

# Counting and Sampling: Algorithms and Complexity

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## Abstract

This report documents the program and the outcomes of Dagstuhl Seminar 22482 “Counting and Sampling: Algorithms and Complexity”. We document the talks presented, covering many advances in the area made over the last five years. As well, we document the progress made by working groups on future projects.

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## 1 Executive Summary

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Counting and sampling problems arise in areas such as statistics (benchmarking statistical tests, or sampling from a posterior distribution) and statistical physics (computing the partition function of a spin system). Computationally, these problems are very different in character from decision or optimisation problems, and their solution requires distinctive techniques. It is natural to treat counting and sampling together in the same Dagstuhl Seminar, as they are closely related computationally: subject to a reasonable side condition, an efficient algorithm for sampling certain combinatorial structures can be used as a black box to approximately count those structures, and vice versa.

Although much attention has been directed towards the complexity of counting and sampling problems, our understanding of them is not as well developed as it is of decision and optimisation problems. This Seminar marks a timely return to the topic, as new ideas have recently been injected into the area, resulting in renewed activity and progress. It is

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particularly satisfying to observe that much of this progress has been in the positive direction, in the form of new efficient algorithms. This is in an area where negative results had become the norm.

The Covid pandemic inevitably left its mark on the meeting. Over five years elapsed between the previous Dagstuhl Seminar on a related topic and the current one. In the meantime, the introduction of a circle of ideas around high-dimensional expanders, spectral expansion and entropy decay has transformed the analysis of Markov chains for sampling, and brought many previously intractable questions within scope of our methods. An unwelcome impact of Covid was to reduce significantly the number of participants. Sadly, it was not possible to invite all the people we would have liked to see at the meeting.

With a view to providing a snapshot of current interests, here is a rough-and-ready breakdown of the presentations against a somewhat arbitrary set of headings.

- *Connections with statistical physics, phase transitions, etc.* Coja-Oghlan, Galanis and Patel,
- *Holant and constraint satisfaction problems.* Backens and Bulatov.
- *Markov chains.* Guo and Miracle.
- *Parameterised complexity of counting problems.* Bressan, Focke, Roth and Wellnitz.
- *Perfect samplers.* Anand and Cannon.
- *Point processes and other geometric connections.* Anari, Jerrum and Pappik.
- *Polynomials associated with graphs, matroids and matrices.* Björklund, Curticapean, Regts.
- *Other.* Göbel, Goldberg, Kaski, Lapinskas.

If nothing else, this rough classification exercise gives an impression of the wide span of current research. Aside from the progress on the analysis of Markov chains mentioned earlier, many other topics have seen advances in the past five years. Examples include: counting small patterns (‘motifs’) in large graphs (networks), sampling structures in regions of phase non-uniqueness, perfect sampling, and weighted counting problems where the weights are complex. It turns out that the latter study shines light on the case of real weights, through an examination of zeros of partition functions in the complex plane. The meeting gave participants a long-awaited chance to review developments over the past five years.

On the organisational front, an innovation (as far as this community is concerned) was the inclusion of a problem session on the first day. This went off quite smoothly, and small working groups formed fairly spontaneously to work on problems during the week. On the final day we heard from the groups a summary of their investigations over the week. Our hope is that sufficient momentum was achieved on some of these problems that groups will continue to work on them beyond the end of the meeting. Indeed, one of the working groups decided to apply to run a workshop on homomorphism counting at ICALP 2023 with this aim in mind. The proposal, by Radu Curticapean and Marc Roth, was accepted, and the workshop, entitled “ADjoint HOMomorphism Counting” (AD HOC) will take place in July 2023. We look forward to being able to report on advances achieved on this and other topics on this website.

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### 3 Overview of Talks

#### 3.1 Lazy Depth-First Sampling of Spin Systems

Konrad Anand (Queen Mary University of London, GB) and Mark R. Jerrum (Queen Mary University of London, GB)

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**Main reference** Konrad Anand, Mark Jerrum: “Perfect Sampling in Infinite Spin Systems Via Strong Spatial Mixing”, *SIAM Journal on Computing*, Vol. 51(4), pp. 1280–1295, 2022.

**URL** <https://doi.org/10.1137/21M1437433>

We present a simple algorithm that perfectly samples configurations from the unique Gibbs measure of a spin system on a potentially infinite graph  $G$ . The sampling algorithm assumes strong spatial mixing together with subexponential growth of  $G$ . It produces a finite window onto a perfect sample from the Gibbs distribution. The run-time is linear in the size of the window, in particular it is constant for each vertex.

#### 3.2 Parallel Discrete Sampling via Continuous Walks

Nima Anari (Stanford University, US)

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**Joint work of** Nima Anari, Yizhi Huang, Tianyu Liu, Thuy-Duong Vuong, Brian Xu, Katherine Yu

We develop a framework for sampling from discrete distributions on the hypercube by sampling from continuous distributions obtained by convolution with spherical Gaussians. We show that for well-studied families of discrete distributions, the result of the convolution is well-conditioned log-concave, whenever the Gaussian’s variance is above an  $O(1)$  threshold. We plug in off-the-shelf continuous sampling methods into our framework to obtain novel discrete sampling algorithms. Additionally, we introduce and study a crucial notion of smoothness for discrete distributions that we call transport-stability, that we use to control the propagation of error in our framework. We expect transport-stability to be of independent interest, as we connect it to constructions of optimally mixing local random walks and concentration inequalities.

As our main application, we resolve open questions on parallel sampling of distributions which admit parallel counting. We show that determinantal point processes can be sampled via RNC algorithms, that is in time  $\text{polylog}(n)$  using  $\text{poly}(n)$  processors. For a wider class of distributions, we show our framework yields Quasi-RNC sampling, that is sampling in time  $\text{polylog}(n)$  using  $n^{O(\log n)}$  processors. This wider class includes random Eulerian tours in digraphs.

