Algebras and Representation Theory (2024) 27:667–678
https://doi.org/10.1007/s10468-023-10229-7

A Note on the Global Dimension of Shifted Orders

Özgür Esentepe[1,2]

Received: 8 April 2022 / Accepted: 18 August 2023 / Published online: 22 September 2023

Abstract
We consider the dominant dimension of an order over a Cohen-Macaulay ring in the category of centrally Cohen-Macaulay modules. There is a canonical tilting module in the case of positive dominant dimension and we give an upper bound on the global dimension of its endomorphism ring.

Keywords Cohen-Macaulay modules · Dominant dimension · Tilting

Mathematics Subject Classification (2010) 13C14 · 16G50 · 18G05 · 18G20

1 Introduction

In a recent article "*Special tilting modules for algebras with positive dominant dimension*" [5], Matthew Pressland and Julia Sauter study what they advertise in their title. They call such tilting modules *shifted modules* and they study their *shifted algebras*. One particular theorem they prove is the following.

Theorem *([5], Proposition 2.13) Let Γ be a finite dimensional algebra with dominant dimension d and let $0 \leq k \leq d$. Then, for the k-shifted algebra B_k of Γ, we have*

$$\operatorname{gldim} B_k \leq \operatorname{gldim} \Gamma.$$

The aim of this note is to show that this theorem holds true in the setting of Cohen-Macaulay representation theory. In our setting, we replace finite dimensional algebras over a field with orders over a Cohen-Macaulay local ring with canonical module. Therefore, we replace the module category of an algebra with the category of Cohen-Macaulay modules over an order. We study the dominant dimension in this category. This idea appears also in [1] where the authors characterize orders of finite lattice type and in [4] where the author generalizes the *Auslander correspondence* to orders over regular local rings.

Presented by: Peter Littelmann

✉ Özgür Esentepe
ozgur.esentepe@uconn.edu; o.esentepe@leeds.ac.uk

1 Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA

2 School of Mathematics, University of Leeds, Leeds LS2 9JT, UK

The dominant dimension of an algebra is an invariant which intends to measure how far away from being self-injective the given algebra is. In the order case, self-injective algebras correspond to *Gorenstein orders* and our Cohen-Macaulay dominant dimension measures how far away from being a Gorenstein order a given order over a Cohen-Macaulay local ring is.

Before stating our main theorem, let us recall some definitions from Cohen-Macaulay representation theory. Let R be a Cohen-Macaulay local ring with canonical module ω_R and Λ be a module finite R-algebra. We say that a right Λ-module M is Cohen-Macaulay if it is maximal Cohen-Macaulay as an R-module and we denote the category of Cohen-Macaulay Λ-modules by $\mathrm{CM}(\Lambda)$ and we denote by $\mathrm{CM}(\Lambda)$ the category of Cohen-Macaulay Λ-modules. We say that Λ is an *R-order* if $\Lambda \in \mathrm{CM}(\Lambda)$. Notice that if R is a field, then an order Λ is nothing but a finite dimensional algebra and every finitely generated Λ-module is Cohen-Macaulay. In this sense, Cohen-Macaulay representation theory can be thought as a generalization of the representation theory of finite dimensional algebras.

The functor $D := \mathrm{Hom}_R(-, \omega_R)$ gives us an exact duality $\mathrm{CM}(\Lambda) \leftrightarrow \mathrm{CM}(\Lambda^{\mathrm{op}})$. We denote by ω the bimodule $D\Lambda$ and call it the *canonical module of* Λ. Considered as a right module, it is an additive generator for the subcategory of Cohen-Macaulay injective modules. That is, if $\mathrm{Ext}^i_\Lambda(M, X) = 0$ for all $M \in \mathrm{CM}(\Lambda)$, then X is a direct summand of ω^n for some positive integer n. Following the convention in commutative algebra, an order Λ is said to be a *Gorenstein order* if ω is a projective Λ-module. Hence, Gorenstein orders can be thought as a generalization of self-injective algebras.

Our exact duality D allows us to construct Cohen-Macaulay injective coresolutions in $\mathrm{CM}(\Lambda)$ by dualizing projective resolutions of modules in $\mathrm{CM}(\Lambda^{\mathrm{op}})$. If we assume that Λ is semi-perfect, then we can talk about minimal CM-injective resolutions and this gives us a well-defined notion of Cohen-Macaulay injective dimension. We say that Λ has *CM-dominant dimension* ℓ if ℓ is the smallest number in a minimal CM-injective coresolution

$$0 \to \Lambda \to I^0 \to \ldots \to I^\ell \to \ldots$$

such that I^ℓ is not projective (or ∞ if such ℓ does not exist). We denote the CM-dominant dimension of Λ by $\mathrm{CMdomdim}(\Lambda)$.

When an order over a Cohen-Macaulay local ring has positive Cohen-Macaulay dominant dimension, there is a special family of tilting modules. We call their endomorphism rings *shifted orders*. While in general derived equivalence preserves finiteness of global dimension, the exact value of the global dimension is not preserved. We prove that the global dimension of the shifted orders can not be greater than the original order.

