



# $s$ -Club Cluster Vertex Deletion on interval and well-partitioned chordal graphs<sup>☆</sup>

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## ABSTRACT

In this paper, we study the computational complexity of  $s$ -CLUB CLUSTER VERTEX DELETION. Given a graph,  $s$ -CLUB CLUSTER VERTEX DELETION ( $s$ -CVD) aims to delete the minimum number of vertices from the graph so that each connected component of the resulting graph has a diameter at most  $s$ . When  $s = 1$ , the corresponding problem is popularly known as CLUSTER VERTEX DELETION (CVD). We provide a faster algorithm for  $s$ -CVD on *interval graphs*. For each  $s \geq 1$ , we give an  $O(n(n+m))$ -time algorithm for  $s$ -CVD on interval graphs with  $n$  vertices and  $m$  edges. In the case of  $s = 1$ , our algorithm is a slight improvement over the  $O(n^3)$ -time algorithm of Cao et al. (2018), and for  $s \geq 2$ , it significantly improves the state-of-the-art running time ( $O(n^4)$ ).

We also give a polynomial-time algorithm to solve CVD on *well-partitioned chordal graphs*, a graph class introduced by Ahn et al. (WG 2020) as a tool for narrowing down complexity gaps for problems that are hard on chordal graphs, and easy on split graphs. Our algorithm relies on a characterisation of the optimal solution and on solving polynomially many instances of the WEIGHTED BIPARTITE VERTEX COVER. This generalises a result of Cao et al. (2018) on split graphs. We also show that for any even integer  $s \geq 2$ ,  $s$ -CVD is NP-hard on well-partitioned chordal graphs.

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## 1. Introduction

Detecting “highly-connected” parts or “clusters” of a complex system is a fundamental research topic in network science [28,38] with numerous applications in computational biology [6,12,30,34,35], machine learning [5], image processing [37], etc. In a graph-theoretic approach, a complex system or a network is often viewed as an undirected graph  $G$  that consists of a set of *vertices*  $V(G)$  representing the atomic entities of the system and a set of *edges*  $E(G)$  representing a binary relationship among the entities. A *cluster* is often viewed as a dense subgraph (often a *clique*), and *partitioning* a graph into such clusters is one of the main objectives of *graph-based data clustering* [6,13,33].

Ben-Dor et al. [6] and Shamir et al. [33] observed that the clusters of certain networks might be retrieved by making a small number of modifications in the network. These modifications may be required to account for the errors introduced

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during the construction of the network. In graph-theoretic terms, the objective is to modify (e.g. edge deletion, edge addition, vertex deletion) a given input graph as little as possible so that each component of the resulting graph is a cluster. When deletion of vertices is the only valid operation on the input graph, the corresponding clustering problem falls in the category of *vertex deletion* problems, a core topic in algorithmic graph theory. Many classic optimisation problems like MAXIMUM CLIQUE, MAXIMUM INDEPENDENT SET, VERTEX COVER are examples of vertex deletion problems. In this paper, we study popular vertex deletion problems called CLUSTER VERTEX DELETION and its generalisation  $s$ -CLUB CLUSTER VERTEX DELETION, both being important in the context of graph-based data clustering.

Given a graph  $G$ , the objective of CLUSTER VERTEX DELETION (CVD) is to delete a minimum number of vertices so that the remaining graph is a set of disjoint cliques. Below we give a formal definition of CVD.

**CLUSTER VERTEX DELETION (CVD)**

**Input:** An undirected graph  $G$ , and an integer  $k$ .

**Output:** YES, if there is a set  $S$  of vertices with  $|S| \leq k$ , such that each component of the graph induced by  $V(G) \setminus S$  is a clique. NO, otherwise.

The term CLUSTER VERTEX DELETION was coined by Gramm et al. [19] in 2004. However, NP-hardness of CVD, even on planar graphs and bipartite graphs, follows from the seminal works of Yannakakis [39] and Lewis & Yannakakis [24] from four decades ago. Since then many researchers have proposed *parameterised algorithms* and *approximation algorithms* for CVD on general graphs [3,8,15–17,20,31,36,40]. In this paper, we focus on polynomial-time solvability of CVD on special classes of graphs.

Cao et al. [9] gave polynomial-time algorithms for CVD on *interval* graphs (see Definition 2) and *split* graphs. Chakraborty et al. [10] gave a polynomial-time algorithm for CVD on *trapezoid* graphs. However, much remains unknown: Chakraborty et al. [10] pointed out that computational complexity of CVD on *planar bipartite* graphs and *cocomparability* graphs is unknown. Cao et al. [9] asked if CVD can be solved on chordal graphs (graphs with no induced cycle of length greater than 3) in polynomial-time.

Ahn et al. [1] introduced *well-partitioned chordal* graphs (see Definition 1) as a tool for narrowing down complexity gaps for problems that are hard on chordal graphs, and easy on split graphs. Since several problems (for example, transversal of longest paths and cycles, tree 3-spanner problem, geodetic set problem) which are either hard or open on chordal graphs become polynomial-time solvable on well-partitioned chordal graphs [1], the computational complexity of CVD on well-partitioned chordal graphs is a well-motivated open question.

In this paper, we also study a generalisation of CVD known as  $s$ -CLUB CLUSTER VERTEX DELETION ( $s$ -CVD). In many applications, the equivalence of cluster and clique is too restrictive [2,4,29]. For example, in protein networks where proteins are the vertices and the edges indicate the interaction between the proteins, a more appropriate notion of clusters may have a diameter (maximum length of the shortest paths) of more than 1 [4]. Therefore researchers have defined the notion of  *$s$ -clubs* [4,11,26]. An  $s$ -club is a graph with *diameter* at most  $s$ . The objective of  $s$ -CLUB CLUSTER VERTEX DELETION ( $s$ -CVD) is to delete the minimum number of vertices from the input graph so that all connected components of the resultant graph are an  $s$ -club. Below we give a formal definition of  $s$ -CVD.

**$s$ -CLUB CLUSTER VERTEX DELETION ( $s$ -CVD)**

**Input:** An undirected graph  $G$ , and integers  $k$  and  $s$ .

**Output:** YES, if there is a set  $S$  of vertices with  $|S| \leq k$ , such that each component of the graph induced by  $V(G) \setminus S$  has diameter at most  $s$ . NO, otherwise.

Schäfer [32] introduced the notion of  $s$ -CVD and gave a polynomial-time algorithm for  $s$ -CVD on trees. Researchers have studied the particular case of 2-CVD as well [14,25]. In general,  $s$ -CVD remains NP-hard on planar bipartite graphs for each  $s \geq 2$ , APX-hard on split graphs for  $s = 2$  [10] (contrasting the polynomial-time solvability of CVD on split graphs). Combination of the ideas of Cao et al. [9] and Schäfer [32], provides an  $O(n^8)$ -time algorithm for  $s$ -CVD on trapezoid graphs (intersection graphs of trapezoids between two horizontal lines) with  $n$  vertices [10]. This algorithm can be modified to give an  $O(n^4)$ -time algorithm for  $s$ -CVD on interval graphs with  $n$  vertices.

**General notations:** For a graph  $G$ , let  $V(G)$  and  $E(G)$  denote the set of vertices and edges, respectively. For a vertex  $v \in V(G)$ , the set of vertices adjacent to  $v$  is denoted by  $N(v)$  and  $N[v] = N(v) \cup \{v\}$ . For  $S \subseteq V(G)$ , let  $G - S$  be an induced graph obtained by deleting the vertices in  $S$  from  $V(G)$ . For two sets  $S_1, S_2$ , let  $S_1 - S_2$  denote the set obtained by deleting the elements of  $S_2$  from  $S_1$ . The set  $S_1 \Delta S_2$  denotes  $(S_1 \cup S_2) - (S_1 \cap S_2)$ . In this work, while writing equations, we use the symbol  $G$  and  $V(G)$  interchangeably when the meaning is clear from the context. For example, in certain cases, set operations like,  $G \cap S$  (or  $G - S$ ) for a set  $S$  implies the set of vertices  $V(G) \cap S$  (or  $V(G) - S$ ). This difference would be clear to the reader from the context.

## 2. Our contributions

In this section, we state our results formally. We start with the definition of well-partitioned chordal graphs as given in [1].

**Definition 1** ([1]). A connected graph  $G$  is a well-partitioned chordal graph if there exists a partition  $\mathcal{P}$  of  $V(G)$  and a tree  $\mathcal{T}$  having  $\mathcal{P}$  as a vertex set such that the following hold.

- (a) Each part  $X \in \mathcal{P}$  is a clique in  $G$ .
- (b) For each edge  $XY \in E(\mathcal{T})$ , there exist  $X' \subseteq X$  and  $Y' \subseteq Y$  such that edge set of the bipartite graph  $G[X, Y]$  is  $X' \times Y'$ .
- (c) For each pair of distinct  $X, Y \in V(\mathcal{T})$  with  $XY \notin E(\mathcal{T})$ , there is no edge between a vertex in  $X$  and a vertex in  $Y$ .

The tree  $\mathcal{T}$  is called a *partition tree* of  $G$ , and the elements of  $\mathcal{P}$  are called its *bags* or *nodes* of  $\mathcal{T}$ .

Our first result is on CVD for well-partitioned chordal graphs, which generalises a result of Cao et al. [9] for split graphs. We prove the following theorem in Section 3.

**Theorem 1.** *Given a connected well-partitioned chordal graph  $G$  and its partition tree, there is an  $O(m^2n)$ -time algorithm to solve CVD on  $G$ , where  $n$  and  $m$  are the number of vertices and edges, respectively.*

Since a partition tree of a well-partitioned chordal graph can be obtained in polynomial time [1], the above theorem adds CVD to the list of problems that are open on chordal graphs but admits polynomial-time algorithm on well-partitioned chordal graphs.

Our algorithm relies on a characterisation of the solution set. We show that the optimal solution of a well-partitioned chordal graph with  $m$  edges can be obtained by finding minimum weighted vertex cover [23] of  $m$  many weighted bipartite graphs with weights at most  $n$ . Then standard *Max-flow* based algorithms [22,23,27] from the literature yield Theorem 1. On the negative side, we prove the following theorem in Section 5.

**Theorem 2.** *Unless the Unique Games Conjecture is false, for any even integer  $s \geq 2$ , there is no  $(2 - \epsilon)$ -approximation algorithm for  $s$ -CVD on well-partitioned chordal graphs.*

Our third result is a faster algorithm for  $s$ -CVD on *interval graphs*.

**Definition 2.** A graph  $G$  is an interval graph if there is a collection  $\mathcal{I}$  of intervals on the real line such that each vertex of the graph can be mapped to an interval and two intervals intersect if and only if there is an edge between the corresponding vertices in  $G$ . The set  $\mathcal{I}$  is an interval representation of  $G$ .

We prove the following theorem in Section 4.

**Theorem 3.** *For each  $s \geq 1$ , there is an  $O(n(n + m))$ -time algorithm to solve  $s$ -CVD on interval graphs with  $n$  vertices and  $m$  edges.*

We note that our techniques deviate significantly from the ones in the previous literature [9,10,32]. We show that the optimal solution (for  $s$ -CVD on interval graphs) must be one of “four types” and the optimum for each of the “four types” can be found by solving  $s$ -CVD on  $O(m + n)$  many induced subgraphs. Furthermore, we exploit the “linear” structure of interval graphs to ensure that the optimal solution in each case can be found in  $O(n)$ -time. Our result significantly improves the state-of-the-art running time ( $O(n^4)$ , see [10]) for  $s$ -CVD on interval graphs.