### 3.3 Holant clones and approximation of holant problems

*Miriam Backens (University of Birmingham, GB)*

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**Joint work of** Miriam Backens, Leslie Ann Goldberg

**Main reference** Miriam Backens, Leslie Ann Goldberg: “Holant Clones and the Approximability of Conservative Holant Problems”, *ACM Trans. Algorithms*, Vol. 16(2), pp. 23:1–23:55, 2020.

**URL** <https://doi.org/10.1145/3381425>

Holant problems are a generalisation of counting constraint satisfaction problems, equivalent to the problem of fully contracting a tensor network built from some fixed family of tensors. Generalising relational and functional clones, the holant clone of a set of constraint functions contains all functions that can be simulated from the original set via gadgets.

I will discuss a result about approximation of holant problems that employs the formalism of holant clones, and talk about some further work in progress.

### 3.4 The Fine-Grained Complexity of Computing the Tutte Polynomial of a Linear Matroid

*Andreas Björklund (Lund, SE)*

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**Joint work of** Andreas Björklund and Petteri Kaski

**Main reference** Andreas Björklund, Petteri Kaski: The Fine-Grained Complexity of Computing the Tutte Polynomial of a Linear Matroid. *SODA 2021*: 2333–2345.

**URL** <https://dl.acm.org/doi/10.5555/3458064.3458203>

We show that computing the Tutte polynomial of a linear matroid of dimension  $k$  on  $k^{O(1)}$  points over a field of  $k^{O(1)}$  elements requires  $k^{\Omega(k)}$  time unless the #ETH— a counting extension of the Exponential Time Hypothesis of Impagliazzo and Paturi [CCC 1999] due to Dell et al. [ACM TALG 2014]—is false.

### 3.5 Linear and sublinear algorithms for sampling graphlets in large graphs

*Marco Bressan (University of Milan, IT)*

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**Main reference** Marco Bressan: “Efficient and near-optimal algorithms for sampling connected subgraphs”, in Proc. of the STOC '21: 53rd Annual ACM SIGACT Symposium on Theory of Computing, Virtual Event, Italy, June 21–25, 2021, pp. 1132–1143, ACM, 2021.

**URL** <https://doi.org/10.1145/3406325.3451042>

A fundamental primitive in modern graph mining is sampling connected subgraphs on  $k$  vertices (also known as  $k$ -graphlets) from a graph  $G$ . For a long time, no good algorithm was known for sampling  $k$ -graphlets uniformly at random; the best algorithms available could sample only approximately. In this talk I will present algorithms for sampling  $k$ -graphlets uniformly or eps-uniformly from an arbitrary  $n$ -vertex graph  $G$  with preprocessing time linear or even sublinear in  $G$  and sampling time logarithmic or even constant in  $G$ .

### 3.6 Complexity classification of counting graph homomorphisms modulo a prime number

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**Joint work of** Andrei A. Bulatov, Amirhosein Kazemina

Counting graph homomorphisms and its generalizations such as the Counting Constraint Satisfaction Problem (CSP), its variations, and counting problems in general have been intensively studied since the pioneering work of Valiant. While the complexity of exact counting of graph homomorphisms (Dyer and Greenhill, 2000) and the counting CSP (Bulatov, 2013, and Dyer and Richerby, 2013) is well understood, counting modulo some natural number has attracted considerable interest as well. In their 2015 paper Faben and Jerrum suggested a conjecture stating that counting homomorphisms to a fixed graph  $H$  modulo a prime number is hard whenever it is hard to count exactly, unless  $H$  has automorphisms of certain kind. In this paper we confirm this conjecture. As a part of this investigation we develop techniques that widen the spectrum of reductions available for modular counting and apply to the general CSP rather than being limited to graph homomorphisms.

### 3.7 Fast and Perfect Sampling of Subgraphs and Polymer Systems

*Sarah Cannon (Claremont McKenna College, US)*

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**Joint work of** Antonio Blanca, Sarah Cannon, Will Perkins

**Main reference** Antonio Blanca, Sarah Cannon, Will Perkins: “Fast and Perfect Sampling of Subgraphs and Polymer Systems”, in Proc. of the Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2022, September 19-21, 2022, University of Illinois, Urbana-Champaign, USA (Virtual Conference), LIPIcs, Vol. 245, pp. 4:1–4:18, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022.

**URL** <https://doi.org/10.4230/LIPIcs.APPROX/RANDOM.2022.4>

We give an efficient perfect sampling algorithm for weighted, connected induced subgraphs (or graphlets) of rooted, bounded degree graphs. Our algorithm utilizes a vertex-percolation process with a carefully chosen rejection filter and works under a percolation subcriticality condition. We show that this condition is optimal in the sense that the task of (approximately) sampling weighted rooted graphlets becomes impossible in finite expected time for infinite graphs and intractable for finite graphs when the condition does not hold. We apply our sampling algorithm as a subroutine to give near linear-time perfect sampling algorithms for polymer models and weighted non-rooted graphlets in finite graphs, two widely studied yet very different problems. This new perfect sampling algorithm for polymer models gives improved sampling algorithms for spin systems at low temperatures on expander graphs and unbalanced bipartite graphs, among other applications.

### 3.8 The random 2-SAT partition function

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**Joint work of** Amin Coja-Oghlan, Dimitris Achlioptas, Max Hahn-Klimroth, Joon Lee, Noëla Müller, Manuel Penschuck, Guangyan Zhou

**Main reference** Dimitris Achlioptas, Amin Coja-Oghlan, Max Hahn-Klimroth, Joon Lee, Noëla Müller, Manuel Penschuck, Guangyan Zhou: “The number of satisfying assignments of random 2-SAT formulas”, *Random Struct. Algorithms*, Vol. 58(4), pp. 609–647, 2021.

