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The Hierarchical Relations among Mathematical Competencies: From Fundamental Numeracy to Complex Mathematical Skills

Chang Xu[1, 2]

Sabrina Di Lonardo Burr[3, 4]

Jo-Anne LeFevre[2, 4]

[1] School of Psychology, Queen's University Belfast, Northern Ireland, United Kingdom

[2] Department of Psychology, Carleton University, Ottawa, Canada

[3] Department of Psychology, University of British Columbia, Vancouver, Canada

[4] Department of Cognitive Science, Carleton University, Ottawa, Canada

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Author Note

Chang Xu: https://orcid.org/0000-0002-6702-3958

Sabrina Di Lonardo Burr: https://orcid.org/0000-0001-6338-9621

Jo-Anne LeFevre: https://orcid.org/0000-0002-1927-7734

To promote transparency and openness, anonymized data and stimuli used in the present study are freely available for download at Open Science Framework (https://osf.io/46nzg/).

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Co-corresponding Author:
Chang Xu, Ph.D.
Email Address: c.xu@qub.ac.uk
School of Psychology
Queen's University Belfast
David Keir Building, 18-30 Malone Road, Belfast, BT9 5BN

Abstract

Mathematical competencies can be conceptualized as layers of knowledge, with numeracy skills as the foundational core and more complex mathematical skills as the additional layers over the core. In this study we tested an expanded *Hierarchical Symbol Integration* (HSI) model by examining the hierarchical relations among mathematical skills. Undergraduate students (N = 236) completed order judgment, simple arithmetic, fraction arithmetic, algebra, and verbal working memory tasks. In a series of hierarchical multiple regressions, we found support for the hierarchical model: Additive skills (i.e., addition and subtraction) predicted unique variance in multiplicative skills (i.e., multiplication and division); multiplicative skills predicted unique variance in fraction arithmetic; and fraction skills predicted unique variance in algebra. These results support the framework of the HSI model in which mathematical competencies are related hierarchically, capturing the increasing complexity of symbolic mathematical skills.

Abstract Word Count: 136

Keywords: Symbol Integration, Arithmetic, Fractions, Algebra, Mathematics, Adults

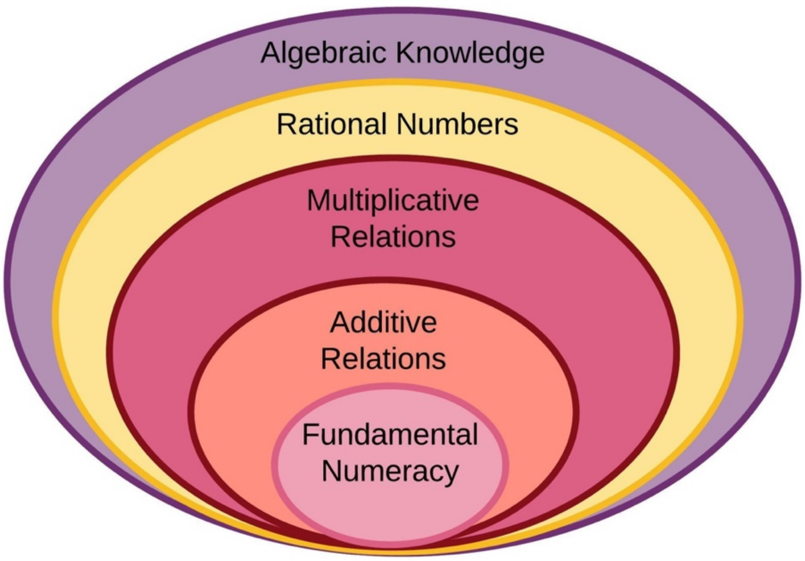
Public Statement [2-3 sentences]: In this study, we test a hierarchical model of mathematics skills and find that adults' addition and subtraction skills predict their multiplication and division skills, their multiplication and division skills predict their fraction performance, and fraction performance predicts algebra. These findings support the value of learners developing strong foundational skills in mathematics to support success in more advanced mathematics skills.

The Hierarchical Relations among Mathematical Competencies: From Fundamental Numeracy to Complex Mathematical Skills

Mathematics depends on an abstract symbol system which represents a complex set of numerical and functional relations (Lyons et al., 2016; Merkley & Ansari, 2016; Núñez, 2017; Thompson & Saldanha, 2003; Xu et al., 2019). These relations can include, for example, cardinal knowledge (e.g., 2 is bigger than 1 and smaller than 3), ordinal knowledge (e.g., 3 follows 2 and comes after 1), and arithmetic knowledge (e.g., $2 + 1 = 3$; $3 – 1 = 2$; $2 \times 3 = 6$; $6 \div 2 = 3$). By building a strong foundation of numerical knowledge, students can integrate more advanced and abstract relations into their hierarchy of knowledge, acquiring skills that support complex mathematics involving, for example, rational numbers and algebraic knowledge (Barbieri et al., 2021; Booth & Newton, 2012; Douglas et al., 2020; Xu et al., 2019). Integration refers to the process of making connections among existing and new knowledge within a single, unified system (Siegler & Chen, 2008). On this view, the development of mathematical competence is a hierarchical process, in which basic numeracy skills provide the building blocks for more advanced knowledge (Hiebert, 1988; Siegler & Lortie-Forgues, 2014; Xu & LeFevre, 2021). In the present study, we investigated this hierarchy of mathematical competencies (see Figure 1). Within this hierarchy, the layers represent core numeral knowledge that is critical for more complex mathematical skills.

Figure 1

Model Representing the Hierarchical Relations among Mathematical Competencies



Fundamental Numeracy

Cardinal and ordinal relations among numbers are fundamental components of symbolic number knowledge (Lyons et al., 2016). Cardinal relations are often assessed by having participants indicate which of two presented digits is numerically larger, to capture individual differences in number processing (De Smedt et al., 2009; Moyer & Landauer, 1967; Schneider et al., 2017). In contrast, ordinal relations are often assessed by having participants judge whether a set of digits is ordered (e.g., 1 2 3 is in order whereas 1 3 2 is not; Lyons & Beilock, 2011; Lyons et al., 2016; Vos et al., 2017; Xu et al., 2019). For children, symbolic number comparison is the best predictor of arithmetic in Grade 1, however, from Grade 2 onwards, symbolic order judgments become increasingly more predictive of arithmetic (Lyons et al., 2014; Sasanguie & Vos, 2018; Xu & LeFevre, 2021). For adults, performance on symbolic order judgments fully mediates the relation between performance on symbolic number comparisons and arithmetic (Lyons & Beilock, 2011; Morsanyi et al., 2017; Sasanguie et al., 2017; Xu et al., 2019).

