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JOYCE STRUCTURES ON SPACES OF QUADRATIC DIFFERENTIALS I

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Abstract. Consider the space parameterising curves of genus g equipped with a quadratic differential with simple zeroes. We use the geometry of isomonodromic deformations to construct a complex hyperkähler structure on the total space of its tangent bundle. This gives a new class of examples of the Joyce structures introduced in [17] in relation to Donaldson-Thomas theory. Our results relate to recent work in mathematical physics on supersymmetric gauge theories of class $S[A_1]$.

1. Introduction

Consider the moduli space $\mathcal{M}_C(E,\Phi)$ parameterising $\mathrm{SL}_2(\mathbb{C})$ Higgs bundles on a complex projective curve C [36]. It has a hyperkähler structure, and a map to the Hitchin base (parameterising quadratic differentials on C) whose general fibres are abelian varieties $(S^1)^{2d}$. In this paper we study a moduli space $\mathcal{M}(C,E,\nabla,\Phi)$ which behaves in some ways like a complexification of $\mathcal{M}_C(E,\Phi)$. This space parameterises the data of a curve C, together with a bundle E equipped with both a flat connection ∇ and a Higgs field Φ. We show that it has a meromorphic complex hyperkähler structure, and a map to the generic Hitchin base (parameterising curves equipped with a quadratic differential) with general fibre $(\mathbb{C}^*)^{2d}$.

Our interest in these complexified moduli spaces arises from a general programme which attempts to encode the Donaldson-Thomas (DT) invariants of a CY_3 (three-dimensional Calabi-Yau) triangulated category $\mathcal{D}$ in a geometric structure on its space of stability conditions $\mathrm{Stab}(\mathcal{D})$. The relevant geometry is a kind of non-linear Frobenius structure, and was christened a Joyce structure in [17] in honour of the paper [44] where the main ingredients were first identified. In later work with Strachan [20] it was shown that a Joyce structure defines a complex hyperkähler structure on the total space of the tangent bundle. However, the procedure for producing a Joyce structure from the DT invariants is highly conjectural, and requires solving a family of non-linear Riemann-Hilbert (RH) problems determined by the DT invariants [15]. At present there are no existence or uniqueness results for these problems, and even defining them requires strong conditions on the growth of the invariants.

The aim of this paper is to test these ideas for an interesting class of CY_3 triangulated categories $\mathcal{D}(g,m)$. We refer to these categories as being of class $S[A_1]$, because their DT invariants coincide with the BPS invariants of the corresponding supersymmetric gauge theories. They are indexed by a genus $g \geq 0$ and a collection of positive integers $m = \{m_1, \cdots, m_l\}$. When $l > 0$ they are defined via quivers with potential associated to marked bordered surfaces [47], and also arise as Fukaya categories of non-compact CY threefolds $Y(g, m)$ fibered over complex curves [56]. When $l = 0$ the appropriate categories were constructed by Haiden [35].

In the cases $l > 0$ the stability spaces of the categories $\mathcal{D}(g, m)$ were studied by the author with Smith [19]. Assuming for simplicity that all $m_i \geq 3$, the main result is an isomorphism of complex orbifolds

$$\operatorname{Stab} \mathcal{D}(g, m) / \operatorname{Aut} \mathcal{D}(g, m) \cong \operatorname{Quad}(g, m). \tag{1}$$

The left-hand side of this relation is the quotient of a distinguished connected component of the stability space by the group of exact $\mathbb{C}$-linear auto-equivalences preserving this component. The right-hand side is the moduli space of curves of genus g equipped with a quadratic differential having poles of orders m_i and simple zeroes. These results were extended to the more difficult $l = 0$ setting by Haiden [35].

We focus here on the holomorphic case $l = 0$, and use the moduli spaces $\mathcal{M}(C, E, \nabla, \Phi)$ to construct a Joyce structure on the quotient $M = \operatorname{Quad}(g, m)$. We then investigate various geometric properties of this Joyce structure. In particular we study its twistor space, and formulate the notion of a good Lagrangian, which seems to be a promising approach to defining "physicist's slices" in stability space.

We leave for future research the problem of linking these Joyce structures to the non-linear RH problems defined by the DT theory of the categories $\mathcal{D}(g, m)$. This would probably be more easily done in the cases $l > 0$, using the Fock-Goncharov cluster structure on the wild character variety [28] following the approach of Gaiotto, Moore and Neitzke [32]. To carry out this programme, one would first need to generalise the results of this paper to the meromorphic setting.

1.1. **Some related work.** Physicists have long understood that the BPS invariants of certain classes of supersymmetric field theories can be encoded in quaternionic metrics on associated spaces of vacua. In the case of $N = 2$ supersymmetric gauge theories in four dimensions, Gaiotto, Moore and Neitzke [31] showed how to construct the corresponding twistor space using non-linear RH problems determined by the BPS invariants. They illustrated their theory by a detailed analysis [32] in the case of theories of class $S[A_1]$. The analogous constructions in string theory have been studied by Alexandrov, Pioline and others [2, 3].

These physical ideas directly inspired the contents of this paper, but it is important to note two differences. Firstly, our story is complexified: our spaces have twice the dimension of their physical counterparts, and are equipped with complex, rather than real, hyperkähler structures. Secondly, we are working with the conformal limit [30]; this leads to simpler geometry, which in particular cases can even be written down explicitly [18]. The relationship between the two approaches has recently been clarified [4], and there are detailed studies from both points-of-view in the case of the resolved conifold [5, 6, 7, 16].

When applied to theories of class $S[A_1]$, this general story makes contact with a rich area of mathematics involving moduli spaces of vector bundles with connections on complex curves. This relies on a fundamental link between non-linear RH problems and the monodromy of differential equations which can be traced back to the work of Sibuya [55], the analytic bootstrap of Voros [59], and then on through the ODE/IM correspondence [26] to more recent work involving resurgence in quantum mechanics [38], and relations to cluster algebras [39].

Partial solutions to the non-linear RH problems relevant to categories of class $S[A_1]$ have been obtained by Allegretti using pencils of opers on Riemann surfaces [8, 9]. A complete family of solutions has been given in the particular case $g = 0$, $m = \{7\}$ by considering opers with apparent singularities [18]. Similar pencils of opers appear in the works [22, 23] under the name quantum curves. In the special 'uncoupled' case when the intersection form vanishes, Iwaki and Kidwai [40, 41] constructed solutions to the RH problems using topological recursion.

1.2. **Summary of contents.** The text switches between the abstract setting of a general Joyce structure and the particular examples relevant to the categories of class $S[A_1]$. The main contents are as follows:

- We begin in Section 2 with the definition of a Joyce structure. This can be expressed either in terms of flat pencils of non-linear symplectic connections, or via complex hyperkähler structures, and we explain the relationship between these two approaches.

- The heart of the paper is Section 3, where we construct the Joyce structures relevant to categories of class $S[A_1]$. These "complexified Hitchin systems" parameterise collections (C, E, ∇, Φ) of a compact complex curve C, and a holomorphic vector bundle E equipped with both a connection ∇ and a Higgs field Φ.
- Section 4 introduces the notion of a good Lagrangian submanifold in the base of a Joyce structure. In the case of categories of class $S[A_1]$ we show that the submanifold of quadratic differentials on a fixed curve C is an example of a good Lagrangian. In general these submanifolds provide a promising approach to defining "physicist's slices" in stability space.

- In Section 5 we recall the definition of the twistor space $p\colon Z \to \mathbb{P}^1$ of a complex hyperkähler manifold, and discuss the special features arising in the case of a Joyce structure. This allows us to give a new perspective on the conjectural construction of Joyce structures via non-linear RH problems from [17].

There are several aspects of the story which we leave for future research:

- Some of the proofs in Section 3 are rather sketchy, and we adopt a rather cavalier approach to the various moduli spaces that appear. A complete treatment would require the language of stacks, and a more careful proof of some of the results, particularly the crucial Theorem 3.3. Full details of these constructions will appear in [21]. It would also be interesting to generalise the material of Section 3 to the case of quadratic differentials with a positive numbers of poles $l > 0$.
- We do not prove that the Joyce structures we construct give solutions to the RH problems defined by the DT theory of the corresponding category $\mathcal{D}(g, m)$. In the case $l > 0$ this seems possible following the approach of Gaiotto, Moore and Neitzke [32], and using existing results in exact WKB analysis [50, 51]. The holomorphic case $l = 0$ would seem torequire new ideas, since the appropriate co-ordinate systems on the character variety associated to saddle-free differentials don't appear to be known.

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2. Joyce structures

This section introduces the geometric structures that will appear throughout the rest of the paper. They can be described either in terms of flat pencils of symplectic, non-linear connections, or via complex hyperkähler structures. A large class of examples will be described in Section 3.

2.1. **Pencils of non-linear connections.** Let $\pi\colon X \to M$ be a holomorphic submersion of complex manifolds. There is a short exact sequence of vector bundles

$$0 \longrightarrow V(\pi) \xrightarrow{i} T_X \xrightarrow{\pi_*} \pi^*(T_M) \longrightarrow 0, \tag{2}$$

where $V(\pi) = \ker(\pi_*)$ is the sub-bundle of vertical tangent vectors. Recall that a (non-linear or Ehresmann) connection on π is a splitting of this sequence, given by a map of bundles $h\colon \pi^*(T_M) \to T_X$ satisfying $\pi_* \circ h = 1$.

Consider the special case in which $\pi\colon X = T_M \to M$ is the total space of the tangent bundle of M. There is then a canonical isomorphism $\nu\colon \pi^*(T_M) \to V(\pi)$ identifying the vertical tangent vectors in the bundle with the bundle itself, and we set $v = i \circ \nu$.

$$\begin{tikzcd} 0 \arrow[r] & V(\pi) \arrow[r, "i"] & T_X \arrow[r, "\pi_*"] & \pi^*(T_M) \arrow[r] \arrow[l, bend right, "h_\epsilon"'] \arrow[ll, bend left, "\nu"] & 0 \end{tikzcd} \tag{3}$$

Definition 2.1. *A ν-pencil of connections on $\pi\colon X = T_M \to M$ is a family of connections of the form $h_\epsilon = h + \epsilon^{-1}v$ parameterised by $\epsilon^{-1} \in \mathbb{C}$. We say that the pencil is flat if each connection h_ϵ is flat.*

Suppose that M is equipped with a holomorphic symplectic form $\omega \in H^0(M, \wedge^2 T_M^*)$. This induces a translation-invariant symplectic form ω_m on each fibre $X_m = \pi^{-1}(m) \subset X$. We say that a connection on π is symplectic if the locally-defined parallel transport maps $X_{m_1} \to X_{m_2}$ preserve these forms. Note that if one of the connections in a ν-pencil $h_\epsilon = h + \epsilon^{-1}v$ is symplectic then they all are.

