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Supplemental material: An invertible map between Bell non-local and contextuality scenarios

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I. THE MAP

To specify the map in fully generality it will be useful for us to be able to describe Bell and contextuality scenarios in which the different measurements have different numbers of outcomes. Thus, we redefine our tuples that denote the scenarios as follows.

We use a tuple $(\boldsymbol{A},\boldsymbol{B},X,Y)$ to denote a two-party Bell scenario in which Alice (Bob) has X (Y) inputs and given an input $x\in[X]$ $(y\in[Y])$ she (he) can obtain one of A_x (B_y) possible outcomes, where A_x (B_y) are the entries of the X-tuple $\boldsymbol{A}$ (Y-tuple $\boldsymbol{B}$). We use the notation $\|\boldsymbol{A}\|=\sum_x A_x$ and $\|\boldsymbol{B}\|=\sum_y B_y$.

Given a Bell scenario $(\boldsymbol{A},\boldsymbol{B},X,Y)$, a correlation is given by a vector $p\in\mathbb{R}^{\|\boldsymbol{A}\|\|\boldsymbol{B}\|}$, with entries $p(a,b|x,y)$. A correlation p is in the quantum set, $\mathcal{C}_{qs}$, if there exist separable Hilbert spaces $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{B}}$, positive-operator-valued measures $M^x=\{M^x_a\}_{a\in[A_x]}$ for all $x\in[X]$ on $\mathcal{H}_{\mathrm{A}}$ and $N^y=\{N^y_b\}_{b\in[B_y]}$ for all $y\in[Y]$ on $\mathcal{H}_{\mathrm{B}}$, and a density operator (positive semidefinite operator with unit trace) ρ on $\mathcal{H}_{\mathrm{A}}\otimes\mathcal{H}_{\mathrm{B}}$ such that

$$p(a,b|x,y)=\mathrm{Tr}\left(M^x_a\otimes N^y_b\rho\right). \tag{I.1}$$

A correlation p is in the no-signalling set if it satisfies the no-signalling constraints

$$\sum_b p(a,b|x,y)=\sum_b p(a,b|x,y') \quad \forall a,x,y,y' \tag{I.2}$$

$$\sum_a p(a,b|x,y)=\sum_a p(a,b|x',y) \quad \forall b,y,x,x'. \tag{I.3}$$

A correlation p is local if there exists a measurable space (Λ,Σ), a probability measure $\mu:\Sigma\to[0,1]$, and local probability distributions $l^{\mathrm{A}}(a|x,E)$ and $l^{\mathrm{B}}(b|y,E)$ satisfying $\sum_a l^{\mathrm{A}}(a|x,E)=\sum_b l^{\mathrm{B}}(b|y,E)=1$ for all $x\in[X]$, $y\in[Y]$ and non-empty $E\in\Sigma$, such that

$$p(a,b|x,y)=\int_\Lambda l^{\mathrm{A}}(a|x,\lambda)l^{\mathrm{B}}(b|y,\lambda)\mathrm{d}\mu(\lambda). \tag{I.4}$$

We identify a prepare-and-measure contextuality scenario with X preparations satisfying equivalences $\mathcal{OE}_P$ and Y measurements where measurement $y\in[Y]$ has B_y outcomes by the tuple $(X,Y,\boldsymbol{B},\mathcal{OE}_P)$, where $\boldsymbol{B}$ is a Y-tuple with yth element B_y.

A behaviour q is in the set of *contextual* behaviours if for every equivalence of the form

$$\sum_x \alpha_x P_x \simeq \sum_x \beta_x P_x, \tag{I.5}$$

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in $\mathcal{OE}_P$ the behaviour satisfies

$$\sum_x \alpha_x q(b|x,y) = \sum_x \beta_x q(b|x,y) \quad \forall b, y. \tag{I.6}$$

A behaviour q is in the quantum set, $\mathcal{Q}$, if there exists a separable Hilbert space $\mathcal{H}$, POVMs $\{N_b^y\}_{b\in[B_y]}$ for $y \in [Y]$ on $\mathcal{H}$ satisfying $\mathcal{OE}_M$ and density operators ρ_x on $\mathcal{H}$ satisfying $\sum_x \alpha_x \rho_x = \sum_x \beta_x \rho_x$ for every equivalence of the form in Eq. (I.5) in $\mathcal{OE}_P$ such that

$$q(b|x,y) = \mathrm{Tr}(N_b^y \rho_x)\,. \tag{I.7}$$

A behaviour q is in the non-contextual set if there exists a measurable space (Λ, Σ), probability measures $\mu_x : \Sigma \to [0,1]$ for all $x \in [X]$ satisfying $\sum_x \alpha_x \mu_x(E) = \sum_x \beta_x \mu_x(E)$ for every equivalence relation of the form (I.5) in $\mathcal{OE}_P$ and so-called *response functions* $\xi_y(b|\cdot)$ for all $b \in [B_y]$ and $y \in [Y]$ on Λ, and $\sum_b \xi(b|E) = 1$ for all $E \in \Sigma$, such that

$$q(b|x,y) = \int_\Lambda \xi_y(b|\lambda)\mathrm{d}\mu_x(\lambda)\,. \tag{I.8}$$

Now we can define the map Γ on the full set of non-signalling correlations. Notice that if there is an outcome $a|x$ that never occurs for Alice in the correlation p then $p_{\mathrm{A}}(a|x) = 0$ and the corresponding preparation $P_{a|x}$ of Bob's system does not appear in the preparation equivalences $\mathcal{NS}(p_{\mathrm{A}})$ (since it would have coefficient zero). Consequently, the choice of preparation $P_{a|x}$ would be unconstrained in both contextual and non-contextual theories. As a result the correlation p could be mapped to one of many possibilities. Since we wish to map each non-signalling correlation to a single behaviour in a contextual scenario, we will not include these unconstrained preparations in the contextuality scenario.

A correlation p in a Bell scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$ is mapped to a behaviour q in the contextuality scenario $(\|\mathcal{A}^p\|, Y, \boldsymbol{B}, \mathcal{NS}(p_{\mathrm{A}}))$, where $\|\mathcal{A}^p\| = \sum_{x\in[X]}|\mathcal{A}_x^p|$ and $\mathcal{A}_x^p = \{a : p_{\mathrm{A}}(a|x) > 0\}$ for all $x \in [X]$, and where the preparation equivalences are given by

$$\sum_{a\in[A_1]} p_{\mathrm{A}}(a|1)P_{a|1} \simeq \cdots \simeq \sum_{a\in[A_X]} p_{\mathrm{A}}(a|X)P_{a|X}, \tag{I.9}$$

which we encode as a vector $\mathcal{NS}(p_{\mathrm{A}})$ in the Cartesian product $\mathbb{R}^{A_1} \times \cdots \times \mathbb{R}^{A_X}$ where the x-th vector has a-th element $p_{\mathrm{A}}(a|x)$. If a preparation $P_{a|x}$ has coefficient zero we will say $P_{a|x}$ does not appear in $\mathcal{NS}(p_{\mathrm{A}})$.

The mapping is given by

$$\Gamma : [p, (\boldsymbol{A}, \boldsymbol{B}, X, Y)] \mapsto [q, (\|\mathcal{A}^p\|, Y, \boldsymbol{B}, \mathcal{NS}(p_{\mathrm{A}}))] \tag{I.10}$$

where

$$q(b|[a|x], y) = \frac{p(a,b|x,y)}{p_{\mathrm{A}}(a|x)} \tag{I.11}$$

for $a \in \mathcal{A}_x^p$, $b \in [B_y]$, $x \in [X]$ and $y \in [Y]$.

For the inverse mapping, let $A_x \in \mathbb{N}$ for all $x \in [X]$ and some $X \in \mathbb{N}$, then consider some $\hat{p}_{\mathrm{A}}(a|x) \geq 0$ for all $a \in [A_x]$ and $x \in [X]$ such that $\sum_{a\in A_x} \hat{p}_{\mathrm{A}}(a|x) = 1$ for all $x \in [X]$. Then, let $\mathcal{NS}(\hat{p}_{\mathrm{A}}) \in \mathbb{R}^{A_1}\times\cdots\times\mathbb{R}^{A_X}$ be such that the a-th element of the x-th vector is $\hat{p}_{\mathrm{A}}(a|x)$. We can now define a contextuality scenario $(Z, Y, \boldsymbol{B}, \mathcal{NS}(\hat{p}_{\mathrm{A}}))$ where the number of preparations, Z, is the number of non-zero elements in all the vectors of $\mathcal{NS}(\hat{p}_{\mathrm{A}})$. Here $\mathcal{NS}(\hat{p}_{\mathrm{A}})$ encodes preparation equivalences as described in Eq. (I.9). The inverse mapping Γ^{-1} takes a behaviour q in this contextuality scenario to a correlation p in the Bell scenario $(\boldsymbol{A} = (A_1, \ldots, A_X), \boldsymbol{B}, X, Y)$ and is given by

$$\Gamma^{-1} : [q, \big(Z, Y, \boldsymbol{B}, \mathcal{NS}(\hat{p}_{\mathrm{A}})\big)] \mapsto [p, (\boldsymbol{A}, \boldsymbol{B}, X, Y)] \tag{I.12}$$

where

$$p(a,b|x,y) = \hat{p}_{\mathrm{A}}(a|x)q(b|[a|x], y) \tag{I.13}$$

for $b \in [B_y]$, $x \in [X]$ and $y \in [Y]$. When the scenarios are already specified, we will refer to the q in Eq. (I.11) as $\Gamma(p)$ and similarly, the p in Eq. (I.13) as $\Gamma^{-1}(q)$.

