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ON CALDERON'S PROBLEM FOR THE CONNECTION LAPLACIAN

RAVIL GABDURAKHMANOV AND GERASIM KOKAREV

ABSTRACT. We consider Calderón's problem for the connection Laplacian on a real-analytic vector bundle over a manifold with boundary. We prove a uniqueness result for this problem when all geometric data are real-analytic, recovering the topology and geometry of a vector bundle up to a gauge transformation and an isometry of the base manifold.

1. STATEMENT AND DISCUSSION OF RESULTS

1.1. **Introduction.** The purpose of this paper is to prove a uniqueness result for the Calderón inverse problem for the connection Laplacian on a vector bundle. Our main hypothesis is that the geometry of a vector bundle, that is a connection, a compatible inner product, and a Riemannian metric on the base manifold, are real-analytic. This Calderón problem is motivated by the Aharonov-Bohm effect that says that different gauge equivalence classes of electromagnetic potentials have different physical effects that can be detected by experiments. Thus, our uniqueness result shows that different gauge equivalence classes of connections have different boundary data, that is such classes are detectable by boundary measurements.

We discuss some of the related literature on this problem in due course, but now say a few words about the classical Calderón problem. Recall that the anisotropic Calderón problem asks whether one can read off the conductivity matrix of a medium from electrical voltage and current measurements on the boundary. In dimension greater than two this problem is equivalent to recovering a Riemannian metric on a compact manifold with boundary from the Dirichlet and Neumann data of harmonic functions. A classical result by Lassas and Uhlmann [13], see also [14, 15], solves this problem when a Riemannian metric is real-analytic. In more detail, it says that the topology and geometry of a real-analytic Riemannian manifold with boundary can be recovered from the Dirichlet-to-Neumann map for the Laplace-Beltrami operator. The main result of this paper can be viewed as a version of the Lassas-Uhlmann theorem in the setting of vector bundles, which allows us to recover additional topological and geometric structures. We proceed with the statement of related hypotheses and conclusions in more detail.

1.2. **Main result.** Let (M_i, g_i), where $i = 1, 2$, be two connected compact Riemannian manifolds with boundary, and let E_i be vector bundles over M_i. We assume that each E_i is equipped with a connection ∇^i and a *Euclidean structure*, that is a compatible inner product $\langle \cdot, \cdot \rangle_{E_i}$ in the sense that

$$D_X \langle u, v \rangle_{E_i} = \langle \nabla^i_X u, v \rangle_{E_i} + \langle u, \nabla^i_X v \rangle_{E_i}$$

for any smooth sections u and v of E_i, and any vector field X on M_i. For open subsets $\Sigma_i \subset \partial M_i$ we denote by Λ_{Σ_i} the corresponding Dirichlet-to-Neumann maps defined on compactly

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supported sections of $E_i|_{\Sigma_i}$ by the taking the normal derivative of the harmonic extension, that is the solution of the equation

$$\Delta^{E_i} u = 0, \qquad u|_{\partial M_i} = s,$$

where s is a section of $E_i|_{\partial M_i}$ supported in Σ_i, and Δ^{E_i} is the connection Laplacian on E_i, see Section 2 for details.

Let $\phi : E_1|_{\Sigma_1} \to E_2|_{\Sigma_2}$ be a morphism of vector bundles that covers a diffeomorphism $\psi : \Sigma_1 \to \Sigma_2$, that is $\pi_2 \circ \phi = \psi \circ \pi_1$, where π_i is the projection map for E_i, $i = 1,2$. We say that such a morphism ϕ *intertwines* with the Dirichlet-to-Neumann maps Λ_{Σ_i} if

$$\phi \circ \Lambda_{\Sigma_1}(s) \circ \psi^{-1} = \Lambda_{\Sigma_2}(\phi \circ s \circ \psi^{-1})$$

for any smooth section s of $E_1|_{\Sigma_1}$. Recall that a vector bundle isomorphism is called *Euclidean*, if it preserves Euclidean structures. Our main result is the following theorem.

Theorem 1.1. *Let $(M_i, g_i, E_i, \nabla^i)$, where $i = 1,2$, be two Euclidean real-analytic vector bundles defined over connected compact real-analytic Riemannian manifolds with boundary, equipped with real-analytic connections. Suppose that $\dim M_i \geqslant 3$ for each $i = 1,2$, and for some open subsets $\Sigma_i \subset \partial M_i$ there exists a real-analytic Euclidean vector bundle isomorphism $\phi : E_1|_{\Sigma_1} \to E_2|_{\Sigma_2}$ that intertwines with the corresponding Dirichlet-to-Neumann operators Λ_{Σ_1} and Λ_{Σ_2}. Then the vector bundles E_1 and E_2 are isomorphic, and moreover, there exists a real-analytic Euclidean vector bundle isomorphism $\Phi : E_1 \to E_2$ that covers an isometry $\Psi : (M_1, g_1) \to (M_2, g_2)$, such that $\Phi^* \nabla^2 = \nabla^1$ and $\Phi|_{\Sigma_1} = \phi$.*

We note that the presence of Euclidean structures on vector bundles E_i in Theorem 1.1 plays an auxiliary, but important role. In more detail, neither the connection Laplacian nor the associated Dirichlet-to-Neumann operator depend on them, see Section 2. On the other hand, we do not know whether the assumption that the isomorphism ϕ in Theorem 1.1 is Euclidean, and the conclusion that so is its extension Φ, can be dropped.

It is an open problem whether the conclusions in Theorems 1.1 hold for arbitrary smooth geometric data, connections and Riemannian metrics, on vector bundles. Under different conditions a similar problem has been considered by Cekić [2, 3]. Let us also mention that in [12] the authors consider the Calderón problem for the wave operator of the connection Laplacian on Hermitian vector bundles, and obtain conclusions similar to the ones in Theorem 1.1 without any hypotheses on the geometry of vector bundles.

Note that the hypothesis on the dimension of the base manifolds M_i in our results is essential for the conclusions to hold. In dimension two the connection Laplacian behaves differently when a Riemannian metric on the base changes conformally, see Section 2, and the Riemannian metric on the base can not be recovered. However, as our next theorem shows, a vector bundle with a connection are still determined by the corresponding Dirichlet-to-Neumann operator. To state this result we first need to fix more specific notation.

As is well-known, the conformal class of any smooth metric g on a compact connected surface M determines a real-analytic structure, that is an atlas formed by isothermal charts of g. We say that a pair (M, c), where c is a conformal class of Riemannian metrics on M, is a *real-analytic surface with boundary*, if the real-analytic structure on M is determined by the conformal class c, and the boundary is real-analytic with respect to this structure. Let E_i, where $i = 1,2$, be vector bundles over M. Suppose that for an open set $\Sigma \subset \partial M$ there is a morphism of vector bundles $\phi : E_1|_\Sigma \to E_2|_\Sigma$ that covers the identity map on the base, that is $\pi_2 \circ \phi = \pi_1$, where π_i is the projection map for E_i, $i = 1,2$. Similarly to the notation used above, we say that such

a morphism ϕ *intertwines* with the Dirichlet-to-Neumann maps $\Lambda_{i,\Sigma}$ *relative to the conformal class* c, if

$$\phi \circ \Lambda_{1,\Sigma}(s) = \Lambda_{2,\Sigma}(\phi \circ s)$$

for any smooth section s of $E_1|_\Sigma$, where the Dirichlet-to-Neumann operators are defined using some metric $g \in c$. It is straightforward to see that this relation does not depend on a metric g in a fixed conformal class c, used to define the Dirichlet-to-Neumann operators, but depends on the conformal class c. The following theorem is the version of our main result for vector bundles over surfaces.

Theorem 1.2. *Let (M,c) be a compact connected real-analytic surface with boundary, and let E_1 and E_2 be two Euclidean real-analytic vector bundles over M equipped with real-analytic connections ∇^1 and ∇^2 respectively. Suppose that for some open subset $\Sigma \subset \partial M$ there exists a real-analytic Euclidean vector bundle isomorphism $\phi : E_1|_\Sigma \to E_2|_\Sigma$ that covers the identity map and intertwines with the corresponding Dirichlet-to-Neumann operators $\Lambda_{1,\Sigma}$ and $\Lambda_{2,\Sigma}$ relative to c. Then the bundles E_1 and E_2 are isomorphic, and moreover, there exists a real-analytic Euclidean vector bundle isomorphism $\Phi : E_1 \to E_2$ that covers the identity map of M, such that $\Phi^*\nabla^2 = \nabla^1$ and $\Phi|_\Sigma = \phi$.*

Related results on the Calderón problem for the connection Laplacian on vector bundles over surfaces can be found in [1, 10], where the authors are concerned with recovering a connection on a fixed vector bundle. Although our hypothesis that the related data are real-analytic might be more restrictive than the ones used in the literature, we do not know any similar uniqueness results for the topology of a vector bundle. It is likely that Theorem 1.2 can be improved, and the topology of the base surface together with the conformal class of metrics on it can also be identified. We plan to address this problem in a future work.

The proofs of Theorems 1.1 and 1.2 build on an elegant idea in [14]. Using Green kernels for the connection Laplacian, we construct immersions of our vector bundles into some function space, and recover the geometry and topology from their images. We believe that some technical details of our argument in the setting of vector bundles might be of independent interest, and the improvements give a more streamlined proof of the original results in [14]. A few generalisations of our results are possible. First, Theorem 1.1 can be extended to vector bundles over non-compact complete manifolds with compact boundaries. Second, the conclusions continue to hold for the Dirichlet-to-Neumann operators associated with Schrodinger operators, that is connection Laplacians with symmetric real-analytic potentials, see [7]. It is likely that the hypothesis that potentials are real-analytic can be significantly weakened, but we have made no attempt in pursuing this direction. In [7] the first-named author also obtains analogous results for the Jacobi operator, and applies them to the Calderón problem for the harmonic map equation.

1.3. **Organisation of the paper.** The paper is organised in the following way. In Section 2 we introduce notation and recall background material on the connection Laplacian and its Green kernel. Here we also recall the results that relate the symbol of the Dirichlet-to-Neumann operator to the geometry on and near the boundary. Section 3 is the main body of the paper. Here we define the immersions of vector bundles via Green kernels to some function space, describe its properties, and prove main results. In Section 4 we outline the version of the main argument for vector bundles over surfaces, which gives the proof of Theorem 1.2. In the last section we collect proofs of auxiliary statements.

2. PRELIMINARIES

2.1. **Notation.** Let (M,g) be a connected compact Riemannian manifold with boundary, and let E be a vector bundle over M. We suppose that E is equipped with a connection ∇^E and a Euclidean structure, that is an inner product $\langle\cdot,\cdot\rangle_E$ compatible with ∇^E. The latter induces a natural L_2-product on the space of smooth sections $\Gamma(E)$ by the formula

$$(u,v)_2 = \int\limits_M \langle u_x, v_x\rangle_E dVol_g(x),$$

where Vol_g is the volume measure on M. Below we use the notation $\Gamma(V)$ for the space of smooth sections of a vector bundle V over M. We also have a natural L_2-product on $\Gamma(T^*M\otimes E)$, defined by the formula

$$(\alpha,\beta)_2 = \int\limits_M trace_g\langle \alpha_x(\cdot),\beta_x(\cdot)\rangle_E dVol_g(x),$$

where $trace_g$ is the trace (contraction) with respect to a metric tensor g on M.

