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TAU FUNCTIONS FROM JOYCE STRUCTURES

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Abstract. We argued in [13] that, when a certain sub-exponential growth property holds, the Donaldson-Thomas invariants of a 3-Calabi-Yau triangulated category can be used to define a geometric structure on its space of stability conditions. In this paper we show how to associate a generating function to these geometric structures which we call the τ-function. In the case of the derived category of the resolved conifold this function reproduces the non-perturbative topological string partition function of [12], and in the case of the Joyce structures of class $S[A_1]$ constructed in [17] we obtain isomonodromic τ-functions.

1. Introduction

This paper is the continuation of a programme which attempts to encode the Donaldson-Thomas (DT) invariants of a CY_3 triangulated category $\mathcal{D}$ in a geometric structure on the space of stability conditions $M = \mathrm{Stab}(\mathcal{D})$. The relevant geometry is a kind of non-linear Frobenius structure, and was christened a Joyce structure in [13] in honour of the paper [28] where the main ingredients were first discovered. In later work with Strachan [16] it was shown that a Joyce structure can be re-expressed in terms of a complex hyperkähler structure on the total space of the tangent bundle $X = T_M$. The twistor space $p\colon Z \to \mathbb{P}^1$ associated to this hyperkähler structure will play a key role in this paper.

The procedure for producing Joyce structures from DT invariants is conjectural, and requires solving a family of non-linear Riemann-Hilbert (RH) problems [11] involving maps from the complex plane into a torus $(\mathbb{C}^*)^n$ with prescribed jumps across a collection of rays. These problems are only defined if the DT invariants of the category satisfy a sub-exponential growth condition. Under this assumption, Kontsevich and Soibelman [30] gave a construction of a less rigid structure, which is essentially the germ of our twistor space over $0 \in \mathbb{P}^1$.

It was discovered in [12] that when $\mathcal{D}$ is the derived category of coherent sheaves on the resolved conifold, the solutions to the associated RH problems can be repackaged in terms of a single function, which can moreover be viewed as a non-perturbative A-model topological string partition function, since its asymptotic expansion coincides with the generating function for the Gromov-Witten invariants. This function was introduced in a rather ad hoc way however, and it was unclear how to extend its definition to more general settings.

The aim of this paper is to formulate a general definition of such generating functions and study their properties. We associate to a Joyce structure on a complex manifold M a canonically defined section of a line bundle over $X = T_M$. Given choices of certain additional data this section can be expressed as a locally-defined function $\tau\colon X \to \mathbb{C}^*$ which we call the τ-function. In the case of the derived category of the resolved conifold, and for appropriate choices of the additional data, the restriction of this τ-function to a natural section $M \subset T_M$ coincides with the non-perturbative partition function obtained in [12].

There is an interesting class of CY_3 triangulated categories $\mathcal{D} = \mathcal{D}(g, m)$ defined by a genus $g \geq 0$ and a non-empty collection of pole orders $m = (m_1, \cdots, m_k)$. We refer to these as categories of class $S[A_1]$, since they are closely related to the four-dimensional supersymmetric gauge theories of the same name. They can be defined using quivers with potential associated to triangulations of marked bordered surfaces [31], or via Fukaya categories of certain non-compact Calabi-Yau threefolds [37]. The relevant threefolds $Y(g, m)$ are fibered over a Riemann surface C, and are described locally by an equation of the form $y^2 + uv = Q(x)$.

It was shown in [15] that the space of stability conditions on the category $\mathcal{D}(g, m)$ is the moduli space of pairs (C, Q) consisting of a Riemann surface C of genus g, equipped with a quadratic differential Q with poles of order $(m_1, \cdots, m_k)$ and simple zeroes. The general story described above then leads one to look for a natural Joyce structure on this space. In the case of differentials without poles this was constructed in [17], and the generalisation to meromorphic differentials will appear in the forthcoming work [39]. The key ingredient in these constructions is the existence of isomonodromic families of bundles with connections.

In several examples, a non-perturbative completion of the B-model topological string partition function of the threefold $Y(g, m)$ is known to be related, via the Nekrasov partition function of the associated class $S[A_1]$ theory, to an isomonodromic τ-function [8, 9, 10]. It therefore becomes natural to try to relate the τ-function associated to a Joyce structure of class $S[A_1]$ to an isomonodromic τ-function. That something along these lines should be true was suggested by the work of Teschner and collaborators [19, 20]. The definition of the Joyce structure τ-function was then reverse-engineered using the work of Bertola and Korotkin [5] on moduli-dependence of isomonodromic τ-functions, and by comparing the explicit description of the Joyce structure in the Painlevé I case [14] with the paper of Lisovyy and Roussillon [32].

The contents of the paper are as follows:

- We begin in Section 2 by recalling the definition of a Joyce structure on a complex manifold M from [13] and the associated complex hyperkähler structure [16] on the total space $X = T_M$. We also review the Joyce structures associated to theories of class $S[A_1]$ constructed in [17].
- The geometry of a Joyce structure is often clearer when viewed through the lens of the associated twistor space $p\colon Z \to \mathbb{P}^1$, which also plays an essential role in the definition of the τ-function. In Section 3 we introduce the basic definitions, and describe some additional structures which are present in the case of Joyce structures of class $S[A_1]$.
- In Section 4 we show how a Joyce structure whose base M is a cotangent bundle leads to a time-dependent Hamiltonian system. The definition requires the choice of a Lagrangian submanifold of the twistor fibre Z_∞. In the class $S[A_1]$ setting we relate the resulting Hamiltonian systems to the isomonodromy equations.
- Section 5 contains the general definition of the τ-function associated to a Joyce structure. It depends on the choice of certain additional data, namely symplectic potentials on the twistor fibres Z_0, Z_1 and Z_∞. We show that when restricted to various loci it produces generating functions for certain natural symplectic maps. We also show that in the setting of Section 4 it defines a τ-function in the usual sense of Hamiltonian systems.
- In Section 6 we consider τ-functions associated to uncoupled BPS structures. When restricted to a section of the projection $\pi\colon X \to M$ we show that our definition reproduces the τ-functions defined in [11]. In particular this applies to the non-perturbative partition function of [12].
- In Section 7 we consider the Joyce structure arising from the DT theory of the A_2 quiver. This was constructed in [14] using the monodromy map for the deformed cubic oscillator. In this case we show that our τ-function coincides with the τ-function associated to Painlevé I, extended as a function of moduli exactly as described in [32].

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Conventions. We work throughout in the category of complex manifolds and holomorphic maps. All symplectic forms, metrics, bundles, connections, sections etc., are holomorphic. We call a holomorphic map of complex manifolds étale if it is a local homeomorphism.

2. Joyce structures

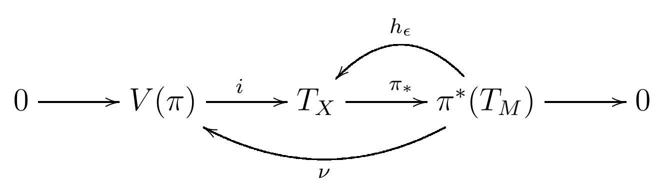
In this section we introduce the geometric structures that will appear throughout the rest of the paper. They can be described either in terms of flat pencils of symplectic non-linear connections [13], or via complex hyperkähler structures as in [16]. Most of this material is standard in the twistor-theory literature, see for example [18, 21], and goes back to the work of Plebański [36]. In the last part we briefly describe a class of examples of Joyce structures on spaces of quadratic differentials, following the treatment in [17].

2.1. **Pencils of non-linear connections.** Let $\pi\colon X \to M$ be a holomorphic submersion of complex manifolds. There is a short exact sequence of vector bundles

$$0 \longrightarrow V(\pi) \xrightarrow{\ i\ } T_X \xrightarrow{\ \pi_*\ } \pi^*(T_M) \longrightarrow 0, \qquad (1)$$

where $V(\pi) = \ker(\pi_*)$ is the sub-bundle of vertical tangent vectors. Recall that a (non-linear or Ehresmann) connection on π is a splitting of this sequence, given by a map of bundles $h\colon \pi^*(T_M) \to T_X$ satisfying $\pi_* \circ h = 1$.

Consider the special case in which $\pi\colon X = T_M \to M$ is the total space of the tangent bundle of M. There is then a canonical isomorphism $\nu\colon \pi^*(T_M) \to V(\pi)$ identifying the vertical tangent vectors in the bundle with the bundle itself, and we set $v = i \circ \nu$.



Definition 2.1. *A ν-pencil of connections on $\pi\colon X = T_M \to M$ is a family of connections of the form $h_\epsilon = h + \epsilon^{-1} v$ parameterised by $\epsilon^{-1} \in \mathbb{C}$. We say that the pencil is flat if each connection h_ϵ is flat.*

Suppose now that M is equipped with a symplectic form $\omega \in H^0(M, \wedge^2 T_M^*)$. This induces a translation-invariant symplectic form ω_m on each fibre $X_m = \pi^{-1}(m) \subset X$. We say that a connection on π is symplectic if the locally-defined parallel transport maps $X_{m_1} \to X_{m_2}$ preserve these forms. Note that if one of the connections in a ν-pencil $h_\epsilon = h + \epsilon^{-1} v$ is symplectic then they all are.

2.2. **Expression in co-ordinates.** Let $n = 2d$ be the complex dimension of M. Given local co-ordinates $(z_1, \cdots, z_n)$ on M there are associated linear co-ordinates $(\theta_1, \cdots, \theta_n)$ on the tangent spaces $T_{M,p}$ obtained by writing a tangent vector in the form $\sum_i \theta_i \cdot \partial/\partial z_i$. We thus get induced local co-ordinates (z_i, θ_j) on the space $X = T_M$. We always assume that the co-ordinates z_i are Darboux, in the sense that

$$\omega = \tfrac{1}{2} \sum_{p,q} \omega_{pq} \cdot dz_p \wedge dz_q, \tag{2}$$

with ω_{pq} a constant skew-symmetric matrix. We denote by η_{pq} the inverse matrix.

Given a symplectic ν-pencil $h_\epsilon = h + \epsilon^{-1} v$ we can write

$$v_i = v\Big(\frac{\partial}{\partial z_i}\Big) = \frac{\partial}{\partial \theta_i}, \qquad h_i = h\Big(\frac{\partial}{\partial z_i}\Big) = \frac{\partial}{\partial z_i} + \sum_{p,q} \eta_{pq} \cdot \frac{\partial W_i}{\partial \theta_p} \cdot \frac{\partial}{\partial \theta_q}, \tag{3}$$

for locally-defined functions W_i on X. The ν-pencil is flat precisely if we can take $W_i = \partial W/\partial \theta_i$ for a single function W, which moreover satisfies Plebański's second heavenly equations

$$\frac{\partial^2 W}{\partial \theta_i \partial z_j} - \frac{\partial^2 W}{\partial \theta_j \partial z_i} = \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial^2 W}{\partial \theta_j \partial \theta_q}. \tag{4}$$

This function $W(z_i, \theta_j)$ will be called the Plebański function.

For later use note that the covector fields dual to the basis of vector fields (3) are

$$h^j = dz^j, \qquad v^j = d\theta^j + \sum_{r,s} \eta^{jr} \cdot \frac{\partial^2 W}{\partial \theta_r \partial \theta_s} \cdot dz^s, \tag{5}$$

so that $(h^j, v_i) = 0 = (v^j, h_i)$ and $(h^j, h_i) = \delta_{ij} = (v^j, v_i)$.

2.3. **Complex hyperkähler structures.** By a complex hyperkähler structure on a complex manifold X we mean the data of a non-degenerate symmetric bilinear form $g\colon T_X \otimes T_X \to \mathscr{O}_X$, together with endomorphisms $I, J, K \in \operatorname{End}(T_X)$ satisfying the quaternion relations

$$I^2 = J^2 = K^2 = IJK = -1, \tag{6}$$

which preserve the form g, and which are parallel with respect to the Levi-Civita connection. Such structures have appeared before in the literature, often under different names. We define the closed 2-forms on X

$$\Omega_I(w_1,w_2)=g(I(w_1),w_2), \qquad \Omega_\pm(w_1,w_2)=g((J\pm iK)(w_1),w_2). \tag{7}$$

Then Ω_I is a symplectic form, but $\Omega_\pm$ are degenerate.

Consider, as in Section 2.1, the tangent bundle $X=T_M$ of a complex manifold M, and let $h_\epsilon=h+\epsilon^{-1}v$ be a ν-pencil of connections on the projection $\pi\colon X\to M$. The direct sum decomposition $T_X=\operatorname{im}(h)\oplus\operatorname{im}(v)$ gives an identification $T_X=\mathbb{C}^2\otimes_{\mathbb{C}}\pi^*(T_M)$. On the other hand, the complexification of the quaternions $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}$ can be identified with the algebra $\operatorname{End}_{\mathbb{C}}(\mathbb{C}^{\oplus 2})$. Putting these two observations together we obtain endomorphisms $I,J,K\in\operatorname{End}(T_X)$, given in terms of the vector fields (3) by the expressions[1]

$$I(h_i)=i\cdot h_i, \qquad J(h_i)=-v_i, \qquad K(h_i)=iv_i, \tag{8}$$

$$I(v_i)=-i\cdot v_i, \qquad J(v_i)=h_i, \qquad K(v_i)=ih_i, \tag{9}$$

A similar argument shows that combining a symplectic form ω on M with the standard symplectic form on $\mathbb{C}^2$ defines a metric g on X given by

$$g(h_i,h_j)=0, \qquad g(h_i,v_j)=\tfrac{1}{2}\omega_{ij}, \qquad g(v_i,v_j)=0. \tag{10}$$

The following result goes back to Plebański [36], and was proved in the form stated here in [17].