**URL** <https://doi.org/10.1002/rsa.20993>

The random 2-SAT problem was the first random constraint satisfaction problem whose satisfiability threshold could be pinpointed precisely [1, 2]. The satisfiability threshold turns out to be determined by the appearance of certain local structures called “bicycles”. In this talk I address the more difficult but no less fundamental problem of calculating the (exponential order of the) number of satisfying assignments within the satisfiable phase. The main result rigorously establishes a prediction from statistical physics called the “replica symmetric ansatz” [3, 4]. The resulting formula does not boil down to a simple algebraic expression, but rather involves a stochastic fixed point problem that mimics the Belief Propagation message passing algorithm. Nonetheless, the formula can be evaluated numerically within arbitrary precision.

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### 3.9 Immanants and determinants

*Radu Curticapean (IT University of Copenhagen, DK)*

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**Main reference** Radu Curticapean: “A full complexity dichotomy for immanant families”, in *Proc. of the STOC '21: 53rd Annual ACM SIGACT Symposium on Theory of Computing*, Virtual Event, Italy, June 21–25, 2021, pp. 1770–1783, ACM, 2021.

**URL** <https://doi.org/10.1145/3406325.3451124>

Immanants are matrix functionals that generalize determinants and permanents by allowing general irreducible characters of the symmetric group as permutations weights rather than merely the sign function (which yields the determinant) or the all-ones function (which yields the permanent). In this talk, we give an introduction to immanants and describe a recent classification of their complexity. In a second part, we give a simple proof that shows how determinants can be expressed in terms of homomorphism counts from cycle covers, leaving open similar expressions for general immanants.

### 3.10 Counting small induced subgraphs with hereditary properties

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**Joint work of** Jacob Focke, Marc Roth

**Main reference** Jacob Focke, Marc Roth: “Counting small induced subgraphs with hereditary properties”, in Proc. of the STOC ’22: 54th Annual ACM SIGACT Symposium on Theory of Computing, Rome, Italy, June 20 – 24, 2022, pp. 1543–1551, ACM, 2022.

**URL** <https://doi.org/10.1145/3519935.3520008>

We study the computational complexity of the problem  $\#\text{IndSub}(\Phi)$  of counting  $k$ -vertex induced subgraphs of a graph  $G$  that satisfy a graph property  $\Phi$ . Our main result establishes an exhaustive and explicit classification for all hereditary properties, including tight conditional lower bounds under the Exponential Time Hypothesis (ETH): If a hereditary property  $\Phi$  is true for all graphs, or if it is true only for finitely many graphs, then  $\#\text{IndSub}(\Phi)$  is solvable in polynomial time. Otherwise,  $\#\text{IndSub}(\Phi)$  is  $\#\text{W}[1]$ -complete when parameterised by  $k$ , and, assuming ETH, it cannot be solved in time  $f(k) \cdot |G|^{\sigma(k)}$  for any function  $f$ . This classification features a wide range of properties for which the corresponding detection problem (as classified by Khot and Raman [TCS 02]) is tractable but counting is hard. Moreover, even for properties which are already intractable in their decision version, our results yield significantly stronger lower bounds for the counting problem. As additional result, we also present an exhaustive and explicit parameterised complexity classification for all properties that are invariant under homomorphic equivalence. By covering one of the most natural and general notions of closure, namely, closure under vertex-deletion (hereditary), we generalise some of the earlier results on this problem. For instance, our results fully subsume and strengthen the existing classification of  $\#\text{IndSub}(\Phi)$  for monotone (subgraph-closed) properties due to Roth, Schmitt, and Wellnitz [FOCS 20]. A full version of our paper, containing all proofs, is available at <https://arxiv.org/abs/2111.02277>.

### 3.11 Metastability for the ferromagnetic Potts model

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**Joint work of** Amin Coja-Oghlan, Andreas Galanis, Leslie Ann Goldberg, Jean Bernoulli Ravelomanana, Daniel Stefankovic, Eric Vigoda

**Main reference** Amin Coja-Oghlan, Andreas Galanis, Leslie Ann Goldberg, Jean Bernoulli Ravelomanana, Daniel Stefankovic, Eric Vigoda: “Metastability of the Potts ferromagnet on random regular graphs”, CoRR, Vol. abs/2202.05777, 2022.

**URL** <https://arxiv.org/abs/2202.05777>

We study the performance of Markov chains for the  $q$ -state ferromagnetic Potts model on random regular graphs. While the cases of the grid and the complete graph are by now well-understood, the case of random regular graphs has resisted a detailed analysis and, in fact, even analysing the properties of the Potts distribution has remained elusive. It is conjectured that the performance of Markov chains is dictated by metastability phenomena, i.e., the presence of “phases” (clusters) in the sample space where Markov chains with local update rules, such as the Glauber dynamics, are bound to take exponential time to escape, and therefore cause slow mixing. The phases that are believed to drive these metastability phenomena in the case of the Potts model emerge as local, rather than global, maxima of the so-called Bethe functional, and previous approaches of analysing these phases based on optimisation arguments fall short of the task.

Our first contribution is to detail the emergence of the two relevant phases for the  $q$ -state Potts model on the  $d$ -regular random graph for all integers  $q, d \geq 3$ , and establish that for an interval of temperatures, delineated by the uniqueness and a broadcasting threshold on the  $d$ -regular tree, the two phases coexist (as possible metastable states). The proofs are based on a conceptual connection between spatial properties and the structure of the Potts distribution on the random regular graph, rather than complicated moment calculations. This significantly refines earlier results by Helmuth, Jenssen, and Perkins who had established phase coexistence for a small interval around the so-called ordered-disordered threshold (via different arguments) that applied for large  $q$  and  $d \geq 5$ .