Recall that the *connection Laplacian* Δ_g^E is a second order differential operator on sections of E defined by the formula

$$\Delta_g^E = -trace_g\nabla^2, \tag{2.1}$$

where $\nabla^2 : \Gamma(E) \to \Gamma(\otimes^2 T^*M\otimes E)$ is the natural second derivative determined by the Levi-Civita connection on M and the connection ∇^E on E. Equivalently, it can be defined as the composition $(\nabla^E)^*\nabla^E$, where $(\nabla^E)^*$ is the formally adjoint operator with respect to the L_2-products on $\Gamma(E)$ and $\Gamma(T^*M\otimes E)$ defined above, see [5, Section 1]. Note that although the second definition uses the inner product on E, by relation (2.1) the operator Δ_g^E does not depend on such an inner product.

Now we briefly discuss the behaviour of the connection Laplacian when a Riemannian metric or a connection change. First, if $\tilde g = \exp(2\varphi)g$ is another Riemannian metric, then a direct computation shows that

$$\Delta_{\tilde g}^E s = \exp(-2\varphi)(\Delta_g^E s - (n-2)\nabla_Z^E s),$$

where $Z = \operatorname{grad}\varphi$ is the gradient vector field with respect to a metric g, and n is the dimension of M. In particular, when $n=2$, the operator Δ_g^E is conformally covariant, and a section $s\in\Gamma(E)$ is harmonic or not with respect to g and $\tilde g$ simultaneously. Second, consider another connection $\tilde\nabla^E$ on E, and denote by $\tilde\Delta_g^E$ the corresponding connection Laplacian. Recall that a vector bundle isomorphism $\Phi : E\to E$ is called a *gauge equivalence* if $\Phi^*\tilde\nabla^E = \nabla^E$, that is

$$\nabla_X^E s = \Phi^{-1}\circ\tilde\nabla_X^E(\Phi\circ s)$$

for any section $s\in\Gamma(E)$. In a local frame for E, this relation is equivalent to

$$\omega = \gamma^{-1}\tilde\omega\gamma + \gamma^{-1}d\gamma,$$

where ω and $\tilde\omega$ are the connection matrices of ∇^E and $\tilde\nabla^E$ respectively, and γ is the matrix of Φ. A straightforward calculation shows that the connection Laplacians of gauge equivalent connections are related by the formula

$$\Delta_g^E = \Phi^{-1}\circ\tilde\Delta_g^E(\Phi\circ s) \tag{2.2}$$

for any section $s\in\Gamma(E)$. These properties determine the behaviour of other quantities closely related to the Laplacian, such as its Green kernel and Dirichlet-to-Neumann operator. We recall the necessary background material on them below.

2.2. **The Green kernel for the connection Laplacian.** Choosing local coordinates on M and a local frame for E, it is straightforward to see that the equation

$$\Delta^E u = w, \quad \text{where } w \in \Gamma(E), \tag{2.3}$$

takes the form of an elliptic system of the second order differential equations, whose principal symbol $\sigma(x,\xi) \in \mathscr{L}(T_xM, T_xM)$ equals $|\xi|_g^2 \mathrm{Id}$, where $\xi \in T_x^*M$, $x \in M$. In particular, this system is *strongly elliptic in the sense of Petrowsky*, which is the strongest notion of ellipticity for systems, and by the results in [11, 18, 19] there exists a Green matrix locally around every point. For a brief overview of various notions of ellipticity for systems of differential equations and related results we refer to [20]. In our notation the existence of the Green matrix means that for any point $p \in M$ there exists a neighbourhood U_p and a smooth matrix-valued function $G(x,y)$, where $x, y \in U_p$, $x \neq y$, such that

$$\int_M \langle G(x,y), \Delta^E w(y) \rangle_{y,E} dVol_g(y) = w(x) \tag{2.4}$$

for any section $w \in \Gamma(E)$ whose support lies in U_p. Above we view $\langle G(x,y), \cdot \rangle_{y,E}$ as a linear operator from the fibre E_y to the fibre E_x. For the sequel we need the existence of a similar object globally on M.

Below by $E \boxtimes E$ we denote the so-called external tensor product, that is the vector bundle over $M \times M$ whose fibre over a point (x,y) equals $E_x \otimes E_y$. With some abuse of notation, for any vectors $v_x \in E_x$, $u_y^1, u_y^2 \in E_y$ we write $\langle v_x \otimes u_y^1, u_y^2 \rangle_{y,E}$ for the vector $\langle u_y^1, u_y^2 \rangle_E v_x \in E_x$. Thus, if $G(x,y)$ is viewed as an element in $E_x \otimes E_y$, the notation $\langle G(x,y), \Delta^E w(y) \rangle_{y,E}$ in relation (2.4) can be understood as an element in E_x. We also use the notation $\mathscr{D}(E)$ for a subspace of $\Gamma(E)$ formed by smooth sections whose supports lie in the interior of M.

Definition 2.1. A smooth section G of the vector bundle $E \boxtimes E$ defined away from the diagonal $\mathrm{diag}(M) = \{(x,x) \in M \times M\}$ is called the *Dirichlet Green kernel* if:

(i) the integral of the function $y \mapsto |G(x,y)|_{E \otimes E}$ is finite for all $x \in M$;
(ii) relation (2.4) holds for all $w \in \mathscr{D}(E)$;
(iii) $G(x,y) = 0$ for all $x \in M$, $y \in \partial M$, $x \neq y$.

It is straightforward to see that the Dirichlet Green kernel occurs as the kernel of a linear operator that sends a section $w \in \Gamma(E)$ to the solution of equation (2.3) with the Dirichlet boundary condition $u|_{\partial M} = 0$. Note that the connection Laplacian Δ^E on sections of E naturally induces the operator Δ_y^E on the sections of $E \boxtimes E$ that sends $v_x \otimes u_y$ to $v_x \otimes \Delta^E u_y$ for any smooth sections $v, u \in \Gamma(E)$. In this notation the condition (ii) in the definition means that

$$\Delta_y^E G(x, \cdot) = \delta_x$$

for any $x \in M$. For the sequel we note that the combination of the Sobolev embedding together with standard regularity theory shows that the section $y \mapsto G(x,y)$ lies in the Sobolev space $W^{2-k,2}(E)$ for any $k > n/2$, $x \in M$, where $n = \dim M$.

The following statement can be found in [7]; it is a consequence of local existence results in [11, 18, 19].

Proposition 2.1. *Let (M,g,E) be a Euclidean vector bundle over a compact Riemannian manifold with boundary, equipped with a compatible connection ∇^E. Then there exists a unique Dirichlet Green kernel on E. Besides, it is symmetric in the sense that*

$$G(y,x) = \tau_{x,y} G(x,y) \qquad \textit{for all } x, y \in M,$$

where $\tau_{x,y} : E_x \otimes E_y \to E_y \otimes E_x$ is a natural isomorphism that sends $v_x \otimes u_y$ to $u_y \otimes v_x$.

Now suppose that (M, g, E) is a real-analytic Euclidean vector bundle over a compact real-analytic Riemannian manifold M with boundary, equipped with a real-analytic compatible connection ∇^E. Then locally the equation

$$\Delta^E_y G(x,y) = 0, \quad \text{where} \quad y \in M \backslash \{x\},$$

takes the form of a system of differential equations with analytic coefficients, and by [21, 11] we conclude that the Green kernel is real-analytic in the second variable y. More generally, we shall often use the fact that a section of E that is harmonic on an open set is automatically real-analytic on this set.

Let $\tilde{\nabla}^E$ be another connection on a fixed vector bundle E that is gauge equivalent to ∇^E, that is there exists a vector bundle isomorphism $\Phi : E \to E$ such that $\Phi^* \tilde{\nabla}^E = \nabla^E$. Denote by $\tilde{G}$ the Green kernel corresponding to the connection Laplacian $\tilde{\Delta}^E$. Using relation (2.2), it is straightforward to see that $\tilde{G} = \Phi^{\boxtimes} \circ G$, where $\Phi^{\boxtimes} : E \boxtimes E \to E \boxtimes E$ is an isomorphism that equals $\Phi_x \otimes \Phi_y$ on each fibre $E_x \otimes E_y$, where $x, y \in M$. In addition, as expected, when dimension of M equals two, the Green kernel is invariant under conformal changes of a metric on M.

2.3. **The Dirichlet-to-Neumann operator and its symbol.** As follows from standard theory [8], see also a discussion in [7], the Dirichlet problem

$$\Delta^E u = 0, \qquad u|_{\partial M} = s, \tag{2.5}$$

has a unique solution for any smooth section s of $E|_{\partial M}$. For a given open set $\Sigma \subset \partial M$ the *Dirichlet-to-Neumann operator* $\Lambda_\Sigma : \mathscr{D}(E|_\Sigma) \to \mathscr{D}(E|_\Sigma)$ is defined on smooth compactly supported in Σ sections by the formula

$$\Lambda_\Sigma(s) = \nabla^E_{\partial/\partial \mathbf{n}} u,$$

where u is the solution to Dirichlet problem (2.5), and $\mathbf{n}$ is the outward unit normal vector to ∂M. Note that if $\Sigma = \partial M$, and Λ is the corresponding Dirichlet-to-Neumann operator, then for any $\Sigma \subset \partial M$ the operator Λ_Σ is precisely the restriction of Λ to the space of sections whose supports lie in Σ.

Following [23], it is straightforward to show, see [7], that the Dirichlet-to-Neumann operator Λ defined above is an elliptic pseudodifferential operator of first order. In more detail, in local coordinated on ∂M that trivialise $E|_{\partial M}$ the operator Λ takes the form of a pseudodifferential operator whose symbol $p(x,\xi)$, where $x \in M$, $\xi \in T^*_x M$, has an asymptotic expansion

$$p(x,\xi) \sim \sum_{j \leqslant 1} p_j(x,\xi),$$

where $p_j(x,\xi)$ is a homogeneous matrix-valued function of degree j in ξ, and $j \in \mathbb{Z}$. The asymptotic expansion above is understood in the sense that for any integer $N \geqslant 0$ the matrix-valued function $p(x,\xi) - \sum_{j=-N}^{1} p_j(x,\xi)$ lies in the class $S^{1-N}_{1,0}$, see [22, 24] for details.

Recall that any local coordinates $(x^1, \dots, x^{n-1})$ on Σ can be extended to the so-called *boundary normal coordinates* $(x^1, \dots, x^{n-1}, x^n)$ on M. The latter are defined by the conditions that the equation $x^n = 0$ describes the boundary ∂M and each curve $t \mapsto (x^1, \dots, x^{n-1}, t)$ is a unit speed geodesic orthogonal to the boundary. Similarly, any local frame $\{s_l\}$ for $E|_{\partial M}$ can be extended to the so-called *boundary normal frame* $\{\bar{s}_l\}$ for E such that

$$\nabla^E_{\partial/\partial x^n} \bar{s}_l = 0 \quad \text{and} \quad \bar{s}_l|_\Sigma = s_l,$$

where $l = 1, \dots, \operatorname{rank} E$.

For the sequel we need the following result that gives a more precise information about the coefficients $p_j(x,\xi)$ and their derivatives. Its proof follows the general strategy used in [17], and can be found in [6]. In a slightly different notation it also appears in [3].