## 3. Polynomial time algorithm for CVD on well-partitioned chordal graphs

In this section, we shall give a polynomial-time algorithm to solve CVD on well-partitioned chordal graphs. In the first subsection, we present the main ideas of our algorithm and describe our techniques for proving Theorem 1.

### 3.1. Overview of the algorithm

Let  $G$  be a well-partitioned chordal graph with a partition tree  $\mathcal{T}$  rooted at an arbitrary node. For a node  $X$ , let  $\mathcal{T}_X$  be the subtree rooted at  $X$  and  $G_X$  be the subgraph of  $G$  induced by the vertices in the nodes of  $\mathcal{T}_X$ . For two adjacent nodes  $X, Y$  of  $\mathcal{T}$ , the *boundary of  $X$  with respect to  $Y$*  is the set  $bd(X, Y) = \{x \in X : N(x) \cap Y \neq \emptyset\}$  and the set  $E(X, Y) = \{xy | x \in X, y \in Y \text{ and } xy \in E(G)\}$ . Each edge  $e \in E(X, Y)$  is called as an  $(X, Y)$ -edge. For a node  $X$ ,  $P(X)$  denotes the parent of  $X$  in  $\mathcal{T}$ . We denote minimum CVD sets of  $G_X$  and  $G_X - bd(X, P(X))$  as  $OPT(G_X)$  and  $OPT(G_X - bd(X, P(X)))$ , respectively. We shall use the above notations extensively in describing our algorithm and proofs.

Our dynamic programming-based algorithm traverses  $\mathcal{T}$  in a post-order fashion and for each node  $X$  of  $\mathcal{T}$ , computes  $OPT(G_X)$  and  $OPT(G_X - bd(X, P(X)))$ . A set  $S$  of vertices is a *CVD set* of  $G$  if  $G - S$  is disjoint union of cliques. At the heart of our algorithm lies a characterisation of CVD sets of  $G_X$ , showing that any CVD set of  $G_X$  can be exactly one of two types,

namely,  $X$ -CVD set or  $(X, Y)$ -CVD set where  $Y$  is a child of  $X$  (see [Definitions 4](#) and [5](#)). Informally, for a node  $X$ , a CVD set is an  $X$ -CVD set if it contains  $X$  or removing it from  $G_X$  creates a cluster all of whose vertices are from  $X$ . On the contrary, a CVD set is an  $(X, Y)$ -CVD set if its removal creates a cluster intersecting both  $X$  and  $Y$ , where  $Y$  is a child of  $X$ . In [Lemma 4](#), we formally show that any CVD set of  $G_X$  must be one of the above two types.

To compute a minimum  $X$ -CVD set, first, we construct a weighted bipartite graph  $\mathcal{H}$  which is defined in [Section 3.3](#) and show that a minimum weighted vertex cover of  $\mathcal{H}$  can be used to construct a minimum  $X$ -CVD set of  $G$  (see [Eqs. \(3\), \(4\), \(5\), \(6\)](#)). Then in [Section 3.4](#), we show that the subroutine for finding minimum  $X$ -CVD sets can be used to get a minimum  $(X, Y)$ -CVD set for each child  $Y$  of  $X$ . Finally, in [Section 3.5](#), we combine our tools and give an  $O(m^2n)$ -time algorithm to find a minimum CVD set of a well-partitioned chordal graph  $G$  with  $n$  vertices and  $m$  edges.

### 3.2. Definitions and lemma

In this section, we introduce some definitions and prove the lemma that facilitates the construction of a polynomial-time algorithm for finding a minimum CVD set of well-partitioned chordal graphs.

**Definition 3.** A cluster  $C$  of a graph  $G$  is a connected component that is isomorphic to a complete graph.

**Definition 4.** Let  $G$  be a well-partitioned chordal graph,  $\mathcal{T}$  be its partition tree, and  $X$  be the root node of  $\mathcal{T}$ . A CVD set  $S$  of  $G$  is an “ $X$ -CVD set” if either  $X \subseteq S$  or  $G - S$  contains a cluster  $C \subseteq X$ .

**Definition 5.** Let  $G$  be a well-partitioned chordal graph,  $\mathcal{T}$  be its partition tree, and  $X$  be the root node of  $\mathcal{T}$ . Let  $Y$  be a child of  $X$ .

A CVD set  $S$  is a “ $(X, Y)$ -CVD set” if  $G - S$  has a cluster  $C$  such that  $C \cap X \neq \emptyset$  and  $C \cap Y \neq \emptyset$ .

**Lemma 4.** Let  $S$  be a CVD set of  $G$ . Then exactly one of the following holds.

- (a) The set  $S$  is a  $X$ -CVD set.
- (b) There is exactly one child  $Y$  of  $X$  in  $\mathcal{T}$  such that  $S$  is an  $(X, Y)$ -CVD set of  $G$ .

**Proof.** If  $X \subseteq S$  or  $G - S$  has a cluster contained in  $X$ , then  $S$  is an  $X$ -CVD set. Otherwise,  $X^* = (G - S) \cap X \neq \emptyset$  and since  $X^*$  is a clique,  $G - S$  must contain a cluster  $C$  such that  $X^* \subset V(C) \not\subseteq X$ . Therefore,  $C$  should intersect with at least one child of  $X$ . Let  $Y_1, Y_2$  be children of  $X$ . If both  $V(C) \cap Y_1 \neq \emptyset$  and  $V(C) \cap Y_2 \neq \emptyset$ , then  $C$  is not a cluster because  $Y_1$  and  $Y_2$  are non-adjacent nodes of  $\mathcal{T}$ . Hence  $C$  intersects exactly one child of  $X$ .  $\square$

### 3.3. Finding minimum $X$ -CVD sets

In this section, we prove the following theorem.

**Theorem 5.** Let  $G$  be a connected well-partitioned chordal graph and  $\mathcal{T}$  be a partition tree of  $G$  rooted at  $X$ . Assume for each node  $Y \in V(\mathcal{T}) - \{X\}$  both  $OPT(G_Y)$  and  $OPT(G_Y - bd(Y, P(Y)))$  are given. Then a minimum  $X$ -CVD set of  $G$  can be computed in  $O(|E(G)| \cdot |V(G)|)$  time.

For the remainder of this section, we denote by  $G$  a fixed well-partitioned chordal graph with a partition tree  $\mathcal{T}$  rooted at  $X$ . Let  $X_1, X_2, \dots, X_t$  be the children of  $X$ . The main idea behind our algorithm for finding minimum  $X$ -CVD set of  $G$  is to construct an auxiliary vertex weighted bipartite graph  $\mathcal{H}$  with at most  $|V(G)|$  vertices such that the (minimum) vertex covers of  $\mathcal{H}$  can be used to construct (minimum)  $X$ -CVD set. Below we describe the construction of  $\mathcal{H}$ .

Let  $\mathcal{B} = \{bd(X_i, X) : i \in [t]\}$ . The vertex set of  $\mathcal{H}$  is  $X \cup \mathcal{B}$  and the edge set of  $\mathcal{H}$  is defined as

$$E(\mathcal{H}) = \{uB : u \in X, B \in \mathcal{B}, \forall v \in B, uv \in E(G)\} \tag{1}$$

The weight function on the vertices of  $\mathcal{H}$  is defined as follows. For each vertex  $u \in X$ , define  $w(u) = 1$  and for each set  $B \in \mathcal{B}$  where  $B = bd(X_j, X)$ , define

$$w(B) = |B| + |OPT(G_{X_j} - B)| - |OPT(G_{X_j})| \tag{2}$$

**Remark 1.** Since  $B \cup OPT(G_{X_j} - B)$  is a CVD set of  $G_{X_j}$ , we have  $|OPT(G_{X_j})| \leq |B| + |OPT(G_{X_j} - B)|$  and therefore  $w(B) \geq 0$ .

Below we show how minimum weighted vertex covers of  $\mathcal{H}$  can be used to compute the minimum  $X$ -CVD set of  $G$ . For an  $X$ -CVD set  $Z$  of  $G$ , define  $Cov(Z) = (X \cap Z) \cup \{B \in \mathcal{B} : B \subseteq Z\}$ .

**Lemma 6.** Let  $Z$  be an  $X$ -CVD set of  $G$ . Then  $Cov(Z)$  is a vertex cover of  $\mathcal{H}$ .

**Proof.** Assume that  $Cov(Z)$  is not a vertex cover of  $\mathcal{H}$ . Then there exists at least one edge  $e = uB$  in  $\mathcal{H} - Cov(Z)$ . Hence from the definition of  $Cov(Z)$ , we infer that  $u \in X - Z$  and  $B \not\subseteq Z$ . Let  $C_u$  be the cluster of  $G - Z$  that contains the vertex  $u$ . Since  $X$  is a clique,  $X - Z \subseteq C_u$ . Observe that since  $uB$  is an edge of  $\mathcal{H}$ , there exists a vertex  $w \in B$  such that  $uw \in E(G)$ . Then the definition of partition tree  $\mathcal{T}$  and  $B$  implies that all vertices of  $B$  are contained in  $N(u)$ . Since  $B \not\subseteq Z$  it follows that there exists at least one vertex  $v \in B$  in  $G - Z$  such that  $uv \in E(G - Z)$  and hence  $v \in C_u$ . Therefore, the cluster  $C_u$  intersects the child of  $X$  that contains  $B$ , which contradicts the assumption that  $Z$  is an  $X$ -CVD set of  $G$  (see definition of  $X$ -CVD set).  $\square$

For a vertex cover  $D$  of  $\mathcal{H}$ , define

$$S_1(D) = D \cap X \tag{3}$$

$$S_2(D) = \bigcup_{\substack{B \in D \cap \mathcal{B} \\ B = bd(X_i, X)}} B \cup OPT(G_{X_i} - bd(X_i, X)) \tag{4}$$

$$S_3(D) = \bigcup_{\substack{B \in \mathcal{B} - D \\ B = bd(X_i, X)}} OPT(G_{X_i}) \tag{5}$$

$$Sol(D) = S_1(D) \cup S_2(D) \cup S_3(D) \tag{6}$$

Note that, by definition  $S_i(D) \cap S_j(D) = \emptyset$ ,  $1 \leq i < j \leq 3$ . We have the following lemma.

**Lemma 7.** Let  $D$  be a vertex cover of  $\mathcal{H}$ . Then  $Sol(D)$  is an  $X$ -CVD set of  $G$ .

**Proof.** Suppose for the sake of contradiction that  $Sol(D)$  is not an  $X$ -CVD set of  $G$ . First, assume  $Sol(D)$  is not a CVD set of  $G$ . Then there exists an induced path  $P = uvw$  in  $G - Sol(D)$ . Consider the following cases.