Theorem 2.2. *The data (g,I,J,K) associated to a ν-pencil $h_\epsilon=h+\epsilon^{-1}v$ as above gives rise to a complex hyperkähler structure on X precisely if the ν-pencil is flat and symplectic.* □

In terms of the covector fields h^i,v^j dual to h_i,v_j, the closed 2-forms (7) are

$$\Omega_I=\tfrac{i}{2}\cdot\sum_{p,q}\omega_{pq}\cdot v^p\wedge h^q, \tag{11}$$

$$\Omega_+=\tfrac{1}{2}\sum_{p,q}\omega_{pq}\cdot h^p\wedge h^q, \qquad \Omega_-=\tfrac{1}{2}\sum_{p,q}\omega_{pq}\cdot v^p\wedge v^q. \tag{12}$$

Using the expressions (5) these become

$$\Omega_+=\tfrac{1}{2}\sum_{p,q}\omega_{pq}\cdot dz_p\wedge dz_q, \qquad 2i\Omega_I=\sum_{p,q}\omega_{pq}\cdot d\theta_p\wedge dz_q, \tag{13}$$

[1] Compared to [16] we have changed the signs of I and K, and divided the metric by 2.

$$\Omega_- = \tfrac{1}{2}\sum_{p,q} \omega_{pq} \cdot d\theta_p \wedge d\theta_q + \sum_{p,q} \frac{\partial^2 W}{\partial \theta_p \partial \theta_q} \cdot d\theta_p \wedge dz_q + \sum_{p,q} \frac{\partial^2 W}{\partial z_p \partial \theta_q} \cdot dz_p \wedge dz_q, \tag{14}$$

where we used the heavenly equation (4) to obtain the last term in (14).

2.4. **Joyce structures.** A Joyce structure [13, 16] on a complex manifold M can be viewed as a kind of non-linear Frobenius structure. It consists of the following data:

(a) a symplectic form on M,
(b) an integral affine structure on M,
(c) a $\mathbb{C}^*$-action on M,
(d) a non-linear symplectic connection h on the projection $\pi\colon X = T_M \to M$,

subject to various compatibility axioms discussed in full in [17, Section 2.4]. We now briefly summarise these axioms in terms of local co-ordinates.

Recall that an integral affine structure on M is the data of a flat, torsion-free (linear) connection ∇^0 on the tangent bundle T_M, together with a parallel sublattice $T_M^{\mathbb{Z}} \subset T_M$ of maximal rank. The Joyce structure axioms first imply that we can take local Darboux co-ordinates z_i on M as in Section 2.2 such that this lattice is spanned by the tangent vectors $\partial/\partial z_i$. Moreover we can assume that the Euler vector field generating the $\mathbb{C}^*$-action is

$$E = \sum_i z_i \cdot \frac{\partial}{\partial z_i}, \tag{15}$$

and that the matrix $\eta_{pq}/2\pi i$ is integral.[2]

The symplectic connection h defines a ν-pencil of connections $h_\epsilon = h + \epsilon^{-1} v$. The most important constraint is that the connections h_ϵ are all flat. Writing the connection h in the form (3) defines a Plebański function W satisfying the heavenly equations (4). The remaining axioms of the Joyce structure are then equivalent to the following symmetry conditions:

(i) $W(z_1, \cdots, z_n, \theta_1 + 2\pi i k_1, \cdots, \theta_n + 2\pi i k_n) = W(z_1, \cdots, z_n, \theta_1, \cdots, \theta_n)$ for $k_i \in \mathbb{Z}$,
(ii) $W(tz_1, \cdots, tz_n, \theta_1, \cdots, \theta_n) = t^{-1} \cdot W(z_1, \cdots, z_n, \theta_1, \cdots, \theta_n)$ for $t \in \mathbb{C}^*$,
(iii) $W(z_1, \cdots, z_n, -\theta_1, \cdots, -\theta_n) = -W(z_1, \cdots, z_n, \theta_1, \cdots, \theta_n)$.

The $\mathbb{C}^*$-action on M induces an action on the tangent bundle $X = T_M$ in the natural way, which we combine with the inverse scaling action on the linear fibres of $\pi\colon X \to M$. In co-ordinates this means that the generating vector fields for the $\mathbb{C}^*$-action on X and M are given

[2] The quotient by $2\pi i$ introduced here is another small change from [16].

by the same formula (15). We will denote them both by the symbol E. Writing $\mathcal{L}$ for the Lie derivative, the condition (ii) together with the formulae (13) and (14) give

$$\mathcal{L}_E(\Omega_+) = 2\Omega_+, \qquad \mathcal{L}_E(\Omega_I) = \Omega_I, \qquad \mathcal{L}_E(\Omega_-) = 0. \tag{16}$$

The periodicity condition (i) ensures that the ν-pencil of connections h_ϵ on the map $\pi\colon X \to M$ descend to the quotient

$$X^\# = T_M^\# = T_M/(2\pi i)\, T_M^{\mathbb{Z}}, \tag{17}$$

which is a $(\mathbb{C}^*)^n$-bundle over M. The associated complex hyperkähler structure on X also descends to $X^\#$.

Remarks 2.3.

(i) Given local co-ordinates on an open subset $U \subset M$, it is really the second derivatives $\partial^2 W/\partial\theta_i\partial\theta_j$ that are well-defined on $\pi^{-1}(U) \subset X$ and satisfy the above symmetry properties. The function W itself is only well-defined up to the addition of expressions of the form $a(z) + \sum_j b_j(z)\theta_j$. We can fix these integration constants by imposing the parity property (iii) above, and the condition that the derivatives $\partial W/\partial\theta_j$ vanish when all $\theta_i = 0$. One can then check [13, Remark 4.2] that the properties (i)–(iii) hold on the nose.

(ii) It turns out that in many interesting examples, the ν-pencil $h_\epsilon\colon \pi^*(T_M) \to T_X$, or equivalently the hyperkähler structure (g, I, J, K), has poles on X. When expressed in terms of local co-ordinates as in Section 2.2, this means that the second derivatives of the Plebański function $\partial^2 W/\partial\theta_i\partial\theta_j$ are meromorphic functions. When it is necessary to be precise about this, we will refer to the resulting structures as meromorphic Joyce structures. Note that such poles in the Joyce structure could lead to complications in relation to Remark (i).

(iii) Rather than assuming the existence of a $\mathbb{C}^*$-action it might be better to consider only the associated Euler vector field E, as in the theory of Frobenius manifolds. The above definition means that we sometimes have to quotient by a discrete group to get a space M which carries a genuine Joyce structure, see for example Remark 7.1. For Joyce structures arising in DT theory the expected picture is clear: there should be a genuine $\mathbb{C}^*$-action on the space of stability conditions after quotienting by the action of the shift functors $[2k]$.

2.5. **Joyce structures of class $S[A_1]$.** The main class of known examples of Joyce structures are related to supersymmetric gauge theories of class $S[A_1]$, and were constructed in the paper

[17], to which we refer for further details. The base M parameterises pairs (C, Q) consisting of an algebraic curve C of some fixed genus $g \geq 2$, and a quadratic differential $Q \in H^0(C, \omega_C^{\otimes 2})$ with simple zeroes. The generalisation to the case of meromorphic quadratic differential with poles of fixed orders will be treated in the forthcoming work [39].

There is a natural $\mathbb{C}^*$-action on M which rescales the quadratic differential Q with weight 2. For each point $(C, Q) \in M$ there is a branched double cover $p\colon \Sigma \to C$ defined via the equation $y^2 = Q(x)$, and equipped with a covering involution $\sigma\colon \Sigma \to \Sigma$. Taking periods of the form $y\,dx$ on Σ identifies the tangent space $T_{(C,Q)}M$ with the anti-invariant cohomology group $H_1(\Sigma, \mathbb{C})^-$. The intersection pairing on $H_1(\Sigma, \mathbb{C})^-$ then induces a symplectic form on M, and the integral homology groups $H_1(\Sigma, \mathbb{Z})^-$ defines an integral affine structure $T_M^{\mathbb{Z}} \subset T_M$.

The usual spectral correspondence associates to a σ-anti-invariant line bundle L on Σ a rank 2 vector bundle $E = p_*(L)$ on C with a Higgs field Φ. A key ingredient in [17] is an extension of this correspondence which relates anti-invariant connections ∂ on L to connections ∇ on E. Given this, we can view the space $X^\#$ appearing in (17) as parameterising the data (C, E, ∇, Φ). The pencil of non-linear connections h_ϵ is then obtained by requiring that the monodromy of the connection $\nabla - \epsilon^{-1}\Phi$ is constant as the pair (C, Q) varies.

To explain the construction in a little more detail, let us fix a parameter $\epsilon \in \mathbb{C}^*$ and contemplate the following diagram of moduli spaces:

$$\begin{array}{ccccc}
 & & \mathcal{M}(C,E,\nabla,\Phi) & & \\
 & \swarrow^{\alpha} & & \searrow^{\beta_\epsilon} & \\
\mathcal{M}(C,Q,L,\partial) & & & & \mathcal{M}(C,Q,E,\nabla_\epsilon) \xrightarrow{\ \rho'\ } \mathcal{M}(C,E,\nabla_\epsilon) \\
\downarrow{\scriptstyle \pi_3} & & & & \downarrow{\scriptstyle \pi_2} \qquad\qquad\qquad \downarrow{\scriptstyle \pi_1} \\
\mathcal{M}(C,Q) & & \xleftrightarrow{\ =\ } & & \mathcal{M}(C,Q) \xrightarrow{\ \rho\ } \mathcal{M}(C)
\end{array} \tag{18}$$

Each moduli space parameterises the indicated objects, and the maps ρ, ρ' and π_i are the obvious projections. The map α is the above-mentioned extension of the spectral correspondence, and the map β_ϵ is defined by the rule

$$\beta_\epsilon(C, E, \nabla, \Phi) = (C, -\det(\Phi), E, \nabla - \epsilon^{-1}\Phi). \tag{19}$$

An important point is that α is birational, and β_ϵ is generically étale.

Given a point $(C, Q) \in M$, an anti-invariant line bundle with connection (L, ∂) on the spectral curve Σ has an associated holonomy representation $H_1(\Sigma, \mathbb{Z})^- \to \mathbb{C}^*$. This determines

(L,∂) up to an action of the group of 2-torsion line bundles on C. We therefore obtain an étale map from $\mathcal{M}(C,Q,L,\partial)$ to the space $X^{\#}$. The isomonodromy connection on the map π_1 is a flat symplectic connection whose leaves consist of connections (E,∇_ϵ) with fixed monodromy. Pulling this connection through (18) gives a family of non-linear symplectic connections h_ϵ on the projection $\pi\colon X^{\#}\to M$, which can be shown to form a ν-pencil. This gives rise to a meromorphic Joyce structure on M, with the poles arising from the fact that the maps α and β_ϵ are only generically étale.

3. Twistor space

In this section we recall the definition of the twistor space of a complex hyperkähler structure, and discuss the extra properties it has in the case of a Joyce structure. Note that we tacitly assume throughout that the Joyce structure is well-defined and holomorphic on the whole space $X=T_M$. Poles in the Joyce structure could cause further complications which would need to be discussed separately.

The final three subsections are rather tentative. We discuss further structures on twistor spaces of Joyce structures which appear naturally in the examples of class $S[A_1]$. They relate to choices of symplectic potentials on the twistor fibres Z_0, Z_1 and Z_∞, and are highly relevant to the definition of the τ-function in Section 5. It would be interesting to study these structures in further examples, and examine the extent to which they can be defined in general.

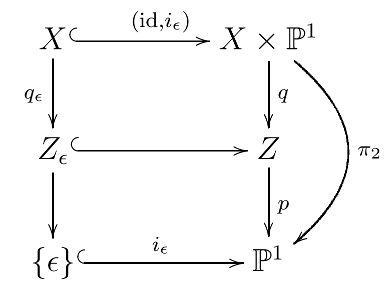
3.1. **Definition of twistor space.** Take notation as in Section 2. Thus M is a complex symplectic manifold, and $h_\epsilon = h+\epsilon^{-1}v$ is a ν-pencil of flat, symplectic connections on $\pi\colon X=T_M\to M$. There is an associated complex hyperkähler structure (g,I,J,K) on X, and corresponding closed 2-forms Ω_I and $\Omega_\pm$.

Definition 3.1. *The twistor space Z is the quotient of $X\times\mathbb{P}^1$ by the integrable distribution*

$$\operatorname{im}(sv+th)=\ker((J+iK)s^2+2istI+t^2(J-iK)), \tag{20}$$

where $[s:t]$ are homogeneous co-ordinates on $\mathbb{P}^1$.

We denote by $q\colon X\times\mathbb{P}^1\to Z$ the quotient map. In making the above definition we identify a vector field on X with the associated vertical vector field on the projection $\pi_2\colon X\times\mathbb{P}^1\to\mathbb{P}^1$. There is an induced projection $p\colon Z\to\mathbb{P}^1$ satisfying $p\circ q=\pi_2$.



We will use the affine co-ordinate $\epsilon = t/s$ and denote by $Z_\epsilon = p^{-1}(\epsilon) \subset Z$ the twistor fibre over $\epsilon \in \mathbb{P}^1$, and $q_\epsilon \colon X \to Z_\epsilon$ the quotient map. For $\epsilon^{-1} \in \mathbb{C}$ the fibre Z_ϵ is the quotient of X by the integrable distribution $\operatorname{im}(h_\epsilon)$, whereas $Z_0 = M$ is the quotient by the vertical sub-bundle $\operatorname{im}(v) = \ker \pi_*$. Each point $x \in X$ determines a section of the map p

$$\sigma_x \colon \mathbb{P}^1 \to Z, \qquad \epsilon \mapsto q(x, \epsilon), \tag{21}$$

whose image is a rational curve $\mathbb{P}^1 \subset Z$ known as a twistor line.

