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# An Algorithmic Framework for Locally Constrained Homomorphisms<sup>\*</sup>

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**Abstract.** A homomorphism  $\phi$  from a guest graph  $G$  to a host graph  $H$  is locally bijective, injective or surjective if for every  $u \in V(G)$ , the restriction of  $\phi$  to the neighbourhood of  $u$  is bijective, injective or surjective, respectively. We prove a number of new FPT, W[1]-hard and paraNP-complete results for the corresponding decision problems LBHOM, LIHOM and LSHOM by considering a hierarchy of parameters of the guest graph  $G$ . In this way we strengthen several existing results. For our FPT results, we develop a new algorithmic framework that involves a general ILP model. We also use our framework to prove FPT results for the ROLE ASSIGNMENT problem, which originates from social network theory and is closely related to locally surjective homomorphisms.

**Keywords:** (locally constrained) graph homomorphism · parameterized complexity · fracture number

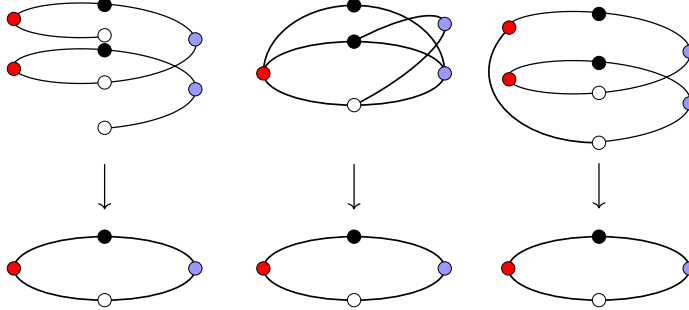
## 1 Introduction

A *homomorphism* from a graph  $G$  to a graph  $H$  is a mapping  $\phi : V(G) \rightarrow V(H)$  such that  $\phi(u)\phi(v) \in E(H)$  for every  $uv \in E(G)$ . Graph homomorphisms generalise graph colourings (using a complete graph for  $H$ ) and have been intensively studied over a long period of time, both from a structural and an algorithmic perspective. We refer to the textbook of Hell and Nešetřil [51] for a further introduction.

We write  $G \rightarrow H$  if there exists a homomorphism from  $G$  to  $H$ ; here,  $G$  is called the *guest graph* and  $H$  is the *host graph*. We denote the corresponding decision problem by HOM, and if  $H$  is fixed, that is, not part of the input, we write  $H$ -HOM. For graphs  $H$  without self-loops, the renowned Hell-Nešetřil dichotomy [49] states that  $H$ -HOM is

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**Fig. 1.** Left: an example of a locally injective homomorphism which is not locally surjective. Middle: an example of a locally surjective homomorphism which is not locally injective. Right: an example of a locally bijective homomorphism.

polynomial-time solvable if  $H$  is bipartite, and NP-complete otherwise. We denote the vertices of  $H$  by  $1, \dots, |V(H)|$  and call them *colours*. The reason for doing this is that graph homomorphisms generalise graph colourings: there exists a homomorphism from a graph  $G$  to the complete graph on  $k$  vertices if and only if  $G$  is  $k$ -colourable.

Instead of fixing the host graph  $H$ , one can also restrict the structure of the guest graph  $G$  by bounding some graph parameter. A classical result states that HOM is polynomial-time solvable when the guest graph  $G$  has bounded treewidth [20,42]. The *core* of a graph  $G$  is the subgraph  $F$  of  $G$  such that  $G \rightarrow F$  and there is no proper subgraph  $F'$  of  $F$  with  $G \rightarrow F'$  (the core is unique up to isomorphism [50]). Dalmau, Kolaitis and Vardi [23] proved that the HOM problem is polynomial-time solvable even if the core of the guest graph  $G$  has bounded treewidth. This result was strengthened by Grohe [47], who proved that if  $\text{FPT} \neq \text{W}[1]$ , then HOM can be solved in polynomial time if and only if this condition holds.

### 1.1 Locally Constrained Homomorphisms

We are interested in three well-studied variants of graph homomorphisms that occur after placing constraints on the neighbourhoods of the vertices of the guest graph  $G$ . Consider a homomorphism  $\phi$  from a graph  $G$  to a graph  $H$ . We say that  $\phi$  is *locally injective*, *locally bijective* or *locally surjective* for a vertex  $u \in V(G)$  if the restriction  $\phi_u : N_G(u) \rightarrow N_H(\phi(u))$  of  $\phi$  is injective, bijective or surjective, respectively. Here,  $N_G(u) = \{v \mid uv \in E(G)\}$  denotes the (open) neighbourhood of a vertex  $u$  in a graph  $G$ . We say that  $\phi$  is *locally injective*, *locally bijective* or *locally surjective* if  $\phi$  is locally injective, locally bijective or locally surjective for every  $u \in V(G)$ . We denote the existence of these *locally constrained* homomorphisms by  $G \xrightarrow{\text{E}} H$ ,  $G \xrightarrow{\text{L}} H$  and  $G \xrightarrow{\text{S}} H$ , respectively; see Figure 1 for some examples.

The three locally constrained variants have been well studied in several settings over a long period of time. For example, locally injective homomorphisms are also known as

*partial graph coverings* and are used in telecommunications [34], in distance constrained labelling [33] and as indicators of the existence of homomorphisms of derivative graphs [66]. Locally bijective homomorphisms originate from topological graph theory [5,65] and are more commonly known as *graph coverings*. They are used in distributed computing [2,3,8] and in constructing highly transitive regular graphs [6]. Locally surjective homomorphisms are sometimes called *colour dominations* [60]. They have applications in distributed computing [16,17] and in social science [31,71,74,76]. In the latter context they are known as *role assignments*, as we will explain in more detail below. (see Section 1.2).

We study the following three decision problems that take two graphs  $G$  and  $H$  as input and ask if there exists a homomorphism from  $G$  to  $H$  of one of the three local kinds.

Locally Bijective Homomorphism (LBHOM)

**Input:** Graphs  $G$  and  $H$ .

**Question:** Does  $G \xrightarrow{E} H$  hold?

Locally Injective Homomorphism (LIHOM)

**Input:** Graphs  $G$  and  $H$ .

**Question:** Does  $G \xrightarrow{I} H$  hold?

Locally Surjective Homomorphism (LSHOM)

**Input:** Graphs  $G$  and  $H$ .

**Question:** Does  $G \xrightarrow{S} H$  hold?

As before, we use the notation  $H$ -LBHOM,  $H$ -LIHOM and  $H$ -LSHOM in the case when the host graph  $H$  is fixed.

Out of the three problems, only the complexity of  $H$ -LSHOM has been completely classified, both for general graphs and bipartite graphs [38]. We refer to a series of papers [1,7,34,36,57,58,63] for polynomial-time solvable and NP-complete cases of  $H$ -LBHOM and  $H$ -LIHOM; see also the survey by Fiala and Kratochvíl [35]. Some more recent results include sub-exponential algorithms for  $H$ -LBHOM,  $H$ -LIHOM and  $H$ -LSHOM on string graphs [68], complexity results for the list version of  $H$ -LSHOM [29] and complexity results for  $H$ -LBHOM for host graphs  $H$  that are multigraphs and/or have semi-edges [10,11,12,59].

In our paper we assume that both  $G$  and  $H$  are part of the input. We note a fundamental difference between locally injective homomorphisms on the one hand and locally bijective and surjective homomorphisms on the other. Namely, for connected graphs  $G$  and  $H$ , we must have  $|V(G)| \geq |V(H)|$  if  $G \xrightarrow{E} H$  or  $G \xrightarrow{S} H$  (see Lemma 1 in Section 2.3). In contrast,  $H$  might be arbitrarily larger than  $G$  if  $G \xrightarrow{I} H$  holds. For example, if we let  $G$  be a complete graph and  $H$  be a graph without self-loops, then  $G \xrightarrow{I} H$  holds if and only if  $H$  contains a clique on at least  $|V(G)|$  vertices.

The above difference is also reflected in the complexity results for the three problems under input restrictions. In fact, LIHOM is closely related to the SUBGRAPH ISOMORPHISM problem and is usually the hardest problem. For example, LBHOM is GRAPH ISOMORPHISM-complete on chordal guest graphs, but polynomial-time solvable on interval guest graphs and LSHOM is NP-complete on chordal guest graphs, but polynomial-time solvable on proper interval guest graphs [48]. In contrast, LIHOM is NP-complete even on complete guest

graphs  $G$ , which follows from a reduction from the CLIQUE problem via the aforementioned equivalence:  $G \xrightarrow{L} H$  holds if and only if  $H$  contains a clique on at least  $|V(G)|$  vertices.

Finally, we emphasize that the aforementioned polynomial-time result on HOM for guest graphs  $G$  with a core of bounded treewidth [20,42] does not carry over to any of the three locally constrained homomorphism problems. Indeed, LBHOM, LSHOM and LIHOM are NP-complete for guest graphs  $G$  of path-width at most 5, 4 and 2, respectively [19] (all three problems are polynomial-time solvable if  $G$  is a tree [19,39]). It is also known that LBHOM [56], LSHOM [60] and LIHOM [34] are NP-complete even if  $G$  is cubic and  $H$  is the complete graph  $K_4$  on four vertices, but polynomial-time solvable if  $G$  has bounded treewidth and one of the two graphs  $G$  or  $H$  has bounded maximum degree [19].

## 1.2 An Application: Role Assignments

Locally surjective homomorphisms from a graph  $G$  to a graph  $H$  are known as  $H$ -role assignments in social network theory. We will include this topic in our investigation and provide some brief context.

Suppose that we are given a social network of individuals whose properties we aim to characterise. Can we assign each individual a role such that individuals with the same role relate in the same way to other individuals with some role, using exactly  $h$  different roles in total?

To formalise the above question, we model the network as a graph  $G$ , where vertices represent individuals and edges represent the existence of a relationship between two individuals. We now ask whether  $G$  has an  $h$ -role assignment, that is, a function  $f$  that assigns each vertex  $u \in V(G)$  a role  $f(u) \in \{1, \dots, h\}$ , such that  $f(V(G)) = \{1, \dots, h\}$  and for every two vertices  $u$  and  $v$ , if  $f(u) = f(v)$  then  $f(N_G(u)) = f(N_G(v))$ .

Role assignments were introduced by White and Reitz [76] as *regular equivalences* and were called *role colourings* by Everett and Borgatti [31]. We observe that two adjacent vertices  $u$  and  $v$  may have the same role, that is,  $f(u) = f(v)$  is allowed (so role assignments are not proper colourings). Hence, a connected graph  $G$  has an  $h$ -role assignment if and only if  $G \xrightarrow{S} H$  for some connected graph  $H$  with  $|V(H)| = h$ , as long as we allow  $H$  to have self-loops (while we assume that  $G$  is a graph with no self-loops). The corresponding decision problem is the following:

ROLE ASSIGNMENT

**Input:** A graph  $G$  and an integer  $h$ .

**Question:** Does  $G$  have an  $h$ -role assignment?

If  $h$  is fixed, then we denote the problem  $h$ -ROLE ASSIGNMENT. Whereas 1-ROLE ASSIGNMENT is trivial, 2-ROLE ASSIGNMENT is NP-complete [74]. In fact,  $h$ -ROLE ASSIGNMENT is NP-complete even for the following classes of graphs: planar graphs ( $h \geq 2$ ) [72], cubic graphs ( $h \geq 2$ ) [73], bipartite graphs ( $h \geq 3$ ) [70], chordal graphs ( $h \geq 3$ ) [52] and split graphs ( $h \geq 4$ ) [25]. Very recently, Pandey, Raman and Sahlo [69] gave an  $n^{\mathcal{O}(h)}$ -time algorithm for ROLE ASSIGNMENT on general graphs and an  $f(h)n^{\mathcal{O}(1)}$ -time algorithm on forests.

### 1.3 Our Focus

We continue the line of study in [19] and focus on the following research question:

*For which parameters of the guest graph do LBHOM, LSHOM and LIHOM become fixed-parameter tractable?*

We will also apply our new techniques towards answering this question for the ROLE ASSIGNMENT problem. We first introduce some additional terminology. A graph parameter  $p$  *dominates* a parameter  $q$  if there is a function  $f$  such that  $p(G) \leq f(q(G))$  for every graph  $G$ . If  $p$  dominates  $q$  but  $q$  does not dominate  $p$ , then  $p$  is *more powerful* than  $q$ . We denote this by  $p \triangleright q$ . If  $p$  dominates  $q$  and  $q$  dominates  $p$ , then  $p$  and  $q$  are *equivalent*. If neither  $p$  dominates  $q$  nor  $q$  dominates  $p$ , then  $p$  and  $q$  are *incomparable*.

Given the paraNP-hardness results on LBHOM, LSHOM and LIHOM for graph classes of bounded path-width [19], we will naturally consider a range of graph parameters that are less powerful than path-width. In this way we aim to increase our understanding of the (parameterized) complexity of LBHOM, LSHOM and LIHOM.

For an integer  $c \geq 1$ , a  $c$ -*deletion set* of a graph  $G$  is a subset  $S \subseteq V(G)$  such that every connected component of the graph  $G \setminus S$  has at most  $c$  vertices.<sup>6</sup> The  $c$ -*deletion set number*  $ds_c(G)$  of  $G$  is the minimum size of a  $c$ -deletion set in  $G$ . If  $c = 1$  we obtain the *vertex cover number*  $vc(G)$  of  $G$ . The  $c$ -deletion set number is closely related to the *fracture number*  $fr(G)$ , introduced by Dvořák et al. [28], which is the minimum  $k$  such that  $G$  has a  $k$ -deletion set of size at most  $k$ . For a graph  $G$ , it holds that  $fr(G) \leq ds_c(G)$  if  $c \leq fr(G) - 1$ , and  $ds_c(G) \leq fr(G)$  if  $c \geq fr(G)$ . Hence, in particular it holds for every integer  $c \geq 1$  that  $fr(G) \leq \max\{c, ds_c(G)\}$ . As we can take the [graph formed as the disjoint union](#) of arbitrarily many complete graphs on  $c + 1$  vertices (which has an arbitrarily large  $c$ -deletion set number while its fracture number is  $c + 1$ ), this inequality shows that for fixed  $c$ ,  $fr \triangleright ds_c$ . However, if  $c$  is not fixed, then  $fr$  and  $c + ds_c$  are equivalent, as in that case  $c + ds_c(G) \leq 2fr(G)$  and  $fr(G) \leq c + ds_c(G)$  holds for every graph  $G$ .

The fracture number is equivalent to several other well-studied graph parameters, such as the *vertex integrity*, introduced by Barefoot, Entringer and Swart [4] or the *safe number*, introduced by Fujita, MacGillivray and Sakuma [44]. The vertex integrity of a graph  $G$  is the minimum value  $|X| + n^c(G \setminus X)$  over all sets  $X \subseteq V(G)$ , where  $n^c(G \setminus X)$  denotes the size of a largest connected component of  $G \setminus X$ . Hence, the equivalence between the fracture number and the vertex integrity follows directly from their definitions, whereas the equivalence between the safe number and vertex integrity (and thus fracture number) is shown by Fujita and Furuya [43].

The *feedback vertex set number*  $fv(G)$  of a graph  $G$  is the size of a smallest set  $S$  such that  $G \setminus S$  is a forest. We write  $tw(G)$ ,  $pw(G)$ ,  $td(G)$  and  $n(G)$  for the treewidth, path-width, tree-depth<sup>7</sup> and number of vertices of a graph  $G$ , respectively; see [67] for more information. It is known that

$$tw \triangleright pw \triangleright td \triangleright fr \triangleright ds_c(\text{fixed } c) \triangleright vc \triangleright n,$$

<sup>6</sup> The graph  $G \setminus S$  is the graph obtained from  $G$  by deleting all vertices of  $S$ ; see Section 2 for any undefined terminology in this section.

<sup>7</sup> See Section 2 for the definitions of treewidth, path-width and tree-depth.

where the second relationship is proven in [9] and the others follow immediately from their definitions (see also Section 2.2). It is readily seen that

$$\text{tw} \triangleright \text{fv} \triangleright \text{ds}_2$$

and moreover, that  $\text{fv}$  is incomparable with the parameters  $\text{pw}$ ,  $\text{td}$ ,  $\text{fr}$  and, for every fixed  $c \geq 3$ ,  $\text{ds}_c$ ; consider, for example, a tree  $T$  of arbitrarily large path-width, whilst  $\text{fv}(T) = 0$ , and consider also the disjoint union  $G$  of arbitrarily many triangles, which has an arbitrarily large feedback vertex set number, but  $\text{ds}_c(G) = 0$  for every  $c \geq 3$ .

guest graph parameter	LIHOM	LBHOM	LSHOM
$ V(G) $	XP, W[1]-hard [26]	FPT	FPT
vertex cover number	XP (Theorem 5), W[1]-hard	FPT	FPT
$c$ -deletion set number (fixed $c$ )	paraNP-c ( $c \geq 2$ ) (Theorem 6)	FPT	FPT
fracture number	paraNP-c	FPT (Theorem 2)	FPT (Theorem 2)
tree-depth	paraNP-c	paraNP-c (Theorem 7)	paraNP-c (Theorem 7)
path-width	paraNP-c [19]	paraNP-c [19]	paraNP-c [19]
treewidth	paraNP-c	paraNP-c	paraNP-c
maximum degree	paraNP-c [34]	paraNP-c [56]	paraNP-c [60]
treewidth plus maximum degree	XP, W[1]-hard	XP [19]	XP [19]
feedback vertex set number	paraNP-c	paraNP-c (Theorem 8)	paraNP-c (Theorem 8)

**Table 1.** The results in purple are our new results and are annotated with the corresponding theorem numbers. The results in black are either known results, some of which are now also implied by our new results, or follow immediately from other results in the table. In particular, LIHOM is W[1]-hard with parameter  $|V(G)|$ , as CLIQUE is W[1]-hard when parameterized by the clique number [26], so we can let  $G$  be the complete graph in this case.

## 1.4 Our Results

We prove a number of new parameterized complexity results for LBHOM, LSHOM and LIHOM by considering some property of the guest graph  $G$  as the parameter. In particular, we consider the graph parameters introduced in Section 1.3.

Our two main results, proven in Section 4, show that LBHOM and LSHOM are fixed-parameter tractable parameterized by  $c + \text{ds}_c$ , or equivalently, the fracture number  $\text{fr}(G)$ , of the guest graph  $G$ . In the same section, we also prove that ROLE ASSIGNMENT is FPT when parameterized by the fracture number. Recall that  $\text{td} \triangleright \text{fr}$ . However, assuming  $\text{P} \neq \text{NP}$ , the FPT results for LBHOM and LSHOM involving the fracture number cannot be strengthened to the tree-depth of the guest graph. Namely, we prove in Section 6 that LBHOM and LSHOM are paraNP-complete when parameterized by the tree-depth of the guest graph. Recall also that  $\text{pw} \triangleright \text{td}$ . Hence, these paraNP-completeness results imply the known paraNP-completeness results for path-width of the guest graph [19].

In Section 6 we also prove that LBHOM and LSHOM are paraNP-complete when parameterized by the feedback vertex set number of the guest graph. In fact, this result and the paraNP-hardness for tree-depth motivated us to consider the fracture number as a natural graph parameter for obtaining an FPT-algorithm.

For a fixed integer  $k \geq 1$ , the  $k$ -FOLDCOVER problem is the restriction of LBHOM to input pairs  $(G, H)$  where  $|V(G)| = k|V(H)|$ . This problem was introduced as the  $k$ -GRAPH

COVERING problem by Bodlaender [8]. The aforementioned paraNP-completeness result of [19] on LBHOM for pairs  $(G, H)$  where  $G$  has path-width 5 holds even for 3-FOLDCOVER. In fact, this was the first proof for showing that  $k$ -FOLDCOVER is NP-complete for some fixed integer  $k \geq 1$ .<sup>8</sup> Similarly, we will show in Section 6 that the two paraNP-completeness results for LBHOM that we described above also hold even for the 3-FOLDCOVER problem.