1.  $X \cap \{u, v, w\} = \emptyset$ . Then there must exist a child  $Y$  of  $X$  such that  $u, v, w$  are vertices of  $G_Y$ . If  $B = bd(Y, X) \in D$ , then by Eqs. (4) and (6),  $Sol(D)$  contains  $B \cup OPT(G_Y - B)$ . But then  $B \cup OPT(G_Y - B)$  is not a CVD set of  $G_Y$ , a contradiction. If  $B \notin D$ , then by Eqs. (5) and (6),  $Sol(D)$  contains  $OPT(G_Y)$ . But then  $OPT(G_Y)$  is not a CVD set of  $G_Y$ , also a contradiction.
2. Otherwise, there always exist two adjacent vertices  $z_1, z_2$  such that  $\{z_1, z_2\} \subset \{u, v, w\}$  and  $z_1 \in X$  and  $z_2 \in Y$ , where  $Y$  is a child of  $X$ . Observe that  $z_2 \in B = bd(Y, X)$  and therefore  $z_1$  is adjacent to  $B$  in  $\mathcal{H}$ . Since  $\{z_1, z_2\} \cap Sol(D) = \emptyset$ ,  $\mathcal{H} - D$  contains the edge  $z_1B$ , contradicting the fact that  $D$  is a vertex cover of  $\mathcal{H}$ .

Now assume that  $Sol(D)$  is a CVD set but not an  $X$ -CVD set. Then there must exist a cluster  $C$  in  $G - Sol(D)$  that contains an  $(X, Y)$ -edge  $uv$  where  $u \in X$  and  $v \in bd(Y, X)$ . Therefore  $u \notin D$  and  $B = bd(Y, X) \notin D$ . Then  $\mathcal{H} - D$  contains the edge  $uB$ , contradicting the fact that  $D$  is a vertex cover of  $\mathcal{H}$ .  $\square$

A minimum weighted vertex cover  $D$  of  $\mathcal{H}$  is also *minimal* if no proper subset of  $D$  is a vertex cover of  $\mathcal{H}$ . The minimality restriction avoids the inclusion of redundant vertices with weight 0 in the minimum vertex cover.

**Observation 2.** Let  $D$  be a minimal minimum weighted vertex cover of  $\mathcal{H}$ . For any  $i \in [t]$ , either  $bd(X, X_i) \subseteq D$  or  $bd(X_i, X) \in D$ , but not both.

**Proof.** First assume  $bd(X, X_i) \not\subseteq D$  and  $B = bd(X_i, X) \notin D$ . Observe that, the neighbourhood of  $B$  in  $\mathcal{H}$  is  $bd(X, X_i)$ . Since  $bd(X, X_i) \not\subseteq D$ , there must exist a vertex  $u \in (bd(X, X_i) - D) \subseteq X - D$ . Then it follows that  $uB$  is an edge of  $\mathcal{H} - D$ . This contradicts the fact that  $D$  is a vertex cover of  $\mathcal{H}$ .

Now assume that both  $bd(X, X_i) \subseteq D$  and  $B = bd(X_i, X) \in D$ . Since  $\{x : xB \in E(\mathcal{H})\} = bd(X, X_i)$  the set  $D - \{B\}$  is also a vertex cover of  $\mathcal{H}$ , a contradiction.  $\square$

From now on,  $D$  denotes a minimal minimum weighted vertex cover of  $\mathcal{H}$ , and  $Z$  denotes a fixed but arbitrary  $X$ -CVD set of  $G$ . Our goal is to show that  $|Sol(D)| \leq |Z|$ . We need some more notations and observations.

First, we define the following four sets  $I_1, I_2, I_3, I_4$ . (Recall that  $X_1, X_2, \dots, X_t$  are children of the root  $X$  of the partition tree  $\mathcal{T}$  of  $G$ .)

$$I_1 = \{i \in [t] : bd(X, X_i) \subseteq Sol(D) \text{ and } bd(X, X_i) \subseteq Z\} \tag{7}$$

$$I_2 = \{i \in [t] : bd(X, X_i) \subseteq Sol(D) \text{ and } bd(X, X_i) \not\subseteq Z\} \tag{8}$$

$$I_3 = \{i \in [t] - (I_1 \cup I_2) : bd(X_i, X) \subseteq Sol(D) \text{ and } bd(X_i, X) \subseteq Z\} \tag{9}$$

$$I_4 = \{i \in [t] - (I_1 \cup I_2) : bd(X_i, X) \subseteq Sol(D) \text{ and } bd(X_i, X) \not\subseteq Z\} \tag{10}$$

Note that  $I_1 \cup I_2 \cup I_3 \cup I_4 = [t]$  and  $(I_1 \cup I_2) \cap (I_3 \cup I_4) = \emptyset$ . We have the following observations on the sets  $I_i$ ,  $1 \leq i \leq 4$ .

**Observation 3.** The sets  $I_1, I_2, I_3, I_4$  form a partition of  $[t]$ .

**Proof.** From the definition of  $I_i, 1 \leq i \leq 4$ , it is clear that  $I_i \cap I_j = \emptyset, i \neq j$ . Assume that there exists an  $i \in [t]$  such that  $i \notin I_1 \cup I_2$ . Hence,  $bd(X, X_i) \not\subseteq Sol(D) \cap X = D \cap X$ . Then by [Observation 2](#),  $bd(X_i, X) \in D$  and by [Eq. \(4\)](#) the set of vertices  $bd(X_i, X) \subseteq Sol(D)$ . Therefore each  $i \in [t] - (I_1 \cup I_2)$  either belongs to the set  $I_3$  or  $I_4$ .  $\square$

**Lemma 8.** Let  $D$  be a vertex cover of  $\mathcal{H}$  and  $Sol(D)$  be an  $X$ -CVD set of  $G$  defined as in [Eq. \(6\)](#). For the sets  $I_i, 1 \leq i \leq 4$  defined by [Eqs. \(7\)–\(10\)](#), the following holds.

- (i)  $\bigcup_{i \in I_1 \cup I_2} bd(X, X_i) = S_1(D)$
- (ii)  $\bigcup_{i \in I_3 \cup I_4} bd(X_i, X) \cup OPT(G_{X_i} - bd(X_i, X)) = S_2(D)$
- (iii)  $\bigcup_{i \in I_1 \cup I_2} OPT(G_{X_i}) = S_3(D)$ .

**Proof.** First note that  $S_1(D) = D \cap X = Sol(D) \cap X$  (by definition of  $Sol(D)$ ). On the other hand, by definition of  $I_1$  and  $I_2$  we have  $\bigcup_{i \in I_1 \cup I_2} bd(X, X_i) \subseteq Sol(D)$ . Moreover,  $\bigcup_{i \in I_1 \cup I_2} bd(X, X_i) \subseteq X$ . Therefore,  $\bigcup_{i \in I_1 \cup I_2} bd(X, X_i) \subseteq Sol(D) \cap X = S_1(D)$ .

Now to prove the other side,  $S_1(D) \subseteq \bigcup_{i \in I_1 \cup I_2} bd(X, X_i)$ , suppose for the sake of contradiction that there exists a vertex  $v \in S_1(D) - \bigcup_{i \in I_1 \cup I_2} bd(X, X_i)$ . Let  $J = \{j : v \in bd(X, X_j)\}$ . Since  $J \cap (I_1 \cup I_2) = \emptyset$ , by definition of  $I_1$  and  $I_2$ , for each  $j \in J, bd(X, X_j) \not\subseteq Sol(D) \cap X = D \cap X$ . Hence by [Observation 2](#),  $bd(X_j, X) \in D$ , for all  $j \in J$ . Therefore,  $D - \{v\}$  is also a vertex cover of  $\mathcal{H}$ , contradicting the minimality of  $D$ . By [Observation 3](#),  $I_3 \cup I_4 = [t] - I_1 \cup I_2$ . Moreover, by the definition of  $I_1$  and  $I_2$ , for each  $i \in [t] - I_1 \cup I_2$  the set  $bd(X, X_i) \not\subseteq Sol(D) \cap X = D \cap X$ . Hence by [Observation 2](#), we have  $bd(X_i, X) \in D$  for each  $i \in I_3 \cup I_4$  and  $bd(X_i, X) \notin D, i \in I_1 \cup I_2$ . Thus it follows from [Observation 3](#) and the definition of  $S_2$  and  $S_3$  that  $\bigcup_{i \in I_3 \cup I_4} bd(X_i, X) \cup OPT(G_{X_i} - bd(X_i, X)) = S_2(D)$  and  $\bigcup_{i \in I_1 \cup I_2} OPT(G_{X_i}) = S_3(D)$ .  $\square$

Based on the set  $I_1$ , we construct two sets  $D_1$  and  $Z_1$  from  $Sol(D)$  and  $Z$ , respectively, which are defined as follows.

$$D_1 = \bigcup_{i \in I_1} bd(X, X_i) \cup (Sol(D) \cap G_{X_i}) \tag{11}$$

$$Z_1 = \bigcup_{i \in I_1} bd(X, X_i) \cup (Z \cap G_{X_i}) \tag{12}$$

**Lemma 9.**  $|D_1| \leq |Z_1|$ .

**Proof.** From the definition of  $Sol(D)$  and [Eq. \(3\)](#), for all  $i \in I_1$ , we infer that  $bd(X, X_i) \subseteq D$ . Hence, by [Observation 2](#),  $bd(X_i, X) \notin D, i \in I_1$  and from [Eq. \(5\)](#),  $Sol(D) \cap G_{X_i} = OPT(G_{X_i})$ . Since for each distinct  $i, j \in I_1, G_{X_i} \cap G_{X_j} = \emptyset$  and  $|Z \cap G_{X_i}| \geq |OPT(G_{X_i})|$ , by the definitions of  $D_1$  and  $Z_1$  we have  $|D_1| \leq |Z_1|$ .  $\square$

Based on the set  $I_2$ , we construct the following two sets  $D_2 \subseteq Sol(D)$  and  $Z_2 \subseteq Z$ .

$$D_2 = \bigcup_{i \in I_2} bd(X, X_i) \cup (Sol(D) \cap G_{X_i}) - \bigcup_{i \in I_1} bd(X, X_i) \tag{13}$$

$$Z_2 = \bigcup_{i \in I_2} bd(X_i, X) \cup (Z \cap (G_{X_i} - bd(X_i, X))) \tag{14}$$

By the definition of the set  $I_2$ , the set of vertices  $bd(X, X_i) \not\subseteq Z$ , for all  $i \in I_2$ . By [Lemma 6](#), recall that there exists a vertex cover,  $Cov(Z)$  of  $\mathcal{H}$  corresponding to every  $X$ -CVD-set  $Z$ . Since  $bd(X, X_i) \not\subseteq Z$  and thus  $bd(X, X_i) \not\subseteq Cov(Z)$ , it is implicit in [Observation 2](#) that  $bd(X_i, X) \in Cov(Z)$ . Hence  $bd(X_i, X) \subseteq Z$  and the set  $Z_2 \subseteq Z$ .