The twistor space $p \colon Z \to \mathbb{P}^1$ comes equipped with a twisted relative symplectic form. More precisely, there is a unique section Ω of the line bundle $\bigwedge^2 T_Z^* \otimes p^*(\mathscr{O}_{\mathbb{P}^1}(2))$ such that

$$q^*(\Omega) = s^2\Omega_+ + 2ist\Omega_I + t^2\Omega_-. \tag{22}$$

Restricting Ω to a twistor fibre Z_ϵ gives a complex symplectic form Ω_ϵ, well-defined up to multiplication by a nonzero constant. For $\epsilon^{-1} \in \mathbb{C}$ we fix this scale by taking

$$q_\epsilon^*(\Omega_\epsilon) = \epsilon^{-2}\Omega_+ + 2i\epsilon^{-1}\Omega_I + \Omega_-. \tag{23}$$

We equip the twistor fibre $Z_0 = M$ with the symplectic form $\Omega_0 = \omega$. Thus we have relations

$$q_0^*(\Omega_0) = \Omega_+, \qquad q_\infty^*(\Omega_\infty) = \Omega_-. \tag{24}$$

Remark 3.2. To obtain a well-behaved twistor space we cannot simply take the space of leaves of the foliation in Definition 3.1. Rather, we should consider the holonomy groupoid, which leads to the analytic analogue of a Deligne-Mumford stack [33]. We will ignore this subtlety here, since we only really use Z as a convenient language to describe objects which can easily be defined directly on X. For example, a symplectic form on the twistor fibre Z_ϵ is nothing but a closed 2-form on X whose kernel is equal to $\operatorname{im}(h_\epsilon) \subset T_X$.

3.2. **Twistor space of a Joyce structure.** Let us now consider a Joyce structure on a complex manifold M, and the associated twistor space $p \colon Z \to \mathbb{P}^1$. In particular there is a $\mathbb{C}^*$-action on M, and an induced action on X as discussed in Section 2.4. Taking the standard action of $\mathbb{C}^*$ on $\mathbb{P}^1$ rescaling ϵ with weight 1, we can then consider the diagonal action on $X \times \mathbb{P}^1$. It follows from the conditions (16) that this descends to an action on Z. The map $p\colon Z \to \mathbb{P}^1$ is then $\mathbb{C}^*$-equivariant.

We can use the $\mathbb{C}^*$-action to trivialise the map $p\colon Z \to \mathbb{P}^1$ over the open subset $\mathbb{C}^* \subset \mathbb{P}^1$. We obtain a commutative diagram

$$\begin{array}{ccccc} Z_1 \times \mathbb{C}^* & \xrightarrow{\ m\ } & p^{-1}(\mathbb{C}^*) & \hookrightarrow & Z \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \mathbb{C}^* & \xleftrightarrow{\ =\ } & \mathbb{C}^* & \hookrightarrow & \mathbb{P}^1 \end{array} \tag{25}$$

where the map m is defined by the action of $\mathbb{C}^*$ on Z.

Note that, unlike in the case of real hyperkähler manifolds, there is no requirement for an involution of the twistor space Z lifting the antipodal map on $\mathbb{P}^1$. In particular, there need be no relation between the twistor fibres Z_0 and Z_∞. There are therefore essentially three distinct twistor fibres: Z_0, Z_1 and Z_∞, of which $Z_0 = M$ is the base of the Joyce structure. The $\mathbb{C}^*$-action on X induces $\mathbb{C}^*$-actions on the fibres Z_0 and Z_∞.

Let us introduce 1-forms on X via the formulae

$$\alpha_+ = i_E(\Omega_+), \qquad \alpha_I = i_E(\Omega_I), \qquad \alpha_- = i_E(\Omega_-). \tag{26}$$

The relations (16) together with the Cartan formula imply that

$$d\alpha_+ = 2\Omega_+, \qquad d\alpha_I = \Omega_I, \qquad d\alpha_- = 0. \tag{27}$$

These relations will play an important role in what follows.

The form α_+ descends to the twistor fibre Z_0. To see this, note that if v is a vector field contracted by q_0, then

$$\mathcal{L}_v(\alpha_+) = i_v d(\alpha_+) + d i_v(\alpha_+) = 2 i_v(\Omega_+) - d i_E i_v(\Omega_+) = 0, \qquad \mathcal{L}_v(d\alpha_+) = \mathcal{L}_v(2\Omega_+) = 0, \tag{28}$$

since $i_v(\Omega_+) = \mathcal{L}_v(\Omega_+) = 0$. Thus we can write $\alpha_+ = q_0^*(\alpha_0)$ for some form α_0 on Z_0, and we see that the symplectic form Ω_0 is exact, with canonical symplectic potential $\frac{1}{2}\alpha_0$. This potential is moreover homogeneous of weight 2 for the induced $\mathbb{C}^*$-action on Z_0, since

$$\mathcal{L}_E(\alpha_+) = i_E d(\alpha_+) + d i_E(\alpha_+) = 2\alpha_+ + d i_E i_E(\Omega_+) = 2\alpha_+. \tag{29}$$

Similarly, the form α_I provides a canonical symplectic potential for the symplectic form Ω_I on X and satisfies $\mathcal{L}_E(\alpha_I) = \alpha_I$. In co-ordinates we have

$$\alpha_+ = \sum_{p,q} \omega_{pq} z_p \, dz_q, \qquad 2i\alpha_I = \sum_{p,q} \omega_{pq} z_p \, d\theta_q. \tag{30}$$

3.3. **Joyce function.** The argument of the previous section shows that the form $\alpha_- = i_E(\Omega_-)$ is the pullback $q_\infty^*(\alpha_\infty)$ of a $\mathbb{C}$-invariant form α_∞ on the twistor fibre Z_∞. However, this form does not give rise to a symplectic potential on Z_∞, and in fact $d\alpha_\infty = 0$. Let us instead consider a locally-defined function F on Z_∞ satisfying $dF = -\alpha_\infty$. Note that $-F$ is a Hamiltonian generating function for the $\mathbb{C}^*$-action on Z_∞, and is therefore invariant for this action. We call such a function F, or its pullback to X, a Joyce function.[3]

To write an explicit expression for the Joyce function choose a local system of co-ordinates on X as in Section 2.2, and let $W = W(z_i, \theta_j)$ be the corresponding Plebański function.

Lemma 3.3. *The locally-defined function on X defined by the expression*

$$F(z_i, \theta_j) = v(E)(W) = \sum_q z_q \frac{\partial W}{\partial \theta_q} \tag{31}$$

satisfies $dF = -i_E(\Omega_\infty)$ and descends to Z_∞.

Proof. The formula (14) gives

$$i_E(\Omega_-) = -\sum_{p,q} z_q \frac{\partial^2 W}{\partial \theta_p \partial \theta_q} d\theta_p + \sum_{p,q} \frac{\partial^2 W}{\partial z_p \partial \theta_q}(z_p\, dz_q - z_q\, dz_p) \tag{32}$$

$$= -\sum_{p,q} z_q \frac{\partial^2 W}{\partial \theta_p \partial \theta_q} d\theta_p - \sum_{p,q} z_q \frac{\partial^2 W}{\partial z_p \partial \theta_q} dz_p - \sum_q \frac{\partial W}{\partial \theta_q} dz_q = -\sum_p \Big(\frac{\partial F}{\partial \theta_p} d\theta_p + \frac{\partial F}{\partial z_p} dz_p\Big), \tag{33}$$

where we used the homogeneity property of W in the form $\sum_q z_q \, \partial W/\partial z_q = -W$. To see that F descends to the quotient of X by the horizontal vector fields for the connection $h = h_\infty$, note that if u is such a vector field, then

$$u(F) = i_u(dF) = -i_u i_E(\Omega_-) = i_E i_u(\Omega_-) = 0, \tag{34}$$

since Ω_- is pulled back from Z_∞. □

Remark 3.4. There are some subtleties hiding in the above proof. Recall from Remark 2.3(i) that even after the local co-ordinates on M are chosen, the function W is only well-defined up to addition of functions of the form $a(z) + \sum_j b_j(z)\theta_j$. Thus in principle the expression (31) is only well-defined up to the addition of functions pulled back from M. However, if the Joyce structure has no poles then these integration constants can be fixed, and once this is done the function W is indeed homogeneous of weight -1 in the variables z_i, and the above proof

[3] In [13] we used the term Joyce function as a synonym for the Plebański function W. Following [3] we now prefer to use it for the function introduced here, which was also considered by Joyce [28].

applies. When fixing the integration constants we require that the derivatives $\partial W/\partial\theta_j$ vanish along the locus when all $\theta_i = 0$. As we will see in the next section, this locus is the inverse image of a single distinguished point in Z_∞. Thus the expression (31) is the unique choice of Joyce function which vanishes at this point.

3.4. **Distinguished point of Z_∞.** In this section it will be important that the Joyce structure on $X = T_M$ has no poles on the zero section $M \subset T_M$. We also assume that the space M is connected.

Lemma 3.5. *The map $q_\infty\colon X \to Z_\infty$ contracts the zero-section $M \subset X = T_M$ to a point $0 \in Z_\infty$. This point is fixed by the $\mathbb{C}^*$-action.*

Proof. Note that the parity property $W(z_i, -\theta_j) = -W(z_i, \theta_j)$ and the formula (3) implies that along the zero-section $M \subset X = T_M$ we have $h_i = \partial/\partial z_i$. The first claim follows immediately from this. The second claim holds because the $\mathbb{C}^*$-action on X preserves the zero-section. □

The operator $J\colon T_X \to T_X$ maps v_i to h_i and hence identifies the normal bundle to the zero-section $M \subset T_M$ with the tangent bundle T_M. The derivative of the quotient map $q_\infty\colon X \to Z_\infty$ identifies this normal bundle with the trivial bundle with fibre $T_{Z_\infty,0}$. The combination of these two maps gives isomorphisms

$$T_{M,p} \xrightarrow{J} N_{M\subset X,p} \xrightarrow{dq_\infty} T_{Z_\infty,0}, \tag{35}$$

and hence a flat connection on the tangent bundle T_M. This is the linear Joyce connection from [13, Section 7], and appeared in the original paper of Joyce [28]. In co-ordinates it is given by the formula

$$\nabla^J_{\frac{\partial}{\partial z_i}}\Big(\frac{\partial}{\partial z_j}\Big) = -\sum_{l,m} \eta_{lm} \cdot \frac{\partial^3 W}{\partial\theta_i\,\partial\theta_j\,\partial\theta_l}\Big|_{\theta=0} \cdot \frac{\partial}{\partial z_m}. \tag{36}$$

The following result follows immediately from the definitions.

Lemma 3.6. *The weight space decomposition for the action of $\mathbb{C}^*$ on $T_{Z_\infty,0}$ defines via the identification* (35) *a decomposition $T_M \cong \bigoplus_{i\in\mathbb{Z}} V_i$ into ∇^J-flat sub-bundles $V_i \subset T_M$.* □

If the distinguished point $0 \in Z_\infty$ is an isolated fixed point for the $\mathbb{C}^*$ action, there are some additional consequences described in the following result. An example when this condition holds is the Joyce structure associated to the DT theory of the A_2 quiver. This example was treated in detail in [14] where all quantities described below were computed explicitly. We review this material in Section 7 below.

Lemma 3.7. *Suppose the point $0 \in Z_\infty$ is an isolated fixed point for the action of $\mathbb{C}^*$.*

(i) *The Hessian of the Joyce function F defines a non-degenerate symmetric bilinear form on $T_{Z_\infty,0}$. Via the identification (35) this induces a metric on M whose Levi-Civita connection is the linear Joyce connection ∇^J.*

(ii) *The positive and negative weight spaces of the $\mathbb{C}^*$-action on $T_{Z_\infty,0}$ define via the identification (35) a decomposition $T_M = V_- \oplus V_+$ into ∇^J-flat sub-bundles. These are Lagrangian for the symplectic form Ω_0.*

Proof. Part (i) is immediate from the result of Lemma 3.3 that F is the moment map for the $\mathbb{C}^*$-action on Z_∞. For part (ii), note that since Ω_- is $\mathbb{C}^*$-invariant, the positive and negative weight spaces in $T_{Z_\infty,0}$ are Lagrangian for the form Ω_-. The result then follows by noting that the operator J exchanges the forms $\Omega_\pm$, so the identification (35) takes the form Ω_+ to Ω_-. $\square$

The metric g of Lemma 3.7 is given in co-ordinates by the formula

$$g\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\Big) = \frac{\partial^2 F}{\partial \theta_i \partial \theta_j}\Big|_{\theta=0}. \tag{37}$$

This is the Joyce metric of [13, Section 7], which also appeared in the original paper [28].

3.5. **Cotangent bundle structure on Z_0.** By a cotangent bundle structure on a complex symplectic manifold (M, ω) we mean the data of a complex manifold B and an open embedding $M \subset T^*_B$, such that ω is the restriction of the canonical symplectic form on T^*_B. We denote by $\rho\colon M \to B$ the induced projection map, and by $\lambda \in H^0(M, T^*_M)$ the restriction of the Liouville 1-form.

Given local co-ordinates $(t_1, \cdots, t_d)$ on B, there are induced linear co-ordinates $(s_1, \cdots, s_d)$ on the cotangent spaces $T^*_b B$ obtained by writing a 1-form as $\sum_i s_i \, dt_i$. In the resulting co-ordinates (s_i, t_i) on M we have

$$\omega = \sum_i dt_i \wedge ds_i, \qquad \lambda = \sum_i s_i \, dt_i.$$

Note that $d(-\lambda) = \omega$. When $M = Z_0$ is the base of a Joyce structure there is a $\mathbb{C}^*$-action on M satisfying $\mathcal{L}_E(\omega) = 2\omega$. It is then natural to seek a cotangent bundle structure $M \subset T^*_B$ such that $\mathcal{L}_E(\lambda) = 2\lambda$. Note that this implies that the $\mathbb{C}^*$-action on M preserves the distribution of vertical vector fields for ρ, since this coincides with the kernel of λ. It follows that there is a

$\mathbb{C}^*$-action on B, and that the $\mathbb{C}^*$-action on M is given by the combining the induced action on T^*_B with a rescaling of the fibres with weight 2.