Theorem *Suppose that Λ is an n-canonical QF-3 order of finite global dimension over a Cohen-Macaulay local ring R with canonical module. Then, the ℓ^{th}-shifted order Γ_ℓ has global dimension*

$$\mathrm{gldim}\Gamma_\ell \leq \mathrm{gldim}\Lambda.$$

2 Dominant Dimension

We have seen that Gorenstein orders are in some sense generalizations of self-injective algebras. Dominant dimension is a measure of how far away an algebra is from being a self-injective algebra. The pioneers of the theory include Nakayama, Tachikawa, Müller and others. The dominant dimension of an algebra A is the minimal number ℓ in a minimal injective resolution

$$0 \to A \to I^0 \to \ldots \to I^\ell \to \ldots$$

such that I^ℓ is not projective. If such a number does not exist, we say that the dominant dimension is infinite. For self-injective algebras, the dominant dimension is infinite. The converse of this statement is the famous Nakayama conjecture: It is an open problem whether infinite dominant dimension implies that the algebra A is self-injective.

In this paper, motivated by the previous section, we will consider the dominant dimension in the category of Cohen-Macaulay modules.

2.1 Setting

Throughout this paper, we assume that R is a Cohen-Macaulay local ring of Krull dimension d with canonical module and Λ is an R-order. When Λ has CM-injective dimension n, we say Λ is an n-canonical order. We will tacitly assume that $0 \neq n < \infty$. By Π, we denote a Λ-module such that $\operatorname{add}\Pi = \operatorname{add}\Lambda \cap \operatorname{add}\omega$. That is, we let Π be an additive generator for the subcategory of CM-projective-injective Λ-modules. We will also assume for convenience that Λ is semiperfect and therefore minimal resolutions exist. This can be guaranteed, for instance, by assuming that R is complete.

Definition 2.2 We say that Λ has *CM-dominant dimension* ℓ if ℓ is the smallest number in a minimal CM-injective coresolution

$$0 \to \Lambda \to I^0 \to \ldots \to I^\ell \to \ldots$$

such that I^ℓ is not projective (or ∞ if such ℓ does not exist). We denote the CM-dominant dimension of Λ by $\operatorname{CMdomdim}(\Lambda)$.

Note that this definition only proves useful in the noncommutative setting. Indeed, in the commutative case, we have $\operatorname{pd}_R \omega_R = 0$ if and only if R is Gorenstein. In this case, the Cohen-Macaulay dominant dimension is ∞. Otherwise, that is if R is a non-Gorenstein Cohen-Macaulay local ring, then $\operatorname{pd}_R \omega_R = \infty$ as a consequence of the Auslander-Buchsbaum formula and the CM-dominant dimension is zero.

It is immediate to see that Gorenstein orders have infinite Cohen-Macaulay dominant dimension. We shall now consider a nontrivial noncommutative example.

2.2 A Nontrivial Example

Let k be an infinite field and consider $R = k[[x, y, z, u, v]]/I$ where I is the ideal generated by the 2×2 minors of the generic matrix

$$\begin{bmatrix} x & y & u \\ y & z & v \end{bmatrix}.$$

This ring is an example of a scroll. Scrolls have nice properties and geometric interpretations for which refer to [3]. In particular, our ring R is an integrally closed Cohen-Macaulay normal domain of Krull dimension 3 and it is a toric isolated singularity. If we consider the $\mathbb{Z}$-graded algebra

$$S = k[[X_1, X_2, X_3, X_4]] = \bigoplus_{i \in \mathbb{Z}} S_i$$

with deg $X_1 = 2$, deg $X_2 = 1$ and deg $X_3 = \deg X_4 = -1$, then we can identify R with S_0. As an application of the theory of almost split sequences, Auslander and Reiten proved in [2, Theorem 2.1] that R has finite Cohen-Macaulay type: it has only finitely many indecomposable maximal Cohen-Macaulay modules up to isomorphism. Using the notation from [9, Proposition 16.12] and the above identification, the indecomposable maximal Cohen-Macaulay modules are S_1, $S_0 \cong R$, $S_{-1} \cong \omega_R$, S_{-2} and a rank two module M. Let us set notation: We put $X = R \oplus \omega$ and $\Lambda = \mathrm{End}_R(X)$. Then, Λ is an R-order. We put

$$\begin{aligned} D &= \mathrm{Hom}_R(-, \omega_R) : \mathrm{CM}(\Lambda) \to \mathrm{CM}(\Lambda^{\mathrm{op}}) \\ F &= \mathrm{Hom}_R(X, -) : \mathrm{mod}R \to \mathrm{mod}\Lambda \\ G &= \mathrm{Hom}_R(-, X) : \mathrm{mod}R \to \mathrm{mod}\Lambda^{\mathrm{op}}. \end{aligned}$$

Note that F restricts to an equivalence $\mathrm{add}X \cong \mathrm{proj}\Lambda$ and G restricts to an equivalence $\mathrm{add}X \cong \mathrm{proj}\Lambda^{\mathrm{op}}$. We are now ready to start. The first thing we do is to understand CM-injective Λ-modules. To do so, we compute ω.

$$\omega = D\Lambda = D\mathrm{Hom}_R(X, X) = DG(X) = DG(R) \oplus DG(\omega_R).$$

On the other hand, we have

$$DG(R) = D\mathrm{Hom}_R(R, X) = DX = \mathrm{Hom}_R(X, \omega_R) = F(\omega_R)$$

and

$$\begin{aligned} DG(\omega_R) = D\mathrm{Hom}_R(\omega, X) &= D\mathrm{Hom}_R(S_{-1}, S_0 \oplus S_{-1}) \\ &= D(S_1 \oplus R) = S_2 \oplus S_{-1} \\ &= \mathrm{Hom}_R(S_0 \oplus S_{-1}, S_{-2}) = \mathrm{Hom}_R(X, S_{-2}) = F(S_{-2}). \end{aligned}$$

Therefore, we conclude that $\omega = F(\omega_R \oplus S_{-2})$. In other words, the only indecomposable CM-injective Λ-modules are $\mathrm{Hom}_R(X, \omega_R)$ and $\mathrm{Hom}_R(X, S_{-2})$ up to isomorphism as F is fully faithful.