**Lemma 10.**  $|D_2| \leq |Z_2|$ .

**Proof.** By arguments similar to that in the proof of [Lemma 9](#), for  $i \in I_2, Sol(D) \cap G_{X_i} = OPT(G_{X_i})$ . Hence,  $D_2 = \bigcup_{i \in I_2} bd(X, X_i) \cup OPT(G_{X_i}) - \bigcup_{i \in I_1} bd(X, X_i)$ . Suppose for contradiction that  $|D_2| > |Z_2|$ . Then by the definitions of  $D_2$  and  $Z_2$  we have

$$\left| \bigcup_{i \in I_2} bd(X, X_i) \cup OPT(G_{X_i}) - \bigcup_{i \in I_1} bd(X, X_i) \right| > \left| \bigcup_{i \in I_2} bd(X_i, X) \cup (Z \cap (G_{X_i} - bd(X_i, X))) \right|$$

Since  $X \cap V(G_{X_i}) = \emptyset$ , for all  $1 \leq i \leq t$  and  $|Z \cap (G_{X_i} - bd(X_i, X))| \geq OPT(G_{X_i} - bd(X_i, X))$ , we can rewrite the above inequality as follows.

$$\left| \bigcup_{i \in I_2} bd(X, X_i) - \bigcup_{i \in I_1} bd(X, X_i) \right| > \left| \bigcup_{i \in I_2} bd(X_i, X) \right| + \left| \bigcup_{i \in I_2} OPT(G_{X_i} - bd(X_i, X)) \right| - \left| \bigcup_{i \in I_2} OPT(G_{X_i}) \right|.$$

That is,

$$\left| \bigcup_{i \in I_2} bd(X, X_i) - \bigcup_{i \in I_1} bd(X, X_i) \right| > \sum_{i \in I_2} (|bd(X_i, X)| + |OPT(G_{X_i} - bd(X_i, X))| - |OPT(G_{X_i})|).$$

By Eq. (2),  $|bd(X_i, X)| + |OPT(G_{X_i} - bd(X_i, X))| - |OPT(G_{X_i})| = w(bd(X_i, X))$  and hence,

$$\left| \bigcup_{i \in I_2} bd(X, X_i) - \bigcup_{i \in I_1} bd(X, X_i) \right| > \sum_{i \in I_2} w(bd(X_i, X)) \tag{15}$$

Recall that  $D$  is a minimal minimum weighted vertex cover of  $\mathcal{H}$ . By Lemma 8 we have  $\bigcup_{i \in I_2} bd(X, X_i) \subseteq D$  and hence for each  $i \in I_2$ , the vertex  $B = bd(X_i, X) \notin D$  by Observation 2. Now we show that if we delete the vertices in  $\bigcup_{i \in I_2} bd(X, X_i) - \bigcup_{i \in I_1} bd(X, X_i)$  from  $D$  and add the set of vertices  $\{bd(X_i, X) : i \in I_2\}$  then we get a vertex cover of smaller weight for  $\mathcal{H}$  by inequality (15), a contradiction.

**Claim 1.** Let  $D_1$  be a set of vertices obtained from  $D$  by deleting the vertices in  $\bigcup_{i \in I_2} bd(X, X_i) - \bigcup_{i \in I_1} bd(X, X_i)$  and by adding the set of vertices  $\{bd(X_i, X) : i \in I_2\}$ . Then,  $D_1$  is a vertex cover of  $\mathcal{H}$ .

*Proof of Claim:* Assume that there exists an edge  $uB \in E(\mathcal{H} - D_1)$  where  $B = bd(X_j, X)$ , for some  $j \in [t]$ . Since  $bd(X_j, X) \notin D_1$ , by the definition of  $D_1$  (given above) observe that  $bd(X_j, X) \notin D$  and  $j \notin I_2$ . Note that the neighbourhood of  $bd(X_j, X)$  in  $\mathcal{H}$  is  $bd(X, X_j)$  and hence  $u \in bd(X, X_j)$ . Since  $D$  is a vertex cover of  $\mathcal{H}$ , we have  $bd(X, X_j) \subseteq D$ . Now we show that  $j \notin I_1$ : By definition of  $D_1$  we have  $\bigcup_{i \in I_1} bd(X, X_i) \cap D \subseteq D_1$ . Since  $u \in bd(X, X_j)$  and  $bd(X, X_j) \subseteq D$ , if  $j \in I_1$  then the vertex  $u$  remains in  $D_1$ . Thus no such edge  $uB$  exists in  $\mathcal{H} - D_1$ . Therefore, we infer that  $j \notin I_1$ . Since  $j \notin I_1 \cup I_2$ , from Lemma 8 we have  $bd(X, X_j) \not\subseteq D \cap X$ . Hence there exists a vertex  $w \in bd(X, X_j)$  such that  $w \in \mathcal{H} - D$ . Moreover, by the definition of partition tree  $\mathcal{T}$  and  $bd(X, X_j)$  the edge  $wB \in E(\mathcal{H} - D)$ . This contradicts the assumption that  $D$  is a vertex cover of  $\mathcal{H}$ . ■

This completes the proof of the Lemma. □

Based on the set  $I_3$ , we construct the following two sets  $D_3 \subseteq Sol(D)$  and  $Z_3 \subseteq Z$ .

$$D_3 = \bigcup_{i \in I_3} bd(X_i, X) \cup OPT(G_{X_i} - bd(X_i, X)) \tag{16}$$

$$Z_3 = \bigcup_{i \in I_3} bd(X_i, X) \cup (Z \cap (G_{X_i} - bd(X_i, X))) \tag{17}$$

**Lemma 11.**  $|D_3| \leq |Z_3|$ .

**Proof.** Since  $|Z \cap (G_{X_i} - bd(X_i, X))| \geq OPT(G_{X_i} - bd(X_i, X))$ , by the definitions of  $D_3$  and  $Z_3$  we have  $|D_3| \leq |Z_3|$ . □

Based on the set  $I_4$ , we construct the following two sets  $D_4 \subseteq Sol(D)$  and  $Z_4 \subseteq Z$ .

$$D_4 = \bigcup_{i \in I_4} bd(X_i, X) \cup OPT(G_{X_i} - bd(X_i, X)) \tag{18}$$

$$Z_4 = \bigcup_{i \in I_4} bd(X, X_i) \cup (Z \cap (G_{X_i})) - \bigcup_{i \in I_1} bd(X, X_i) \tag{19}$$

By the definition of the set  $I_4$ , the set of vertices  $bd(X_i, X) \not\subseteq Z, i \in I_4$ . By Lemma 6, recall that there exists a vertex cover,  $Cov(Z)$  of  $\mathcal{H}$  corresponding to every  $X$ -CVD-set  $Z$ . Since  $bd(X_i, X) \not\subseteq Z, i \in I_4$ , by definition of  $Cov(Z)$  we have  $bd(X_i, X) \notin Cov(Z)$  and hence it is implicit in Observation 2 that  $bd(X, X_i) \subseteq Cov(Z)$ . Hence  $bd(X, X_i) \subseteq Z$  and the set  $Z_4 \subseteq Z$ .

**Lemma 12.**  $|D_4| \leq |Z_4|$ .

**Proof.** Suppose for contradiction that  $|D_4| > |Z_4|$ . Then by the definitions of  $D_4$  and  $Z_4$  we have

$$\left| \bigcup_{i \in I_4} (bd(X_i, X) \cup OPT(G_{X_i} - bd(X_i, X))) \right| > \left| \bigcup_{i \in I_4} bd(X, X_i) \cup (Z \cap V(G_{X_i})) - \bigcup_{i \in I_1} bd(X, X_i) \right|$$

Since  $G_{X_i} \cap G_{X_j} = \emptyset$  for  $i, j \in I_4$  and  $Z \cap V(G_{X_i}) \geq OPT(G_{X_i})$ , we have

$$\begin{aligned} & \left| \bigcup_{i \in I_4} bd(X_i, X) \right| + \left| \bigcup_{i \in I_4} OPT(G_{X_i} - bd(X_i, X)) \right| - \left| \bigcup_{i \in I_4} OPT(G_{X_i}) \right| \\ & > \left| \bigcup_{i \in I_4} bd(X, X_i) - \bigcup_{i \in I_1} bd(X, X_i) \right| \end{aligned}$$

Note that by Eq. (2),  $|bd(X_i, X)| + |OPT(G_{X_i} - bd(X_i, X))| - |OPT(G_{X_i})| = w(bd(X_i, X))$  and hence,

$$\sum_{i \in I_4} w(bd(X_i, X)) > \left| \bigcup_{i \in I_4} bd(X, X_i) - \bigcup_{i \in I_1} bd(X, X_i) \right| \tag{20}$$

Recall that  $D$  is a minimal minimum weighted vertex cover of  $\mathcal{H}$ . Observe that by definition of  $I_1$  and  $Sol(D)$ , the set  $\bigcup_{i \in I_1} bd(X, X_i) \subseteq D$ . Now we show that if we delete the vertices in  $\{bd(X_i, X) : i \in I_4\}$  from  $D$  and adding the set of vertices  $\bigcup_{i \in I_4} bd(X, X_i) - \bigcup_{i \in I_1} bd(X, X_i)$ , then we get a vertex cover of smaller weight for  $\mathcal{H}$  by inequality (20), a contradiction: By definition of  $I_1$  and  $Sol(D)$ , the set  $\bigcup_{i \in I_1} bd(X, X_i) \subseteq D$ . Hence by the addition of the vertices  $\bigcup_{i \in I_4} bd(X, X_i) - \bigcup_{i \in I_1} bd(X, X_i)$  to  $D$  we have the neighbourhood of each deleted vertex  $bd(X_i, X)$  in  $D$ .  $\square$

**Lemma 13.**  $Sol(D) = \bigsqcup_{i=1}^4 D_i$  and for each distinct  $i, j \in [4]$ ,  $Z_i \cap Z_j = \emptyset$ .

**Proof.** By Observation 3 it follows from the definition that for  $1 \leq i \neq j \leq 4$ , the sets  $D_i \cap D_j = \emptyset$  and  $Z_i \cap Z_j = \emptyset$ .

Now we show that  $Sol(D) = D_1 \cup D_2 \cup D_3 \cup D_4$ . First consider the set  $D_1 \cup D_2 = \bigcup_{i \in I_1 \cup I_2} (bd(X, X_i) \cup (Sol(D) \cap G_{X_i}))$ . By Lemma 8,  $\bigcup_{i \in I_1 \cup I_2} bd(X, X_i) = S_1(D)$  and  $\bigcup_{i \in I_1 \cup I_2} (Sol(D) \cap G_{X_i}) = S_3(D)$ . Hence  $D_1 \cup D_2 = S_1(D) \cup S_3(D)$ .

Now consider the set  $D_3 \cup D_4 = \bigcup_{i \in I_3 \cup I_4} (bd(X_i, X) \cup OPT(G_{X_i} - bd(X_i, X)))$ . Hence by Lemma 8,  $D_3 \cup D_4 = S_2(D)$ . Therefore, the definition of  $Sol(D)$  (Eq. (6)) implies  $Sol(D) = \bigsqcup_{i=1}^4 D_i$ .  $\square$

**Proof of Theorem 5.** Using Lemma 13, we have that  $|Sol(D)| \leq |Z_1 \cup Z_2 \cup Z_3 \cup Z_4| \leq |Z|$ . Hence,  $Sol(D)$  is a minimum  $X$ -CVD set of  $G$ . Furthermore,  $\mathcal{H}$  has at most  $|V(G)|$  vertices and  $|E(G)|$  edges. Therefore minimum weighted vertex cover of  $\mathcal{H}$  can be found in  $O(|V(G)| \cdot |E(G)|)$ -time [23,27] and  $Sol(D)$  can be computed in total of  $O(|V(G)| \cdot |E(G)|)$ -time.  $\square$

Below we give a short pseudocode of our algorithm to find a minimum  $X$ -CVD set of  $G$ .