Consider the Joyce structures of class $S[A_1]$ of Section 2.5. The base $M = \mathcal{M}(C,Q)$ has a natural cotangent bundle structure, with $B = \mathcal{M}(C)$ being the moduli space of curves of genus g, and $\rho\colon M \to B$ the obvious projection $\rho\colon \mathcal{M}(C,Q) \to \mathcal{M}(C)$. Indeed, the tangent spaces to $\mathcal{M}(C)$ are the vector spaces $\mathcal{T}_C\mathcal{M}(C) = H^1(C,T_C)$, and Serre duality gives $H^0(C,\omega_C^{\otimes 2}) = H^1(C,T_C)^*$. Thus T^*_B parameterises pairs (C,Q) of a curve C together with a quadratic differential $Q \in H^0(C,\omega_C^{\otimes 2})$, and $M \subset T^*_B$ is the open subset where Q has simple zeroes. We note in passing that in this case the fibres of ρ have a highly non-trivial compatibility with the Joyce structure: in the language of [17, Section 4] they are good Lagrangians.

Remark 3.8. It is not clear whether the base $M = Z_0$ of a general Joyce structure admits a natural cotangent bundle structure. In the case when the distinguished point $0 \in Z_\infty$ is an isolated fixed point for the $\mathbb{C}^*$-action, one possibility is to define the projection $\rho\colon M \to B$ by quotienting by the integrable distribution $V_- \subset T_M$ of Lemma 3.7. Somewhat remarkably, in the A_2 example discussed in Section 7, this procedure leads to the same natural projection $\rho\colon M \to B$ described in the previous paragraph.

3.6. **Cluster-type structure on Z_1.** A crucial part of the definition of the τ-function in Section 5 will be the existence of collections of preferred co-ordinate systems $(x_1, \cdots, x_n)$ on the twistor fibre Z_1. These should be Darboux in the sense that

$$\Omega_1 = \tfrac{1}{2}\sum_{i,j} \omega_{ij} \cdot dx_i \wedge dx_j, \tag{38}$$

for some constant skew-symmetric matrix ω_{ij}. Although it will not be needed in Section 5, these co-ordinates are also expected to have the following asymptotic property: given local co-ordinates $(z_1, \cdots, z_n)$ centered at a generic point $m \in M$ as in Section 2.2, there should be a particular choice of preferred co-ordinates $(x_1, \cdots, x_n)$ on Z_1 such that

$$x_i(\epsilon^{-1}z_1, \cdots, \epsilon^{-1}z_n, \theta_1, \cdots, \theta_n) \sim -\epsilon^{-1}z_i + \theta_i \tag{39}$$

as $\epsilon \to 0$ in the half-plane $\mathrm{Re}(\epsilon) > 0$.

When the Joyce structure is constructed via the DT RH problems of [13], the required systems of preferred co-ordinates x_i are given directly by the solutions to the RH problems, and the property (39) holds by definition. Thus for the Joyce structures of interest in DT theory there is no problem finding this additional data. It is still interesting however to ask whether such distinguished co-ordinate systems exist for general Joyce structures. A heuristic explanation for why this might be the case can be found in [17, Section 5.3]. This question is also closely related to the contents of [30, Section 5].

Consider the Joyce structures of class $S[A_1]$ of Section 2.5. Choose a reference surface S_g of genus g, set $G = \mathrm{PGL}_2(\mathbb{C})$, and define

$$MCG(g) = \pi_0(\mathrm{Diff}^+(S_g)), \qquad \mathcal{X}(g) = \mathrm{Hom}_{\mathrm{grp}}(\pi_1(S_g), G)/G. \tag{40}$$

Then the mapping class group $\mathrm{MCG}(g)$ acts on the character stack $\mathcal{X}(g)$ in the usual way, and taking the monodromy of the connection $\nabla - \epsilon^{-1}\Phi$ defines a map

$$\mu_\epsilon \colon \mathcal{M}(C, E, \nabla, \Phi) \to \mathcal{X}(g)/\,\mathrm{MCG}(g), \tag{41}$$

which by definition is constant on the leaves of the connection h_ϵ. This yields an étale map

$$\mu_\epsilon \colon Z_\epsilon \to \mathcal{X}(g)/\,\mathrm{MCG}(g). \tag{42}$$

Suppose we instead consider Joyce structures on moduli spaces of meromorphic quadratic differentials with fixed pole orders [39]. Then the character stack in the above discussion should be replaced by a wild character stack parameterising framed local systems on a marked bordered surface. The preferred co-ordinate systems are expected to be the logarithms of Fock-Goncharov co-ordinates [22], and are indexed by ideal triangulations of this surface. The asymptotic property (39) should then follow from exact WKB analysis. In the case of quadratic differentials on $\mathbb{P}^1$ with a single pole of order 7 this story is treated in detail in [14].

3.7. **Lagrangian submanifolds of Z_∞.** At various points in what follows it will be convenient to choose a $\mathbb{C}^*$-invariant Lagrangian submanifold $R \subset Z_\infty$ in the twistor fibre at infinity. For example in the next section we show that the choice of such a Lagrangian, together with a cotangent bundle structure on Z_0, leads to a time-dependent Hamiltonian system. Unfortunately, for the Joyce structures arising from the DT RH problems of [13], the geometry of the twistor fibre Z_∞ is currently quite mysterious, since it relates to the behaviour of solutions to the RH problems at $\epsilon = \infty$. So it is not yet clear whether such a Lagrangian submanifold can be expected to exist in general.

When discussing the twistor fibre Z_∞ for Joyce structures of class $S[A_1]$ it is important to distinguish the case of holomorphic quadratic differentials treated in [17] from the extension to meromorphic quadratic differentials [39]. In the holomorphic setting, the argument leading to the local homeomorphism (42) applies also when $\epsilon = \infty$, because the non-linear connection h_ϵ

is obtained by fixing the monodromy of the connection $\nabla_\epsilon = \nabla - \epsilon^{-1}\Phi$, and these connections are of the same type for all $\epsilon^{-1} \in \mathbb{C}$. Moreover, since the $\mathbb{C}^*$-action on X rescales the Higgs field Φ while leaving C and (E, ∇) fixed, the induced action on Z_∞ is trivial.

In the meromorphic case, one should combine the data of the curve C with the irregular type of the connection at the singularities to give the notion of wild curve. Then in order for all the connections $\nabla - \epsilon^{-1}\Phi$ to lie on the same wild curve it is necessary for the poles of ∇ to have roughly half the order of those of Φ. This means that the direct analogue of the map (42) takes values in a wild character stack of a strictly lower dimension, and the fibres of this map are not well-understood at present. This relates to the problem of understanding the behaviour of the Fock-Goncharov co-ordinates of the connection $\nabla - \epsilon^{-1}\Phi$ in the limit as $\epsilon \to \infty$. Similarly, since the $\mathbb{C}^*$-action on X now changes the wild curve, the induced action on Z_∞ is non-trivial, as can be seen explicitly in the example of Section 7.

4. Hamiltonian systems

In this section we show how to use a Joyce structure to define a time-dependent Hamiltonian system. The construction depends on two of the pieces of data discussed in Section 3: a cotangent bundle structure on Z_0 and a Lagrangian submanifold $R \subset Z_\infty$. We also explain how, for Joyce structures of class $S[A_1]$, this construction gives rise to a Hamiltonian description of the isomonodromy equations for curves of arbitrary genus.

4.1. **Time-dependent Hamiltonian systems.** A time-dependent Hamiltonian system consists of the following data:

(i) a submersion $f\colon Y \to B$ with a relative symplectic form $\Omega \in H^0(Y, \wedge^2 T^*_{Y/B})$,

(ii) a flat, symplectic connection k on f,

(iii) a section $\varpi \in H^0(Y, f^*(T^*_B))$.

For a nice exposition of this definition see [6, Section 5]. Note that Boalch works in the real C^∞ setting, whereas we assume, as elsewhere in the paper, that all structures are holomorphic. Note also that Boalch assumes that $Y = M \times B$ is a global product, with M a fixed symplectic manifold and k the canonical connection on the projection $f\colon M \times B \to B$. We can always reduce to this case locally.

For each vector field $u \in H^0(B, T_B)$ there is an associated function

$$H_u = (f^*(u), \varpi)\colon Y \to \mathbb{C}. \tag{43}$$

There is then a pencil k_ϵ of symplectic connections on f defined by

$$k_\epsilon(u) = k(u) + \epsilon^{-1} \cdot \Omega^\sharp(H_u). \tag{44}$$

The system is called strongly-integrable if these connections are all flat.

These definitions becomes more familiar when expressed in local co-ordinates. If we take co-ordinates t_i on the base B, which we can think of as times, and k-flat Darboux co-ordinates (q_i, p_i) on the fibres of f, we can write $\varpi = \sum_i H_i \, dt_i$ and view the functions $H_i \colon Y \to \mathbb{C}$ as time-dependent Hamiltonians. The connection k_ϵ is then given by the flows

$$k_\epsilon\Big(\frac{\partial}{\partial t_i}\Big) = \frac{\partial}{\partial t_i} + \frac{1}{\epsilon} \cdot \sum_j \Big(\frac{\partial H_i}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H_i}{\partial q_j}\frac{\partial}{\partial p_j}\Big). \tag{45}$$

The condition that the system is strongly-integrable is that

$$\sum_{r,s} \Big(\frac{\partial H_i}{\partial q_r} \cdot \frac{\partial H_j}{\partial p_s} - \frac{\partial H_i}{\partial q_s} \cdot \frac{\partial H_j}{\partial p_r}\Big) = 0, \qquad \frac{\partial H_i}{\partial t_j} = \frac{\partial H_j}{\partial t_i}. \tag{46}$$

Suppose given a strongly-integrable time-dependent Hamiltonian system in the above sense. Let $L \subset Y$ denote a leaf of the foliation k_1. The restriction $\varpi|_L$ is then closed, so we can write $\varpi|_L = d\log(\tau_L)$ for some locally-defined function $\tau_L \colon L \to \mathbb{C}^*$. In terms of co-ordinates, since the projection $\pi \colon L \to B$ is a local isomorphism, we can lift t_i to co-ordinates on L, whence we have

$$\frac{\partial}{\partial t_i} \log(\tau_L) = H_i|_L. \tag{47}$$

A τ-function in this context is a locally-defined function $\tau \colon Y \to \mathbb{C}^*$ whose restriction $\tau|_L$ to each leaf $L \subset Y$ satisfies (47). Note that this definition only specifies τ up to multiplication by the pullback of an arbitrary function on the space of leaves.

4.2. **Hamiltonian systems from Joyce structures.** Let M be a complex manifold equipped with a Joyce structure. Thus there is a ν-pencil of flat, symplectic connections $h_\epsilon = h + \epsilon^{-1} v$ on the projection $\pi \colon X = T_M \to M$, and closed 2-forms Ω_I and $\Omega_\pm$ on X. We denote by $p \colon Z \to \mathbb{P}^1$ the associated twistor space. Suppose also given:

(i) a cotangent bundle structure $M \subset T_B^*$,

(ii) a Lagrangian submanifold $R \subset Z_\infty$.

For the notion of a cotangent bundle structure see Section 3.5. We denote by $\rho\colon M \to B$ the induced projection, and $\beta \in H^0(M, \rho^*(T_B^*))$ the tautological section. If we take local co-ordinates t_i on B, and extend them to canonical co-ordinates (s_i, t_i) on M as in Section 3.5, then $\beta = \sum_i s_i \cdot \rho^*(dt_i)$. Note that β is almost the same as the Liouville form $\lambda \in H^0(M, T_M^*)$ appearing in Section 3.5: they correspond under the inclusion $\rho^*(T_B^*) \hookrightarrow T_M^*$ induced by ρ.

Set $Y = q_\infty^{-1}(R) \subset X$, and denote by $i\colon Y \hookrightarrow X$ the inclusion. There are maps

$$Y \overset{i}{\hookrightarrow} X \xrightarrow{\pi} M \xrightarrow{\rho} B \tag{48}$$

Define $p\colon Y \to M$ and $f\colon Y \to B$ as the composites $p = \pi \circ i$ and $f = \rho \circ \pi \circ i$. We make the following transversality assumption:

($\star$) For each $b \in B$ the restriction of $q_1\colon X \to Z_1$ to the fibre $f^{-1}(b) \subset Y \subset X$ is étale.

The following result will be proved in the next section.

Theorem 4.1. *Given the above data there is a strongly-integrable time-dependent Hamiltonian system on the map $f\colon Y \to B$ uniquely specified by the following conditions:*

(i) *the relative symplectic form Ω is induced by the closed 2-form $i^*(2i\Omega_I)$ on Y;*

(ii) *for each $\epsilon \in \mathbb{C}^*$ the connection k_ϵ on $f\colon Y \to B$ satisfies*

$$\operatorname{im}(k_\epsilon) = T_Y \cap \operatorname{im}(h_\epsilon) \subset T_X; \tag{49}$$

(iii) *the Hamiltonian form is $\varpi = p^*(\beta) \in H^0(Y, f^*(T_B^*))$.*

To make condition (iii) more explicit, take local co-ordinates t_i on B, and extend to local co-ordinates (s_i, t_i) on M, and (t_i, q_j, p_j) on Y, as above. Then $\varpi = \sum_j p^*(s_j) \cdot f^*(dt_j)$ and

$$k_\epsilon\Big(\frac{\partial}{\partial t_i}\Big) = \frac{\partial}{\partial t_i} + \frac{1}{\epsilon} \cdot \sum_j \bigg(\frac{\partial p^*(s_i)}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial p^*(s_i)}{\partial p_j}\frac{\partial}{\partial q_j}\bigg). \tag{50}$$

The main non-trivial claim is that the connections k_ϵ are all flat, so that the conditions (46) hold for the Hamiltonians $H_i = p^*(s_i)$.

4.3. **Proof of Theorem 4.1.** Take a point $y \in Y \subset X$ with $\pi(y) = m \in M$ and set $b = \rho(m)$. Given a tangent vector $u \in T_{B,b}$ let us choose a lift $w \in T_{M,m}$ satisfying $\rho_*(w) = u$. Using the maps $h, v\colon \pi^*(T_M) \to T_X$ as in Section 2.1 we then obtain tangent vectors $h_\epsilon(w), v(w) \in T_{X,y}$. Recall that $h_\epsilon(w) = \epsilon^{-1}v(w) + h(w)$, and note that $h(w) \subset T_{Y,y}$, since $Y = q_\infty^{-1}(R)$, and q_∞ contracts the leaves of $h = h_\infty$. This implies that for any $\epsilon \in \mathbb{C}^*$ the following two conditions are equivalent:

(a) $v(w) \in T_{Y,y} \subset T_{X,y}$,

(b) $h_\epsilon(w) \in T_{Y,y} \subset T_{X,y}$.