Proposition 2.2. *Let (M,g,E) be a vector bundle over a compact Riemannian manifold with boundary, equipped with a connection ∇^E. Suppose that $n=\dim M \geqslant 3$, and let $\Sigma \subset \partial M$ be an open set that trivialises $E|_{\partial M}$. For local coordinates $(x^1,\dots,x^{n-1})$ on Σ and a local frame $\{s_\ell\}$ of E over Σ let $p_j(x,\xi)$, where $j \leqslant 1$, be a full symbol of the Dirichlet-to-Neumann operator Λ. Then the Taylor series of the metric tensor g and the connection matrix ω of ∇^E in the boundary normal coordinates $(x^1,\dots,x^{n-1},x^n)$ and the boundary normal frame $\{\bar{s}_\ell\}$ at a point x on the boundary are determined by explicit formulae in terms of the matrix-valued functions $p_j(x,\xi)$, where $j \leqslant 1$, and their derivatives at x.*

Proposition 2.2 is an important initial ingredient in the proof of our main result, Theorem 1.1. In particular, it says that the symbol of the Dirichlet-to-Neumann operator determines the metric and connection on the boundary. Moreover, if these data are real-analytic, then the symbol determines them in a neighbourhood of the boundary. It is worth noting that although, in general, the connection Laplacian has a zero order term in a fixed frame, it does not have a natural conformal gauge invariance property, similar to the one for the conformal Laplacian, see [4, 16]. This can be seen from the explicit formula for the zero term in boundary normal coordinates and boundary normal frame in [6]. In particular, the phenomenon described in [4, Theorem 8.4] does not occur for the connection Laplacian, and the statement of Proposition 2.2 is indeed natural.

In dimension two, the formulae in [3, 6] do not allow us to determine the normal derivatives of the metric tensor. However, the Taylor series of the connection matrix can still be recovered from the Dirichlet-to-Neumann operator, assuming that the metric tensor together with all its derivatives is known. More precisely, the following version of Proposition 2.2 holds, see [6].

Proposition 2.3. *Let (M,g,E) be a vector bundle over a compact Riemannian surface with boundary, equipped with a connection ∇^E, and let $\Sigma \subset \partial M$ be an open set that trivialises $E|_{\partial M}$. For a local coordinate x^1 on Σ and a local frame $\{s_\ell\}$ of E over Σ let $p_j(x,\xi)$, where $j \leqslant 1$, be a full symbol of the Dirichlet-to-Neumann operator Λ. Then the Taylor series of the connection matrix ω of ∇^E in the boundary normal coordinates (x^1,x^2) and the boundary normal frame $\{\bar{s}_\ell\}$ at a point x on the boundary are determined by explicit formulae in terms of the matrix-valued functions $p_j(x,\xi)$, where $j \leqslant 1$, the metric tensor g, and their derivatives at x.*

Now let (M_i,E_i,∇^i), where $i=1,2$, be two vector bundles over compact Riemannian manifolds with boundary (M_i,g_i), equipped with connections ∇^i. Suppose that these data are *gauge equivalent* in the following sense: there exists a vector bundle isomorphism $\Phi : E_1 \to E_2$ that covers an isometry $\Psi : (M_1,g_1) \to (M_2,g_2)$ such that $\Phi^*\nabla^2 = \nabla^1$. Then, using relation (2.2) it is straightforward to conclude that the corresponding Dirichlet-to-Neumann operators Λ_1 and Λ_2 intertwine, that is

$$\phi \circ \Lambda_1(s) \circ \psi^{-1} = \Lambda_2(\phi \circ s \circ \psi^{-1})$$

for any smooth section s of $E_1|_{\partial M_1}$, where $\psi = \Psi|_{\partial M_1}$ and $\phi = \Phi|_{\partial M_1}$.

Analysing the formulae for the symbols of Λ_1 and Λ_2, see [7], one can show that the converse holds on the boundary.

Proposition 2.4. *Let (M_i,g_i,E_i,∇^i), where $i=1,2$, be two Euclidean smooth vector bundles defined over connected compact Riemannian manifolds with boundary. Suppose that for some*

open subsets $\Sigma_i \subset \partial M_i$ there exists a vector bundle isomorphism $\phi : E_1|_{\Sigma_1} \to E_2|_{\Sigma_2}$ that intertwines with the corresponding Dirichlet-to-Neumann operators Λ_{Σ_1} and Λ_{Σ_2}. Then the isomorphism ϕ is a gauge equivalence, $\phi^*\nabla^2 = \nabla^1$, and covers an isometry $\psi : (\Sigma_1, g_1) \to (\Sigma_2, g_2)$.

The statement above can be viewed as the boundary version of Theorem 1.1. Note that there is no restriction on dimension of M in Proposition 2.4. Similar results continue to hold for the Dirichlet-to-Neumann operator associated with the connection Laplacian with a symmetric real-analytic potential. We refer to [7] for precise statements and a related discussion.

3. Immersions by Green kernels

3.1. **The construction of immersions.** Let E be a Euclidean real-analytic vector bundle over a connected compact real-analytic manifold M with boundary, equipped with a real-analytic connection ∇^E. In this section we assume that $n = \dim M \geqslant 3$, and describe how one can reconstruct E from the Dirichlet-to-Neumann operator Λ_Σ, where $\Sigma \subset \partial M$ is an open subset. Our argument develops the ideas from [14] to the setting of vector bundles, and we aim to make the related technical details to be rather explicit.

Fix a point $p \in \Sigma$. First, note that we may consider M as a subset of a larger real-analytic manifold $\tilde{M}$. More precisely, choosing boundary normal coordinates $(x^1, \dots, x^n)$ around p, we may identify a neighbourhood of p in M with the Euclidean half-ball

$$B^+(0,\rho) = \{(x^1, \dots, x^n) \in B^n(0,\rho) : x^n \geqslant 0\},$$

where $B^n(0,\rho)$ is an open Euclidean ball of radius $\rho > 0$ in $\mathbb{R}^n$. Then, as the manifold $\tilde{M}$ one can take the manifold obtained by gluing $B^n(0,\rho)$ to M such that points in $B^+(0,\rho)$ are identified with points in M by means of boundary normal coordinates. Below by U we denote the open set $\tilde{M} \backslash \bar{M}$. For the sequel it is important to note that the set U does not really depend on M. In other words, if there are two manifolds M_i of the same dimension and two points $p_i \in \Sigma_i \subset \partial M_i$, where $i = 1, 2$, then choosing a sufficiently small $\rho > 0$ we may assume that the sets $\tilde{M}_1 \backslash \bar{M}_1$ and $\tilde{M}_2 \backslash \bar{M}_2$ coincide.

It is straightforward to see that a real-analytic metric g on M extends to a real-analytic metric $\tilde{g}$ on $\tilde{M}$, if ρ is sufficiently small. Similarly, the above construction shows that a real-analytic vector bundle E over M extends to a real-analytic vector bundle $\tilde{E}$ over $\tilde{M}$ such that $\tilde{E}\big|_U$ is trivial. Making $\rho > 0$ smaller if necessary, we may also assume that a real-analytic Euclidean structure and a real-analytic connection ∇ on E extend to an inner product and a connection $\tilde{\nabla}$ on $\tilde{E}$. If the former were compatible on E, then by unique continuation so are the latter on $\tilde{E}$. Below by $\tilde{G}$ we denote the Dirichlet Green kernel on $\tilde{E}$.

Denote by $\mathscr{E}$ the trivial vector bundle $\tilde{E}\big|_U$. For a given integer $\ell < 2 - n/2$, where n is the dimension of M, we define the map $\mathscr{G} : \tilde{E} \to W^{\ell,2}(\mathscr{E})$ by setting

$$\tilde{E}_x \ni v_x \longmapsto \langle v_x, \tilde{G}(x,\cdot)\rangle_{x,\tilde{E}} \in W^{\ell,2}(\mathscr{E}), \tag{3.1}$$

where $x \in \tilde{M}$. The condition on ℓ guarantees that the space $W_0^{2-\ell,2}(\mathscr{E})$ embeds into the Hölder space $C^{0,\alpha}(\mathscr{E})$ for some $\alpha > 0$, and hence, the dual space $W^{\ell-2,2}(\mathscr{E})$ contains the delta function. Then, by elliptic regularity we conclude that $\tilde{G}(x,\cdot)$ lies in $W^{\ell,2}(\mathscr{E})$. In addition, it is straightforward to show that

$$\big|\tilde{G}(x_1,\cdot) - \tilde{G}(x_2,\cdot)\big|_{W^{\ell,2}} \leqslant C_1 \operatorname{dist}(x_1,x_2)^\alpha$$

for some constant $C_1 > 0$, where for simplicity we may assume that the points x_1 and $x_2 \in \tilde{M}$ lie in the same chart. Thus, we conclude that the map $\mathscr{G}$ is continuous. Similarly, we have the following statement.

Lemma 3.1. *Let ℓ be an integer such that $\ell < 1 - n/2$. Then the map $\mathscr{G} : \tilde{E} \to W^{\ell,2}(\mathscr{E})$ defined by* (3.1) *is C^1-smooth.*

For the sake of completeness we prove Lemma 3.1 in Section 5. It allows us to study the map $\mathscr{G}$ from a viewpoint of differential geometry. As we shall see below, the map $\mathscr{G}$ is a linear embedding on each fibre $\tilde{E}_x$ for $x \notin \partial\tilde{M}$, and collapses the fibre to the origin for $x \in \partial\tilde{M}$. Further, it maps the base manifold $\tilde{M}$, viewed as the image of the zero section, to zero in $W^{\ell,2}(\mathscr{E})$. To avoid these degeneracies we often restrict it to the open set $\tilde{E}^0$ obtained by considering $\tilde{E}$ on the interior of $\tilde{M}$ and removing the zero section. The following statement shows that the map $\mathscr{G}$ is well-behaved on $\tilde{E}^0$.

Lemma 3.2. *Let ℓ be an integer such that $\ell < 1 - n/2$. Then the map $\mathscr{G} : \tilde{E} \to W^{\ell,2}(\mathscr{E})$ defined by* (3.1) *is a linear embedding on each fibre $\tilde{E}_x$, where $x \notin \partial\tilde{M}$. Moreover, it is an injective immersion on the set $\tilde{E}^0$, obtained by removing the image of the zero section from $\tilde{E}$ over the interior of $\tilde{M}$.*

Note that the image of the total space $\tilde{E}$ under $\mathscr{G}$ can be viewed as the cone whose link is the image of the subset $S_1\tilde{E}$ that is formed by vectors of unit length. Then the image of $\mathscr{G}(\tilde{E}^0)$ is precisely the set obtained by removing the origin from this cone. By Lemmas 3.1 and 3.2 it is straightforward to see that the set $\mathscr{G}(\tilde{E}^0)$ is a C^1-smooth submanifold of $W^{\ell,2}(\mathscr{E})$. The main idea behind the proof of Theorem 1.1 is to recover the topology and geometry of $\tilde{E}$ from this image.

We end this discussion with a lemma that describes another property of the image of $\mathscr{G}$.

Lemma 3.3. *For given two distinct points q_1 and q_2 in the interior of $\tilde{M}$ let $\mathscr{W}$ be the direct sum $\mathscr{G}((\tilde{E})_{q_1}) \oplus \mathscr{G}((\tilde{E})_{q_2})$, viewed as a subspace of $W^{\ell,2}(\mathscr{E})$. Suppose that for some point $x \in \tilde{M}$ the intersection $\mathscr{G}((\tilde{E})_x) \cap \mathscr{W}$ is non-trivial. Then the point x has to coincide with one of the points q_1 or q_2.*

Proofs of Lemmas 3.2 and 3.3 appear in Section 5. We continue with a discussion of the main result, Theorem 1.1.