Based on our new structural understanding of the model, our second contribution is to obtain metastability results for two classical Markov chains for the Potts model. We first complement recent fast mixing results for Glauber dynamics by Blanca and Gheissari below the uniqueness threshold, by showing an exponential lower bound on the mixing time above the uniqueness threshold. Then, we obtain tight results even for the non-local and more elaborate Swendsen-Wang chain, where we establish slow mixing/metastability for the whole interval of temperatures where the chain is conjectured to mix slowly on the random regular graph. The key is to bound the conductance of the chains using a random graph “planting” argument combined with delicate bounds on random-graph percolation.

### 3.12 Instability of contention resolution protocols

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**Joint work of** Leslie Ann Goldberg, John Lapinskas

**Main reference** Leslie Ann Goldberg, John Lapinskas: “Instability of backoff protocols with arbitrary arrival rates”, CoRR, Vol. abs/2203.17144, 2022.

**URL** <https://doi.org/10.48550/arXiv.2203.17144>

A backoff protocol is a simple and elegant randomised algorithm for communicating in a Multiple Access Channel. Aldous conjectured in 1987 that, for any positive arrival rate, every backoff protocol is unstable. I will report on new work with John Lapinskas towards proving this conjecture. (This will appear in SODA 2023.)

### 3.13 Towards derandomising Markov chain Monte Carlo

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**Joint work of** Heng Guo, Weiming Feng, Chunyang Wang, Jiaheng Wang, Yitong Yin

**Main reference** Weiming Feng, Heng Guo, Chunyang Wang, Jiaheng Wang, Yitong Yin: “Towards derandomising Markov chain Monte Carlo”, CoRR, Vol. abs/2211.03487, 2022.

**URL** <https://doi.org/10.48550/arXiv.2211.03487>

We present a new framework to derandomise certain Markov chain Monte Carlo (MCMC) algorithms. As in MCMC, we first reduce counting problems to sampling from a sequence of marginal distributions. For the latter task, we introduce a method called coupling towards the past that can, in logarithmic time, evaluate one or a constant number of variables from a stationary Markov chain state. Since there are at most logarithmic random choices, this leads

to very simple derandomisation. We provide two applications of this framework, namely efficient deterministic approximate counting algorithms for hypergraph independent sets and hypergraph colourings, under local lemma type conditions matching, up to lower order factors, their state-of-the-art randomised counterparts.

### 3.14 Analysis of the survival time of the SIRS process via expansion

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**Joint work of** Tobias Friedrich, Andreas Göbel, Nicolas Klodt, Martin S. Krejca, Marcus Pappik

**Main reference** Tobias Friedrich, Andreas Göbel, Nicolas Klodt, Martin S. Krejca, Marcus Pappik: “Analysis of the survival time of the SIRS process via expansion”, arXiv, 2022.

**URL** <https://doi.org/10.48550/ARXIV.2205.02653>

We study the SIRS process, a continuous-time Markov chain modelling the spread of infections on graphs. In this process, vertices are either susceptible, infected, or recovered. Each infected vertex becomes recovered at rate 1 and infects each of its susceptible neighbours independently at rate  $\lambda$ , and each recovered vertex becomes susceptible at a rate  $\rho$ , which we assume to be independent of the graph size. A central quantity of the SIRS process is the time until no vertex is infected, known as the *survival time*. The survival time of the SIRS process is studied extensively in a variety of contexts. Surprisingly though, to the best of our knowledge, no rigorous theoretical results exist so far. This is even more surprising given that for the related SIS process, mathematical analysis began in the 70s and continues to this day.

We address this imbalance by conducting the first theoretical analyses of the SIRS process on various graph classes via their expansion properties. Our analyses assume that the graphs start with at least one infected vertex and no recovered vertices. Our first result considers stars, which have poor expansion. We prove that the expected survival time of the SIRS process on stars is at most polynomial in the graph size for *any* value of  $\lambda$ . This behaviour is fundamentally different from the SIS process, where the expected survival time is exponential already for small infection rates. Due to this property, for the SIS process, stars constitute an important sub-structure for proving an expected exponential survival time of more complicated graphs. For the SIRS process, this argument is not sufficient.

Our main result is an exponential lower bound of the expected survival time of the SIRS process on expander graphs. Specifically, we show that on expander graphs  $G$  with  $n$  vertices, degree close to  $d$ , and sufficiently small spectral expansion, the SIRS process has expected survival time at least exponential in  $n$  when  $\lambda \geq c/d$  for a constant  $c > 1$ . This result is complemented by established results for the SIS process, which imply that the expected survival time of the SIRS process is at most logarithmic in  $n$  when  $\lambda \leq c/d$  for a constant  $c < 1$ . Combined, our result shows an almost-tight threshold behaviour of the expected survival time of the SIRS process on expander graphs. Additionally, our result holds even if  $G$  is a subgraph. This allows, for the SIRS process, the use of expanders as sub-structures for lower bounds, similar to stars in the SIS process. Notably, our result implies an almost-tight threshold for Erdős–Rényi graphs and a regime of exponential survival time for hyperbolic random graphs, one of the most popular graph models, as it incorporates many properties found in real-world networks. The proof of our main result draws inspiration from Lyapunov functions used in mean-field theory to devise a two-dimensional potential function and applying a negative-drift theorem to show that the expected survival time is exponential.

### 3.15 Counting vertices of integral polytopes defined by facets

Mark R. Jerrum (Queen Mary University of London, GB)

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**Joint work of** Heng Guo (Edinburgh) and Mark Jerrum

**Main reference** Heng Guo, Mark Jerrum: “Counting vertices of integral polytopes defined by facets”, *Discrete and Computational Geometry*, July 2022.

**URL** <https://doi.org/10.1007/s00454-022-00406-8>

We present a number of complexity results concerning the problem of counting vertices of an integral polytope defined by a system of linear inequalities. The focus is on polytopes with small integer vertices, particularly 0/1-polytopes and half-integral polytopes (ones whose vertices are contained in  $\{0, 1\}^n$  and  $\{0, \frac{1}{2}, 1\}^n$ , respectively). Such polytopes are ubiquitous in the field of combinatorial optimisation.