Let us now compute a minimal CM-injective resolution of Λ. We will start with a projective resolution of ω. We have a short exact sequence

$$0 \to R \to \omega_R^{\oplus 2} \to S_{-2} \to 0.$$

If we apply F to this exact sequence, we get

$$0 \to FR \to F\omega_R^{\oplus 2} \to FS_{-2} \to \mathrm{Ext}^1_R(X, R) \cong 0.$$

Since $F\omega_R$ is CM-injective, this short exact sequence yields

$$0 \to FR \to F\omega_R^{\oplus 3} \to F\omega_R \oplus FS_{-2} \cong \omega \to 0$$

which is a minimal projective resolution of the Λ-module ω. Now, applying D to this resolution, we get the following CM-injective resolution of Λ in $\mathrm{CM}(\Lambda^{op})$:

$$0 \to \Lambda \to DF\omega_R^{\oplus 3} \to DFR \to 0.$$

We have seen that the only indecomposable CM-injective Λ-modules are $F\omega_R$ and FS_{-2}. Therefore, FR is not a CM-injective Λ-module and DFR is not a projective Λ^{op}-module which shows us that the CM-dominant dimension of Λ is 1.

2.3 Cohen-Macaulay Nakayama Conjecture

We have said that the dominant dimension of a finite dimensional algebra measures how far away it is from being a self-injective algebra: If the algebra is self-injective, then the dominant dimension is infinite. We have mentioned that the converse of this statement is the Nakayama conjecture. It is easy to see from our definition of Cohen-Macaulay dominant dimension, Gorenstein orders have infinite CM-dominant dimension. Indeed, the canonical module of a Gorenstein order is projective and projective modules and CM-injective modules coincide. The Cohen-Macaulay Nakayama conjecture states, then, that if an order over a Cohen-Macaulay local ring has infinite dominant dimension, then it must be a Gorenstein order. The following proposotion shows that if one can find a counterexample to the Cohen-Macaulay Nakayama conjecture, then it gives a counterexample to the original Nakayama conjecture.

Proposition 2.5 *Let Λ be as in Setting Section 2.1 and x be a central nonzerodivisor on Λ. Then,*

$$\mathrm{CMdomdim}(\Lambda) = \mathrm{CMdomdim}(\Lambda/x\Lambda).$$

Proof It is standard that if the depth of Λ is d as an R-module, then the depth of $\Lambda/x\Lambda$ as an R/xR-module is $d-1$. That is, $\Lambda/x\Lambda$ is an R/xR-order and our definitions make sense.

We know that if P is a projective Λ-module, then P/xP is a projective $\Lambda/x\Lambda$-module. We also have that the canonical module of R/xR is isomorphic to $\omega_R/x\omega_R$. From here, we can conclude that

$$\omega_{\Lambda/x\Lambda} = \mathrm{Hom}_{R/xR}(\Lambda/x\Lambda, \omega_{R/xR}) \cong \omega_\Lambda/x\omega_\Lambda.$$

Now, we can complete the proof by taking a projective resolution of ω_Λ as a Λ^{op}-module and tensoring it with $\Lambda/x\Lambda$ since by doing so still gives us an exact seqeunce and we have already observed that CM-projective-injective Λ-modules go to CM-projective-injective $\Lambda/x\Lambda$-modules. □

Corollary 2.6 *Let Λ be as in Setting Section 2.1 and let $\mathbf{x}$ be a regular sequence of length d on Λ. Then, we have*

$$\mathrm{CMdomdim}(\Lambda) = \mathrm{domdim}(\Lambda/\mathbf{x}\Lambda).$$

Hence a counterexample to the Cohen-Macaulay Nakayama conjecture would imply a counterexample to the original Nakayama conjecture.

Remark 2.7 It is well-known in the representation theory of finite dimensional algebras that the Nakayama conjecture holds true for those algebras which have finite finitistic dimension. Similar arguments also can be made for orders over Cohen-Macaulay local rings. When Λ is an n-canonical order over a Cohen-Macaulay local ring of Krull dimension d, there is an inequality version of Auslander-Buchsbaum formula which was proved by Stangle in their thesis [6]: for any Λ-module X with finite projective dimension, we have

$$d \leq \mathrm{pd}_\Lambda X + \mathrm{depth} X \leq d + n.$$

Hence n-canonical orders have finite finitistic dimension. Therefore, if one wants to find a counterexample to the Nakayama conjecture, then they need to look at orders over which the canonical module has infinite projective dimension.