3.2. **The main result.** Now let E_i be two real-analytic vector bundles over real-analytic manifolds M_i, where $i = 1, 2$, and suppose that for some open sets $\Sigma_i \subset \partial M_i$ there exists a vector bundle isomorphism $\phi : E_1|_{\Sigma_1} \to E_2|_{\Sigma_2}$ that intertwines with the Dirichlet-to-Neumann operators Λ_{Σ_1} and Λ_{Σ_2}. Suppose that ϕ covers a diffeomorphism $\psi : \Sigma_1 \to \Sigma_2$, that is $\pi_2 \circ \phi = \psi \circ \pi_1$. For a fixed point $p_1 \in \Sigma_1$ we set $p_2 = \psi(p_1)$, and choose local coordinates on the Σ_i's around these points that are related by ψ. Note that by Proposition 2.2 the metrics g_i coincide in such coordinates. Thus, making the Σ_i's smaller if necessary, we see that the map $\psi : \Sigma_1 \to \Sigma_2$ is an isometry. Since the metrics are real-analytic, by Proposition 2.2 we also conclude that their extensions $\tilde{g}_i$ coincide in neighbourhoods of the points p_i in $\tilde{M}_i$. In other words, the isometry $\psi : \Sigma_1 \to \Sigma_2$ extends to a real-analytic isometry $\Psi : W_1 \to W_2$, defined by identifying boundary normal coordinates, where W_i is a neighbourhood of the point p_i in $\tilde{M}_i$. In the sequel, we also identify the sets $W_1 \backslash \bar{M}_1$ and $W_2 \backslash \bar{M}_2$, and denote them by U.

Similarly, choosing orthonormal frames related by ϕ, we may identify the trivialisations of $E_1|_{\Sigma_1}$ and $E_2|_{\Sigma_2}$ around the points p_1 and $p_2 = \psi(p_1)$. They extend to trivial vector bundles, which we may assume are defined over W_1 and W_2. Following the discussion above, we obtain vector bundles $\tilde{E}_1$ and $\tilde{E}_2$, defined over $\tilde{M}_1$ and $\tilde{M}_2$ respectively, and equipped with inner products that are extensions of the original ones on E_1 and E_2. Making W_1 and W_2 smaller, if

necessary, we may also assume that boundary normal frames of $\tilde{E}_1$ and $\tilde{E}_2$ are defined on W_1 and W_2 respectively. Then, the isomorphism ϕ extends to the Euclidean isomorphism

$$\Phi : \tilde{E}_1\big|_{W_1} \to \tilde{E}_2\big|_{W_2},$$

defined by identifying the corresponding boundary normal frames. Note that Φ covers the isometry $\Psi : W_1 \to W_2$. By Proposition 2.2 the real-analytic connection matrices of ∇^1 and ∇^2 coincide in such frames, and we conclude that the isomorphism Φ is a gauge equivalence, that is $\Phi^*\tilde{\nabla}^2 = \tilde{\nabla}^1$. We continue to use the notation $\mathscr{E}$ for the vector bundles $\tilde{E}_i\big|_U$.

Below by $\tilde{E}_i^0$ we denote the vector bundles $\tilde{E}_i$ with removed zero sections over the interiors of $\tilde{M}_i$, where $i = 1,2$. By the hypotheses of Theorem 1.1 the dimensions of the base manifolds M_i coincide; we denote this integer by n. Theorem 1.1 is a consequence of the following statement.

Theorem 3.4. *Under the hypotheses of Theorem 1.1, consider the maps $\mathscr{G}_i : \tilde{E}_i \to W^{\ell,2}(\mathscr{E})$ defined by* (3.1)*, where $i = 1,2$, and ℓ is an integer such that $\ell < 1 - n/2$. Suppose that the vector bundle isomorphism $\Phi : \mathscr{E} \to \mathscr{E}$, described above, intertwines with the $\mathscr{G}_i$'s, that is*

$$\mathscr{G}_2 \circ \Phi = \Phi \circ \mathscr{G}_1 \qquad \text{on } \mathscr{E}. \tag{3.2}$$

Then the images $\mathscr{G}_2(\tilde{E}_2^0)$ and $\Phi \circ \mathscr{G}_1(\tilde{E}_1^0)$ coincide as subsets in $W^{\ell,2}(\mathscr{E})$, and the map $\mathscr{G}_2^{-1} \circ \Phi \circ \mathscr{G}_1 : \tilde{E}_1^0 \to \tilde{E}_2^0$ extends to a real-analytic vector bundle isomorphism $J : \tilde{E}_1 \to \tilde{E}_2$ that covers an isometry $j : \tilde{M}_1 \to \tilde{M}_2$ such that $J^*\tilde{\nabla}^2 = \tilde{\nabla}^1$.

Now we show how Theorem 3.4 implies Theorems 1.1.

Proof of Theorem 1.1. First, since the vector bundle isomorphism $\phi : E_1|_{\Sigma_1} \to E_2|_{\Sigma_2}$ preserves inner products on E_1 and E_2, then so does its extension $\Phi : \mathscr{E} \to \mathscr{E}$. This statement follows directly from the definition of Φ as an isomorphism that identifies boundary normal frames. Now we claim that the conclusion of Theorem 3.4 implies Theorem 1.1. Indeed, by relation (3.2), we see that the vector bundle isomorphism $J : \tilde{E}_1 \to \tilde{E}_2$ coincides with Φ on the set $\tilde{E}_1\big|U$, and the isometry $j : \tilde{M}_1 \to \tilde{M}_2$ coincides with Ψ on U. Thus, they are genuine extensions of the isomorphism ϕ and the isometry ψ from the boundary, and satisfy the conclusions of Theorem 1.1. Since Φ preserves the inner products, we conclude that the products $\langle\cdot,\cdot\rangle_{\tilde{E}_1}$ and $J^*\langle\cdot,\cdot\rangle_{\tilde{E}_2}$ coincide on $\tilde{E}_1\big|U$, and hence, by unique continuation coincide everywhere on $\tilde{E}_1$. Thus, the isomorphism J preserves inner products, and its restriction to E_1 satisfies all conclusions of Theorem 1.1.

For a proof of Theorem 1.1 we need to prove relation (3.2), that is the vector bundle isomorhism $\Phi : \mathscr{E} \to \mathscr{E}$ intertwines with the immersions $\mathscr{G}_i$'s. Since Φ preserves Euclidean structures, for the latter it is sufficient to show that

$$\tilde{G}_2(\Psi(x),\Psi(y)) = \Phi^{\boxtimes}\tilde{G}_1(x,y) \qquad \text{for all } (x,y) \in U \times U.$$

Choosing coordinates on W_1 and W_2 related by Ψ, we may assume that $\Psi : W_1 \to W_2$ is the identity. Similarly, choosing local trivialisations of the $\tilde{E}_i\big|_{W_i} \simeq \mathscr{E}$ related by Φ, we assume that so is Φ. Thus, it remains to show that the Green matrices $\tilde{G}_1$ and $\tilde{G}_2$, viewed as sections of the trivial bundle $\mathscr{E} \boxtimes \mathscr{E}$, coincide. For classical Green functions, that is when the rank of $\mathscr{E}$ equals one, this statement is well known, see [14, Lemma 2.1]. It is a consequence of standard regularity theory together with uniqueness of Dirichlet Green functions. Below we outline a version of this argument in our setting.

First, since under our assumptions the isomorphism Φ is the identity on $\mathscr{E}$, the hypothesis in Theorem 1.1 means that the Dirichlet-to-Neumann operators Λ_1 and Λ_2 restricted to sections

supported in $W_1 \cap \partial M_1$ and $W_2 \cap \partial M_2$ respectively, coincide. Pick a point $x \in U$, and for a non-zero vector v_x in the fibre $\mathscr{E}_x$ consider a solution s to the Dirichlet problem

$$\Delta^{E_2} s = 0, \qquad s|_{\partial M_2} = \langle v_x, \tilde{G}_1(x,\cdot)\rangle_{x,\tilde{E}_1},$$

on M_2. We define a continuous section $\tilde{s}$ of $\tilde{E}_2$ away from x by extending s as $\langle v_x, \tilde{G}_1(x,\cdot)\rangle_{x,\tilde{E}_1}$ on $U\backslash\{x\}$. Note that the section $\langle v_x, \tilde{G}_1(x,\cdot)\rangle_{x,\tilde{E}_1}$ solves the Dirichlet problem

$$\Delta^{E_1} s = 0, \qquad s|_{\partial M_1} = \langle v_x, \tilde{G}_1(x,\cdot)\rangle_{x,\tilde{E}_1},$$

and since the Dirichlet-to-Neumann operators coincide, we conclude that so do the normal derivatives of s and $\langle v_x, \tilde{G}_1(x,\cdot)\rangle_{x,\tilde{E}_1}$ on the boundary $W_2 \cap \partial M_2$. Thus, the section $\tilde{s}$ is C^1-smooth, and the standard application of Green's formulae shows that $\tilde{s}$ is weakly harmonic on $\tilde{M}_2\backslash\{x\}$, and hence, is smooth. Since a vector $v_x \in \mathscr{E}_x$ is arbitrary, and the Euclidean structures agree, this construction yields a smooth section $H(x,y) \in (\tilde{E}_2)_x \otimes (\tilde{E}_2)_y$ such that:

- $\Delta_y^{E_2} H(x,y) = 0$ for $y \in M_2$;
- $H(x,y) = \tilde{G}_1(x,y)$ for $y \in U$, $y \neq x$;
- $H(x,y) = 0$ for $y \in \partial\tilde{M}_2$.

In particular, we see that $\Delta_y^{E_2} H(x,\cdot) = \delta_x$ on $\tilde{M}_2$, and the standard argument used to prove uniqueness of the Dirichlet Green kernel shows that $H(x,y)$ coincides with the Dirichlet Green kernel $\tilde{G}_2(x,y)$ for all $y \in \tilde{M}_2$. Thus, the Green matrices $\tilde{G}_1(x,\cdot)$ and $\tilde{G}_2(x,\cdot)$ indeed coincide on the set $U\backslash\{x\}$, and we are done. □

3.3. **Proof of Theorem 3.4.** We start with outlining the general strategy of a proof. Let $B_1 \subset \tilde{M}_1$ be the largest connected open set containing the fixed point $p_1 \in \Sigma_1$ and such that for any $x \in B_1$ there exists a unique $j(x) \in \tilde{M}_2$ such that the images of fibres $\Phi \circ \mathscr{G}_1((\tilde{E}_1)_x)$ and $\mathscr{G}_2((\tilde{E}_2)_{j(x)})$ coincide and the operator

$$J_x = \mathscr{G}_2^{-1} \circ \Phi \circ \mathscr{G}_1 : (\tilde{E}_1)_x \longrightarrow (\tilde{E}_2)_{j(x)} \tag{3.3}$$

is an isometry with respect to the inner products on the fibres. Note that if the subspaces $\Phi \circ \mathscr{G}_1((\tilde{E}_1)_x)$ and $\mathscr{G}_2((\tilde{E}_2)_{j(x)})$ coincide, by Lemma 3.2 the map J_x, defined in (3.3), is automatically an isomorphism of the fibres, and defines a fibre preserving map $J : \tilde{E}_1\big|_{B_1} \to \tilde{E}_2$.

First, we claim that the set B_1 contains the neighbourhood W_1 of p_1 constructed above. Indeed, since Φ intertwines with the maps $\mathscr{G}_1$ and $\mathscr{G}_2$ on $\mathscr{E}$, we have

$$\Phi_y \langle v_x, \tilde{G}_1(x,y)\rangle_x = \langle \Phi_{\Psi(x)} v_{\Psi(x)}, \tilde{G}_2(\Psi(x),\Psi(y))\rangle_x \tag{3.4}$$

for all $x,y \in U$, and $v_x \in (\tilde{E}_1)_x$. Choosing a real-analytic non-zero section v on W_1, since both sides in the relation above are real-analytic, we conclude that this relation continues to hold for all $x \in W_1$, $y \in U$. Since v_x may take arbitrary values, it is straightforward to see that for any $x \in W_1$ we may choose $\Psi(x)$ as the point $j(x)$ in the definition of the set B_1. Indeed, relation (3.4) implies that for any $x \in W_1$ the operator J_x coincides with $\Phi : (\tilde{E}_1)_x \to (\tilde{E}_2)_{\Psi(x)}$, which is an isometry by its own definition. By Lemma 3.2 it is a unique point that satisfies this condition. Thus, the set W_1 indeed lies in B_1.