What is the source of the relations among cardinal, ordinal, and arithmetic processing? Lyons and Ansari (2015) posit that the relation between order judgments and arithmetic is driven by fluent recognition of familiar number relations, an ability that is presumably important for developing both skilled ordinal processing and arithmetic (Dubinkina et al., 2021; Lyons & Ansari, 2015; Sella et al., 2021). In contrast, Orrantia et al. (2019) highlight the extensive overlap between cardinal (number comparisons) and ordinal processing (order judgments) in adults' arithmetic, suggesting that both processes draw on underlying symbolic magnitude associations. Other researchers have suggested that both familiarity and magnitude associations may influence performance on the order judgment task, depending on the task demands (e.g., stimulus characteristics; Vos et al., 2021) and individuals' strategy use (Dubinkina et al., 2021; Muñez et al., 2021; Vogel et al., 2021; Vos et al., 2021). Extending the latter views, strong correlations between order judgments and arithmetic may reflect the extent to which individuals have developed an integrated knowledge system (Xu et al., 2019).

Additive and Multiplicative Relations in Whole Number Arithmetic

Arithmetic with natural numbers is an expansion of the network of number symbol relations and thus is built upon fundamental numeracy knowledge (Núñez, 2017; Lyons et al., 2016). Arithmetic fluency (speed and accuracy of solutions) relies directly on the accessibility of specific associations that need to be managed or differentially activated depending on the situation (Ashcraft, 1982; Campbell, 1994, 1995; Verguts & Fias, 2005). In the absence of arithmetic fluency, people may rely on procedural solutions to solve simple arithmetic problems (LeFevre et al., 1996). Notably, procedural solutions rely on more basic cardinal or ordinal relations. For example, a procedural solution for 9 + 4 might involve knowing that 9 is 1 less than 10, and then solving 10 + 4 – 1. In contrast, direct retrieval of the solution to 9 + 4 involves learning the specific connections between operands and answers. Direct retrieval of arithmetic facts may involve processes that are like those involved in recognizing that 3 4 5 is an ascending sequence whereas 3 5 4 is not (Dubinkina et al., 2021). On this view, both order judgments and direct retrieval of arithmetic facts rely heavily on accessing familiar symbol-symbol relations from the knowledge system (Lyons et al., 2012; Reynvoet & Sasanguie, 2016; Xu et al., 2019).

Among arithmetic relations, additive and multiplicative operations are complementary, both procedurally and conceptually, such that any subtraction or division problem can be transformed into a complementary addition or multiplication problem, and vice versa (e.g., $3 + 4 = 7$; $7 – 3 = 4$; $5 \times 6 = 30$; $30 \div 5 = 6$; Robinson, 2017). Experience with the complementary operations can strengthen the connections underlying both operations. For example, practicing addition problems facilitates their subtraction complements (Buckingham, 1927; Campbell & Agnew, 2009) and practicing division problems facilitates their multiplication complements (Campbell & Alberts, 2009; De Brauwer & Fias, 2011). Thus, the complementary conceptual relation between additive operations (i.e., addition and subtraction) and between multiplicative operations (i.e., multiplication and division) enhances the integration of these operations into the knowledge system. Additive operations involve representing quantities as collections of units of one whereas multiplicative operations involve representing quantities as units of units (Clark & Kamii, 1996; Harel & Confrey, 1994; Nunes et al., 2016; Steffe, 1992; Steffe & Olive, 2010). That is, multiplicative representations are constructed based on existing additive representations (Steffe, 1992).

Rational Numbers

Fraction knowledge is built on a foundation of multiplicative reasoning (i.e., multiplication and division; Mack, 1993; Nunes et al., 2016; Steffe & Olive, 2010; Thompson &

Saldanha, 2003). Fractions represent a ratio between two relative sizes, measured flexibly in units of each other (Clark & Kamii, 1996; Harel & Confrey, 1994; Thompson & Saldanha, 2003). For example, the idea that Richard's pizza is $\frac{1}{2}$ as large as Ben's pizza is the same as the idea that Ben's pizza is 2 times as large as Richard's pizza. On this view, multiplicative reasoning supports the conceptual understanding of fractions. The relations between multiplicative skills, especially division, and fraction skills are well-established (Hansen et al., 2015; Namkung et al., 2018; Sidney & Alibali, 2017; Siegler et al., 2012; Siegler & Pyke, 2013; Stelzer et al., 2021; Xu et al., 2022). However, the hierarchical relations among additive skills, multiplicative skills, and fraction knowledge have rarely been studied explicitly.

Algebraic Knowledge

Knowledge of algebra is built on an understanding of rational numbers, including fractions (see Bush & Karp, 2013 for a review). Moreover, both fraction arithmetic and algebraic problem solving require flexible manipulation of symbols (Wu, 2001). For example, knowledge of equivalence allows students to convert $4x = 3$ to $x = \frac{3}{4}$ (Newton et al., 2020). Thus, fluency with fraction arithmetic is required for algebraic problem solving. As students advance to each subsequent level of mathematics, the list of algebraic generalizations that are dependent on fractional constructs grows (Brown & Quinn, 2007). For example, students need to form equivalent equations and manipulate fractions to solve systems of linear and quadratic equations. Thus, insufficient understanding of basic fraction concepts and procedures will impede the learning of algebraic concepts. Supporting this view, fraction knowledge predicted performance on algebraic tasks for both adults (Hurst & Cordes, 2018) and children (Barbieri et al., 2021; Booth & Newton, 2012; Booth et al., 2014; DeWolf et al., 2015; Siegler et al., 2012).