2.4 Auslander Correspondence

In his seminal work in 1970s, Auslander proves that there exists a one-to-one correspondence (up to appropriate equivalences) between representation finite algebras Λ with $\mathrm{mod}\Lambda = \mathrm{add}M$ and the so-called Auslander algebras Γ with $\mathrm{gldim}\Gamma \leq 2 \leq \mathrm{domdim}\Gamma$. The correspondence is given by $M \mapsto \Gamma = \mathrm{End}_\Lambda(M)$. Then, in the beginning of 21st century, Iyama proves a much more general version of this in [4]. Iyama replaces additive generators with cluster tilting objects in order to obtain his correspondence and also generalizes Auslander's correspondence to orders over regular local rings. To establish this, he introduces Auslander-type (m, n) conditions. In the case where the base ring is a regular local ring and the order is an isolated singularity, this condition can be translated as follows: given a minimal CM-injective coresolution $0 \to \Lambda \to I_0 \to$, we say that Λ satisfies the (m, n)-condition if the projective dimension of I_i is less than $m - d$ for any $i < n - d$. Hence, in Iyama's language, the CM-dominant dimension of Λ is at least ℓ if and only if Λ satisfies the $(d + 1, d + \ell)$-condition.

Unfortunately, when one moves from regular local rings to Cohen-Macaulay local rings, there are some subtleties appearing. We are now going to examine the CM-dominant dimension of endomorphism rings of some modules that satisfy certain conditions and point out (one of) these subtleties.

Let Λ be an order over a Cohen-Macaulay local ring and M a Cohen-Macaulay Λ-module. Once again, we put $D = \mathrm{Hom}_R(-, \omega_R)$ which gives us an exact duality $\mathrm{CM}(\Lambda) \leftrightarrow \mathrm{CM}(\Lambda^{\mathrm{op}})$. We start with a projective resolution $\mathbb{P}$ of the Λ^{op}-module DM and by dualizing it, we get a CM-injective resolution $D\mathbb{P} : 0 \to \omega^{n_0} \to \omega^{n_1} \to \ldots$ of the Λ-module M. Then, by applying $\mathrm{Hom}_\Lambda(M, -)$ to this complex, we consider the complex $\mathrm{Hom}_R(M, D\mathbb{P})$ which may have cohomology.

By carefully tracking left and right module structures, one can show that the cohomology of this complex is isomorphic to $\mathrm{Ext}^*_\Lambda(DM, DM) \cong \mathrm{Ext}^*_\Lambda(M, M)$ and therefore, the vanishing of these Ext-modules gives us extra information. In particular, we know that if $\mathrm{Ext}^i(M, M) = 0$ for $i = 1, \ldots, d - 2$, then we have an exact sequence

$$0 \to \Gamma := \mathrm{End}_\Lambda(M) \to \mathrm{Hom}_\Lambda(M, \omega)^{n_0} \to \mathrm{Hom}_\Lambda(M, \omega)^{n_1} \to \ldots \to \mathrm{Hom}_\Lambda(M, \omega)^{n_{d-1}} \tag{1}$$

which consequently shows us that Γ has depth d and thus is maximal Cohen-Macaulay as an R-module (hence an order).

Assume, now, also that M contains Λ and ω as a direct summand. This would happen, for instance, if M is a (higher) cluster-tilting module in $\mathrm{CM}(\Lambda)$. Then, by standard homological arguments, we can show that M_Γ is a projective Γ_Γ-module. Therefore, $\mathrm{Hom}_R(M, \omega_R) \cong \mathrm{Hom}_\Lambda(M, \omega)$ is a CM-injective Γ-module. Similarly, one can show that $\mathrm{Hom}_R(M, \omega_R)$ is a direct summand of ${}_\Gamma\Gamma$ meaning that the modules appearing in (1) are CM-projective-injective Γ-modules.

So far, we have established that if (1) can be completed to a CM-injective resolution of Λ, then Γ has dominant dimension at least d. However, while this is not a problem in, say, the representation theory of finite dimensional algebras, it may be a problem in our case. Indeed, if the cokernel of the map $\mathrm{Hom}_\Lambda(M, \omega)^{n_{d-2}} \to \mathrm{Hom}_\Lambda(M, \omega)^{n_{d-1}}$ is not Cohen-Macaulay, then we can not extend this sequence to an injective resolution inside $\mathrm{CM}(\Lambda)$. Hence, one needs to be careful to deal with the Auslander correspondence problem in the general Cohen-Macaulay setting.

3 Shifted Orders

In this section, we keep assuming Setting Section 2.1. We start with an R-order Λ of CM-dominant dimension at least ℓ so that the minimal CM-injective coresolution of Λ is of the form

$$0 \to \Lambda \to \Pi^0 \to \Pi^1 \to \ldots \to \Pi^{\ell-2} \to \Pi^{\ell-1} \to \ldots \quad (2)$$

with Π^j's CM-projective-injective for $j = 0, 1, \ldots, \ell - 1$. We say that Λ is a *QF-3 order* if the CM-dominant dimension of Λ is at least 1. The letters QF come from quasi-Frobenius rings and QF-3 rings were introduced by Robert Thrall as a generalization of quasi-Frobenius rings [8], see also [7].

We denote by K_ℓ the cokernel of $\Pi^{\ell-2} \to \Pi^{\ell-1}$. It is clear by definition that the projective dimension of K_ℓ is at most ℓ.