In Section 5 we prove that LIHOM is in XP and W[1]-hard when parameterized by the vertex cover number of the guest graph, or equivalently, the  $c$ -deletion set number for  $c = 1$ . However, we show that the XP result for LIHOM cannot be generalised to hold for  $c \geq 2$ . In fact, in Section 5, we will give a full complexity dichotomy. Namely, we determine the computational complexity of LIHOM on graphs with  $c$ -deletion set number at most  $k$  for every fixed pair of integers  $c$  and  $k$ .

Table 1 summarizes the new and known results for LBHOM, LSHOM and LIHOM.

### 1.5 Algorithmic Framework

Our FPT results for LBHOM, LSHOM and ROLE ASSIGNMENT and our XP result for LIHOM are proven via a new algorithmic framework (described in detail in Section 3). This framework involves a reduction to an integer linear program (ILP). We emphasize that in our framework the host graph  $H$  is *not* fixed, but part of the input. This is in contrast to other frameworks that include the locally constrained homomorphism problems (and that consequently work for more powerful graph parameters), such as the framework of locally checkable vertex partitioning problems [13,75] or the framework of Gerber and Kobler [46] based on (feasible) interval degree constraint matrices.

The main ideas behind our algorithmic ILP framework are as follows. Let  $G$  and  $H$  be the guest and host graphs, respectively. First, we observe that if  $G$  has a  $c$ -deletion of size at most  $k$  and there is a locally surjective homomorphism from  $G$  to  $H$ , then  $H$  must also have a  $c$ -deletion set of size at most  $k$ . However, it does not suffice to compute  $c$ -deletion sets  $D_G$  and  $D_H$  for  $G$  and  $H$ , guess a partial homomorphism  $h$  from  $D_G$  to  $D_H$ , and use the structural properties of  $c$ -deletion sets to decide whether  $h$  can be extended to a desired homomorphism from  $G$  to  $H$ . This is because a homomorphism from  $G$  to  $H$  does not necessarily map  $D_G$  to  $D_H$ . Moreover, even if it did, vertices in  $G \setminus D_G$  can still be mapped to vertices in  $D_H$ . Consequently, components of  $G \setminus D_G$  can still be mapped to more than one component of  $H \setminus D_H$ .

The above makes it difficult to decompose the homomorphism from  $G$  to  $H$  into small independent parts. To overcome this challenge, we prove that there are small sets  $D_G$  and  $D_H$  of vertices in  $G$  and  $H$ , respectively, such that every locally surjective homomorphism from  $G$  to  $H$  satisfies:

1. the pre-image of  $D_H$  is a subset of  $D_G$ ,
2.  $D_H$  is a  $c'$ -deletion set for  $H$  for some  $c'$  bounded in terms of only  $c + k$ , and
3. all but at most  $k$  components of  $G \setminus D_G$  have at most  $c$  vertices, whilst the treewidth of the remaining (and possibly large) components is bounded in terms of  $c + k$ .

<sup>8</sup> The proof can easily be adapted to hold for every  $k \geq 3$ , as observed by Klavík [55], while the cases  $k = 1$  (which is equivalent to GRAPH ISOMORPHISM) and  $k = 2$  are still open.



As  $D_G$  and  $D_H$  are small, we can enumerate all possible homomorphisms from some subset of  $D_G$  to  $D_H$ . Condition 2 allows us to show that any locally surjective homomorphism from  $G$  to  $H$  can be decomposed into locally surjective homomorphisms from a small set of components of  $G \setminus D_G$  (plus  $D_G$ ) to one component of  $H \setminus D_H$  (plus  $D_H$ ). This enables us to formulate the question of whether a homomorphism from a subset of  $D_G$  to  $D_H$  can be extended to a desired homomorphism from  $G$  to  $H$  in terms of an ILP. Finally, Condition 3 allows us to efficiently compute the possible parts of the decomposition, that is, which (small) sets of components of  $G \setminus D_G$  can be mapped to which components of  $H \setminus D_H$ .

## 2 Preliminaries

We use standard notation from graph theory, as can be found in e.g. [24]. Let  $G$  be a graph. We denote the vertex set and edge set of  $G$  by  $V(G)$  and  $E(G)$ , respectively. Let  $X \subseteq V(G)$  be a set of vertices of  $G$ . The *subgraph of  $G$  induced by  $X$* , denoted  $G[X]$ , is the graph with vertex set  $X$  and edge set  $\{uv \in E(G) \mid u, v \in X\}$ . When the underlying graph is clear from the context, we will sometimes refer to an induced subgraph simply by its set of vertices. We use  $G \setminus X$  to denote the subgraph of  $G$  induced by  $V(G) \setminus X$ . Similarly, for  $Y \subseteq E(G)$  we let  $G \setminus Y$  be the subgraph of  $G$  obtained by deleting all edges in  $Y$  from  $G$ .

For a graph  $G$  and a vertex  $u \in V(G)$ , we let  $N_G(u) = \{v \mid uv \in E(G)\}$  and  $N_G[v] = N_G(u) \cup \{u\}$  denote the open and closed neighbourhood of  $u$  in  $G$ , respectively. We let  $\Delta(G)$  be the maximum degree of  $G$ . Recall that we assume that the guest graph  $G$  does not contain self-loops, while the host graph  $H$  is permitted to have self-loops. In this case, by definition,  $u \in N_H(u)$  if  $uu \in E(H)$ .

### 2.1 Parameterized Complexity

In parameterized complexity [22,27,40], the complexity of a problem is studied not only with respect to the input size, but also with respect to some problem parameter(s). A *parameterized problem*  $Q$  is a subset of  $\Omega^* \times \mathbb{N}$ , where  $\Omega$  is a fixed alphabet. Each instance of  $Q$  is a pair  $(I, \kappa)$ , where  $\kappa \in \mathbb{N}$  is called the *parameter*. A parameterized problem  $Q$  is *fixed-parameter tractable* (FPT) if there is an algorithm, called an *FPT-algorithm*, that decides whether an input  $(I, \kappa)$  is a member of  $Q$  in time  $f(\kappa) \cdot |I|^{\mathcal{O}(1)}$ , where  $f$  is a computable function and  $|I|$  is the size of the input instance. The class FPT denotes the class of all fixed-parameter tractable parameterized problems.

A parameterized problem  $Q$  is *FPT-reducible* to a parameterized problem  $Q'$  if there is an algorithm, called an *FPT-reduction*, that transforms each instance  $(I, \kappa)$  of  $Q$  into an instance  $(I', \kappa')$  of  $Q'$  in time  $f(\kappa) \cdot |I|^{\mathcal{O}(1)}$ , such that  $\kappa' \leq g(\kappa)$  and  $(I, \kappa) \in Q$  if and only if  $(I', \kappa') \in Q'$ , where  $f$  and  $g$  are computable functions. By *FPT-time*, we denote time of the form  $f(\kappa) \cdot |I|^{\mathcal{O}(1)}$ , where  $f$  is a computable function. A problem  $\Pi$  is in  $W[1]$  if it is FPT-reducible to INDEPENDENT SET parameterized by the solution size and  $W[1]$ -hard if the latter problem is FPT-reducible to  $\Pi$ ; it is easy to verify that  $FPT \subseteq W[1]$ . As an analogue to the conjecture that  $P \neq NP$ , it is widely believed that  $FPT \neq W[1]$ .

The class XP contains all parameterized problems that can be solved in  $\mathcal{O}(|I|^{f(\kappa)})$  time, where  $f$  is a computable function. The class **paraNP** is the class of parameterized problems that can be solved by non-deterministic algorithms in  $f(\kappa) \cdot |I|^{\mathcal{O}(1)}$  time, where  $f$  is a computable function.

## 2.2 Graph Parameters

A *tree-decomposition*  $\mathcal{T}$  of a graph  $G$  is a pair  $(T, \chi)$ , where  $T$  is a tree and  $\chi$  is a function that assigns each tree node  $t$  a set  $\chi(t) \subseteq V(G)$  of vertices such that the following conditions hold:

- (i) For every edge  $uv \in E(G)$ , there is a tree node  $t$  such that  $u, v \in \chi(t)$ .
- (ii) For every vertex  $v \in V(G)$ , the set of tree nodes  $t$  with  $v \in \chi(t)$  induces a non-empty subtree of  $T$ .

The sets  $\chi(t)$  are called *bags* of the decomposition  $\mathcal{T}$  and  $\chi(t)$  is the bag associated with the tree node  $t$ . The *width* of a tree-decomposition  $(T, \chi)$  is the size of a largest bag minus 1. The *treewidth*  $\text{tw}(G)$  of  $G$  is the minimum width over all tree-decompositions of  $G$ . If  $T$  is a path, then we obtain the notions of path-decomposition and *path-width*.

For a rooted forest  $F$ , the *closure*  $C(F)$  is the graph with vertex set  $V(F)$  such that two vertices  $u$  and  $v$  are adjacent in  $C(F)$  if and only if  $u$  is an ancestor of  $v$  in  $F$ . We say that  $F$  is a *tree-depth decomposition* of a graph  $G$  if  $G$  is a subgraph of  $C(F)$ . The *depth* of  $F$  is equal to the height of  $F$  plus 1. The *tree-depth*  $\text{td}(G)$  of  $G$  is the minimum depth over all tree-depth decompositions of  $G$ .

We also need the following (well-known) fact on  $c$ -deletion sets.

**Proposition 1 ([61]).** *Let  $G$  be a graph and let  $k$  and  $c$  be natural numbers. Then, deciding whether  $G$  has a  $c$ -deletion set of size at most  $k$  is fixed-parameter tractable parameterized by  $k + c$ .*

**Definition 1.** *A  $(k, c)$ -extended deletion set for a graph  $G$  is a set  $D \subseteq V(G)$  such that:*

- every component of  $G \setminus D$  either has at most  $c$  vertices or has a  $c$ -deletion set of size at most  $k$  and
- at most  $k$  components of  $G \setminus D$  have more than  $c$  vertices.

The following proposition summarizes some known relationships between the parameters we consider.

**Proposition 2 ([67]).** *Let  $G$  be a graph and let  $k$  and  $c$  be natural numbers. Then:*

- if  $G$  has a  $c$ -deletion set of size at most  $k$ , then  $\text{td}(G) \leq k + c$ .
- if  $G$  has a  $(k', c)$ -extended deletion set of size at most  $k$ , then  $\text{td}(G) \leq k' + k + c$ .
- $\text{tw}(G) \leq \text{td}(G)$ .

### 2.3 Locally Constrained Homomorphisms

Recall that we allow self-loops for the host graph  $H$ , but not for the guest graph  $G$  (see also Section 1). Here we show some basic properties of locally constrained homomorphisms.

**Lemma 1.** *Let  $G$  and  $H$  be non-empty connected graphs and let  $\phi$  be a locally surjective homomorphism from  $G$  to  $H$ . Then  $\phi$  is surjective.*

*Proof.* Suppose not, and let  $C$  be the set of vertices in  $V(H) \setminus \phi(V(G))$ . Note that  $C \neq \emptyset$  (because otherwise  $\phi$  is surjective) and  $\phi(V(G)) \neq \emptyset$  (because  $G$  is non-empty). Because  $H$  is connected, there is an edge  $uv \in E(H)$  such that  $u \in V(C)$  and  $v \in \phi(V(G))$ . But then, the mapping  $\phi_x : N_G(x) \rightarrow N_H(v)$  is not surjective for any vertex  $x \in \phi^{-1}(v)$ .  $\square$

**Lemma 2.** *Let  $G$  and  $H$  be non-empty connected graphs with a homomorphism  $\phi$  from  $G$  to  $H$  and let  $I \subseteq \phi(V(G))$ . Let  $P = \phi^{-1}(I)$  and  $\phi_R = \phi|_P$ . If  $\phi$  is a locally injective, surjective or bijective homomorphism, then  $\phi_R$  is a locally injective, surjective or bijective homomorphism, respectively, from  $G[P]$  to  $H[I]$ .*

*Proof.* Clearly,  $\phi_R$  is a homomorphism from  $G[P]$  to  $H[I]$  and since  $\phi_R$  is a restriction of  $\phi$ , it follows that if  $\phi$  is locally injective, then so is  $\phi_R$ . It remains to show that if  $\phi$  is locally surjective, then so is  $\phi_R$ . Suppose, for contradiction, that  $\phi$  is locally surjective, but  $\phi_R$  is not. Then there is a vertex  $v \in P$  such that  $\phi_R(N_G(v) \cap P) \subsetneq N_H(\phi_R(v)) \cap I$ . However, since  $\phi$  does not map any vertex in  $V(G) \setminus P$  to a vertex of  $I$ , it follows that  $\phi(N_G(v)) \cap I \subsetneq N_H(\phi(v)) \cap I$ , so  $\phi(N_G(v)) \neq N_H(\phi(v))$ . Thus  $\phi$  is not surjective, a contradiction.  $\square$

**Lemma 3.** *Let  $G$  and  $H$  be graphs, let  $D \subseteq V(G)$ , and let  $\phi$  be a homomorphism from  $G$  to  $H$ . Then, for every component  $C_G$  of  $G \setminus D$  such that  $\phi(C_G) \cap \phi(D) = \emptyset$ , there is a component  $C_H$  of  $H \setminus \phi(D)$  such that  $\phi(C_G) \subseteq C_H$ . Moreover, if  $\phi$  is locally injective/surjective/bijective, then  $\phi|_{D \cup C_G}$  is a homomorphism from  $G' = G[D \cup C_G]$  to  $H' = H[\phi(D) \cup C_H]$  that is locally injective/surjective/bijective for every  $v \in V(C_G)$ .*

*Proof.* Suppose for a contradiction that this is not the case. Then, there is a component  $C_G$  of  $G \setminus D$  and an edge  $uv \in E(C_G)$  such that  $\phi(u)$  and  $\phi(v)$  are in different components of  $H \setminus \phi(D)$ . Therefore,  $\phi(u)\phi(v) \notin E(H)$ , contradicting our assumption that  $\phi$  is a homomorphism.

Towards showing the second statement, first note that  $\phi_R := \phi|_{D \cup C_G}$  is a homomorphism from  $G'$  to  $H'$ . Moreover,  $N_G[v] = N_{G'}[v]$  for every vertex  $v \in V(C_G)$ , so if  $\phi$  is locally injective/surjective/bijective for a vertex  $v \in V(C_G)$ , then so is  $\phi_R$ .  $\square$

The following lemma is a basic but crucial observation showing that if  $G \xrightarrow{s} H$  and  $G$  has a small  $c$ -deletion set, then so does  $H$ .

**Lemma 4.** *Let  $G$  and  $H$  be non-empty connected graphs, let  $D \subseteq V(G)$  be a  $c$ -deletion set for  $G$ , and let  $\phi$  be a locally surjective homomorphism from  $G$  to  $H$ . Then  $\phi(D)$  is a  $c$ -deletion set for  $H$ .*

*Proof.* Suppose not, then there is a component  $C_H$  of  $H \setminus \phi(D)$  such that  $|C_H| > c$ . By Lemma 1, it follows that  $\phi$  is surjective and therefore  $\phi^{-1}(C_H)$  is **non-empty**. Let  $v \in \phi^{-1}(C_H)$ . Then  $v \notin D$  and therefore  $v$  is in some component  $C_G$  of  $G \setminus D$ . Lemma 3 implies that  $\phi_R = \phi|_{D \cup C_G}$  is a homomorphism from  $G[D \cup C_G]$  to  $H[\phi(D) \cup C_H]$  that is locally surjective for every  $v \in V(C_G)$ .

Now  $|V(C_G)| < |V(C_H)|$ , so there must be a vertex in  $V(C_H) \setminus \phi_R(C_G)$ . Because  $C_H$  is connected, there is an edge  $xy \in E(C_H)$  such that  $x \in V(C_H) \setminus \phi_R(C_G)$  and  $y \in \phi_R(V(C_G))$ . But then, the mapping  $\phi_z : N_G(z) \rightarrow N_H(y)$  is not surjective for any vertex  $z \in \phi_R^{-1}(y)$ .  $\square$

## 2.4 Integer Linear Programming

Given a set  $\mathcal{X}$  of variables and a set  $\mathcal{C}$  of linear constraints (i.e. inequalities) over the variables in  $\mathcal{X}$  with integer coefficients, the task in the feasibility variant of *integer linear programming* (ILP) is to decide whether there is an assignment  $\alpha : \mathcal{X} \rightarrow \mathbb{Z}$  of the variables satisfying all constraints in  $\mathcal{C}$ . We will use the following well-known result by Lenstra [62].

**Proposition 3 ([32,41,53,62]).** *ILP is FPT parameterized by the number of variables.*

## 3 Our Algorithmic Framework

In this section we present our main algorithmic framework that will allow us to show that LSHOM, LBHOM and ROLE ASSIGNMENT are FPT, parameterized by  $k + c$  when the guest graph has  $c$ -deletion set number at most  $k$ .

To illustrate the main ideas behind our framework, let us first explain these ideas for the examples of LSHOM and LBHOM. In this case we are given  $G$  and  $H$  and we know that  $G$  has a  $c$ -deletion set of size at most  $k$ . Because of Lemma 4, it then follows that if  $(G, H)$  is a yes-instance of LSHOM or LBHOM, then  $H$  also has a  $c$ -deletion set of size at most  $k$ . Informally, our next step, which is given in Section 3.1, is to compute a small (i.e. with size bounded by a function of  $k + c$ ) set  $\Phi$  of partial locally surjective homomorphisms such that

- (1) every locally surjective homomorphism from  $G$  to  $H$  augments some  $\phi_P \in \Phi$ , and
- (2) for every  $\phi_P \in \Phi$ , the domain of  $\phi_P$  is a  $(k, c)$ -extended deletion set of  $G$  and the co-domain of  $\phi_P$  is a  $c'$ -deletion set of  $H$ , where  $c'$  is bounded by a function of  $k + c$ .

Here and in what follows, we say that a function  $\phi : V(G) \rightarrow V(H)$  *augments* (or is an *augmentation* of) a partial function  $\phi_P : W_G \rightarrow W_H$ , where  $W_G \subseteq V(G)$  and  $W_H \subseteq V(H)$  if  $v \in W_G \Leftrightarrow \phi(v) \in W_H$  and  $\phi|_{W_G} = \phi_P$ . This allows us to reduce our problems to (boundedly many) subproblems of the following form: Given a  $(k, c)$ -extended deletion set  $D_G$  for  $G$ , a  $c'$ -deletion set  $D_H$  for  $H$ , and a locally surjective (respectively, bijective) homomorphism  $\phi_P$  from  $D_G$  to  $D_H$ , find a locally surjective homomorphism  $\phi$  from  $G$  to  $H$  that augments  $\phi_P$ .

In Section 3.2 we will then show how to formulate this subproblem as an integer linear program and in Section 3.3 we will show that we can efficiently construct and solve the ILP for this subproblem. Importantly, our ILP formulation will allow us to solve a much

more general problem, where the host graph  $H$  is not explicitly given, but defined in terms of a set of linear constraints. We will then exploit this in Section 4 to solve not only LSHOM and LBHOM, but also the ROLE ASSIGNMENT problem.

### 3.1 Partial Homomorphisms for the Deletion Set

For a graph  $G$  and  $m \in \mathbb{N}$  we let  $\Delta_G^m := \{v \in V(G) \mid \deg_G(v) \geq m\}$ . Our aim is to show in Lemma 7 that there is a small set  $\Phi$  of partial homomorphisms such that every locally surjective (respectively, bijective) homomorphism from  $G$  to  $H$  augments some  $\phi_P \in \Phi$  and, for every  $\phi_P \in \Phi$ , the domain of  $\phi_P$  is a  $(k, c)$ -extended deletion set for  $G$  of size at most  $k$  and its co-domain is a  $c'$ -deletion set of size at most  $k$  for  $H$ . The main idea behind finding this set  $\Phi$  is to consider the set of high degree vertices in  $G$  and  $H$ , that is, the sets  $\Delta_G^{k+c}$  and  $\Delta_H^{k+c}$ .

We start with the following lemma.