2.2. **Expression in co-ordinates.** We denote the complex dimension of M by $n = 2d$. Given local co-ordinates $(z_1, \cdots, z_n)$ on M there are associated linear co-ordinates $(\theta_1, \cdots, \theta_n)$ on the tangent spaces $T_{M,p}$ obtained by writing a tangent vector in the form $\sum_i \theta_i \cdot \partial/\partial z_i$. We thus get induced local co-ordinates (z_i, θ_j) on the space $X = T_M$. We always assume that z_i are Darboux, so that

$$\omega = \frac{1}{2} \sum_{p,q} \omega_{pq} \cdot dz_p \wedge dz_q, \tag{4}$$

with ω_{pq} a constant skew-symmetric matrix. We denote by η_{pq} the inverse matrix.

Given a symplectic ν-pencil $h_\epsilon = h + \epsilon^{-1}v$ we can write

$$v\Big(\frac{\partial}{\partial z_i}\Big) = \frac{\partial}{\partial \theta_i}, \qquad h\Big(\frac{\partial}{\partial z_i}\Big) = \frac{\partial}{\partial z_i} + \sum_{p,q} \eta_{pq} \cdot \frac{\partial W_i}{\partial \theta_p} \cdot \frac{\partial}{\partial \theta_q}, \tag{5}$$

for (non-unique) functions $W_i\colon X \to \mathbb{C}$. The pencil is flat precisely if we can take $W_i = \partial W/\partial \theta_i$ for a single function $W\colon X \to \mathbb{C}$, which moreover satisfies Plebański's second heavenly equations

$$\frac{\partial^2 W}{\partial \theta_i \partial z_j} - \frac{\partial^2 W}{\partial \theta_j \partial z_i} = \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial^2 W}{\partial \theta_j \partial \theta_q}. \tag{6}$$

The function $W(z_i, \theta_j)$ will be called the Plebański function.

The symplectic form ω defines an isomorphism $\omega^\flat\colon T_M \to T_M^*$. Pulling back the canonical symplectic form on T_M^* via this isomorphism gives a symplectic form

$$(\omega^\flat)^*(\Omega) = \sum_{p,q} \omega_{pq} \cdot dz_p \wedge d\theta_q \tag{7}$$

on X which will appear in Theorem 2.2 below.

2.3. **Complex hyperkähler structure.** By a complex hyperkähler structure on a complex manifold X we mean the data of a non-degenerate symmetric bilinear form $g\colon T_X \otimes T_X \to \mathscr{O}_X$, together with endomorphisms $I, J, K \in \operatorname{End}(T_X)$ satisfying the quaternion relations

$$I^2 = J^2 = K^2 = IJK = -1, \tag{8}$$

which preserve the form g, and which are parallel with respect to the holomorphic Levi-Civita connection. Such structures have appeared before in the literature, often under different names [13, 27, 42, 43].

Consider as in Section 2.1 the situation in which $X = T_M$ is the tangent bundle of a complex manifold M equipped with a holomorphic symplectic form ω. We define the closed holomorphic 2-forms on X

$$\Omega_I(w_1, w_2) = g(I(w_1), w_2), \qquad \Omega_\pm(w_1, w_2) = g((J \pm iK)(w_1), w_2). \tag{9}$$

Then Ω_I is a symplectic form, but $\Omega_\pm$ are degenerate.

Theorem 2.2. *Let $h_\epsilon = h + \epsilon^{-1} v$ be a flat, symplectic ν-pencil of connections on the map $\pi\colon X \to M$. Then there is a unique complex hyperkähler structure on X such that*

$$I \circ v = i \cdot v, \qquad I \circ h = -i \cdot h, \qquad J \circ v = h, \tag{10}$$

$$\Omega_- = \pi^*(\omega), \qquad 2i\Omega_I = (\omega^\flat)^*(\Omega). \tag{11}$$

Moreover, any complex hyperkähler structure on X satisfying (11) *arises in this way.*

Proof. Consider a flat, symplectic ν-pencil $h_\epsilon = h + \epsilon^{-1} v$. The images of the maps h, v define complementary sub-bundles $H, V \subset T_X$ so the formulae (10) uniquely define operators I, J satisfying $I^2 = J^2 = -1$. Defining $K = IJ$ we also have $K^2 = -1$. The second relation of

(11) then uniquely defines a non-degenerate bilinear form g on X, but the condition that g is symmetric is non-trivial, being equivalent to the relation

$$\Omega_I(I(w_1), I(w_2)) = \Omega_I(w_1, w_2). \tag{12}$$

Using (10), and the fact that V is Lagrangian for Ω_I, this is equivalent to the statement that H is also Lagrangian. Plugging (5) into (7) this holds precisely if $\partial W_i/\partial\theta_j = \partial W_j/\partial\theta_i$, which is part of the condition that the connections h_ϵ are flat. Note that H, V are now isotropic for g, and

$$2g(h(w_1), v(w_2)) = \omega(w_1, w_2). \tag{13}$$

It follows that I, J, K preserve g, and that the first relation of (11) also holds.

The operator $I_\epsilon = I + \epsilon^{-1}(J + iK)$ on T_M satisfies $I_\epsilon^2 = -1$ and has $+i$ eigenspace V and $-i$ eigenspace $H_\epsilon = \operatorname{im}(h_\epsilon)$. It defines an almost complex structure on the underlying real manifold of M, which extended to the complexified tangent bundle has $-i$ eigenspace $\bar{V} \oplus H_\epsilon$. The fact that h_ϵ is flat and holomorphic then implies that I_ϵ is integrable. If we knew the form $\Omega_\epsilon(-,-) = g(I_\epsilon -, -)$ was closed, a standard argument from Kähler geometry would show that the complex structure I_ϵ is parallel. Knowing this for all $\epsilon^{-1} \in \mathbb{C}$ we could then deduce that I, J, K are parallel, and hence that (g, I, J, K) is a complex hyperkähler structure.

Since Ω_ϵ is a linear combination of the forms Ω_-, Ω_I, which are clearly closed, and Ω_+, it remains to check that this last form is closed. Since the connection h is flat, we can take local co-ordinates (z_i, w_j) on X, with z_i pulled back from M as before, so that H is spanned by the vectors $\partial/\partial z_i$. Since $\ker(\Omega_+) = H$, in these co-ordinates there is an expression

$$\Omega_+ = \sum_{i,j} f_{ij}(z, w)\, dw_i \wedge dw_j. \tag{14}$$

Note that the restriction of Ω_+ to a fibre of $\pi\colon X \to M$ is closed, and coincides with the relative symplectic form induced by the symplectic form ω on M. Thus we see that $d\Omega_+ = 0$ precisely if $f_{ij}(z, w)$ is independent of z, and this holds by the assumption that h is symplectic.

Conversely, given a complex hyperkähler structure on X satisfying (11), the $+i$ eigenspace of I coincides with $V = \ker(\pi_*)$, and we can define a connection h by declaring its image to be the $-i$ eigenspace. We then obtain a ν-pencil of connections $h_\epsilon = h + \epsilon^{-1}v$. The complex structures I, J, K are parallel, hence so too is I_ϵ. This implies that I_ϵ is integrable, and hence that the connection h_ϵ is flat. Finally, by the argument above, since Ω_+ is closed we find that the connection h is symplectic. □

Remark 2.3. Theorem 2.2 is actually more general than it first appears, in that, at least locally, any complex hyperkähler structure arises in this way. Indeed, given a complex hyperkähler structure on a manifold Y, we can locally construct a map $\pi\colon Y \to M$ by quotienting by the integrable distribution $\ker(\Omega_-)$. Then, setting $X = T_M$, it is not hard to show [20, Section 2.3] that there is a local isomorphism $X \cong Y$ such that the induced hyperkähler structure on X satisfies the relations (11).

2.4. **Joyce structures.** Consider again a flat ν-pencil h_ϵ of symplectic connections on $\pi\colon X = T_M \to M$ as in Section 2.1. We now discuss three additional properties which hold in the examples we will consider. These properties were axiomatised in [17, 20] in the notion of a Joyce structure.

(i) *Periodicity.* Recall that an integral affine structure on M is the data of a flat, torsion-free (linear) connection ∇^0 on the tangent bundle T_M, together with a parallel sublattice $T_M^{\mathbb{Z}} \subset T_M$ of maximal rank. The quotient

$$X^{\#} = T_M^{\#} = T_M/(2\pi i)T_M^{\mathbb{Z}} \tag{15}$$

is then a $(\mathbb{C}^*)^n$-bundle over M. We ask that the connections h_ϵ on $\pi\colon X \to M$ descend via the quotient map $X \to X^{\#}$. The inverse of the symplectic form ω defines a Poisson structure $\eta\colon T_M^* \times T_M^* \to \mathscr{O}_M$. We also ask that η takes integral values on the dual lattice $(T_M^{\mathbb{Z}})^* \subset T_M^*$.

(ii) *Homogeneity.* Suppose given a $\mathbb{C}^*$-action on M, and denote by $m_t\colon M \to M$ the action of $t \in \mathbb{C}^*$. We lift the action to $X = T_M$ by the formula

$$m_t\colon v \in T_pM \mapsto (m_t)_*(t^{-1}v) \in T_{m_t(p)}M, \tag{16}$$

and ask that the connection $h\colon \pi^*(T_M) \to T_X$ is $\mathbb{C}^*$-equivariant. In terms of the hyperkähler structure of Section 2.3 this is the condition

$$\mathcal{L}_E(g) = g, \qquad \mathcal{L}_E(I) = 0, \qquad \mathcal{L}_E(J \pm iK) = \mp(J \pm iK), \tag{17}$$

where E is the generating vector field for the $\mathbb{C}^*$-action on X, and $\mathcal{L}$ denotes the Lie derivative. We also ask that the $\mathbb{C}^*$-action (16) on $X = T_M$ preserves the sublattice $T_M^{\mathbb{Z}} \subset T_M$, and hence descends to the quotient $X^{\#} = T_M^{\#}$.

(iii) *Involution.* There is an involution ι of the space $X = T_M$ which preserves the projection $\pi\colon X \to M$ and acts by -1 on the fibres. Given a vector field w on M we ask that the vector field $h(\pi^*(w))$ on X is invariant under ι. In terms of the hyperkähler structure the condition is that I is ι-invariant, whereas g, J, K are ι-anti-invariant.

Take local co-ordinates $(z_1, \cdots, z_n)$ on M which are flat for ∇^0, and also integral, in the sense that the vector fields $\partial/\partial z_i$ are sections of $T_M^{\mathbb{Z}} \subset T_M$. The matrix η_{pq} is then integral.