Our main aim is to show that the set B_1 coincides with $\tilde{M}_1$. Once this statement is proved, we shall show that the map $J : \tilde{E}_1 \to \tilde{E}_2$, defined on each fibre by relation (3.3), is a vector bundle isomorphism that satisfies the conclusions of the theorem.

Suppose the contrary, $B_1 \neq \tilde{M}_1$. Then there exists a point $x_1 \in \partial B_1$ that lies in the interior of $\tilde{M}_1$, that is $x_1 \notin \partial\tilde{M}_1$. Since $W_1 \subset B_1$, the point x_1 lies in the complement $\tilde{M}_1\backslash W_1$, and in particular, we see that $x_1 \notin \bar{U}$.

Step 1. First, we claim that the map J can be extended to the fibre $(\tilde{E}_1)_{x_1}$ over x_1.

Lemma 3.5. *Let $x_1 \in \partial B_1$ be a point such that $x_1 \notin \partial \tilde{M}_1$. Then there exists a unique point x_2 in the interior of $\tilde{M}_2$ such that the images of fibres $\Phi \circ \mathscr{G}_1((\tilde{E}_1)_{x_1})$ and $\mathscr{G}_2((\tilde{E}_2)_{x_2})$ coincide, and the corresponding operator J_{x_1} is an isometry. Moreover, for any non-zero vector $v_{x_1} \in (\tilde{E}_1)_{x_1}$ there exists a unique non-zero vector $w_{x_2} \in (\tilde{E}_2)_{x_2}$ such that*

$$\Phi \circ \mathscr{G}_1(v_{x_1}) = \mathscr{G}_2(w_{x_2}), \qquad |v_{x_1}|_{\tilde{E}_1} = |w_{x_2}|_{\tilde{E}_2},$$

and for any converging sequence $v_{p_k} \to v_{x_1}$, where $p_k \to x_1$, we have $J(v_{p_k}) \to w_{x_2}$ as $k \to +\infty$.

Proof. Let $p_k \in B_1$ be a sequence of points that converges to the point $x_1 \in \partial B_1$, and q_k the corresponding sequence of points such that the images of fibres $\Phi \circ \mathscr{G}_1((\tilde{E}_1)_{p_k})$ and $\mathscr{G}_2((\tilde{E}_2)_{q_k})$ coincide. Since $\tilde{M}_2$ is compact, then choosing a subsequence, which we denote by the same symbol q_k, we may assume that $q_k \to q_0 \in \tilde{M}_2$ as $k \to +\infty$. For a non-zero vector $v_{x_1} \in (\tilde{E}_1)_{x_1}$ pick a sequence $v_k \in (\tilde{E}_1)_{p_k}$ that converges to v_{x_1}, and let $w_k \in (\tilde{E}_2)_{q_k}$ be the corresponding sequence such that

$$\Phi \circ \mathscr{G}_1(v_k) = \mathscr{G}_2(w_k) \quad \text{and} \quad |v_k|_{\tilde{E}_1} = |w_k|_{\tilde{E}_2}.$$

Since the sequence w_k is bounded, we may assume, again after choosing a subsequence, that w_k converges to some vector $w_{q_0} \in (\tilde{E}_2)_{q_0}$ as $k \to +\infty$. It is straightforward to see that the norm of w_{q_0} equals the one of v_{x_1}. Now for a proof of the lemma it remains to show that $q_0 \notin \partial \tilde{M}_2$. If the latter holds, then we may take q_0 as x_2, and the statement follows directly by continuity of $\Phi \circ \mathscr{G}_1$ and $\mathscr{G}_2$. The uniqueness of the point x_2 and the vector w_{x_2} is a consequence of Lemma 3.2.

Suppose the contrary, $q_0 \in \partial \tilde{M}_2$. Then by continuity we obtain

$$\Phi \circ \mathscr{G}_1(v_{x_1}) = \lim \Phi \circ \mathscr{G}_1(v_k) = \lim \mathscr{G}_2(w_k) = \mathscr{G}_2(w_{q_0}).$$

Since the point x_1 lies in the interior of $\tilde{M}_1$, by Lemma 3.2 the left-hand side above is non-zero, while since $q_0 \in \partial \tilde{M}_2$, the right-hand side vanishes. Thus, we arrive at a contradiction. □

Step 2. Now we analyse the images $\mathscr{R}_i$ of the maps $\mathscr{G}_i$ in $W^{\ell,2}(\mathscr{E})$, where $i = 1, 2$. Take a non-zero vector $v_{x_1} \in (\tilde{E}_1)_{x_1}$, and let $x_2 \in \tilde{M}_2$ and $w_{x_2} \in (\tilde{E}_2)_{x_2}$ be a point and a vector respectively that satisfy the conclusions of Lemma 3.5. In particular, the vectors $\Phi \circ \mathscr{G}_1(v_{x_1})$ and $\mathscr{G}_2(w_{x_2})$ coincide in $W^{\ell,2}(\mathscr{E})$, and we denote this value by u. By Lemma 3.2 we see that locally the sets $\Phi(\mathscr{R}_1)$ and $\mathscr{R}_2$ are submanifolds in $W^{\ell,2}(\mathscr{E})$, whose tangent spaces can be viewed as the images of the differentials $D(\Phi \circ \mathscr{G}_1)$ and $D\mathscr{G}_2$. Combining this with Lemma 3.5, we conclude that the tangent spaces $T_u\Phi(\mathscr{R}_1)$ and $T_u\mathscr{R}_2$ coincide as subspaces in $W^{\ell,2}(\mathscr{E})$. Using the inverse function theorem we may view $\Phi(\mathscr{R}_1)$ and $\mathscr{R}_2$ locally near u as graphs of smooth functions defined on an open subset in

$$\mathscr{V} = T_u\Phi(\mathscr{R}_1) = T_u\mathscr{R}_2.$$

In more detail, let Π be the orthogonal projection onto $\mathscr{V}$ in $W^{\ell,2}(\mathscr{E})$, and consider the map

$$\Pi \circ \mathscr{G}_2 : \tilde{E}_2 \to \mathscr{V}, \qquad v_x \longmapsto \Pi(\langle v_x, \tilde{G}_2(x, \cdot)\rangle).$$

By Lemma 3.2, its differential is an isomorphism near u, and hence, there exists a C^1-smooth inverse map $H_2 : \mathscr{O} \to \tilde{E}_2$, defined in the neighbourhood $\mathscr{O}$ of $\Pi(u)$ in $\mathscr{V}$. Then, it is straightforward to see that near u the image $\mathscr{R}_2$ is the graph of the map

$$F_2 : \mathscr{O} \to \mathscr{V}^\perp, \qquad \upsilon \longmapsto \mathscr{G}_2(H_2(\upsilon)) - \upsilon,$$

where $\mathscr{V}^\perp$ is the orthogonal complement of $\mathscr{V}$ in $W^{\ell,2}(\mathscr{E})$. Similarly, one shows that there exists a C^1-smooth map $H_1 : \mathscr{O} \to \tilde{E}_1$, which we may assume is defined on the same set $\mathscr{O}$, such that near u the image $\Phi(\mathscr{R}_1)$ is the graph of the map

$$F_1 : \mathscr{O} \to \mathscr{V}^\perp, \qquad \upsilon \longmapsto \Phi \circ \mathscr{G}_1(H_1(\upsilon)) - \upsilon.$$

From this construction we see that the vectors $v_{x_1} \in (\tilde{E}_1)_{x_1}$ and $w_{x_2} \in (\tilde{E}_2)_{x_2}$ are precisely the images $H_1 \circ \Pi(u)$ and $H_2 \circ \Pi(u)$, and moreover, the isomorphism J has the form $H_2 \circ H_1^{-1}$ on the open subset

$$\Omega_1 = H_1(\mathscr{O}) \cap \tilde{\pi}_1^{-1}(B_1) \subset \tilde{E}_1, \tag{3.5}$$

where $\tilde{\pi}_1 : \tilde{E}_1 \to \tilde{M}_1$ is the vector bundle projection.

For the sequel we need the following lemma.

Lemma 3.6. *The maps $H_i : \mathscr{O} \to \tilde{E}_i$ constructed above, where $i = 1,2$, are real-analytic in a neighbourhood of $\Pi(u)$ in $\mathscr{V}$. In particular, there exists a neighbourhood of v_{x_1} in $\tilde{E}_1$ such that the map $H_2 \circ H_1^{-1}$ is real-analytic on it.*

Proof. Choosing an orthonormal basis (φ_i) in $\mathscr{V}$, where $i = 1, \dots, m$, we may identify the vector space $\mathscr{V}$ with $\mathbb{R}^m$. First, we claim that the map $\Pi \circ \mathscr{G}_2 : \tilde{E}_2 \to \mathscr{V} \simeq \mathbb{R}^m$ is real-analytic in a neighbourhood of w_{x_2}, that is the coordinate functions, given by products

$$(\Pi \circ \mathscr{G}_2, \varphi_i)_{\ell,2} = (\mathscr{G}_2, \varphi_i)_{\ell,2}, \qquad \text{where} \quad i = 1, \dots, m,$$

and $(\cdot,\cdot)_{\ell,2}$ stands for the scalar product in $W^{\ell,2}(\mathscr{E})$, are real-analytic. By definition of $\mathscr{G}_2$ for the latter it is sufficient to show that the sections

$$x \longmapsto (\tilde{G}_2(x,\cdot), \varphi_i)_{\ell,2} \in E_x \qquad \text{where} \quad i = 1, \dots, m,$$

are real-analytic in a neighbourhood of x_2. Let $f_i \in W_0^{-\ell,2}(\mathscr{E})$ be a vector dual to φ_i, that is such that $\varphi_i(s) = (s, f_i)_{-\ell,2}$ for any $s \in W_0^{-\ell,2}(\mathscr{E})$. Since the canonical map $f \mapsto (\cdot, f)_{-\ell,2}$ preserves scalar products, we conclude that

$$(\tilde{G}_2(x,\cdot), \varphi_i)_{\ell,2} = \int_U \langle \tilde{G}_2(x,y), f_i(y) \rangle_{y,\tilde{E}_2} dVol_g(y).$$

Recall that the point x_1 does not lie in the closure $\bar{U} \subset \tilde{M}_2$. Then, by properties of the Green kernel, it is straightforward to see that the integral on the right-hand side above defines a harmonic section in any neighbourhood of x_1 that is disjoint with U. As was discussed in Section 2, any harmonic section is real-analytic under our hypotheses, and we conclude that so is the integral above. Thus, the coordinate functions $(\mathscr{G}_2, \varphi_i)_{\ell,2}$ are real-analytic in a neighbourhood of x_1 for all $i = 1, \dots, m$. Further, we conclude that the map H_2, as the inverse map to $\Pi \circ \mathscr{G}_2$, is also real-analytic in a neighbourhood of $\Pi(u)$.

A similar argument shows that the maps $\Pi \circ \Phi \circ \mathscr{G}_1$ and H_1 are real-analytic as well. Hence, the map $H_2 \circ H_1^{-1}$ is real-analytic as the composition of real-analytic maps. □

Step 3. Now we claim that the images of $\Phi(\mathscr{R}_1)$ and $\mathscr{R}_2$ coincide around the point u. This is the consequence of the following lemma.