**Lemma 5.** *Let  $G$  be a graph. If  $G$  has a  $c$ -deletion set of size at most  $k$ , then the set  $\Delta_G^{k+c}$  is a  $kc(k+c)$ -deletion set of size at most  $k$ . Furthermore, every subset of  $\Delta_G^{k+c}$  is a  $(k - |D|, c)$ -extended deletion set of  $G$ .*

*Proof.* Let  $D$  be a  $c$ -deletion set of  $G$  of size at most  $k$ . Then every vertex  $v \in V(G) \setminus D$  has degree at most  $k + c - 1$ , as each of its neighbours lies either in its own component of  $G \setminus D$  or in  $D$ . Hence  $\Delta_G^{k+c} \subseteq D$  and therefore  $|\Delta_G^{k+c}| \leq k$ . Let  $C_1, \dots, C_m$  be the components of  $G \setminus D$  that contain a vertex adjacent to a vertex in  $D \setminus \Delta_G^{k+c}$ . Since  $|D \setminus \Delta_G^{k+c}| \leq k$  and every vertex in  $D \setminus \Delta_G^{k+c}$  has degree at most  $k + c - 1$ , we find that  $m \leq k(k + c - 1)$  and  $|C_1 \cup \dots \cup C_m \cup (D \setminus \Delta_G^{k+c})| \leq kc(k + c - 1) + k \leq kc(k + c)$ . Since every component in  $G \setminus \Delta_G^{k+c}$  is either contained in a component of  $G \setminus D$  or contained in  $C_1 \cup \dots \cup C_m \cup (D \setminus \Delta_G^{k+c})$ , we find that  $\Delta_G^{k+c}$  is a  $kc(k + c)$ -deletion set.

Let  $D' \subseteq \Delta_G^{k+c} \subseteq D$ . We will show that  $D'$  is a  $(k - |D'|, c)$ -extended deletion set of  $G$ . The components of  $G \setminus D'$  that contain no vertices from  $D \setminus D'$  are components of  $G \setminus D$  and thus have size at most  $c$ . Consider a component  $C$  of  $G \setminus D'$  that contains at least one vertex from  $D \setminus D'$ . Let  $D_C = V(C) \cap (D \setminus D')$ . Every component of  $C \setminus D_C$  is a component of  $G \setminus D$  and thus has size at most  $c$ . Moreover,  $D_C$  has size at most  $|D \setminus D'| \leq k - |D'|$ .

We conclude that every component of  $G \setminus D'$  either has size at most  $c$  or has a  $c$ -deletion set of size at most  $k - |D'|$ . Furthermore, since there are at most  $k - |D'|$  vertices in  $\Delta_G^{k+c} \setminus D'$ , and every component of  $G \setminus D'$  that has size larger than  $c$  must contain a vertex of  $\Delta_G^{k+c}$ , it follows that there are at most  $k - |D'|$  components of  $G \setminus D'$  that have size larger than  $c$ . This completes the proof.  $\square$

We use Lemma 5 in the proof of our next lemma, which shows that every locally surjective (respectively, bijective) homomorphism from  $G$  to  $H$  has to augment a locally surjective (respectively, bijective) homomorphism from some induced subgraph of  $G[\Delta_G^{k+c}]$  to  $H[\Delta_H^{k+c}]$ . Intuitively, this holds because for every locally surjective homomorphism, only vertices of high degree in  $G$  can be mapped to a vertex of high degree in  $H$  and every vertex in  $H$  must have a pre-image in  $G$ .

**Lemma 6.** *Let  $G$  and  $H$  be non-empty connected graphs such that  $G$  has a  $c$ -deletion set of size at most  $k$ . If there is a locally surjective homomorphism  $\phi$  from  $G$  to  $H$ , then there is a set  $D \subseteq \Delta_G^{k+c}$  and a locally surjective homomorphism  $\phi_P$  from  $G[D]$  to  $H[\Delta_H^{k+c}]$  such that  $\phi$  augments  $\phi_P$ . If  $\phi$  is locally bijective, then  $D = \Delta_G^{k+c}$  and  $\phi_P$  is a locally bijective homomorphism.*

*Proof.* Observe that for a locally surjective homomorphism  $\phi$  from  $G$  to  $H$ , the inequality  $\deg_G(v) \geq \deg_H(\phi(v))$  holds for every  $v \in V(G)$ ; moreover the equality  $\deg_G(v) = \deg_H(\phi(v))$  holds in the locally bijective case. Since  $\phi$  is surjective by Lemma 1, this implies that  $\phi(\Delta_G^{k+c}) \supseteq \Delta_H^{k+c}$  (and if  $\phi$  is locally bijective, then  $\phi(\Delta_G^{k+c}) = \Delta_H^{k+c}$ ). Let  $D = \phi^{-1}(\Delta_H^{k+c})$ , so  $D \subseteq \Delta_G^{k+c}$  (note that  $D = \Delta_G^{k+c}$  if  $\phi$  is locally bijective). Now  $\phi|_D$  is a surjective map from  $D$  to  $\Delta_H^{k+c}$ . Furthermore,  $\phi(\Delta_G^{k+c} \setminus D) \cap \phi(D) = \phi(\Delta_G^{k+c} \setminus D) \cap \Delta_H^{k+c} = \emptyset$ . Moreover, for every  $v \in V(G) \setminus \Delta_G^{k+c}$ ,  $\phi(v) \notin \Delta_H^{k+c} = \phi|_D(D)$ , since  $\deg_G(v) \geq \deg_H(\phi(v))$ . Furthermore,  $\phi|_D$  is a homomorphism from  $G[D]$  to  $H[\Delta_H^{k+c}]$  because  $\phi$  is a homomorphism. Additionally,  $\phi|_D$  is locally surjective (respectively, bijective) by Lemma 2.  $\square$

We are now ready to show that we can easily compute all possible pre-images of  $\Delta_H^{k+c}$  in any locally surjective (respectively bijective) homomorphism from  $G$  to  $H$ .

**Lemma 7.** *Let  $G$  and  $H$  be non-empty connected graphs, and let  $k$  and  $c$  be two non-negative integers. For any  $D \subseteq \Delta_G^{k+c}$ , we can compute the set  $\Phi_D$  of all locally surjective (respectively, bijective) homomorphisms  $\phi_P$  from  $G[D]$  to  $H[\Delta_H^{k+c}]$  in  $\mathcal{O}(|D|^{|D|+2})$  time. Furthermore,  $|\Phi_D| \leq |D|^{|D|}$ .*

*Proof.* Let  $D \subseteq \Delta_G^{k+c}$  and suppose there is a surjective map  $\phi_P : D \rightarrow \Delta_H^{k+c}$ . Then for every vertex  $v \in \Delta_H^{k+c}$ , there must be a vertex  $x \in D$  such that  $\phi_P(x) = v$ . Therefore  $|\Delta_H^{k+c}| \leq |D|$ , so if this condition fails, then we can immediately return that  $\Phi_D = \emptyset$ .

Otherwise, for each vertex of  $D$ , there are  $|\Delta_H^{k+c}| \leq |D|$  possible choices for where a map  $\phi_P : D \rightarrow \Delta_H^{k+c}$  could map this vertex. We can list all of the at most  $|D|^{|D|}$  resulting maps in  $\mathcal{O}(|D|^{|D|})$  time, and for each such map, we can check whether it is a locally surjective (respectively, bijective) homomorphism in  $\mathcal{O}(|D|^2)$  time.  $\square$

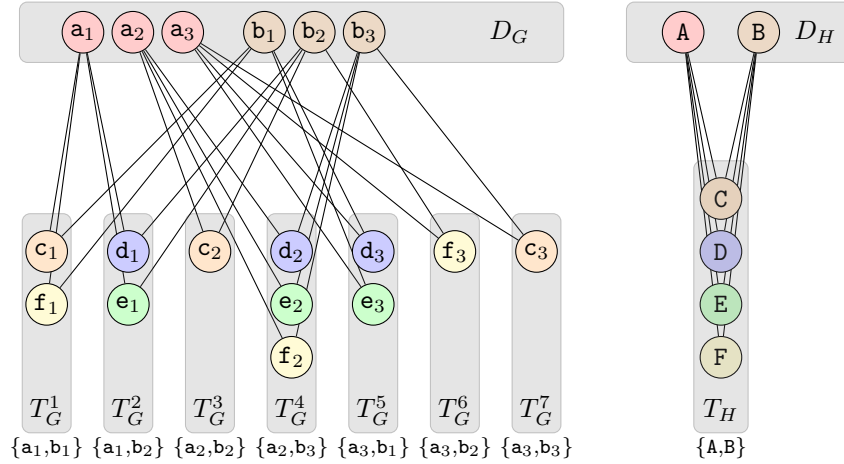
### 3.2 ILP Formulation

In this section, we will show how to formulate the subproblem obtained in the previous subsection in terms of an ILP instance. More specifically, we will show that the following problem can be formulated in terms of an ILP: given a partial locally surjective (respectively, bijective) homomorphism  $\phi_P$  from some induced subgraph  $D_G$  of  $G$  to some induced subgraph  $D_H$  of  $H$ , can this be augmented to a locally surjective (respectively, bijective) homomorphism from  $G$  to  $H$ ? See Figure 2 for an illustration of the subproblem for the simpler case when  $D_G$  is a vertex cover of  $G$  and we are looking for a locally bijective homomorphism. Moreover, we will actually show that for this to work, the host graph  $H$  does not need to be given explicitly, but can instead be defined by a certain system of linear constraints.

We now sketch the main ideas behind our translation to ILP. We postpone the formal definitions of notions introduced below until later. Suppose that there is a locally surjective

(respectively, bijective) homomorphism  $\phi$  from  $G$  to  $H$  that augments  $\phi_P$ . Because  $\phi$  augments  $\phi_P$ , [Lemma 3](#) implies that  $\phi$  maps every component  $C_G$  of  $G \setminus V(D_G)$  entirely to some component  $C_H$  of  $H \setminus V(D_H)$ , moreover,  $\phi|_{V(D_G) \cup V(C_G)}$  is already locally surjective (respectively, bijective) for every vertex  $v \in V(C_G)$ .

Our aim now is to describe  $\phi$  in terms of its parts consisting of locally surjective (respectively, bijective) homomorphisms from extensions of  $D_G$  in  $G$ , that is, sets of components of  $G \setminus D_G$  plus  $D_G$ , to simple extensions of  $D_H$  in  $H$ , that is, single components of  $H \setminus D_H$  plus  $D_H$ .



$$\begin{array}{l}
 \text{Ext}_G^1 : T_G^1 + \quad T_G^3 + \quad T_G^7 \quad (x_{\text{Ext}_G^1 T_H} = 1) \rightarrow \text{C} \\
 \text{Ext}_G^2 : \quad T_G^2 + \quad T_G^4 + \quad T_G^5 \quad (x_{\text{Ext}_G^2 T_H} = 2) \rightarrow \text{D, E} \\
 \text{Ext}_G^3 : T_G^1 + \quad T_G^4 + \quad T_G^6 \quad (x_{\text{Ext}_G^3 T_H} = 1) \rightarrow \text{F}
 \end{array}$$

**Fig. 2.** A locally bijective homomorphism from a graph  $G$  (left) to a graph  $H$  (right), augmenting the partial homomorphism  $\phi_P$  mapping the vertices of the vertex cover  $D_G$  into  $D_H = \{A, B\}$ . The  $i$ th vertex of  $G$  mapped to some vertex  $X$  of  $H$  is denoted  $x_i$ . Vertices in  $G \setminus D_G$  are grouped by type (e.g.  $\{c_1\}$  and  $\{f_1\}$  have type  $T_G^1$ ), each  $T_G^i$  is characterised by the neighbours of its vertices, recalled below each column. Vertices in  $H \setminus D_H$  all have the same type  $T_H$ . Rows  $\text{Ext}_G^i$  are extensions that can be minimally  $\phi_P$ -B-mapped to  $T_H$ . This, in particular, means that each  $a_i$  and  $b_i$  must have a neighbour in  $\text{Ext}_G^i$ . This is because otherwise  $a_i$  or  $b_i$  is not mapped locally bijective to its image  $A$  or  $B$ , because  $A$  and  $B$  have a neighbor in  $T_H$  but the particular  $a_i$  or  $b_i$  has no neighbor in  $\text{Ext}_G^i$ . Note that the neighbour can then later be used as a pre-image of any vertex in  $\{C, D, E, F\}$ . Using  $\text{Ext}_G^1$  once (for colour C),  $\text{Ext}_G^2$  twice (for colours D and E) and  $\text{Ext}_G^3$  once (for colour F) yields the given locally bijective homomorphism, and it can be verified that each  $a_i$  and  $b_i$  indeed has all four colours in its neighbourhood.

The main difficulty of doing the above comes from the fact that we need to ensure that  $\phi$  is locally surjective (respectively, bijective) for every  $d \in D_G$  and not only for the vertices

within the components of  $G \setminus D_G$ . This is why we need to describe the parts of  $\phi$  using sets of components of  $G \setminus D_G$  and not just single components. However, as we will show, it will suffice to consider only minimal extensions of  $D_G$  in  $G$ , where an extension is minimal if no subset of it allows for a locally surjective (respectively bijective) homomorphism from it to some simple extension of  $D_H$  in  $H$ . The fact that we only need to consider minimal extensions is important for showing that we can compute the set of all possible parts of  $\phi$  efficiently (see Section 3.3). Having shown this, we can create an ILP that has one variable  $x_{\text{EC}_G \text{ES}_H}$  for every minimal extension  $\text{EC}_G$  and every simple extension  $\text{ES}_H$  such that there is a locally surjective (respectively, bijective) homomorphism from  $\text{EC}_G$  to  $\text{ES}_H$  that augments  $\phi_P$ . The value of the variable  $x_{\text{EC}_G \text{ES}_H}$  now corresponds to the number of **times  $\phi$  maps a minimal extension isomorphic to  $\text{EC}_G$  to a simple extension isomorphic to  $\text{ES}_H$  that augments  $\phi_P$** . We can then use linear constraints on these variables to ensure that:

- $H$  contains exactly the right number of extensions isomorphic to  $\text{ES}_H$  required by the assignment for  $x_{\text{EC}_G \text{ES}_H}$ ,
- $G$  contains exactly the right number of minimal extensions isomorphic to  $\text{EC}_G$  required by the assignment for  $x_{\text{EC}_G \text{ES}_H}$  (if  $\phi$  is locally bijective),
- $G$  contains at least the number of minimal extensions isomorphic to  $\text{EC}_G$  required by the assignment for  $x_{\text{EC}_G \text{ES}_H}$  (if  $\phi$  is locally surjective), and
- for every simple extension  $\text{ES}_G$  of  $G$  that is not yet used in any part of  $\phi$ , there is a homomorphism from  $\text{ES}_G$  to some simple extension of  $D_H$  in  $H$  that augments  $\phi_P$  and is locally surjective for every vertex in  $\text{ES}_G \setminus D_G$  (if  $\phi$  is locally surjective).

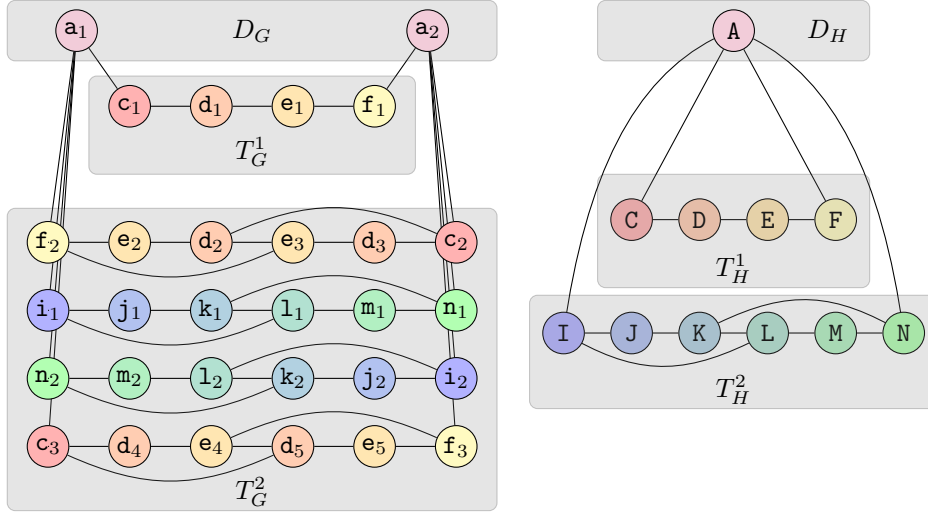
Together, these constraints ensure that there is a locally surjective (respectively, bijective) homomorphism  $\phi$  from  $G$  to  $H$  that augments  $\phi_P$ . See also [Figure 3](#) for an illustration of the main ideas.

We are now ready to formalise these ideas. To do so, we need the following additional notation. Given a graph  $D$ , an *extension of  $D$*  is a graph  $\text{EC}$  containing  $D$  as an induced subgraph. It is *simple* if  $\text{Ext} \setminus D$  is connected, and *complex* in general. Given two extensions  $\text{EC}_1, \text{EC}_2$  of  $D$ , we write  $\text{EC}_1 \sim_D \text{EC}_2$  if there is an isomorphism  $\tau$  from  $\text{EC}_1$  to  $\text{EC}_2$  with  $\tau(d) = d$  for every  $d \in D$ . Then  $\sim_D$  is an equivalence relation. Let the *types* of  $D$ , denoted  $\mathcal{T}_D$ , be the set of equivalence classes of  $\sim_D$  of simple extensions of  $D$ . We write  $\mathcal{T}_D^c$  to denote the set of types of  $D$  of size at most  $|D| + c$ , so

$$|\mathcal{T}_D^c| \leq \sum_{i=0}^c 2^{\binom{|D|+i}{2} - \binom{|D|}{2}} \leq c 2^{\binom{|D|+c}{2}}.$$

Given a complex extension  $\text{EC}$  of  $D$ , let  $C$  be a connected component of  $\text{EC} \setminus D$ . Then  $C$  has type  $T \in \mathcal{T}_D$  if  $\text{EC}[D \cup C] \sim_D T$  (depending on the context, we also say that the extension  $\text{EC}[D \cup C]$  has type  $T$ ). The *type-count* of  $\text{EC}$  is the function  $\text{tc}_{\text{EC}} : \mathcal{T}_D \rightarrow \mathbb{N}$  such that  $\text{tc}_{\text{EC}}(T)$  for  $T \in \mathcal{T}_D$  is the number of connected components of  $\text{EC} \setminus D$  with type  $T$  (in particular if  $\text{EC}$  is simple, the type-count is 1 for  $\text{EC}$  and 0 for other types). Note that two extensions are equivalent **under  $\sim_D$**  if and only if they have the same type-counts; **in particular, this** implies that there is an isomorphism  $\tau$  between the two extensions satisfying  $\tau(d) = d$  for every  $d \in D$ . We write  $\text{EC} \preceq \text{EC}'$  if  $\text{tc}_{\text{EC}}(T) \leq \text{tc}_{\text{EC}'}(T)$  for all types  $T \in \mathcal{T}_D$ . If  $\text{EC}$  is an extension of  $D$ , we write  $\mathcal{T}_D(\text{EC}) = \{T \in \mathcal{T}_D \mid \text{tc}_{\text{EC}}(T) \geq 1\}$  for





$$\begin{aligned} \text{Ext}_G^1 : 2 \times T_G^1 & & \text{Ext}_G^2 : T_G^1 + T_G^2 & & \text{Ext}_G^3 : 2 \times T_G^2 \\ x_{\text{Ext}_G^1, T_H^1} = 0 & & x_{\text{Ext}_G^2, T_H^1} = 1 & & x_{\text{Ext}_G^3, T_H^1} = 0 & & x_{\text{Ext}_G^3, T_H^2} = 1 \end{aligned}$$

**Fig. 3.** A locally surjective homomorphism from a graph  $G$  (left) to a graph  $H$  (right), where  $D_G$  is a 6-deletion set. The extensions  $\text{Ext}_G^1, \text{Ext}_G^2, \text{Ext}_G^3$  can be minimally  $\phi_P$ -S-mapped to  $T_H^1$ ; only  $\text{Ext}_G^3$  can also be minimally  $\phi_P$ -S-mapped to  $T_H^2$ . Furthermore,  $T_G^1$  and  $T_G^2$  can each be weakly  $\phi_P$ -S-mapped to some type in  $H$  (respectively,  $T_H^1$  and  $T_H^2$ ). Using pair  $(\text{Ext}_G^2, T_H^1)$  and  $(\text{Ext}_G^3, T_H^2)$  once is sufficient to ensure that the mapping is locally surjective for each  $\mathbf{a}_i$ .

the set of types of  $\mathbf{EC}$  and  $\mathcal{E}_D(\mathbf{EC})$  for the set of simple extensions of  $D$  in  $\mathbf{EC}$ . Moreover, for  $T \in \mathcal{T}_D$ , we write  $\mathcal{E}_D(\mathbf{EC}, T)$  for the set of simple extensions of  $D$  in  $\mathbf{EC}$  having type  $T$ .