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**Algorithm 1:** Pseudocode to find a minimum  $X$ -CVD set of a well-partitioned chordal graph

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**Input :** A well-partitioned chordal graph  $G$ , a partition tree  $\mathcal{T}$  of  $G$  rooted at the node  $X$ , for each node  $Y \in \mathcal{T} - \{X\}$  both  $OPT(G_Y)$  and  $OPT(G_Y - bd(Y, P(Y)))$  are given as part of input

**Output:** A minimum  $X$ -CVD set

- 1 Construct a weighted bipartite graph  $\mathcal{H}$  as described in Equations (1) and (2);
  - 2 Find a minimum weighted vertex cover  $D$  of  $\mathcal{H}$ ;
  - 3 Construct the sets  $S_1(D)$ ,  $S_2(D)$ ,  $S_3(D)$  and  $Sol(D)$  as described in Equations (3), (4), (5) and (6), respectively;
  - 4 **return**  $Sol(D)$
- 

### 3.4. Finding minimum $(X, Y)$ -CVD set of well-partitioned chordal graphs

In this section, we prove the following theorem.

**Theorem 14.** Let  $G$  be a well-partitioned chordal graph;  $\mathcal{T}$  be a partition tree of  $G$  rooted at  $X$ ;  $Y$  be a child of  $X$ . Moreover, for each  $Z \in V(\mathcal{T}) - \{X\}$ , assume both  $OPT(G_Z)$  and  $OPT(G_Z - bd(Z, P(Z)))$  are given ( $P(Z)$  denotes the parent of  $Z$  in  $\mathcal{T}$ ). Then a minimum  $(X, Y)$ -CVD set of  $G$  can be computed in  $O(|E(G)|^2 \cdot |V(G)|)$  time.

For the remainder of this section, the meaning of  $G$ ,  $\mathcal{T}$ ,  $X$  and  $Y$  will be as given in Theorem 14. For an  $(X, Y)$ -edge  $e$ , we say that a minimum  $(X, Y)$ -CVD set  $A$  “preserves” the edge  $e$  if  $G - A$  contains the edge  $e$ . Let  $e \in E(X, Y)$  be an  $(X, Y)$ -edges of  $G$ . Then to prove Theorem 14, we use Theorem 15. First, we show how to construct a minimum  $(X, Y)$ -CVD



set  $S_e$  that preserves the edge  $e \in E(X, Y)$  and prove [Theorem 15](#). Clearly, a minimum  $(X, Y)$ -CVD set  $S$  of  $G$  is the one that satisfies  $|S| = \min_{e \in E(X, Y)} |S_e|$ . Therefore, [Theorem 14](#) will follow directly from [Theorem 15](#). The remainder of this section is devoted to prove [Theorem 15](#).

**Theorem 15.** *Assuming the same conditions as in [Theorem 14](#), for  $e \in E(X, Y)$ , a minimum  $(X, Y)$ -CVD set of  $G$  that preserves  $e$  can be computed in  $O(|E(G)| \cdot |V(G)|)$  time.*

First, we need the following observation about the partition trees of well-partitioned chordal graphs, which is easy to verify.

**Observation 4.** *Let  $G$  be a well-partitioned chordal graph with a partition tree  $\mathcal{T}$ . Let  $X, Y$  be two adjacent nodes of  $\mathcal{T}$  such that  $X \cup Y$  induces a complete subgraph in  $G$  and  $\mathcal{T}'$  be the tree obtained by contracting the edge  $XY$  in  $\mathcal{T}$ . Now associate the newly created node with the subset of vertices  $(X \cup Y)$  and retain all the other nodes of  $\mathcal{T}$  and their associated subsets in  $\mathcal{T}'$  as it is. Then  $\mathcal{T}'$  is also a partition tree of  $G$ .*

We begin building the machinery to describe our algorithm for finding a minimum  $(X, Y)$ -CVD of  $G$  that preserves an  $(X, Y)$ -edge  $ab$ . Observe that any  $(X, Y)$ -CVD set that preserves the edge  $ab$  must contain the set  $(N[a] \Delta N[b])$  as a subset. (Otherwise, the connected component of  $G - S$  containing  $ab$  would not be a cluster, a contradiction).

Let  $H$  denote the graph  $G - (N[a] \Delta N[b])$ . Now consider the partition  $\mathcal{Q}$  defined as  $\{Z - (N[a] \Delta N[b]) : Z \in V(\mathcal{T})\}$ . Now construct a graph  $\mathcal{F}$  whose vertex set is  $\mathcal{Q}$  and two vertices  $Z_1, Z_2$  are adjacent in  $\mathcal{F}$  if there is an edge  $uv \in E(H)$  such that  $u \in Z_1$  and  $v \in Z_2$ . Observe that  $\mathcal{F}$  is a forest.

Now we have the following observation that relates the connected components of  $H$  with that of  $\mathcal{F}$ .

**Observation 5.** *There is a bijection  $f$  between the connected components of  $H$  and the connected components of  $\mathcal{F}$ , such that for a component  $C$  of  $H$ ,  $f(C)$  is the partition tree of  $C$ . Moreover, we can choose the root node of  $f(C)$  as a subset of some node in  $\mathcal{T}$ .*

**Proof.** Recall that  $\mathcal{Q}$  is a partition of  $V(H)$ , and the graph  $\mathcal{F}$  is a forest. Let  $A$  be a connected component of  $H$ . We have the following cases.

1.  $A$  contains the vertices  $a$  and  $b$ . Observe that  $\mathcal{Q}$  contains two sets  $Z_1 = bd(X, Y)$  and  $Z_2 = bd(Y, X)$ . Hence,  $Z_1$  and  $Z_2$  are adjacent vertices in  $\mathcal{F}$ . Now define  $f(A)$  to be the subgraph of  $\mathcal{F}$  that contains  $Z_1$  and  $Z_2$ . Clearly,  $f(A)$  is a partition tree of  $A$ , and we can make the root node of  $f(A)$  as  $bd(X, Y)$  which is a subset of  $X$ , the root node of  $\mathcal{T}$ .
2.  $A$  contains a vertex  $v$  such that there exists an edge  $e = uv$  with  $u \in N[a] \setminus N[b]$  in  $G$ . In this case, observe that  $v$  must lie in some node  $Z$  of  $\mathcal{T}$  and  $Z - (N[a] \Delta N[b]) \in V(\mathcal{Q})$ . Now define  $f(A)$  to be the subgraph of  $\mathcal{F}$  that contains  $Z - (N[a] \Delta N[b])$ . Clearly,  $f(A)$  is a partition tree of  $A$  and make  $Z - (N[a] \Delta N[b])$  as the root node of  $f(A)$ .
3.  $A$  contains a vertex  $v$  such that there exists an edge  $e = uv$  with  $u \in N[b] \setminus N[a]$  in  $G$ . In this case, observe that  $v$  must lie in some node  $Z$  of  $\mathcal{T}$  and  $Z - (N[a] \Delta N[b]) \in V(\mathcal{Q})$ . Hence,  $f(A)$  can be defined as in Case 2.

Observe that any connected component of  $H$  belongs to one of the above cases and each connected component of  $\mathcal{F}$  contains at most one node corresponding to a set  $Z - (N[a] \Delta N[b])$ .  $\square$

Consider the connected component  $H^*$  of  $H$  which contains  $a$  and  $b$  and let  $\mathcal{F}' = f(H^*)$  where  $f$  is the function given by [Observation 5](#). Observe that the root  $R'$  of  $\mathcal{F}'$  is actually  $bd(X, Y)$ . Moreover,  $R'$  has a child  $R''$ , which is actually  $bd(Y, X)$ . Observe that  $R' \cup R''$  induces a complete subgraph in  $H^*$ . Hence, due to [Observation 4](#), the tree  $\mathcal{F}^*$  obtained by contracting the edge  $R'R''$  is a partition tree of  $H^*$ . Moreover,  $R^* = R' \cup R'' = bd(X, Y) \cup bd(Y, X)$  is the root node of  $\mathcal{F}^*$ . Recall that our objective is to find a minimum  $(X, Y)$ -CVD set that preserves the edge  $ab$ . We have the following lemma.

**Lemma 16.** *Let  $H^*, H_1, H_2, \dots, H_{k'}$  be the connected components of  $H$ . Let  $S^*$  be a minimum  $(R^*)$ -CVD set of  $H^*$ ,  $S_0 = (N[a] \Delta N[b])$ , and for each  $j \in [k']$ , let  $S_j$  denote a minimum CVD set of  $H_j$ . Then  $(S_0 \cup S_1 \cup S_2 \cup \dots \cup S_{k'} \cup S^*)$  is a minimum  $(X, Y)$ -CVD set of  $G$  that preserves the edge  $ab$ .*

**Proof.** Observe that any vertex adjacent to  $a$  or  $b$  lies in  $R^*$ . Since  $S^*$  is a minimum  $(R^*)$ -CVD set,  $S^* \cap \{a, b\} = \emptyset$  and therefore  $H^* - S^*$  has a cluster that contains the edge  $ab$ . Hence  $S_0 \cup S_1 \cup S_2 \cup \dots \cup S_{k'} \cup S^*$  is an  $(X, Y)$ -CVD set that preserves the edge  $ab$ .

Let  $Z$  be any  $(X, Y)$ -CVD set of  $G$  that preserves the edge  $ab$ . For any vertex  $u \in S_0$ , observe that  $a, b, u$  induces a path of length 3. Hence,  $S_0 \subseteq Z$ . Let  $C$  be a connected component of  $G - S_0$ . Observe that  $Z \cap V(C)$  must be a CVD set of  $C$ . Therefore, for each  $i \in [k']$ ,  $|Z \cap V(H_i)| \geq |S_i|$ .

Since  $Z$  is an  $(X, Y)$ -CVD set of  $G$  that preserves the edge  $ab$ ,  $\{a, b\} \cap Z = \emptyset$ . Since  $a, b$  are vertices of  $H^*$ ,  $(Z \cap V(H^*)) \cap \{a, b\} = \emptyset$ . Now suppose  $(Z \cap V(H^*))$  is not an  $(R^*)$ -CVD set of  $H^*$ . Then due to [Lemma 4](#),  $(Z \cap V(H^*))$  must be a  $(R^*, R)$ -CVD set of  $G^*$  for some child  $R$  of  $R^*$  in  $\mathcal{T}^*$ . Hence, there exists a  $(R^*, R)$ -edge  $cd$  which is preserved by  $(Z \cap V(H^*))$ . Without loss of generality, assume  $c \in R^*$  and  $d \in R$ . Observe that  $d$  is adjacent to neither  $a$  nor  $b$ . Hence,  $a, c, d$  induce a path of length 3 in  $H^* - (Z \cap V(H^*))$ , a contradiction. Hence  $|Z \cap V(H^*)| \geq |S^*|$ . Therefore  $|Z| \geq |S_0 \cup S_1 \cup S_2 \cup \dots \cup S_{k'} \cup S^*|$ .  $\square$

**Lemma 16** provides a way to compute a minimum  $(X, Y)$ -CVD set of  $G$  that preserves the edge  $ab$ . Clearly, the set  $S_0 = (N[a] \Delta N[b])$  can be computed in polynomial time. The following observation provides a way to compute a minimum CVD set of all connected components that are different from  $H^*$ .

