For each natural number $n = 1, 2, \ldots$ there is a unique (up to isomorphism) simple n-dimensional $\mathcal{A}_2$-module.*
2. *Let $\mathcal{A}_3 = U(\mathcal{H}_3)$. Then*
   (a) *$Z(\mathcal{A}_3) = K[z]$.*
   (b) *$\mathcal{A}_3/(z) \simeq K[x, y]$ is a polynomial algebra and $\mathcal{A}_3(z - \lambda) \simeq A_1$ is the Weyl algebra for all $\lambda \in K^{\times}$.*
   (c) *$\{K_{\mu,\delta} := \mathcal{A}_3/(x - \mu, y - \delta, z) \mid \mu, \delta \in K\}$ is the set of isomorphism classes of simple finite dimensional $\mathcal{A}_3$-modules and $\dim(K_{\mu,\delta}) = 1$ and $\mathrm{ann}_{Z(\mathcal{A}_3)}(K_{\mu,\delta}) = zK[z]$ for all $\mu, \delta \in K$.*
3. *Let $\mathcal{A}_4 = U(\mathcal{N})/(c - 1) = A_1 \otimes K[x']$. Then*
   (a) *$Z(\mathcal{A}_n) = K[x']$ where $x' = x + \frac{1}{2}z^2$.*
   (b) *All simple $\mathcal{A}_4$-modules are infinite dimensional.*
   (c) *For all $\lambda \in K, \mathcal{A}_4/(x' - \lambda) \simeq A_1$, the Weyl algebra.*
4. *Let $\mathcal{A}_5 = U(\mathfrak{n}_2 \times Kz) = U(\mathfrak{n}_2) \otimes K[z]$. Then*
   (a) *$Z(\mathcal{A}_5) = K[z]$.*
   (b) *For all $\lambda \in K, \mathcal{A}_5/(z - \lambda) \simeq U(\mathfrak{n}_2) \simeq K[x][y; \sigma]$ is a skew polynomial algebra where $\sigma(x) = x + 1$.*
   (c) *$\{K_\mu(\lambda) := \mathcal{A}_5/(x - \mu, y, z - \lambda) \mid \lambda, \mu \in K\}$ is the set of isomorphism classes of simple finite dimensional $\mathcal{A}_5$-modules and $\dim(K_\mu(\lambda)) = 1$ for all $\mu, \lambda \in K$, and $\mathrm{ann}_{Z(\mathcal{A}_5)}(K_\mu(\lambda)) = (z - \lambda)K[z]$ for all $\lambda, \mu \in K$.*
5. *Let $\mathcal{A}_6 := U(\mathcal{M})/(c - 1) \simeq A_1[y, \sigma]$ where $A_1 = K\langle x, z | [x, z] = 1\rangle$ is the Weyl algebra, $\sigma(x) = x + 1$ and $\sigma(z) = z$. Then*
   (a) *$Z(\mathcal{A}_6) = K$.*
   (b) *There is no simple finite dimensional $\mathcal{A}_6$-modules.*

Proof. 1. Statement 1 is a well-known fact.

2.(a) The statement (a) is a well-known (and easy to prove) fact.

(b) The statement (b) is obvious.

(c) The statement (c) follows from the statement (b) and the fact then the Weyl algebra A_1 is a simple infinite dimensional algebra (and therefore, all simile A_1-modules are infinite dimensional).

3. (a) $Z(\mathcal{A}_4) = Z(A_1 \otimes K[x']) = Z(A_1) \otimes Z(K[x']) = K \otimes K[x'] = K[x']$ since $Z(A_1) = K$.

(b),(c): The statement (c) is obvious. The statement (b) follows from the statement (c).

4. (a) $Z(\mathcal{A}_5) = Z(U(\mathfrak{n}_2 \times Kz)) = Z(U(\mathfrak{n}_2) \otimes K[z]) = Z(U(\mathfrak{n}_2)) \otimes K[z] = K[z]$.

(b) The statement (b) follows from the statement (a).

(c) The statement (c) follows from the statement (b). In more detail, the algebra

$$U(\mathfrak{n}_2) \simeq K[x][y; \sigma]$$

is a skew polynomial algebra. The element y is a regular normal element ($yU(\mathfrak{n}_2) = U(\mathfrak{n}_2)y$). The localization of the algebra $U(\mathfrak{n}_2)$ at the powers of the element y is a simple skew polynomial ring

$$B_1 = K[x][y, y^{-1}; \sigma].$$

If M is a simple $U(\mathfrak{n}_2)$-module the either $yM = 0$ or otherwise the map

$$y : M \to M, \quad m \mapsto ym$$

is a bijection and therefore, the $U(\mathfrak{n}_2)$-module, M is also a simple B_1-module, and so

$$\dim(M) = \infty.$$

So, if M is a simple finite dimensional $U(\mathfrak{n}_2)$-module then $yM = 0$, and so M is a simple finite dimensional module over the polynomial algebra $U(\mathfrak{n}_2)/(y) \simeq K[x]$. Now, the statement (b) is obvious.

5. (a) The centralize $C(y)$ of the element y in $\mathcal{A}_6$ is equal to the polynomial algebra $K[y, z]$ (since $\sigma(x) = x + 1$ and $\sigma(z) = z$). Similarly, the centralize $C(x)$ of x is the polynomial algebra $K[x]$. Since

$$Z(\mathcal{A}_6) \subseteq C(x) \cap C(y) = K[x] \cap K[y, z] = K,$$

we must have $Z(\mathcal{A}_6) = K$.

(b) Since the Weyl algebra A_1 is a subalgebra of $\mathcal{A}_6$, every nonzero $\mathcal{A}_6$-module is also a nonzero A_1-module. The Weyl algebra A_1 is a simple infinite dimensional algebra. So, all nonzero A_1-modules are infinite dimensional, and the statement (b) follows. $\square$

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