Suppose a polytope  $P$  is defined by linear inequalities  $Ax \leq b$ . If the matrix  $A$  is ‘totally unimodular’ then  $P$  is guaranteed to be integral; many integral polytopes that are encountered in practice arise in this way. Network matrices and their transposes are particular kinds of totally unimodular matrices. Our main results are the following.

- Counting the vertices of a 0/1-polytope exactly is #P complete, even when restricted to the case when  $A$  is a network matrix or the transpose of one. (This is nothing more than an observation, given that the problems of counting perfect matchings or independent sets in a bipartite graph are both #P-complete.)
- Approximately counting the vertices of a half-integral polytope is NP-hard, as witnessed by the ‘perfect 2-matching polytope’.
- The vertex-counting problem for 0/1-polytopes defined by transposes of network matrices is equivalent, under approximation-preserving reductions, to counting independent sets of a bipartite graph (#BIS). No efficient approximation algorithm is known for #BIS, but neither is approximating #BIS known to be NP-hard. Many natural counting problems are known to be equivalent to #BIS under polynomial-time approximation-preserving reductions.
- For a natural subclass of polytopes defined by network matrices, it is possible to approximate the number of vertices in polynomial time. The complexity for network matrices in general is, however, unknown,

### 3.16 Nearly optimal independence oracle algorithms for edge estimation in hypergraphs

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**Joint work of** John Lapinskas, Holger Dell, Kitty Meeks

**Main reference** Holger Dell, John Lapinskas, Kitty Meeks: “Nearly optimal independence oracle algorithms for edge estimation in hypergraphs”, *CoRR*, Vol. abs/2211.03874, 2022.

**URL** <https://doi.org/10.48550/arXiv.2211.03874>

We study a query model of computation in which an  $n$ -vertex  $k$ -hypergraph can be accessed only via its independence oracle or via its colourful independence oracle, and each oracle query may incur a cost depending on the size of the query. In each of these models, we obtain oracle algorithms to approximately count the hypergraph’s edges, and we unconditionally prove that no oracle algorithm for this problem can have significantly smaller worst-case oracle cost than our algorithms.

### 3.17 Iterated Decomposition of Biased Permutations Via New Bounds on the Spectral Gap of Markov Chains

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**Joint work of** Sarah Miracle, Amanda Pascoe Streib, Noah Streib

**Main reference** Sarah Miracle, Amanda Pascoe Streib, Noah Streib: “Iterated Decomposition of Biased Permutations via New Bounds on the Spectral Gap of Markov Chains”, in Proc. of the Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2020, August 17-19, 2020, Virtual Conference, LIPIcs, Vol. 176, pp. 3:1–3:21, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2020.

**URL** <https://doi.org/10.4230/LIPIcs.APPROX/RANDOM.2020.3>

We study a nearest-neighbor Markov chain over biased permutations of  $[n]$ . We build on previous work that analyzed the spectral gap of the chain when  $[n]$  is partitioned into  $k$  classes. There, the authors iteratively decomposed the nearest neighbor chain into simpler chains, but incurred a multiplicative penalty of  $n^{-2}$  for each application of the decomposition theorem. We introduce a new decomposition theorem which allows us to avoid this penalty (in certain cases) and prove the first inverse-polynomial bound on the spectral gap of the chain when  $k$  is as large as  $\Theta(n/\log n)$ . The previous best known bound assumed  $k$  was at most a constant.

### 3.18 Discretization-based algorithms for repulsive Gibbs point processes

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**Joint work of** Marcus Pappik, Tobias Friedrich, Andreas Göbel, Maximilian Katzmann, and Martin S. Krejca

**Main reference** Tobias Friedrich, Andreas Göbel, Maximilian Katzmann, Martin S. Krejca, Marcus Pappik: “Using random graphs to sample repulsive Gibbs point processes with arbitrary-range potentials”, CoRR, Vol. abs/2204.01793, 2022.

**URL** <https://doi.org/10.48550/arXiv.2204.01793>

Gibbs point processes are a popular way to model particle distributions of fluids and gasses in Euclidean space. Similar to spin systems on graphs, which might be seen as their discrete counterparts, sampling the Gibbs distribution and computing its normalizing constant, the partition function, of such point processes is highly relevant. However, until recently, very few rigorous computational results existed. In this talk, we give a brief introduction to Gibbs point processes and present recent algorithmic results. In particular, we focus on discretization-based algorithms, which reduce the algorithmic tasks at hand to related problems for discrete spin systems, making use of the rich literature in that area. Our main focus will be on a recent approach that employs hard-core models on carefully constructed families of geometric random graphs to obtain sampling and approximation algorithms for Gibbs point processes. This results in efficient algorithms for arbitrary repulsive pair potentials  $\phi$  up to a fugacity of  $\lambda < e/C_\phi$ , where  $C_\phi$  is the temperedness constant of  $\phi$ .

### 3.19 Sampling from the low temperature ferromagnetic Potts model via flows

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**Joint work of** Viresh Patel, Jeroen Huijben, and Guus Regts

**Main reference** Jeroen Huijben, Viresh Patel, Guus Regts: “Sampling from the low temperature Potts model through a Markov chain on flows”, *Random Struct. Algorithms*, Vol. 62(1), pp. 219–239, 2023.

**URL** <https://doi.org/10.1002/rsa.21089>

I will discuss how one can (approximately and quickly) sample configurations from the ferromagnetic Potts model with underlying graph  $G$  at low temperatures using a Markov chain on flows. The use of flows allows one to work with certain types of graphs that can have unbounded degree.

### 3.20 Trustworthy Monte Carlo

*Petteri Kaski*

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**Joint work of** Juha Harviainen, Mikko Koivisto, Petteri Kaski

**Main reference** Juha Harviainen, Mikko Koivisto, Petteri Kaski: “Trustworthy Monte Carlo”, in *Proc. of the Advances in Neural Information Processing Systems*, 2022.