Relations between Basic and Complex Mathematics Skills

The hierarchy of mathematical competence is built through integration of number relations: Learners make connections between number concepts to construct a more advanced system (Hiebert, 1988; ; Siegler & Chen, 2008; Xu et al., 2019). Integration of mathematical competence occurs when the acquired knowledge becomes fully accessible within the knowledge system so that solvers can use it flexibly to solve problems (Ashcraft, 1982). There are several theories of arithmetic that focus on associations, including the *Associative Network* model (Ashcraft, 1982), the *Distribution of Associations* model (Siegler, 1988), the *Network Interference* model (Campbell, 1994), the *Interacting Neighbours* model (Verguts & Fias, 2005), and the *Identical Elements* model (Rickard, 2005). Expanding on these theories, the *Hierarchical Symbol Integration* model (HSI; Xu et al., 2019) focuses on symbolic number relations across a variety of mathematical tasks. According to this model, symbolic number knowledge becomes increasingly integrated as mathematical skills develop.

Evidence for the Hierarchical Symbol Integration Model

The HSI model is based on the assumption that symbolic representations are central to mathematical processing (Deacon, 1997; Lyons et al., 2012; Núñez, 2017). The model was first tested with adults, with Xu et al. (2019) finding that cardinal, ordinal, and multi-digit arithmetic knowledge became increasingly integrated as numerical skill increased. For example, multi-digit arithmetic fluency fully mediated the relations between symbolic order judgments and more complex mathematics (i.e., fraction and algebra arithmetic, and word problem solving) for more-skilled adults, suggesting that they relied on an integrated knowledge system among cardinal, ordinal, and arithmetic skills. In contrast, for less-skilled adults, the mediation was partial: both order judgments and multi-digit arithmetic fluency were related to the more complex mathematical measures, suggesting a less integrated knowledge system. Xu et al. concluded that the basic cardinal and ordinal skills are not replaced, per se, but are integrated with the higher-level arithmetic skills, becoming the core determinant of individual differences in more complex mathematical skills. Thus, the increasing complexity of symbolic knowledge preserves the hierarchical relations among them.

Beyond the original HSI study, empirical evidence supporting the assumptions of the HSI model has come from research with children. For example, in Grade 1 (age 6 to 7 years), students rely mainly on cardinal knowledge to order numbers (e.g., to order 1 7 5, children compare all possible pairs; Xu & LeFevre, 2021). By Grade 2 (age 7 to 8 years), however, students can fluently recognize familiar number sequences (e.g., 2 3 4), contributing to the integration of ordinal and cardinal knowledge such that both contribute to their developing addition skills (Lyons et al., 2014; LeFevre et al., 1991; Sasanguie & Vos, 2018; Xu & LeFevre, 2021). For students in Grades 2 to 3 (ages 8 to 9 years), additive knowledge (addition and subtraction) become integrated, and thus later-learned subtraction predicts multiplication in Grade 3 (Xu et al., 2021). For students in Grade 4 (age 9 to 10 years), later-learned division, rather than other whole number arithmetic skills (i.e., addition, subtraction, or multiplication) has been shown to predict unique variance in students' knowledge of fraction notation (Xu et al., 2022). Thus, the development of these hierarchical relations starts in the early years and is continually refined as students encounter more advanced mathematical concepts.

The Present Research

Based on the HSI model, we propose that numerical and mathematical relations are nested within a hierarchy, progressing from fundamental numeracy skills to whole number arithmetic skills to more complex skills (see Figure 1). Referring to Figure 1, we propose that undergraduate students' foundational numeracy skills will predict their whole number arithmetic skills. Whole number arithmetic skills are expected to predict rational number knowledge (i.e., fractions), which in turn will predict algebra knowledge.

For adults, researchers have found correlations between basic and more complex mathematics skills (Goffin & Ansari, 2016; Orrantia et al., 2019; Siegler et al., 2012), however, few researchers have focused on the hierarchical relations within a theoretical framework (Cirino et al., 2016; Douglas et al., 2020; Xu et al., 2019). Thus, the goal of the present study was to expand the HSI model to test the hierarchical relations among ordinal, arithmetic, rational number, and algebraic skills. Although these skills are all introduced to students before high school and the hierarchy is developed through years of experience with mathematics, there is evidence to suggest that the hierarchy is preserved among skilled adults. For example, university students show considerable variability in their mastery at all levels: cardinal (Maloney et al., 2011), ordinal (Lyons & Beilock, 2011), basic arithmetic (LeFevre et al., 2015), fractions (Powell & Nelson, 2021), and algebra (Cirino et al., 2016; Douglas et al., 2020). Thus, individual differences at each level of the hierarchy and the use of foundational skills, such as arithmetic, to solve more complex mathematical problems lead us to believe that the hierarchy is maintained in adulthood.

To evaluate performance within each level of the hierarchy, students completed order judgments (fundamental numeracy), whole number arithmetic (addition, subtraction, multiplication, and division), fraction arithmetic (rational numbers), and algebra tasks. To tap into the different aspects of arithmetic, two latent factors were extracted: A factor extracted from addition and subtraction, labelled *additive skills* and a factor extracted from multiplication and division, labelled *multiplicative skills*. Based on the theoretical model (see Figure 1), we tested three regression models, each of which assessed expected relations among the different levels of the hierarchy.

We had three hypotheses. First, we expected additive skills to succeed ordinal skills in predicting unique variance in multiplicative skills. Second, we expected multiplicative skills to succeed ordinal and additive skills in predicting unique variance in fraction arithmetic. Third, we expected fraction skills to succeed ordinal, additive and multiplicative skills in predicting unique variance in algebra knowledge. Notably, when testing these hierarchical relations, we included a measure of working memory (digit backward span) in the analyses to account for individual differences related to controlling, regulating, and actively maintaining relevant information. Working memory is a predictor of individual differences in many mathematical tasks (see reviews in Allen et al., 2019; DeStefano & LeFevre, 2004; Friso-van den Bos et al., 2015; LeFevre et al., 2005; Peng et al., 2016; Raghubar et al., 2010).

Method

Participants

The study was approved by the Carleton University Research Ethics Board. Students were recruited from introductory Psychology or Cognitive Science courses and received partial course credit for their participation. Initially, 344 undergraduate students accepted the invitation to participate. However, 108 participants were excluded who met one of the following criteria: 1) responded “No” to the question at the end of the study “Did you honestly put your best effort into your responses?” ($n = 70$) or 2) did not complete all mathematical tasks ($n = 38$). After these exclusions, data were retained for 236 participants ($M_{age} = 19.7$ years, $SD = 3.1$; 168 women, 60 men, 3 non-binary and 5 did not disclose gender). Participants were all undergraduates enrolled in their first ($n = 129$), second ($n = 58$), third ($n = 21$), or fourth ($n = 28$) year of study. All participants spoke English, with 83.5% identifying English as their first language. After English, the most common first languages reported were Chinese (3.8%), Arabic (3.4%), and French (2.1%). The remaining 7.2% of participants reported other first languages with a frequency of less than 1%.