**Observation 6.** *Let  $A$  be a connected component of  $H$  which is different from  $H^*$ . Then a minimum CVD set of  $A$  can be computed in polynomial time.*

**Proof.** Recall that  $A$  was obtained by deleting  $(N[a] \Delta N[b])$  from  $G$ , and that  $\mathcal{T}$  is the partition tree of  $G$  where the root is  $X$ . Due to **Observation 5**, there is a function  $f$  between the connected components of  $H$  and the connected components of  $\mathcal{F}$  such that  $f(A)$  is the partition tree of  $A$  and there is a node  $R \in \mathcal{T}$  such that the vertices in the root node of  $f(A)$  are a subset of  $R$ . Now consider the following cases.

1. Consider the case when  $\{a, b\} \cap bd(P(R), R) = \emptyset$ . This implies no vertex of  $R$  is adjacent to  $a$  or  $b$ . Moreover, since  $A$  is different from  $H^*$ ,  $bd(P(R), R) \cap (bd(X, Y) \cup bd(Y, X)) = \emptyset$ . This further implies that, either  $bd(P(R), R) \subseteq N[a] - N[b]$  or  $bd(P(R), R) \subseteq N[b] - N[a]$ . In either case,  $R \cap (N[a] \cup N[b]) = \emptyset$ . This implies  $R$  is a node of  $\mathcal{T}$  distinct from  $X$  such that  $A$  is isomorphic to  $G_R$ . Hence, due to the assumption given in **Theorem 15**,  $OPT(G_R - bd(R, P(R)))$  is known and therefore a minimum CVD set of  $A$  is known.
2. Consider the case when there is a vertex  $z \in \{a, b\}$  such that  $z \in bd(P(R), R)$ . Let  $z'$  be the vertex among  $a$  and  $b$  distinct from  $z$ . Since  $A$  is different from  $H^*$ ,  $z' \notin bd(P(R), R)$ . Hence,  $bd(R, P(R)) \subset N(z)$  and therefore  $bd(R, P(R)) \subset (N[a] \Delta N[b])$ . This implies that  $R$  is a node of  $\mathcal{T}$  distinct from  $X$  such that  $A$  is isomorphic to  $G_R - bd(R, P(R))$ . Hence, due to the assumption given in **Theorem 15**,  $OPT(G_R - bd(R, P(R)))$  is known and therefore a minimum CVD set of  $A$  is known.

Clearly, distinguishing between the above cases takes  $O(|E(G)|)$  time. This completes the proof.  $\square$

Let  $H_1, H_2, \dots, H_{k'}$  be the connected components of  $H$ , all different from  $H^*$ . Applying **Observation 6** repeatedly on each component, it is possible to obtain, for each  $j \in [k']$ , a minimum CVD set  $S_j$  of  $H_j$ . The following observation provides a way to compute a minimum  $(R^*)$ -CVD set of  $H^*$ .

**Observation 7.** *Let  $R$  be a child of  $R^*$  in  $\mathcal{F}^*$ . Then both  $OPT(H_R^*)$  and  $OPT(H_R^* - bd(R, R^*))$  are known.*

**Proof.** Since no vertex of  $R$  is adjacent to  $a$  or  $b$  in  $G$ , there must exist a node  $Q \in \mathcal{T}$  such that the vertices in the node  $Q$  are the same as that in  $R$ ,  $\mathcal{T}_Q = \mathcal{T}_R^*$  and  $G_Q = H_R^*$ . Moreover,  $bd(R, R^*) = bd(Q, P(Q))$ , where  $P(Q)$  is the parent of  $Q$  in  $\mathcal{T}$ . Hence, due to the assumption given in **Theorem 15**, both  $OPT(H_R^*)$  and  $OPT(H_R^* - bd(R, R^*))$  are known.  $\square$

Due to **Observation 7** and **Theorem 5**, it is possible to compute a minimum  $(R^*)$ -CVD set  $S^*$  of  $H^*$  in  $O(|V(G)| \cdot |E(G)|)$  time. Now due to **Lemma 16**, we have that  $(S_0 \cup S_1 \cup S_2 \cup \dots \cup S_{k'} \cup S^*)$  is a minimum  $(X, Y)$ -CVD set of  $G$  that preserves the edge  $ab$ . This completes the proof of **Theorem 15** and therefore of **Theorem 14**. In Algorithm 2, we give a short pseudocode of our algorithm to find a minimum  $(X, Y)$ -CVD set of  $G$  that preserves an  $(X, Y)$ -edge  $ab$ . Using Algorithm 2, in Algorithm 3 we provide a short pseudocode to find a minimum  $(X, Y)$ -CVD set of  $G$ .

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**Algorithm 2:** Pseudocode to find a minimum  $(X, Y)$ -CVD set of a well-partitioned chordal graph that preserves an  $(X, Y)$ -edge

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**Input :** A well-partitioned chordal graph  $G$ , a partition tree  $\mathcal{T}$  of  $G$  rooted at the node  $X$ , a child node  $Y$ , an  $(X, Y)$ -edge  $ab$ , for each node  $Z \in \mathcal{T} - \{X\}$  both  $OPT(G_Z)$  and  $OPT(G_Z - bd(Z, P(Z)))$  are given as part of input

**Output:** A minimum  $(X, Y)$ -CVD set of  $G$  that preserves the edge  $ab$

- 1 Construct the set  $S_0 = (N[a] \Delta N[b])$ ;
  - 2 Construct the graph  $H = G - (N[a] \Delta N[b])$ ;
  - 3 Let  $H^*$  be the connected component of  $H$  containing  $a$  and  $b$ . Let  $H_1, H_2, \dots, H_{k'}$  be the remaining connected components of  $H$ .
  - 4 **for**  $i = 1$  to  $k'$  **do**
  - 5     Compute a minimum CVD set  $S_i$  of  $H_i$  (**Observation 6**);
  - 6 Find the partition tree of  $\mathcal{T}^*$  of  $G^*$  whose root is  $X^* = bd(X, Y) \cup bd(Y, X)$ ;
  - 7 Compute a minimum  $(X^*)$ -CVD set  $S^*$  of  $G^*$  using **Algorithm 1**;
  - 8  $Sol = S_0 \cup S_1 \cup S_2 \cup \dots \cup S_{k'} \cup S^*$ ;
  - 9 **return**  $Sol$ ;
- 

**Algorithm 3:** Pseudocode to find a minimum  $(X, Y)$ -CVD set of a well-partitioned chordal graph.

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**Input :** A well-partitioned chordal graph  $G$ , a partition tree  $\mathcal{T}$  of  $G$  rooted at the node  $X$ , a child node  $Y$ , for each node  $Z \in \mathcal{T} - \{X\}$  both  $OPT(G_Z)$  and  $OPT(G_Z - bd(Z, P(Z)))$  are given as part of input

**Output:** A minimum  $(X, Y)$ -CVD set of  $G$ .

- 1 For each  $(X, Y)$ -edge  $e$ , compute a minimum  $(X, Y)$ -CVD set that preserves the edge  $e$  using **Algorithm 2**;
  - 2 Let  $S$  be a set among all  $S_e$ 's that has the least cardinality;
  - 3 **return**  $S$ ;
-

### 3.5. Main algorithm

From now on,  $G$  denotes a fixed well-partitioned chordal graph with a partition tree  $\mathcal{T}$  whose vertex set is  $\mathcal{P}$ , a partition of  $V(G)$ . We will process  $\mathcal{T}$  in the post-order fashion, and for each node  $X$  of  $\mathcal{T}$ , we give a dynamic programming algorithm to compute both  $OPT(G_X)$  and  $OPT(G_X - bd(X, P(X)))$  where  $P(X)$  is the parent of  $X$  (when exists) in  $\mathcal{T}$ . Due to [Observation 4](#), we can assume that  $bd(X, P(X)) \subsetneq X$ . In the remaining section,  $X$  is a fixed node of  $\mathcal{T}$ ,  $A$  has a fixed value (which is either  $\emptyset$  or  $bd(X, P(X))$ ),  $G_X^A$  denotes the graph  $G_X - A$ . Since well-partitioned chordal graphs are closed under vertex deletion,  $G_X^A$  is a well partitioned chordal graph which may be disconnected. Now consider the partition  $\mathcal{P}^A$  defined as  $\{Y - A : Y \in V(\mathcal{T}_X)\}$ . Observe that, apart from the set  $X$  all other sets of the partitions  $\mathcal{P}$  have remained in  $\mathcal{P}^A$ . Now construct a graph  $\mathcal{T}'$  whose vertex set is the partition sets of  $\mathcal{P}^A$  and two vertices  $X, Y$  are adjacent in  $\mathcal{T}'$  if there is an edge  $uv \in E(G_X^A)$  such that  $u \in X$  and  $v \in Y$  (since the graph induced by the union of the sets in  $\mathcal{P}^A$  is  $G_X^A$ , the definition of  $\mathcal{T}'$  is valid). Now we have the following observation whose proof is similar to that of [Observation 5](#).

**Observation 8.** *There is a bijection  $f$  between the connected components of  $G_X^A$  and the connected components of  $\mathcal{T}'$ , such that for a component  $C$  of  $G_X^A$ ,  $f(C)$  is a partition tree of  $C$ , and the root of  $f(C)$  is a child of  $X$ .*

Since the vertices of  $X - A$  induce a clique in  $G_X^A$ , there exists at most one component  $G^*$  in  $G_X^A$  that contains a vertex from  $X - A$ . Due to [Observation 5](#) there exists a unique connected component  $f(G^*) = \mathcal{T}^*$  of  $\mathcal{T}'$ , which is a partition tree of  $G^*$ . Let the remaining connected components of  $G_X^A$  be  $G_1, G_2, \dots, G_k$  and for each  $i \in [k]$ , let  $f(G_i) = \mathcal{T}_i$  and  $X_i$  is the root of  $\mathcal{T}_i$ . Let  $X^*$  denote the root node of  $\mathcal{T}^*$  and  $X_1^*, X_2^*, \dots, X_t^*$  be the children of  $X^*$  in  $\mathcal{T}^*$ . We have the following observation.

**Observation 9.** *For each  $j \in [t]$ , there is a child  $Y_j$  of  $X$  in  $\mathcal{T}$  such that  $Y_j = X_j^*$  and  $G_{Y_j} = G_{X_j^*}^*$ .*

**Proof.** Observe that the root of  $\mathcal{T}^*$  is  $X^* = X - A$ . Since  $A \subsetneq X$ , any child of  $X^*$  must be a child of  $X$ .  $\square$

We have the following lemma.