Lemma 3.7. *The maps $F_i : \mathscr{O} \to \mathscr{V}^\perp$ constructed above, where $i = 1,2$, coincide in a neighbourhood of $\Pi(u)$ in $\mathscr{V}$.*

Proof. Fix an orthonormal basis (φ_j) in $\mathscr{V}^\perp$, where $j = 1,2,\ldots,\infty$. For a proof of the lemma it is sufficient to show that the coordinate functions $(F_1,\varphi_j)_{\ell,2}$ and $(F_2,\varphi_j)_{\ell,2}$ coincide for all $j = 1,2,\ldots,\infty$, where $(\cdot,\cdot)_{\ell,2}$ is the scalar product in $W^{\ell,2}(\mathscr{E})$. Note that

$$(F_2,\varphi_j)_{\ell,2}(\upsilon) = (\mathscr{G}_2,\varphi_j)_{\ell,2}\circ H_2(\upsilon) - (\upsilon,\varphi_j)_{\ell,2} \tag{3.6}$$

for any $\upsilon \in \mathscr{O}$. The argument used in the proof of Lemma 3.6 shows that the function $(\mathscr{G}_2,\varphi_j)_{\ell,2}$ is real-analytic in some neighbourhood of v_{x_1}, and by Lemma 3.6 we also know that the map H_2 is real-analytic in a neighbourhood of $\Pi(u)$. Since the second term on the right-hand side of (3.6) is linear in υ, we conclude that the function $(F_2,\varphi_j)_{\ell,2}$ is real-analytic in a neighbourhood of $\Pi(u)$, which we may also denote by $\mathscr{O}$. This statement holds for all values $j = 1,2,\ldots,\infty$, with the same set $\mathscr{O}$.

Similarly, one shows that all functions

$$(F_1,\varphi_j)_{\ell,2}(\upsilon) = (\Phi\circ\mathscr{G}_1,\varphi_j)_{\ell,2}\circ H_1(\upsilon) - (\upsilon,\varphi_j)_{\ell,2} \tag{3.7}$$

are also real-analytic on the same set $\mathscr{O}$. Without loss of generality, we may assume that the open set $\mathscr{O}$ is connected. Now by the choice of the point x_1, we know that the maps $\Phi\circ\mathscr{G}_1$ and $\mathscr{G}_2\circ J$ coincide on an open subset $\Omega_1 \subset \tilde{E}_1$, defined in (3.5), whose closure contains v_{x_1}. Recall that the map J coincides with $H_2\circ H_1^{-1}$ on Ω_1, and hence, the maps $\Phi\circ\mathscr{G}_1\circ H_1$ and $\mathscr{G}_2\circ H_2$ coincide on $H_1^{-1}(\Omega_1) \subset \mathscr{O}$. Combining the latter with relations (3.6) and (3.7), we conclude that the real-analytic functions $(F_1,\varphi_j)_{\ell,2}$ and $(F_2,\varphi_j)_{\ell,2}$ coincide on an open subset $H_1^{-1}(\Omega_1) \subset \mathscr{O}$, and hence, by unique continuation coincide on $\mathscr{O}$ for all $j = 1,2,\ldots,\infty$. Thus, we are done. □

Due to conical structure of the images $\Phi(\mathscr{R}_1)$ and $\mathscr{R}_2$, from the above we conclude that there are conical neighbourhoods of u, that is neighbourhoods invariant under multiplication by $t > 0$, that coincide. In fact, as the following lemma shows, even a stronger statement holds.

Lemma 3.8. *There is a neighbourhood O_1 of the point $x_1 \in \tilde{M}_1$ such that for any $x \in O_1$ there exists $z \in \tilde{M}_2$ such that the images of fibres $\Phi\circ\mathscr{G}_1((\tilde{E}_1)_x)$ and $\mathscr{G}_2((\tilde{E}_2)_z)$ coincide.*

Proof. Choose a neighbourhood O_1 of $x_1 \in \tilde{M}_1$ such that $O_1 \subset \tilde{\pi}_1\circ H_1(\mathscr{O})$, where $\tilde{\pi}_1 : \tilde{E}_1 \to \tilde{M}_1$ is the vector bundle projection. We intend to show that for any $x \in O_1$ there exists $z \in \tilde{M}_2$ such that the image $\Phi\circ\mathscr{G}_1((\tilde{E}_1)_x)$ lies in $\mathscr{G}_2((\tilde{E}_2)_z)$. Since these images are vector spaces of the same dimension, the statement of the lemma follows immediately.

First, for a given point $x \in O_1$ and a vector $v_x \in (\tilde{E}_1)_x$, the considerations above show that the image $\Phi\circ\mathscr{G}_1(v_x)$ lies in the set $\mathscr{G}_2(CH_2(\mathscr{O}))$, where $CH_2(\mathscr{O})$ is a conical open set,

$$CH_2(\mathscr{O}) = \{tw \in \tilde{E}_2 :\ t \in \mathbb{R},\ t > 0,\ \text{and } w \in H_2(\mathscr{O})\}.$$

Thus, there exists $z \in \tilde{M}_2$ such that $\Phi\circ\mathscr{G}_1(v_x)$ lies in $\mathscr{G}_2((\tilde{E}_2)_z)$. We claim that for any $w_x \in (\tilde{E}_1)_x$ its image $\Phi\circ\mathscr{G}_1(w_x)$ lies in the same subspace $\mathscr{G}_2((\tilde{E}_2)_z)$.

Suppose the contrary, that is there exists a non-zero vector $w_x \in (\tilde{E}_1)_x$ such that its image $\Phi\circ\mathscr{G}_1(w_x)$ lies in $\mathscr{G}_2((\tilde{E}_2)_y)$, where $z \neq y$. Then, we see that

$$\Phi\circ\mathscr{G}_1(w_x - v_x) \in \mathscr{G}_2((\tilde{E}_2)_z) \oplus \mathscr{G}_2((\tilde{E}_2)_y).$$

Since the vectors v_x and w_x are different, arguing as above, we may find another point $q \in \tilde{M}_2$ such that $\Phi\circ\mathscr{G}_1(w_x - v_x)$ lies in $\mathscr{G}_2((\tilde{E}_2)_q)$. Now by Lemma 3.3 we conclude that the point q coincides with either z or y, and in each case it is straightforward to arrive at a contradiction. For example, if $q = z$, we immediately conclude that the vector

$$\Phi\circ\mathscr{G}_1(w_x) = \Phi\circ\mathscr{G}_1(w_x - v_x) + \Phi\circ\mathscr{G}_1(v_x)$$

lies in the image $\mathscr{G}_2((\tilde{E}_2)_z)$, and by Lemma 3.2, the points z and y coincide. □

The last lemma shows that the operator

$$J_x = \mathscr{G}_2^{-1} \circ \Phi \circ \mathscr{G}_1 : (\tilde{E}_1)_x \longrightarrow (\tilde{E}_2)_z$$

is defined for all x in a neighbourhood O_1 of the point x_1, and by the discussion in Step 2, has to coincide with the map $H_2 \circ H_1^{-1}$ in a neighbourhood $H_1(\mathscr{O}) \cap \tilde{\pi}_1^{-1}(O_1)$ of a given point $v_{x_1} \in (\tilde{E}_1)_{x_1}$. Without loss of generality we may assume that the last set is connected. Since the map $H_2 \circ H_1^{-1}$ is real-analytic, and is an isometry on the open subset

$$\Omega_1 \cap H_1(\mathscr{O}) \cap \tilde{\pi}_1^{-1}(O_1),$$

where Ω_1 is given by (3.5), by unique continuation we conclude that it is an isometry on $H_1(\mathscr{O}) \cap \tilde{\pi}_1^{-1}(O_1)$. Performing this argument for all points v_{x_1} from the unit sphere in the fibre $(\tilde{E}_1)_{x_1}$, we conclude that the operator J_x is an isometry for all x in a neighbourhood of x_1. This immediately yields a contradiction with the assumption $B_1 \neq \tilde{M}_1$, since the point $x_1 \in \tilde{M}_1$ has been chosen on the boundary ∂B_1. Thus, we conclude that the set B_1 coincides with the whole manifold $\tilde{M}_1$.

Step 4. Now we collect final conclusions. First, relation (3.3) defines the fibre preserving map $J : \tilde{E}_1 \to \tilde{E}_2$. By the argument in Step 2 we see that locally it can be written in the form $H_2 \circ H_1^{-1}$, and hence, is smooth, and by Lemma 3.6 is real-analytic. Since by definition it is an isomorphism on each fibre, we conclude that it is a real-analytic vector bundle isomorphism. In particular, it covers a real analytic map $j : \tilde{M}_1 \to \tilde{M}_2$. Since it is an isometry on each fibre, it is a Euclidean vector bundle isomorphism.

Note that the isomorphism J coincides with the isomorphism Φ on fibres over $W_1 \subset \tilde{M}_1$. Since the latter is a gauge equivalence, the connections $J^*\tilde{\nabla}^2$ and $\tilde{\nabla}^1$ coincide on W_1, and since they are real-analytic and $\tilde{M}_1$ is connected, they coincide on $\tilde{M}_1$. Similarly, the map $j : \tilde{M}_1 \to \tilde{M}_2$ coincides with the isometry Ψ on W_1, that is the real-analytic metrics $j^*\tilde{g}_2$ and $\tilde{g}_1$ coincide on W_1, and hence, they coincide everywhere on $\tilde{M}_1$. Thus, the vector bundle isomorphism J is indeed a gauge equivalence that covers an isometry. □

4. Vector bundles over surfaces

4.1. **Proof of Theorem 1.2.** First, note that we may choose a real-analytic metric $g \in c$. Indeed, viewing M as a domain in a closed surface, by the uniformisation theorem, it is straightforward to see that there is a constant Gauss curvature metric $g \in c$. In isothermal coordinates such a metric is real-analytic, and hence, is real-analytic with respect to the real-analytic atlas determined by c. Throughout the rest of the section we always use this metric on M; first, to define the Dirichlet-to-Neumann operators, and second, to describe the immersions by Green kernels. The construction below follows closely the lines in Section 3, but uses Proposition 2.3 instead of Proposition 2.2.

Fix a point $p \in \Sigma \subset \partial M$. We may view the surface M as a subset of the larger surface $\tilde{M}$, obtained by gluing the Euclidean disc $D(0,\rho)$ to M such that the points in the half-disc are identified with the points in M by means of boundary normal coordinates centred at p. Since these coordinates, constructed using our metric g, lie in the fixed real-analytic atlas, we conclude that this construction yields the real-analytic structure on $\tilde{M}$, extending the one on M. Making ρ smaller, if necessary, we may assume that the metric g extends to a real-analytic metric $\tilde{g}$ on $\tilde{M}$. Below by W we denote the neighbourhood of p in $\tilde{M}$ that corresponds to the disc $D(0,\rho)$.

Let E_1 and E_2 be two real-analytic vector bundles over M, and suppose that for a given open set $\Sigma \subset \partial M$ there exists a vector bundle isomorphism

$$\phi : E_1|_\Sigma \to E_2|_\Sigma$$

that covers the identity map of Σ and intertwines with the Dirichlet-to-Neumann operators $\Lambda_{1,\Sigma}$ and $\Lambda_{2,\Sigma}$. Making W smaller, if necessary, we can choose orthonormal frames defined over $W \cap \Sigma$ that are related by ϕ. They identify trivialisations of $E_1|_\Sigma$ and $E_2|_\Sigma$, and extending the latter as trivial vector bundles over W, we obtain vector bundles $\tilde{E}_1$ and $\tilde{E}_2$ over $\tilde{M}$. Further, identifying boundary normal frames of $\tilde{E}_1$ and $\tilde{E}_2$, we extend ϕ to a Euclidean isomorphism

$$\Phi : E_1|_W \to E_2|_W \,. \tag{4.1}$$

By Proposition 2.3 the real-analytic connection matrices of ∇^1 and ∇^2 coincide in such frames, and we conclude that Φ is a gauge equivalence.