If we are simply given a behaviour in a contextuality scenario of the right type[1] we have a choice of Bell scenario into which to map. This choice stems from being able to consider a correlation in a Bell scenario as a correlation from a larger scenario where some outcomes never occur. For example, consider a contextuality scenario with five preparations $Q_1, \dots, Q_5$ and preparation equivalences $\frac{1}{2}Q_1 + \frac{1}{2}Q_2 \simeq \frac{1}{4}Q_3 + \frac{3}{4}Q_4 \simeq Q_5$, and two measurements with two outcomes each. The simplest choice would be to map to a Bell scenario with three measurements for Alice, the first two having two outcomes each and the third having one (trivial) outcome. This choice corresponds to thinking of the preparations as $Q_1 = P_{1|1}$, $Q_2 = P_{2|1}$, $Q_3 = P_{1|2}$, $Q_4 = P_{2|2}$ and $Q_5 = P_{1|3}$ which results from choosing $\mathcal{NS}(p_A) = (1/2, 1/2) \times (1/4, 3/4) \times (1)$.

However, an equally valid choice is to take $\mathcal{NS}(p_A) = (0, 1/2, 0, 0, 1/2) \times (1/4, 3/4, 0) \times (0, 1)$ which means we would think of the preparations as $Q_1 = P_{2|1}$, $Q_2 = P_{5|1}$, $Q_3 = P_{1|2}$, $Q_4 = P_{2|2}$ and $Q_5 = P_{2|3}$. The resulting Bell scenario is $\big((5, 3, 2), (2, 2), 3, 2\big)$. In this case the behaviours from this contextuality scenario will map to the correlations in the Bell scenario with marginals given by the coefficients in the equivalences, e.g. $p_{\mathrm{A}}(1|2) = 1/4$, and in which all Alice's outcomes that do not have a corresponding preparation never occur, e.g. $p_{\mathrm{A}}(1|1) = 0$. It follows from the results of the present manuscript that if the image of a behaviour in the first Bell scenario is a local, quantum or non-signalling correlation, then the image in the second Bell scenario will also be local, quantum or non-signalling, respectively.

II. QUANTUM CASE: CONTEXTUALITY TO BELL

In this section we will show that Γ^{-1} always takes a quantum behaviour q in a scenario $(Z, Y, \boldsymbol{B}, \mathcal{NS}(\hat{p}_{\mathrm{A}}))$ to a quantum correlation p. In the case of the simplest Bell scenario some similar arguments are described in Refs. [28] and [29]. Consider a contextuality scenario $(Z, Y, \boldsymbol{B}, \mathcal{NS}(\hat{p}_{\mathrm{A}}))$ as described in the previous section, where $\hat{p}_{\mathrm{A}}(a|x) \geq 0$ for $a \in [A_x]$ and $x \in [X]$ and $\mathcal{NS}(\hat{p}_{\mathrm{A}}) \in \mathbb{R}^{A_1} \times \cdots \times \mathbb{R}^{A_X}$ has a-th element in the x-th vector $\hat{p}_{\mathrm{A}}(a|x)$ and where Z is the number of strictly positive elements of the vectors in $\mathcal{NS}(\hat{p}_{\mathrm{A}})$. We will also denote by $\hat{\mathcal{A}}_x$ the subset of $[A_x]$ such that $\hat{p}_{\mathrm{A}}(a|x) > 0$. Let q be a quantum behaviour in this scenario with a realisation, that is

$$q(b|[a|x], y) = \mathrm{Tr}(\rho_{a|x} N_b^y)\,, \tag{II.1}$$

for some density operators $\rho_{a|x}$ for all $a \in \hat{\mathcal{A}}_x$ and $x \in [X]$, and some POVMs $\{N_b^y\}_b$ for all $b \in [B_y]$ and $y \in [Y]$ on a Hilbert space $\mathcal{H}$, where $\sum_{a \in \hat{\mathcal{A}}_x} \hat{p}_{\mathrm{A}}(a|x)\rho_{a|x} = \rho_{\mathrm{B}}$ for all $x \in [X]$ and some density operator ρ_{B}.

We will show that the correlation

$$p(a, b|x, y) = \hat{p}_{\mathrm{A}}(a|x) q(b|[a|x], y) \tag{II.2}$$

for $a \in [A_x]$, $b \in [B_y]$, $x \in [X]$ and $y \in [Y]$ in the Bell scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$ has a quantum realisation.

Now, we want to find POVMs $M^x = \{M_a^x\}_{a \in [A_x]}$ on a Hilbert space $\mathcal{H}_{\mathrm{A}}$ and a density operator ρ on $\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$ such that if Alice measures the POVM M^x on system A then with probability $\hat{p}_{\mathrm{A}}(a|x)$ (from our operational equivalences $\mathcal{NS}(\hat{p}_{\mathrm{A}})$) she sees outcome a and Bob is left with the state $\rho_{a|x}$ (from our quantum realisation in the contextuality scenario) for $a \in \hat{\mathcal{A}}_x$ and outcome M_a^x never occurs for $a \in [A_x] \backslash \hat{\mathcal{A}}_x$. In other words, we are looking for a way for Alice to steer Bob's system into the assemblage given by the states $\rho_{a|x}$.

Mathematically, we want

$$\mathrm{Tr}_{\mathrm{A}}\left(M_a^x \otimes \mathbb{I}\rho\right) = \begin{cases} \hat{p}_{\mathrm{A}}(a|x)\rho_{a|x} & \text{if } a \in \hat{\mathcal{A}}_x \\ 0 & \text{otherwise}\,, \end{cases} \tag{II.3}$$

since then, if Alice measures M^x on system A and Bob measures N^y (also from the quantum realisation in the contextuality scenario) on system B when the system AB is in state ρ we find

$$p(a, b|x, y) = \mathrm{Tr}(M_a^x \otimes N_b^y \rho) = \begin{cases} \hat{p}_{\mathrm{A}}(a|x)\,\mathrm{Tr}(N_b^y \rho_{a|x}) = \hat{p}_{\mathrm{A}}(a|x) q(b|[a|x], y) & \text{if } a \in \hat{\mathcal{A}}_x \\ 0 = \hat{p}_{\mathrm{A}}(a|x) q(b|[a|x], y) & \text{otherwise}\,, \end{cases} \tag{II.4}$$

[1] A scenario with no measurement equivalences and preparation equivalences given by multiple decompositions of one hypothetical preparation in which each preparation only appears once.

and we have a quantum realisation for our Bell correlation. Our construction of M^x and ρ is based on the Schrödinger–HJW theorem [16], in particular, on the infinite dimensional argument given by Navascués et al. [22, Lemma 4].

Since ρ_{B} is a density operator, it has a spectral decomposition given by a countable sum

$$\rho_{\mathrm{B}} = \sum_n \lambda_n |n\rangle\langle n|, \tag{II.5}$$

where $\{|n\rangle : n \in \mathbb{N}\}$ is a set of orthonormal vectors of $\mathcal{H}$, and the positive eigenvalues λ_n of ρ_{B} satisfy $\sum_n \lambda_n = 1$. Note that we have excluded any zero eigenvalues from this decomposition. Let $\mathcal{H}_{\mathrm{B}}$ be the support of ρ_{B} and $\Pi_{\mathrm{B}} = \sum_n |n\rangle\langle n|$ be the projection onto this closed subspace of $\mathcal{H}$. Thus, we have that $\{|n\rangle : n \in \mathbb{N}\}$ forms an orthonormal basis for $\mathcal{H}_{\mathrm{B}}$ and $\Pi_{\mathrm{B}}\rho_{\mathrm{B}}\Pi_{\mathrm{B}} = \rho_{\mathrm{B}}$. Furthermore, we have $\Pi_{\mathrm{B}}\rho_{a|x}\Pi_{\mathrm{B}} = \rho_{a|x}$ for all $a \in \hat{\mathcal{A}}_x$ and $x \in [X]$ since we have $1 = \mathrm{Tr}(\Pi_{\mathrm{B}}\rho_{\mathrm{B}}\Pi_{\mathrm{B}}) = \sum_{a' \in \hat{\mathcal{A}}_x} \hat{p}_{\mathrm{A}}(a'|x)\,\mathrm{Tr}\left(\Pi_{\mathrm{B}}\rho_{a'|x}\Pi_{\mathrm{B}}\right)$ which implies $\mathrm{Tr}(\Pi_{\mathrm{B}}\rho_{a|x}\Pi_{\mathrm{B}}) = 1$ and therefore $\Pi_{\mathrm{B}}\rho_{a|x}\Pi_{\mathrm{B}} = \rho_{a|x}$ for all $a \in \hat{\mathcal{A}}_x$.

First, we define the state for the realisation of the quantum behaviour p. Let $\mathcal{H}_{\mathrm{A}} = \mathcal{H}_{\mathrm{B}}$, then the state is given by $|\Psi\rangle = \sum_n \sqrt{\lambda_n}\,|n\rangle|n\rangle \in \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$. This series converges since $\sum_n \left\|\sqrt{\lambda_n}\,|n\rangle|n\rangle\right\|^2 = \sum_n \lambda_n = 1$.

Second, we define the effects M_a^x for $a \in \hat{\mathcal{A}}_x$ of the POVM in the quantum realisation of $p(a,b|x,y)$. To do so, we define the operators

$$M_{a,K}^x = \left[\sum_{m=1}^K \frac{1}{\sqrt{\lambda_m}} |m\rangle\langle m|\right] \hat{p}_{\mathrm{A}}(a|x)\rho_{a|x}^T \left[\sum_{n=1}^K \frac{1}{\sqrt{\lambda_n}} |n\rangle\langle n|\right] \tag{II.6}$$

on $\mathcal{H}_{\mathrm{A}}$, where the transpose is taken in the eigenbasis of ρ_{B}, and $K \in \mathbb{N}$. From now on, we assume that $\mathcal{H}_{\mathrm{A}}$ is infinite dimensional, since the finite-dimensional case is simpler. We will show that the operator M_a^x on $\mathcal{H}_{\mathrm{A}}$ given by $M_a^x|\psi\rangle = \lim_{K\to\infty} M_{a,K}^x|\psi\rangle$ is a bounded linear operator on $\mathcal{H}_{\mathrm{A}}$ for all $a \in \hat{\mathcal{A}}_x$ and $x \in [X]$, and these operators will form the effects of the POVMs in the quantum realisation.