**Lemma 17.**  $OPT(G_X^A) = \left( \bigsqcup_{i=1}^k OPT(G_{X_i}) \right) \sqcup OPT(G^*)$

**Proof.** The lemma follows directly from the fact that  $G_{X_1}, G_{X_2}, \dots, G_{X_k}$  and  $G^*$  are connected components of  $G_X^A$ .  $\square$

Due to [Observation 8](#),  $OPT(G_{X_i})$  is already known for each  $i \in [k]$ . Due to [Lemma 4](#), any CVD set  $S$  of  $G^*$  is either a  $(X^*)$ -CVD set or there exists a unique child  $Y$  of  $X^*$ , such that  $S$  is a  $(X^*, Y)$ -CVD set of  $G^*$ . By [Theorem 5](#), it is possible to compute a minimum  $(R^*)$ -CVD set  $S_0$  of  $G^*$  in polynomial time. Due to [Observation 9](#), for any node  $Y$  of  $\mathcal{T}^*$  which is different from  $X^*$ , both  $OPT(G_Y)$  and  $OPT(G_Y - bd(Y, P(Y)))$  are known, where  $P(Y)$  is the parent of  $Y$  in  $\mathcal{T}^*$ . Hence, by [Theorem 14](#) for each child  $X_i^*, i \in [t]$ , computing a minimum  $(X^*, X_i^*)$ -CVD set  $S_i$  is possible in  $O(|V(G_{X_i^*}^*)| \cdot |E(G_{X_i^*}^*)|^2)$  time. Let  $S^* \in \{S_0, S_1, S_2, \dots, S_t\}$  be a set with the minimum cardinality. Due to [Lemma 4](#),  $S^*$  is a minimum CVD set of  $G^*$  that can be obtained in  $O(m^2n)$  time. Finally, due to [Lemma 17](#), we have a minimum CVD set of  $G_X^A$ .

## 4. $O(n(n + m))$ -Time algorithm for s-CVD on interval graphs

In this section, we shall give an  $O(n(n + m))$ -time algorithm to solve s-CVD on interval graph  $G$  with  $n$  vertices and  $m$  edges. For a set  $X \subseteq V(G)$ , if each connected component of  $G - X$  is an s-club, then we call  $X$  as an s-club vertex deleting set (s-CVD set). In the next section, we present the main ideas of our algorithm to find a minimum cardinality s-CVD set of an interval graph.

### 4.1. Overview of the algorithm

The heart of our algorithm lies in a characterisation of s-CVD sets of an interval graph. We show (in [Lemma 18](#)) that any s-CVD set must be one of four types, defined in [Definitions 9–12](#). Hence, the problem boils down to computing a minimum s-CVD set of each type. To do this, first we arrange the maximal cliques in the order of its *Helly region*. Let  $Q_1, Q_2, \dots, Q_k$  be the ordering of the cliques. Then for each  $1 \leq a \leq k$ , we find a minimum cardinality s-CVD set of the graph  $G[1, a]$  which is the subgraph induced by the vertices in  $(Q_1 \cup Q_2 \cup \dots \cup Q_a)$ . Moreover, to facilitate future computations we also find a minimum s-CVD set of the graph  $G[1, a] - A$  where  $A = Q_a \cap Q_b$  for some  $a < b \leq k$ . The trick is to show that, by solving s-CVD on  $O(n + m)$  many different “induced subgraphs” of  $G$ , it is possible to solve s-CVD on  $G$ . In other words, by solving  $O(n + m)$  many different subproblems, it is possible to solve s-CVD on  $G$ . Moreover, it is possible to solve a subproblem in  $O(n)$  time. In [Section 4.2](#), we define four types of s-CVD sets and state that any optimal solution must be one of those four types. In [Section 4.4](#), we give a sketch of our algorithm and analyse the time complexity in [Section 4.5](#).

### 4.2. Definitions and main lemma

Let  $G$  denote a connected interval graph with  $n$  vertices and  $m$  edges. The set  $\mathcal{I}$  denotes a fixed interval representation of  $G$  where the endpoints of the representing intervals are distinct. Let  $l(v)$  and  $r(v)$  denote the left and right endpoints, respectively, of an interval corresponding to a vertex  $v \in V(G)$ . Then the interval assigned to the vertex  $v$  in  $\mathcal{I}$  is denoted by  $I(v) = [l(v), r(v)]$ .

Observe that, intervals on a real line satisfy the Helly property; hence, for each maximal clique  $Q$  of  $G$  there is an interval  $I = \bigcap_{v \in Q} I(v)$ . We call  $I$  as the *Helly region* corresponding to the maximal clique  $Q$ . Let  $Q_1, Q_2, \dots, Q_k$  denote the set of maximal cliques of  $G$  ordered with respect to their *Helly regions*  $I_a, 1 \leq a \leq k$  on the real line. That is,  $I_1 < I_2 < \dots < I_k$ . Observe that, for any two integers  $a, b$  we have  $I_a \cap I_b = \emptyset$  as both  $Q_a$  and  $Q_b$  are maximal cliques. Moreover, for any  $a \leq b \leq c$  if a vertex  $v \in Q_a \cap Q_c$ , then  $v \in Q_b$ .

With respect to an ordering of maximal cliques  $Q_1, Q_2, \dots, Q_k$  of  $G$ , we define the following. In the following discussions, we consider  $Q_0$  as an empty set.

**Definition 6.**

- (i) For integers  $a, b$  where  $1 \leq a < b \leq k$ , let  $S_a^b = Q_a \cap Q_b$ .
- (ii) For an integer  $a$ , let  $\mathcal{S}(Q_a) = \{S_a^b : a < b \leq k \text{ and } S_a^b \neq S_a^{b'}, a < b' < b\} \cup \emptyset$ . (Note that, the members of the set  $\mathcal{S}(Q_a)$  are distinct.)
- (iii) For  $A \in \mathcal{S}(Q_a)$ , let  $Y_A^a = (Q_a - Q_{a-1}) - A$ .
- (iv) For a vertex  $v \in V(G)$ , the index  $q_v^- = \min\{a : v \in Q_a\}$ . That is, the minimum integer  $a$  such that  $v$  belongs to the maximal clique  $Q_a$ .
- (v) For a vertex  $v \in V(G)$ , the index  $q_v^+ = \max\{a : v \in Q_a\}$ . That is, the maximum integer  $a$  such that  $v$  belongs to the maximal clique  $Q_a$ .
- (vi) For a vertex  $v \in V(G)$ ,  $N_{\text{left}}(v) = \{u : u \in N(v) \text{ and } l(u) < l(v)\}$ .

We use the following observation to prove our main lemma.

**Observation 10.** Let  $X \subseteq V(G)$  and  $u, v$  be two vertices with  $r(u) < l(v)$  such that  $u$  and  $v$  lie in different connected components in  $G - X$ . Then there exists an integer  $a$  with  $q_u^+ \leq a < q_v^-$ , such that  $S_a^{a+1} \subseteq X$ .

**Proof.** Let  $\mathcal{C}$  be the set of all connected components of  $G - X$ . For a connected component  $C \in \mathcal{C}$ , define  $\hat{r}(C) = \max\{r(v) : v \in C\}$  and  $\hat{l}(C) = \min\{l(v) : v \in C\}$ . Note that the interval  $[\hat{l}(C), \hat{r}(C)] = \bigcup_{v \in V(C)} I(v)$  and we call it as the *span*( $C$ ). Observe that for two distinct connected components  $C, C' \in \mathcal{C}$  we have  $\text{span}(C) \cap \text{span}(C') = \emptyset$ . Therefore,  $\mathcal{C}$  can be ordered with respect to the order in which the span of components appears on the real line. Let  $C_1, \dots, C_x$  be this ordering. We define  $\text{gap}(C_i, C_{i+1}) = (\hat{r}(C_i), \hat{l}(C_{i+1}))$ ,  $1 \leq i \leq x - 1$ . Note that any vertex whose corresponding interval contains a point in  $\text{gap}(C_i, C_{i+1})$  should be a member of  $X$ : otherwise that vertex belongs to another component in between  $C_i$  and  $C_{i+1}$  (by definition of  $\text{gap}(C_i, C_{i+1})$ ) which contradicts the ordering of components. Let  $C^u = C_t$  and  $C^v = C_{t'}$  denote the connected components of  $G - X$  that contain  $u$  and  $v$ , respectively. Since  $r(u) < l(v)$ , we have  $t < t'$ .

Let  $p \in V(G)$  be such that  $r(p) = \max\{r(w) : w \in V(G), r(w) < \hat{l}(C^v)\}$ . Now take  $a = q_p^+$ , the maximum index  $i$  such that  $p \in Q_i, 1 \leq i \leq k$ . For the index  $a$ , we will show that  $q_u^+ \leq a < q_v^-$  and  $S_a^{a+1} \subseteq X$ .

(i)  $q_u^+ \leq a < q_v^-$ : It is immediate from the definition of  $r(p)$  that  $r(u) \leq \hat{r}(C^u) \leq r(p)$  and  $r(p) < \hat{l}(C^v) \leq l(v)$ . Since  $r(p) < l(v)$ , observe that the *Helly region* corresponding to the clique containing the vertex  $p$  comes before that of  $v$  on the real line. Moreover, since the maximal cliques are numbered with respect to the order in which their *Helly regions* appear on the real line, we can infer that  $q_p^+ = a < q_v^-$ . Similarly, since  $r(u) \leq r(p)$ , by similar arguments as above, we have  $q_u^+ \leq a$ . Therefore we have proved  $q_u^+ \leq a < q_v^-$ .

(ii)  $S_a^{a+1} \subseteq X$ : Consider the component  $C_{t'-1}$  which comes in the immediate left of  $C^v$  in the ordering of the components in  $\mathcal{C}$ . Since  $r(p) < \hat{l}(C^v)$ , the *Helly region* of  $Q_a$  ends before  $\text{span}(C^v)$ . Observe that  $r(p) \geq \hat{r}(C_{t'-1})$ . Moreover, the *Helly region* of  $Q_{a+1}$  starts after that of  $Q_a$ . Since  $p \notin Q_{a+1}$  by definition of  $a$  it follows that *Helly region* of  $Q_{a+1}$  is after the  $\text{span}(C_{t'-1})$ . Therefore, the intervals corresponding to those vertices common to both  $Q_a$  and  $Q_{a+1}$  contain some points of  $\text{gap}(C_{t'-1}, C^v)$ . This implies  $S_a^{a+1} \subseteq X$ .  $\square$

For two integers  $a, b$  with  $1 \leq a \leq b \leq k$ , let  $G[a, b]$  denote the subgraph induced by the set  $\{Q_a \cup Q_{a+1} \cup \dots \cup Q_b\}$ .

**Definition 7.** For an induced subgraph  $H$  of  $G$ , a vertex  $v \in V(H)$  and an integer  $a$ , let  $L_H(a, v)$  denote the set of vertices in  $H$  that lie at distance  $a$  from  $v$  in  $H$ .

In the remainder of this section, we use the notation  $L_H(s + 1, v)$  where  $H = G[1, a] - A$  for some integer  $a$  and  $v \in Y_A^a$  (see Definition 6, (iii)) several times.

**Definition 8.** For an integer  $a$ ,  $1 \leq a \leq k - 1$  and a set  $A \in \mathcal{S}(Q_a)$  consider the induced subgraph  $H = G[1, a] - A$  and the sub-interval representation  $\mathcal{I}' \subseteq \mathcal{I}$  of  $H$ . We define the “frontal component” of the induced graph as the connected component of  $G[1, a] - A$  containing the vertex with the rightmost endpoint in  $\mathcal{I}'$ .