The fact that the lifted $\mathbb{C}^*$-action preserves $T_M^{\mathbb{Z}}$ implies that, after affine shifts of the z_i, we can assume the $\mathbb{C}^*$-action on M rescales the co-ordinates z_i with weight 1. The induced action (16) on X leaves the co-ordinates θ_i unchanged, so

$$E = \sum_i z_i \cdot \frac{\partial}{\partial z_i}. \tag{18}$$

The above properties then imply the following symmetries of the Plebański function:

$$W(z_1, \cdots, z_n, \theta_1 + 2\pi i k_1, \cdots, \theta_n + 2\pi i k_n) = W(z_1, \cdots, z_n, \theta_1, \cdots, \theta_n) \text{ for } k_i \in \mathbb{Z}, \tag{19}$$

$$W(tz_1, \cdots, tz_n, \theta_1, \cdots, \theta_n) = t^{-1} \cdot W(z_1, \cdots, z_n, \theta_1, \cdots, \theta_n) \text{ for } t \in \mathbb{C}^*, \tag{20}$$

$$W(z_1, \cdots, z_n, -\theta_1, \cdots, -\theta_n) = -W(z_1, \cdots, z_n, \theta_1, \cdots, \theta_n). \tag{21}$$

So far we have assumed that our Joyce structure is well-defined on the full space $X = T_M$. It turns out that in the examples described in Section 3 below, the connection h has poles. Thus there is an effective divisor D on X, and a map $h\colon \pi^*(T_M) \to T_X(D)$ such that the composite $\pi_*(D) \circ h\colon \pi^*(T_M) \to \pi^*(T_M)(D)$ is the canonical inclusion. When expressed in terms of local co-ordinates as in Section 2.2, this means that the Plebański function W is a meromorphic function. When it is necessary to be precise about this, we will refer to the resulting structures as meromorphic Joyce structures.

Remark 2.4. It was explained in [20, Section 3.2] that the involution property (iii) above implies that, when restricted to the zero-section $M \subset X = T_M$, the holomorphic Levi-Civita connection of the complex hyperkähler structure on X induces a flat, torsion-free connection ∇^J on the tangent bundle of M. This connection was referred to in [17, Section 7] as the linear Joyce connection, and is given in co-ordinates by

$$\nabla^J_{\frac{\partial}{\partial z_i}}\Big(\frac{\partial}{\partial z_j}\Big) = \sum_{p,q} \eta_{qp} \cdot \frac{\partial^3 W}{\partial\theta_i \partial\theta_j \partial\theta_p}\Big|_{\theta=0} \cdot \frac{\partial}{\partial z_q}. \tag{22}$$

Note however, that locating the poles of the Joyce structures of Section 3 is a subtle problem, and in particular it is not clear whether the structure is regular along the zero-section $M \subset X = T_M$: see Remark 3.6 for more on this. Thus the question as to whether the connection ∇^J is well-defined becomes an interesting one. Note that in the one example that has been computed in detail [18], the connection ∇^J is indeed well-defined, and turns out to be quite natural.

3. Joyce structures on spaces of quadratic differentials

In this section we describe a class of examples of the Joyce structures introduced in Section 2. Our treatment will be rather brief in places: full details and proofs will appear in [21]. The bases M parameterise compact curves C of some fixed genus $g \geq 2$, equipped with a quadratic differential $Q \in H^0(C, \omega_C^{\otimes 2})$ with simple zeroes. These examples should form part of a larger class whose bases parameterise quadratic differentials with poles of fixed orders $m = \{m_1, \cdots, m_l\}$. All these examples are expected to arise via the formalism of [15, 17] applied to the corresponding CY_3 categories $\mathcal{D}(g, m)$ of class $S[A_1]$, although we shall not prove that here. The interpretation of the spaces of quadratic differentials in terms of stability conditions can be found in [19, 35].

3.1. **Quadratic differentials.** We begin by reviewing some standard material on moduli spaces of quadratic differentials. Let $\mathcal{M}(g)$ denote the moduli space of complex projective curves of genus $g \geq 2$. It has complex dimension $d = 3g - 3$. Since the tangent spaces are $\mathcal{T}_C\mathcal{M}(g) = H^1(C, T_C)$, and Serre duality gives $H^0(C, \omega_C^{\otimes 2}) = H^1(C, T_C)^*$, the cotangent bundle $T^*\mathcal{M}(g)$ parameterises pairs (C, Q) of a curve C, together with a quadratic differential $Q \in H^0(C, \omega_C^{\otimes 2})$. Let $M \subset T^*\mathcal{M}(g)$ be the open subset of pairs (C, Q) for which Q has simple zeroes.

Given a point $(C, Q) \in M$ we consider the curve $\Sigma \subset T^*C$ defined by the equation $y^2 = Q$. There is a canonical map $p\colon \Sigma \to C$, and a covering involution $\sigma\colon \Sigma \to \Sigma$ mapping $(x, y) \mapsto (x, -y)$. Since Q has simple zeroes, Σ is smooth, and since Q has at least one zero, Σ is connected. Riemann-Hurwitz gives $g(\Sigma) = 4g - 3$.

We define the (anti-)invariant homology groups

$$H_1(\Sigma, \mathbb{Z})^{\pm} = \{\gamma \in H_1(\Sigma, \mathbb{Z}) : \sigma^*(\gamma) = \pm\gamma\},$$

and similarly for the cohomology groups $H^1(\Sigma, \mathbb{Z})^{\pm}$, $H^1(\Sigma, \mathbb{C})^{\pm}$, etc. There is a short exact sequence of free abelian groups

$$0 \longrightarrow H_1(\Sigma, \mathbb{Z})^- \longrightarrow H_1(\Sigma, \mathbb{Z}) \xrightarrow{p_*} H_1(C, \mathbb{Z}) \longrightarrow 0, \tag{23}$$

the map p_* being surjective because p is ramified. Taking maps into $\mathbb{Z}$ shows that the image of $p^*\colon H^1(C, \mathbb{Z}) \to H^1(\Sigma, \mathbb{Z})$ coincides with the subgroup $H^1(\Sigma, \mathbb{Z})^+$.

The anti-invariant homology group $H_1(\Sigma, \mathbb{Z})^-$ is free of rank $n = 2d = 6g-6$. The restriction of the intersection pairing $\langle -, - \rangle$ is skew-symmetric and non-degenerate. We also consider the extended anti-invariant homology group

$$\tilde{H}^1(\Sigma, \mathbb{Z})^- := \operatorname{Hom}_{\mathbb{Z}}(H_1(\Sigma, \mathbb{Z})^-, \mathbb{Z}) = H^1(\Sigma, \mathbb{Z})/H^1(C, \mathbb{Z}). \tag{24}$$

Introduce the vector bundle $\mathcal{H} \to M$ whose fibre over a point (C, Q) is the vector space $H^1(\Sigma, \mathbb{C})^-$. It is equipped with a flat Gauss-Manin connection ∇^{GM}, and a parallel lattice of maximal rank $\mathcal{H}^{\mathbb{Z}} \subset \mathcal{H}$ whose fibres are the groups $\tilde{H}^1(\Sigma, \mathbb{Z})^-$. The dual bundle $\mathcal{H}^*$ has fibres $H_1(\Sigma, \mathbb{C})^-$ and contains the dual lattice $(\mathcal{H}^{\mathbb{Z}})^*$ with fibres $H_1(\Sigma, \mathbb{Z})^-$. The intersection form defines a parallel skew-symmetric form on $\mathcal{H}^*$ which takes integral values on $(\mathcal{H}^{\mathbb{Z}})^*$.

For each point $(C, Q) \in M$, the tautological 1-form $y\,dx$ on T^*C restricts to a 1-form $\lambda \in H^0(\Sigma, \omega_\Sigma)$ satisfying $\lambda^{\otimes 2} = p^*(Q)$ and $\sigma^*(\lambda) = -\lambda$. The associated de Rham cohomology class is an element of $H^1(\Sigma, \mathbb{C})^-$, and the resulting map

$$\delta\colon M \to \mathcal{H}, \qquad (C, Q) \mapsto [\lambda] \in H^1(\Sigma, \mathbb{C})^- \tag{25}$$

is a holomorphic section of the bundle $\mathcal{H}$.

Theorem 3.1 ([58])**.** *The covariant derivative of δ with respect to the Gauss-Manin connection defines an isomorphism $\nabla^{\mathrm{GM}}(\delta)\colon T_M \to \mathcal{H}$.*

Taking a basis $(\gamma_1, \cdots, \gamma_n) \subset H_1(\Sigma, \mathbb{Z})^-$ at some point $(C, Q) \in M$ and extending to nearby points using the Gauss-Manin connection gives locally-defined functions on M

$$z_i = Z(\gamma_i) = ([\lambda], \gamma_i) = \int_{\gamma_i} \sqrt{Q}, \qquad 1 \leqslant i \leqslant n. \tag{26}$$

Theorem 3.1 is then the statement that these functions are local co-ordinates on M. Note that the associated linear co-ordinates $\theta_i = (dz_i, -)$ on the fibres of the bundle $T_M \to M$ considered in Section 2.2 correspond, under the isomorphism of Theorem 3.1, to the functions on the fibres of the bundle $\mathcal{H} \to M$ given by pairing with the classes γ_i.

We can use the isomorphism of Theorem 3.1 to transfer the data $(\nabla^{GM}, \mathcal{H}^{\mathbb{Z}})$ from the bundle $\mathcal{H}$ to the tangent bundle T_M. This gives an integral affine structure $(\nabla_0, T_M^{\mathbb{Z}})$ on M for which the periods (26) are flat, integral co-ordinates. The intersection form on $\mathcal{H}^*$ induces a parallel non-degenerate Poisson structure $\eta\colon T_M^* \times T_M^* \to \mathcal{O}_M$, which takes integral values on the lattice $(T_M^{\mathbb{Z}})^* \subset T_M^*$. The inverse of η is then a symplectic form $\omega \in H^0(M, \wedge^2 T_M)$.

3.2. **Spectral correspondence.** Fix a point $(C, Q) \in M$, and denote by $R \subset \Sigma$ the branch divisor of the double cover $p\colon \Sigma \to C$. It has degree $4g - 4$. There is a short exact sequence

$$0 \longrightarrow p^*(\omega_C) \xrightarrow{\ i\ } \omega_\Sigma \longrightarrow \mathcal{O}_R \longrightarrow 0. \tag{27}$$

By a branched connection on a line bundle L on Σ we simply mean a meromorphic connection $\partial\colon L \to L \otimes \omega_\Sigma(R)$ with regular singularities along R.