Procedure

Participants were recruited through an online system. In the study description, participants were told they would be completing demographic information, math, and memory tasks. Once they logged into the system, they received a link to the study which was created using the Gorilla online data collection tool (https://app.gorilla.sc/). Electronic informed consent was obtained, followed by participants completing the demographic questionnaire. Participants completed the measures in the order they are listed below. The study took up to 60 minutes to complete. During the study, entertaining memes (i.e., a humorous photo combined with text) appeared on the screen at three time points. The memes signalled to participants that they could take a break for up to three minutes before continuing with the remaining tasks.

Measures

Detailed information about the stimuli for each of the presented tasks and the dataset used for this paper are available on the Open Science Framework (Xu et al., 2023).

Digit Span Backward

This task was included as a cognitive control measure of verbal working memory (adapted from WISC-V; Wechsler, 2014). At the beginning of the task, participants were instructed that they needed a quiet space or to wear headphones to complete this task. When they were ready, they pressed the “PLAY” button to hear the recorded numbers. They were instructed to type the numbers backward with no spaces between them in a text box (e.g., if they heard 6-2-

5-3, they should type 3526). Participants completed two practice trials of 2-digit spans with feedback. The experimental trials consisted of 12 sequences, with two trials for each span ranging from 3-digit span to 8-digit span. The task was discontinued if the participants got both trials incorrect for a given span. The split-half (odd/even) reliability was calculated based on the subscores of the first and second trials, Cronbach's $\alpha = .85$.

Order Judgment

Order judgment was assessed using a task similar to the one described in Lyons and Beilock (2009). Notably, because our study was administered online, instead of presenting participants with a page of sequences, we adapted the task so that participants were only presented with one trial at a time. Participants were presented with three numbers and had to decide whether sequences were in ascending order as fast as possible. If all three numbers were in ascending order (e.g., 2 3 4 or 6 8 9), they were asked to select the check mark on the right-hand side of the screen by pressing "P" on the keyboard. If the three numbers were not in ascending order (e.g., 3 2 4 or 9 6 8), they were asked to select the cross mark on the left-hand side by pressing "Q" on the keyboard. A fixation cross appeared in the middle of the screen for 500 ms, followed by a blank screen for 500 ms, and then the stimuli. After three practice trials with feedback, 24 sequences (with digits ranging from 1 to 9) were presented to participants in random order. Participants who did not respond after three seconds on any given trial were presented with the next trial. Response time (RT) was recorded from the time that the three numbers appeared on the screen to the time that the participants pressed one of the two response keys. Half of these sequences had adjacent numbers (count-list sequences, e.g., 1 2 3, 1 3 2), and the other half had non-adjacent numbers (neutral sequences, e.g., 1 4 6, 1 6 4). Furthermore, half of the sequences were presented in order (e.g., 2 3 4, 2 4 7), and the other half of the sequences were not presented in order (e.g., 3 2 4, 4 2 7). The internal reliability based on RT for correct trials was .90.

Arithmetic Fluency

Forty-eight problems were presented in four blocks of 12 trials each of addition, subtraction, multiplication, and division. Participants were instructed not to use a calculator or paper and pencil but to respond as accurately and quickly as possible. A fixation cross appeared in the middle of the screen for 500 ms, followed by a blank screen for 500 ms, and then the stimuli. Within each operation, participants started with two practice trials, followed by 12 problems. Each was presented horizontally and the order of presentation was randomized across participants. Addition problems consisted of single-digit numbers (e.g., 2 + 4; 7 + 6). Subtraction problems consisted of a mixture of single- and double-digit minuends and single-digit subtrahends (e.g., 5 – 3; 13 – 4). Multiplication problems consisted of single-digit numbers (e.g., 3 × 2, 8 × 9). Division problems consisted of a mixture of single- and double-digit dividends and single-digit divisors (e.g., 8 ÷ 2, 63 ÷ 7).

Participants were instructed to type their answer in the provided textbox and to press the "ENTER" key to proceed to the next trial. Response time (RT) was recorded from the time that the arithmetic problem appeared on the screen to the time that the participants pressed the "ENTER" key. If participants did not respond to a trial within five seconds, the software automatically advanced to the next trial. Notably, 14.3% of the participants reported they used the number pad of the keyboard to enter the answers, however, there was no significant difference in performance between participants who did and did not use the number pad on any of the arithmetic operations, *p*s > .05. The internal reliabilities based on RT for correct trials for addition, subtraction, multiplication, and division were .80, .78, .80, and .87, respectively.

Fraction Arithmetic

Participants were asked to choose one of four potential answers for 16 fraction arithmetic trials (4 addition, 4 subtraction, 4 multiplication, and 4 division), presented one at a time in a fixed order. For each operation, two problems had common denominators. All fractions had single-digit numerators and denominators. Foil answers were created based on the misconceptions and errors noted in Di Lonardo Burr et al. (2020): They were either the result of a common arithmetic mistake, the result of a common math misconception, or the result of applying the correct procedure for a different operation. Participants were encouraged to use paper and pencil to solve the problems. Participants were instructed to press “A”, “B”, “C”, or “D”, depending on which response they wanted to select. After they pressed the key, the software advanced to the next trial. Proportion correct scores were used in subsequent analyses. The internal reliabilities based on the accuracy of the individual trials was .86.

Algebra

Participants were asked to solve 10 algebra problems presented in a random order, one at a time. Problems were adapted from the Mathematics Knowledge subtest of the Armed Services Vocational Aptitude Battery (ASVAB; U.S. Department of Defense, 1984). The ASVAB is a multi-aptitude test given to Americans who are interested in enlisting in the armed forces. Outside of the military, it has been used as a proxy measure of intelligence (Herrnstein & Murray, 1994). The problems were presented in multiple choice format. Items required middle- and high-school level knowledge of algebra, such as factoring expressions, solving for x in an algebraic equation, and simplifying expressions and exponents. Participants were encouraged to use paper and pencil to solve the problem. Participants were instructed to press “A”, “B”, “C”, or “D”, depending on which response they wanted to select. After they pressed the key, the software advanced to the next trial. Proportion correct scores were used in the subsequent analyses. The split-half reliability was calculated based on the subscores of the odd and even trials, Cronbach's $\alpha = .63$. The moderate reliability is reflective of the varying difficulty of the problems. Across the 10 items, the mean proportion correct ranged from .64 to .85, indicating considerable variability across items.