We have $M_{a,K}^x \geq 0$ for all $K \in \mathbb{N}$ since $M_{a,K}^x = S^\dagger \hat{p}_{\mathrm{A}}(a|x)\rho_{a|x}^T S$, where $S = \sum_{m=1}^K \frac{1}{\sqrt{\lambda_m}}|m\rangle\langle m|$, and $\hat{p}_{\mathrm{A}}(a|x)\rho_{a|x}^T$ is positive semidefinite (since transposition is a positive map). Further, we have that $M_{a,K}^x \leq \mathbb{I}$, since

$$\begin{aligned}
\sum_a M_{a,K}^x &= \left[\sum_{m=1}^K \frac{1}{\sqrt{\lambda_m}} |m\rangle\langle m|\right] \rho_{\mathrm{B}}^T \left[\sum_{n=1}^K \frac{1}{\sqrt{\lambda_n}} |n\rangle\langle n|\right] \\
&= \left[\sum_{m=1}^K \frac{1}{\sqrt{\lambda_m}} |m\rangle\langle m|\right] \sum_{j=1}^\infty \lambda_j |j\rangle\langle j| \left[\sum_{n=1}^K \frac{1}{\sqrt{\lambda_n}} |n\rangle\langle n|\right] \\
&= \sum_{m=1}^K |m\rangle\langle m| \leq \mathbb{I}.
\end{aligned} \tag{II.7}$$

Thus, $M_{a,K}^x$ is a positive semidefinite bounded linear operator on $\mathcal{H}_{\mathrm{A}}$ for all $K \in \mathbb{N}$, $a \in \hat{\mathcal{A}}_x$, and $x \in [X]$.

Consider the vector subspace $\mathcal{F}$ of $\mathcal{H}_{\mathrm{A}}$ consisting of finite linear combinations of the eigenvectors of ρ_{B}, i.e. vectors of the form $\sum_{n=1}^L c_n|n\rangle$, for some $L \in \mathbb{N}$. The limit $\lim_{K\to\infty} M_{a,K}^x|\psi\rangle$ converges in $\mathcal{H}_{\mathrm{A}}$ for every element $|\psi\rangle$ of the subspace $\mathcal{F}$, since there exists some $K_\psi \in \mathbb{N}$ such that $\lim_{K\to\infty} M_{a,K}^x|\psi\rangle = M_{a,K_\psi}^x|\psi\rangle = M_a^x|\psi\rangle$. Thus, we find that on $\mathcal{F}$, $\|M_a^x|\psi\rangle\| = \left\|M_{a,K_\psi}^x|\psi\rangle\right\| \leq \||\psi\rangle\|$ and, hence, M_a^x is a bounded linear operator on $\mathcal{F}$. Since $\mathcal{F}$ is a dense subspace of $\mathcal{H}_{\mathrm{A}}$, we have that M_a^x has a unique linear extension to $\mathcal{H}_{\mathrm{A}}$. This extension is defined as follows: given a vector $|\psi\rangle \in \mathcal{H}_{\mathrm{A}}$ let $(|\psi_j\rangle)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ such that $(|\psi_j\rangle)_j \to |\psi\rangle$ as $j \to \infty$. Then we define $M_a^x|\psi\rangle = \lim_{j\to\infty} M_a^x|\psi_j\rangle$. This limit exists since the sequence $(M_a^x|\psi_j\rangle)_j$ is Cauchy which can be seen by the following argument. The sequence $(|\psi_j\rangle)$ is Cauchy, therefore for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\||\psi_m\rangle - |\psi_n\rangle\| < \epsilon$ for all $m, n > N$. Thus, we have that $\|M_a^x|\psi_m\rangle - M_a^x|\psi_n\rangle\| \leq \||\psi_m\rangle - |\psi_n\rangle\| < \epsilon$ since $|\psi_m\rangle - |\psi_n\rangle \in \mathcal{F}$.

Finally, we set $M_a^x = 0$ for the remaining values of a, i.e. for $a \in [A_x]\backslash\hat{\mathcal{A}}_x$ and verify that Eq. (II.3) holds, that is, we have a quantum realisation of our correlation $p(a,b|x,y)$, given by the POVMs $M^x = \{M_a^x\}$ on $\mathcal{H}_{\mathrm{A}}$ for Alice and $N^y = \{N_b^y\}$ on $\mathcal{H}_{\mathrm{B}}$ for Bob and the quantum state $|\Psi\rangle = \sum_n \sqrt{\lambda_n}\,|n\rangle|n\rangle \in$

$\mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{B}$. Clearly, for $a \in [\mathcal{A}_x] \backslash \hat{\mathcal{A}}_x$ we have $\mathrm{Tr}_\mathrm{A}(M_a^x \otimes \mathbb{I} \left|\Psi\right\rangle \left\langle\Psi\right|) = 0$. Then, for $a \in \hat{\mathcal{A}}_x$ we have

$$\begin{aligned}
\mathrm{Tr}_\mathrm{A}(M_a^x \otimes \mathbb{I} \left|\Psi\right\rangle \left\langle\Psi\right|) &= \sum_{j,m,n} \sqrt{\lambda_m \lambda_j} \left\langle j\right| \otimes \mathbb{I} \left(M_a^x \otimes \mathbb{I} \left|m\right\rangle \left\langle n\right| \otimes \left|m\right\rangle \left\langle n\right|\right) \left|j\right\rangle \otimes \mathbb{I} \\
&= \sum_{j,m} \sqrt{\lambda_m \lambda_j} \left\langle j\right| \lim_{K\to\infty} M_{a,K}^x \left|m\right\rangle \left|m\right\rangle \left\langle j\right| \\
&= \sum_{j,m} \sqrt{\lambda_m \lambda_j} \sum_{k=1}^{\infty} \frac{p_\mathrm{A}(a|x)}{\sqrt{\lambda_m \lambda_k}} \left\langle j|k\right\rangle \left\langle n\right| \rho_{a|x}^T \left|m\right\rangle \left|m\right\rangle \left\langle j\right| \\
&= \sum_{j,m} \hat{p}_\mathrm{A}(a|x) \left\langle m\right| \rho_{a|x} \left|j\right\rangle \left|m\right\rangle \left\langle j\right| = \hat{p}_\mathrm{A}(a|x) \rho_{a|x} \,.
\end{aligned} \tag{II.8}$$

III. QUANTUM CASE: BELL TO CONTEXTUALITY

Conversely, consider a quantum correlation, p, from a bipartite Bell scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$ given by

$$p(a,b|x,y) = \mathrm{Tr}(M_a^x \otimes N_b^y \rho) \,, \tag{III.1}$$

for some POVMs $M^x = \{M_a^x\}$ on a Hilbert space $\mathcal{H}_\mathrm{A}$ and $N^y = \{N_b^y\}$ on a Hilbert space $\mathcal{H}_\mathrm{B}$ and a density operator ρ on $\mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{B}$. Denote Alice's marginal probabilities by $p_\mathrm{A}(a|x) = \mathrm{Tr}(M_a^x \otimes \mathbb{I}\rho)$ and Bob's reduced states after outcome a of measurement x of Alice with $p_\mathrm{A}(a|x) \neq 0$ by

$$\rho_{a|x} = \frac{\mathrm{Tr}_\mathrm{A}(M_a^x \otimes \mathbb{I}\rho)}{p_\mathrm{A}(a|x)} \,. \tag{III.2}$$

Recalling that we denote the subset of $[A_x]$ such that $p_\mathrm{A}(a|x) > 0$ by $\mathcal{A}_x^p$, we have that $\sum_{a\in\mathcal{A}_x^p} p_\mathrm{A}(a|x)\rho_{a|x} = \mathrm{Tr}_\mathrm{A}(\rho)$ for all $x \in [X]$, therefore the density operators $\rho_{a|x}$ satisfy the equivalences $\mathcal{NS}(p_\mathrm{A})$ given in Eq. (I.9). Taking $\rho_{a|x}$ as preparation $P_{a|x}$ and N^y as the y-th measurement in the contextuality scenario $(\|\mathcal{A}^p\|, Y, \boldsymbol{B}, \mathcal{NS}(p_\mathrm{A}))$ results in the behaviour

$$q(b|[a|x], y) = \mathrm{Tr}(N_b^y \rho_{a|x}) = \frac{\mathrm{Tr}(M_a^x \otimes N_b^y \rho)}{p_\mathrm{A}(a|x)} = \frac{p(a,b|x,y)}{p_\mathrm{A}(a|x)} \,. \tag{III.3}$$

Thus, q is a quantum behaviour in the contextuality scenario $(\|\mathcal{A}^p\|, Y, \boldsymbol{B}, \mathcal{NS}(p_\mathrm{A}))$.

IV. LOCAL AND NON-CONTEXTUAL CASE

Let $p(a,b|x,y)$ be a local correlation in a Bell scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$. Then, there exists a measurable space (Λ, Σ), a probability measure $\mu : \Sigma \to [0,1]$ and local probability distributions $l^\mathrm{A}(a|x,E)$ and $l^\mathrm{B}(b|y,E)$ satisfying $\sum_a l^\mathrm{A}(a|x,E) = \sum_b l^\mathrm{B}(b|y,E) = 1$ for all $x \in [X]$, $y \in [Y]$ and non-empty $E \in \Sigma$ such that

$$p(a,b|x,y) = \int_\Lambda l^\mathrm{A}(a|x,\lambda) l^\mathrm{B}(b|y,\lambda) \mathrm{d}\mu(\lambda). \tag{IV.1}$$

We now construct a non-contextual ontological model that yields the behaviour

$$q(b|[a|x], y) = \frac{p(a,b|x,y)}{p_\mathrm{A}(a|x)} \tag{IV.2}$$

in the contextuality scenario $(\|\mathcal{A}^p\|, Y, \boldsymbol{B}, \mathcal{NS}(p_\mathrm{A}))$. Note that the marginals of Alice, $p_\mathrm{A}(a|x) = \sum_b p(a,b|x,y)$, can now be expressed as $p_\mathrm{A}(a|x) = \int_\Lambda l^\mathrm{A}(a|x,\lambda)\mathrm{d}\mu(\lambda)$.