The square-root of $p^*(Q)$ defines a section $\phi\colon \mathscr{O}_\Sigma \to p^*(\omega_C)$ with simple zeroes on R. This should not be confused with the differential $\lambda\colon \mathscr{O}_\Sigma \to \omega_\Sigma$ considered above, which is the composite $\lambda = i\circ\phi$, and has double zeroes on R. Using ϕ as a meromorphic gauge transformation we can transfer the trivial connection on $\mathscr{O}_\Sigma$ to a branched connection $\partial_{\rm can}$ on the line bundle $p^*(\omega_C)$. This connection has simple poles with residues -1 at the points of R. We call a branched connection anti-invariant if

$$(L,\partial_L)\otimes\sigma^*(L,\partial_L)\cong(p^*(\omega_C),\partial_{\rm can}). \tag{28}$$

The well-known $\mathrm{SL}_2(\mathbb{C})$ spectral curve correspondence [36, Section 8], [12, Section 3] relates

(i) line bundles L on the double cover $p\colon \Sigma\to C$ such that $L\otimes\sigma^*(L)\cong p^*(\omega_C)$,

(ii) rank 2 vector bundles E on C, with $\det(E)\cong\mathscr{O}_C$, equipped with a Higgs field $\Phi\colon E\to E\otimes\omega_C$ with $\mathrm{tr}(\Phi)=0$ and $\det(\Phi)=-Q$.

A line bundle L on Σ is sent to the bundle $E=p_*(L)$ equipped with the Higgs field Φ which is the push-forward of the map $1\otimes\phi\colon L\to L\otimes p^*(\omega_C)$. In the reverse direction, L is obtained via the eigen-decomposition of $p^*(\Phi)$ on the pullback $p^*(E)$.

It is less well known that the spectral curve correspondence can be extended to include connections on the bundles L and $E=p_*(L)$. More precisely, the extension relates

(i) anti-invariant branched connections ∂ on the line bundle L,

(ii) connections ∇ on E inducing the trivial connection on $\det(E)\cong\mathscr{O}_C$.

This construction was discussed in [25, Section 3.2] and also appears in [10].

There is a short exact sequence

$$0\longrightarrow p^*(E)\longrightarrow L\oplus\sigma^*(L)\longrightarrow\mathscr{O}_R\longrightarrow 0. \tag{29}$$

The local calculation in the next subsection shows that given a connection ∇ on E, we can uniquely extend the pullback $p^*(\nabla)$ to a meromorphic connection $\tilde\nabla$ on the direct sum $L\oplus\sigma^*(L)$ with simple poles along the divisor R. Taking the component of $\tilde\nabla$ along L then gives the required branched connection ∂.

To prove that (L,∂) is anti-invariant in the sense defined above, take determinants of (29) to get a map $f\colon\mathscr{O}_\Sigma\to L\otimes\sigma^*(L)$. Since f is an isomorphism away from R, it follows for degree reasons that $L\otimes\sigma^*(L)\cong p^*(\omega_C)$ and that f is multiplication by a nonzero scalar multiple of ϕ. Then $\det(\tilde\nabla)=\partial\otimes 1+1\otimes\sigma^*(\partial)$ is related to the trivial connection $\det(p^*(\nabla))$ by the meromorphic gauge change ϕ, and hence coincides with $\partial_{\rm can}$.

Remark 3.2. The story we are about to describe is really about the gauge group $\mathrm{PGL}_2(\mathbb{C})$ rather than $\mathrm{SL}_2(\mathbb{C})$. For various reasons it is convenient to present this in the form $\mathrm{SL}_2(\mathbb{C})/\{\pm1\}$

rather than $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^*$. This means that we impose the trivial determinant conditions on our vector bundles appropriate for $\mathrm{SL}_2(\mathbb{C})$, and then quotient our moduli spaces by a finite group $J^2(C) \cong \{\pm 1\}^{2g}$. The alternative would be to drop the trivial determinant conditions and quotient by the group $J^\sharp(C) \cong (\mathbb{C}^*)^{2g}$ of all line bundles with connection on C.

3.3. **Local computation at a branch-point.** Take notation as in the last section and consider a σ-invariant neighbourhood $U \subset \Sigma$ of a branch-point $p \in R$. Choose a local co-ordinate $w\colon U \to \mathbb{C}$ satisfying $\sigma^*(w) = -w$, and hence $w(p) = 0$. Choose also a local non-vanishing section $s \in H^0(U, L)$. In terms of the basis of sections $(s, \sigma^*(s))$ we can write the induced connection $\tilde{\nabla}$ on $L \oplus \sigma^*(L)$ in the form

$$\tilde{\nabla} = d + \begin{pmatrix} \alpha(w) & \beta(w) \\ \gamma(w) & \delta(w) \end{pmatrix} dw, \tag{30}$$

where $\alpha, \beta, \gamma, \delta\colon U \to \mathbb{C}$ are meromorphic functions defined in a neighbourhood of $0 \in \mathbb{C}$, and regular away from 0. The invariance of the pullback connection on $p^*(E)$ shows that $\gamma(w) = -\beta(-w)$ and $\delta(w) = -\alpha(-w)$.

The sequence (29) shows that a section of $p^*(E)$ over U is determined by sections of L and $\sigma^*(L)$ which agree at the branch-points of p. Now consider

$$\tilde{\nabla}_{\frac{\partial}{\partial w}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha(w) + \beta(w) \\ -\alpha(-w) - \beta(-w) \end{pmatrix}, \tag{31}$$

$$\tilde{\nabla}_{\frac{\partial}{\partial w}} \begin{pmatrix} w \\ -w \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + w \begin{pmatrix} \alpha(w) - \beta(w) \\ \alpha(-w) - \beta(-w) \end{pmatrix}. \tag{32}$$

As derivatives of regular sections of $p^*(E)$, these expressions are also regular sections of $p^*(E)$. It follows that the leading order behaviour of α, β at $w = 0$ takes the form

$$\alpha(w) = -\frac{1}{2w} + c + O(w), \qquad \beta(w) = \frac{1}{2w} - c + O(w), \tag{33}$$

for some element $c \in \mathbb{C}$. In particular $\tilde{\nabla}$ has simple poles on R. The induced branched connection on L is given by $\partial = d + \alpha(w)dw$.

3.4. **Diagram of moduli spaces.** We use the following notation:

- C is a complex projective curve of genus g,
- $Q \in H^0(C, \omega_C^{\otimes 2})$ is a quadratic differential on C with simple zeroes,
- $p\colon \Sigma \to C$ is the double cover defined by the quadratic differential Q,
- E is a rank 2 vector bundle on C with trivial determinant,
- Φ is a trace-free Higgs field on E such that $\det(\Phi)$ has simple zeroes,

- ∇ is a connection on E inducing the trivial connection on $\det(E)$,
- L is a line bundle on Σ such that $L \otimes \sigma^*(L) \cong p^*(\omega_C)$,
- ∂ is an anti-invariant branched connection on L.

$$\begin{array}{ccccc} & & \mathcal{M}(C,E,\nabla,\Phi) & & \\ & \overset{\alpha}{\swarrow} & & \overset{\beta_\epsilon}{\searrow} & \\ \mathcal{M}(C,Q,L,\partial) & & & & \mathcal{M}(C,Q,E,\nabla_\epsilon) \xrightarrow{\ \rho'\ } \mathcal{M}(C,E,\nabla_\epsilon) \\ \downarrow{\scriptstyle \pi_3} & & & & \downarrow{\scriptstyle \pi_2} \qquad\qquad\qquad \downarrow{\scriptstyle \pi_1} \\ \mathcal{M}(C,Q) & & \xleftrightarrow{\quad = \quad} & & \mathcal{M}(C,Q) \xrightarrow{\ \rho\ } \mathcal{M}(C) \end{array} \tag{34}$$

Let us fix a parameter $\epsilon^{-1} \in \mathbb{C}$ and contemplate the diagram (34). Each moduli space $\mathcal{M}(\cdots)$ parameterises the indicated objects, and the maps ρ, ρ' and π_i are the obvious projections. Note that $\mathcal{M}(C) = \mathcal{M}(g)$ and $\mathcal{M}(C,Q) = M$ are the moduli spaces appearing in Section 3.1.

The map α is the extension of the usual spectral curve construction discussed in the previous subsection, and $\beta(\epsilon)$ is defined by the rule

$$\beta_\epsilon(C,E,\nabla,\Phi) = (C, -\det(\Phi), E, \nabla - \epsilon^{-1}\Phi). \tag{35}$$

A complete proof of the following result will appear in [21].

Theorem 3.3. (i) *The map α is birational,*

(ii) *The map β_ϵ is generically étale.*

Sketch proof. For (i), suppose we have two connections on E giving rise to the same connection on L. In terms of the local computation of Section 3.3 this means we have two possible β_i with the same α, and in particular, the same $c \in \mathbb{C}$. Then the difference $\beta_2(w) - \beta_1(w)$ is regular and vanishes at the branch-point $w = 0$. Globally the difference $(\beta_2(w) - \beta_1(w))dw$ corresponds to a section $\sigma^*(L) \to L \otimes \omega_\Sigma$ vanishing along R. But Riemann-Roch shows that $\chi(L \otimes \sigma^*(L^*) \otimes \omega_\Sigma(-R)) = 0$, and it follows that for generic L any such section is zero.

For (ii), note first that the source of β_ϵ may be equivalently viewed as $\mathcal{M}(C,E,\nabla_\epsilon,\Phi)$. Thus the fibre of β_ϵ over a point (C,Q,E,∇_ϵ) parameterises Higgs fields Φ on the bundle E for which $\det(E) = -Q$. It follows from the results of [52] that, when restricted to the open locus where E is very stable, the map β_ϵ is finite. Since the space $\mathcal{M}(C,E,\nabla,\Phi)$ is smooth at all points where E is stable, the result follows from generic smoothness. $\square$

3.5. **Prym variety.** Fix a point $(C, Q) \in M$ and let $p\colon \Sigma \to C$ be the associated double cover. We denote by $J(C)$ and $J(\Sigma)$ the Jacobians of the curves C and Σ. Set

$$J(\Sigma)^- = \{M \in J(\Sigma) : M \otimes \sigma^*(M) \cong \mathscr{O}_\Sigma\}, \qquad J^2(C) = \{P \in J(C) : P^{\otimes 2} \cong \mathscr{O}_C\}. \tag{36}$$

The pullback map $p^*\colon J(C) \to J(\Sigma)$ is injective [49, Section 3], and we identify $J(C)$ with its image. The Prym variety is defined by either of the quotients

$$P(\Sigma) = J(\Sigma)/J(C) = J(\Sigma)^-/J^2(C). \tag{37}$$

To see that these quotients are the same, consider the map $j\colon J(\Sigma)^- \to J(\Sigma)/J(C)$ induced by the inclusion $J(\Sigma)^- \subset J(\Sigma)$. Then j is surjective, because for any $M \in J(\Sigma)$ we can write $M = N^{\otimes 2}$ and then

$$M = (N \otimes \sigma^*(N^*)) \otimes (N \otimes \sigma^*(N)). \tag{38}$$

The first factor clearly lies in $J(\Sigma)^-$, and it is proved in [49, Section 3] that the second lies in $J(C)$. The kernel of j is the intersection $J(C) \cap J(\Sigma)^- \subset J(\Sigma)$, and since any element $M \in J(C)$ satisfies $\sigma^*(M) = M$, this coincides with $J^2(C)$.