A *target description* is a tuple  $(D_H, c, \text{CH})$  where  $D_H$  is a graph,  $c$  is an integer and  $\text{CH}$  is a set of linear constraints over variables  $x_T$  for every  $T \in \mathcal{T}_{D_H}^c$ . A type-count for  $D_H$  is an integer assignment of the variables  $x_T$ . A graph  $H$  satisfies the target description  $(D_H, c, \text{CH})$  if it is an extension of  $D_H$ ,  $\text{tc}_H(T) = 0$  for  $T \notin \mathcal{T}_{D_H}^c$ , and setting  $x_T = \text{tc}_H(T)$  for all  $T \in \mathcal{T}_{D_H}^c$  satisfies all constraints in  $\text{CH}$ . Note that target descriptions can be easily used to specify any graph  $H$  that has deletion set  $D_H$  into components of size at most  $c$  by using  $\text{CH}$  to specify the number of components of each type. However, allowing for arbitrary linear equations instead of specifying the graph  $H$  explicitly is much more flexible, and we will show how to employ this flexibility for the **ROLE ASSIGNMENT** problem in Section 4.

In what follows, we assume that the following are given: the graphs  $D_G, D_H$ , an extension  $G$  of  $D_G$ , a target description  $\mathcal{D} = (D_H, c, \text{CH})$ , and a locally surjective (respectively, bijective) homomorphism  $\phi_P : D_G \rightarrow D_H$ . Let  $\mathbf{EC}_G$  be an extension of  $D_G$  with  $\mathbf{EC}_G \preceq G$  and let  $T_H \in \mathcal{T}_{D_H}^c$ ; note that we only consider  $T_H \in \mathcal{T}_{D_H}^c$ , because we assume that  $T_H$  is a type of a simple extension of a graph  $H$  that satisfies the target description  $\mathcal{D}$ . We say that  $\mathbf{EC}_G$  can be

- *weakly  $\phi_P$ -S-mapped* to **the** type  $T_H$  if there exists an augmentation  $\phi : \mathbf{EC}_G \rightarrow T_H$  of  $\phi_P$  such that  $\phi$  is locally surjective for every  $v \in \mathbf{EC}_G \setminus D_G$ ;
- *$\phi_P$ -S-mapped* (respectively,  *$\phi_P$ -B-mapped*) to **the** type  $T_H$  if there exists an augmentation  $\phi : \mathbf{EC}_G \rightarrow T_H$  of  $\phi_P$  such that  $\phi$  is locally surjective (respectively, locally bijective)
- *minimally  $\phi_P$ -S-mapped* (respectively, *minimally  $\phi_P$ -B-mapped*) to **the type**  $T_H$  if  $\mathbf{EC}_G$  can be  $\phi_P$ -S-mapped (respectively,  $\phi_P$ -B-mapped) to  $T_H$  and no other extension  $\mathbf{EC}'_G$  with  $\mathbf{EC}'_G \preceq \mathbf{EC}_G$  can be  $\phi_P$ -S-mapped (respectively,  $\phi_P$ -B-mapped) to  $T_H$ .

Note that in the case of locally bijective homomorphisms, the two notions  $\phi_P$ -B-mapped and minimally  $\phi_P$ -B-mapped coincide. We also define the following sets:

- the set  $\mathbf{wSM} = \mathbf{wSM}(G, \mathcal{D}, \phi_P)$  consists of all pairs  $(T_G, T_H)$  such that  $T_G \in \mathcal{T}_{D_G}(G)$  can be weakly  $\phi_P$ -S-mapped to  $T_H \in \mathcal{T}_{D_H}^c$ ;
- the set  $\mathbf{SM} = \mathbf{SM}(G, \mathcal{D}, \phi_P)$  consists of all pairs  $(\mathbf{EC}_G, T_H)$  with  $\mathbf{EC}_G \preceq G$  and  $T_H \in \mathcal{T}_{D_H}^c$  such that  $\mathbf{EC}_G$  can be minimally  $\phi_P$ -S-mapped to  $T_H$ ; and
- the set  $\mathbf{BM} = \mathbf{BM}(G, \mathcal{D}, \phi_P)$  consists of all pairs  $(\mathbf{EC}_G, T_H)$  with  $\mathbf{EC}_G \preceq G$  and  $T_H \in \mathcal{T}_{D_H}^c$  such that  $\mathbf{EC}_G$  can be minimally  $\phi_P$ -B-mapped to  $T_H$ .

See [Figure 3](#) for an illustration of these notions.

We now build a set of linear constraints. To this end, besides variables  $x_T$  for  $T \in T_H$ , we introduce variables  $x_{\mathbf{EC}_G T_H}$  for each  $(\mathbf{EC}_G, T_H) \in \mathbf{SM}$  (respectively  $\mathbf{BM}$ ).

$$\begin{aligned}
\text{(S1)} \quad & \sum_{(\mathbf{EC}_G, T_H) \in \mathbf{SM}} \text{tc}_{\mathbf{EC}_G}(T_G) * x_{\text{Ext}_G T_H} \leq \text{tc}_G(T_G) \text{ for every } T_G \in \mathcal{T}_{D_G}(G), \\
\text{(B1)} \quad & \sum_{(\mathbf{EC}_G, T_H) \in \mathbf{BM}} \text{tc}_{\mathbf{EC}_G}(T_G) * x_{\mathbf{EC}_G T_H} = \text{tc}_G(T_G) \text{ for every } T_G \in \mathcal{T}_{D_G}(G), \\
\text{(S2)} \quad & \sum_{\mathbf{EC}_G : (\mathbf{EC}_G, T_H) \in \mathbf{SM}} x_{\mathbf{EC}_G T_H} = x_{T_H} \text{ for every } T_H \in \mathcal{T}_{D_H}, \\
\text{(B2)} \quad & \sum_{\mathbf{EC}_G : (\mathbf{EC}_G, T_H) \in \mathbf{BM}} x_{\mathbf{EC}_G T_H} = x_{T_H} \text{ for every } T_H \in \mathcal{T}_{D_H}, \text{ and} \\
\text{(S3)} \quad & \sum_{(T_G, T_H) \in \mathbf{wSM}} x_{T_H} \geq 1 \text{ for every } T_G \in \mathcal{T}_{D_G}(G).
\end{aligned}$$

We refer to [Figure 3](#) for an illustration.

**Lemma 8.** *Let  $D_G$  and  $D_H$  be graphs, let  $G$  be an extension of  $D_G$  and let  $\mathcal{D} = (D_H, c, \text{CH})$  be a target description. Moreover, let  $\phi_P : V(D_G) \rightarrow V(D_H)$  be a locally surjective (respectively, bijective) homomorphism from  $D_G$  to  $D_H$ . There exists a graph  $H$  satisfying  $\mathcal{D}$  and a locally surjective (respectively, bijective) homomorphism  $\phi$  from  $G$  to  $H$  augmenting  $\phi_P$  if and only if the equation system  $(\text{CH}, \text{S1}, \text{S2}, \text{S3})$  (respectively,  $(\text{CH}, \text{B1}, \text{B2})$ ) admits a solution.*

*Proof.* Towards showing the forward direction of the claim, let  $H$  be a graph satisfying  $\mathcal{D} = (D_H, c, \text{CH})$  and let  $\phi$  be a locally surjective (respectively, bijective) homomorphism that augments  $\phi_P$ .

Consider  $T_H \in \mathcal{T}_{D_H}(H)$  and let  $\mathbf{ES}_H \in \mathcal{E}_{D_H}(H, T_H)$ . Let  $W_\phi(\mathbf{ES}_H) = G[\phi^{-1}(\mathbf{ES}_H)]$ ; note that  $D_G \subseteq V(W_\phi(\mathbf{ES}_H))$  and therefore  $W_\phi(\mathbf{ES}_H)$  is a (possibly) complex extension of

$D_G$ . Then because of Lemma 2, we obtain that  $\phi_R = \phi|_{W_\phi(\text{ES}_H)}$  is a locally surjective (respectively, bijective) homomorphism from  $W_\phi(\text{ES}_H)$  to  $\text{ES}_H$  that augments  $\phi_P$ . Moreover, because of Lemma 3, it follows that  $W_\phi(\text{ES}_H) \setminus D_G$  is the union of a set of components of  $G - D_G$ . Therefore,  $W_\phi(\text{ES}_H)$  can be  $\phi_P$ -S-mapped (respectively,  $\phi_P$ -B-mapped) to  $T_H$ . Moreover, if  $W_\phi(\text{ES}_H)$  can be  $\phi_P$ -S-mapped to  $T_H$ , then  $W_\phi(\text{ES}_H)$  also contains a subgraph  $W_\phi^{\min}(\text{ES}_H)$  induced by  $D_G$  and a subset of components of  $W_\phi(\text{ES}_H) \setminus D_G$  that can be minimally  $\phi_P$ -S-mapped to  $T_H$ . Note that  $W_\phi^{\min}(\text{ES}_H)$  is not uniquely defined. However, the concrete choice for  $W_\phi^{\min}(\text{ES}_H)$  does not matter. Informally, this is because  $W_\phi^{\min}(\text{ES}_H)$  ensures that some extension is  $\phi$ -S-mapped to  $\text{ES}_H$  while the remaining components in  $W_\phi(\text{ES}_H)$  can still be weakly  $\phi$ -S-mapped to  $\text{ES}_H$ .

Let  $X_T = \{x_{T_H} : T_H \in \mathcal{T}_{D_H}(H)\}$ ,  $X_M = \{x_{\text{EC}_G T_H} : (\text{EC}_G, T_H) \in \text{SM}\}$  (respectively,  $X_M = \{x_{\text{EC}_G T_H} : (\text{EC}_G, T_H) \in \text{BM}\}$ ), and  $X = X_T \cup X_M$ . Let  $\alpha : X \rightarrow \mathbb{N}$  be defined by setting:

- $\alpha(x_{T_H}) = \text{tc}_H(T_H)$  and
- $\alpha(x_{\text{EC}_G T_H}) = |\{\text{Ext}_H \in \mathcal{E}_{D_H}(H, T_H) : \text{tc}_{\text{EC}_G} = \text{tc}_{W_\phi^{\min}(\text{ES}_H)}\}|$  (in the locally surjective case)
- $\alpha(x_{\text{EC}_G T_H}) = |\{\text{ES}_H \in \mathcal{E}_{D_H}(H, T_H) : \text{tc}_{\text{EC}_G} = \text{tc}_{W_\phi(\text{ES}_H)}\}|$  (in the locally bijective case)

We claim that the assignment  $\alpha$  satisfies the equation system (CH, S1, S2, S3) (respectively, the equation system (CH, B1, B2)). Because  $H$  satisfies  $\mathcal{D}$ , it follows that  $\alpha$  satisfies CH.

We start by showing the claim for the locally surjective case. Towards showing that (S3) is satisfied, consider a type  $T_G \in \mathcal{T}_{D_G}(G)$  and let  $\text{ES}_G \in \mathcal{E}_{D_G}(G, T_G)$ . Then, because of Lemma 3, the mapping  $\phi|_{\text{ES}_G}$  maps  $\text{ES}_G$  to some type  $T_H \in \mathcal{T}_{D_H}(H)$  and shows that  $\text{ES}_G$  can be weakly  $\phi_P$ -S-mapped to  $T_H$ .

Towards showing (S1), let  $T_G \in \mathcal{T}_{D_G}(G)$ . Because of Lemma 3, every simple extension  $\text{ES}_G \in \mathcal{E}_{D_G}(G, T_G)$  satisfies  $\phi(\text{ES}_G) \subseteq \text{ES}_H$  for some simple extension  $\text{ES}_H$  of  $D_H$ . In other words  $\text{ES}_G$  is contained in the pre-image of exactly one simple extension  $\text{ES}_H$ , showing that every simple extension  $\text{ES}_G \in \mathcal{E}_{D_G}(G, T_G)$  is counted at most once on the left side of the inequality in (S1) and therefore the left side is at most  $\text{tc}_G(T_G)$ .

Towards showing (S2), let  $T_H \in \mathcal{T}_{D_H}$ . Then, because  $V(W_\phi^{\min}(\text{ES}_H)) \cap V(W_\phi^{\min}(\text{ES}'_H)) = D_G$ , i.e.,  $W_\phi^{\min}(\text{ES}_H)$  and  $W_\phi^{\min}(\text{ES}'_H)$  share no components, for every two distinct  $\text{ES}_H, \text{ES}'_H \in \mathcal{E}_{D_H}(H, T_H)$ , we obtain:

$$\begin{aligned}
& \sum_{\text{EC}_G : (\text{EC}_G, T_H) \in \text{SM}} \alpha(x_{\text{EC}_G T_H}) \\
&= \sum_{\text{EC}_G : (\text{EC}_G, T_H) \in \text{SM}} |\{\text{ES}_H \in \mathcal{E}_{D_H}(H, T_H) : \text{tc}_{\text{EC}_G} = \text{tc}_{W_\phi^{\min}(\text{ES}_H)}\}| \\
&= \text{tc}_H(T_H) \\
&= \alpha(x_{T_H}),
\end{aligned}$$

as required.

Finally, if  $\phi$  is locally bijective, we only have to show (B1) and (B2), which can be shown similarly to (S1) and (S2), **respectively**. That is, (B1) can be shown similarly to (S1) by using the additional observation that due to the definition of  $\alpha$  in terms of  $W_\phi(\mathbf{ES}_H)$  instead of  $W_\phi^{\min}(\mathbf{ES}_H)$ , every simple extension  $\mathbf{ES}_G$  also occurs in the pre-image of at least one simple extension  $\mathbf{ES}_H$ . Moreover, (B2) can be shown in the same manner as (S2), since  $V(W_\phi^{\min}(\mathbf{ES}_H)) \cap V(W_\phi^{\min}(\mathbf{ES}'_H)) = D_G$  also holds for every two distinct  $\mathbf{ES}_H, \mathbf{ES}'_H \in \mathcal{E}_{D_H}(H, T_H)$ .

Towards showing the reverse direction, let  $\alpha : X \rightarrow \mathbb{N}$  be an assignment satisfying the equation system (CH, S1, S2, S3) (respectively, the equation system (CH, B1, B2)). Let  $H$  be the unique graph consisting of  $D_H$  and  $\alpha(x_{T_H})$  extensions of  $D_H$  of type  $T_H$  for every  $T_H \in \mathcal{T}_{D_H}$ . Then  $H$  satisfies  $(D_H, c, \text{CH})$  and  $\text{tc}_H(T_H) = \alpha(x_{T_H})$ .

We now define a function  $\phi : V(G) \rightarrow V(H)$ , which will be a locally surjective (respectively, locally bijective) homomorphism that augments  $\phi_P$  as follows.

Let  $\mathcal{A}$  be the multiset containing each pair  $(\mathbf{EC}_G, T_H) \in \text{SM}$  (respectively, BM) exactly  $\alpha(x_{\mathbf{EC}_G T_H})$  times. Because of (S2) (respectively, (B2)), there is a bijection  $\gamma_{T_H}$  between  $\mathcal{A}_{T_H} = \{\mathbf{EC}_G : (\mathbf{EC}_G, T_H) \in \mathcal{A}\}$  and  $\mathcal{E}_{D_H}(H, T_H)$  for every  $T_H \in \mathcal{T}_{D_H}(H)$ . Let  $\gamma$  be the bijection between  $\mathcal{A}$  and the extensions  $\mathcal{E}_{D_H}(H)$  given by  $\gamma((\mathbf{EC}_G, T_H)) = \gamma_{T_H}(\mathbf{EC}_G)$ .

**Since** the proof now diverges quite significantly for the locally surjective and locally bijective cases, we start by **giving** the proof for the former case and then show how to adapt the proof in latter (easier) case.

Because of (S1), there is a function  $\beta$  from  $\mathcal{A}$  to the complex extensions of  $G$  such that:

- $\text{tc}_{\beta((\mathbf{EC}_G, T_H))} = \text{tc}_{\mathbf{EC}_G}$  for every  $(\mathbf{EC}_G, T_H) \in \mathcal{A}$ ,
- $\beta(A) \cap \beta(A') = D_G$  for every two distinct  $A$  and  $A'$  in  $\mathcal{A}$ .

Let  $A = (\mathbf{EC}_G, T_H) \in \mathcal{A}$ . Because  $A \in \text{SM}$ , there is a locally surjective homomorphism  $\phi_A$  from  $\beta(A)$  to  $\gamma(A)$  that augments  $\phi_P$ . Let  $\mathcal{E}_A$  be the set of simple extensions  $\mathbf{ES}_G$  in  $\mathcal{E}_{D_G}(G)$  for which there is an  $A \in \mathcal{A}$  such that  $\mathbf{ES}_G$  is an induced subgraph of  $\beta(A)$ . Moreover, let  $\bar{\mathcal{E}}_A$  be the set of all remaining simple extensions in  $\mathcal{E}_{D_G}(G)$ , i.e. the set of all simple extensions  $\mathbf{ES}_G$  in  $\mathcal{E}_{D_G}(G) \setminus \mathcal{E}_A$ . Consider a simple extension  $\mathbf{ES}_G$  in  $\bar{\mathcal{E}}_A$ . Then, because of (S3), there is a  $T_H \in \mathcal{T}_{D_H}(H)$  and a corresponding simple extension  $\mathbf{ES}_H \in \mathcal{E}_{D_H}(H, T_H)$  such that there is a homomorphism  $\phi_{\mathbf{ES}_G}$  from  $\mathbf{ES}_G$  to  $\mathbf{ES}_H$  that augments  $\phi_P$ , which is locally surjective for every  $v \in V(\mathbf{ES}_G - D_G)$ . We are now ready to define  $\phi : V(G) \rightarrow V(H)$ . That is, we set  $\phi(v)$  to be equal to:

- $\phi_P(v)$  if  $v \in D_G$ ,
- $\phi_{\text{Ext}_G}(v)$  if  $v \in V(\mathbf{ES}_G)$  for some simple extension  $\mathbf{ES}_G \in \bar{\mathcal{E}}_A$ , and
- $\phi_A(v)$  if  $v \in V(\mathbf{ES}_G - D_G)$  for some  $\mathbf{ES}_G = \beta(A)$  and  $A \in \mathcal{A}$ .

It remains to show that  $\phi$  is a locally surjective homomorphism from  $G$  to  $H$  that augments  $\phi_P$ . Clearly,  $\phi$  augments  $\phi_P$  by definition and because  $\phi_{\mathbf{ES}_G}$  does so too for every simple extension  $\mathbf{ES}_G$  in  $\bar{\mathcal{E}}_A$ , as does  $\phi_A$  for every  $A \in \mathcal{A}$ .

Moreover,  $\phi$  is a homomorphism, because every edge  $\{u, v\} \in E(G)$  is contained in  $G[\mathbf{ES}_G]$  for some simple extension  $\mathbf{ES}_G$  in  $\mathcal{E}_{D_G}(G)$  and  $\phi$  maps  $\mathbf{ES}_G$  according to some homomorphism  $\phi_{\text{Ext}_G}$  (if  $\mathbf{ES}_G \in \bar{\mathcal{E}}_A$ ) or some homomorphism  $\phi_A$  (otherwise). For basically the same reason, namely because every  $\phi_{\mathbf{ES}_G}$  and every  $\phi_A$  is locally surjective for every vertex in  $V(G) \setminus D_G$ , we have that  $\phi$  is locally surjective for every vertex  $v \in V(G) \setminus D_G$ .