Scoring for Speeded Tasks

Performance (response times and accuracy) on the speeded tasks (order judgment, arithmetic fluency) was converted to adjusted response times according to a linear integrated speed-accuracy calculation:

$$RT_{adj} = RT_{correct} + PE \times [SD_{RT}/SD_{PE}]$$

$RT_{correct}$ is the mean response time on the correct trials and percentage error (PE) is weighted by the ratio of the standard deviations of the correct RT and percentage of error (Vandierendonck, 2017).

Results

Descriptive Statistics

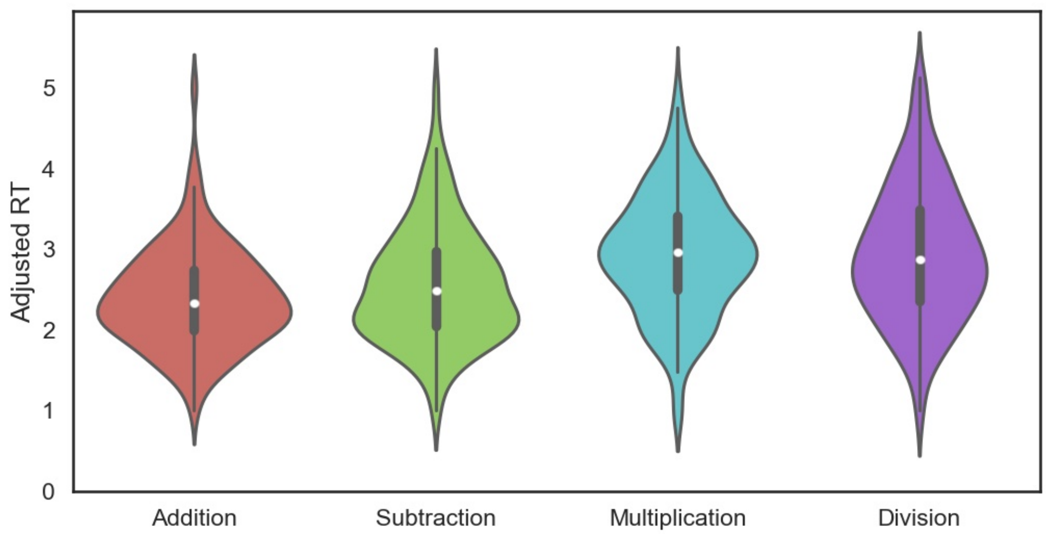
Descriptive statistics and correlations among variables are shown in Table 1. The distributions of the scores varied across the measures, as shown in the violin plots for the tasks in Figures 2 and 3. In particular, on average, students did well on the fraction arithmetic and algebra tasks. However, as shown in Figure 3, there was considerable variability in scores. Outliers were identified as those values that were three times beyond the interquartile range in either direction using boxplots for two speeded tasks: order judgment ($n = 1$) and addition ($n = 1$). Sensitivity analyses with and without these outliers showed the same patterns of results, and thus all the data were included in the final analyses.

Violin plots for adjusted response times for the four arithmetic tasks are shown in Figure 2. The adjusted response times for each operation were analyzed in a repeated measures analysis of variance (ANOVA). Performance varied with operation, $F(2.73, 639.74) = 117.26$, $p < .001$, $\eta_p^2 = .33$. We conducted post hoc pairwise comparisons using the Bonferroni adjustment. Participants performed better on addition than on the other three operations, *p*s < .001, and better on subtraction than on multiplication and division, *p*s < .001, but no difference was found between multiplication and division, $p = .999$.

As shown in Figure 2, the patterns of distribution for additive skills (addition and subtraction) were similar, and the patterns of distribution for multiplicative skills (multiplication and division) were similar, with more variable performance among individuals for the latter than the former. Because the arithmetic scores were highly intercorrelated, we created latent additive and multiplicative factors using principal component analysis. The latent additive factor, *additive skills*, was comprised of two items (addition and subtraction) that accounted for 88.6% of the variance, with factor loadings of .94 and an eigenvalue of 1.8. The latent multiplicative factor, *multiplicative skills*, was comprised of two items (multiplication and division) that accounted for 85.6% of the variance, with factor loadings of .93 and an eigenvalue of 1.7.

Figure 2

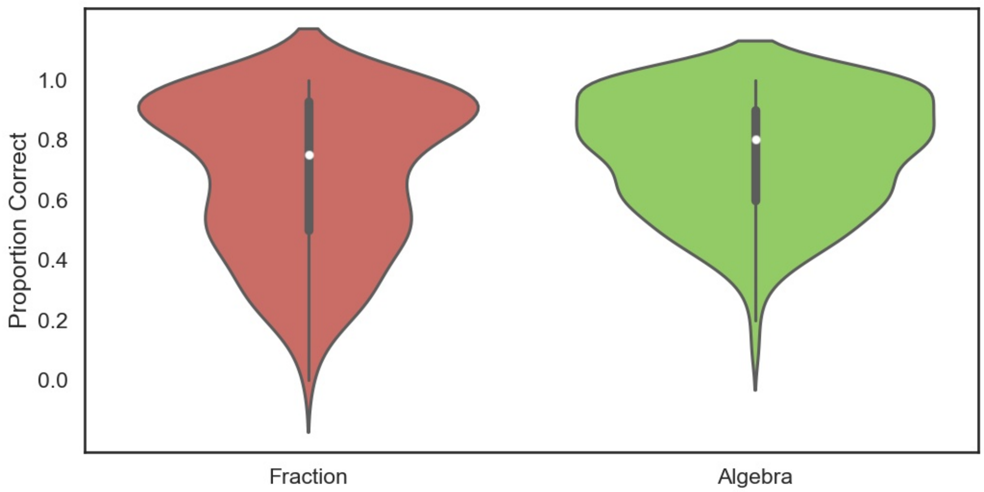
Violin Plots of Adjusted Response Times (seconds) for Addition, Subtraction, Multiplication and Division Tasks (N = 236)



Note. The white dot is the median, the black bar in the center of the plot shows the interquartile range, and the thin black bar shows the range of scores.