We also consider the spaces $J^\sharp(C)$ and $J^\sharp(\Sigma)$ of line bundles with connection. We can again identify $J^\sharp(C)$ with the image of the pullback map $p^*\colon J^\sharp(C) \to J^\sharp(\Sigma)$. We set

$$J^\sharp(\Sigma)^- = \{(M, \partial) \in J^\sharp(\Sigma) : (M, \partial) \otimes \sigma^*(M, \partial) \cong (\mathscr{O}_\Sigma, d)\}. \tag{39}$$

Note that if (N, ∂_N) is a line bundle with connection on C, and $P^{\otimes 2} \cong N$, then there is a unique connection ∂_P such that $(P, \partial_P)^{\otimes 2} \cong (N, \partial_N)$. In particular each $P \in J^2(C)$ has a unique connection ∂_P satisfying $(P, \partial_P)^{\otimes 2} = (\mathscr{O}_C, d)$, and we can therefore identify $J^2(C)$ with a subgroup of $J^\sharp(\Sigma)$. We define

$$P^\sharp(\Sigma) = J^\sharp(\Sigma)/J^\sharp(C) = J^\sharp(\Sigma)^-/J^2(C), \tag{40}$$

with a similar argument as before showing that these two quotients are equal.

The exact sequence (23) shows that

$$\tilde{H}^1(\Sigma, \mathbb{C}^*)^- := \operatorname{Hom}_\mathbb{Z}(H_1(\Sigma, \mathbb{Z})^-, \mathbb{C}^*) \cong H^1(\Sigma, \mathbb{C}^*)/H^1(C, \mathbb{C}^*). \tag{41}$$

The Riemann-Hilbert isomorphism $J^\sharp(\Sigma) \cong H^1(\Sigma, \mathbb{C}^*)$ then induces an isomorphism

$$P^\sharp(\Sigma) \cong \tilde{H}^1(\Sigma, \mathbb{C}^*)^-. \tag{42}$$

Let $J^\sharp_{br}(\Sigma)$ denote the space of line bundles L on Σ equipped with anti-invariant branched connections ∂. The group $J^2(C) \subset J^\sharp(\Sigma)$ acts on this space by tensor product, and we set $P^\sharp_{br}(\Sigma) = J^\sharp_{br}(\Sigma)/J^2(C)$.

Lemma 3.4. *There is a canonical isomorphism* $P^\sharp_{br}(\Sigma) \cong P^\sharp(\Sigma)$.

Proof. Tensor product gives $J^\sharp_{br}(\Sigma)$ the structure of a torsor over $J^\sharp(\Sigma)^-$, so choosing a point $(L_0, \partial_0) \in J^\sharp_{br}(\Sigma)$ gives a non-canonical identification of the two spaces

$$(L, \partial) \in J^\sharp_{br}(\Sigma) \mapsto (L, \partial) \otimes (L_0, \partial_0)^{-1} \in J^\sharp(\Sigma)^-. \tag{43}$$

This then descends to the quotients by $J^2(C)$. To obtain a canonical bijection, take a spin structure $\omega_C^{1/2}$ on C and let ∂_0 be the unique branched connection on $L_0 = p^*(\omega_C^{1/2})$ satisfying $(L_0, \partial_0)^{\otimes 2} = (p^*(\omega_C), \partial_{\text{can}})$. Since $\omega_C^{1/2}$ is uniquely defined up to the action of $J^2(C)$, the resulting isomorphism $P^\sharp_{br}(\Sigma) \cong P^\sharp(\Sigma)$ is canonically defined. □

Introduce the bundle $\mathcal{H}^\# = \mathcal{H}/(2\pi i)\mathcal{H}^\mathbb{Z}$ over M with fibres $\tilde{H}^1(\Sigma, \mathbb{C}^*)^-$. Under the isomorphism of Theorem 3.1 it corresponds to the quotient $T_M^\# = T_M/(2\pi i)T_M^\mathbb{Z}$ from Section 2.4. The following diagram can now be tacked onto the left-hand side of (34)

$$\begin{array}{ccccc} T_M^\# & \xrightarrow{\nabla^{\mathrm{GM}}(\delta)} & \mathcal{H}^\# & \xleftarrow{\ \tau\ } & \mathcal{M}(C, Q, L, \partial) \\ \downarrow{\scriptstyle \pi_5} & & \downarrow{\scriptstyle \pi_4} & & \downarrow{\scriptstyle \pi_3} \\ M & \xleftrightarrow{=} & M & \xleftrightarrow{=} & \mathcal{M}(C, Q) \end{array} \tag{44}$$

Fibrewise, the map τ is the composite of the isomorphism of Lemma 3.4 with the Riemann-Hilbert isomorphism (42).

3.6. **Joyce structure.** Returning to the diagram (34), there is a flat symplectic connection on the map π_1, known as the isomonodromy connection [14], whose horizontal leaves are families of connections with constant monodromy.

The right-hand square in (34) is Cartesian so we can pull back the isomonodromy connection to obtain a connection on π_2. Using Theorem 3.3 we can further transfer the connection to a dense open subset of π_3. Note that the bundle of groups $J^2(C)$ over $\mathcal{M}(C)$ acts by tensor product on the upper row of the diagram (34). Tensoring (E, ∇) by an element $(P, \partial_P) \in J^2(C)$ multiplies the monodromy by a homomorphism $H_1(C, \mathbb{Z}) \to \{\pm 1\}$. This action clearly preserves the isomonodromy connection, and the connection on π_3 therefore descends along the map τ appearing in (44). Continuing across this diagram, we finally obtain a connection h_ϵ on a dense open subset of $\pi_5 \colon T_M^\# \to M$. Of course, this can be equivalently viewed as a connection on $\pi \colon T_M \to M$ which is invariant under translation by the lattice $T_M^\mathbb{Z} \subset T_M$.

Theorem 3.5. *The connections h_ϵ extend to a meromorphic ν-pencil of flat, symplectic connections on $\pi\colon X = T_M \to M$.*

Proof. Using the fact that the spaces $\mathcal{M}(C,Q,L,\partial)$ and $\mathcal{M}(C,Q,E,\nabla_\epsilon)$ are smooth, and the maps α, β_ϵ are algebraic, it is easy to see that the connections h_ϵ defined above on an open subset of X, extend to a meromorphic connection on the whole space. These connections are flat and symplectic because they are pullbacks of the isomonodromy connection. It remains to prove that $h_\epsilon = h + \epsilon^{-1}v$ where $h = h_\infty$.

Consider the automorphism

$$r_\epsilon\colon \mathcal{M}(C,E,\nabla,\Phi) \to \mathcal{M}(C,E,\nabla,\Phi), \qquad (C,E,\nabla,\Phi) \mapsto (C,E,\nabla + \epsilon^{-1}\Phi,\Phi). \tag{45}$$

Then $\beta_\epsilon = \beta_\infty \circ r_{-\epsilon}$ and it follows that $h_\epsilon = (r_\epsilon)_* \circ h$ as maps of bundles $\pi^*(T_M) \to T_X$. It will be enough to show that $(r_\epsilon)_* = \mathrm{id} + \epsilon^{-1}(v \circ \pi_*)$.

Recall the tautological differential $\lambda \in H^0(\Sigma, \omega_\Sigma)$ of Section 3.1 whose periods (26) around a basis of cycles $(\gamma_1, \cdots, \gamma_n) \subset H_1(\Sigma, \mathbb{Z})^-$ define local flat integral co-ordinates z_i on M. After transferring along the birational map α the map r_ϵ becomes

$$r_\epsilon\colon \mathcal{M}(C,Q,L,\partial) \to \mathcal{M}(C,Q,L,\partial), \qquad (C,Q,L,\partial) \mapsto (C,Q,L,\partial + \epsilon^{-1}\lambda). \tag{46}$$

Thus r_ϵ is the operation of tensoring $(L, \partial_L) \in J^\sharp_{br}(\Sigma)$ with $(\mathscr{O}_\Sigma, d + \epsilon^{-1}\lambda) \in J^\sharp(\Sigma)^-$. Note that the monodromy of this second connection around a cycle γ_i is just $\exp(\epsilon^{-1}z_i)$. We now transfer the automorphism r_ϵ across the diagram (44). The map σ takes a point of the space $\mathcal{M}(C,Q,L,\partial)$ to the monodromy of the product (43). Thus taking fibre co-ordinates θ_i as in Section 2.2, we finally arrive at the automorphism of $X = T_M$ given in local co-ordinates by $\theta_i \mapsto \theta_i + \epsilon^{-1}z_i$, and the claim follows. □

Applying Theorem 2.2 we obtain a complex hyperkähler structure on a dense open subset of X. Since the maps α and β_ϵ are algebraic, it extends to a meromorphic hyperkähler structure on X. We leave the reader to check in detail the further features discussed in Section 2.4. The periodicity property clearly holds by construction. The action of $t \in \mathbb{C}^*$ and the involution ι are defined on the space $\mathcal{M}(C,E,\nabla,\Phi)$ by

$$t \cdot (C,E,\nabla,\Phi) = (C,E,\nabla,t\,\Phi), \qquad \iota(C,E,\nabla,\Phi) = (C,E,\nabla,-\Phi). \tag{47}$$

3.7. **Further comments.** We collect here a few remarks on the above construction.

Remark 3.6. In view of Remark 2.4 it is interesting to understand the properties of the Joyce structure in a neighbourhood of the zero-section of the bundle $\pi\colon T_M \to M$. This submanifold $M \subset X$ is the fixed locus of the involution ι of Section 2.4, and when transferred across the diagram (44) corresponds to the multi-section of π_3 consisting of points satisfying $(L, \partial_L)^{\otimes 2} =$

$(p^*(\omega_C), \partial_{\rm can})$. The bundle $E = p_*(L)$ associated to such points fits into a non-split short exact sequence

$$0 \longrightarrow \omega_C^{1/2} \longrightarrow E \longrightarrow \omega_C^{-1/2} \longrightarrow 0 \tag{48}$$

with $\omega_C^{1/2}$ a spin structure on C. This bundle is uniquely defined up to the action of $J^2(C)$, and isomorphism classes of connections on E are in bijection with $\mathrm{PGL}_2(\mathbb{C})$-opers on C, or in another language, complex projective structures. Understanding the properties of the Joyce structure at these points is difficult however, because they lie in the exceptional locus of the birational map α.