We select the ontic state space (Λ, Σ) and each preparation $P_{a|x}$ for $a \in \mathcal{A}_x^p$ and $x \in [X]$ is given by the measure

$$\mu_{a|x}(E) = \frac{l^\mathrm{A}(a|x,E)\mu(E)}{p_\mathrm{A}(a|x)} \tag{IV.3}$$

on Λ. These are indeed probability measures on Λ, since

$$\int_\Lambda \mathrm{d}\mu_{a|x}(\lambda) = \frac{1}{p_\mathrm{A}(a|x)} \int_\Lambda l^\mathrm{A}(a|x,\lambda)\mathrm{d}\mu(\lambda) = 1\,. \tag{IV.4}$$

Furthermore, the measures $\mu_{a|x}$ satisfy the operational equivalences $\mathcal{NS}(p_\mathrm{A})$, since

$$\sum_a p_\mathrm{A}(a|x)\mu_{a|x}(E) = \mu(E), \tag{IV.5}$$

for all $E \in \Sigma$ and $x \in [X]$.

The response function for each measurement M^y for $y \in [Y]$ is given by $\xi_y(b|E) = l^\mathrm{B}(b|y,E)$. Now, for all $a \in \mathcal{A}^p_x$ and $x \in [X]$ we find that

$$q(b|[a|x],y) = \int_\Lambda \xi_y(b|\lambda)\mathrm{d}\mu_{a|x}(\lambda) = \int_\Lambda l^\mathrm{B}(b|y,\lambda)\frac{l^\mathrm{A}(a|x,\lambda)}{p_\mathrm{A}(a|x)}\mathrm{d}\mu(\lambda) = \frac{p(a,b|x,y)}{p_\mathrm{A}(a|x)}\,. \tag{IV.6}$$

Conversely, consider a non-contextual behaviour $q(b|[a|x],y)$ in a contextuality scenario $(Z,Y,\boldsymbol{B},\mathcal{NS}(\hat{p}_\mathrm{A}))$ where $\hat{p}_\mathrm{A}(a|x) \geq 0$ for $a \in [A_x]$ and $x \in [X]$ and $\mathcal{NS}(\hat{p}_\mathrm{A}) \in \mathbb{R}^{A_1} \times \cdots \times \mathbb{R}^{A_X}$ has a-th element in the x-th vector $\hat{p}_\mathrm{A}(a|x)$ and where Z is the number of strictly positive elements of the vectors in $\mathcal{NS}(\hat{p}_\mathrm{A})$. We will also denote by $\hat{\mathcal{A}}_x$ the subset of $[A_x]$ such that $\hat{p}_\mathrm{A}(a|x) > 0$.

Then, there exists a measurable space (Λ,Σ), probability measures $\mu_{a|x}$ for all $a \in \hat{\mathcal{A}}_x$ and $x \in [X]$ and $\xi_y(b|\cdot)$ for all $b \in [B_y]$ and $y \in [Y]$ on Λ satisfying $\sum_a \hat{p}_\mathrm{A}(a|x)\mu_{a|x}(E) = \mu(E)$ for all $x \in [X]$ and $\sum_b \xi(b|E) = 1$ for all $E \in \Sigma$ such that

$$\int_\Lambda \xi_y(b|\lambda)\mathrm{d}\mu_{a|x}(\lambda) = q(b|[a|x],y)\,. \tag{IV.7}$$

Note that we can assume that $\mu(E) > 0$ for all non-empty $E \in \Sigma$.

We will construct a local hidden variable model for the correlation

$$p(a,b|x,y) = \begin{cases} \hat{p}_\mathrm{A}(a|x)q(b|[a|x],y) & \text{if } a \in \hat{\mathcal{A}}_x \\ 0 & \text{otherwise}\,, \end{cases} \tag{IV.8}$$

for $b \in [B_y]$, $x \in [X]$ and $y \in [Y]$. Let

$$l^\mathrm{A}(a|x,E) = \begin{cases} \frac{\hat{p}_\mathrm{A}(a|x)\mu_{a|x}(E)}{\mu(E)} & \text{if } a \in \hat{\mathcal{A}}_x \text{ and } E \text{ non-empty} \\ 0 & \text{otherwise,} \end{cases} \tag{IV.9}$$

and $l^\mathrm{B}(b|y,E) = \xi_y(b|E)$. Then, for $a \in \hat{\mathcal{A}}_x$, we find

$$\begin{aligned} p(a,b|x,y) &= \int_\Lambda l^\mathrm{A}(a|x,\lambda)l^\mathrm{B}(b|y,\lambda)\mathrm{d}\mu(\lambda) \\ &= \int_\Lambda \hat{p}_\mathrm{A}(a|x)\xi_y(b|\lambda)\mathrm{d}\mu_{a|x}(\lambda) = \hat{p}_\mathrm{A}(a|x)q(b|[a|x],y)\,, \end{aligned} \tag{IV.10}$$

and for $a \in [A_x]\backslash\hat{\mathcal{A}}_x$, we have $p(a,b|x,y) = 0$.

V. NON-SIGNALLING AND CONTEXTUAL CASE

Given a non-signalling correlation p in a Bell scenario $(\boldsymbol{A},\boldsymbol{B},X,Y)$ we find that q defined by Eq. (I.11) is in the set of contextual behaviours in the contextuality scenario $(\|\mathcal{A}^p\|,Y,\boldsymbol{B},\mathcal{NS}(p_\mathrm{A}))$ since

$$\sum_{a\in\mathcal{A}^p_x} p_\mathrm{A}(a|x)q(b|[a|x],y) = \sum_{a\in\mathcal{A}^p_x} p(a,b|x,y) = \sum_{a\in[A_x]} p(a,b|x,y) = p_\mathrm{B}(b|y)\,, \tag{V.1}$$

for all $b \in [B_y]$, $x \in [X]$ and $y \in [Y]$.

Conversely, given a behaviour in the contextual set of a scenario $(Z, Y, \boldsymbol{B}, \mathcal{NS}(\hat{p}_{\text{A}}))$ we find that the correlation p in Eq. (I.13) is non-signalling in the Bell scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$ since

$$\sum_{a\in[A_x]} p(a,b|x,y) = \sum_{a\in\hat{\mathcal{A}}_x} q(b|[a|x],y)\hat{p}_{\text{A}}(a|x) = Q(b|y)\,, \tag{V.2}$$

for some $Q(b|y) \in \mathbb{R}$ (recalling that $\hat{\mathcal{A}}_x$ denotes the subset of $[A_x]$ such that $\hat{p}_{\text{A}}(a|x) > 0$), and

$$\sum_{b\in[B_y]} p(a,b|x,y) = \sum_{b\in[B_y]} q(b|[a|x],y)\hat{p}_{\text{A}}(a|x) = \hat{p}_{\text{A}}(a|x)\,, \tag{V.3}$$

for all for all $b \in [B_y]$ and $y \in [Y]$.

VI. LIMITATIONS OF THE MAP

Preparation equivalences in the form of Eq. (14) of the main text generally may involve a single preparation appearing in multiple mixtures, for example, see how preparation P_1 appears in all three mixtures in Eq. (VI.1) below. Such equivalences do not arise from the no-signalling constraint in a remote-preparation scenario. In this section we will demonstrate with an explicit example how treating the multiple instances of a single preparation as different preparations in order to apply our map can result in local correlations being mapped to contextual behaviours.

Preparation equivalences do not have one unique expression. For example, the equivalence

$$\frac{1}{2}(P_1 + P_2) \simeq \frac{1}{3}(P_1 + P_3 + P_4)\,, \tag{VI.1}$$

can be expressed as

$$\frac{1}{4}P_1 + \frac{3}{4}P_2 \simeq \frac{1}{2}(P_3 + P_4)\,. \tag{VI.2}$$

Relabelling these preparations can now yield an equivalence in the form in Eq. (I.9), since each preparation only appears once. In fact, any single equivalence $\sum_{a=1}^{Z} p_{a,1}P_a \simeq \sum_{a=1}^{Z} p_{a,2}P_a$ can be expressed such that each preparation only features once, like in the case of Eqs. (VI.1) and (VI.2). This rearrangement is achieved by subtracting $p_{a,x}P_a$ from both sides and renormalising for all a, where $x \in \{1,2\}$ is such that $p_{a,x} = \min\{p_{a,1}, p_{a,2}\}$.

Performing this procedure for each of the individual equivalences in Eq. (14) of the main text will in general lead to equivalences given by decompositions of multiple different hypothetical preparations. For example,

$$\mathcal{OH} = \left\{\frac{1}{2}(P_1 + P_2) \simeq \frac{1}{3}(P_1 + P_3 + P_4) \simeq \frac{1}{5}P_1 + \frac{2}{5}(P_3 + P_5)\right\} \tag{VI.3}$$

becomes

$$\begin{aligned} &\frac{1}{4}P_1 + \frac{3}{4}P_2 \simeq \frac{1}{2}(P_3 + P_4) \text{ and} \\ &\frac{3}{8}P_1 + \frac{5}{8}P_2 \simeq \frac{1}{2}(P_3 + P_5)\,. \end{aligned} \tag{VI.4}$$

Consider then the contextuality scenario $H = (5, 2, 2, \mathcal{OH})$, where the preparation equivalences $\mathcal{OH}$ are given in Eq. (VI.3). By considering multiple instances of a repeated preparation, e.g. P_3, as distinct preparations, we can embed the behaviours from the scenario H in a scenario H' which can be mapped to a Bell scenario. To do so, we first subtract $\frac{1}{5}P_1$ from each hypothetical preparation and renormalise to arrive at the relations

$$\frac{3}{8}P_1 + \frac{5}{8}P_2 \simeq \frac{1}{6}P_1 + \frac{5}{12}(P_3 + P_4) \simeq \frac{1}{2}(P_3 + P_5) \tag{VI.5}$$

Next, we can treat the two instances of P_1 and the two instances of P_3 as different preparations (re-interpreting the second instance of P_1 as P_6 and the second instance of P_3 as P_7) to embed the behaviours into $H' = (7, 2, 2, \mathcal{OH}')$ where

$$\mathcal{OH}' = \left\{ \frac{3}{8}P_1 + \frac{5}{8}P_2 \simeq \frac{1}{6}P_6 + \frac{5}{12}(P_3 + P_4) \simeq \frac{1}{2}(P_7 + P_5) \right\}. \tag{VI.6}$$

Under relabelling, these preparation equivalences are of the form $\mathcal{NS}(p_{\mathrm{A}})$. Explicitly, we can map a behaviour q in the scenario H to a behaviour q' in the scenario H' by setting $q'(b|x, y) = q(b|x, y)$ for $1 \leq x \leq 5$, and $q'(b|6, y) = q(b|1, y)$ and $q'(b|7, y) = q(b|3, y)$. However, although under this embedding non-contextual and quantum behaviours remain non-contexual and quantum, respectively, it is possible for a contextual behaviour to become non-contextual due to the relaxation of preparation equivalences.