Towards showing that  $\phi$  is also locally surjective for every  $d \in D_G$ , let  $n_H$  be any neighbour of  $\phi(d)$  in  $H$ . If  $n_H \in \phi_P(D_G)$ , then there is a neighbour  $n_G$  of  $d$  in  $D_G$  with  $\phi(n_G) = n_H$ , because  $\phi_P$  is locally surjective. If on the other hand  $n_H \in V(\mathbf{ES}_H - D_H)$  for some  $\mathbf{ES}_H \in \mathcal{E}_{D_H}(H, T_H)$  with  $T_H \in \mathcal{T}_{D_H}(H)$ , then there is a neighbour  $n_G$  of  $d$  in  $\beta(\gamma^{-1}(T_H))$  with  $\phi(n_G) = n_H$ , because  $\phi$  (restricted to  $\beta(\gamma^{-1}(T_H))$ ) is a locally surjective homomorphism from  $\beta(\gamma^{-1}(T_H))$  to  $\mathbf{ES}_H$ .

This completes the proof for the locally surjective case. We now complete the proof for the locally bijective case. First note that because of (B1), the function  $\beta$  from  $\mathcal{A}$  to the complex extensions of  $G$  is bijective. Moreover, if  $A = (\mathbf{EC}_G, T_H) \in \mathcal{A}$ , then because  $A \in \mathbf{BM}$ , there is a locally bijective homomorphism  $\phi_A$  from  $\beta(A)$  to  $\gamma(A)$  that augments  $\phi_P$ . This now allows us to directly define  $\phi : V(G) \rightarrow V(H)$ . That is, we set  $\phi(v)$  to be equal to:

- $\phi_P(v)$  if  $v \in D_G$  and
- $\phi_A(v)$  if  $v \in V(\mathbf{EC}_G - D_G)$  for some  $\mathbf{EC}_G = \beta(A)$  and  $A \in \mathcal{A}$ .

It remains to show that  $\phi$  is a locally bijective homomorphism from  $G$  to  $H$  that augments  $\phi_P$ . Note that we can assume that  $\phi$  is already a locally surjective homomorphism that augments  $\phi_P$ , using the same arguments as for the locally surjective case. Thus it only remains to show that  $\phi$  is also locally injective for every  $d \in D_G$ . Suppose not, then there are two distinct neighbours  $n_G$  and  $n'_G$  that are mapped to the same neighbour  $n_H$  of  $\phi(d)$  in  $H$ . This is clearly not possible if both  $n_G$  and  $n'_G$  are in  $D_G$  because  $\phi_P$  is locally bijective on  $D_G$ . Moreover, this can also not be the case if exactly one of  $n_G$  and  $n'_G$  is in  $D_G$ , because then  $n_H \in V(D_H)$ , but because  $\phi$  augments  $\phi_P$ , the other cannot be mapped to  $D_H$ . Therefore, we can assume that  $n_G$  and  $n'_G$  are outside of  $D_G$ . Let  $\mathbf{ES}_H \in \mathcal{E}_{D_H}(H)$  be the simple extension containing  $n_H$ . Then,  $n_G$  and  $n'_G$  must be mapped by  $\phi_{\gamma^{-1}(\mathbf{ES}_H)}$ , but this is not possible because  $\phi_{\gamma^{-1}(\mathbf{ES}_H)}$  is locally bijective.  $\square$

### 3.3 Constructing and Solving the ILP

The main aim of this section is to show the following theorem; see Definition 1 for a formal description of a  $(k, c)$ -extended deletion.

**Theorem 1.** *Let  $G$  be a graph, let  $D_G$  be a  $(k, c)$ -extended deletion set (respectively, a  $c$ -deletion set) of size at most  $k$  for  $G$ , let  $\mathcal{D} = (D_H, c', \mathbf{CH})$  be a target description and let  $\phi_P : D_G \rightarrow D_H$  be a locally surjective (respectively, bijective) homomorphism from  $D_G$  to  $D_H$ . Deciding whether there is a locally surjective (respectively bijective) homomorphism that augments  $\phi_P$  from  $G$  to any graph satisfying  $\mathbf{CH}$  is FPT parameterized by  $k + c + c'$ .*

In order to prove Theorem 1, we need to show that we can construct and solve the ILP instance given in the previous section. The main ingredient for the proof of Theorem 1 is Lemma 12, which shows that we can efficiently compute the sets  $\mathbf{wSM}$ ,  $\mathbf{SM}$ , and  $\mathbf{BM}$ . We start by showing that the set  $\mathcal{T}_{D_G}(G)$  can be computed efficiently and has small size.

**Lemma 9.** *For a graph  $G$  and a  $(k, c)$ -extended deletion set  $D_G$  of size at most  $k$  for  $G$ , it holds that  $\mathcal{T}_{D_G}(G)$  has size at most  $k + c2^{\binom{|D_G|+c}{2}}$ . Moreover, the problem of computing  $\mathcal{T}_{D_G}(G)$  and  $\mathbf{tc}_G$  for a given graph  $G$  with  $(k, c)$ -extended deletion set  $D_G$  is FPT parameterized by  $|D_G| + k + c$ .*

*Proof.* Because  $|\mathcal{T}_G(G) \setminus \mathcal{T}_G^c| \leq k$  and  $|\mathcal{T}_G^c| \leq c2^{\binom{|D_G|+c}{2}}$ , we obtain that  $|\mathcal{T}_{D_G}(G)| \leq k + c2^{\binom{|D_G|+c}{2}}$ . Moreover, we can compute  $\mathcal{T}_{D_G}(G)$  starting from the empty set and adding a simple extension  $G[D_G \cup C]$  for some component  $C$  of  $G \setminus D_G$  if  $G[D_G \cup C]$  is not equivalent with respect to  $\sim_{D_G}$  to any element already added to  $\mathcal{T}_{D_G}(G)$ . Note that checking whether  $G[D_G \cup C] \sim_D G[D_G \cup C']$  for two components  $C$  and  $C'$  of  $G \setminus D_G$  is FPT parameterized by  $|D_G| + k + c$ , because  $G[D_G \cup C]$  has treewidth at most  $|D_G| + k + c$  for every component  $C$  of  $G \setminus D_G$  (because of Proposition 2) and graph isomorphism is FPT parameterized by treewidth [64]. The same procedure can now also be used to compute all the non-zero entries of the function  $\text{tc}_G$  (i.e. the entries where  $\text{tc}_G(T) \neq 0$ ), which provides us with a compact representation of  $\text{tc}_G$ .  $\square$

The following lemma is crucial for computing the sets SM and BM that are required to construct the ILP instance. Informally, we will show that if  $(\mathbf{EC}_G, \mathbf{ES}_H) \in \text{SM}$  (or  $(\mathbf{EC}_G, \mathbf{ES}_H) \in \text{BM}$ ), then  $\mathbf{EC}_G$  consists of only boundedly many (in terms of some function of the parameters) components, which will allow us to enumerate all possibilities for  $\mathbf{EC}_G$  in FPT-time.

**Lemma 10.** *Let  $D_G$  and  $D_H$  be graphs and let  $\phi_P$  be a locally surjective (respectively, locally bijective) homomorphism from  $D_G$  to  $D_H$ . Moreover, let  $\mathbf{EC}_G$  be an extension of  $D_G$  that can be minimally  $\phi_P$ -S-mapped (respectively, minimally  $\phi_P$ -B-mapped) to an extension  $\mathbf{ES}_H$  of  $D_H$ . Then,  $\mathbf{EC}_G \setminus D_G$  consists of at most  $|D_G||\mathbf{ES}_H \setminus D_H|$  components.*

*Proof.* We first show the statement of the lemma for the case when  $\phi_P$  is locally surjective and therefore  $\mathbf{EC}_G$  can be minimally  $\phi_P$ -S-mapped to  $\mathbf{ES}_H$ . Let  $\phi : V(\mathbf{EC}_G) \rightarrow V(\mathbf{ES}_H)$  be a locally surjective homomorphism that augments  $\phi_P$  and that exists because  $\mathbf{EC}_G$  can be  $\phi_P$ -S-mapped to  $\mathbf{ES}_H$ . Let  $\mathbf{EC}'_G$  be an extension of  $\mathbf{EC}_G$  with  $\mathbf{EC}'_G \preceq \mathbf{EC}_G$ . Then, because of Lemma 3, it follows that  $\phi|_{\mathbf{EC}'_G}$  is a homomorphism from  $\mathbf{EC}'_G$  to  $\mathbf{ES}_H$  that is locally surjective for every  $v \in \mathbf{EC}'_G \setminus D_G$ . Therefore,  $\phi|_{\mathbf{EC}'_G}$  is a locally surjective homomorphism from  $\mathbf{EC}'_G$  to  $\mathbf{ES}_H$  if and only if  $\mathbf{EC}'_G$  is such that  $\phi|_{\mathbf{EC}'_G}$  is locally surjective for every  $d \in D_G$ . That is, for every  $d \in D_G$  and every neighbour  $n_H$  of  $\phi(d)$  in  $\mathbf{ES}'_H$ , there has to exist a neighbour  $n_G$  of  $d$  in  $\mathbf{EC}'_G$  such that  $\phi(n_G) = n_H$ . Since this holds if  $n_H \in D_H$  (because  $\phi_P$  is a locally surjective homomorphism from  $D_G$  to  $D_H$ ), we can assume that the above only has to hold for every  $d \in D_G$  and  $n_H \in \mathbf{ES}_H \setminus D_H$ .

Because  $\phi$  is a locally surjective homomorphism from  $\mathbf{EC}_G$  to  $\mathbf{ES}_H$ , it follows that for every  $d \in D_G$  and every neighbour  $n_H$  of  $\phi(d)$  in  $\mathbf{ES}_H$ , there is a component, say  $C_{d,n_H}$ , containing a neighbour  $n_G$  of  $d$  in  $\mathbf{EC}_G$  such that  $\phi(n_G) = \phi(n_H)$ ; note that because  $\phi$  augments  $\phi_P$ , it follows that  $n_G \notin D_G$  because  $n_H \notin D_H$ .

Let  $\mathbf{EC}'_G$  be the extension of  $D_G$  consisting of  $D_G$  and all components  $C_{d,n_H}$  for every  $d \in D$  and  $n_H \in \mathbf{ES}_H \setminus D_H$  as above. Then,  $\phi|_{\mathbf{EC}'_G}$  is a locally surjective homomorphism from  $\mathbf{EC}'_G$  to  $\mathbf{ES}_H$  and since  $\mathbf{EC}_G$  is minimally  $\phi_P$ -S-mapped to  $\mathbf{ES}_H$  and  $\mathbf{EC}'_G \preceq \mathbf{EC}_G$ , it follows that  $\mathbf{EC}'_G = \mathbf{EC}_G$ . However,  $\mathbf{EC}'_G \setminus D_G$  consists of at most one component for every  $d \in D_G$  and every  $n_H \in \mathbf{ES}_H \setminus D_H$  and therefore it consists of at most  $|D_G||\mathbf{ES}_H \setminus D_H|$  components. This concludes the proof for the case when  $\phi_P$  is locally surjective.

It remains to show the statement of the lemma for the case when  $\phi_P$  is locally bijective and  $\mathbf{EC}_G$  is minimally  $\phi_P$ -B-mapped to  $\mathbf{ES}_H$ . Let  $\phi : V(\mathbf{EC}_G) \rightarrow V(\mathbf{ES}_H)$  be a locally bijective homomorphism that augments  $\phi_P$  and that exists because  $\mathbf{EC}_G$  can be  $\phi_P$ -B-mapped to  $\mathbf{ES}_H$ . Because  $\phi$  is locally bijective, it is also locally surjective and therefore

we can obtain the components  $C_{d,n_H}$  of  $\mathbf{EC}_G \setminus D_G$  for  $d \in D_H$  and  $n_H \in \mathbf{ES}_H \setminus D_H$  using the same arguments as in the case when  $\phi$  was locally surjective. As before, let  $\mathbf{EC}'_G$  be the extension of  $D_G$  containing all components  $C_{d,n_H}$ . Then, as we showed above,  $\phi|_{\mathbf{EC}'_G}$  is a locally surjective homomorphism from  $\mathbf{EC}'_G$  to  $\mathbf{ES}_H$ . Moreover,  $\phi|_{\mathbf{EC}'_G}$  is also locally injective, because so is  $\phi$ . Therefore,  $\phi|_{\mathbf{EC}'_G}$  is a locally bijective homomorphism from  $\mathbf{EC}'_G$  to  $\mathbf{ES}_H$ . The latter implies, as  $\mathbf{EC}_G$  can be minimally  $\phi_P$ -B-mapped to  $\mathbf{ES}_H$ , that  $\mathbf{EC}_G = \mathbf{EC}'_G$ . This concludes the proof of the lemma, because  $\mathbf{EC}'_G$  consists of at most  $|D_G||\mathbf{ES}_H \setminus D_H|$  components.  $\square$

The following proposition is a slight generalisation of [19, Theorem 4] and will allow us to efficiently decide whether an extension  $\mathbf{EC}_G$  can be (weakly) S-mapped (respectively, B-mapped) to some extension  $\mathbf{ES}_H$ .

**Lemma 11 ([19, Theorem 4]).** *Let  $G$  and  $H$  be graphs and let  $\phi_P : D_G \rightarrow D_H$  be a locally surjective (respectively, bijective) homomorphism from  $D_G$  to  $D_H$  for some subgraphs  $D_G$  of  $G$  and  $D_H$  of  $H$ . Then deciding whether there is a locally surjective (respectively, bijective) homomorphism from  $G$  to  $H$  that augments  $\phi_P$  can be achieved in  $\mathcal{O}(|V(G)|(|V(H)2^{\Delta(H)})^{\text{tw}(G)}2^{\text{tw}(G)}\Delta(H)))$  time and is therefore FPT parameterized by  $\text{tw}(G) + |V(H)|$ .*

*Proof.* In [19, Theorem 4], the authors provided an algorithm that, given a graph  $G$  and a graph  $H$ , decides in  $\mathcal{O}(|V(G)|(|V(H)2^{\Delta(H)})^{\text{tw}(G)}2^{\text{tw}(G)}\Delta(H)))$  time whether there is a locally surjective homomorphism from  $G$  to  $H$ . The algorithm uses a standard dynamic programming approach on a tree decomposition of  $G$ , and it is straightforward to verify that the algorithm can be adapted with only minor modifications to an algorithm using the same run-time that decides whether there is a locally bijective homomorphism from  $G$  to  $H$ . Similarly, it is straightforward to adapt their algorithm to the case that one is additionally given a locally surjective (respectively, bijective) homomorphism  $\phi_P$  from some induced subgraph  $D_G$  of  $G$  to some induced subgraph  $D_H$  of  $H$  and one only looks for a locally surjective (respectively, bijective) homomorphism from  $G$  to  $H$  that augments  $\phi_P$ .  $\square$

The following corollary now follows directly from Lemma 11 and the definition of (weakly) S-mapped (respectively, B-mapped).

**Corollary 1.** *Let  $D_G$  and  $D_H$  be graphs and let  $\phi_P$  be a locally surjective (respectively, bijective) homomorphism from  $D_G$  to  $D_H$ . Let  $\mathbf{EC}_G$  be an extension of  $D_G$  having treewidth at most  $\omega$  and let  $\mathbf{ES}_H$  be an extension of  $D_H$ . Then, testing whether  $\mathbf{EC}_G$  can be weakly  $\phi_P$ -S-mapped,  $\phi_P$ -S-mapped, or  $\phi_P$ -B-mapped to  $\mathbf{ES}_H$  is FPT parameterized by  $\omega + |\mathbf{ES}_H|$ .*

We are now ready to show that we can efficiently compute the sets wSM, SM, and BM, which is the last crucial step towards constructing the ILP instance.

**Lemma 12.** *Let  $G$  be a graph, let  $D_G$  be a  $(k, c)$ -extended deletion set (respectively, a  $c$ -deletion set) of size at most  $k$  for  $G$ , let  $\mathcal{D} = (D_H, c', \text{CH})$  be a target description and let  $\phi_P$  be a locally surjective (respectively, bijective) homomorphism from  $D_G$  to  $D_H$ . Then, the sets  $\text{wSM} = \text{wSM}(G, \mathcal{D}, \phi_P)$  and  $\text{SM} = \text{SM}(G, \mathcal{D}, \phi_P)$  (respectively, the set  $\text{BM} = \text{BM}(G, \mathcal{D}, \phi_P)$ ) can be computed in FPT-time parameterized by  $k + c + c'$  and  $|\text{SM}|$*

(respectively,  $|\text{BM}|$ ) is bounded by a function depending only on  $k + c + c'$ . Moreover, the number of variables in the equation system  $(\text{CH}, S1, S2, S3)$  (respectively,  $(\text{CH}, B1, B2)$ ) is bounded by a function depending only on  $k + c + c'$ .

*Proof.* We only show the lemma for the set SM, since the proof for the set wSM can be seen as a special case and the proof for the set BM is identical. Let  $(\text{EC}_G, T_H) \in \text{SM}$ . Then,  $\text{EC}_G$  is an extension of  $D_G$  with  $\text{EC}_G \preceq G$ ,  $T_H \in \mathcal{T}_{D_H}^{c'}$ , and  $\text{EC}_G$  can be minimally  $\phi_P$ -S-mapped to  $T_H$ . Because  $\text{EC}_G$  can be minimally  $\phi_P$ -S-mapped to  $\text{ES}_H$ , Lemma 10 implies that  $\text{EC}_G \setminus D_G$  consists of at most  $\ell = |D_G| |\text{ES}_H \setminus D_H|$  components and, because  $\text{EC}_G \preceq G$ , these are also components of  $G \setminus D_G$ . Therefore, there are at most  $(|\mathcal{T}_{D_G}(G)|)^\ell$  non-isomorphic possibilities for  $\text{EC}_G$ , which together with Lemma 9 and the facts that  $\ell \leq kc'$  and  $|\mathcal{T}_{D_H}^{c'}| \leq c'2^{\binom{k+c'}{2}}$  shows that  $|\text{SM}| \leq (|\mathcal{T}_{D_G}(G)|)^\ell |\mathcal{T}_{D_H}^{c'}| \leq (k + c2^{\binom{k+c}{2}})^\ell (c'2^{\binom{k+c'}{2}})$ . Therefore,  $|\text{SM}|$  is bounded by a function depending only on  $k + c + c'$ . Towards showing that we can compute SM in FPT-time parameterized by  $k + c + c'$ , first note that the set  $\mathcal{T}_{D_G}(G)$  can be computed in FPT-time parameterized by  $k + c$  using Lemma 9. Similarly, the set  $\mathcal{T}_{D_H}^{c'}$  can be computed in FPT-time parameterized by  $k + c'$  using the same idea as in Lemma 9. This now allows us to compute the set  $\mathcal{A}$  containing all non-isomorphic possibilities for  $\text{EC}_G$ , i.e. the set of all extensions  $\text{EC}_G$  of  $D_G$  with  $\text{EC}_G \preceq G$  and  $\sum_{T_G \in \mathcal{T}_{D_G}(G)} \text{tc}_{\text{EC}_G}(T_G) \leq \ell$  in FPT-time parameterized by  $k + c + c'$ , i.e. in time at most  $(|\mathcal{T}_{D_G}(G)|)^\ell$ . But then, SM is equal to the set of all pairs  $(\text{EC}_G, \text{ES}_H) \in \mathcal{A} \times \mathcal{T}_{D_H}^{c'}$  such that  $\text{EC}_G$  can be minimally  $\phi_P$ -S-mapped to  $\text{ES}_H$ . Moreover, for every such pair  $(\text{EC}_G, \text{ES}_H)$  we can test in FPT-time parameterized by  $k + c + c'$  whether  $\text{EC}_G$  can be  $\phi_P$ -S-mapped to  $\text{ES}_H$  using Corollary 1, because the treewidth of  $\text{EC}_G$  is at most  $k + c$  (Proposition 2). Therefore, we can compute SM by enumerating all pairs  $(\text{EC}_G, \text{ES}_H) \in \mathcal{A} \times \mathcal{T}_{D_H}^{c'}$ , testing for each of them whether  $\text{EC}_G$  can be  $\phi_P$ -S-mapped to  $\text{ES}_H$  using Corollary 1, and keeping only those pairs  $(\text{EC}_G, \text{ES}_H)$  such that  $\text{EC}_G$  can be  $\phi_P$ -S-mapped to  $\text{ES}_H$  and  $\text{EC}_G$  is inclusion-wise minimal among all pairs  $(\text{EC}'_G, \text{ES}_H)$ .  $\square$

We are now ready to prove the main result of this subsection.