Lemma 3.1 *The Λ-module K_ℓ is a Cohen-Macaulay Λ-module.*

Proof If the CM-injective dimension of Λ is finite, then let X be the last nonzero CM-injective module in the coresolution (2). In particular, X is a maximal Cohen-Macaulay R-module and by applying the depth lemma successively, we see that K_ℓ is maximal Cohen-Macaulay as an R-module. In case Λ has infinite CM-injective dimension, then K_ℓ is the dth syzygy of a Λ-module. Once again, the depth lemma does the trick. $\square$

Recall that a Λ-module T is called an *ℓ-tilting* module if

(1) the projective dimension of T is at most ℓ,
(2) there is an exact sequence $0 \to \Lambda \to t_0 \to t_1 \to \ldots \to t_\ell \to 0$ with $t_0, \ldots, t_\ell \in \operatorname{add} T$,
(3) there are no self-extensions of T in the sense that $\operatorname{Ext}^{>0}_\Lambda(T, T) = 0$.

Note that if T is a tilting Λ-module, then Λ and $\operatorname{End}_\Lambda(T)$ are derived equivalent. The following lemma is about extensions with CM-projective-injective middle terms.

Lemma 3.2 *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of maximal Cohen-Macaulay Λ-modules with B CM-projective-injective. Then, for every $i > 0$, we have*

$$\operatorname{Ext}^i_\Lambda(A, A) \cong \operatorname{Ext}^i_\Lambda(C, C).$$

Proof If we apply $\operatorname{Hom}_\Lambda(-, A)$ to the short exact sequence, we get a long exact sequence

$$\ldots \to \operatorname{Ext}^i_\Lambda(B, A) \to \operatorname{Ext}^i_\Lambda(A, A) \to \operatorname{Ext}^{i+1}_\Lambda(C, A) \to \operatorname{Ext}^{i+1}_\Lambda(B, A) \to \ldots$$

The outer terms vanish as B is a projective Λ-module and therefore we see that

$$\operatorname{Ext}^i_\Lambda(A, A) \cong \operatorname{Ext}^{i+1}_\Lambda(C, A).$$

If we apply $\operatorname{Hom}_\Lambda(C, -)$ to the short exact sequence, we get a long exact sequence

$$\ldots \to \operatorname{Ext}^i_\Lambda(C, B) \to \operatorname{Ext}^i_\Lambda(C, C) \to \operatorname{Ext}^{i+1}_\Lambda(C, A) \to \operatorname{Ext}^{i+1}_\Lambda(C, B) \to \ldots$$

Once again, the outer terms vanish as B is a CM-injective Λ-module. So, we have isomorphisms

$$\operatorname{Ext}^i_\Lambda(C, C) \cong \operatorname{Ext}^{i+1}_\Lambda(C, A).$$

Combining the two isomorphisms, we get the result. $\square$

Lemma 3.3 *The Λ-module $T_\ell = K_\ell \oplus \Pi$ is an ℓ-tilting Λ-module.*

Proof The first two conditions of a tilting module hold by our definition of T_ℓ. So, we will show the third condition. Note that we have $\mathrm{Ext}^i_\Lambda(\Pi, -) = \mathrm{Ext}^i_\Lambda(-, \Pi) = 0$ for $i > 0$ since Π is CM-projective-injective. Thus, it is enough to show $\mathrm{Ext}^i_\Lambda(K_\ell, K_\ell) = 0$. Note also that by Lemma 3.2, it is enough to show $\mathrm{Ext}^i_\Lambda(K_1, K_1) = 0$ since we have a short exact sequence

$$0 \to K_j \to \Pi^j \to K_{j+1} \to 0$$

for every $1 \leq j \leq \ell - 1$.

For $i \geq 2$, we have $\mathrm{Ext}^i_\Lambda(K_1, K_1) = 0$ as K_1 has projective dimension 1. To show that $\mathrm{Ext}^1_\Lambda(K_1, K_1)$ vanishes, we apply $\mathrm{Hom}_\Lambda(K_1, -)$ to $0 \to \Lambda \to \Pi^0 \to K_1 \to 0$. We get a long exact sequence

$$\ldots \to \mathrm{Ext}^1_\Lambda(K_1, \Pi^0) \to \mathrm{Ext}^1_\Lambda(K_1, K_1) \to \mathrm{Ext}^2_\Lambda(K_1, \Lambda) \to \ldots.$$

The outside terms vanish as Π^0 is CM-injective and projective dimension of K_1 is at most 1. □

Proposition 3.4 *The endomorphism ring $\Gamma_\ell = \mathrm{End}_\Lambda(T_\ell)$ is again an R-order.*

Proof We need to show that $\mathrm{End}_\Lambda(T_\ell)$ is maximal Cohen-Macaulay as an R-module. Let us start with the direct sum decomposition $\mathrm{End}_\Lambda(T_\ell) \cong \mathrm{Hom}_\Lambda(\Pi, T_\ell) \oplus \mathrm{Hom}_\Lambda(K_\ell, T_\ell)$. We know that Π is a projective Λ-module which tells us that $\mathrm{Hom}_\Lambda(\Pi, T_\ell) \in \mathrm{add}_\Lambda T_\ell$. Hence, $\mathrm{Hom}_\Lambda(\Pi, T_\ell)$ is maximal Cohen-Macaulay since so is T_ℓ. Therefore, it is enough to show that $\mathrm{Hom}_\Lambda(K_\ell, T_\ell)$ is maximal Cohen-Macaulay. We further decompose this module as

$$\mathrm{Hom}_\Lambda(K_\ell, T_\ell) \cong \mathrm{Hom}_\Lambda(K_\ell, K_\ell) \oplus \mathrm{Hom}_\Lambda(K_\ell, \Pi).$$