Indeed, we now give an explicit example of a contextual behaviour in H that becomes a non-contextual behaviour in H' using the above notation and mapping. Using the procedure in Ref. [Section IV,10] and the vertex enumeration software package `lrs` [23], we found all the facet inequalities defining the non-contextual polytope in H. The polytope has 60 facets, one of which is given by the inequality

$$-q(1|1, 2) - 3q(1|2, 1) + 2q(1|3, 1) + 2q(1|3, 2) + 2 \geq 0. \tag{VI.7}$$

An explicit contextual behaviour violating this inequality is given by

$$\begin{aligned} q_c &= \big(q_c(1|1,1), q_c(1|1,2), q_c(1|2,1), q_c(1|,2,2), \ldots, q_c(1|5,2)\big) \\ &= \left(\frac{19}{200}, \frac{1}{2}, \frac{127}{200}, \frac{1}{2}, \frac{19}{200}, \frac{19}{200}, \frac{181}{200}, \frac{181}{200}, \frac{77}{100}, \frac{181}{200}, \frac{19}{200}, \frac{1}{2}, \frac{19}{200}, \frac{19}{200}\right), \end{aligned} \tag{VI.8}$$

where $q_c(2|x, y) = 1 - q_c(1|x, y)$. In particular, this behaviour violates Eq. (VI.7) by $-\frac{1}{40}$, and thus it is contextual.

Let us now map q_c above to a behaviour q'_c in the scenario H', that is, we have

$$q'_c = \left(\frac{19}{200}, \frac{1}{2}, \frac{127}{200}, \frac{1}{2}, \frac{19}{200}, \frac{19}{200}, \frac{181}{200}, \frac{181}{200}, \frac{77}{100}, \frac{181}{200}, \frac{19}{200}, \frac{1}{2}, \frac{19}{200}, \frac{19}{200}, \frac{19}{200}, \frac{1}{2}, \frac{19}{200}, \frac{19}{200}\right). \tag{VI.9}$$

This behaviour is non-contextual, which can be shown by constructing an explicit non-contextual model, that is, a measurable space (Λ, Σ), probability measures $\mu_x : \Sigma \to [0, 1]$ and $\xi_y(b|.)$ response functions such that

$$q'_c(b|x, y) = \int_\Lambda \xi_y(b|\lambda) \mathrm{d}\mu_x(\lambda) \quad \forall b, x, y. \tag{VI.10}$$

Let us take the discrete measurable space $\Lambda = \{1, 2, 3, 4\}$ with the usual σ-algebra Σ of the power sets. We define the response functions

$$\xi_1(1|1) = \xi_1(1|3) = 0, \quad \xi_1(1|2) = \xi_1(1|4) = 1 \tag{VI.11}$$

$$\xi_2(1|1) = \xi_2(1|2) = 0, \quad \xi_2(1|3) = \xi_2(1|4) = 1 \tag{VI.12}$$

with $\xi_y(2|\lambda) = 1 - \xi_y(1|\lambda)$ for all y and λ. Furthermore, we define the probability measures via their values $\mu_x(\{\lambda\})$ given by

$$\mu = \begin{pmatrix} \frac{1}{2} & \frac{73}{200} & \frac{1671}{2000} & 0 & 0 & \frac{81}{200} & \frac{133}{160} \\ \frac{81}{200} & 0 & \frac{139}{2000} & \frac{19}{200} & \frac{23}{100} & \frac{1}{2} & \frac{59}{800} \\ 0 & \frac{27}{200} & \frac{139}{2000} & \frac{19}{200} & \frac{19}{200} & \frac{19}{200} & \frac{59}{800} \\ \frac{19}{200} & \frac{1}{2} & \frac{51}{2000} & \frac{81}{100} & \frac{27}{40} & 0 & \frac{17}{800} \end{pmatrix}, \tag{VI.13}$$

where the rows are indexed by λ and the columns are indexed by x, i.e. $\mu_{\lambda,x} = \mu_x(\{\lambda\})$. It is a straightforward computation to verify Eq. (VI.10) with these choices. Thus, the contextual behaviour q_c in the scenario H is mapped to a non-contextual behaviour q'_c in the scenario H'.

VII. PROOF OF COROLLARY 1

Suppose there exists an algorithm to decide whether any behaviour belongs to the quantum set in any given contextuality scenario. Then, given any correlation p in a Bell scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$ one can decide whether q given by Eq. (I.11) belongs to the quantum set in the contextuality scenario $(\|\mathcal{A}^p\|, Y, \boldsymbol{B}, \mathcal{NS}(p_{\mathrm{A}}))$. Since $q \in \mathcal{Q}$ if and only if $p \in \mathcal{C}_{qs}$, one could therefore decide the membership problem for the set of quantum spatial correlations, however this problem is known to be undecidable [14, 24].

VIII. PROOF OF COROLLARY 2

If any behaviour in $\mathcal{Q}$ in any contextuality scenario could be realised with finite dimensional quantum systems, then the construction in Sec. II would give a finite dimensional quantum realisation of any correlation in a Bell scenario, which is known not to exist [25].

IX. PROOF OF COROLLARY 3

For the proofs of Corollaries 3 and 4, we will find it useful to remove probability zero or one outcomes of Alice from a correlation. To do so we will map a given correlation in a Bell scenario to one in a scenario with fewer inputs and/or outputs in which Alice's marginal probabilities are strictly between zero and one. We now describe this map and show how it preserves the closure $\mathcal{C}_{qa}$ of the set of quantum spatial correlations $\mathcal{C}_{qs}$.

Let $\hat{p} \in \mathcal{C}_{qa}$ be a correlation in a Bell scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$ such that $\hat{p}_{\mathrm{A}}(a|x) = 0$ for all $a > a'_x$ for each $x \in [X]$ (note that if there are some zeroes in Alice's marginals, we can always relabel Alice's outcomes such that these zeroes appear at $a > a'_x$, since relabelling is a symmetry of $\mathcal{C}_{qa}$). Furthermore, let Alice's outcomes be completely deterministic for inputs $x > X'$. Since $\hat{p} \in \mathcal{C}_{qa}$, there exists a sequence of correlations $(\hat{p}_j)_j \subset \mathcal{C}_{qa}$ with finite dimensional quantum realisations such that $\hat{p}_j \to \hat{p}$ as $j \to \infty$, i.e, for each $j \in \mathbb{N}$ we have

$$\hat{p}_j(a, b|x, y) = \langle\psi_j| M_a^{x,j} \otimes N_b^{y,j} |\psi_j\rangle \tag{IX.1}$$

for some separable Hilbert spaces $\mathcal{H}_{\mathrm{A}}^j$ and $\mathcal{H}_{\mathrm{B}}^j$, unit vectors $|\psi_j\rangle \in \mathcal{H}_{\mathrm{A}}^j \otimes \mathcal{H}_{\mathrm{B}}^j$ and projective measurements $\{M_a^{x,j}\}$ and $\{N_b^{y,j}\}$ on $\mathcal{H}_{\mathrm{A}}^j$ and $\mathcal{H}_{\mathrm{B}}^j$, respectively.

Now, consider the Bell scenario $(\boldsymbol{A'}, \boldsymbol{B}, X', Y)$ in which we have removed all of Alice's inputs that give a deterministic outcome in $\hat{p}$, i.e. $x > X'$ and all the outputs $a|x$ for Alice such that $\hat{p}_{\mathrm{A}}(a|x) = 0$, i.e. $a > a'_x$. Therefore, $\boldsymbol{A'}$ has elements $A'_x = a'_x$. Define a correlation $\tau(\hat{p}) = p'$ in this scenario by $p'(a, b|x, y) = \hat{p}(a, b|x, y)$ for all $a \in [A'_x]$, $b \in [B_y]$, $x \in [X']$ and $y \in [Y]$.

Lemma 1. *A correlation $\hat{p}$ is in the set $\mathcal{C}_{qa}$ of a Bell scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$ if and only if $p' = \tau(\hat{p})$ is a correlation in the set $\mathcal{C}_{qa}$ of the Bell scenario $(\boldsymbol{A'}, \boldsymbol{B}, X', Y)$*

Proof. Firstly, it is clear that removing all the deterministic inputs $x > X'$ from each correlation in the sequence Eq. (IX.1) leaves a sequence of correlations $p_j(a, b|x, y) = \hat{p}_j(a, b|x, y)$ for all $a \in [A_x]$, $b \in [B_y]$, $x \in [X']$ and $y \in [Y]$ with a quantum realisation that tend to a correlation p defined by $p(a, b|x, y) = \hat{p}(a, b|x, y)$ for all $a \in [A_x]$, $b \in [B_y]$, $x \in [X']$ and $y \in [Y]$.