Figure 3

Violin Plots of Proportion Correct Scores for Fraction Arithmetic and Algebra Tasks (N = 236)



Note. The white dot is the median, the black bar in the center of the plot shows the interquartile range, and the thin black bar shows the range of scores.

As shown in Table 1, most measures were significantly correlated with each other, with one exception: digit span backward was not significantly correlated with subtraction despite similar effect sizes for the correlations between digit span and the other three arithmetic measures. The correlations with digit span and the more complex measures had a medium effect size (Funder & Ozer, 2019).

Given that the additive and multiplicative arithmetic skills were highly correlated (see Table 1), we examined multicollinearity among the latent variables. A variance inflation factor (VIF) of 10 or more and/or a tolerance of 0.2 or less indicates an issue with multicollinearity (Field, 2013). In the regression models where both additive and multiplicative factors were included as predictors, multicollinearity was not detected among the latent factors (VIFs < 2.3; tolerance > 0.4) and thus was not a concern in the subsequent analyses.

Table 1

Descriptive Statistics and Correlations Among Variables (N = 236)

Variable	*M*	*SD*	*Skew*	1	2	3	4	5	6	7
1. Digit Span Backward [a]	6.81	2.80	0.07	-						
2. Order Judgment [b]	1.21	0.32	1.34	-.15*	-					
3. Addition [b]	2.41	0.61	0.92	-.16*	.47***	-				
4. Subtraction [b]	2.59	0.71	0.78	-.11	.46***	.77***	-			
5. Multiplication [b]	2.99	0.72	0.26	-.14*	.35***	.63***	.61***	-		
6. Division [b]	2.97	0.81	0.42	-.15*	.45***	.68***	.68***	.71***	-	
7. Fraction Arithmetic [c]	.68	.25	-0.50	.27***	-.24**	-.32***	-.39***	-.49***	-.47***	-
8. Algebra [c]	.76	.20	-0.53	.28**	-.17*	-.25***	-.29***	-.40***	-.38***	.61***

Notes. [a] Total correct; [b] Adjusted RT in seconds; [c] Proportion Correct.

* $p < .05$, ** $p < .01$, *** $p < .001$

Hierarchical Multiple Regression Analyses

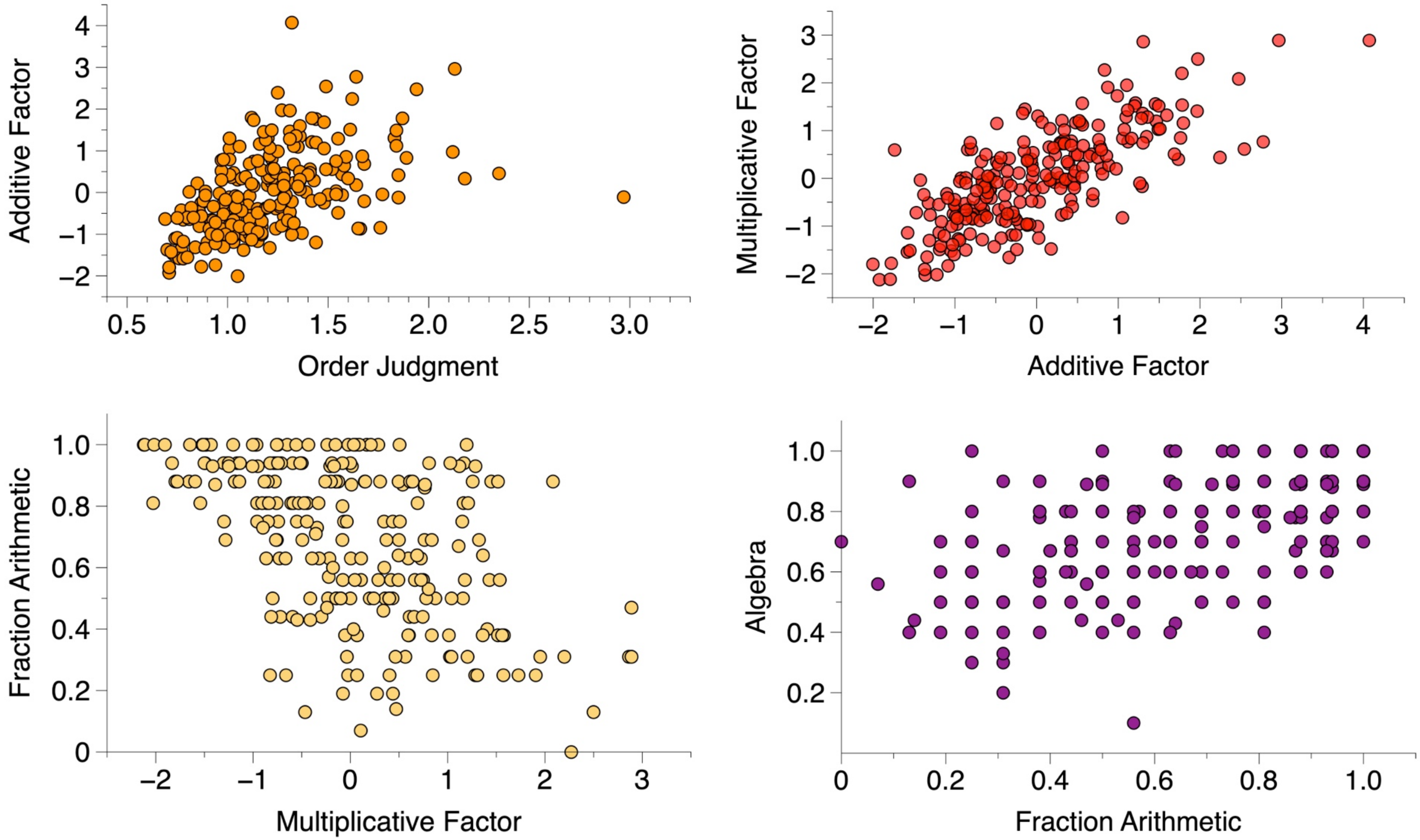
Hierarchical multiple regression analyses were conducted to examine the unique contribution of predictors to each of the mathematical outcomes. Across all regression models, working memory was included as a control measure. In the first model, we considered ordinal and additive skills as predictors of multiplicative skills. In the second model, we considered ordinal, additive and multiplicative skills as predictors of fraction arithmetic. In the last model, we considered ordinal, additive, multiplicative and fraction arithmetic skills as predictors of algebra. The scatterplots in Figure 4 of the relations between the adjacent levels of the hierarchy show that there was substantial variability in performance for all mathematics measures.