Thus when (b) holds for some $\epsilon \in \mathbb{C}^*$ it holds for all such ϵ.

Consider next the transversality statement

$(\star)_\epsilon$ For each $b \in B$ the restriction of $q_\epsilon \colon X \to Z_\epsilon$ to the fibre $f^{-1}(b) \subset Y \subset X$ is étale.

This is equivalent to the existence of a unique lift $w \in T_{M,m}$ satisfying condition (b), since it is equivalent to the statement that $h_\epsilon^{-1}(T_{Y,y}) \cap \ker(\rho_*) = \{0\} \subset T_{M,m}$. Thus since we assumed $(\star)_\epsilon$ for $\epsilon = 1$, it holds for all $\epsilon \in \mathbb{C}^*$.

We can now connections k_ϵ on $\pi \colon Y \to B$ for all $\epsilon \in \mathbb{C}^*$ by setting $k_\epsilon(u) = h_\epsilon(w)$, where w is the unique lift satisfying condition (b). In more geometric terms, since the connection h_ϵ is flat, the condition $(\star)_\epsilon$ ensures that the map $(f, q_\epsilon) \colon Y \to B \times Z_\epsilon$ is étale, and the connection k_ϵ is then pulled back from the trivial connection on the projection $B \times Z_\epsilon \to B$. In particular k_ϵ is flat.

The closed 2-form $q_1^*(\Omega_1)$ defines a relative symplectic form on f since the restriction of $q_1 \colon X \to Z_1$ to each fibre $f^{-1}(b) \subset Y \subset X$ is étale. Recall the identity of closed 2-forms

$$q_\epsilon^*(\Omega_\epsilon) = \epsilon^{-2} q_0^*(\Omega_0) + 2i\epsilon^{-1}\Omega_I + q_\infty^*(\Omega_\infty) \tag{51}$$

from Section 3.1. On restricting to Y the last term on the right-hand side vanishes, since $R \subset Z_\infty$ is Lagrangian. On further restricting to a fibre $f^{-1}(b) \subset Y$ the first term also vanishes, since $p^{-1}(b) \subset M$ is Lagrangian. Thus for $\epsilon \in \mathbb{C}^*$ the forms $\epsilon \cdot q_\epsilon^*(\Omega_\epsilon)$ define the same relative symplectic form Ω on f, and this is also induced by $2i\Omega_I$. Note that the the kernel of the restriction $\Omega_\epsilon|_Y$ clearly contains the subspace $\operatorname{im}(k_\epsilon)$, and hence coincides with it. This implies that the connection k_ϵ on f is symplectic [26, Theorem 4].

Observe next that there is a commutative diagram

$$\begin{array}{ccc} T_X & \xrightarrow{-(2i\Omega_I)_\flat} & T_X^* \\ {\scriptstyle v}\uparrow & & \uparrow{\scriptstyle \pi^*} \\ T_M & \xrightarrow{\omega_\flat} & T_M^* \end{array} \tag{52}$$

since the relations (8)-(9) show that for tangent vectors w_1 to M and w_2 to X

$$-2i\Omega_I(v(w_1), w_2) = -g(2iIv(w_1), w_2) = -2g(v(w_1), w_2) = g((J+iK)h(w_1), w_2) \tag{53}$$

$$= \Omega_+(h(w_1), w_2) = (\pi^*\omega)(h(w_1), w_2) = \omega(w_1, \pi_*(w_2)). \tag{54}$$

The fact that the closed 2-form $i^*(2i\Omega_I)$ induces a relative symplectic form on the map $f\colon Y \to B$ is the statement that the composite of the bundle maps

$$\ker(f_*) \hookrightarrow T_Y \xrightarrow{(2i\Omega_I)_\flat} T^*_Y \longrightarrow T^*_Y/f^*(T^*_B) \tag{55}$$

is an isomorphism. The relative Hamiltonian flow r corresponding to a function $H\colon Y \to \mathbb{C}$ is then the unique vertical vector field on the map $f\colon Y \to B$ which is mapped to dH viewed as a section of $T^*_Y/f^*(T^*_B)$. In symbols we can write $(2i\Omega_I)_\flat(r) = dH + f^*(\alpha)$ for some covector field α on B.

Consider the canonical section $\varpi = p^*(\beta) \in H^0(Y, f^*(T^*_B))$. Note that given a vector field $u \in H^0(B, T_B)$ the corresponding Hamiltonian $H_u = (f^*(u), \varpi) = p^*(\rho^*(u), \beta)$ on Y is pulled back from M. Let us lift u to a vector field w on M as above. Then $k_\epsilon(u) = h_\epsilon(w) = h(w) + \epsilon^{-1} \cdot v(w)$. We claim that $v(w)$ is the relative Hamiltonian flow r defined by the function H_u. To prove this we must show that the 1-form $(2i\Omega_I)_\flat(v(w)) - dH_u$ on Y is a pullback from B. By the commutative diagram (52) this is equivalent to showing that the 1-form $\omega_\flat(w) - d(\rho^*(u), \beta)$ on M is a pullback from B.

This final step is perhaps most easily done in local co-ordinates (s_i, t_i) on M as above in which $\beta = \sum_i s_i dt_i$ and $\omega = \sum_i dt_i \wedge ds_i$. If we take $u = \frac{\partial}{\partial t_i}$ then $(\rho^*(u), \beta) = s_i$ and the lift w has the form $w = \frac{\partial}{\partial t_i} + \sum_j a_j \frac{\partial}{\partial s_j}$ for locally-defined functions $a_j\colon M \to \mathbb{C}$. But then $\omega_\flat(w) = ds_i - \sum_j a_j\, dt_j$, and since the t_i are pulled back via ρ this proves the claim.

We have now defined the symplectic connections k_ϵ for $\epsilon \in \mathbb{C}^*$ and proved the relation (45). We can then define a symplectic connection $k = k_\infty$ by the same relation. Since the k_ϵ are flat for all $\epsilon \in \mathbb{C}^*$ the relations (46) hold, and it follows that k is also flat.

4.4. **Hamiltonian systems from Joyce structures of class $S[A_1]$.** In this section we consider the Hamiltonian system of Theorem 4.1 in the case of the Joyce structures of class $S[A_1]$ of Section 2.5. Recall that a crucial feature of the construction of these Joyce structures is the isomondromy connection on the map

$$\pi_1\colon \mathcal{M}(C, E, \nabla_1) \to \mathcal{M}(C). \tag{56}$$

Note that to construct a Hamiltonian system on this map we need another 'base' connection. In the notation of Section 4, the isomonodromy connection is k_1, but we also need the connection k_∞ before we can write the equation (44). This issue is often a little hidden in the literature because in many examples there is a natural choice for the reference connection k_∞ which is then taken without further comment. It is discussed explicitly in [27], and is also mentioned for example in [7, Remark 7.1].

One way to get a base connection k_∞ on the map (56) is to choose for each bundle E a distinguished 'reference' connection ∇_∞. Then k_∞ can be taken to the isomonodromy connection for the family of connections (E, ∇_∞). Note that once we have chosen the reference connection ∇_∞ we get a whole pencil of connections $\nabla_\epsilon = \nabla_\infty - \epsilon^{-1}\Phi$ obtained by setting $\Phi = \nabla_\infty - \nabla_1$. Many interesting examples of isomonodromic systems in the literature involve bundles with meromorphic connections on a genus 0 curve. Since the generic such bundle E is trivial, it is then natural to take $\nabla_\infty = d$. But for bundles with connection on higher genus curves there is no such canonical choice.

Consider the specific setting of Section 2.5 involving bundles E with holomorphic connections on higher genus curves. Recall also the character stack $\mathcal{X}(g)$ and mapping class group $\mathrm{MCG}(g)$ as defined in (40). The subspace of the quotient $\mathcal{X}(g)/\mathrm{MCG}(g)$ consisting of monodromy representations of connections on a fixed bundle E is known to be Lagrangian. Let us choose another such Lagrangian $R \subset \mathcal{X}(g)/\mathrm{MCG}(g)$. Then for a generic bundle E we can expect these two Lagrangians to meet in a finite set of points, and so locally on the moduli of bundles we can define ∇_∞ by insisting that its monodromy lies in R.

We can now apply the construction of Section 4.2 to these example. There is a canonical cotangent bundle structure on Z_0 as discussed in Section 3.5. In Section 3.7 it was explained that the twistor fibre Z_∞ has an étale map to the quotient $\mathcal{X}(g)/\mathrm{MCG}(g)$, so the Lagrangian R determines by pullback a Lagrangian in Z_∞ which we also denote by R. We obtain a diagram

$$\begin{array}{ccccc} Y = q_\infty^{-1}(R) & \xhookrightarrow{\;i\;} & \mathcal{M}(C,E,\nabla,\Phi) & \xrightarrow{\;\beta_1\;} & \mathcal{M}(C,E,\nabla_1) \\ & {\scriptstyle f}\searrow & \downarrow & \swarrow{\scriptstyle \pi_1} & \\ & & \mathcal{M}(C) & & \end{array} \tag{57}$$

where the map β_1 is defined by setting $\nabla_1 = \nabla - \Phi$.

The isomonodromy connection defines a flat connection h_1 on the map π_1. The Hamiltonian system of Theorem 4.1 defines a whole pencil of flat connections k_ϵ on the map f. The transversality assumption ensures that the map $\beta_1 \circ i$ is étale, and the pullback of the connection h_1 then coincides with k_1. Thus by choosing the Lagrangian $R \subset Z_\infty$ and using it to define reference connections, we have upgraded the isomonodromy connection to a Hamiltonian system.

5. The τ-function

In this section we define the τ-function associated to a Joyce structure on a complex manifold M, and discuss some of its basic properties. It is most naturally viewed as the unique up-to-scale section of a flat line bundle on $X = T_M$. By choosing a section of this line bundle it becomes a locally-defined function on X, and by pulling back via the multiplication map $m\colon X \times \mathbb{C}^* \to X$ it can be viewed as also depending on a parameter $\epsilon \in \mathbb{C}^*$.

5.1. **Definition of the τ-function.** Let M be a complex manifold equipped with a Joyce structure as above, and let $p\colon Z \to \mathbb{P}^1$ be the associated twistor space. We set $\Theta_I = i_E(\Omega_I)$, so that as in Section 3.2 we have $d\Theta_I = \Omega_I$. Recall the identity of closed 2-forms on X

$$q_1^*(\Omega_1) = q_0^*(\Omega_0) + 2i\Omega_I + q_\infty^*(\Omega_\infty). \tag{58}$$

We start by giving the definition of the τ-function in explicit local form. The geometrically-minded reader is encouraged to read the next section first.

Definition 5.1. *Choose locally-defined symplectic potentials:*

(i) Θ_0 *on* Z_0 *satisfying* $d\Theta_0 = \Omega_0$ *and* $\mathcal{L}_E(\Theta_0) = 2\Theta_0$,

(ii) Θ_1 *on* Z_1 *satisfying* $d\Theta_1 = \Omega_1$,

(iii) Θ_∞ *on* Z_∞ *satisfying* $d\Theta_\infty = \Omega_\infty$ *and* $\mathcal{L}_E(\Theta_\infty) = 0$.

Then the corresponding τ-function is the locally-defined function on X uniquely specified up to multiplication by constants by the relation

$$d\log(\tau) = q_0^*(\Theta_0) + 2i\Theta_I + q_\infty^*(\Theta_\infty) - q_1^*(\Theta_1). \tag{59}$$

By pulling back τ via the multiplication map $m\colon \mathbb{C}^* \times X \to X$ we can view it as a function also of $\epsilon \in \mathbb{C}^*$. Restricted to the slice $\{\epsilon\} \times X$ it then satisfies

$$d\log(\tau) = \epsilon^{-2} q_0^*(\Theta_0) + 2i\epsilon^{-1}\Theta_I + q_\infty^*(\Theta_\infty) - q_\epsilon^*(\Theta_\epsilon), \tag{60}$$

where $\Theta_\epsilon = m_{\epsilon^{-1}}^*(\Theta_1)$, and we used the relation $\mathcal{L}_E(\Theta_I) = \Theta_I$. Although the extra parameter ϵ is redundant, it is frequently useful to introduce it, for example so as to expand τ as an asymptotic series.

Remark 5.2. Given two symplectic potentials $\Theta_0^{(i)}$ the resulting functions $\tau^{(i)}$ will be related by

$$\log \tau^{(2)} - \log \tau^{(1)} = \epsilon^{-2} \cdot q_0^*(G), \tag{61}$$

where G is a locally-defined function on Z_0 satisfying $dG = \Theta_0^{(2)} - \Theta_0^{(1)}$. Similarly, if we consider two symplectic potentials $\Theta_\infty^{(i)}$ the resulting functions $\tau^{(i)}$ will be related by

$$\log \tau^{(2)} - \log \tau^{(1)} = q_\infty^*(G), \tag{62}$$

for some locally-defined function G on Z_∞. In either case the function G is independent of ϵ. Thus if we are interested in the asymptotic expansion of τ in the limit $\epsilon \to 0$, it is the choice of Θ_1 in Definition 5.1 which is the most important, since the choices of Θ_0 and Θ_∞ will only effect the coefficients of ϵ^{-2} and ϵ^0 respectively.

5.2. **Global perspective.** Suppose that there exist line bundles with connection (L_0, ∇_0), (L_1, ∇_1) and $(L_\infty, \nabla_\infty)$ on the twistor fibres Z_0, Z_1 and Z_∞, with curvature forms Ω_0, Ω_1 and Ω_∞ respectively. For a review of such pre-quantum line bundles see Appendix A. The identity (58) shows that the connection

$$\big(q_0^*(\nabla_0) \otimes 1 \otimes 1\big) + \big(1 \otimes q_\infty^*(\nabla_\infty) \otimes 1\big) - \big(1 \otimes 1 \otimes q_1^*(\nabla_1)\big) + 2i\Theta_I$$

on the line bundle $q_0^*(L_0) \otimes q_\infty^*(L_\infty) \otimes q_1^*(L_1)^{-1}$ is flat. There is therefore a unique flat section up to scale, which we call the τ-section.