As in the proof of Theorem 1.1, by U we denote the open set $\tilde{M}\backslash\bar{M}$, and by $\mathscr{E}$ the trivial vector bundle over U, which coincides with both restrictions of $\tilde{E}_1$ and $\tilde{E}_2$ to U. We use the same notation $\tilde{E}_i^0$ for the vector bundle $\tilde{E}_i$ with the removed zero section over the interior of $\tilde{M}$. Let $\tilde{G}_i$ be the Dirichlet Green kernel for the connection Laplacian on $\tilde{E}_i$ with respect to the metric $\tilde{g}$ on $\tilde{M}$. For each $i = 1,2$ we define the map $\mathscr{G}_i : \tilde{E}_i \to W^{-1,2}(\mathscr{E})$ by setting

$$\tilde{E}_x \ni v_x \longmapsto \langle v_x, \tilde{G}_i(x,\cdot)\rangle_{x,\tilde{E}} \in W^{-1,2}(\mathscr{E}), \tag{4.2}$$

where $x \in \tilde{M}$. It satisfies the conclusions in Lemmas 3.1-3.3.

The repetition of the argument in Section 3 shows that Theorem 1.2 is a consequence of the following statement.

Theorem 4.1. *Under the hypotheses of Theorem 1.2, consider the maps $\mathscr{G}_i : \tilde{E}_i \to W^{-1,2}(\mathscr{E})$ defined by* (4.2)*, where $i = 1,2$. Suppose that the vector bundle isomorphism $\Phi : \mathscr{E} \to \mathscr{E}$, described above, intertwines with the $\mathscr{G}_i$'s, that is*

$$\mathscr{G}_2 \circ \Phi = \Phi \circ \mathscr{G}_1 \qquad \text{on } \mathscr{E}.$$

Then the images $\mathscr{G}_2(\tilde{E}_2^0)$ and $\Phi \circ \mathscr{G}_1(\tilde{E}_1^0)$ coincide as subsets in $W^{-1,2}(\mathscr{E})$, and the map $\mathscr{G}_2^{-1} \circ \Phi \circ \mathscr{G}_1 : \tilde{E}_1^0 \to \tilde{E}_2^0$ extends to a real-analytic vector bundle isomorphism $J : \tilde{E}_1 \to \tilde{E}_2$ that covers the identity map of M and such that $J^*\tilde{\nabla}^2 = \tilde{\nabla}^1$.

Assuming that the construction described above is the partial case of the setup in Section 3, such that the extended isomorphism Φ in (4.1) covers the identity map, Theorem 4.1 can actually be derived from Theorem 3.4. More precisely, the only statement that needs to be checked is that the isomorphism J in Theorem 3.4 covers the identity map of M. The latter is a consequence of the unique continuation property of real-analytic maps, since J coincides with Φ over W and the latter covers the identity map.

However, under the hypotheses of Theorem 1.2, and consequently, of Theorem 4.1, the original argument in Section 3 can be simplified, and for reader's convenience we prefer to outline this below, highlighting the differences.

4.2. **Outline of the proof of Theorem 4.1.** Let $B \subset \tilde{M}$ be the largest connected open set containing the fixed point $p \in \Sigma$ and such that for any $x \in B$ the images of the fibres $\Phi \circ \mathscr{G}_1((\tilde{E}_1)_x)$ and $\mathscr{G}_2((\tilde{E}_2)_x)$ coincide. Then, by Lemma 3.2 the composition

$$J_x = \mathscr{G}_2^{-1} \circ \Phi \circ \mathscr{G}_1 : (\tilde{E}_1)_x \longrightarrow (\tilde{E}_2)_x \tag{4.3}$$

is an isomorphism of the fibres, and defines a fibre preserving map $J : \tilde{E}_1\big|_B \to \tilde{E}_2$. Repeating the argument in Section 3, we see that $W \subset B$. The aim is to show that B coincides with the interior of $\tilde{M}$. Note that unlike in the proof of Theorem 3.4 we do not ask J_x to be an isometry on the fibres. This is related to the fact that Step 1 of the proof is trivial in this case, since we are not trying to determine the topology of M.

Suppose the contrary, $B \neq \tilde{M}$, and pick a point $\bar{x} \in \partial B$ that lies in the interior of M. By continuity of $\Phi \circ \mathscr{G}_1$ and $\mathscr{G}_2$, together with Lemma 3.2, we conclude that the map J extends to $\bar{x}$. Then there exist non-zero vectors $v_{\bar{x}} \in (\tilde{E}_1)_{\bar{x}}$ and $w_{\bar{x}} \in (\tilde{E}_2)_{\bar{x}}$ such that the images $\Phi \circ \mathscr{G}_1(v_{\bar{x}})$ and $\mathscr{G}_2(w_{\bar{x}})$ coincide. Denoting this vector by u, we can repeat the argument in the proof of Theorem 3.4, see Step 2 and Lemma 3.7 in Step 3, to conclude that the images of $\Phi \circ \mathscr{G}_1$ and $\mathscr{G}_2$ coincide around u. Then, we have the following version of Lemma 3.8.

Lemma 4.2. *There is a connected neighbourhood O of the point $\bar{x} \in \tilde{M}$ such that for any $x \in O$ the images of fibres $\Phi \circ \mathscr{G}_1((\tilde{E}_1)_x)$ and $\mathscr{G}_2((\tilde{E}_2)_x)$ coincide.*

Proof. Following the notation and the line of argument in Section 3, we choose a neighbourhood $O \subset \tilde{\pi}_1 \circ H_1(\mathscr{O})$ of $\bar{x} \in \tilde{M}$. For a given point $x \in O$ and a vector $v_x \in H_1(\mathscr{O})$ denote by w_z the vector $H_2 \circ H_1^{-1}(v_x)$. It satisfies the relation

$$\Phi \circ \mathscr{G}_1(v_x) = \mathscr{G}_2(w_z),$$

and we claim that the point z has to coincide with x. Indeed, this follows from the fact that the map $H_2 \circ H_1^{-1}$ coincides with J on the open set $\mathscr{O} \cap \Omega_1$, where Ω_1 is defined by relation (3.5), and in our setup J covers the identity map on the base, that is

$$\tilde{\pi}_2 \circ H_2 \circ H_1^{-1} = \tilde{\pi}_1 \qquad \text{on} \qquad \mathscr{O} \cap \Omega_1. \tag{4.4}$$

By Lemma 3.6 the map $H_2 \circ H_1^{-1}$ is real-analytic, and so are the vector bundle projections $\tilde{\pi}_1$ and $\tilde{\pi}_2$. Thus, by unique continuation we conclude that relation (4.4) holds on $\mathscr{O}$, and hence,

$$z = \tilde{\pi}_2(w_z) = \tilde{\pi}_1(v_x) = x.$$

Now by linearity, it is straightforward to see that the image of the fibre $\Phi \circ \mathscr{G}_1((\tilde{E}_1)_x)$ lies in $\mathscr{G}_2((\tilde{E}_2)_x)$. Since these images are vector subspaces of the same dimension, we are done. □

The last lemma yields a contradiction, and we conclude that $B = \tilde{M}$. Thus, relation (4.3) defines the fibre preserving map $J : \tilde{E}_1 \to \tilde{E}_2$, which covers the identity map of M. Other remaining properties of J are obtained by repeating the argument at the end of Section 3.

5. Proofs of auxiliary results

5.1. **Proof of Lemma 3.1.** Since the map $\mathscr{G} : \tilde{E} \to W^{\ell,2}(\mathscr{E})$ is linear on each fibre, for a proof of the lemma it is sufficient to show that the map that sends a point $x \in \tilde{M}$ to the function $\tilde{G}(x,\cdot)$, viewed as an element in the Sobolev space $W^{\ell,2}$, is smooth. Below we assume that x ranges in a chart on $\tilde{M}$ where the vector bundle $\tilde{E}$ is trivial. First, we claim that for any section $\varphi \in W_0^{-\ell,2}(\mathscr{E})$ the section

$$\tilde{\psi}(x) = \int_{\tilde{M}} \langle \tilde{G}(x,y), \tilde{\varphi}(y) \rangle_y dVol(y),$$

is differentiable, where $\tilde{\varphi}$ is an extension of φ by zero, and for any $h \in \mathbb{R}^n$ the linear functional

$$\varphi \longmapsto \sum_i h_i \frac{\partial}{\partial x^i} \tilde{\psi}(x) = \sum_i h_i \frac{\partial}{\partial x^i} \int_{\tilde{M}} \langle \tilde{G}(x,y), \tilde{\varphi}(y) \rangle_y dVol(y) \tag{5.1}$$

defines an element in $W^{\ell,2}(\mathscr{E})$. Indeed, by standard theory the section $\tilde{\psi}$ can be viewed as the solution to the Dirichlet problem

$$\Delta^{\tilde{E}} \tilde{\psi} = \tilde{\varphi}, \qquad \tilde{\psi}|_{\partial \tilde{M}} = 0,$$

in the Sobolev space $W_0^{-\ell+2,2}$, and since $\ell < 1 - n/2$, it lies in the Holder space $C^{1,\alpha}$ for some $\alpha > 0$. To show that the functional defined by (5.1) lies in $W^{\ell,2}(\mathscr{E})$, it is sufficient to show that

$$\left| \sum_i h_i \frac{\partial}{\partial x^i} \tilde{\psi}(x) \right| \leqslant C\,|h|\,|\varphi|_{-\ell,2}$$

for some constant $C > 0$, where $|\cdot|_{-\ell,2}$ stands for the Sobolev norm. The latter is a direct consequence of the inequality $|\tilde{\psi}|_{-\ell+2,2} \leqslant C'\,|\tilde{\varphi}|_{-\ell,2}$, which follows from standard theory, together with the Sobolev embedding theorem. Thus, we obtain the linear operator

$$L_x : \mathbb{R}^n \ni h \longmapsto \sum_i h_i \frac{\partial}{\partial x^i} \tilde{G}(x,\cdot) \in W^{\ell,2}(\mathscr{E}),$$

and claim that it is the differential of the map $x \mapsto \tilde{G}(x,\cdot)$. In other words, we claim that for any $\varepsilon > 0$ the inequality

$$\left|\tilde{G}(x+h,\cdot) - \tilde{G}(x,\cdot) - L_x(h)\right|_{\ell,2} \leqslant \varepsilon\,|h|$$

holds, for any $h \in \mathbb{R}^n$ such that $|h| < \delta$ for an appropriate $\delta > 0$. In the notation above, for the latter it is sufficient to show that

$$\left| \tilde{\psi}(x+h) - \tilde{\psi}(x) - \sum_i h_i \frac{\partial}{\partial x^i} \tilde{\psi}(x) \right| \leqslant C\,|h|^{1+\alpha}\,|\varphi|_{-\ell,2} \tag{5.2}$$

for some positive constants C and α, and arbitrary $h \in \mathbb{R}^n$. Recall the so-called Hadamard formula for C^1-smooth functions:

$$\tilde{\psi}(x+h) - \tilde{\psi}(x) = \sum_i \gamma_i(x) h_i, \quad \text{where} \quad \gamma_i(x) = \int_0^1 \frac{\partial \tilde{\psi}}{\partial x^i}(x+th)\,dt,$$

and we use a trivialisation of $\tilde{E}$ to view sections around x as vector functions. Using this relation, we obtain

$$\begin{aligned}
\left| \tilde{\psi}(x+h) - \tilde{\psi}(x) - \sum_i h_i \frac{\partial}{\partial x^i} \tilde{\psi}(x) \right| &= \left| \sum_i (\gamma_i(x) - \frac{\partial}{\partial x^i} \tilde{\psi}(x)) h_i \right| \\
\leqslant |h| \left(\int_0^1 |D\tilde{\psi}(x+th) - D\tilde{\psi}(x))|^2\,dt \right)^{1/2} &\leqslant |\tilde{\psi}|_{C^{1,\alpha}}\,|h|^{1+\alpha} \leqslant C''\,|\tilde{\psi}|_{-\ell+2,2}\,|h|^{1+\alpha} \\
&\leqslant C'C''\,|\tilde{\varphi}|_{-\ell,2}\,|h|^{1+\alpha} \leqslant C\,|\varphi|_{-\ell,2}\,|h|^{1+\alpha},
\end{aligned}$$

where in the second inequality we estimate the integral via the Holder norm and $|h|^\alpha$, and in the third we use the Sobolev embedding theorem. Thus, relation (5.2) is demonstrated, and we conclude that the map $x \mapsto \tilde{G}(x,\cdot)$ is differentiable. Finally, for a proof that it is smooth, it remains to show that the map $x \mapsto L_x$ is continuous. The latter is a consequence of the inequality

$$\left| \sum_i (\frac{\partial}{\partial x^i} \tilde{\psi}(x_1) - \frac{\partial}{\partial x^i} \tilde{\psi}(x_2)) h_i \right| \leqslant C\,|h|\,|x_1 - x_2|^\alpha\,|\varphi|_{-\ell,2}$$

for some positive constants C and α, which can be proved in a fashion similar to the one above. Thus, we are done. □