We then proceed by removing all the zero probability outcomes of one input $x^* \in [X']$ of Alice, by mapping p to a correlation p^* in the scenario $(\boldsymbol{A}^*, \boldsymbol{B}, X', Y)$, where $A^*_{x^*} = a'_{x^*}$ and $A^*_x = A_x$ for all $x \neq x^*$. We take $p^*(a, b|x, y) = p(a, b|x, y)$ for all $a \in [A^*_x]$, $b \in [B_y]$, $x \in [X']$ and $y \in [Y]$. Let $\Pi^j = \sum_{a \leq A^*_{x^*}} M_a^{x,j}$ and $\mathcal{H}_{\mathrm{A}}^{j*}$ be the support of Π^j. Observe that, denoting the identity operator on $\mathcal{H}_{\mathrm{A}}^j$ ($\mathcal{H}_{\mathrm{B}}^j$) by $\mathbb{I}_{\mathrm{A}^j}$ ($\mathbb{I}_{\mathrm{B}^j}$), we have that

$$0 = \sum_{a > A^*_{x^*}} p_{\mathrm{A}}(a|x^*) = \lim_{j\to\infty} \langle\psi_j| (\mathbb{I}_{\mathrm{A}^j} - \Pi^j) \otimes \mathbb{I}_{\mathrm{B}^j} |\psi_j\rangle = \lim_{j\to\infty} \left(1 - \langle\psi_j| \Pi^j \otimes \mathbb{I}_{\mathrm{B}^j} |\psi_j\rangle\right), \tag{IX.2}$$

which gives

$$\lim_{j\to\infty} \langle\psi_j| \Pi^j \otimes \mathbb{I}_{\mathrm{B}^j} |\psi_j\rangle = 1. \tag{IX.3}$$

We may then define the states

$$|\psi_j^*\rangle = \frac{\Pi^j \otimes \mathbb{I}_{\mathrm{B}^j} |\psi_j\rangle}{\sqrt{\langle\psi_j| \Pi^j \otimes \mathbb{I}_{\mathrm{B}^j} |\psi_j\rangle}} \in \mathcal{H}_{\mathrm{A}}^{j*} \otimes \mathcal{H}_{\mathrm{B}}^{j} , \tag{IX.4}$$

where without loss of generality we can assume that the denominator is strictly positive for all $j \in \mathbb{N}$ since it tends to one in the limit $j \to \infty$.

It also follows from Eq. (IX.3) that (embedding $|\psi_j^*\rangle$ into $\mathcal{H}_{\mathrm{A}}^{j} \otimes \mathcal{H}_{\mathrm{B}}^{j}$)

$$\lim_{j\to\infty} \langle\psi_j|\psi_j^*\rangle = \lim_{j\to\infty} \frac{\langle\psi_j| \Pi^j \otimes \mathbb{I}_{\mathrm{B}^j} |\psi_j\rangle}{\sqrt{\langle\psi_j| \Pi^j \otimes \mathbb{I}_{\mathrm{B}^j} |\psi_j\rangle}} = 1 . \tag{IX.5}$$

Observe that we can find unit vectors $|\psi_j^\perp\rangle \in \mathcal{H}_{\mathrm{A}}^{j} \otimes \mathcal{H}_{\mathrm{B}}^{j}$ orthogonal to $|\psi_j\rangle$ such that (again for the embedding)

$$|\psi_j^*\rangle = \alpha_j |\psi_j\rangle + \beta_j |\psi_j^\perp\rangle \tag{IX.6}$$

for each $j \in \mathbb{N}$. Note that we can choose the $|\psi_j^\perp\rangle$ such that $\beta_j \in \mathbb{R}$, and we have that $\alpha_j = \langle\psi_j| \Pi^j \otimes \mathbb{I}_{\mathrm{B}^j} |\psi_j\rangle \in \mathbb{R}$, which also implies $\beta_j = \sqrt{1-\alpha_j^2}$. By Eq. (IX.5), we find that $\lim_{j\to\infty} \alpha_j = 1$ and hence, also $\lim_{j\to\infty} \sqrt{1-\alpha_j^2} = 0$.

Now, consider the sequence of quantum spatial correlations in the scenario $(\boldsymbol{A}^*, \boldsymbol{B}, X', Y)$ given by

$$p_j^*(a,b|x,y) = \langle\psi_j^*| R_a^{x,j} \otimes N_b^{y,j} |\psi_j^*\rangle , \tag{IX.7}$$

for all $a \in [A_x^*]$, $b \in [B_y]$, $x \in [X']$ and $y \in [Y]$, where the operators $R_a^{x,j} = \Pi^j M_a^{x,j} \Pi^j$ form projective measurements on the subspace $\mathcal{H}_{\mathrm{A}}^{j*}$ of $\mathcal{H}_{\mathrm{A}}^{j}$ since they satisfy $\sum_{a\in[A_x^*]} R_a^{x,j} = \sum_{a\in[A_x^*]} \Pi^j M_a^{x,j} \Pi^j = \Pi^j \mathbb{I}_{\mathrm{A}^j} \Pi^j = \mathbb{I}_{\mathrm{A}^{*j}}$.

We can now evaluate the limit of our sequence of correlations using the expression in Eq. (IX.6):

$$\begin{aligned} \lim_{j\to\infty} p_j^*(a,b|x,y) &= \lim_{j\to\infty} \langle\psi_j^*| R_a^{x,j} \otimes N_b^{y,j} |\psi_j^*\rangle \\ &= \lim_{j\to\infty} \langle\psi_j^*| M_a^{x,j} \otimes N_b^{y,j} |\psi_j^*\rangle \\ &= \lim_{j\to\infty} \alpha_j^2 \langle\psi_j| M_a^{x,j} \otimes N_b^{y,j} |\psi_j\rangle \\ &\quad + 2\mathrm{Re}\left(\alpha_j \sqrt{1-\alpha_j^2} \langle\psi_j^\perp| M_a^{x,j} \otimes N_b^{y,j} |\psi_j\rangle\right) \\ &\quad + (1-\alpha_j^2)^2 \langle\psi_j^\perp| M_a^{x,j} \otimes N_b^{y,j} |\psi_j^\perp\rangle . \end{aligned} \tag{IX.8}$$

Since we have $0 \leq M_a^{x,j} \otimes N_b^{y,j} \leq \mathbb{I}_{\mathrm{A}^j\mathrm{B}^j}$, both $\left|\langle\psi_j^\perp| M_a^{x,j} \otimes N_b^{y,j} |\psi_j^\perp\rangle\right|$ and $\left|\langle\psi_j^\perp| M_a^{x,j} \otimes N_b^{y,j} |\psi_j\rangle\right|$ are bounded in the unit interval. It follows that the final two summands of the last expression in Eq. (IX.8) tend to zero in the limit due to the factor of $\sqrt{1-\alpha_j^2}$. Remembering that $\alpha_j \to 1$ as $j \to \infty$, we are left with

$$\lim_{j\to\infty} p_j^*(a,b|x,y) = \lim_{j\to\infty} \langle\psi_j| M_a^{x,j} \otimes N_b^{y,j} |\psi_j\rangle = p(a,b|x,y) , \tag{IX.9}$$

for all $a \in [A_x^*]$, $b \in [B_y]$, $x \in [X']$ and $y \in [Y]$. This argument can be applied iteratively for each $x \in [X']$ to show that the correlation $p' = \tau(\hat{p})$ is a member of the set $\mathcal{C}_{qa}$ in the Bell scenario $(\boldsymbol{A}', \boldsymbol{B}, X', Y)$.

Conversely, given any correlation $p' \in \mathcal{C}_{qa}$ in a scenario $(\boldsymbol{A}', \boldsymbol{B}, X', Y)$ we can embed the correlation in a scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$ in which Alice has more inputs and/or outputs via the map

$$\hat{p}(a,b|x,y) = \begin{cases} p'(a,b|x,y) & \text{for } a \in [A_x'],\ b \in [B_y],\ x \in [X'],\ y \in [Y] \\ 1 & \text{for } a = 1 \text{ and } X' < x \leq X \\ 0 & \text{otherwise.} \end{cases} \tag{IX.10}$$

If $p' \in \mathcal{C}_{qa}$ then there exists a sequence of correlations p'_j which tend to p' and have quantum realisations. This sequence can be transformed to a sequence of quantum spatial correlations in the scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$ which tends to $\hat{p}$ by adding zero operators to the POVMs for the additional probability zero outcomes in the existing inputs of Alice and adding POVMs given by the identity operator followed by zero operators for the additional deterministic settings. □

Remark 1. *One can analogously show that a correlation $\hat{p}$ is in the set $\mathcal{C}_{qs}$ of a Bell scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$ if and only if $p' = \tau(\hat{p})$ is a correlation in the set $\mathcal{C}_{qs}$ of the Bell scenario $(\boldsymbol{A'}, \boldsymbol{B}, X', Y)$ with an argument that follows the proof of Lemma 1 but with the simplification of not having to consider limits of sequences of correlations.*

Now, let p be a correlation in a Bell scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$ that is contained in the closure $\mathcal{C}_{qa}$ of the set $\mathcal{C}_{qs}$ of quantum spatial correlations but that is not contained in the set $\mathcal{C}_{qs}$ itself, i.e. $p \in \mathcal{C}_{qa} \backslash \mathcal{C}_{qs}$ [24]. It follows from Lemma 1 and Remark 1 that $p' = \tau(p) \in \mathcal{C}_{qa} \backslash \mathcal{C}_{qs}$ in the scenario $(\boldsymbol{A'}, \boldsymbol{B}, X', Y)$. First, we will construct a sequence $(p^n)_{n\in\mathbb{N}}$ of correlations in $\mathcal{C}_{qs}$ in the scenario $(\boldsymbol{A'}, \boldsymbol{B}, X', Y)$ converging to p' such that every element of the sequence has the same marginals for Alice as p', i.e. $p^n_{\mathrm{A}} = p'_{\mathrm{A}}$ for all $n \in \mathbb{N}$. Since the correlations p^n will all have the same marginals for Alice, they will each be mapped to a behaviour $q^n = \Gamma(p^n)$ in the same single contextuality scenario $(\|\mathcal{A}'^p\|, Y, \boldsymbol{B}, \mathcal{NS}(p'_{\mathrm{A}}))$ where $\|\mathcal{A}'^p\| = \sum_{x\in[X']} |\mathcal{A}^p_x| = \sum_{x\in[X']} \left|\mathcal{A}^{p'}_x\right|$.

Next, we will show that this sequence of behaviours converges to $q = \Gamma(p')$, meaning q is in the closure $\overline{\mathcal{Q}}$ of the set of quantum behaviours. Finally, it follows that $q \notin \mathcal{Q}$ since otherwise we could construct a quantum realisation of the correlation $p' = \Gamma^{-1}(q)$, via the method in Sec. II. Thus, we have that $q \in \overline{\mathcal{Q}} \setminus \mathcal{Q}$.