Since Π is a projective module, we have $\mathrm{Hom}_\Lambda(K_\ell, \Pi)$ is a summand in $\mathrm{Hom}_\Lambda(K_\ell, \Lambda) \cong \mathrm{Hom}_R(K, \omega_R)$ which is a maximal Cohen-Macaulay module. So, it is enough to show that $\mathrm{Hom}_\Lambda(K_\ell, K_\ell)$ is maximal Cohen-Macaulay. We will do this by showing that $\mathrm{Hom}_\Lambda(K_j, K_\ell)$ is maximal Cohen-Macaulay for every $1 \leq j \leq \ell$.

We start with the short exact sequence $0 \to \Lambda \to \Pi^0 \to K_1 \to 0$ defining K_1 and we apply $\mathrm{Hom}_\Lambda(-, K_\ell)$ to it to get an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(K_1, K_\ell) \to \mathrm{Hom}_\Lambda(\Pi^0, K_\ell) \to \mathrm{Hom}_\Lambda(\Lambda, K_\ell) \to \mathrm{Ext}^1_\Lambda(K_1, K_\ell).$$

The rightmost term is zero. Indeed, K_1 is the $(\ell-1)^{th}$ syzygy of K_ℓ as a Λ-module. Therefore, $\mathrm{Ext}^1_\Lambda(K_1, K_\ell) \cong \mathrm{Ext}^\ell_\Lambda(K_\ell, K_\ell) \cong 0$ by the previous lemma. We have that Π^0 and Λ both projective. So, both $\mathrm{Hom}_\Lambda(\Pi^0, K_\ell)$ and $\mathrm{Hom}_\Lambda(\Lambda, K_\ell)$ are in $\mathrm{add}_\Lambda K_\ell$. Thus, they are both maximal Cohen-Macaulay. By the depth lemma, $\mathrm{Hom}_\Lambda(K_1, K_\ell)$ is also maximal Cohen-Macaulay. Now, assume that the result holds for $j < \ell$. We have a short exact sequence

$$0 \to K_j \to \Pi^j \to K_{j+1} \to 0$$

to which we apply $\mathrm{Hom}_\Lambda(-, K_\ell)$. We get an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(K_{j+1}, K_\ell) \to \mathrm{Hom}_\Lambda(\Pi^j, K_\ell) \to \mathrm{Hom}_\Lambda(K_j, K_\ell) \to \mathrm{Ext}^1_\Lambda(K_{j+1}, K_\ell).$$

By the same argument as above, the rightmost term is zero and the second term is maximal Cohen-Macaulay. By the induction hypothesis, the third terms is also maximal Cohen-Macaulay. Therefore, by the depth lemma, $\mathrm{Hom}_\Lambda(K_{j+1}, K_\ell)$ is maximal Cohen-Macaulay which finishes the proof. □

Definition 3.5 We call the module T_ℓ the ℓ^{th}*-shifted module* of Λ and $\Gamma_\ell = \mathrm{End}_\Lambda(T_\ell)$ the ℓ^{th}*-shifted order*.

We choose the terminology after Matthew Pressland and Julia Sauter [5, Definition 2.5].

Lemma 3.6 *The ℓ^{th}-shifted module T_ℓ has CM-injective dimension $n - \ell$.*

Proof The ℓ^{th}-shifted module is the direct sum of a CM-projective-injective with the cokernel K_ℓ in Lemma 3.1. If $0 \to \Lambda \to \Pi^0 \to \Pi^1 \to \ldots$ is a CM-projective-injective coresolution of Λ, then $0 \to K_\ell \to \Pi^\ell \to \ldots$ is a CM-injective coresolution of K. □

3.1 Towards the Main Theorem

For the purposes of the next three lemmas and the following proposition, we will assume that Λ is an n-canonical R-order, M is a maximal Cohen-Macaulay Λ-module with CM-injective dimension m. Let us denote by $K^{-,b}(\mathrm{add}M)$ the homotopy category of complexes of Λ-modules with terms in $\mathrm{add}_\Lambda M$, bounded below and with finitely many non-vanishing cohomology groups. We consider a complex $X = (X^i, d^i) \in K^{-,b}(\mathrm{add}M)$ with vanishing nonnegative cohomology and put $L_j = \ker d^j$ for $j \leq 0$. We also let $L_1 = \mathrm{im} d^0$ and $L_2 = X^1/\mathrm{im} d^0$. Hence, we have short exact sequences

$$0 \to L_j \to X^j \to L_{j+1} \to 0 \tag{3}$$

for $j \leq 1$. Assume that $\mathrm{Ext}^i_\Lambda(M, M) = 0$ for any $i > 0$.