**URL** <https://openreview.net/forum?id=jglXPY6gH-1>

Building on work of Williams (CCC’16) and Björklund & Kaski (PODC’16) on fine-grained noninteractive proof systems in the context of deterministic counting problems, we study verifiable randomized approximation schemes for hard counting problems such as the permanent. We show that sample-average-based Monte Carlo estimators such as the Godsil-Gutman and the Chien-Rasmussen-Sinclair estimators for the permanent admit verifiable randomized approximation with verifier complexity scaling essentially as the square root of the prover/estimator complexity.

[This work is to appear in NeurIPS’22.]

### 3.21 Approximating the chromatic polynomial is as hard as computing it exactly

*Guus Regts (University of Amsterdam, NL)*

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**Joint work of** Guus Regts, Ferenc Bencs, Jeroen Huijben

**Main reference** Ferenc Bencs, Jeroen Huijben, Guus Regts: “Approximating the chromatic polynomial is as hard as computing it exactly”, *CoRR*, Vol. abs/2211.13790, 2022.

**URL** <https://doi.org/10.48550/arXiv.2211.13790>

In this talk I will explain that for any non-real algebraic number  $q$ , approximately computing the absolute value of the chromatic polynomial evaluated at  $q$  is as hard as computing it exactly and hence is #P-hard. The proof is based on constructing series-parallel gadgets that “implement” a dense set of edge interactions and is inspired by Sokal’s result saying that chromatic roots are dense in the complex plane.

### 3.22 Counting Small Directed Subgraphs, Parameterised by the Outdegree

Marc Roth (University of Oxford, GB)

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**Joint work of** Marco Bressan, Matthias Lanzinger, Marc Roth

**Main reference** Marco Bressan, Matthias Lanzinger, Marc Roth: “The Complexity of Pattern Counting in Directed Graphs, Parameterised by the Outdegree”, CoRR, Vol. abs/2211.01905, 2022.

**URL** <https://doi.org/10.48550/arXiv.2211.01905>

We study the problem of counting the copies of a small directed pattern graph  $H$  in a large directed host graph  $G$ . Motivated by the recent surge on pattern counting in degenerate graphs, we focus on host graphs with small outdegree  $d(G)$ . Formally, we choose  $|H| + d(G)$  as the problem parameter and ask for which classes of patterns the problem is fixed-parameter tractable.

This talk presents a complete parameterised complexity classification of the problem and provides an overview of the technical challenges in proving this result – among others, those challenges include a careful analysis of a variety of width measures on hypergraphs encoding the reachability structure of directed graphs.

### 3.23 Tight Complexity Bounds for Counting Generalized Dominating Sets in Bounded-Treewidth Graphs

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**Joint work of** Jacob Focke, Dániel Marx, Fionn Mc Inerney, Daniel Neuen, Govind S. Sankar, Philipp Schepper, Philip Wellnitz

**Main reference** Jacob Focke, Dániel Marx, Fionn Mc Inerney, Daniel Neuen, Govind S. Sankar, Philipp Schepper, Philip Wellnitz: “Tight Complexity Bounds for Counting Generalized Dominating Sets in Bounded-Treewidth Graphs”, CoRR, Vol. abs/2211.04278, 2022.

**URL** <https://doi.org/10.48550/arXiv.2211.04278>

We investigate how efficiently a well-studied family of domination-type problems can be solved on bounded-treewidth graphs. For sets  $\sigma, \rho$  of non-negative integers, a  $(\sigma, \rho)$ -set of a graph  $G$  is a set  $S$  of vertices such that  $|N(u) \cap S| \in \sigma$  for every  $u \in S$ , and  $|N(v) \cap S| \in \rho$  for every  $v \notin S$ . The problem of finding a  $(\sigma, \rho)$ -set (of a certain size) unifies standard problems such as INDEPENDENT SET, DOMINATING SET, INDEPENDENT DOMINATING SET, and many others.

For all pairs of finite or cofinite sets  $(\sigma, \rho)$ , we determine (under standard complexity assumptions) the best possible value  $c_{\sigma, \rho}$  such that there is an algorithm that counts  $(\sigma, \rho)$ -sets in time  $c_{\sigma, \rho}^{\text{tw}} \cdot n^{O(1)}$  (if a tree decomposition of width  $\text{tw}$  is given in the input). Let  $s_{\text{top}}$  denote the largest element of  $\sigma$  if  $\sigma$  is finite, or the largest missing integer  $+1$  if  $\sigma$  is cofinite;  $r_{\text{top}}$  is defined analogously for  $\rho$ . Surprisingly,  $c_{\sigma, \rho}$  is often significantly smaller than the natural bound  $s_{\text{top}} + r_{\text{top}} + 2$  achieved by existing algorithms [van Rooij, 2020]. Toward defining  $c_{\sigma, \rho}$ , we say that  $(\sigma, \rho)$  is  $m$ -structured if there is a pair  $(\alpha, \beta)$  such that every integer in  $\sigma$  equals  $\alpha \bmod m$ , and every integer in  $\rho$  equals  $\beta \bmod m$ . Then, setting

- $c_{\sigma, \rho} = s_{\text{top}} + r_{\text{top}} + 2$  if  $(\sigma, \rho)$  is not  $m$ -structured for any  $m \geq 2$ ,
- $c_{\sigma, \rho} = \max\{s_{\text{top}}, r_{\text{top}}\} + 2$  if  $(\sigma, \rho)$  is 2-structured, but not  $m$ -structured for any  $m \geq 3$ , and  $s_{\text{top}} = r_{\text{top}}$  is even, and
- $c_{\sigma, \rho} = \max\{s_{\text{top}}, r_{\text{top}}\} + 1$ , otherwise,

we provide algorithms counting  $(\sigma, \rho)$ -sets in time  $c_{\sigma, \rho}^{\text{tw}} \cdot n^{O(1)}$ . For example, for the EXACT INDEPENDENT DOMINATING SET problem (also known as PERFECT CODE) corresponding to  $\sigma = \{0\}$  and  $\rho = \{1\}$ , this improves the  $3^{\text{tw}} \cdot n^{O(1)}$  algorithm of van Rooij to  $2^{\text{tw}} \cdot n^{O(1)}$ .