Consider the correlation $p^{\mathrm{int}}(a, b|x, y) = p'_{\mathrm{A}}(a|x) \frac{1}{B_y}$. We demonstrate that this correlation is in the relative interior of the local polytope (and, hence, in the relative interior of $\mathcal{C}_{qs}$) as follows. The vertices of the local polytope are exactly the correlations V that admit an expression

$$V(a, b|x, y) = v^{\mathrm{A}}(a|x) v^{\mathrm{B}}(b|y)\,, \tag{IX.11}$$

for two deterministic conditional probability distributions v^{A} and v^{B}. In any polytope of unconstrained conditional probability distributions $r(c|z)$ over some variables $c \in [C]$ and $z \in [Z]$ for some $C, Z \in \mathbb{N}$, any point on the boundary contains at least one zero element, i.e. $r(c|z) = 0$ for some $c \in [C]$ and $z \in [Z]$. Both distributions $p^{\mathrm{int}}_{\mathrm{A}}(a|x) = p'_{\mathrm{A}}(a|x)$ and $p^{\mathrm{int}}_{\mathrm{B}}(b|y) = \frac{1}{B_y}$ are entirely non-zero and, therefore, are in the relative interiors of their respective polytopes.

It follows that $p^{\mathrm{int}}_{\mathrm{A}}(a|x)$ and $p^{\mathrm{int}}_{\mathrm{B}}(b|y)$ admit convex decompositions $p^{\mathrm{int}}_{\mathrm{A}}(a|x) = \sum_{a,x} \alpha_{a,x} v^{\mathrm{A}}(a|x)$ and $p^{\mathrm{int}}_{\mathrm{B}}(b|y) = \sum_{b,y} \beta_{b,y} v^{\mathrm{B}}(b|y)$ where $\alpha_{a,x} > 0$, $\beta_{b,y} > 0$ for all $a \in [A'_x]$, $b \in [B_y]$, $x \in [X]$ and $y \in [Y]$ and $\sum_{a,x} \alpha_{a,x} = \sum_{b,y} \beta_{b,y} = 1$. Thus, we find that $p^{\mathrm{int}}(a, b|x, y)$ also admits a convex decomposition in which all of the vertices of the local polytope have a strictly non-zero coefficient, namely,

$$p^{\mathrm{int}}(a, b|x, y) = \sum_{a,b,x,y} \alpha_{a,x} \beta_{b,y} v^{\mathrm{A}}(a|x) v^{\mathrm{B}}(b|y)\,, \tag{IX.12}$$

showing that p^{int} is in the relative interior of the local polytope.

Additionally, we have that $p^{\mathrm{int}}_{\mathrm{A}} = p'_{\mathrm{A}}$. Define $p^n = \frac{1}{n} p^{\mathrm{int}} + \left(1 - \frac{1}{n}\right) p'$. Each term of the sequence has the same marginals for Alice, $p^n_{\mathrm{A}} = p'_{\mathrm{A}}$, and is in the relative interior of $\mathcal{C}_{qa}$, since it is a mixture of a point in $\mathcal{C}_{qa}$ and a point in the relative interior of $\mathcal{C}_{qa}$. It follows that $(p^n)_{n\in\mathbb{N}}$ is a sequence of points in $\mathcal{C}_{qs}$ that converge to p'.

Now, consider the image $(q^n)_{n\in\mathbb{N}}$ of the sequence $(p^n)_{n\in\mathbb{N}}$ under our map Γ [see Eq. (I.11)] in the contextuality scenario $(\|\mathcal{A}'^p\|, Y, \boldsymbol{B}, \mathcal{NS}(p'_{\mathrm{A}}))$, noting that all points in the sequence are mapped to the same contextuality scenario since they have the same marginals $p'_{\mathrm{A}}(a|x)$ for Alice. Since each $p^n \in \mathcal{C}_{qs}$, we have that each $q^n \in \mathcal{Q}$. Furthermore, we have that for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $\|p^n - p'\|_1 < \epsilon$ for all $n \geq N_\epsilon$. Thus, letting $\epsilon' = \min_{a,x}\{p'_{\mathrm{A}}(a|x)\}\epsilon$, we have that for all $n \geq N_{\epsilon'}$

$$\begin{aligned}
\|q^n - q\|_1 &= \sum_{a,b,x,y} |q^n(b|[a|x],y) - q(b|[a|x],y)| \\
&= \sum_{a,b,x,y} \left| \frac{p^n(a,b|x,y)}{p'_{\mathrm{A}}(a|x)} - \frac{p'(a,b|x,y)}{p'_{\mathrm{A}}(a|x)} \right| \\
&\le \max_{a,x} \left\{ \frac{1}{p'_{\mathrm{A}}(a|x)} \right\} \sum_{a,b,x,y} |p^n(a,b|x,y) - p'(a,b|x,y)| \\
&< \max_{a,x} \left\{ \frac{1}{p'_{\mathrm{A}}(a|x)} \right\} \epsilon' = \frac{\epsilon'}{\min_{a,x}\{p'_{\mathrm{A}}(a|x)\}} = \epsilon\,,
\end{aligned} \tag{IX.13}$$

and we find that $(q^n) \subset \mathcal{Q}$ converges to q, meaning $q \in \overline{\mathcal{Q}}$.

On the other hand, we have that $q \notin \mathcal{Q}$ since otherwise we could construct a quantum realisation of the correlation $p' = \Gamma^{-1}(q)$, via the method in Sec. II. Therefore, it follows that that $q \in \overline{\mathcal{Q}} \setminus \mathcal{Q}$ and $\mathcal{Q}$ is not closed.

X. PROOF OF COROLLARY 4

We require two results from the literature. Firstly, it is known that the following weak-membership problem for $\mathcal{C}_{qa} = \overline{\mathcal{C}_{qs}}$ is undecidable [14]:

[WMEM] given a correlation p in a Bell scenario $(\boldsymbol{A}, \boldsymbol{B}, X, Y)$ and $\varepsilon > 0$ decide whether $p \in \mathcal{C}_{qa}$ or $p \notin \mathcal{C}_{qa}^{\varepsilon} = \{p : \|p - p'\|_1 \le \varepsilon \text{ for some } p' \in \mathcal{C}_{qa}\}$ with the promise that either $p \in \mathcal{C}_{qa}$ or $p \notin \mathcal{C}_{qa}^{\varepsilon}$,

where $\|.\|_1$ is the ℓ_1-norm.

Secondly, it is known [14] that for any $\delta > 0$ there exists an algorithm (FIN) that verifies that a correlation $p \in \mathcal{C}_{qa}^{\delta}$ and halts for any correlation $p \in \mathcal{C}_{qa}^{\delta}$. The input of this algorithm is a fixed correlation p. At the d-th step, (FIN:d), of the algorithm a finite set of quantum correlations, $\{\tilde{p}_n^d\}_n$, is constructed such that these correlations are realisable when $\dim(\mathcal{H}_{\mathrm{A}}) = \dim(\mathcal{H}_{\mathrm{B}}) = d$, and moreover, for any correlation, $\tilde{p}$ that is also realisable with such Hilbert spaces we have $\left\|\tilde{p} - \tilde{p}_n^d\right\|_1 < \delta$ for some $\tilde{p}_n^d$. This can be achieved, because the set of quantum correlations achievable in a fixed dimension is compact. Then, the distance $\left\|p - \tilde{p}_n^d\right\|_1$ is calculated for all n. If this distance is less than δ for some n the algorithm returns $p \in \mathcal{C}_{qa}^{\delta}$. Otherwise, the algorithm proceeds to (FIN:$d+1$). Since the closure of the set of quantum correlations realisable in some finite dimension is the same as $\mathcal{C}_{qa}$ [26, 27], it follows that if $p \in \mathcal{C}_{qa}^{\delta}$ then the algorithm halts for some finite d, otherwise it does not halt.

Now, suppose there exists a hierarchy of SDPs wherein each level, j, decides whether a behaviour q in a contextuality scenario $(Z, Y, \boldsymbol{B}, \mathcal{NS}(\hat{p}_{\mathrm{A}}))$ is in a superset $\mathcal{Q}_j$ of $\mathcal{Q}$ or not, and these supersets converge to $\mathcal{Q}$ as j tends to infinity (i.e. $\mathcal{Q} = \cap_{\mathbb{N}} \mathcal{Q}_j$). Under this hypothesis, we will construct an algorithm that decides the problem [WMEM], and therefore reach a contradiction.

We can now give the algorithm that decides [WMEM]. Given a correlation p with the promise that either $p \in \mathcal{C}_{qa}$ or $p \notin \mathcal{C}_{qa}^{\varepsilon}$:

Step (1): If p_{A} is deterministic return $p \in \mathcal{C}_{qa}$ and halt.
Otherwise, relabel Alice's inputs such that any inputs giving a deterministic outcome are labelled with the highest values $x > X'$ in $[X]$ and for each $x \in [X']$ relabel the outcomes such that any zeroes in p_A are for outcomes $a > a'_x$ for some $a'_x \in \mathbb{N}$ and (we retain the notation p for this relabelling), and map p to the correlation $p' = \tau(p)$ in the Bell scenario $(\boldsymbol{A}', \boldsymbol{B}, X', Y)$ such that p' has marginals p'_{A} strictly between zero and one (see Sec. IX).

Step (2): Map p' to $q = \Gamma(p')$ in the contextuality scenario $(\|\mathcal{A}'^p\|, Y, \boldsymbol{B}, \mathcal{NS}(p'_{\mathrm{A}}))$, where $\|\mathcal{A}'^p\| = \sum_{x\in[X']} |\mathcal{A}_x^p| = \sum_{x\in[X']} \left|\mathcal{A}_x^{p'}\right|$—see Eqs. (I.10) and (I.11)..