Lemma 3.8 *With the notation as above, we have that L_j is a maximal Cohen-Macaulay Λ-module for every $j \leq 2 - d$.*

Proof The proof is by applying the depth lemma to the short exact sequences (3). □

Lemma 3.9 *Suppose $t \leq 2$. If $i - t \geq m + d - 1$, then $\mathrm{Ext}^i_\Lambda(L_t, M) = 0$.*

Proof Applying $\mathrm{Hom}_\Lambda(-, M)$ to the short exact sequences (3) yields long exact sequences

$$\ldots \to \mathrm{Ext}^i_\Lambda(X^j, M) \to \mathrm{Ext}^i_\Lambda(L_j, M) \to \mathrm{Ext}^{i+1}_\Lambda(L_{j+1}, M) \to \mathrm{Ext}^{i+1}_\Lambda(X^j, M) \to \ldots$$

The outside two terms vanish for $i > 0$ as $X^j \in \mathrm{add}M$ and we assumed that $\mathrm{Ext}^i_\Lambda(M, M)$ vanishes for $i > 0$. So, we have isomorphisms

$$\mathrm{Ext}^i_\Lambda(L_j, M) \cong \mathrm{Ext}^{i+1}_\Lambda(L_{j+1}, M)$$

for all $i > 0$ and $j \leq 1$. This gives us the vanishings

$$\mathrm{Ext}^1_\Lambda(L_{j-m}, M) \cong \ldots \cong \mathrm{Ext}^m_\Lambda(L_{j-1}, M) \cong \mathrm{Ext}^{m+1}_\Lambda(L_j, M) \cong 0$$

and

$$0 \cong \mathrm{Ext}^{m+1}_\Lambda(L_j, M) \cong \mathrm{Ext}^{m+2}_\Lambda(L_{j+1}, M) \cong \ldots \cong \mathrm{Ext}^{m+2-j}_\Lambda(L_1, M)$$

for $j \leq 2 - d$. In other words, if $i - t = m - 1 - j$, then $\mathrm{Ext}^i_\Lambda(L_t, M) = 0$. This is because we have $L_j \in \mathrm{CM}(\Lambda)$ for $j \leq 2 - d$ and M has CM-injective dimension m. In particular, we have $\mathrm{Ext}^i_\Lambda(L_t, M) = 0$ provided $i - t \geq m + d - 1$. □

Lemma 3.10 *If $i - t \geq m + d - 1$, then, we have $\mathrm{Ext}^i_\Lambda(L_t, L_{j+1}) \cong \mathrm{Ext}^{i+1}_\Lambda(L_t, L_j)$ for any $j \leq 1$.*

Proof By applying $\mathrm{Hom}_\Lambda(L_t, -)$ to the short exact sequences (3), we get a long exact sequence

$$\ldots \to \mathrm{Ext}^i_\Lambda(L_t, X^j) \to \mathrm{Ext}^i_\Lambda(L_t, L_{j+1}) \to \mathrm{Ext}^{i+1}_\Lambda(L_t, L_j) \to \mathrm{Ext}^{i+1}_\Lambda(L_t, X^j) \to \ldots.$$

When $i - t \geq m + d - 1$, the outer terms vanish by Lemma 3.9. Hence, we have the desired isomorphisms. □

Proposition 3.11 *Assume that Λ has finite global dimension. If $t \leq 2 - d - m$, then $\mathrm{Ext}^1_\Lambda(L_t, L_{j+1}) = 0$ for any $j \leq 1$. In particular, we have*

$$\mathrm{Ext}^1_\Lambda(L_{2-d-m}, L_{1-d-m}) = 0.$$

Proof The condition $t \leq 2 - d - m$ is equivalent to the condition $1 - t \leq m + d - 1$. So, we can apply Lemma 3.10 to get

$$\mathrm{Ext}^1_\Lambda(L_t, L_{j+1}) \cong \mathrm{Ext}^2_\Lambda(L_t, L_j) \cong \ldots \cong \mathrm{Ext}^n_\Lambda(L_t, L_{j+2-n}) \cong \mathrm{Ext}^{n+1}_\Lambda(L_t, L_{j+1-n}).$$

Since $t \leq 2 - d - m \leq 2 - d$, we get that $L_t \in \mathrm{CM}(\Lambda)$. Therefore, its projective dimension is bounded above by n. This gives us that $\mathrm{Ext}^{n+1}_\Lambda(L_t, L_{j+1-n})$ vanishes which finishes the proof. □

Corollary 3.12 *Let Λ be an n-canonical R-order of finite global dimension and M be a Λ-module such that $\mathrm{Ext}^i_\Lambda(M, M) = 0$ for all $i > 0$. For any $X \in K^{-,b}(\mathrm{add} M)$ with vanishing non-negative cohomology, we have $X = Y \oplus Z$ where Y is acyclic and $Z^i = 0$ for all $i < 1 - m - d$ where m is the CM-injective dimension of M.*

Proof With the notation as above, we have short exact sequences

$$0 \to L_j \to X^j \to L_{j+1} \to 0$$

and we now know that $\mathrm{Ext}^1_\Lambda(L_{2-d-m}, L_{1-d-m}) = 0$ so that the sequence

$$0 \to L_{1-d-m} \to X^{1-d-m} \to L_{2-d-m} \to 0$$

splits and we have a decomposition $X^{1-d-m} \cong L_{1-d-m} \oplus L_{2-d-m}$. In particular, we have that $L_{1-d-m}, L_{2-d-m} \in \mathrm{CM}(\Lambda)$. Therefore, choosing Y and Z as follows give us the desired direct sum decomposition.