**Theorem 1 (restated).** *Let  $G$  be a graph, let  $D_G$  be a  $(k, c)$ -extended deletion set (respectively, a  $c$ -deletion set) of size at most  $k$  for  $G$ , let  $\mathcal{D} = (D_H, c', \text{CH})$  be a target description and let  $\phi_P : D_G \rightarrow D_H$  be a locally surjective (respectively, bijective) homomorphism from  $D_G$  to  $D_H$ . Then, deciding whether there is a locally surjective (respectively bijective) homomorphism that augments  $\phi_P$  from  $G$  to any graph satisfying CH is FPT parameterized by  $k + c + c'$ .*

*Proof.* We first compute the sets wSM and SM (respectively, the set BM), which because of Lemma 12 can be achieved in FPT-time parameterized by  $k + c + c'$ . This now allows us to construct the ILP instance  $\mathcal{I}$  given by the equation system  $(\text{CH}, S1, S2, S3)$  (respectively, the equation system  $(\text{CH}, B1, B2)$ ) in FPT-time parameterized by  $k + c + c'$ . Moreover, because the number of variables in  $\mathcal{I}$  is bounded by a function of  $k + c + c'$  and we can employ Proposition 3 to solve  $\mathcal{I}$  in FPT-time parameterized by  $k + c + c'$ . Finally, because of Lemma 8, it follows that  $\mathcal{I}$  has a solution if and only if there is a locally surjective (respectively bijective) homomorphism that augments  $\phi_P$  from  $G$  to any graph satisfying CH, which completes the proof of the theorem.  $\square$



## 4 Applications of Our Algorithmic Framework

We are ready to show the main results of our paper, which can be obtained as an application of our framework given in the previous section. Our first result implies that LSHOM and LBHOM are FPT parameterized by the fracture number of the guest graph.

**Theorem 2.** *LSHOM and LBHOM are FPT parameterized by  $k + c$ , where  $k$  and  $c$  are such that the guest graph  $G$  has a  $c$ -deletion set of size at most  $k$ .*

*Proof.* Let  $G$  and  $H$  be non-empty connected graphs such that  $G$  has a  $c$ -deletion set of size at most  $k$ . Let  $D_H = H[\Delta_H^{k+c}]$ . We first verify whether  $H$  has a  $c$ -deletion set of size at most  $k$  using Proposition 1. *If this is not the case, then we can return that there is no locally surjective (and therefore also no bijective) homomorphism from  $G$  to  $H$  because of Lemma 4.* Therefore, we can assume in what follows that  $H$  also has a  $c$ -deletion set of size at most  $k$ , which together with Lemma 5 implies that  $V(D_H)$  is a  $kc(k+c)$ -deletion set of size at most  $k$  for  $H$ . Therefore, using Lemma 9, we can compute  $\text{tc}_H$  in FPT-time parameterized by  $k+c$ . This now allows us to obtain a target description  $\mathcal{D} = (D_H, c', \text{CH})$  with  $c' = kc(k+c)$  for  $H$ , i.e.  $\mathcal{D}$  is satisfied only by the graph  $H$ , by adding the constraint  $x_T = \text{tc}_H(T_H)$  to CH for every simple extension type  $T_H \in \mathcal{T}_{D_H}^c$ ; note that  $\mathcal{T}_{D_H}^c$  can be computed in FPT-time parameterized by  $k+c$  by Lemma 9.

Because of Lemma 6, we obtain that there is a locally surjective (respectively, bijective) homomorphism  $\phi$  from  $G$  to  $H$  if and only if there is a set  $D \subseteq \Delta_G^{k+c}$  and a locally surjective (respectively, bijective) homomorphism  $\phi_P$  from  $D_G = G[D]$  to  $D_H$  such that  $\phi$  augments  $\phi_P$ . Therefore, we can solve LSHOM by checking, for every  $D \subseteq \Delta_G^{k+c}$  and every locally surjective homomorphism  $\phi_P$  from  $D_G = G[D]$  to  $D_H$ , whether there is a locally surjective homomorphism from  $G$  to  $H$  that augments  $\phi_P$ . Note that there are at most  $2^k$  subsets  $D$  and because of Lemma 7, we can compute the set  $\Phi_D$  for every such subset in  $\mathcal{O}(k^{k+2})$  time. Furthermore, due to Lemma 5,  $D$  is a  $(k - |D|, c)$ -extended deletion set of size at most  $k$  for  $G$ . Therefore, for every  $D \subseteq \Delta_G^{k+c}$  and  $\phi_P \in \Phi_D$ , we can use Theorem 1 to decide in FPT-time, parameterized by  $k+c$  (because  $c' = kc(k+c)$ ), if there is a locally surjective (respectively, bijective) homomorphism from  $G$  to a graph satisfying  $\mathcal{D}$  that augments  $\phi_P$ . As  $H$  is the only graph satisfying  $\mathcal{D}$ , we proved the theorem.  $\square$

The proof of our next theorem is similar to the proof of Theorem 2. The major difference is that  $H$  is not given. Instead, we use Theorem 1 for a selected set of target descriptions. Each of these target descriptions enforces that graphs satisfying it have to be connected and have precisely  $h$  vertices, where  $h$  is part of the input for the ROLE ASSIGNMENT problem. Furthermore, we ensure that every graph  $H$  satisfying the requirements of ROLE ASSIGNMENT satisfies at least one of the selected target descriptions. The size of the set of considered target descriptions depends only on  $c$  and  $k$ , as it suffices to consider any small graph  $D_H$  and types of small simple extensions of  $D_H$ .

**Theorem 3.** *ROLE ASSIGNMENT is FPT parameterized by  $k + c$ , where  $k$  and  $c$  are such that  $G$  has a  $c$ -deletion set of size at most  $k$ .*

*Proof.* Let  $G$  be a non-empty connected graph such that  $G$  has a  $c$ -deletion set of size at most  $k$  and let  $h \geq 1$  be an integer.

In order to use Theorem 1 in this case, we need to ensure that the target descriptions used enforce that  $H$  is connected and has  $h$  vertices. Therefore, for a fixed graph  $D$  on at most  $k$  vertices, we let  $\text{CON}_D$  be the set of all minimal sets  $S \subseteq \mathcal{T}_D^{k+c}$  such that any extension  $H$  of  $D$ , that contains exactly the types in  $S$ , is connected.

Since  $|\mathcal{T}_D^{k+c}|$  is bounded by  $(2k+c)2^{\binom{2k+c}{2}}$ , we can compute  $\text{CON}_D$  by considering every  $S \subseteq \mathcal{T}_D^{k+c}$  and checking whether an extension  $T \in \mathcal{T}_D$  of  $D$  containing precisely the types in  $S$  is connected. Since  $|V(T)| \leq k + (k+c) \cdot |S|$  and checking connectivity takes linear time (using BFS or DFS) we can compute  $\text{CON}_D$  in time depending only on  $k$  and  $c$ . For  $S \in \text{CON}_D$ , we set  $\text{CH}_S$  to be the set of equations containing  $x_T \geq 1$  for every  $T \in S$  and  $|V(D_H)| + \sum_{T \in \mathcal{T}_{D_H}^c} (|V(T)| - |V(D_H)|) * x_T = h$ . Note that for  $D$  and  $S \in \text{CON}_D$ , any graph  $H$  satisfying the target description  $(D, c+k, \text{CH}_S)$  is connected and has  $h$  vertices.

If there is a connected graph  $H$  on  $h$  vertices and a locally surjective homomorphism  $\phi$  from  $G$  to  $H$ , then by Lemma 6 there is a set  $D \subseteq \Delta_G^{k+c}$  and a locally surjective homomorphism  $\phi_P$  from  $D_G = G[D]$  to  $D_H = H[\Delta_H^{k+c}]$  such that  $\phi$  augments  $\phi_P$ . Note that by Lemmas 4 and 5,  $D_H$  is a  $(k+c)$ -deletion set of size at most  $k$ . This implies firstly that  $D_H$  is a graph on at most  $k$  vertices. Secondly,  $H$  is an extension of  $D_H$  such that  $\text{tc}_H(T) = 0$  for  $T \notin \mathcal{T}_{D_H}^{c+k}$  and, since  $H$  is also connected and has  $h$  vertices,  $H$  satisfies the target description  $(D_H, c+k, \text{CH}_S)$  for at least one  $S \in \text{CON}_{D_H}$ .

Therefore, we can solve the ROLE ASSIGNMENT problem by checking for every  $D \subseteq \Delta_G^{k+c}$ , every graph  $D_H$  on no more than  $k$  vertices, every  $S \in \text{CON}_{D_H}$  and every locally surjective homomorphism  $\phi_P$  from  $D_G = G[D]$  to  $D_H$ , whether there is a graph  $H$  satisfying the target description  $(D_H, k+c, \text{CH}_S)$  and a locally surjective homomorphism from  $G$  to  $H$  that augments  $\phi_P$ . Note that there are at most  $2^k$  subsets  $D$ . Furthermore, there are at most  $k2^{\binom{k}{2}}$  graphs on at most  $k$  vertices and for each we can compute  $\text{CON}_D$  in time depending only on  $k$  and  $c$ . For each such graph  $D_H$ , there are at most  $|\text{CON}_{D_H}| \leq 2^{(2k+c)2^{\binom{2k+c}{2}}}$  subsets  $S$  to consider. Lastly, because of Lemma 7, for every  $D \subseteq \Delta_G^{k+c}$  and any graph  $D_H$  on no more than  $k$  vertices, we can compute the set of locally surjective homomorphisms  $\phi_P$  from  $G[D]$  to  $D_H$  in time  $\mathcal{O}(k^{k+2})$  time and there are at most  $|D|^{|D|}$   $\phi_P$  to consider.

By Lemma 5,  $D$  is a  $(k-|D|, c)$ -extended deletion set of size at most  $k$  for  $G$ . Therefore, for every  $D \subseteq \Delta_G^{k+c}$ , every graph  $D_H$  on no more than  $k$  vertices, every  $S \in \text{CON}_{D_H}$  and every locally surjective homomorphism  $\phi_P$  from  $D_G = G[D]$  to  $D_H$ , we can employ Theorem 1 to decide in FPT-time parameterized by  $k+c$ , whether there is a graph  $H$  satisfying  $(D_H, c+k, \text{CH}_S)$  and a locally surjective homomorphism from  $G$  to  $H$  that augments  $\phi_P$ . This completes the proof.  $\square$

## 5 Locally Injective Homomorphisms

The following result is well known. We include a proof for completeness.

**Theorem 4 (Folklore).** *LIHOM is  $W[1]$ -hard parameterized by  $|V(G)|$ . In particular, it is  $W[1]$ -hard for all structural parameters of  $G$ .*

*Proof.* Let  $G$  be a complete graph on  $k$  vertices, and let  $H$  be an arbitrary graph. There exists a locally injective homomorphism  $\phi$  from  $G$  to  $H$  if and only if  $H$  contains a clique  $K$  on  $k$  vertices. Indeed, for the forward direction, pick  $K$  to be the image of  $V(G)$  under  $\phi$ . Then  $|K| = |V(G)| = k$  by the local injectivity of  $\phi$ , and  $K$  is a clique. For the reverse direction, let  $\phi$  be any bijection between  $V(G)$  and  $K$ . The result follows from the fact that CLIQUE is W[1]-hard.  $\square$

The locally injective case is more difficult in our setting since, in general, surjectivity helps to transfer structural parameters on  $G$  to similar structures on  $H$  (for example, in LSHOM and LBHOM the image of a deletion set is also a deletion set by Lemma 4). In LIHOM however, and even in the restricted case of graphs with bounded vertex cover number, no such property can be used to help find the image of a vertex cover, and exponential-time enumerations appear to be necessary. On the positive side, once a [partial homomorphism](#) from a vertex cover of  $G$  to  $H$  has been found, our ILP framework can still be applied to map the remaining vertices in FPT-time. This leads to an XP-algorithm for vertex cover number (Theorem 5). Interestingly, this result does not extend to  $c$ -deletion set number for  $c > 1$ : even if the mapping of the deletion set can be guessed, the fact that the non-trivial remaining components must be mapped to distinct subgraphs of  $H$  makes the problem difficult (see Theorem 6).

**Theorem 5.** *LIHOM is in XP parameterized by the vertex cover number of  $G$ .*

*Proof.* As for the surjective and bijective cases, we employ a two-step algorithm that first computes a suitable vertex cover, [which can be done in time  \$1.2738^{\text{vc}\(G\)}\$  \[21\]](#), and guesses the image of the vertex cover and a partial homomorphism. Second, the algorithm finds a solution of an ILP which defines how to map the remaining vertices. The ILP only requires FPT-time, however the first step needs an exhaustive enumeration of subsets of  $H$  (in the injective case, the image of a vertex cover does not have to be a vertex cover), hence the XP running time.

We use the definitions of *types* and *extensions* from Section 3.2. Note that for any connected graph  $G$  with vertex cover  $D_G$  the connected components of  $G \setminus D_G$  are single vertices. We can thus define the type of a vertex  $v$  of  $G \setminus D_G$  to be the type of the simple extension  $G[\{v\} \cup D_G]$ . Note that two vertices  $u, v$  from  $G \setminus D_G$  have the same type if they have the same neighbours in  $D_G$ . Hence, there are at most  $2^{|D_G|}$  types in  $G$ .

In the following, let  $G$  be the given connected guest graph, and let  $H$  be the host graph. The main property of locally injective homomorphisms we use in this proof is that the size of the pre-image of any vertex of  $H$  can be bounded by twice the vertex cover number of  $G$ . This follows from the following claims.

**Claim 1.** *For any locally injective homomorphism  $\phi : G \rightarrow H$  and every  $h \in V(H)$  no two vertices in the pre-image  $\phi^{-1}(h)$  share a neighbour.*

*Proof of Claim.* Assume that  $\phi : G \rightarrow H$  is a locally injective homomorphism and there are vertices  $h \in V(H)$ ,  $v, v' \in \phi^{-1}(h)$  and  $u \in V(G)$  such that  $u$  is adjacent to both  $v$  and  $v'$ . But then the restriction of  $\phi$  to  $N_G(u)$  is not injective contradicting the assumption.  $\diamond$

Since for every vertex cover  $D_G$  and every vertex  $u \in V(G)$ , either  $u \in D_G$  or every neighbour of  $u$  is in  $D_G$  we observe the following.

**Claim 2.** *If  $R \subseteq V(G)$  is a subset with the property that no two vertices  $u, v \in R$  share a neighbour then  $|R| \leq 2 \cdot \text{vc}(G)$ .*

We use the property that pre-images have bounded size of locally injective homomorphisms to guess a partial homomorphism we aim to augment as well as determining which types of complex extensions our ILP needs to consider. In the following we show that picking suitable pre-images for the vertices of the host graph  $H$  yields a locally injective homomorphism.

Assume that  $D_G$  is a vertex cover of  $G$  and  $D_H \subseteq V(H)$ . We further fix  $\phi_P : D_G \rightarrow D_H$  to be a partial homomorphism such that for every  $h \in D_H$  no two vertices  $u, v \in \phi_P^{-1}(h)$  share a neighbour in  $G$ . We say that a (possibly empty) subset  $R$  of  $V(G \setminus D_G)$  is a *candidate pre-image* of a vertex  $h \in V(H \setminus D_H)$  if the following two conditions hold:

1.  $\phi_P(N_G(R)) \subseteq N_H(h) \cap D_H$  and
2. any two vertices in  $R$  do not share a neighbour.

**Claim 3.** *There is a locally injective homomorphism from  $G$  to  $H$  augmenting  $\phi_P$  if and only if there is a partition  $\{R_h \mid h \in V(H)\}$  of  $V(G)$  such that  $R_h = \phi_P^{-1}(h)$  for every  $h \in D_H$ , and  $R_h$  is a candidate pre-image for every  $h \in V(H) \setminus D_H$ .*

*Proof of Claim.* First assume that  $\phi : V(G) \rightarrow V(H)$  is a locally injective homomorphism from  $G$  to  $H$  augmenting  $\phi_P$ . Define  $R_h = \phi^{-1}(h)$  for every  $h \in V(H)$ . Since  $\phi$  is a mapping, we get that  $\{R_h \mid h \in V(H)\}$  is a partition of  $V(G)$ . Since  $\phi$  augments  $\phi_P$ , we know that  $\phi_P^{-1}(h) = \phi^{-1}(h)$ . The latter implies that  $\phi_P^{-1}(h) = R_h$  for every  $h \in D_H$ . Finally, let  $h \in V(H) \setminus D_H$ . As  $\phi$  augments  $\phi_P$  the set  $R_h$  must be disjoint from  $D_G$ . Hence,  $N_G(R_h) \subseteq D_G$  and  $\phi_P(N_G(R_h)) \subseteq D_H$ . Additionally, since  $\phi$  is a homomorphism,  $\phi(N_G(R_h)) = \phi_P(N_G(R_h)) \subseteq N_H(h)$  implying 1. Hence, by Claim 1, we find that  $R_h$  is a candidate pre-image.

On the other hand, assume that  $\{R_h \mid h \in V(H)\}$  is a partition of  $V(G)$  such that  $R_h = \phi_P^{-1}(h)$  for every  $h \in D_H$  and  $R_h$  is a candidate pre-image for every  $h \in V(H) \setminus D_H$ . Define a mapping  $\phi : V(G) \rightarrow V(H)$  where  $\phi(v) = h$  if  $v \in R_h$ . Note that  $\phi$  is well defined as  $\{R_h \mid h \in V(H)\}$  is a partition of  $V(G)$ . We now argue that  $\phi$  is a homomorphism, i.e., we argue that for any two vertices  $u, v \in V(G)$  it holds that if  $uv \in E(G)$  then  $\phi(u)\phi(v) \in E(H)$ . For  $u, v \in D_G$  this directly follows from  $\phi_P$  being a homomorphism. For  $u \in D_G$  and  $v \notin D_G$  it follows from Condition 1 while  $uv \notin E(G)$  for any  $u, v \notin D_G$ . We are left to prove that  $\phi$  is locally injective.

Let  $v \in V(G)$  and  $u, u' \in N_G(v)$  be any vertices. We prove that  $\phi(u) \neq \phi(u')$ . If  $u, u' \in D_G$ , then  $u, u'$  share a neighbour which implies  $\phi_P(u) \neq \phi_P(u')$  by choice of  $\phi_P$ . If  $u \in D_G$  and  $u' \notin D_G$ , then  $\phi(u) \in D_H$  and  $\phi(u') \notin D_H$  as  $\phi$  is an augmentation of  $\phi_P$ . If  $u, u' \notin D_G$ , then  $u, u'$  share a neighbour, and they cannot be in the same candidate pre-image by Condition 2.  $\diamond$

Let  $H^{D_H}$  be the graph obtained from  $H$  by deleting all edges that are not incident to  $D_H$ . In particular,  $D_H$  is a vertex cover of  $H^{D_H}$ .