Step ($j \ge 3$): Use level j of the SDP hierarchy to decide whether $q \in \mathcal{Q}_j$.
If $q \notin \mathcal{Q}_j$: return $p \notin C_{qa}^{\varepsilon}$ and halt.
If $q \in \mathcal{Q}_j$: run step (FIN:j) of the algorithm (FIN) on p' with $\delta < \varepsilon$.
If (FIN:j) returns $p' \in \mathcal{C}_{qa}^{\delta}$: return $p \in \mathcal{C}_{qa}$ and halt.
Otherwise, perform Step (j+1).

To see that the algorithm would return the correct answer, first, define $\mathcal{Q}^\varepsilon = \{q : \|q - q'\|_1 \leq \varepsilon \text{ for some } q' \in \mathcal{Q}\}$.

Case (1) $p \notin \mathcal{C}_{qa}^\varepsilon$:
Let p'_ε be any behaviour such that $\|p' - p'_\varepsilon\|_1 < \varepsilon$. Then we have that $\left\|p - \tau^{-1}(p'_\varepsilon)\right\|_1 = \|p' - p'_\varepsilon\|_1 < \varepsilon$ and thus $\tau^{-1}(p'_\varepsilon) \notin \mathcal{C}_{qa}$. Therefore, $p'_\varepsilon \notin \mathcal{C}_{qa}$ in the scenario $(\boldsymbol{A'}, \boldsymbol{B}, X', Y)$ and $p' \notin \mathcal{C}_{qa}^\varepsilon$, since there is an ε-ball around p' entirely outside of $\mathcal{C}_{qa}$.

Now we will show that $q = \Gamma(p') \notin \mathcal{Q}^\varepsilon$. Consider any behaviour q_ε such that $\|q_\varepsilon - q\|_1 < \varepsilon$ and let $p_\varepsilon = \Gamma^{-1}(q_\varepsilon)$ be the image of q_ε under the map in Eq. (I.13) (where $\hat{p}_\mathrm{A} = p'_\mathrm{A}$). Then we have

$$\begin{aligned}
\|p_\varepsilon - p'\|_1 &= \sum_{a,b,x,y} |p'_\mathrm{A}(a|x) q_\varepsilon(b|[a|x], y) - p'(a, b|x, y)| \\
&= \sum_{a,b,x,y} |p'_\mathrm{A}(a|x) q_\varepsilon(b|[a|x], y) - p'_\mathrm{A}(a|x) q(b|[a|x], y)| \\
&= \sum_{a,b,x,y} p'_\mathrm{A}(a|x) |q_\varepsilon(b|[a|x], y) - q(b|[a|x], y)| \\
&\leq \sum_{a,b,x,y} |q_\varepsilon(b|[a|x], y) - q(b|[a|x], y)| < \varepsilon\,.
\end{aligned} \tag{X.1}$$

Therefore, we have that $p_\varepsilon \notin \mathcal{C}_{qa}$, which implies $p_\varepsilon \notin \mathcal{C}_{qs}$ and thus $q_\varepsilon = \Gamma(p_\varepsilon) \notin \mathcal{Q}$. We have shown that there is an ε-ball around q entirely outside of $\mathcal{Q}$, thus $q \notin \mathcal{Q}^\varepsilon$.

For a finite level j of the SDP hierarchy we find $q \notin \mathcal{Q}_j$ and hence, at a finite step $(j+1)$ of the algorithm we obtain $p \notin \mathcal{C}_{qa}^\varepsilon$. On the other hand, at no Step (j) will the algorithm return $p \in \mathcal{C}^{qa}$, since we have shown that $p' \notin \mathcal{C}_{qa}^\varepsilon$ in the scenario $(\boldsymbol{A'}, \boldsymbol{B}, X', Y)$, and therefore $\|p' - p''\|_1 > \varepsilon > \delta$ for any correlation p'' realisable with finite dimensional quantum systems.

Case (2) $p \in \mathcal{C}_{qa}$**:**
We have that $p' = \tau(p) \in \mathcal{C}_{qa}$ in the scenario $(\boldsymbol{A'}, \boldsymbol{B}, X', Y)$, and we will show that $q = \Gamma(p') \in \overline{\mathcal{Q}}$ using the same method as in Sec. IX. To do so, we will construct a sequence $(p^n)_{n\in\mathbb{N}}$ of correlations in $\mathcal{C}_{qs}$ converging to p' such that every element of the sequence has the same marginals for Alice as p', i.e. $p^n_\mathrm{A} = p'_\mathrm{A}$ for all $n \in \mathbb{N}$. Since the correlations p^n will all have the same marginals for Alice, they will all be mapped to a behaviour $q^n = \Lambda(p^n)$ in the same contextuality scenario $(\|\mathcal{A}'^p\|, Y, \boldsymbol{B}, \mathcal{NS}(p'_\mathrm{A}))$.

Consider the correlation $p^{\mathrm{int}}(a, b|x, y) = p'_\mathrm{A}(a|x)\frac{1}{B_y}$. We demonstrate that this correlation is in the relative interior of the local polytope (and, hence, in the relative interior of $\mathcal{C}_{qs}$) as follows. The vertices of the local polytope are exactly the correlations V that admit an expression

$$V(a, b|x, y) = v^\mathrm{A}(a|x) v^\mathrm{B}(b|y)\,, \tag{X.2}$$

for two deterministic, conditional probability distributions v^A and v^B. In any polytope of unconstrained conditional probability distributions $r(c|z)$ over some variables $c \in [C]$ and $z \in [Z]$ for some $C, Z \in \mathbb{N}$, any point on the boundary contains at least one zero element, i.e. $r(c|z) = 0$ for some $c \in [C]$ and $z \in [Z]$. Both distributions $p^{\mathrm{int}}_\mathrm{A}(a|x) = p'_\mathrm{A}(a|x)$ and $p^{\mathrm{int}}_\mathrm{B}(b|y) = \frac{1}{B_y}$ are entirely non-zero and, therefore, are in the relative interiors of their respective polytopes.

It follows that $p^{\mathrm{int}}_\mathrm{A}(a|x)$ and $p^{\mathrm{int}}_\mathrm{B}(b|y)$ admit convex decompositions $p^{\mathrm{int}}_\mathrm{A}(a|x) = \sum_{a,x} \alpha_{a,x} v^\mathrm{A}(a|x)$ and $p^{\mathrm{int}}_\mathrm{B}(b|y) = \sum_{b,y} \beta_{b,y} v^\mathrm{B}(b|y)$ where $\alpha_{a,x} > 0$, $\beta_{b,y} > 0$ for all $a \in [A'_x]$, $b \in [B_y]$, $x \in [X]$ and $y \in [Y]$ and $\sum_{a,x} \alpha_{a,x} = \sum_{b,y} \beta_{b,y} = 1$. Thus, we find that $p^{\mathrm{int}}(a, b|x, y)$ also admits a convex decomposition in which all of the vertices of the local polytope have a strictly non-zero coefficient, namely,

$$p^{\mathrm{int}}(a, b|x, y) = \sum_{a,b,x,y} \alpha_{a,x} \beta_{b,y} v^\mathrm{A}(a|x) v^\mathrm{B}(b|y)\,, \tag{X.3}$$

showing that p^{int} is in the relative interior of the local polytope.

Additionally, we have that $p^{\mathrm{int}}_\mathrm{A} = p'_\mathrm{A}$. Define $p^n = \frac{1}{n} p^{\mathrm{int}} + \left(1 - \frac{1}{n}\right) p'$. Each term of the sequence has the same marginals for Alice, $p^n_\mathrm{A} = p'_\mathrm{A}$, and is in the relative interior of $\mathcal{C}_{qa}$, since it is a mixture of a point in $\mathcal{C}_{qa}$ and a point in the relative interior of $\mathcal{C}_{qa}$. It follows that $(p^n)_{n\in\mathbb{N}}$ is a sequence of points in $\mathcal{C}_{qs}$ that converge to p'.

Now consider the image $(q^n)_{n\in\mathbb{N}}$ of the sequence $(p^n)_{n\in\mathbb{N}}$ under our map Γ [see Eq. (I.11)] in the contextuality scenario $(\|\mathcal{A}'^p\|, Y, \boldsymbol{B}, \mathcal{NS}(p'_\mathrm{A}))$, noting that all points in the sequence are mapped to the same contextuality scenario since they have the same marginals $p'_{\mathrm{A}}(a|x)$ for Alice. Since each $p^n \in \mathcal{C}_{qs}$, we have that each $q^n \in \mathcal{Q}$. Furthermore, we have that for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $\|p^n - p'\|_1 < \epsilon$ for all $n \geq N_\epsilon$. Thus, letting $\epsilon' = \min_{a,x}\{p'_{\mathrm{A}}(a|x)\}\epsilon$, we have that for all $n \geq N_{\epsilon'}$

$$\begin{aligned}
\|q^n - q\|_1 &= \sum_{a,b,x,y} |q^n(b|[a|x], y) - q(b|[a|x], y)| \\
&= \sum_{a,b,x,y} \left| \frac{p^n(a,b|x,y)}{p'_{\mathrm{A}}(a|x)} - \frac{p'(a,b|x,y)}{p'_{\mathrm{A}}(a|x)} \right| \\
&\leq \max_{a,x} \left\{ \frac{1}{p'_{\mathrm{A}}(a|x)} \right\} \sum_{a,b,x,y} |p^n(a,b|x,y) - p'(a,b|x,y)| \\
&< \max_{a,x} \left\{ \frac{1}{p'_{\mathrm{A}}(a|x)} \right\} \epsilon' = \frac{\epsilon'}{\min_{a,x}\{p'_{\mathrm{A}}(a|x)\}} = \epsilon \,,
\end{aligned} \tag{X.4}$$

and we find that $(q^n) \subset \mathcal{Q}$ converges to q, meaning $q \in \overline{\mathcal{Q}}$.

Therefore, in this case the algorithm will not return $p \notin \mathcal{C}_{qa}^{\varepsilon}$ in any step j. On the other hand, at some finite step (FIN:d) the algorithm (FIN) will establish $p' \in \mathcal{C}_{qa}^{\delta}$ and our algorithm will return $p \in \mathcal{C}_{qa}$ and halt at Step d.