Hypothesis 1: Relations Among Ordinal, Additive, and Multiplicative Tasks

The first hierarchical linear regression analysis was conducted to examine whether ordinal skills and additive skills account for unique variability in multiplicative skills. The results are presented in Table 2. In Block 1, students' digit backward span predicted 2.2% of the variance in multiplicative skills. In Block 2, performance on the ordinal measure predicted approximately 17.5% of the unique variance in multiplicative skills. In Block 3, performance on digit backward, ordinal skills, and additive skills together predicted approximately 56.5% of the variance in multiplicative performance. In support of Hypothesis 1, only additive skills, not order judgments, predicted unique variance in multiplicative skills.

Figure 4

Scatterplots Showing the Relations Between the Adjacent Hierarchical Levels of Skill



Note. For the speeded tasks that used an integrated response time and accuracy scoring method (speeded order judgment, additive, and multiplicative tasks), lower values indicate better performance.

Table 2

Hierarchical Linear Regression Showing Working Memory, Order Judgment, and Additive Skills Predicting Multiplicative Skills

Variable	*B*	*SE*	*β*	*t*	*p*	Unique r^2
Block 1						
Digit Backward	-0.05	0.02	-.15	-2.31	.022	.022
Block 2						
Digit Backward	-0.03	0.02	-.09	-1.46	.146	.007
Order Judgment	0.00	0.00	.42	7.10	<.001	.175
R^2						.197
Block 3						
Digit Backward	-0.02	0.02	-.05	-1.13	.258	.002
Order Judgment	0.01	0.01	.08	1.58	.115	.005
Additive Skills	0.71	0.05	.70	13.98	<.001	.368
Total R^2						.565

Note. Unique r^2 represents the squared semi-partial correlations within that specific model tested.

Hypothesis 2: Relations Among Ordinal, Additive, Multiplicative, and Fraction Tasks

The second hierarchical linear regression analysis was conducted to examine whether ordinal, additive, and multiplicative skills account for unique variability in fraction arithmetic performance. The results are presented in Table 3. In Block 1, students' digit backward span predicted 6.9% of the variance in fraction arithmetic performance. In Block 2, additive skills predicted approximately 7.2% of the unique variance in fraction arithmetic performance. In Block 3, digit backward, ordinal skills, additive skills, and multiplicative skills together predicted approximately 30.1% of the variance in fraction arithmetic performance. In support of Hypothesis 2, only multiplicative skills, not additive skills, predicted unique variance in fraction arithmetic performance.

Table 3

Hierarchical Linear Regression Showing Working Memory, Ordinal Judgment, Additive Skills, and Multiplicative Skills Predicting the Fraction Arithmetic

Variable	*B*	*SE*	β	*t*	*p*	Unique r^2
Block 1						
Digit Backward	0.02	0.01	.26	4.14	<.001	.069
Block 2						
Digit Backward	0.02	0.01	.22	3.57	<.001	.045
Order Judgment	0.00	0.00	-.05	-0.73	.469	.002
Additive Skills	-0.08	0.02	-.31	-4.52	<.001	.072
R^2						.181
Block 3						
Digit Backward	0.02	0.01	.19	3.38	<.001	.035
Order Judgment	0.00	0.00	-.01	-0.13	.899	.000
Additive Skills	0.15	0.02	.06	0.67	.501	.001
Multiplicative Skills	-0.13	0.02	-.53	-6.30	<.001	.120
Total R^2						.301

Note. Unique r^2 represents the squared semi-partial correlations within that specific model tested.

Hypothesis 3: Relations Among Ordinal, Additive, Multiplicative, Fraction and Algebra Knowledge

The last hierarchical linear regression analysis was conducted to examine whether ordinal skills, additive skills, multiplicative skills, and fraction arithmetic performance account for unique variability in algebra performance. The results are presented in Table 4. In Block 1, students' digit backward span predicted 7.2% of the variance in algebra performance. In Block 2, multiplicative skills predicted approximately 8.8% of the unique variance in algebra performance. In Block 3, digit backward span, ordinal skills, additive skills, multiplicative skills, and fraction arithmetic together predicted approximately 39.2% of the variance in algebra. Both multiplicative skills and fraction arithmetic predicted unique variance in algebra performance. Notably, in support of Hypothesis 3, the unique variance explained by fraction arithmetic in algebra was substantially larger than the unique variance explained by multiplicative skills (17.2% vs. 1.3%).

Table 4

Hierarchical Linear Regression Showing Working Memory, Order Judgment, Additive Skills, Multiplicative Skills, and Fraction Arithmetic Predicting Algebra

Variable	*B*	*SE*	*β*	*t*	*p*	Unique r^2
Block 1						
Digit Backward	0.02	0.00	.27	4.24	<.001	.072
Block 2						
Digit Backward	0.02	0.00	.21	3.61	<.001	.044
Order Judgment	0.00	0.00	.02	0.36	.723	.000
Additive Skills	0.02	0.02	.08	0.83	.407	.002
Multiplicative Skills	-0.09	0.02	-.45	-5.10	<.001	.088
R^2						.219
Block 3						
Digit Backward	0.01	0.00	.12	2.23	.027	.013
Order Judgment	0.00	0.00	.03	0.47	.640	.000
Additive Skills	0.01	0.02	.05	0.58	.562	.000
Multiplicative Skills	-0.04	0.02	-.19	-2.23	.027	.013
Fraction Arithmetic	0.38	0.05	.50	8.06	<.001	.172
Total R^2						.392

Note. Unique r^2 represents the squared semi-partial correlations within that specific model tested.

Discussion

To succeed in many university programs, including those in the social, natural, and life sciences, students need to have functional mathematical skills to support their problem solving, logical reasoning, or statistical computation performance (Council of Canadian Academies, 2015; LeFevre et al., 2017; Mulhern & Wylie, 2004; Thompson et al., 2015). Moreover, regardless of the mathematical requirements of specific university programs, mathematical knowledge can open doors to a variety of educational and employment opportunities (Finnie & Meng, 2006). Thus, it is important to understand the nature and organization of fundamental numerical knowledge that students need to acquire more complex mathematical skills (Siegler et al., 2012; Davis-Kean et al., 2021). In the present research, we proposed and tested an expanded Hierarchical Symbol Integration (HSI) model (Xu et al., 2019), which specifies the hierarchical relations among fundamental number knowledge (i.e., ordinal skills), whole number arithmetic skills (i.e., additive and multiplicative), rational numbers (i.e., fractions), and algebraic knowledge.