Remark 3.7. The map α appearing in Theorem 3.3 can be viewed as an abelianization map for flat connections in the presence of a quadratic differential. It leads to functions $\exp(\theta_i)$ on X given by the monodromy of the connection (43). For the purposes of this remark we should imagine a generalisation of the construction in which the connections are allowed to have poles. Then α, which is a de Rham construction, can be compared with the Betti abelianization procedure of [37, 50], which depends on the choice of a spectral network on C. For suitable spectral networks this leads to the Fock-Goncharov co-ordinates $X_i(\epsilon) = \exp(x_i(\epsilon))$ evaluated at the monodromy of the connection (E, ∇_ϵ). The relation between the functions θ_i and $x_i(\epsilon)$ is discussed in Section 5.3 below, and is expected to give rise to solutions to the RH problems of [15]. In particular, if we take the spectral network to be the WKB triangulation determined by a saddle-free quadratic differential Q, we expect that $x_i(\epsilon) \sim -\epsilon^{-1} z_i + \theta_i$ as $\epsilon \to 0$ in the right-hand half-plane, where the periods z_i are as in (26).

Remark 3.8. Extending the construction of this section to other gauge groups, for example $G = \mathrm{SL}_r(\mathbb{C})$ would require major new ideas. In particular, one would need to replace $\mathcal{M}(C)$ with a space whose cotangent fibres are the Hitchin base $\bigoplus_{i=2}^r H^0(C, \omega^{\otimes k})$. This is a well-known problem in higher Teichmüller theory. The expectation is then that the stability space of a CY_3 category of the type discussed in [34, 57] should be an open subset of the cotangent bundle of this space.

4. Good Lagrangian submanifolds

There are several situations in which spaces of vacua in supersymmetric quantum field theory are expected to be identified with complex Lagrangian submanifolds in the stability space of an associated CY_3 triangulated category. Examples include Coulomb branches in $d = 4$, $N = 2$ gauge theory, and the complex structure and complexified Kähler structure moduli spaces in string theory. In the example of theories of class $S[A_1]$ relevant to Section 3, this Lagrangian in $M = \mathcal{M}(C, Q)$ is cut out by fixing the curve C. It is has been a long-standing question to try to abstractly characterise these "physicist's slices" in stability space.

In this section we define the notion of a good Lagrangian submanifold $B \subset M$ in the base of a Joyce structure. Put briefly, the condition is that the restriction of each non-linear connection h_ϵ on $\pi\colon X = T_M \to M$ to the bundle $\pi\colon T_M|_B \to B$ descends to a connection on the normal bundle $\pi\colon N_B \to B$. An equivalent condition is that the submanifold $T_B \subset X = T_M$ is totally geodesic with respect to the (indefinite) hyperkähler metric g. We then prove that in the setting of Section 3 the submanifolds in $\mathcal{M}(C, Q)$ obtained by fixing the curve C indeed have this property.

4.1. **General definition.** Consider a Joyce structure on a complex manifold M as in Section 2, and a complex Lagrangian submanifold $B \subset M$. Consider the normal bundle $\pi\colon N_B \to B$ fitting into the sequence

$$0 \longrightarrow T_B \xrightarrow{\ i\ } T_M|_B \xrightarrow{\ k\ } N_B \longrightarrow 0. \tag{49}$$

There is a canonical real structure on the tangent bundle T_M whose fixed locus $T_M^{\mathbb{R}} \subset T_M$ is the real span of the integral affine structure $T_M^{\mathbb{Z}} \subset T_M$. We call B non-degenerate if $T_B \cap T_M^{\mathbb{R}}|_B = (0) \subset T_M|_B$. When this holds, the restriction of the map k to the lattice $T_M^{\mathbb{Z}}|_B \subset T_M|_B$ is injective, and we denote its image by $N_B^{\mathbb{Z}} \subset N_B$. The fibres of the projection $\pi\colon N_B/N_B^{\mathbb{Z}} \to B$ are then compact tori $\mathbb{C}^d/\mathbb{Z}^{2d} \cong (S^1)^{2d}$.

Remark 4.1. The real structure on T_M and the symplectic form ω together define a non-degenerate Hermitian form on T_M, given in flat Darboux co-ordinates by

$$\zeta = \sum_{p,q} i\omega_{pq} \cdot dz_p \wedge d\bar{z}_q. \tag{50}$$

As explained in [1, Section 1.3], the complex Lagrangian $B \subset M$ is non-degenerate in the above sense precisely if the restriction of ζ to B is non-degenerate. In this case, B becomes a (possibly indefinite) special Kähler manifold. Via the isomorphism $N_B \cong T_B^*$ induced by ω, the torus fibration $\pi\colon N_B/N_B^{\mathbb{Z}} \to B$ becomes the algebraic integrable system considered in [29, Section 3].

Recall the pencil of connections $h_\epsilon = h + \epsilon^{-1} v$ on the bundle $\pi\colon X = T_M \to M$. For any complex submanifold $B \subset M$, and any $\epsilon^{-1} \in \mathbb{C}$, the connection h_ϵ restricts to a connection $h_\epsilon|_B$ on the bundle $X_B = T_M|_B \to B$.

Definition 4.2. *A complex Lagrangian submanifold $B \subset M$ will be called good if the restricted connection $h_\epsilon|_B$ descends via the map $k\colon X_B \to N_B$ to a connection n on the normal bundle $\pi\colon N_B \to B$.*

To explain this condition in more detail, take $x \in X_B$ with $\pi(x) = b \in B$. The bundle map k defines a map of complex manifolds $k\colon X_B \to N_B$, and we set $y = k(x) \in N_B$.

$$\begin{array}{ccccc} X & \hookleftarrow & X_B & \xrightarrow{\ k\ } & N_B \\ {\scriptstyle\pi}\downarrow & & \downarrow & & \downarrow{\scriptstyle\pi} \\ M & \hookleftarrow & B & \xleftrightarrow{\ =\ } & B \end{array} \tag{51}$$

Given a vector $w \in T_bB \subset T_bM$ the connection h_ϵ defines a lift $h_\epsilon(w) \in T_xX_B \subset T_xX$, and we define $n(w) = k_*(h_\epsilon(w)) \in T_yN_B$. Note that $n(w)$ is independent of ϵ, since $k_*(v(w)) = 0$. The condition of Definition 4.2 is that $n(w)$ depends only on $y \in N_B$, not on the element $x \in X_B$ satisfying $k(x) = y$. When this condition holds the map n defines a connection on $\pi\colon N_B \to B$.

Lemma 4.3. *If $B \subset M$ is a good Lagrangian then the induced connection n on the normal bundle $\pi\colon N_B \to B$ is flat. If B is also non-degenerate then n descends to the quotient $\pi\colon N_B/N_B^{\mathbb{Z}} \to B$.*

Proof. The second statement is obvious from the periodicity property of the Joyce structure. To prove the first, recall that if $f\colon M \to N$ is a map of complex manifolds, and v, w are vector fields on M, N respectively, then v, w are said to be f-related if $f_*(v_m) = w_{f(m)}$ for all $m \in M$. Given vector fields v_1, v_2 on M which are f-related to vector fields w_1, w_2 on N it is easily checked that $[v_1, v_2]$ is f-related to $[w_1, w_2]$. We will apply this to the map $k\colon X_B \to N_B$.

Given a vector field v on B, we can extend it to a vector field on M which we also denote by v. We can then use the connection h_ϵ to lift it to the vector field $h_\epsilon(v)$ on X. The restriction of this vector field to X_B is a vector field on X_B, and is independent of the chosen extension. The good Lagrangian condition states that this vector field on X_B is k-related to a vector field on N_B, which by definition is $n(v)$.

The connection h_ϵ being flat is the condition that for any vector fields v_1, v_2 on M we have $h_\epsilon([v_1, v_2]) = [h_\epsilon(v_1), h_\epsilon(v_2)]$. But then it follows that $h_\epsilon([v_1, v_2])$ is k-related to $[n(v_1), n(v_2)]$, which by definition of n implies that $n([v_1, v_2]) = [n(v_1), n(v_2)]$ and hence that the connection n is flat. □

To express the good Lagrangian condition more concretely, take local Darboux co-ordinates $(z_1, \cdots, z_{2k})$ on M as in Section 2.2, and assume that $B \subset M$ is given by the equations $z_{d+1} = \cdots = z_{2d} = 0$, and that $\omega_{pq} = \pm 1$ if $q - p = \pm d$ and is otherwise zero. Lifting the vector fields $\partial/\partial z_i$ with $1 \leqslant i \leqslant d$ from M to X as in (5) gives

$$v_i = \frac{\partial}{\partial \theta_i}, \qquad h_i = \frac{\partial}{\partial z_i} + \sum_{j=1}^{d} \Big(\frac{\partial^2 W}{\partial \theta_i \partial \theta_{j+d}} \cdot \frac{\partial}{\partial \theta_j} - \frac{\partial^2 W}{\partial \theta_i \partial \theta_j} \cdot \frac{\partial}{\partial \theta_{j+d}} \Big). \tag{52}$$

Applying the projection $k \colon T_M|_B \to N_B$ amounts to setting $\partial/\partial\theta_i = 0$ for $1 \leqslant i \leqslant d$. The condition of Definition 4.2 is then that for $1 \leqslant i \leqslant d$ the result of this projection should be independent of the co-ordinates θ_i for $1 \leqslant i \leqslant d$. This is equivalent to

$$\frac{\partial^3 W}{\partial \theta_i \partial \theta_j \partial \theta_k} = 0 \qquad 1 \leqslant i, j, k \leqslant d \tag{53}$$

along the locus $z_{d+1} = \cdots = z_{2d} = 0$.

Remark 4.4. The equation (53) relies on us having chosen an appropriate system of Darboux co-ordinates z_i on M. If we change these co-ordinates, the function W picks up extra cubic terms in the θ_i variables (see [18, Section 4.2] for explicit formulae). A more invariant statement is therefore that if $w_1, \cdots, w_4$ are vector fields on B, and we consider the associated vertical vector fields $v_i = v(w_i)$ on $X_B \to B$, then the fourth partial derivatives $v_1 v_2 v_3 v_4 (W|_{X_B})$ all vanish.

4.2. **Good Lagrangians for categories of class $S[A_1]$.** Consider the setting of Section 3 in which the base of the Joyce structure is $M = \mathcal{M}(C, Q)$. Let us fix a curve $C \in \mathcal{M}(C)$ and consider the Lagrangian submanifold $B = \mathcal{M}_C(Q) \subset M$ which is the corresponding fibre of the projection $\rho \colon \mathcal{M}(C, Q) \to \mathcal{M}(C)$. Thus $B \subset H^0(C, \omega_C^{\otimes 2})$ parameterises quadratic differentials on C with simple zeroes.