$$Y = \ldots \longrightarrow X^{-1-d-m} \longrightarrow X^{-d-m} \longrightarrow L_{1-d-m} \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

$$Z = \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow L_{2-d-m} \longrightarrow X^{2-d-m} \longrightarrow X^{3-d-m} \longrightarrow \ldots$$

□

Theorem 3.13 *Suppose that Λ is an R-order of finite global dimension and $M \in \mathrm{CM}(\Lambda)$ is an ℓ-tilting module of CM-injective dimension m. Then, we have*

$$\mathrm{gldim}\,\mathrm{End}_\Lambda(M) \leq \ell + m + d.$$

Proof Let us start with putting $\Gamma = \mathrm{End}_\Lambda(M)$. Since Λ is of finite global dimension and Γ is derived equivalent to Λ, we know that Γ also has finite global dimension. We will show that any $X \in \mathrm{mod}\Gamma$ has projective dimension at most $\ell + m + d$.

Let P be a minimal projective resolution of X. Since P is minimal, the width of P is $\mathrm{pd}X + 1$. The equivalence between $D^b(\Lambda)$ and $D^b(\Gamma)$ restricts to an equivalence between $K^{-,b}(\mathrm{add}M)$ and $K^{-,b}(\mathrm{proj}\Gamma)$. The image of $P \in K^{-,b}(\mathrm{proj}\Gamma)$ under this equivalence is $M \otimes_\Gamma P$. Since P is minimal, it has no nonzero acyclic summands and the same holds for $M \otimes_\Gamma P$.

Since M is a tilting Λ-module of projective dimension at most ℓ, it has projective dimension at most ℓ as a Γ-module. Hence,

$$H^i(M \otimes_\Gamma P) = \mathrm{Tor}^\Gamma_{-i}(M, X) = 0$$

for $i < -\ell$. Thus, $M \otimes_\Gamma P[-\ell - 1]$ has no non-negative cohomology and we can apply Corollary 3.12. This gives us a direct sum decomposition $M \otimes_\Gamma P[-\ell-1] = Y \oplus Z$ where Y is acylic and $Z^i = 0$ for $i < 1 - m - d$. But since $M \otimes_\Gamma P$ does not have any nonzero acylic summands, we must have $Y = 0$. Therefore, we actually have $M \otimes_\Gamma P[-1 - \ell] \cong Z$. So, we can conclude that Z^i is only allowed to be nonzero in the interval $1 - d - m \leq i \leq \ell + 1$. Consequently, we have an upper bound on the width of $M \otimes_\Gamma P$. Namely, $\ell + m + d + 1$. This means that the width of P is bounded above by $\ell + m + d + 1$ and thus the projective dimension of X is bounded above by $\ell + m + d$. □

Corollary 3.14 *Suppose that Λ is an n-canonical QF-3 order of finite global dimension. Then, the global dimension of the ℓ^{th}-shifted order Γ_ℓ is bounded above by the global dimension of Λ.*

Proof The ℓ^{th}-shifted module T_ℓ is an ℓ-tilting module in $\mathrm{CM}(\Lambda)$ which has CM-injective dimension $n - \ell$ by Lemma 3.6. Thus by Theorem 3.13, we get that

$$\mathrm{gldim}\Gamma_\ell \leq n + d = \mathrm{gldim}\Lambda.$$

□

Acknowledgements I would like to thank Graham Leuschke for adopting me as a PhD student and introducing me to work of Pressland and Sauter. I also thank Osamu Iyama, Benjamin Briggs, Vincent Gélinas and Louis-Philippe Thibault for many fruitful discussions and the anonymous referee for thoughtful comments that improved not only this paper but also my understanding of the subject.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declaration

Competing interest The authors have no competing interests to declare that are relevant to the content of this article.

References

1. Auslander, M., Roggenkamp, K.W.: A characterization of orders of finite lattice type. Invent. Math. **17**, 79–84 (1972)
2. Auslander, M., Reiten, I.: The Cohen-Macaulay type of Cohen-Macaulay rings. Adv. in Math. **73**(1), 1–23 (1989)
3. Eisenbud, D., Harris, J.: On varieties of minimal degree (a centennial account). In: Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), vol. 46 of Proc. Sympos. Pure Math. pp. 3–13. Amer. Math. Soc., Providence, RI, (1987)
4. Iyama, O.: Auslander correspondence. Adv. Math. **210**(1), 51–82 (2007)
5. Pressland, M., Sauter, J.: Special tilting modules for algebras with positive dominant dimension. Glasg. Math. J. 1–27, (2020)
6. Stangle, J.: Representation Theory of Orders over Cohen-Macaulay Rings. ProQuest LLC, Ann Arbor, MI, 2017. Thesis (Ph.D.)–Syracuse University (2017)
7. Tachikawa, H.: Quasi-Frobenius rings and generalizations. QF − 3 and QF − 1 rings. Lecture Notes in Mathematics, Vol. 351. Springer-Verlag, Berlin-New York (1973). Notes by Claus Michael Ringel
8. Thrall, R.M.: Some generalization of quasi-Frobenius algebras. Trans. Amer. Math. Soc. **64**, 173–183 (1948)
9. Yoshino, Y.: Cohen-Macaulay modules over Cohen-Macaulay rings, vol. 146. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge (1990)