5.2. **Proof of Lemma 3.2.** First, we show that the map $\mathscr{G}$ is a linear embedding on each fibre $\tilde{E}_x$. For otherwise, there exists a point x in the interior of $\tilde{M}$ and a non-zero vector $v_x \in \tilde{E}_x$ such that the product $\langle v_x, \tilde{G}(x,\cdot)\rangle_x$ equals zero in $W^{\ell,2}(\mathscr{E})$. The latter in particular implies that

$$\langle v_x, \tilde{G}(x,y)\rangle_x = 0 \qquad \text{for all} \quad y \in U\backslash\{x\}. \tag{5.3}$$

Since the left-hand side above is real-analytic, we conclude that relation (5.3) continues to hold on $\tilde{M}\backslash\{x\}$. Now let $s \in \mathscr{D}(\tilde{E})$ be a compactly supported section such that $s(x) = v_x$. Then, we obtain

$$0 = \int_{\tilde{M}} \langle\langle v_x, \tilde{G}(x,y)\rangle_x, \Delta^{\tilde{E}} s(y)\rangle_y dVol(y) = \langle v_x, \int_{\tilde{M}} \langle \tilde{G}(x,y), \Delta^{\tilde{E}} s(y)\rangle_y dVol(y)\rangle_x = \langle v_x, v_x\rangle,$$

where we changed the order of operations in independent variables x and y in the second relation, and used the definition of the Dirichlet Green kernel in the third. Thus, we conclude that the vector v_x has to vanish, and the kernel of a linear operator given by (3.1) is trivial, that is the map $\mathscr{G}$ is indeed a linear embedding on each fibre.

A similar argument shows that the map $\mathscr{G}$ is injective everywhere on $\tilde{E}^0$. Indeed, suppose that there exist two points x_1 and x_2 in the interior of $\tilde{M}$ and non-zero vectors v_{x_1} and v_{x_2} in the fibres over them such that $\langle v_{x_1}, \tilde{G}(x_1,\cdot)\rangle$ and $\langle v_{x_2}, \tilde{G}(x_2,\cdot)\rangle$ coincide in $W^{\ell,2}(\mathscr{E})$. Then, it is straightforward to see that

$$\langle v_{x_1}, \tilde{G}(x_1,y)\rangle_x = \langle v_{x_2}, \tilde{G}(x_2,y)\rangle_x \qquad \text{for all} \quad y \in U\backslash\{x_1,x_2\}. \tag{5.4}$$

As above, by unique continuation we may assume that relation (5.4) holds for all $y \in \tilde{M}\backslash\{x_1,x_2\}$. In addition, since the map $\mathscr{G}$ is injective on fibres, we may assume that $x_1 \neq x_2$. Let $s \in \mathscr{D}(\tilde{E})$ be a compactly supported section such that $s(x_1) = v_{x_1}$ and $s(x_2) = 0$. Then, we obtain

$$\begin{aligned}\langle v_{x_1}, v_{x_1}\rangle &= \langle v_{x_1}, \int_{\tilde{M}} \langle \tilde{G}(x_1,y), \Delta^{\tilde{E}} s(y)\rangle_y dVol(y)\rangle_x = \int_{\tilde{M}} \langle\langle v_{x_1}, \tilde{G}(x_1,y)\rangle_x, \Delta^{\tilde{E}} s(y)\rangle_y dVol(y) \\ &= \int_{\tilde{M}} \langle\langle v_{x_2}, \tilde{G}(x_2,y)\rangle_x, \Delta^{\tilde{E}} s(y)\rangle_y dVol(y) = \langle v_{x_2}, \int_{\tilde{M}} \langle \tilde{G}(x_2,y), \Delta^{\tilde{E}} s(y)\rangle_y dVol(y)\rangle_x \\ &= \langle v_{x_2}, s(x_2)\rangle = \langle v_{x_2}, 0\rangle = 0.\end{aligned}$$

Thus, the vector v_{x_1} vanishes, and we arrive at a contradiction.

Finally, to show that the map $\mathscr{G}$ is an immersion we analyse its differential $D_{v_x} : T_{v_x}\tilde{E} \to W^{\ell,2}(\mathscr{E})$. First, note that a connection on the vector bundle $\tilde{E}$ defines the decomposition of the tangent space $T_{v_x}\tilde{E}$ as the direct sum $H_{v_x} \oplus \tilde{E}_x$, where H_{v_x} is the so-called horizontal subspace, see [9]. Since the differential of the projection $\tilde{\pi} : \tilde{E} \to \tilde{M}$ establishes an isomorphism $D_{v_x}\tilde{\pi} : H_{v_x} \to T_x\tilde{M}$, we may view tangent vectors from $T_{v_x}\tilde{E}$ as pairs (X,ξ), where $X \in T_xM$, and $\xi \in \tilde{E}_x$. With these identifications, it is straightforward to show that

$$D_{v_x}\mathscr{G}(X,\xi) = \langle v_x, \nabla_X \tilde{G}(x,\cdot)\rangle_{x,\tilde{E}} + \langle \xi, \tilde{G}(x,\cdot)\rangle_{x,\tilde{E}}, \tag{5.5}$$

where by the covariant derivative $\nabla_X \tilde{G}(x,\cdot)$ we mean the derivative with respect to the variable x on $\tilde{E} \boxtimes \tilde{E}$, that is given by $\nabla_X(u_x \otimes u_y) = \nabla^{\tilde{E}}_X u_x \otimes u_y$. Now choosing appropriate test-sections in the fashion similar to the one above, it is straightforward to show that the differential $D_{v_x}\mathscr{G}$ is injective. In more detail, assume that the right-hand side of relation (5.5) equals zero for some $X \in T_xM$ and $\xi \in \tilde{E}_x$. Then, by unique continuation we may assume that

$$\langle v_x, \nabla_X \tilde{G}(x,y)\rangle_{x,\tilde{E}} + \langle \xi, \tilde{G}(x,y)\rangle_{x,\tilde{E}} = 0 \qquad \text{for all} \quad y \in \tilde{M}\backslash\{x\}. \tag{5.6}$$

Now choosing a compactly supported section $s \in \mathscr{D}(\tilde{E})$ such that $s(x) = \xi$ and $\nabla^{\tilde{E}}_X s\Big|_x = 0$, we obtain

$$\begin{aligned} 0 &= \int_{\tilde{M}} \langle \langle v_x, \nabla_X \tilde{G}(x,y)\rangle_x, \Delta^{\tilde{E}} s(y)\rangle_y dVol(y) + \int_{\tilde{M}} \langle \langle \xi, \tilde{G}(x,y)\rangle_x, \Delta^{\tilde{E}} s(y)\rangle_y dVol(y) \\ &= \langle v_x, \nabla_X \int_{\tilde{M}} \langle \tilde{G}(x,y), \Delta^{\tilde{E}} s(y)\rangle_y dVol(y)\rangle_x + \langle \xi, \int_{\tilde{M}} \langle \tilde{G}(x,y), \Delta^{\tilde{E}} s(y)\rangle_y dVol(y)\rangle_x \\ &= \langle v_x, \nabla^{\tilde{E}}_X s\rangle + \langle \xi, \xi\rangle = 0 + \langle \xi, \xi\rangle. \end{aligned}$$

Thus, the vector $\xi \in \tilde{E}_x$ vanishes, and by relation (5.6) we conclude that the term $\langle v_x, \nabla_X \tilde{G}(x,\cdot)\rangle_x$ equals zero. Now choosing a test-section $s \in \mathscr{D}(\tilde{E})$ such that $\nabla_X s|_x = v_x$, it is straightforward to see that the vector v_x equals zero as well. Thus, the differential $D_{v_x}\mathscr{G}$ is indeed injective, and we are done. □

5.3. **Proof of Lemma 3.3.** Suppose the contrary, the point x does not coincide neither with q_1 nor with q_2. Then there exist non-zero vectors $v_x \in \tilde{E}_x$, $w_{q_1} \in \tilde{E}_{q_1}$, and $w_{q_2} \in \tilde{E}_{q_2}$ such that

$$\langle v_x, \tilde{G}(x,y)\rangle_{x,\tilde{E}} = \langle w_{q_1}, \tilde{G}(q_1,y)\rangle_{q_1,\tilde{E}} + \langle w_{q_2}, \tilde{G}(q_2,y)\rangle_{q_2,\tilde{E}} \tag{5.7}$$

for any $y \in U$. Since both parts of this identity are real-analytic functions of y, we conclude that it continues to hold for all y in the complement of the points x, q_1, and q_2 in $\tilde{M}$. Since x does not coincide neither with q_1 nor with q_2, there exists a smooth section $s \in \mathscr{D}(\tilde{E})$ whose support does not contain q_1 and q_2, and such that $s(x) = v_x$. Then, by definition of the Dirichlet Green kernel we obtain

$$\begin{aligned} \langle v_x, v_x\rangle &= \langle v_x, \int_{\tilde{M}} \langle \tilde{G}(x,y), \Delta^{\tilde{E}} s(y)\rangle_y dVol(y)\rangle_x = \int_{\tilde{M}} \langle \langle v_x, \tilde{G}(x,y)\rangle_x, \Delta^{\tilde{E}} s(y)\rangle_y dVol(y) \\ &= \int_{\tilde{M}} \langle \langle w_{q_1}, \tilde{G}(q_1,y)\rangle_x, \Delta^{\tilde{E}} s(y)\rangle_y dVol(y) + \int_{\tilde{M}} \langle \langle w_{q_2}, \tilde{G}(q_2,y)\rangle_x, \Delta^{\tilde{E}} s(y)\rangle_y dVol(y) \\ &= \langle w_{q_1}, \int_{\tilde{M}} \langle \tilde{G}(q_1,y), \Delta^{\tilde{E}} s(y)\rangle_y dVol(y)\rangle_x + \int_{\tilde{M}} \langle w_{q_2}, \langle \tilde{G}(q_2,y), \Delta^{\tilde{E}} s(y)\rangle_y dVol(y)\rangle_x \\ &= \langle w_{q_1}, s(q_1)\rangle + \langle w_{q_2}, s(q_2)\rangle = 0, \end{aligned}$$

where we used the relation $s(x) = v_x$ in the first equality, identity (5.7) in the third, and the fact that the section s is chosen so that it vanishes at q_1 and q_2 in the last. Thus, we conclude that v_x equals zero and arrive at a contradiction. □

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