Note that if $Q' \in H^0(C, \omega_C^{\otimes 2})$ is a quadratic differential on C, then $p^*(Q')$ vanishes to order two along the branch divisor $R \subset \Sigma$. It follows that the tangent space to B at a point (C, Q) can be identified with $H^0(\Sigma, \omega_\Sigma)^-$ via the map

$$T_b B = H^0(C, \omega_C^{\otimes 2}) \to H^0(\Sigma, \omega_\Sigma)^-, \qquad Q' \mapsto p^*(Q')/2\lambda, \tag{54}$$

where λ is the tautological 1-form on Σ appearing in (25). Under the isomorphism of Theorem 3.1 the sequence (49) then corresponds to the Hodge filtration

$$0 \longrightarrow H^0(\Sigma, \omega_\Sigma)^- \xrightarrow{\ i\ } H^1(\Sigma, \mathbb{C})^- \xrightarrow{\ k\ } H^1(\Sigma, \mathscr{O}_\Sigma)^- \longrightarrow 0. \tag{55}$$

It follows from the isomorphism (42) that the fibers

$$H^1(\Sigma, \mathscr{O}_\Sigma)^- / \tilde{H}^1(\Sigma, \mathbb{Z}) = P^\sharp(\Sigma)/H^0(\Sigma, \omega_\Sigma)^- \tag{56}$$

of the map $\pi \colon N_B/N_B^{\mathbb{Z}} \to B$ are the Prym varieties $P(\Sigma)$ appearing in Section 3.5.

Lemma 4.5. *For each curve C the submanifold $B = \mathcal{M}_C(Q) \subset M = \mathcal{M}(C, Q)$ is a good Lagrangian. The horizontal leaves of the induced meromorphic flat connection on $\pi\colon N_B/N_B^{\mathbb{Z}} \to B$ are defined by the condition that $E = p_*(L)$ is constant.*

Proof. We use the notation $\mathcal{M}_C(Q, L, \partial)$ to denote the space parameterising data (Q, L, ∂) on the fixed curve C, and similarly for $\mathcal{M}_C(Q, E, \nabla_\epsilon)$, etc. Let w be a vector field on $B \subset M$ and let $u = h_\epsilon(w)$ be the lift to a vector field on $X_B \subset X$. Transferring across the diagram (44) we can consider u to be a vector field on $\mathcal{M}_C(Q, L, \partial)$, and we must show that it descends to the space $\mathcal{M}_C(Q, L)$. That is, the flow of the line bundle L under u should be independent of the connection ∂. Passing through the diagram (34) we can view u as a vector field on $\mathcal{M}_C(Q, E, \nabla_\epsilon)$, and we must show that it descends to the space $\mathcal{M}_C(Q, E)$. Now by definition, the connection h_ϵ on the projection π_2 is pulled back from the isomonodromy connection on π_1. Since $\rho_*(w) = 0$, it follows that $\rho'_*(u) = 0$. That is, u is obtained by keeping the pair (E, ∇_ϵ) on C fixed as Q varies with w. It is then clear that u descends to $\mathcal{M}_C(Q, E)$, and the result follows. □

We can give a direct construction of the connection of the Lemma using the diagram of moduli spaces

$$\begin{array}{ccccccc}
N_B/N_B^{\mathbb{Z}} & \xleftarrow{\;\tau\;} & \mathcal{M}_C(Q,L) & \xrightarrow{\;\kappa\;} & \mathcal{M}_C(E,\Phi) & \xrightarrow{\;\xi\;} & \mathcal{M}_C(E)\times\mathcal{M}_C(Q) \\
\downarrow{\scriptstyle\pi} & & \downarrow & & \downarrow & & \downarrow \\
B & \xleftrightarrow{\;=\;} & \mathcal{M}_C(Q) & \xleftrightarrow{\;=\;} & \mathcal{M}_C(Q) & \xleftrightarrow{\;=\;} & \mathcal{M}_C(Q)
\end{array}$$

Here κ is the isomorphism defined by the usual spectral construction sending a line bundle L on Σ to the Higgs bundle (E, Φ) on C. The map ξ is obtained by setting $Q = -\det(\Phi)$. As in the proof of Theorem 3.3 the results of [52] show that ξ is generically étale. Thus we can pullback the trivial connection from the product on the right-hand side to obtain the required meromorphic connection on the left-hand side.

5. Twistor spaces

The geometry of a Joyce structure is often clearer when viewed through the lens of the twistor space of the associated complex hyperkähler manifold [53]. We recall the relevant definitions here and then explain how the $\mathbb{C}^*$-action on the twistor space leads directly to the link with the RH problems of [15, 17].

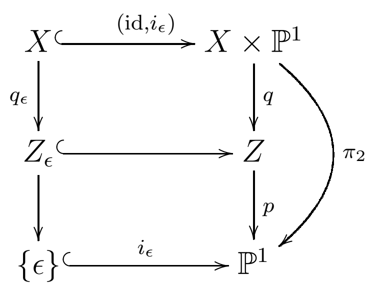
5.1. **Definition.** Consider a Joyce structure on a complex manifold M as in Section 2. Thus $h_\epsilon = h + \epsilon^{-1} v$ is a ν-pencil of flat symplectic connections on $\pi\colon X = T_M \to M$, and (g, I, J, K) is the associated complex hyperkähler structure on X.

Definition 5.1. *The twistor space Z is the quotient of $X \times \mathbb{P}^1$ by the integrable distribution*

$$\operatorname{im}(sv + th) = \ker((J - iK)s^2 - 2istI + t^2(J + iK)), \tag{57}$$

where $[s : t]$ are homogeneous co-ordinates on $\mathbb{P}^1$. We denote by $q\colon X \times \mathbb{P}^1 \to Z$ the quotient map.

In making the above definition we identify a vector field on X with the associated vertical vector field on the projection $\pi_2\colon X \times \mathbb{P}^1 \to \mathbb{P}^1$. Thus there is an induced projection $p\colon Z \to \mathbb{P}^1$ satisfying $p \circ q = \pi_2$.



We will use the affine co-ordinate $\epsilon = t/s$ and denote by $Z_\epsilon = p^{-1}(\epsilon) \subset Z$ the twistor fibre over $\epsilon \in \mathbb{P}^1$. It is the quotient of X by the integrable distribution $\operatorname{im}(h_\epsilon)$. In particular, $Z_0 = M$ is the quotient of X by the vertical sub-bundle $\operatorname{im}(v) = \ker(\pi_*)$. Each point $x \in X$ determines a section

$$\sigma_x\colon \mathbb{P}^1 \to Z, \qquad \epsilon \mapsto q(x, \epsilon), \tag{58}$$

of the map p, whose image is a curve $\mathbb{P}^1 \subset Z$ known as a twistor line.

The twistor space $p\colon Z \to \mathbb{P}^1$ comes equipped with a twisted relative symplectic form. More precisely, there is a unique section of the line bundle $\bigwedge^2 T_Z^* \otimes p^*(\mathscr{O}_{\mathbb{P}^1}(2))$ whose pullback to $X \times \mathbb{P}^1$ is

$$\Omega = s^2\Omega_- - 2ist\Omega_I + t^2\Omega_+. \tag{59}$$

Thus each twistor fibre Z_ϵ has a complex symplectic form, well-defined up to scale.

Remark 5.2. To obtain a well-behaved twistor space we cannot simply take the space of leaves of the foliation in Definition 5.1. Rather, we should consider the holonomy groupoid, which leads to the analytic analogue of a Deligne-Mumford stack [48]. More-or-less by definition, the tangent bundle of Z is then the quotient of that of $X \times \mathbb{P}^1$ by the distribution (57). We will ignore this point here, since we only really use Z as a convenient language to describe objects which can easily be defined directly on X. For example, by a symplectic form on the twistor fibre Z_ϵ as above, we just mean a closed 2-form on X whose kernel is equal to $\operatorname{im}(h_\epsilon)$.

5.2. **Action of $\mathbb{C}^*$.** Consider a $\mathbb{C}^*$-action on M and the induced action (16) on X as in Section 2.4. Taking the standard action of $\mathbb{C}^*$ on $\mathbb{P}^1$ rescaling ϵ with weight 1, we can then consider the diagonal action on $X \times \mathbb{P}^1$. It follows from the conditions (17) that this descends to an action on Z, and the map $p\colon Z \to \mathbb{P}^1$ is then $\mathbb{C}^*$-equivariant.

We can use the $\mathbb{C}^*$-action to trivialise the map $p\colon Z \to \mathbb{P}^1$ over the open subset $\mathbb{C}^* \subset \mathbb{P}^1$. We obtain a commutative diagram

$$\begin{array}{ccccc} Z_1 \times \mathbb{C}^* & \xrightarrow{m} & p^{-1}(\mathbb{C}^*) & \hookrightarrow & Z \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \mathbb{C}^* & \xleftrightarrow{=} & \mathbb{C}^* & \hookrightarrow & \mathbb{P}^1 \end{array} \tag{60}$$

where m is defined by the $\mathbb{C}^*$-action. Note that the induced form $m^*(\Omega)$ on $Z_1 \times \mathbb{C}^*$ coincides up to scale with the pullback $\pi_1^*(\Omega_1)$ along the projection $\pi_1\colon Z_1 \times \mathbb{C}^* \to Z_1$.

Remark 5.3. Unlike in the case of real hyperkähler manifolds, there is no reason to expect an involution of the twistor space Z lifting the antipodal map on $\mathbb{P}^1 = S^2$. In particular, there is no a priori relation between the fibres Z_0 and Z_∞.

Consider the map $y\colon \mathbb{C}^* \times X \to Z_1$ given by

$$y(\epsilon, x) = m_{\epsilon^{-1}} \circ q_\epsilon(x) = q_1 \circ m_{\epsilon^{-1}}(x). \tag{61}$$

Note that, under the trivialisation (60), the twistor line for a point $x \in X$ becomes the section of $\pi_2\colon Z_1 \times \mathbb{C}^* \to \mathbb{C}^*$ given by $\epsilon \mapsto (y(\epsilon, x), \epsilon)$.