Given a section $s_0 \in H^0(U, L_0)$ over an open subset $U \subset Z_0$ we can write $\nabla_0(s_0) = \Theta_0 \cdot s_0$ for a 1-form Θ_0 on U. Then $d\Theta_0 = \Omega_0|_U$, so that Θ_0 is a symplectic potential for Ω_0 on this open subset. This construction defines a bijection between local sections of L_0 up to scale, and local symplectic potentials for Ω_0. Let us take local sections s_0, s_1 and s_∞ of the bundles L_0, L_1 and L_∞ respectively, and let Θ_0, Θ_1 and Θ_∞ be the corresponding symplectic potentials. Then we can write the τ-section in the form $\tau \cdot (s_0 \otimes s_\infty \otimes s_1^{-1})$, and the resulting locally-defined function τ on X will satisfy the equation (59).

We can constrain the possible choices of local sections s_0 and s_∞ using the $\mathbb{C}^*$-actions on the fibres Z_0 and Z_∞. The relations $\mathcal{L}_E(\Omega_0) = 2\Omega_0$ and $\mathcal{L}_E(\Omega_\infty) = 0$ show that the symplectic forms Ω_0 and Ω_∞ are homogeneous for this action. It follows that the action lifts to the pre-quantum line bundles L_0 and L_∞. We can then insist that the local sections s_0 and s_∞ are $\mathbb{C}^*$-equivariant, which in terms of the corresponding symplectic potentials translates into the conditions $\mathcal{L}_E(\Theta_0) = 2\Theta_0$ and $\mathcal{L}_E(\Theta_\infty) = 0$ appearing in Definition 5.1.

Remark 5.3. The existence of the pre-quantum line bundle (L_1, ∇_1) assumed above is equivalent to the condition that the de Rham cohomology class of $\Omega_1/(2\pi i)$ is integral. Note that since the forms Ω_0 and $2i\Omega_I$ are exact, the relation (58) shows that this is also the condition for (L_0, ∇_0) and $(L_\infty, \nabla_\infty)$ to exist.

For Joyce structures arising in DT theory this integrality condition is expected to be a consequence of refined DT theory. Roughly speaking, the twistor fibre Z_1 is covered by preferred Darboux co-ordinate charts whose transition functions are symplectic maps given by time 1 Hamiltonian flows of products of quantum dilogarithms. The cocycle condition is the wall-crossing formula in DT theory. The generating functions for these symplectic maps are given by formulae involving the Rogers dilogarithm, and the cocycle condition for the pre-quantum line bundle (L_1, ∇_1) is then an extension of the wall-crossing formula involving these generating functions. The statement is then that this extended wall-crossing formula follows from the wall-crossing formula in refined DT theory involving automorphisms of quantum tori.

This picture was explained by Alexandrov, Persson and Pioline [2], and by Neitzke [35], and in the case of theories of class $S[A_1]$ plays an important role in the work of Teschner *et al* [19, 20]. The Rogers dilogarithm identities were described in the context of cluster theory by Fock and Goncharov [23, Section 6], and by Kashaev and Nakanishi [29, 35].

5.3. **Choice of symplectic potentials.** To obtain a well-defined τ-function, we need some prescription for choosing the symplectic potentials Θ_0, Θ_1 and Θ_∞, or equivalently the local sections s_0, s_1 and s_∞. We make some general remarks about this here, although some aspects are still unclear.

It was explained in Section 3.2 that setting

$$\Theta_0 = \tfrac{1}{2} i_E(\Omega_0) = \tfrac{1}{2}\sum_{i,j} \omega_{ij} z_i \, dz_j, \tag{63}$$

provides a canonical and global choice for Θ_0. On the other hand, the key to defining Θ_1 is to assume the existence of a distinguished Darboux co-ordinate system $(x_1, \cdots, x_n)$ on the twistor fibre Z_1 as in Section 3.6. We can then take

$$\Theta_1 = \tfrac{1}{2}\sum_{i,j} \omega_{ij} x_i \, dx_j. \tag{64}$$

The choice of Θ_∞ remains quite mysterious in general. One way to side-step this problem is to choose a $\mathbb{C}^*$-invariant Lagrangian $R \subset Z_\infty$ as in Section 3.7, since if we restrict τ to the inverse image $Y = q_\infty^{-1}(R)$ we can then drop the term $q_\infty^*(\Theta_\infty)$ from the definition of the τ-function. It is not quite clear whether this procedure is any more than a convenient trick. In other words, it is not clear whether τ should be viewed as a function on X, depending on a choice of symplectic potential Θ_+, or whether the natural objects are the Lagrangian $R \subset Z_\infty$, and a function τ defined on the corresponding submanifold $Y \subset X$.

In practice, in examples, the above choices of symplectic potentials Θ_0 and Θ_1 do not always give the nicest results. We discuss several possible modifications here. Of course, from the global point-of-view, these variations correspond to expressing the same τ-section in terms of different local sections of the line bundles L_0 and L_1.

5.3.1. *Cotangent bundle.* Suppose we are given a cotangent bundle structure on M as discussed in Section 3.5. As explained there, it is natural to assume that the associated Liouville form λ satisfies $\mathcal{L}_E(\lambda) = 2\lambda$. We can then consider the following three choices of symplectic potential

$$\Theta_0^{\mathrm{L}} = -\lambda, \qquad \Theta_0 = \tfrac{1}{2} i_E(\Omega_0), \qquad \Theta_0^{\mathrm{H}} = i_E(\Omega_0) + \lambda. \tag{65}$$

The justification for the strange-looking primitive Θ_0^{H} will become clear in Section 5.4.4: it is the correct choice to produce τ-functions for the Hamiltonian systems of Section 4. Applying the Cartan formula gives

$$\Theta_0^{\mathrm{H}} - \Theta_0 = \Theta_0 - \Theta^{\mathrm{L}} = \tfrac{1}{2} i_E(\Omega_0) + \lambda = -\tfrac{1}{2} i_E(d\lambda) + \tfrac{1}{2}\mathcal{L}_E(\lambda) = \tfrac{1}{2} d i_E(\lambda). \tag{66}$$

Thus the resulting τ functions differ by the addition of the global function $\frac{1}{2} q_0^*(i_E(\lambda))$.

5.3.2. *Polarisation.* Suppose the distinguished Darboux co-ordinate system $(x_1, \cdots, x_n)$ on Z_1 is polarised, in the sense that the associated matrix ω_{ij} satisfies $\omega_{ij} = 0$ unless $|j - i| = d$, where as before we write $n = 2d$. We can then take as symplectic potential on Z_1

$$\Theta_1^{\mathrm{P}} = \sum_{i=1}^{d} \omega_{i,i+d} \cdot x_i \, dx_{i+d}, \tag{67}$$

This resulting τ-function will be modified by $\frac{1}{2} \sum_i \omega_{i,i+d} \cdot x_i x_{i+d}$.

5.3.3. *Flipping* Θ_I. Given local co-ordinates on X as in Section 2.2 there are in fact two obvious choices of symplectic potential for $2i\Omega_I$, namely

$$2i\Theta_I = \sum_{p,q} \omega_{pq} z_p \, d\theta_q, \qquad 2i\Theta_I' = -\sum_{p,q} \omega_{pq} \theta_p \, dz_q. \tag{68}$$

In Section 5.4.2 it will be convenient to replace $2i\Theta_I$ in the definition of the τ-function with $2i\Theta_I'$. This will change the τ-function by the addition of $K = \sum_{p,q} \omega_{pq} \cdot z_p \theta_q$. Note that $K\colon X \to \mathbb{C}$ is a globally-defined function, since it is the 1-form $i_E(\omega)$ on M considered as a function on $X = T_M$. It does not however descend to the quotient $X^\# = T_M/(2\pi i)\, T_M^{\mathbb{Z}}$.

5.4. **Interpretations of the τ-function.** By restricting to various submanifolds of X the τ-function can be viewed as a generating function in a confusing number of ways.

5.4.1. *Restriction to the zero-section.* Let $j\colon M \hookrightarrow X = T_M$ be the inclusion of the zero-section, defined by setting all co-ordinates $\theta_i = 0$. Since this is the fibre $q_\infty^{-1}(0)$ over the distinguished point $0 \in Z_\infty$ of Section 3.4 we have $j^* q_\infty^*(\Theta_\infty) = 0$. The formula (30) for $\alpha_I = \Theta_I$ shows that also $j^*(\Theta_I) = 0$. The defining relation of the τ-function then implies that

$$d\log(\tau|_M) = \Theta_0 - j^* q_1^*(\Theta_1) \tag{69}$$

is the generating function for the symplectic map $q_1 \circ j\colon M \to Z_1$ with respect to the symplectic potentials Θ_0 and Θ_1.

5.4.2. *Restriction to a fibre of q_0.* Let $j\colon F = T_{M,p} \hookrightarrow X$ be the inclusion of a fibre of the projection $\pi\colon X = T_M \to M$. Restriction to this locus corresponds to fixing the co-ordinates z_i. By (14), the restriction

$$\Omega_F = j^* q_\infty^*(\Omega_+) = \tfrac{1}{2} \sum_{p,q} \omega_{pq} \cdot d\theta_p \wedge d\theta_q \tag{70}$$

is the linear symplectic form on $T_{M,p}$ defined by the symplectic form ω. It follows that $\Theta_F = j^* q_\infty^*(\Theta_+)$ is a symplectic potential for this form.

Let us take the flipped form $2i\Theta_I'$ in the definition of the τ-function as in Section 5.3.3. The forms Θ_- and $2i\Theta_I'$ vanish when restricted to F, so

$$d\log(\tau|_F) = j^* q_\infty^*(\Theta_+) - j^* q_1^*(\Theta_1) \tag{71}$$

is the generating function for the symplectic map $q_1 \circ j\colon F \to Z_1$ with respect to the symplectic potentials Θ_F and Θ_1.

5.4.3. *Restriction to a fibre of f.* Let us consider the setting of Theorem 4.1. Thus we have chosen a cotangent bundle structure $M \subset T_B^*$, with associated projection $\rho\colon Y \to B$, and a $\mathbb{C}^*$-invariant Lagrangian submanifold $R \subset Z_\infty$, and set $Y = q_\infty^{-1}(R)$. Let $j\colon F \hookrightarrow X$ be the inclusion of a fibre $F = f^{-1}(b)$ of the map $f\colon Y \to B$. By Theorem 4.1 (i) the closed 2-form $2i\Omega_I$ restricts to a symplectic form on F. As above, we can drop the term $q_\infty^*(\Theta_\infty)$ from the definition of the τ-function. Let us take the symplectic potential on Z_0 to be $\Theta_0^{\mathrm{L}} = -d\lambda$ as in Section 5.3.1. Then since $\pi(F) \subset \rho^{-1}(b)$ we also have $j^* q_0^*(\Theta_0^{\mathrm{L}}) = 0$. Thus

$$d\log(\tau|_F) = j^*(2i\Theta_I) - j^* q_1^*(\Theta_1) \tag{72}$$

is the generating function for the symplectic map $q_1 \circ j\colon F \to Z_1$ with respect to the symplectic potentials $j^*(2i\Theta_I)$ and Θ_1.

5.4.4. *Hamiltonian system τ-function.* Consider again the setting of Theorem 4.1. As before, we can drop the term $q_\infty^*(\Theta_\infty)$ from the definition of the τ-function. Let us take the symplectic potential on Z_0 to be Θ_0^{H} as defined in Section 5.3.1. Then we have

$$d\log(\tau|_Y) = j^*q_0^*(\lambda) + j^*q_0^*(i_E(\Omega_0)) + j^*(i_E(2i\Omega_I)) - j^*q_1^*(\Theta_1) \tag{73}$$

$$= p^*(\lambda) + j^*(i_E(q_1^*(\Omega_1))) - j^*q_1^*(\Theta_1). \tag{74}$$

Here we used the identity (58), together with the assumption that the Lagrangian $R \subset Z_\infty$ is $\mathbb{C}^*$-invariant, which ensures that $i_E(\Omega_\infty)|_R = 0$. Let $L \subset X$ be a leaf of the connection k_1 on $f\colon Y \to B$. By construction of k_1 this is the intersection of Y with a leaf of the connection h_1 on $\pi\colon X \to M$. Note that if u is a horizontal vector field for the connection h_1 then $i_u i_E(q_1^*(\Omega_1)) = -i_E\, i_u(q_1^*(\Omega_1)) = 0$. Thus

$$d\log(\tau|_L) = p^*(\lambda)|_L. \tag{75}$$

Applying the definition of Section 4.1 it follows that $\tau|_Y$ is a possible choice of τ-function for the strongly-integrable time-dependent Hamiltonian system of Theorem 4.1.

6. Uncoupled BPS structures

In the paper [11] it was found that in certain special cases, solutions to DT RH problems could be encoded by a single generating function, which was denoted τ. The most basic case is the one arising from the DT theory of the doubled A_1 quiver, where the resulting τ-function is a modified Barnes G-function [11, Section 5]. In the case of the DT theory of the resolved conifold τ was shown to be a variant of the Barnes triple sine function [12], and interpreted as a non-perturbative topological string partition function. In this section we show that these τ-functions can be viewed as special cases of the more general definition given above.

6.1. **Uncoupled BPS structures and associated τ-function.** Consider as in [11, Section 5.4] a framed, miniversal family of finite, integral BPS structures over a complex manifold M. At each point $p \in M$ there is a BPS structure consisting of a fixed lattice $\Gamma \cong \mathbb{Z}^{\oplus n}$, with a skew-symmetric form $\langle -,- \rangle$, a central charge $Z_p\colon \Gamma \to \mathbb{C}$, and a collection of BPS invariants $\Omega_p(\gamma) \in \mathbb{Q}$ for $\gamma \in \Gamma$. The miniversal assumption is that the central charges $z_i = Z(\gamma_i)$ of a collection of basis vectors $\gamma_i \in \Gamma$ define local co-ordinates on M. The finiteness assumption ensures that only finitely many $\Omega_p(\gamma)$ are nonzero for any given point $p \in M$, and the integrality condition is that $\Omega_p(\gamma) \in \mathbb{Z}$ for a generic point $p \in M$.