**Claim 4.** *There is a locally injective homomorphism from  $G$  to  $H$  augmenting  $\phi_P$  if and only if there is a locally injective homomorphism from  $G$  to  $H^{D_H}$  augmenting  $\phi_P$ .*

*Proof of Claim.* By Claim 3 there is a locally injective homomorphism from  $G$  to  $H$  augmenting  $\phi_P$  if and only if there is a partition  $\{R_h \mid h \in V(H)\}$  of  $V(G)$  such that  $R_h = \phi_P^{-1}(h)$  for every  $h \in D_H$  and  $R_h$  is a candidate pre-image for every  $h \in V(H) \setminus D_H$ . Observe that  $R_h$  being a candidate pre-image is independent on whether we consider the graph  $H$  or the graph  $H^{D_H}$ . This directly yields the claim using Claim 3.  $\diamond$

We are now ready to describe our algorithm. Since in the first step we want to guess all possible partial homomorphisms we first observe the following. Assume that  $\phi : V(G) \rightarrow V(H)$  is a locally injective homomorphism from  $G$  to  $H$ , that  $D'_G$  is a vertex cover of  $G$  of size  $\text{vc}(G)$ , and that  $D_H$  is the image  $\phi(D'_G)$  of  $D'_G$ . It is not necessarily true that the pre-image  $\phi^{-1}(D_H)$  is  $D'_G$ . Hence our goal is to guess the image  $D_H$  of  $D'_G$  as well as the pre-image  $D_G$  of  $D_H$ . Note that guessing  $D_G$  could be done in FPT time by essentially only considering the type of pre-image of every vertex  $h \in D_H$ . However, we simplify the analysis here and guess  $D_G$  with XP many guesses.

Given two graphs  $G$  and  $H$  as input the algorithm proceeds as follows. Since we can consider any connected component of  $G$  individually, we assume that  $G$  is connected. First we compute any vertex cover  $D'_G$  of  $G$  of size  $\text{vc}(G)$ . We proceed by considering every set  $D_H \subseteq V(H)$  of size at most  $\text{vc}(G)$  and any set  $\{R_h \mid h \in D_H\}$  of pairwise disjoint subsets of  $D(G)$ , for which the following conditions hold:

- $D'_G \subseteq D_G$  where we define  $D_G$  to be  $\bigcup_{h \in D_H} R_h$ ,
- the partial mapping  $\phi_P : D_G \rightarrow D_H$  defined by setting  $\phi_P(v) = h$  whenever  $v \in R_h$  is a homomorphism from  $D_G$  to  $D_H$  and
- for every  $h \in D_H$  no two vertices  $u, v \in R_h$  share a neighbour in  $G$ .

In total we consider  $(|V(H)| + 1)^{\text{vc}(G)}$  subsets  $D_H$  of  $V(H)$ . Furthermore, we only need to consider sets  $\{R_h \mid h \in D_H\}$  for which  $|R_h|$  is at most  $2 \times \text{vc}(G)$  by the condition that no two vertices  $u, v \in R_h$  share a neighbour and Claim 2. Hence, we consider  $(|V(G)| + 1)^{2 \cdot \text{vc}(G)^2}$  such sets and check for each whether the conditions are met. In the next step of the algorithm we decide for every partial homomorphism  $\phi_P$  considered in the first step, whether there is a locally injective homomorphism  $\phi : V(G) \rightarrow V(H)$  from  $G$  to  $H$  augmenting  $\phi_P$ . Note that by Claim 4 there is such a homomorphism if and only if there is a locally injective homomorphism  $\phi : V(G) \rightarrow V(H^{D_H})$  from  $G$  to  $H^{D_H}$  augmenting  $\phi_P$ . We encode the existence of such a homomorphism into an ILP.

Observe that  $D_G$  is a vertex cover of  $G$  of size at most  $2 \cot \text{vc}(G)^2$  (the set  $D_G$  consists of a set  $R_h$  of size at most  $2 \cdot \text{vc}(G)$  for each of the at most  $\text{vc}(G)$  vertices  $h \in D_H$ ) and  $D_H$  is a vertex cover of  $H^{D_H}$ . Types considered in the following are the types of  $G$  and  $H^{D_H}$  with regards to vertex covers  $D_G$  and  $D_H$ . To construct the ILP we first compute  $\text{tc}_G$  and  $\text{tc}_{H^{D_H}}$  in polynomial time. Let IM be the set of pairs  $(\text{EC}_G, T_H)$ , where  $\text{EC}_G \in \mathcal{E}_{D_G}(G)$  is a complex extension of  $D_G$  and  $T_H \in \mathcal{T}_{D_H}(H^{D_H})$  is a simple extension of  $D_H$ , such that  $V(\text{EC}_G \setminus D_G)$  is a candidate pre-image of  $T_H \setminus D_H$ .

Observe that the size of every candidate pre-image is at most  $2 \cdot \text{vc}(G)$  by Condition 2 and Claim 2. Therefore, we get that the size of IM is at most  $|\mathcal{T}_{D_G}(G)|^{2 \cdot \text{vc}(G)} \cdot |\mathcal{T}_{D_H}(H^{D_H})| = 2^{4 \cdot \text{vc}(G)^3} \cdot 2^{\text{vc}(G)}$ . We introduce a variable  $x_{\text{EC}_G T_H}$  for each pair  $(\text{EC}_G, T_H) \in \text{IM}$  and let

$$X = \{x_{\text{EC}_G T_H} : (\text{EC}_G, T_H) \in \text{IM}\}.$$

The variable  $x_{\text{EC}_G T_H}$  represents the number of vertices  $h$  in  $V(H) \setminus D_H$  with type  $T_H$  whose pre-image is  $V(\text{EC}_G) \setminus D_G$ . We introduce two types of constraints (see below). Constraint (I1) enforces that the pre-images  $R_h$  form a partition of  $V(G \setminus D_G)$  by counting vertices of each type in each  $\text{EC}_G$  and checking that the sum corresponds to the count in  $G$ . Constraint (I2) corresponds to the fact that each vertex in  $H$  needs to be assigned a (possibly empty) pre-image (the number of pairs involving a type  $T_H$  must correspond to the type-count of  $T_H$  in  $H$ ).

$$(I1) \quad \sum_{(\text{EC}_G, T_H) \in \text{IM}} \text{tc}_{\text{EC}_G}(T_G) * x_{\text{EC}_G T_H} = \text{tc}_G(T_G) \text{ for every } T_G \in \mathcal{T}_{D_G}(G),$$

$$(I2) \quad \sum_{\text{EC}_G: (\text{EC}_G, T_H) \in \text{IM}} x_{\text{EC}_G T_H} = \text{tc}_H(T_H) \text{ for every } T_H \in \mathcal{T}_{D_H}(H^{D_H}).$$

We accept the input if any of the considered ILP's admits a solution, which can be checked in FPT time.

It remains to show that the algorithm is correct. If there is a locally injective homomorphism from  $G$  to  $H$ , then it must augment one of the partial homomorphisms  $\phi_P : D_G \rightarrow D_H$  considered in the first step. By Claim 3 there is a partition  $\{R_h \mid h \in V(H)\}$ , such that  $R_h$  is a candidate pre-image for every  $h \in V(H) \setminus D_H$ , and hence  $(R_h \cup D_G, D_H \cup \{h\}) \in \text{IM}$ . Hence, we obtain a feasible solution of the constraints (I1) and (I2) considering the following assignment  $\alpha$ . The assignment  $\alpha : X \rightarrow \mathbb{N}$  sets variable  $x_{\text{EC}_G T_H}$  to the number of vertices  $h \in V(H) \setminus D_H$  of type  $T_H$  for whom the extension  $R_h \cup D_G$  has type  $\text{EC}_G$ .

Conversely, let  $\alpha : X \rightarrow \mathbb{N}$  be an assignment which is a solution to the constraints (I1) and (I2) for some partial homomorphism  $\phi_P : D_G \rightarrow D_H$ . For each  $T_H \in \mathcal{T}_{D_H}(H^{D_H})$ , we first partition the vertices of  $V(H^{D_H}) \setminus D_H$  into sets  $S_{\text{EC}_G T_H}$  for every  $(\text{EC}_G, T_H) \in \text{IM}$  such that  $|S_{\text{EC}_G T_H}| = \alpha(x_{\text{EC}_G T_H})$ . This is possible due to constraint (I2). We then choose a partition  $\{R_h \mid h \in V(H^{D_H})\}$  of  $V(G)$  as follows. For each  $h \in D_H$ , we set  $R_h$  to be  $\phi_P^{-1}(h)$ . For  $h \in S_{\text{EC}_G T_H}$ , we choose  $R_h \subseteq V(G) \setminus D_G$  such that  $R_h \cup D_H$  has type  $\text{EC}_G$ . This is possible because of constraint (I1). Since  $\{R_h \mid h \in V(H^{D_H})\}$  is a partition satisfying that  $R_h = \phi_P^{-1}(h)$  for every  $h \in D_H$  and  $R_h$  is a candidate pre-image for every  $h \in V(H) \setminus D_H$ , we conclude that there is a locally injective homomorphism augmenting  $\phi_P$  by Claim 3.  $\square$

**Corollary 2.** *For any constant  $k$ , LIHOM is polynomial-time solvable for graphs  $G$  where  $\text{ds}_1(G)$  is at most  $k$ .*

We actually obtain the following dichotomy for the complexity of LIHOM, where the  $c = 1$ ,  $k \geq 1$  case is already given by Corollary 2.

**Theorem 6.** *Let  $c, k \geq 1$ . Then LIHOM is polynomial-time solvable on guest graphs  $G$  where  $\text{ds}_c(G)$  is at most  $k$  if either  $c = 1$  and  $k \geq 1$  or  $c = 2$  and  $k = 1$ ; otherwise, it is NP-complete.*

Theorem 6 follows from Corollary 2 and the following three lemmas.

**Lemma 13.** *LIHOM is polynomial-time solvable for graphs  $G$  where  $\text{ds}_2(G)$  is at most 1.*

*Proof.* Let  $G$  and  $H$  be connected graphs such that  $G$  has 2-deletion set number at most 1. If  $G$  has a 2-deletion set containing no vertices, then  $G$  contains at most two vertices, in which case we can solve LIHOM in polynomial time. Otherwise, we can find a 2-deletion set  $\{v\}$  in polynomial time by trying all possibilities for  $v$ . Let  $p$  be the number of edges in  $G[N_G(v)]$  and let  $w$  be a vertex of  $H$ . We claim that there is a locally injective homomorphism  $\phi$  from  $G$  to  $H$  such that  $\phi(v) = w$  if and only if  $H[N_H(w)]$  has a matching on at least  $p$  edges and  $d_G(v) \leq d_H(w)$ .

Indeed, if such a locally injective homomorphism  $\phi$  exists, then  $d_G(v) \leq d_H(w)$  because  $\phi$  is locally injective. Furthermore, for every edge  $xy$  in  $G[N_G(v)]$ , the homomorphism  $\phi$  maps the vertices  $x$  and  $y$  to adjacent vertices of  $H[N_H(w)]$ , and since  $\phi$  is locally injective, it cannot map two vertices of  $N_G(v)$  to the same vertex in  $N_H(w)$ . Therefore  $H[N_H(w)]$  must have a matching on at least  $p$  edges.

Now suppose that  $H[N_H(w)]$  has a matching  $M$  on at least  $p$  edges and  $d_G(v) \leq d_H(w)$ . For each edge  $xy$  in  $G[N_G(v)]$ , let  $\phi(x)$  and  $\phi(y)$  be the endpoints of an edge in  $M$  (choosing a different edge of  $M$  for each edge  $xy$ ). For the remaining vertices  $x \in N_G(v)$ , assign the remaining vertices of  $N_H(w)$  arbitrarily, such that no two vertices of  $N_G(v)$  are assigned the same value (this can be done since  $d_G(v) \leq d_H(w)$ ). Let  $\phi(x) = w$  for all remaining vertices of  $G$  (i.e. the vertex  $v$  and all vertices non-adjacent to  $v$  that have a common neighbour with  $v$ ). By construction,  $\phi$  is a locally injective homomorphism from  $G$  to  $H$ .

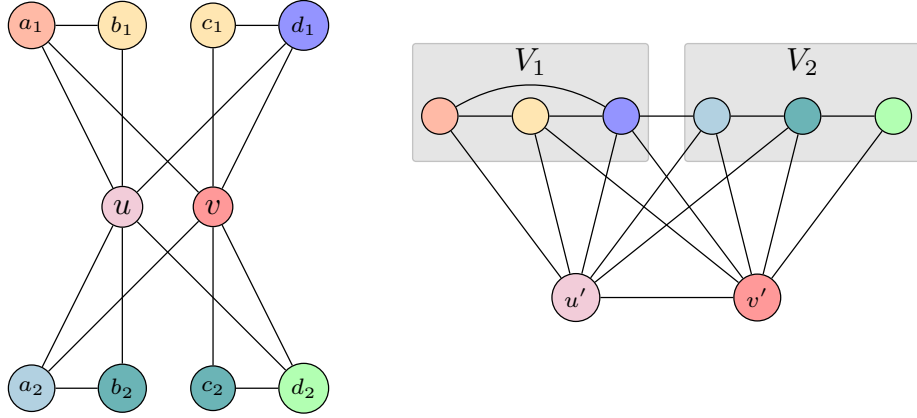
The size of a maximum matching in a graph can be found in polynomial time [30]. Thus, by branching over the possible vertices  $w \in V(H)$ , we obtain a polynomial-time algorithm for LIHOM.  $\square$

For the NP-hardness part of Theorem 6 we use a reduction from  $H'$ -PARTITION, defined in the following, with  $H' = P_3$  in Lemma 14 and  $H' = K_3$  in Lemma 15. Let  $H'$  be a fixed graph on  $h$  vertices. The  $H'$ -PARTITION problem takes as input a graph  $G'$  on  $hn$  vertices and the task is to decide whether the vertex set of  $G'$  can be partitioned into sets  $V_1, \dots, V_n$ , each of size  $h$ , such that  $G'[V_i]$  contains  $H'$  as a subgraph for all  $i \in \{1, \dots, n\}$ . This problem is known to be NP-complete if  $H' \in \{K_3, P_3\}$  [45,54].

**Lemma 14.** *For  $c \geq 2$  and  $k \geq 2$ , LIHOM is NP-hard on graphs  $G$  where  $ds_c(G)$  is  $k$ .*

*Proof.* We first consider the case when  $k = 2$ . Consider an instance  $G'$  of the  $P_3$ -PARTITION problem on  $3n$  vertices, where  $n \geq c$ . We construct a graph  $G$  as follows. For  $i \in \{1, \dots, n\}$ , add vertices  $a_i, b_i, c_i$  and  $d_i$  and edges  $a_i b_i$  and  $c_i d_i$ . Then add vertices  $u$  and  $v$  and make  $u$  adjacent to  $a_i, b_i$  and  $d_i$  and  $v$  adjacent to  $a_i, c_i$  and  $d_i$  for all  $i \in \{1, \dots, n\}$ . Finally, add the edge  $uv$ . Note that  $\{u, v\}$  is a minimum-size  $c$ -deletion set for  $G$  since  $\deg_G(u) = \deg_G(v) > c$ . Now let  $H$  be the graph obtained from  $G'$  by adding two vertices  $u'$  and  $v'$  that are adjacent to all the vertices in  $V(G')$  and to each other. For an illustration of the construction see Figure 4. We claim that there is a locally injective homomorphism  $\phi$  from  $G$  to  $H$  if and only if  $G'$  is a yes-instance of the  $P_3$ -PARTITION problem.

Suppose that  $G'$  is a yes-instance of the  $P_3$ -PARTITION problem and, for  $i \in \{1, \dots, n\}$ , let  $v_i^1, v_i^2, v_i^3$  be the three vertices in  $V_i$ , such that  $v_i^2$  is adjacent to  $v_i^1$  and  $v_i^3$  ( $v_i^1$  may or may not be adjacent to  $v_i^3$ ). Let  $\phi : V(G) \rightarrow V(H)$  be the function such that  $\phi(u) = u'$ ,  $\phi(v) = v'$ , and for  $i \in \{1, \dots, n\}$ ,  $\phi(a_i) = v_i^1$ ,  $\phi(b_i) = \phi(c_i) = v_i^2$  and  $\phi(d_i) = v_i^3$ . Note that  $\phi$  is injective on  $N_G(a_i)$  for every  $i \in [n]$  as  $\phi(u), \phi(v)$  and  $\phi(b_i)$  are pairwise different. For



**Fig. 4.** An instance of LIHOM consisting of the graph  $G$  (left) and the graph  $H$  (right) corresponding to the instance  $G' = (V, E)$  of  $P_3$ -PARTITION, where  $V = [6]$  and  $E = \{12, 13, 23, 34, 45, 56\}$ . As  $G'$  is a yes-instance of the  $P_3$ -PARTITION problem (partition is indicated by gray boxes), there is a locally injective homomorphism from  $G$  to  $H$  which is indicated by colours.

the same reason,  $\phi$  is injective on  $N_G(b_i)$ ,  $N_G(c_i)$  and  $N_G(d_i)$  for every  $i \in [n]$ . To see that  $\phi$  is injective on  $N_G(u)$  observe that  $\phi$  restricted to  $V(G) \setminus \{c_i : i \in [n]\}$  is a bijection. Since  $u$  is not adjacent to any vertex in  $\{c_i : i \in [n]\}$ , this proves that  $\phi$  is injective on  $N_G(u)$ . For  $v$  we conclude with a symmetric argument. Hence,  $\phi$  is a locally injective homomorphism from  $G$  to  $H$ .

Now suppose that  $\phi$  is a locally injective homomorphism from  $G$  to  $H$ . Now  $\deg_G(u) = \deg_G(v) = 3n + 1$ . Since  $H$  has  $3n + 2$  vertices and  $\phi$  is a locally injective homomorphism, it follows that  $\phi(u)$  and  $\phi(v)$  must be universal vertices in  $H$ . By symmetry, we may therefore assume that  $\phi(u) = u'$  and  $\phi(v) = v'$ . Now  $u$  is adjacent to  $v$  and the vertices  $a_i$ ,  $b_i$  and  $d_i$  for all  $i \in \{1, \dots, n\}$ . Similarly,  $v$  is adjacent to  $u$  and the vertices  $a_i$ ,  $c_i$  and  $d_i$  for all  $i \in \{1, \dots, n\}$ . Since  $\deg_G(u) = 3n + 1$ , and  $\phi$  is locally injective, it follows that  $\phi(\{a_i, b_i, d_i \mid i \in \{1, \dots, n\}\}) = V(G')$ . Similarly, since  $\deg_G(v) = 3n + 1$ , it follows that  $\phi(\{a_i, c_i, d_i \mid i \in \{1, \dots, n\}\}) = V(G')$ . Therefore  $\phi(\{b_i \mid i \in \{1, \dots, n\}\}) = \phi(\{c_i \mid i \in \{1, \dots, n\}\})$ . Renumbering the indices of the  $c_i$  and  $d_i$  vertices if necessary, we may therefore assume by symmetry that  $\phi(b_i) = \phi(c_i)$  for all  $i \in \{1, \dots, n\}$ . Now, for all  $i \in \{1, \dots, n\}$ , the vertices  $a_i$  and  $b_i$  are adjacent in  $G$ , so  $\phi(a_i)$  and  $\phi(b_i)$  are adjacent in  $H$ . Furthermore the vertices  $c_i$  and  $d_i$  are adjacent in  $G$ , so  $\phi(c_i) = \phi(b_i)$  and  $\phi(d_i)$  are adjacent in  $H$ . We now set  $V_i = \{\phi(a_i), \phi(b_i), \phi(d_i)\}$  and note that the  $V_i$  sets partition  $V(G')$ , and that  $G'[V_i]$  contains a  $P_3$  subgraph for all  $i \in \{1, \dots, n\}$ . This completes the proof of the case when  $k = 2$ .

To extend the proof to graphs with  $c$ -deletion number  $k > 2$ , we add  $(k - 1)$  universal vertices to  $H$  and replace  $u$  with a  $k$ -clique  $K$  each of whose vertices is adjacent to  $v$  and  $a_i$ ,  $b_i$  and  $d_i$  for all  $i \in \{1, \dots, n\}$ .  $\square$

**Lemma 15.** For  $c \geq 3$  and  $k \geq 1$ , LIHOM is NP-hard on graphs  $G$  where  $ds_c(G)$  is  $k$ .



*Proof.* We first consider the case when  $k = 1$ . Consider an instance  $G'$  of the  $K_3$ -PARTITION problem on  $3n$  vertices, where  $n \geq c$ . Let  $H$  be the graph obtained from  $G'$  by adding a universal vertex  $w$ . Let  $G$  be the graph obtained by taking the disjoint union of  $n$  copies of  $K_3$  and adding a universal vertex  $v$ . Note that  $\{v\}$  forms a minimum-size  $c$ -deletion set for  $G$  since  $\deg_G(v) > c$ . We claim that there is a locally injective homomorphism  $\phi$  from  $G$  to  $H$  if and only if  $G'$  is a yes-instance of the  $K_3$ -PARTITION problem.