Lemma 5.4. *The map $y(\epsilon, x)$ satisfies the equation*

$$\frac{\partial}{\partial \epsilon} y(\epsilon, x) = \left(\frac{1}{\epsilon^2} \cdot \sum_i z_i \cdot \frac{\partial}{\partial \theta_i} + \frac{1}{\epsilon} \cdot \sum_{i,p,q} \eta_{pq} \cdot z_i \cdot \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial}{\partial \theta_q} \right) y(\epsilon, x), \tag{62}$$

Proof. By definition, $y = q_1 \circ m_{\epsilon^{-1}}$ is invariant under the diagonal $\mathbb{C}^*$-action on $X \times \mathbb{C}^*$, and is therefore annhilated by the vector field $\sum_i z_i \cdot \partial/\partial z_i + \epsilon \cdot \partial/\partial \epsilon$. On the other hand, by the definition of the twistor space, since y factors via q_ϵ, it is also annhilated by the vector fields $h_\epsilon(\partial/\partial z_i)$. The result then follows from (5). □

When the periodicity assumption of Section 2.4 holds the complex hyperkähler structure on X descends to the torus bundle (15) over M with fibres

$$T^{\#}_{M,p} = T_{M,p}/T^{\mathbb{Z}}_{M,p} \cong (\mathbb{C}^*)^n. \tag{63}$$

We denote by $Z^{\#}$ the resulting twistor space. There is an étale map $Z \to Z^{\#}$, and the map y descends to a similar map $y\colon \mathbb{C}^* \times X^{\#} \to Z_1^{\#}$.

5.3. **Preferred co-ordinates and Stokes data.** The definition of a Joyce structure was first identified in [17] by considering an analogy between the wall-crossing formula in DT theory and a class of iso-Stokes deformations familiar in the theory of Frobenius manifolds. This analogy can be re-expressed in terms of the geometry of the twistor space $p\colon Z \to \mathbb{P}^1$, and this leads to interesting conjectural properties of Z which we attempt to explain here. The basic point is that there should be preferred systems of co-ordinates on the twistor fibre Z_1, in terms of which the twistor lines, viewed as maps $\mathbb{C}^* \to Z_1$ via the trivialisation (60), have good asymptotic properties as $\epsilon \to 0$.

Fix a point $p \in M$ and consider the map y of Section 5.2, restricted to a single torus fibre (63). Changing the notation slightly, we can view this as a family of maps $y(\epsilon)\colon T^{\#}_{M,p} \to Z_1^{\#}$. Thus for each point $x \in T^{\#}_{M,p}$, the restriction of the corresponding twistor line (58) corresponds under the trivialisation (60) to the map $\epsilon \mapsto y(\epsilon)(x)$. It is then interesting to observe that the equation (62) controlling the variation of $y(\epsilon)$ is formally analogous to the linear equation

$$\frac{d}{d\epsilon} y(\epsilon) = \left(\frac{U}{\epsilon^2} + \frac{V}{\epsilon}\right) y(\epsilon) \tag{64}$$

occurring in the theory of Frobenius manifolds. In the equation (64) the matrices U and V are infinitesimal linear automorphisms of the tangent space T_pM at a point $p \in M$ of a Frobenius manifold, whereas the corresponding quantities in (62) are infinitesimal symplectic automorphisms of the torus $T^{\#}_{M,p}$. This analogy was the main topic of [17], and leads to the link with DT invariants.

Suppose that the matrix U in (64) has distinct eigenvalues $u_i \in \mathbb{C}$. The Stokes rays of the equation (64) are then defined to be the rays $\ell_{ij} = \mathbb{R}_{>0} \cdot (u_i - u_j) \subset \mathbb{C}^*$. A result of Balser, Jurkat and Lutz [11] shows that in any half-plane $H(\varphi) \subset \mathbb{C}^*$ centred on a non-Stokes ray $r = \mathbb{R}_{>0} \cdot \exp(i\pi\varphi) \subset \mathbb{C}^*$, there is a unique fundamental solution $\Phi\colon H(\varphi) \to \mathrm{GL}_n(\mathbb{C})$ to (64) satisfying

$$\Phi(\epsilon) \cdot \exp(U/\epsilon) \to \mathrm{id} \ \text{ as } \epsilon \to 0. \tag{65}$$

Comparing these canonical solutions for half-planes $H(\varphi_\pm)$ centred on small perturbations $r_\pm$ of a Stokes ray $\ell \subset \mathbb{C}^*$ defines the Stokes factors $\mathbb{S}(\ell) \in \mathrm{GL}_n(\mathbb{C})$.

Turning to the equation (62), note that the first term on the right-hand-side is the invariant vector field $U = \sum_i z_i \cdot \partial/\partial\theta_i$ on $T^{\#}_{M,p}$ corresponding to the value $E_p \in T_{M,p}$ of the vector field (18). Exponentiating yields well-defined automorphisms $\exp(U/\epsilon)$ of $T^{\#}_{M,p}$ given in co-ordinates by $\theta_i \mapsto \theta_i + \epsilon^{-1} z_i$. The above analogy then suggests that there should be a countable collection of Stokes rays $\ell \subset \mathbb{C}^*$, and for any half-plane $H(\varphi) \subset \mathbb{C}^*$ centred on a non-Stokes ray $r = \mathbb{R}_{>0} \cdot \exp(i\pi\varphi) \subset \mathbb{C}^*$, a symplectic isomorphism $F\colon Z^{\#}_1 \to T^{\#}_{M,p}$ such that $\Phi(\epsilon) = F \circ y(\epsilon)\colon T^{\#}_{M,p} \to T^{\#}_{M,p}$ satisfies

$$\exp(U/\epsilon) \circ \Phi(\epsilon) \to \mathrm{id} \ \text{ as } \epsilon \to 0. \tag{66}$$

The maps F corresponding to different half-planes $H(\varphi)$ then differ by compositions of symplectic automorphisms $\mathbb{S}(\ell) \in \mathrm{Aut}_\omega(T^{\#}_{M,p})$ associated to rays $\ell \subset \mathbb{C}^*$. These automorphisms $\mathbb{S}(\ell)$ should be viewed as non-linear Stokes factors. (The factors in (65) and (66) appear in different orders because we are working in the group of symplectic automorphisms of $(\mathbb{C}^*)^n$ rather than the opposite group of Poisson automorphisms of $\mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$: compare [17, Section 6.6]).

In practice the above analogy should be considered a guiding principle rather than a precise statement. In particular we cannot expect the map F to be defined on the whole of $Z^{\#}_1$. Nonetheless the basic point is that there should be preferred Darboux co-ordinates $(x_1, \cdots, x_n)$ on open subsets of $Z^{\#}_1$, depending on a point $p \in M$ and a half-plane $H(\varphi) \subset \mathbb{C}^*$, which have the property that

$$x_i(y(\epsilon)) \sim -\epsilon^{-1} z_i + \theta_i + O(\epsilon), \tag{67}$$

as $\epsilon \to 0$ in the half-plane $H(\varphi)$.

Remark 5.5. Let M be the stability space of a suitably-finite CY_3 triangulated category $\mathcal{D}$. The speculative suggestion of [17] (see also [20, Appendix A] for a brief summary) is that there should be a Joyce structure on M with the property that the Stokes automorphisms $\mathbb{S}(\ell)$ are given by

$$\mathbb{S}(\ell)^*(X_\beta) = X_\beta \cdot \prod_{Z(\gamma) \in \ell} (1 + \sigma(\gamma) X_\gamma)^{\Omega(\gamma) \cdot \langle \gamma, \beta \rangle}, \tag{68}$$

where $\Omega(\gamma) \in \mathbb{Z}$ are the DT invariants for the stability condition $p \in M$. Here $X_\gamma = \exp(x_\gamma)$ are characters on the torus $T^{\#}_{M,p}$, and $\sigma\colon \Gamma \to \{\pm 1\}$ is a quadratic refinement of the Euler form $\langle -, - \rangle$.

In [46, Section 5.4], Kontsevich and Soibelman propose a precise (but conjectural) relation between Joyce-like structures and DT invariants along these lines. On the Joyce structure side they consider the restriction of the foliation (57) on $X^{\#} \times \mathbb{P}^1$ to the germ of a neighbourhood of the subset $X^{\#} \times \{0\}$. (Note that their $E_0 \to X$ is our $X^{\#} \to M$). On the DT side they impose growth conditions on the DT invariants ("analytic stability structure") to ensure that the wall-crossing automorphisms (68) are well-defined on analytic open subsets of the tori $T^{\#}_{M,p} \cong (\mathbb{C}^*)^n$. to their Stokes data.

5.4. **Twistor spaces for categories of class $S[A_1]$.** Let us fix a genus $g \geq 2$ and consider the Joyce structure constructed in Section 3. Note that since this is a meromorphic Joyce structure the definition of the twistor space is even more delicate than usual. The most obvious fix is to replace X in Definition 5.1 by the open subset on which the Joyce structure is regular. It would be interesting to try to give a direct geometric construction of the resulting twistor space $p\colon Z^{\#} \to \mathbb{P}^1$. Here we restrict ourselves to a simple remark about the fibres over points $\epsilon \in \mathbb{C}^* \subset \mathbb{P}^1$.

Choose a reference surface S_g of genus g, set $G = \mathrm{PGL}_2(\mathbb{C})$, and define

$$\mathrm{MCG}(g) = \pi_0(\mathrm{Diff}^+(S_g)), \qquad \mathcal{X}(g) = \mathrm{Hom}_{\mathrm{grp}}(\pi_1(S_g), G)/G. \tag{69}$$

Then the mapping class group $\mathrm{MCG}(g)$ acts on the character stack $\mathcal{X}(g)$ in the usual way, and sending a quadruple (C, E, ∇) to the monodromy of the connection ∇ defines a map

$$\mu\colon \mathcal{M}(C, E, \nabla)/J^2(C) \to \mathcal{X}(g)/\,\mathrm{MCG}(g), \tag{70}$$

whose fibres are by definition the horizontal leaves of the isomonodromy connection. Transferring this map across the diagrams (34) and (42), and passing to the quotient by $\mathrm{im}(h_\epsilon)$ for $\epsilon \in \mathbb{C}^*$ yields an étale map

$$\mu\colon Z^{\#}_\epsilon \to \mathcal{X}(g)/\,\mathrm{MCG}(g). \tag{71}$$

It would be interesting to test the speculations of Section 5.3 in this context. Since the differential Q is holomorphic, its precise trajectory structure can be very complicated, and to the best of the author's knowledge the correct co-ordinates on $Z^{\#}_1$ to associate to a general half-plane $H(\varphi)$ are unknown. If we generalise to meromorphic quadratic differentials however the trajectory structure is much simpler, and many of the ingredients required for a rigorous treatment are known.

It is natural to expect that there is a similar map to (71) in which $\mathcal{X}(g)$ is replaced with a space of framed local systems [28]. The preferred Darboux co-ordinates associated to a general half-plane $H(\varphi)$ should then be the Fock-Goncharov co-ordinates for the WKB triangulation determined by the horizontal trajectories of the quadratic differential $e^{-2\pi i\varphi} \cdot Q$. It follows from the work of Gaiotto, Moore and Neitzke [32] and the results of [19] that as φ varies these satisfy the jumps (68) determined by the DT invariants of the corresponding category $\mathcal{D}(g, m)$. The asymptotic property (67) should then follow from results in exact WKB analysis [39, 51]. The special case $g = 0, m = \{7\}$ was treated in full detail in [18].

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