Despite the unusually delicate definition of  $c_{\sigma, \rho}$ , we show that our algorithms are most likely optimal, that is, for any pair  $(\sigma, \rho)$  of finite or cofinite sets where the problem is non-trivial, and any  $\varepsilon > 0$ , a  $(c_{\sigma, \rho} - \varepsilon)^{\text{tw}} \cdot n^{O(1)}$ -algorithm counting the number of  $(\sigma, \rho)$ -sets would violate the Counting Strong Exponential-Time Hypothesis (#SETH). For finite sets  $\sigma$  and  $\rho$ , our lower bounds also extend to the decision version, showing that our algorithms are optimal in this setting as well. In contrast, for many cofinite sets, we show that further significant improvements for the decision and optimization versions are possible using the technique of representative sets.

## 4 Working groups

### 4.1 Counting Functions via Extension Oracles

Marco Bressan (University of Milan, IT), Konrad Anand (Queen Mary University of London, GB), and Holger Dell (Goethe-Universität – Frankfurt am Main, DE)

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**Introduction.** Let  $X = \{1, \dots, n\}$ , and let  $\mathcal{C}$  be a set of functions from  $X$  to some set  $Y$ ; for the sake of this introduction we may assume  $Y = \{0, 1\}$ . A *partial function* from  $X$  to  $Y$  is a function  $\hat{f} : X \rightarrow Y \cup \{\star\}$  where  $\star$  is a special symbol meaning abstention. An *extension oracle* or *consistency oracle*  $O_{\mathcal{C}}$  for  $\mathcal{C}$  takes in input a partial function  $\hat{f}$  from  $X$  to  $Y$  and returns 1 if and only if that function has an extension in  $\mathcal{C}$ , i.e., if  $\exists f \in \mathcal{C}$  such that  $f(x) = \hat{f}(x)$  for all  $x \in \hat{f}^{-1}(Y)$ . We consider the following counting problem: given  $X, Y$  and access to  $O_{\mathcal{C}}$ , compute  $|\mathcal{C}|$  or a good approximation to it. This problem arises for instance in machine learning, where  $\mathcal{C}$  is the concept class, or the version space (the set of concepts consistent with what the learner has seen so far). In this case, an efficient algorithm to compute or estimate  $|\mathcal{C}|$  yields efficient algorithms for, say, the Halving algorithm [2, 1, 5].

**Outcomes.** First, we have highlighted a separation between the consistency oracle and the *independence oracle* for hypergraphs. The independence oracle was recently used by Dell, Lapinskas and Meek [4, 3] in the problem of counting the hyperedges of a hypergraph  $H = (V, E)$ . Clearly, this problem can be cast in our setting by letting  $X = \{1, \dots, |V|\}$ ,  $Y = \{0, 1\}$ , and  $\mathcal{C} = \{\mathcal{K}_e : e \in E\}$ . An *independence oracle* for  $H$  takes in input a subset  $S \subseteq V$  and returns 1 if and only if there exists  $e \in E$  with  $e \subseteq S$ . In their work, the authors consider  $k$ -hypergraphs (ones where every edge has size  $k$ ), and they show that even in that case one may need  $|V|^{\Omega(k)}$  queries to approximate  $|E|$  [4]. In fact it is easy to see that an independence oracle is not sufficient to solve the problem at all if the hypergraph is arbitrary: for instance, an independence oracle will output 1 on every input as long as  $E$  contains all singletons, and therefore any two such hypergraphs are indistinguishable via independence oracles. Instead, a consistency oracle always allows one to compute  $\mathcal{C}$  using  $O(|\mathcal{C}|)$  queries, via a simple exhaustive search tree exploration. Thus in particular one gets a polynomial-time algorithm whenever  $|\mathcal{C}| = n^{O(1)}$ .

Second, we have obtained a construction that proves what follows. For every fixed  $k \geq 1$ , in order to be able to distinguish with non-vanishing probability between  $|\mathcal{C}| = n^k$  and  $|\mathcal{C}| = \Omega(n^{2k})$ , one needs to make a number of calls to the consistency oracle of order:

$$\left(\frac{n}{2k \log n}\right)^{k+1} \quad (1)$$

This construction will likely be the basis for future developments.

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## 4.2 Independent sets of fixed size

Ewan Davies (University of Colorado – Boulder, US), Sarah Cannon (Claremont McKenna College, US), Charlie Carlson (University of Colorado – Boulder, US), Sarah Miracle (University of St. Thomas – St. Paul, US), and Noela Müller (TU Eindhoven, NL)

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**Main reference** Ewan Davies, Will Perkins: “Approximately Counting Independent Sets of a Given Size in Bounded-Degree Graphs”, in Proc. of the 48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference), LIPIcs, Vol. 198, pp. 62:1–62:18, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021.

**URL** https://doi.org/10.4230/LIPIcs.ICALP.2021.62

Recently, a number of algorithmic methods for approximately counting the number of independent sets of a given size  $k = \lfloor \alpha n \rfloor$  in  $n$ -vertex bounded-degree graphs have been discovered. In [2] the sharp hardness threshold in the density  $\alpha$  was uncovered, and the algorithm in the tractable region of densities is based on rejection sampling from the hard-core model. Alternative approaches based local central limit theorems that yield faster algorithms were given in [3].