Let us also assume that all the BPS structures parameterised by M are uncoupled, which means that $\Omega_p(\gamma_i) \neq 0$ for $i = 1, 2$ implies $\langle \gamma_1, \gamma_2 \rangle = 0$. This is a very special assumption, which implies [11, Remark A.4] that the BPS invariants $\Omega_p(\gamma) = \Omega(\gamma)$ are independent of $p \in M$. We can then take a basis $(\gamma_1, \cdots, \gamma_{2d})$, where $n = 2d$ as before, such that $\langle \gamma_i, \gamma_j \rangle = 0$ unless $|j - i| = d$, and such that $\Omega(\gamma) \neq 0$ implies that $\gamma \in \bigoplus_{i=1}^d \mathbb{Z}\gamma_i$. We set $\eta_{ij} = 2\pi i \cdot \langle \gamma_i, \gamma_j \rangle$ and take ω_{ij} to be the inverse matrix. Note that $\omega_{i,i+d} \cdot \eta_{i,i+d} = -1$ for all $1 \leqslant i \leqslant d$.

At each point $p \in M$ there is a DT RH problem depending on the BPS structure, and also on a twisted character $\xi\colon \Gamma \to \mathbb{C}^*$. For $1 \leqslant i \leqslant d$ the expressions $\exp(-\epsilon^{-1} z_i) \cdot \xi(\gamma_i)$ are solutions to this problem. We assume that $\xi(\gamma_i) = 1$ for all $1 \leqslant i \leqslant d$ which then implies that $\Omega(\gamma) \neq 0 \implies \xi(\gamma) = 1$. This amounts to fixing a Lagrangian $R \subset Z_\infty$. Then by [11, Theorem 5.3], the DT RH problem has a unique solution whose components $X_i = \exp(x_i)$ can be written in the form

$$x_i = -\epsilon^{-1} z_i + y_i, \qquad y_i = \sum_{\gamma \in \Gamma} \Omega(\gamma) \cdot \langle \gamma, \gamma_i \rangle \cdot \log \Lambda\bigg(\frac{Z(\gamma)}{2\pi i \epsilon}\bigg), \tag{76}$$

where $\Lambda(w)$ is the modified gamma function

$$\Lambda(w) = \frac{e^w \cdot \Gamma(w)}{\sqrt{2\pi} \cdot w^{w - \frac{1}{2}}}. \tag{77}$$

Note that for $1 \leqslant j \leqslant d$ we have $y_j = 0$, and

$$2\pi i \omega_{j,j+d} \cdot y_{j+d} = -\sum_{k_1, \cdots, k_d \in \mathbb{Z}} \Omega\Big(\sum_{p=1}^d k_p \gamma_p\Big) \cdot k_j \cdot \log \Lambda\bigg((2\pi i \epsilon)^{-1} \sum_{p=1}^d k_p z_p\bigg). \tag{78}$$

This implies the relations

$$\omega_{j,j+d} \cdot \frac{\partial y_{j+d}}{\partial z_i} = \omega_{i,i+d} \cdot \frac{\partial y_{i+d}}{\partial z_j}, \qquad \sum_{i=1}^d z_i \cdot \frac{\partial y_{j+d}}{\partial z_i} + \epsilon \cdot \frac{\partial y_{j+d}}{\partial \epsilon} = 0. \tag{79}$$

The τ-function of [11] was then defined as a locally-defined function $\tau\colon M \to \mathbb{C}^*$ satisfying

$$\frac{\partial}{\partial z_i} \log(\tau) = \omega_{i,i+d} \cdot \frac{\partial y_{i+d}}{\partial \epsilon}, \qquad \frac{\partial}{\partial z_{i+d}} \log(\tau) = 0 \tag{80}$$

for $1 \leqslant i \leqslant d$, and homogeneous under simultaneous rescaling of all z_i and ϵ.

6.2. **Comparison of τ-functions.** We can give M a cotangent bundle structure in which the map $\rho\colon M \to B$ just projects to the co-ordinates $(z_1, \cdots, z_d)$. We then have

$$\lambda = \sum_{i=1}^{d} \omega_{i,i+d} \cdot z_{i+d}\, dz_i, \qquad \Omega_0 = \sum_{i=1}^{d} \omega_{i,i+d} \cdot dz_i \wedge dz_{i+d}. \tag{81}$$

Taking the polarised choice for Θ_1 and the Hamiltonian system choice for Θ_0 we have

$$\Theta_1 = \sum_{j=1}^{d} \omega_{j,j+d} \cdot x_j\, dx_{j+d}, \qquad \Theta_0 = \sum_{j=1}^{d} \omega_{j,j+d} \cdot z_j\, dz_{j+d}. \tag{82}$$

The definition (60) of the τ-function becomes

$$d \log(\tau|_M) = \epsilon^{-1} \cdot \sum_{j=1}^{d} \omega_{j,j+d} \cdot z_j\, dy_{j+d}. \tag{83}$$

Expressing τ as a function of the co-ordinates z_i, we find that for $1 \leqslant i \leqslant d$

$$\frac{\partial}{\partial z_i} \log(\tau|_M) = \epsilon^{-1} \cdot \sum_{j=1}^{d} \omega_{j,j+d} \cdot z_j \frac{\partial y_{j+d}}{\partial z_i}, \qquad \frac{\partial}{\partial z_{i+d}} \log(\tau|_M) = 0. \tag{84}$$

Using the relations (79) this gives

$$\frac{\partial}{\partial z_i} \log(\tau|_M) = \epsilon^{-1} \cdot \sum_{j=1}^{d} \omega_{i,i+d} \cdot z_j \frac{\partial y_{i+d}}{\partial z_j} = -\frac{\partial y_{i+d}}{\partial \epsilon} \tag{85}$$

which coincides with (80).

Thus we see that the τ-functions of [11] are particular examples of the τ-functions introduced here. The non-perturbative topological string partition function for the resolved conifold obtained in [12] also fits into this framework. Although the relevant BPS structures are not finite, they are uncoupled, and the above analysis goes through unchanged.

7. A$_2$ example: cubic oscillators and Painlevé I

This example arises from the DT theory of the A$_2$ quiver. It was studied in detail in [14] to which we refer the reader for further details. We show that with the natural choices of symplectic potentials $\Theta_0, \Theta_1, \Theta_\infty$, the resulting τ-function coincides with the particular extension of the Painlevé I τ-function considered by Lisovyy and Roussillon [32].

7.1. **Deformed cubic oscillators.** The base M of the Joyce structure is the moduli space of quadratic differentials on $\mathbb{P}^1$ which have a single pole of order 7 and simple zeroes. By applying automorphisms of $\mathbb{P}^1$ any such differential can be brought to the form

$$Q_0(x)\,dx^{\otimes 2} = (x^3 + ax + b)\,dx^{\otimes 2} \tag{86}$$

for some $a, b \in \mathbb{C}$. The condition on simple zeroes is then that $4a^3 + 27b^2 \neq 0$. The group $\mu_5 \subset \mathbb{C}^*$ of fifth roots of unity acts on $\mathbb{P}^1$ preserving the form of the expression (86) but modifying (a, b) via the action $(a, b) \mapsto (\zeta^3\,a, \zeta^2\,b)$. The space M is then the quotient by this action. Thus

$$M = \big\{(a, b) \in \mathbb{C}^2 : 4a^3 + 27b^2 \neq 0\big\}/\mu_5. \tag{87}$$

The $\mathbb{C}^*$-action on M is obtained by rescaling the quadratic differential $Q_0(x)dx^{\otimes 2}$ with weight 2. In terms of the above parameterisation it is given explicitly by $t \cdot (a, b) = (t^{4/5}\,a, t^{6/5}\,b)$.

Remark 7.1. In relation to Remark 2.3(iii) note that this $\mathbb{C}^*$-action is not well-defined on the space $\mathbb{C}^2$ with co-ordinates (a, b). Thus one really needs to take the μ_5-quotient in (87) to get a Joyce structure as defined in Section 2.4.

We consider the deformed cubic oscillator

$$y''(x) = Q(x)y(x), \qquad Q(x) = \epsilon^{-2}Q_0(x) + \epsilon^{-1}Q_1(x) + Q_2(x), \tag{88}$$

where the terms in the potential are

$$Q_1(x) = \frac{p}{x-q} + r, \qquad Q_2(x) = \frac{3}{4(x-q)^2} + \frac{r}{2p(x-q)} + \frac{r^2}{4p^2}, \tag{89}$$

depending on $(q, r) \in \mathbb{C}^2$, with p defined implicitly by $p^2 = q^3 + aq + b$. This is gauge equivalent to the pencil of connections $\nabla - \epsilon^{-1}\Phi$ on the trivial rank 2 bundle on $\mathbb{P}^1$ with

$$\nabla = d - \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} \frac{dx}{2p}, \qquad \Phi = \begin{pmatrix} p & x^2 + xq + q^2 + a \\ x - q & -p \end{pmatrix} dx. \tag{90}$$

7.2. **Local co-ordinates.** The double cover Σ corresponding to a point of M is the elliptic curve $y^2 = x^3 + ax + b$. We take a basis of cycles $(\gamma_1, \gamma_2) \subset H_1(\Sigma, \mathbb{Z})$ ordered so that $\gamma_1 \cdot \gamma_2 = 1$. Then $\eta_{12} = 2\pi i$ and $\omega_{12} = -1/2\pi i$. The local period co-ordinates on M are

$$z_i = \int_{\gamma_i} y\,dx = \int_{\gamma_i} (x^3 + ax + b)^{1/2}\,dx, \tag{91}$$

The space of deformed cubic oscillators considered above can be identified with the quotient $X^{\#} = T_M/(2\pi i)\, T_M^{\mathbb{Z}}$. Under this identification the fibre co-ordinates θ_i on $X = T_M$ are

$$\theta_i = -\int_{\gamma_i} \frac{Q_1}{2\sqrt{Q_0}} = -\int_{\gamma_i} \left(\frac{p}{x-q} + r\right)\frac{dx}{2y}. \tag{92}$$

We shall need the periods and quasi-periods of the elliptic curve Σ. They are given by

$$\omega_i = \int_{\gamma_i} \frac{dx}{2y}, \qquad \eta_i = -\int_{\gamma_i} \frac{x\,dx}{2y}, \tag{93}$$

and satisfy the Legendre relation $\omega_2\eta_1 - \omega_1\eta_2 = 2\pi i$. There are relations

$$\frac{\partial}{\partial a} = -\eta_1 \frac{\partial}{\partial z_1} - \eta_2 \frac{\partial}{\partial z_2}, \qquad \frac{\partial}{\partial b} = \omega_1 \frac{\partial}{\partial z_1} + \omega_2 \frac{\partial}{\partial z_2}. \tag{94}$$

In terms of the fibre co-ordinates (θ_a, θ_b) associated to the co-ordinates (a, b) we then have

$$\theta_i = -\eta_i \theta_a + \omega_i \theta_b. \tag{95}$$

The functions (θ_a, θ_b) are given explicitly by

$$\theta_a = -\frac{1}{4}\int_{(q,-p)}^{(q,p)} \frac{dx}{y}, \qquad \theta_b = \frac{1}{4}\int_{(q,-p)}^{(q,p)} \frac{x dx}{y} - r. \tag{96}$$

7.3. **Joyce structure.** The Joyce structure on $X = T_M$ is obtained by taking the connection h_ϵ on $\pi\colon X \to M$ to be the isomonodromy connection for the equation (88). In terms of the co-ordinates (a, b, q, r) this is given by

$$h_\epsilon\left(\frac{\partial}{\partial a}\right) = -\frac{2p}{\epsilon}\frac{\partial}{\partial q} - \frac{q}{\epsilon}\frac{\partial}{\partial r} + \left(\frac{\partial}{\partial a} - \frac{r}{p}\frac{\partial}{\partial q} - \frac{r^2(3q^2+a) - qpr}{2p^3}\frac{\partial}{\partial r}\right), \tag{97}$$

$$h_\epsilon\left(\frac{\partial}{\partial b}\right) = -\frac{1}{\epsilon}\frac{\partial}{\partial r} + \left(\frac{\partial}{\partial b} + \frac{r}{2p^2}\frac{\partial}{\partial r}\right). \tag{98}$$

The Plebański function has the explicit expression

$$W = \frac{1}{4(4a^3+27b^2)p}\Big(2apr^3 - (6aq^2 - 9bq + 4a^2)r^2 - 3p(3b - 2aq)r - 2ap^2\Big). \tag{99}$$

The fact that W is cubic in r corresponds to the good Lagrangian property of [17, Section 4] (see particularly equation (53)).

In the co-ordinates (a, b, q, r) the generator for the $\mathbb{C}^*$-action on X is

$$E = \frac{4a}{5}\frac{\partial}{\partial a} + \frac{6b}{5}\frac{\partial}{\partial b} + \frac{2q}{5}\frac{\partial}{\partial q} + \frac{r}{5}\frac{\partial}{\partial r}. \tag{100}$$

The change of basis (94) gives $\Omega_0 = -(2\pi i)^{-1} \cdot dz_1 \wedge dz_2 = da \wedge db$. Thus

$$i_E(\Omega_0) = -\frac{6b}{5}da + \frac{4a}{5}db. \tag{101}$$

The expressions (94) and (95) and a short calculation using (96) gives

$$2i\Omega_I = -da \wedge d\theta_b + db \wedge d\theta_a = dq \wedge dp + da \wedge dr. \tag{102}$$

It then follows that

$$i_E(2i\Omega_I) = \frac{2q}{5}dp - \frac{3p}{5}dq + \frac{4a}{5}dr - \frac{r}{5}da. \tag{103}$$

7.4. **Twistor fibre Z_∞.** The twistor fibre Z_∞ is the space of leaves of the foliation defined by the connection $h = h_\infty$. The functions

$$\phi_1 = q + \frac{ar}{p}, \qquad \phi_2 = \frac{r}{2p}, \tag{104}$$

descend to Z_∞ because they are constant along the flows (97) and (98) with $\epsilon = \infty$. Moreover, if we restrict to a fibre F of the projection $\pi\colon X = T_M \to M$ by fixing (a, b), then

$$-\frac{1}{2\pi i} \cdot d\theta_1 \wedge d\theta_2\big|_F = d\theta_a \wedge d\theta_b\big|_F = -\frac{dr}{2p} \wedge dq\big|_F = d\phi_1 \wedge d\phi_2\big|_F, \tag{105}$$

so (ϕ_1, ϕ_2) are Darboux co-ordinates on Z_∞.