Interpretations of the Present Findings

According to the HSI model, students' acquisition of mathematics knowledge involves constructing a unified knowledge system, starting with fundamental numeracy skills and progressing to more complex mathematical knowledge (Xu et al., 2019). Moreover, the HSI model posits that knowledge acquisition and efficiency are fully integrated when the inner layers of knowledge are consolidated before moving toward the outer layers (Xu et al., 2021). Accordingly, consistent with other research (Lyons & Beilock, 2011; Reynvoet & Sasanguie, 2016; Sasanguie et al., 2017; Vos et al., 2017; Xu et al., 2019) we found strong correlations among the ordinal and arithmetic measures. However, we also found that additive skills succeeded ordinal skills in predicting unique variance in multiplicative skills, a hierarchical relation that had not been previously tested. These findings extend existing research by showing that the hierarchical symbol network involves the integration of cardinal, ordinal, additive, and multiplicative skills.

Acquiring knowledge of fractions and algebra is critical to mathematical learning, but mastery of these skills is often challenging for students (Bailey et al., 2012; Booth & Newton, 2012; Empson et al., 2011; Siegler & Lortie-Forgues, 2017). Students may have difficulty learning fraction knowledge because they need to restructure their whole-number understanding to accommodate new information about rational numbers (see reviews by Ni & Zhou, 2005; Siegler & Lortie-Forgues, 2017). Misconceptions about fractions can arise even when whole-number knowledge is sufficient because principles and rules that apply to natural numbers are different from those that apply to fractions (Powell & Nelson, 2021; Stafylidou & Vosniadou, 2004; Vosniadou et al., 2008). Ultimately, integrating whole-number and rational-number skills will support students' learning of more advanced mathematical constructs, such as algebra (Barbieri et al., 2021; Booth & Newton, 2012).

Are there consequences if students do not fully integrate their knowledge as they acquire and consolidate each hierarchical layer? Mathematical concepts build upon each other and thus, if students do not master each layer, it is likely that those difficulties will carry over into the next layer (Hiebert, 1988; Siegler & Lortie-Forgues, 2014; Xu & LeFevre, 2021; Xu et al., 2021). For example, students who struggled when they were learning basic arithmetic will continue to face challenges when learning more complex mathematics (Empson et al., 2011). Research shows that some adults do not master arithmetic (LeFevre et al., 2017; Thompson et al., 2015) or rational number skills (Powell & Nelson, 2021), despite the importance of an understanding of the conceptual continuity from whole number arithmetic to rational numbers (Siegler, 2016; Siegler & Braithwaite, 2017). The fundamental properties of arithmetic operations and fractions form the foundations of algebra (Empson et al., 2011; Hackenberg, 2013; Mack, 1993; Nunes et al., 2016; Steffe & Olive, 2010; Thompson & Saldanha, 2003) because algebra is a more abstract, generalized version of arithmetic. That is, algebra focuses on properties that are common to both whole numbers and fractions (Wu, 2001). When students lack the skills to perform fraction arithmetic, learning new algebraic concepts is impeded (Booth & Newton, 2012; Brown & Quinn, 2007). The present research supports the view that individual differences in fluent access to multiplicative skills predicts fraction performance. Moreover, both multiplicative skills and fraction arithmetic skills predicted unique variance in algebra, however, fraction arithmetic predicted a substantially higher amount of unique variance in algebra than multiplicative skills, presumably because not all adults have fully integrated their whole and rational number arithmetic knowledge (Siegler et al., 2011).

Limitations and Future Research

One limitation of this research is that only one task was used to measure most layers of the proposed hierarchy. Although these tasks were carefully chosen based on existing work to reflect the key symbolic processes assumed to be relevant within each layer (i.e., fundamental numeracy, additive and multiplicative arithmetic processes, rational number, and algebraic knowledge), more comprehensive and differentiated assessments of rational number and algebraic knowledge would strengthen the claims of the HSI model. Moreover, we acknowledge that beyond symbolic processing, there are many skills (e.g., proportional reasoning, estimation, problem solving) that are important for success with more advanced mathematics that were not measured in the present study. For example, conceptual understanding of arithmetic principles

(Dubé & Robinson, 2018) and adaptive number knowledge (McMullen et al., 2016, 2017) also predict pre-algebra or algebra skills.

The present study focused on the relations among domain-specific symbolic mathematical skills. Thus, only a single measure of verbal working memory was included as a control variable. Because domain-general skills are correlated with mathematics (De Smedt, 2022), in the future we should consider including additional domain-general measures in our model, such as fluid intelligence (Peng et al., 2019), language skills (Peng et al., 2020), and executive functions (Peng et al., 2016). It is possible that some of the relations shown between mathematical skills could reflect shared reliance on domain-general abilities. Therefore, studies which include a broader assessment of domain-general abilities would allow an even more stringent test of the HSI model.

Finally, because this study was correlational and the data were measured at a single point in time, it does not directly address the question of how various mathematical skills develop. Presumably, the relations among the mathematical skills will show complex and dynamic patterns over time that are not reflected in the current data, which reflect performance of adults after 10 to 15 years of experience with mathematics. In support of the hierarchical model, Xu et al. (2021) found that for children in grades 2 to 3, addition and subtraction become integrated, and later-learned subtraction knowledge predicted multiplication in grade 3. However, longitudinal designs that cover longer learning trajectories are necessary to explore the causal links among the skills and to capture interactions among the different layers during the acquisition process.

Conclusions

Mathematical knowledge is multi-dimensional – students can succeed with some aspects of mathematics but struggle with others. In part, these difficulties may occur because mastery of basic knowledge supports students' learning of more complex skills (Hiebert, 1988; Núñez, 2017; Siegler & Lortie-Forgues, 2014; Xu & LeFevre, 2021). In the present paper, we show evidence for hierarchical relations among fundamental numeracy, arithmetic fluency, rational number, and algebra knowledge among adults. Our findings are consistent with the view that the fundamental numeracy and arithmetic skills work together as integrated predictors of more complex mathematical knowledge.

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