There is a natural Markov chain whose stationary distribution is uniform on independent sets of size  $k$ , namely the *down-up walk* that from a state  $I$  takes a uniform random element  $v \in I$  and moves to a uniform random independent set of size  $k$  containing  $I \setminus \{v\}$ . This was shown using path coupling [1] to mix rapidly at densities up to roughly  $\alpha \sim 1/(2d)$  in graphs of maximum degree  $d$ . The hardness threshold from [2] is slightly larger:  $\alpha_c \sim e/(1+e) \cdot 1/d$ .

This working group focused on the question of whether the down-up walk mixes rapidly up to the hardness threshold. Emerging techniques that seem pertinent include spectral independence and localization schemes, but there is a rather significant obstacle to apply these techniques associated with the global constraint of a fixed size  $k$ . We did not overcome

the main obstacle, and instead explored alternative approaches and familiarised ourselves with topics such as correlation decay, computation trees, and zeros of partition functions for the fixed-size model. One starting point is an idea due to Heng Guo that provides a computation tree for the ratios  $r_k(v) = \Pr_k(v \in I) / \Pr_k(v \notin I)$  where  $\Pr_k$  is over the uniform independent set of size  $k$ , but it remains unclear how to use this insight to study the mixing time of the down-up walk. We thank Guus Regts for an interesting discussion on techniques for finding zero-free regions.

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## 4.3 Fine grained complexity of counting independent sets in bounded degree graphs

Heng Guo (*University of Edinburgh, GB*)

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A key component of the approximate counting algorithm for independent sets in bounded degree graphs by Patel-Regts is a fixed parameter tractable algorithm for exactly counting them with the size being the parameter. The natural question is then if there is a corresponding lower bound. There are various hardness results when the degree bound goes to  $n$ , namely for general graphs. We have worked on reducing from those cases. However, there is a main difficulty, in that for general graphs, the size of the maximum independent set can be arbitrary, and yet for bounded degree graphs it is linear in the number of vertices. This makes such reductions difficult to construct.

#### 4.4 Approximating the number of proper colorings of a planar graph with a large number of colors

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The computational complexity of approximately counting the number of proper colorings with say a million colors of planar graphs is unclear. On the one hand as soon as we replace the number a million by any non-real number close arbitrarily to it and look at the evaluation of the chromatic polynomial at this point, this problem becomes #P-hard on planar graphs, as was recently proved in [1]. On the other hand a planar graph can be colored with 4 colors, which could possibly lead to the suspicion that for some large enough number of colors approximately counting the number of proper colorings should not be computational hard.

We have met in various compositions throughout the week and discussed the problem. Unfortunately we have not really made any progress. Some of the things we looked at include: the use of standard decomposition techniques for planar graphs and reductions to partition function of the ferromagnetic Potts model. The main conclusion from our initial discussions has been that this appears to be a difficult problem for which the current techniques are not powerful enough to make progress. Perhaps a useful question is to see if there exist sub exponential algorithms. For example Nederlof [2] designed an exact  $2^{O(\sqrt{k})}\text{poly}(n)$  algorithm for counting the number of independent sets of size  $k$  in a planar  $n$ -vertex graph.

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#### 4.5 Understanding the Homomorphism Basis

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The goal of this working group was to improve the understanding of the structure of the so-called homomorphism basis of counting problems: Well-known transformations dating back to early works of Lovász allow the expression of a wide range of counting problems (including problems arising in database theory and network sciences) as a finite linear combination of homomorphism counts. For example, it is known that for every graph  $H$  there exists

a finitely supported function  $a$  from graphs to rationals such that, for every graph  $G$  the following is true:

$$\#\text{Sub}(H \rightarrow G) = \sum_F a(F) \cdot \#\text{Hom}(F \rightarrow G), \quad (2)$$

where  $\#\text{Sub}(H \rightarrow G)$  denotes the number of subgraphs of  $G$  that are isomorphic to  $H$ , and  $\#\text{Hom}(F \rightarrow G)$  denotes the number of graph homomorphisms (edge-preserving mappings) from  $F$  to  $G$ . In other words, the problem of counting copies of  $H$  in a graph  $G$  can be cast as computing a finite linear combination of homomorphism counts.

In 2017, Curticapean, Dell and Marx [1] established a remarkable property, sometimes called “complexity monotonicity”, of linear combinations as in (2): They are *exactly* as hard to compute as their hardest term  $\#\text{Hom}(F \rightarrow G)$  with a non-zero coefficient ( $a(F) \neq 0$ ).<sup>1</sup> Since the complexity of computing individual homomorphism counts is reasonably well-understood due to a result of Dalmau and Jonsson [2], this discovery enabled a general strategy for studying the complexity of counting problems: Cast the problem as a linear combination of homomorphism counts and investigate which terms cancel out, i.e., for which graphs  $F$  the coefficient  $a(F)$  becomes 0. In recent years, this strategy has seen significant success in classifying counting problems arising e.g. in database theory [3, 4], subgraph and induced subgraph counting [1, 5, 6], modular counting [7, 8], fine-grained homomorphism counting [12], and pattern counting in degenerate graphs [9, 10, 11].

The purpose of this working group was to gather and unify the many different tools that have been established for analysing the homomorphism basis, to enhance the methods to tackle new problems, and to find common grounds for future collaborations.

## Outcomes

- We discovered a new strategy for analysing the homomorphism basis of induced subgraph counting problems that enabled us to understand the complexity of various instances of the induced subgraph counting problem that have previously been unclassified. We hope that this strategy will enable us in the future to completely resolve the complexity of the induced subgraph counting problem as studied in [5, 6].
- We discovered that the general framework extends to pattern counting problems in hypergraphs (which is not surprising). However, we also found that we need new tools to understand the coefficients in the homomorphism basis, since the established tools do not always generalise to hypergraphs (which is surprising).

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<sup>1</sup> This property is very special to homomorphism counts, as it does not hold for computing linear combinations of general counting problems: For example, computing the number of satisfying assignments plus the number of unsatisfying assignments of an  $n$ -variable CNF is always  $2^n$ , i.e., easy to compute, although computing the individual terms is #P-hard.

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