Since the $\mathbb{C}^*$-action rescales ϕ_1 and ϕ_2 with weights $-\frac{2}{5}$ and $\frac{2}{5}$ respectively, we have

$$i_E(\Omega_\infty) = i_E(d\phi_1 \wedge d\phi_2) = \frac{2}{5}d(\phi_1\phi_2). \tag{106}$$

Thus the Joyce function of Section 3.2, which is well-defined up to the addition of a constant, can be taken to be $V = -2\phi_1\phi_2/5$.

The distinguished fixed point of Z_∞ is defined by $(\phi_1, \phi_2) = (0, \infty)$. It is an isolated fixed point. The linear Joyce connection on M is the one whose flat co-ordinates are (a, b), and the negative and positive weight spaces V_- and V_+ of Lemma 3.7 are spanned by $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial b}$ respectively. The Joyce metric is

$$g = \frac{1}{5} \cdot (da \otimes db + db \otimes da). \tag{107}$$

Remark 7.2. The alternative Plebański function $U\colon X \to \mathbb{C}$ of Appendix B is given by

$$U = \frac{1}{2}\int_{(q,-p)}^{(q,p)} (x^3 + ax + b)^{1/2}\, dx, \tag{108}$$

because a simple calculation using the flows (97) and (98) shows that

$$\frac{\partial U}{\partial b}\bigg|_{\phi} = h\Big(\frac{\partial}{\partial b}\Big)U = -\theta_a, \qquad \frac{\partial U}{\partial a}\bigg|_{\phi} = h\Big(\frac{\partial}{\partial a}\Big)U = \theta_b. \tag{109}$$

It would be interesting to know whether an analogous formula to (108) exists for all Joyce structures of class $S[A_1]$.

7.5. **Hamiltonian system and τ-function.** Setting $\phi_2 = 0$ defines a Lagrangian $R \subset Z_\infty$ whose inverse image $Y = q_\infty^{-1}(L) \subset X$ is the 3-dimensional locus $r = 0$. This corresponds to choosing the reference connection ∇ in (90) to be d.

There is a natural cotangent bundle structure on M for which $\rho\colon M \to B$ is the projection to the co-ordinate a, which is the Painlevé time. The associated Liouville form is $\lambda = b da$. The map $f\colon Y \to B$ of Section 4 is given by $(a, b, q) \mapsto a$. The Hamiltonian form is $\varpi = b da$. By (102) the form $2i\Omega_I$ induces the relative symplectic form $dq \wedge dp$. The horizontal leaves of the connection k_∞ are obtained by varying a while keeping (q, p) fixed.

The twistor fibre Z_1 is the space of framed local systems, and is covered by birational Fock-Goncharov co-ordinate charts $(\exp(x_1), \exp(x_2))$. We take the polarised choice for the symplectic potential Θ_1, and the Hamiltonian system choice for Θ_-. Thus

$$\Theta_0 = i_E(\Omega_-) + \lambda, \qquad \Theta_1 = -(2\pi i)^{-1} x_1 dx_2. \tag{110}$$

Omitting the pullbacks q^* from the notation, the definition of the τ-function reads

$$d\log(\tau) = \epsilon^{-2}\cdot\Theta_0 + \epsilon^{-1}\cdot 2i\Theta_I + \Theta_\infty - \Theta_1. \tag{111}$$

With the above choices this gives

$$d\log(\tau|_Y) = \epsilon^{-2}\Big(-\frac{6b}{5}da + \frac{4a}{5}db + b da\Big) + \epsilon^{-1}\Big(\frac{2q}{5}dp - \frac{3p}{5}dq\Big) + \frac{1}{2\pi i}x_1 dx_2. \tag{112}$$

7.6. **Comparison with the Painlevé τ-function.** We now compare with the Painlevé τ-function computed in [32] and use notation as there. Consider the form ω defined in (3.4). It considers local co-ordinates t, m_a, m_b with m_a, m_b constant under Painlevé flow. Let us re-express ω in terms of the local co-ordinates t, q, p. Using the relations $q_t = p$, $p_t = 6q^2 + t$ and $H_t = -q$ appearing on page 1, we get

$$\frac{\omega}{2} = H dt + \frac{1}{5}\Big(4t dH + 3q_t dq - 2q dq_t - (4tH_t + 3q_t^2 - 2qq_{tt})dt\Big) \tag{113}$$

$$= -\frac{H}{5}dt + \frac{4t}{5}dH + \frac{3p}{5}dq - \frac{2q}{5}dp. \tag{114}$$

Then the authors show that $\Omega = d\omega = 4\pi i d\nu_1 \wedge d\nu_2$ and define τ_{LR} by

$$d\log(\tau_{LR}) = \tfrac{1}{2}\omega - 2\pi i \nu_1 d\nu_2. \tag{115}$$

To compare with the τ-function τ_{TB} of the previous section set $p_{LR} = -2p_{TB}$, $q_{LR} = q_{TB}$, $t = 2a$, $H = 2b$. Here the subscript TB means as in this paper, whereas LR means as appearing in [32]. Also we should set $r = 0$ and $\epsilon = \frac{1}{2}$. The connection $\nabla - \epsilon^{-1}\Phi$ of (90) then becomes gauge equivalent to the system (2.1a) of [32]. To match the monodromy data we set $x_1 = 2\pi i\nu_1$ and $x_2 = -2\pi i \nu_2$. Then in terms of the notation of this paper, the definition (115) becomes

$$d\log\tau_{LR} = -\frac{4b}{5}da + \frac{16a}{5}db - \frac{6p}{5}dq + \frac{4q}{5}dp + \frac{1}{2\pi i}x_1 dx_2, \tag{116}$$

which coincides with (112). It follows that the two τ-functions, which are both well-defined up to multiplication by a nonzero constant, coincide.

Appendix A. Pre-quantum line bundles in the holomorphic setting

The following result on pre-quantum line bundles in the holomorphic setting is standard and well-known, but is so relevant to the definition of the τ-function that it seems worth briefly recalling the proof. As in the rest of the paper, all line bundles, connections, symplectic forms etc., considered will be holomorphic, but to make clear the distinction from the more familiar geometric quantization story we will re-emphasize this at several places.

Theorem A.1. *Let M be a complex manifold equipped with a holomorphic symplectic form Ω. Assume that the de Rham cohomology class $[\Omega]$ satisfies the integrality condition*

$$[\Omega] \in (2\pi i)\cdot H^2(M,\mathbb{Z}) \subset H^2(M,\mathbb{C}). \tag{117}$$

Then there is a holomorphic line bundle with holomorphic connection whose curvature is Ω.

Proof. Consider the following diagram of sheaves of abelian groups on M, in which $d\mathscr{O}$ is the sheaf of closed holomorphic 1-forms, d is the de Rham differential, and the unlabelled arrows are the obvious inclusions.

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\cdot 2\pi i} & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^* \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \xrightarrow{\cdot 2\pi i} & \mathscr{O} & \xrightarrow{\exp} & \mathscr{O}^* \\ & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d\log} \\ & & d\mathscr{O} & \longrightarrow & d\mathscr{O} \end{array} \tag{118}$$

The sheaf of 1-forms satisfying $d\Theta = \Omega$ is a torsor for the sheaf $d\mathscr{O}$, and hence defines an element $\eta \in H^1(M, d\mathscr{O})$. The image of η via the boundary map in the central column is the class $[\Omega] \in H^2(M, \mathbb{C})$. By the integrality assumption, the image of η via the boundary map in the right-hand column is $1 \in H^2(M, \mathbb{C}^*)$. Thus, by the long exact sequence in cohomology for the right-hand column there exist elements $\phi \in H^1(M, \mathscr{O}^*)$ satisfying $d\log(\phi) = \eta$. Such a class ϕ defines a line bundle L on M, and one can then see that L has a holomorphic connection with curvature Ω.

Translating the above discussion into Čech cohomology gives the following. Take a covering of X by open subsets U_i such that all intersections $U_{i_1,\cdots,i_n} = U_{i_1} \cap \cdots \cap U_{i_n}$ are contractible. Choose 1-forms Θ_i on U_i satisfying $d\Theta_i = \Omega|_{U_i}$, and set $\Theta_{ij} = \Theta_i|_{U_{ij}} - \Theta_j|_{U_{ij}}$. Then $d\Theta_{ij} = 0$ and the collection $\{\Theta_{ij}\}$ defines a Cech 1-cocycle for the sheaf $d\mathscr{O}$. On U_{ij} we can now write

$$d\log\phi_{ij} = \Theta_{ij} = \Theta_i|_{U_{ij}} - \Theta_j|_{U_{ij}} \tag{119}$$

for functions $\phi_{ij} \colon U_{ij} \to \mathbb{C}^*$. The integrality assumption implies that, after replacing ϕ_{ij} by $r_{ij} \cdot \phi_{ij}$ for constants $r_{ij} \in \mathbb{C}^*$, we can assume that

$$\phi_{ij}|_{U_{ijk}} \cdot \phi_{jk}|_{U_{ijk}} \cdot \phi_{ki}|_{U_{ijk}} = 1. \tag{120}$$

We then define the line bundle L by gluing the trivial line bundles L_i over U_i using multiplication by ϕ_{ij}. The relations (119) show that the connections $\nabla_i = d + \Theta_i$ on L_i glue to a connection ∇ on L. Since ∇_i has curvature $d\Theta_i = \Omega|_{U_i}$, the glued connection ∇ has curvature Ω. □

Remarks A.2.
(i) The relation (119) can be phrased as the statement that the gluing map ϕ_{ij} for the line bundle L is the exponential generating function relating the symplectic potentials $\Theta_i|_{U_{ij}}$ and $\Theta_j|_{U_{ij}}$ on U_{ij}.
(ii) Suppose $U \subset M$ is a contractible open subset. Then sections $s \in H^0(U, L)$ up to scale are in bijection with symplectic potentials Θ on U. Given s we can write $\nabla(s) = \Theta \cdot s$ with $d\Theta = \Omega$. Conversely, given another symplectic potential Θ' on U we can write $\Theta' - \Theta = d\log(f)$ and hence define a section $s' = f \cdot s$ satisfying $\nabla(s') = \Theta' \cdot s'$.

Appendix B. Another Plebański function

Throughout the paper we described the complex hyperkähler structure on $X = T_M$ in terms of local co-ordinates (z_i, θ_j) using the Plebański function $W(z_i, \theta_j)$. For the sake of completeness we briefly discuss here an alternative generating function, also introduced by Plebański [36]. This function only appears in the body of the paper in Remark 7.2 where we compute it in an interesting example. In the literature the function we have been calling W is usually called the Plebański function of the second kind, whereas the function U introduced below is the Plebański function of the first kind.

Take notation as in Section 2, and introduce a new system of co-ordinates on X by combining local Darboux co-ordinates $(z_1, \cdots, z_n)$ on the twistor fibre $Z_0 = M$ as in Section 2.2, with the pullback of a system of Darboux co-ordinates $(\phi_1, \cdots, \phi_n)$ on the twistor fibre fibre Z_∞. Then by definition

$$\Omega_+ = \frac{1}{2}\sum_{p,q} \omega_{pq} \cdot dz_p \wedge dz_q, \qquad \Omega_- = \frac{1}{2}\sum_{p,q} \omega_{pq} \cdot d\phi_p \wedge d\phi_q. \tag{121}$$

To find an expression for Ω_I, note that since Z_∞ is the quotient of X by the distribution spanned by the vector fields h_i of (3), we can write

$$\frac{\partial}{\partial z_i}\bigg|_\phi = \frac{\partial}{\partial z_i}\bigg|_\theta + \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 W}{\partial\theta_i \partial\theta_p} \cdot \frac{\partial}{\partial\theta_q}, \tag{122}$$

where the subscripts indicate which variables are being held fixed. Then

$$\frac{\partial}{\partial z_r}\bigg|_\phi \Big(\sum_k \omega_{ks}\theta_k\Big) = \frac{\partial^2 W}{\partial\theta_r\partial\theta_s}, \tag{123}$$

and the symmetry of the right-hand side shows that there is then a locally-defined function $U = U(z_i, \phi_j)$ on X satisfying

$$\frac{\partial U}{\partial z_s} = \sum_k \omega_{ks}\theta_k, \qquad \frac{\partial^2 U}{\partial z_r \partial z_s} = \frac{\partial^2 W}{\partial\theta_r\partial\theta_s}. \tag{124}$$

The formula (13) then becomes

$$2i\Omega_I = d\Big(\sum_{p,q} \omega_{pq}\theta_p \, dz_q\Big) = \sum_{p,q} \frac{\partial^2 U}{\partial\phi_p \partial z_q} \cdot d\phi_p \wedge dz_q. \tag{125}$$

Consider the restriction of the form Ω_- to a fibre of the projection $\pi\colon X \to M$. The formula (14) then gives

$$\sum_{p,q} \omega_{pq} \cdot d\theta_p \wedge d\theta_q = \sum_{p,q} \omega_{pq} \cdot d\phi_p \wedge d\phi_q. \tag{126}$$

Using (124) we then find that for all indices i, j

$$\sum_{r,s} \eta_{rs} \cdot \frac{\partial^2 U}{\partial\phi_i \partial z_r} \cdot \frac{\partial^2 U}{\partial\phi_j \partial z_s} = -\omega_{ij}. \tag{127}$$

These are known as Plebański's first heavenly equations.

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