Indeed, suppose there is such a  $\phi$ . Since  $\phi$  is locally injective and the graphs  $G$  and  $H$  each have  $3n$  vertices, the universal vertex  $v$  must be mapped to a universal vertex of  $H$ ; without loss of generality, we may therefore assume that  $\phi(v) = w$ . Since  $v$  and  $w$  are universal vertices of the same degree, it follows that  $\phi$  is a bijection from  $V(G)$  to  $V(H)$ . Every  $K_3$  in the disjoint union part of  $G$  must therefore be mapped to a  $K_3$  in  $H \setminus \{w\} = G'$ . Therefore  $G'$  is a yes-instance of the  $K_3$ -PARTITION problem.

Now suppose that  $G'$  is a yes-instance of the  $K_3$ -PARTITION problem. We let  $\phi(v) = w$ , and map the vertices of each  $K_3$  in the disjoint union part of  $G$  to some  $V_i$  from the  $K_3$ -partition of  $H$ , mapping each  $K_3$  to a different set  $V_i$ . Clearly this is a locally injective homomorphism. This completes the proof of the case when  $k = 1$ . To extend the proof to graphs with  $c$ -deletion number  $k > 1$ , we add  $(k - 1)$  universal vertices to  $G$  and  $H$ .  $\square$

## 6 Bounded Tree-depth and Feedback Vertex Set Number

By Theorem 6, we already obtained paraNP-hardness for LIHOM parameterized by tree-depth or feedback vertex set number. In this section we show that our tractability results for LSHOM and LBHOM cannot be significantly extended, since both problems become paraNP-hard parameterized by tree-depth. Furthermore, the reduction we give here also provides paraNP-hardness for both LSHOM and LBHOM parameterized by the feedback vertex set number. We show this by replacing cycles with stars in the reduction provided in [19] for path-width. This strengthens their result from path-width to tree-depth and feedback vertex set number.

**Theorem 7.** LBHOM, or more specifically, 3-FOLDCOVER, and LSHOM are NP-complete on input pairs  $(G, H)$  where  $G$  has tree-depth at most 6 and  $H$  has tree-depth at most 4.

*Proof.* First note that LBHOM, 3-FOLDCOVER and LSHOM are in NP. To prove NP-hardness for 3-FOLDCOVER and LSHOM we use a reduction from the 3-PARTITION problem. This problem takes as input a multiset  $A$  of  $3m$  integers, denoted in what follows by  $\{a_1, a_2, \dots, a_{3m}\}$ , and a positive integer  $b > 2$ , such that  $\frac{b}{4} < a_i < \frac{b}{2}$  for all  $i \in \{1, \dots, 3m\}$  and  $\sum_{1 \leq i \leq 3m} a_i = mb$ . The task is to determine whether  $A$  can be partitioned into  $m$  disjoint sets  $A_1, \dots, A_m$  such that  $\sum_{a \in A_i} a = b$  for all  $i \in \{1, \dots, m\}$ . Note that the restrictions on the size of each element in  $A$  implies that each set  $A_i$  in the desired partition must contain exactly three elements, which is why such a partition  $A_1, \dots, A_m$  is called a 3-partition of  $A$ . The 3-PARTITION problem is strongly NP-complete [45], that is, it remains NP-complete even if the problem is encoded in unary. **If  $b = 3$ , then 3-PARTITION can be solved in polynomial time (since all  $a_i$ 's are 1). Hence, we assume that  $b > 3$ .**

We first prove NP-hardness for 3-FOLDCOVER, which implies NP-hardness for LBHOM. Given an instance  $(A, b)$  of 3-PARTITION, we construct an instance of 3-FOLDCOVER consisting of connected graphs  $G$  and  $H$  with  $|V(G)| = 3|V(H)|$  as follows. To construct  $G$  we do as follows:

- Take  $3m$  disjoint copies  $S_1, \dots, S_{3m}$  of  $K_{1,b}$  (stars), one for each element of  $A$ . For each  $i \in \{1, \dots, 3m\}$ , the vertices of  $S_i$  are labelled  $c^i, u_1^i, \dots, u_b^i$ , where  $c_i$  is the vertex of degree  $b$  in  $S_i$  (the centre of the star).
- Add two new vertices  $p_j^i$  and  $q_j^i$  for each  $i \in \{1, \dots, 3m\}$ ,  $j \in \{1, \dots, b\}$ , as well as two new edges  $u_j^i p_j^i$  and  $u_j^i q_j^i$ .
- Add three new vertices  $x, y$  and  $z$ .
- Make  $x$  adjacent to the vertices  $p_1^i, p_2^i, \dots, p_{a_i}^i$  and  $q_1^i, q_2^i, \dots, q_{a_i}^i$  for every  $i \in \{1, \dots, 3m\}$ .
- Make  $y$  adjacent to every vertex  $p_j^i$  that is not adjacent to  $x$ .
- Make  $z$  adjacent to every vertex  $q_j^i$  that is not adjacent to  $x$ .

See Figure 5 for an example.

To construct  $H$ , we take  $m$  disjoint copies  $\tilde{S}_1, \dots, \tilde{S}_m$  of  $K_{1,b}$ , where the vertices of each star  $\tilde{S}_i$  are labelled  $\tilde{c}^i, \tilde{u}_1^i, \dots, \tilde{u}_b^i$ . For each  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, b\}$ , we add two vertices  $\tilde{p}_j^i$  and  $\tilde{q}_j^i$  and make both of them adjacent to  $\tilde{u}_j^i$ . Finally, we add a vertex  $\tilde{x}$  and make it adjacent to each of the vertices  $\tilde{p}_j^i$  and  $\tilde{q}_j^i$ . This finishes the construction of  $H$ . **Note that  $|V(G)| = 3|V(H)|$ .** See again Figure 5 for an illustration.

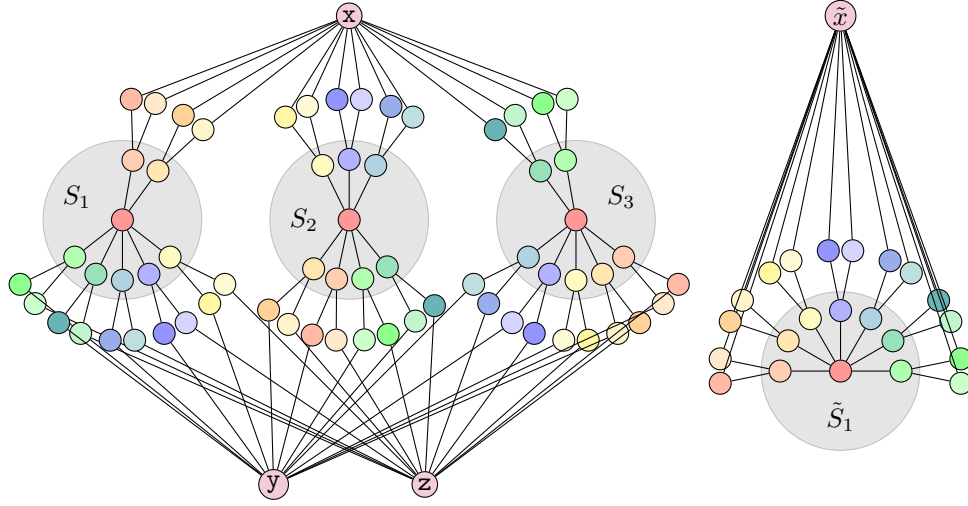
We now show that there exists a locally bijective homomorphism from  $G$  to  $H$  if and only if  $(A, b)$  is a yes-instance of 3-PARTITION.

Let us first assume that there exists a locally bijective homomorphism  $\phi$  from  $G$  to  $H$ . Since  $\phi$  is a degree-preserving mapping, we must have  $\phi(x) = \tilde{x}$ . Moreover, since  $\phi$  is locally bijective, the restriction of  $\phi$  to  $N_G(x)$  is a bijection from  $N_G(x)$  to  $N_H(\tilde{x})$ . Again using the definition of a locally bijective mapping, this time considering the neighbourhoods of the vertices in  $N_H(\tilde{x})$ , we deduce that there is a bijection from the set  $N_G^2(x) := \{u_j^i \mid 1 \leq i \leq 3m, 1 \leq j \leq a_i\}$ , i.e. from the set of vertices in  $G$  at distance 2 from  $x$ , to the set  $N_H^2(\tilde{x}) := \{\tilde{u}_j^k \mid 1 \leq k \leq m, 1 \leq j \leq b\}$  of vertices that are at distance 2 from  $\tilde{x}$  in  $H$ .

For every  $k \in \{1, \dots, m\}$ , we define a set  $A_k \subseteq A$  such that  $A_k$  contains element  $a_i \in A$  if and only if  $\phi(u_1^i) \in \{\tilde{u}_1^k, \dots, \tilde{u}_b^k\}$ . Since  $\phi$  is a bijection from  $N_G^2(x)$  to  $N_H^2(\tilde{x})$ , the sets  $A_1, \dots, A_m$  are disjoint; moreover each element  $a_i \in A$  is contained in exactly one of them. Since  $\phi$  is degree preserving, each  $c^i$  has to be mapped onto a  $\tilde{c}^j$  (**note that we use the assumption  $b > 3$  here**). Additionally, since  $\phi$  is locally bijective, for every  $i \in \{1, \dots, 3m\}$  there is a bijection from  $N_G(c^i) = \{u_1^i, \dots, u_b^i\}$  to  $N_H(\tilde{c}^j) = \{\tilde{u}_1^j, \dots, \tilde{u}_b^j\}$  for the  $j \in \{1, \dots, m\}$  for which  $\phi(c^i) = \tilde{c}^j$ . Combining this and the previous argument implies that  $\sum_{a \in A_i} a = b$  for all  $i \in \{1, \dots, m\}$ . Hence  $A_1, \dots, A_m$  is a 3-partition of  $A$ .

For the reverse direction, suppose that there exists a 3-partition  $A_1, \dots, A_m$  of  $A$ . We define a mapping  $\phi$  as follows. We first set  $\phi(x) = \phi(y) = \phi(z) = \tilde{x}$ . Let  $A_i = \{a_r, a_s, a_t\}$  be any set of the 3-partition. We map the vertices of  $S_r, S_s, S_t$  to the vertices of  $\tilde{S}_i$  in the following way:

- $\phi(c_r) = \phi(c_s) = \phi(c_t) = \tilde{c}_i$ ;
- $\phi(u_j^r) = \tilde{u}_j^i$  for each  $j \in \{1, \dots, b\}$ ;
- $\phi(u_j^s) = \tilde{u}_{a_r+j}^i$  for each  $j \in \{1, \dots, a_s + a_t\}$ ;
- $\phi(u_j^s) = \tilde{u}_{a_r+j-b}^i$  for  $j \in \{a_s + a_t + 1, \dots, b\}$ ;
- $\phi(u_j^t) = \tilde{u}_{a_r+a_s+j}^i$  for each  $j \in \{1, \dots, a_t\}$ ; and
- $\phi(u_j^t) = \tilde{u}_{a_r+j-b}^i$  for  $j \in \{a_t + 1, \dots, b\}$ .



**Fig. 5.** An instance of LBHOM consisting of the graph  $G$  (left) and the graph  $H$  (right) corresponding to the instance  $(A, b)$  of 3-PARTITION, where  $A = \{2, 3, 2\}$  and  $b = 7$ . As  $(A, b)$  is a yes-instance of the 3-PARTITION problem, there is a locally bijective homomorphism from  $G$  to  $H$  which is indicated by colours.

It remains to map the vertices  $p_j^i$  and  $q_j^i$  for each  $i \in \{1, \dots, 3m\}$  and  $j \in \{1, \dots, b\}$ . Let  $p_j^i, q_j^i$  be a pair of vertices in  $G$  that are adjacent to  $x$ , and let  $u_j^i$  be the second common neighbour of  $p_j^i$  and  $q_j^i$ . Suppose that  $\tilde{u}_\ell^k$  is the image of  $u_j^i$ , i.e. suppose that  $\phi(u_j^i) = \tilde{u}_\ell^k$ . Then we map  $p_j^i$  and  $q_j^i$  to  $\tilde{p}_\ell^k$  and  $\tilde{q}_\ell^k$ , respectively. We now consider the neighbours of  $y$  and  $z$  in  $G$ . By construction, the neighbourhood of  $y$  consists of the  $2mb$  vertices in the set  $\{p_j^i \mid a_{i+1} \leq j \leq b\}$ , while  $N_G(z) = \{q_j^i \mid a_{i+1} \leq j \leq b\}$ .

Observe that  $\tilde{x}$ , the image of  $y$  and  $z$ , is adjacent to two sets of  $mb$  vertices: one of the form  $\tilde{p}_\ell^k$ , the other of the form  $\tilde{q}_\ell^k$ . Hence, we need to map half the neighbours of  $y$  to vertices of the form  $\tilde{p}_\ell^k$  and half the neighbours of  $y$  to vertices of the form  $\tilde{q}_\ell^k$  in order to make  $\phi$  a locally bijective homomorphism. The same must be done with the neighbours of  $z$ . For every vertex  $\tilde{u}_\ell^k$  in  $H$ , we do as follows. By construction, exactly three vertices of  $G$  are mapped to  $\tilde{u}_\ell^k$ , and exactly two of these vertices, say  $u_j^i$  and  $u_h^g$ , are at distance 2 from  $y$  in  $G$ . We set  $\phi(p_j^i) = \tilde{p}_\ell^k$  and  $\phi(p_h^g) = \tilde{q}_\ell^k$ . We also set  $\phi(q_j^i) = \tilde{q}_\ell^k$  and  $\phi(q_h^g) = \tilde{p}_\ell^k$ . This completes the definition of the mapping  $\phi$ . For an illustration of the map  $\phi$ , see Figure 5.

Since the mapping  $\phi$  preserves adjacencies, it is a homomorphism. In order to show that  $\phi$  is locally bijective, we first observe that the degree of every vertex in  $G$  is equal to the degree of its image in  $H$ , in particular,

$$d_G(x) = d_G(y) = d_G(z) = d_H(\tilde{x}) = 2mb.$$

From the above description of  $\phi$  we get a bijection between the vertices of  $N_H(\tilde{x})$  and the vertices of  $N_G(v)$  for each  $v \in \{x, y, z\}$ . For every vertex  $p_j^i$  that is adjacent to  $x$  and  $u_j^i$  in  $G$ , its image  $\tilde{p}_\ell^k$  is adjacent to the images  $\tilde{x}$  of  $x$  and  $\tilde{u}_\ell^k$  of  $u_j^i$ . For every vertex  $p_j^i$

that is adjacent to  $y$  (respectively,  $z$ ) and  $u_j^i$  in  $G$ , its image  $\tilde{p}_\ell^k$  or  $\tilde{q}_\ell^k$  is adjacent to  $\tilde{x}$  of  $y$  (respectively,  $z$ ) and  $\tilde{u}_\ell^k$  of  $u_j^i$ . Hence the restriction of  $\phi$  to  $N_G(p_j^i)$  is bijective for every  $i \in \{1, \dots, 3m\}$  and  $j \in \{1, \dots, b\}$ , and the same clearly holds for the restriction of  $\phi$  to  $N_G(q_j^i)$ .

The vertices of each star  $S_i$  are mapped to the vertices of some star  $\tilde{S}_k$  in such a way that the centres are mapped to centres. This, together with the fact that the image  $\tilde{u}_\ell^k$  of every vertex  $u_j^i$  is adjacent to the images  $\tilde{p}_\ell^k$  and  $\tilde{q}_\ell^k$  of the neighbours  $p_j^i$  and  $q_j^i$  of  $u_j^i$ , shows that the restriction of  $\phi$  to  $N_G(u_j^i)$  is bijective for every  $i \in \{1, \dots, 3m\}$  and  $j \in \{1, \dots, b\}$ . Finally, the neighbourhood of  $c^i$  is clearly mapped to the neighbourhood of  $\phi(c^i)$  for every  $i \in \{1, \dots, 3m\}$ . We conclude that  $\phi$  is a locally bijective homomorphism from  $G$  to  $H$ .

In order to show that the tree-depth of  $G$  is at most 6, we observe that removing  $x, y$  and  $z$  yields a forest of depth 2. Similarly,  $H$  has tree-depth 4 since removing  $\tilde{x}$  leaves a tree of depth 2. This completes the proof for 3-FOLDCOVER and therefore LBHOM.

In order to prove NP-hardness for LSHOM we can use the same reduction as for LBHOM. For this we can argue that there is a locally bijective homomorphism from  $G$  to  $H$ , for the graphs  $G$  and  $H$  constructed above, if and only if there is a locally surjective homomorphism from  $G$  to  $H$ . While the one direction is clear, if  $G \xrightarrow{B} H$  then  $G \xrightarrow{S} H$ , for the converse direction we can make use of the following statement due to Kristiansen and Telle [60]:

(\*) If  $G \xrightarrow{S} H$  and  $\text{drm}(G) = \text{drm}(H)$ , then  $G \xrightarrow{B} H$ .

Here  $\text{drm}(G)$ ,  $\text{drm}(H)$  refers to the degree refinement matrix of  $G$  or  $H$  respectively, which is defined as follows. An *equitable partition* of a connected graph  $G$  is a partition of its vertex set into blocks  $B_1, \dots, B_k$  such that every vertex in  $B_i$  has the same number  $m_{i,j}$  of neighbours in  $B_j$ . Then  $\text{drm}(G) = (m_{i,j})$  for  $m_{i,j}$  corresponding to the coarsest equitable partition of  $G$ . We can easily observe that

$$\text{drm}(G) = \text{drm}(H) = \begin{pmatrix} 0 & 0 & 2mb & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix},$$

corresponding to the equitable partitions  $B_1 = \{x, y, z\}$ ,  $B_2 = \{u_j^i \mid i \in \{1, \dots, 3m\}, j \in \{1, \dots, b\}\}$ ,  $B_3 = \{p_j^i, q_j^i \mid i \in \{1, \dots, 3m\}, j \in \{1, \dots, b\}\}$  and  $B_4 = \{c^i \mid i \in \{1, \dots, 3m\}\}$  in  $G$  and a similar equitable partition in  $H$ . Hence by (\*) we find that  $G \xrightarrow{B} H$  if and only if  $G \xrightarrow{S} H$ , completing the proof for LSHOM.  $\square$

**Theorem 8.** LBHOM, or more specifically, 3-FOLDCOVER, and LSHOM are NP-complete on input pairs  $(G, H)$  where  $G$  and  $H$  have feedback vertex set number at most 3 and 1, respectively.

*Proof.* To prove the statement we use the same reductions as in the proof of Theorem 7. This is sufficient, as the set  $\{x, y, z\}$  is a feedback vertex set of  $G$  and the set  $\{\tilde{x}\}$  is a feedback vertex set of  $H$  for graphs  $G$  and  $H$  defined in the proof of Theorem 7.  $\square$

## 7 Conclusions

We presented a fairly comprehensive picture concerning the parameterized complexity of three locally constrained graph homomorphism problems, namely LSHOM, LBHOM, and LIHOM, when parameterized by some property of the guest graph. Our hardness results showed that the fracture number is the most suitable graph parameter of the guest graph for obtaining (parameterized) algorithms for these problems. We developed our algorithms through a general ILP-based framework. Besides the three locally constrained graph homomorphism problems, we also illustrated the applicability of our framework for the ROLE ASSIGNMENT problem. This yielded three FPT results and one XP result in total.

As future research, we aim to extend our ILP-based framework. If successful, this will then also enable us to address the parameterized complexity of other graph homomorphism variants such as quasi-covers [37] and pseudo-covers [15,17,18]. We also recall the open problem from [19]: are LBHOM and LSHOM in FPT when parameterized by the treewidth of the guest graph plus the maximum degree of the guest graph?

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