Note that for an integer  $a$  and  $A \in \mathcal{S}(Q_a)$ , the vertices of  $Y_A^a$ , if any, lie in the *frontal* component of  $G[1, a] - A$ . Below we categorise an  $s$ -CVD set  $X$  of  $G[1, a] - A$  into four types. In the following definitions, we consider an integer  $a$ ,  $1 < a \leq k$  and a set  $A \in \mathcal{S}(Q_a)$ .

**Definition 9.** An  $s$ -CVD set  $X$  of  $G[1, a] - A$  is of “type-1” if  $Y_A^a \subseteq X$ .

**Definition 10.** An  $s$ -CVD set  $X$  of  $H = G[1, a] - A$  is of “type-2” if there is a vertex  $v \in Y_A^a$  such that  $L_H(s + 1, v) \subseteq X$ .

**Definition 11.** An  $s$ -CVD set  $X$  of  $H = G[1, a] - A$  is of “type-3” if there exists an integer  $c$ ,  $1 \leq c < a$  such that  $S_c^{c+1} - A \subseteq X$  and  $G[c + 1, a] - (S_c^{c+1} \cup A)$  is connected and has diameter at most  $s$ .

**Definition 12.** An  $s$ -CVD set  $X$  of  $H = G[1, a] - A$  is of “type-4” if there exists an integer  $c$ ,  $1 \leq c < a$  such that  $S_c^{c+1} - A \subseteq X$  and  $G[c + 1, a] - (S_c^{c+1} \cup A)$  is connected and has diameter exactly  $s + 1$ .

The following lemma is crucial for our algorithm.

**Lemma 18 (Main Lemma).** Consider an integer  $1 \leq a \leq k$  and a set  $A \in \mathcal{S}(Q_a)$ . Then at least one of the following holds:

1. Every connected component of  $G[1, a] - A$  has diameter at most  $s$ .
2. Any  $s$ -CVD set of  $G[1, a] - A$  is of some type- $j$  where  $j \in \{1, 2, 3, 4\}$ .

**Proof.** Assume that the *frontal* component of  $H = G[1, a] - A$  has diameter at least  $s + 1$  and the set  $Y_A^a \neq \emptyset$ . Otherwise, any  $s$ -CVD set  $X$  of  $H$  is of either type-1 or type-2: Type-1 is obvious when  $Y_A^a = \emptyset$  because  $\emptyset \subseteq X$ . If the diameter of *frontal* component is at most  $s$  then the set  $L_H(s + 1, v) = \emptyset$  and hence any  $s$ -CVD set of  $H$  is of type-2.

Let  $H$  have an  $s$ -CVD set  $X$  that is not of type- $j$  for any  $j \in \{1, 2\}$  and  $v$  be a vertex in  $Y_A^a$ . Since  $X$  is not of type-2,  $H - X$  contains a vertex  $u$  such that  $u \in L_H(s + 1, v)$ . Now choose a vertex  $u \in L_H(s + 1, v)$  such that  $q_u^+ = \max\{q_{u'}^+ : u' \in L_H(s + 1, v) - X\}$ .

Let  $X' = X \cup A$ . Then observe that  $G[1, a] - X' = H - X$  and hence,  $u \in V(G[1, a] - X')$ . Since  $X$  is an  $s$ -CVD set of  $H$  and the distance between  $u$  and  $v$  in  $H$  is  $s + 1$ , the vertices  $u$  and  $v$  must lie in different connected components in  $G[1, a] - X'$ . Therefore, by [Observation 10](#), there is an integer  $b$  such that  $S_b^{b+1} \subseteq X'$  and  $q_u^+ \leq b < q_v^-$ . Let  $b$  be the maximum among all  $b'$  such that  $q_u^+ \leq b' < q_v^-$  and  $S_{b'}^{b'+1} \subseteq X'$ . Note that  $S_b^{b+1} \subseteq X'$  implies  $S_b^{b+1} - A \subseteq X$ . To complete the proof, we need the following claim.

**Claim.** Let  $Y$  be a subset of  $H$  such that  $S_b^{b+1} \subseteq Y \subseteq X$  where  $b$  is the maximum among all  $b'$  such that  $S_{b'}^{b'+1} \subseteq X$ . Then  $G[b + 1, a] - (Y \cup A)$  is connected.

*Proof of Claim:* Suppose  $G[b + 1, a] - (Y \cup A)$  is not connected. Let  $Z = Y \cup A$  and  $C_v$  be the connected component of  $G[b + 1, a] - Z$  containing a vertex  $v \in Y_A^a$  (Note that  $Y_A^a \neq \emptyset$ ). Since  $G[b + 1, a] - Z$  is not connected, there exists a vertex  $u' \in G[b + 1, a] - Z$  such that  $u' \notin C_v$ . Since  $u'$  and  $v$  are in different components and  $v \in Y_A^a$ , we have  $q_{u'}^+ < q_v^- = a$ . Note that  $u' \in G - Z$  and  $G - Z$  is also not connected. Then, by [Observation 10](#), there exists an integer  $b^*$  such that  $S_{b^*}^{b^*+1} \subseteq Z$  and  $q_{u'}^+ \leq b^* < q_v^-$ . Since  $u' \in G[b + 1, a] - Z$ , the index  $q_{u'}^+ > b$ . Thus it follows that  $b < q_u^+ \leq b^* < q_v^-$ , which contradicts the maximality of the index  $b$ . ■

Let  $H_b = G[b + 1, a] - (S_b^{b+1} \cup A)$ . Now we show that  $H_b$  has a diameter at most  $s + 1$ . Otherwise,  $H_b$  contains vertices at a distance greater than  $s + 1$  from the vertex  $v$ . Let  $Q_{b''}$  be the highest indexed maximal clique containing a vertex  $x$  such that distance between  $x$  and  $v$  in  $H_b$  is exactly  $s + 2$ . By the definition of  $H_b$ , observe that  $b'' > b$ . Now we show that  $S_{b''}^{b''+1} \subseteq X$  which contradicts the maximality of  $b$  (see the definition of  $b$  defined in the above paragraph).

For that, since  $S_{b''}^{b''+1} \subseteq Q_{b''+1}$ , the maximality of  $b''$  implies that the vertices in  $S_{b''}^{b''+1}$  are at distance  $s + 1$  from  $v$  in  $H_b$ . Note that by the above claim, the induced subgraphs  $H_b$  and  $G[b + 1, a] - (X \cup A)$  are connected. Moreover, since  $X$  is an  $s$ -CVD set of  $H = G[1, a] - A$ , when  $S_b^{b+1} - A \subseteq X$  all vertices at distance greater than  $s + 1$  from the vertex  $v$  in  $H_b$  must be in  $X - (S_b^{b+1} - A)$ . Therefore,  $S_{b''}^{b''+1} \subseteq X$  and  $S_{b''}^{b''+1} \cup A \subseteq X'$ . This contradicts the maximality of  $b$ . If the diameter of  $H_b$  is exactly  $s + 1$ , then  $X$  is of type-4. Otherwise,  $X$  is of type-3. □

### 4.3. Some more observations

Let  $H$  be an induced subgraph of  $G$ , and  $u, v$  be two vertices of  $H$ . The distance between  $u$  and  $v$  in  $H$  is denoted by  $d_H(u, v)$ .

**Observation 11.** Consider two integers  $a, b$  with  $1 \leq a < b \leq k$  and a set  $A \in \mathcal{S}(Q_b)$ . Let  $H = G[1, b] - A$  and  $u, v, w$  be three vertices of  $H$  such that  $\{u, v\} \subseteq Q_b - Q_{b-1}$  and  $w \in Q_a$ . Then  $d_H(u, w) = d_H(v, w)$ .

**Proof.** Suppose for contradiction that  $d_H(u, w) \neq d_H(v, w)$ . Without loss of generality assume that  $d_H(u, w) < d_H(v, w)$ . Let  $P$  be a shortest path between  $u$  and  $w$  in  $H$  and  $u'$  be the vertex in  $P$  which is adjacent to  $u$ . Observe that  $u' \in Q_b \cap Q_{b-1}$  (this is because  $u$  is not intersecting with the Helly region of  $Q_{b-1}$ ,  $a < b$  in the ordering and  $P$  is a shortest path). Therefore  $u'$  is adjacent to  $v$  and  $P' = (P - \{u\}) \cup \{v\}$  is a path between  $v$  and  $w$  such that  $d_H(v, w) \leq |P'| = |P| = d_H(u, w)$ , a contradiction.  $\square$

**Observation 12.** Let  $C_f^H$  be the frontal component of  $H = G[1, a] - A^*$ ,  $A^* \subseteq V(G)$ . Let  $Y_{A^*}^a = (Q_a - Q_{a-1}) - A^*$ . If  $Y_{A^*}^a \neq \emptyset$ , then any vertex  $v \in Y_{A^*}^a$  is an end vertex of a diametral path (a shortest path whose length is equal to the diameter of a graph) of  $C_f^H$ .

**Proof.** Suppose that  $Y_{A^*}^a \neq \emptyset$  and a vertex  $v \in Y_{A^*}^a$  is not an end vertex of a diametral path of  $C_f^H$ . Let  $P$  be a diametral path of  $C_f^H$  and  $x, y$  be the end vertices. Observe that neither  $x$  nor  $y$  is in  $Y_{A^*}^a$ . Without loss of generality, assume that  $q_x^- \leq q_y^-$ . Let  $P'$  be a shortest path between  $x$  and  $v$  where  $v \in Y_{A^*}^a$ . Since  $P$  have the maximum size among the shortest paths and  $P'$  is not a diametral path, we have  $|P'| < |P|$ . Since  $v \in Y_{A^*}^a$  and  $x, y \notin Y_{A^*}^a$  we have  $a = q_v^- > q_y^- \geq q_x^-$ . Hence the path  $P'$  contains a vertex  $w$  such that  $w \neq v$  and  $q_w^- \leq q_y^- \leq q_w^+$  (That is, any path from  $v$  to  $x$  should cross the cliques containing  $y$ ). This implies  $w$  is a neighbour of  $y$  and there exists a path  $P''$  between  $x$  and  $y$  via  $w$  such that  $|P''| \leq |P'|$  (the path  $P''$  is obtained by adding the edge  $wy$  to the subpath from  $x$  to  $w$  in  $P'$ ). Since  $|P'| < |P|$ , this contradicts the assumption that  $P$  is a shortest path between  $x$  and  $y$ . Therefore, there exists at least one vertex  $v \in Y_{A^*}^a$ , which is an end vertex of a diametral path of  $C_f^H$ . Then by **Observation 11**, each vertex in  $Y_{A^*}^a$  is an end vertex of a diametral path of  $C_f^H$ .  $\square$

#### 4.4. The algorithm

Our algorithm constructs a table  $\Psi$  iteratively whose cells are indexed by two parameters. For an integer  $a$ ,  $1 \leq a \leq k$  and a set  $A \in \mathcal{S}(Q_a)$ , the cell  $\Psi[a, A]$  contains a minimum s-CVD set of  $G[1, a] - A$ . Clearly,  $\Psi[k, \emptyset]$  contains a minimum s-CVD set of  $G$ .

Now we start the construction of  $\Psi$ . Since  $G[1, 1]$  is a clique, we set  $\Psi[1, A] = \emptyset$  for all  $A \in \mathcal{S}(Q_1)$ .

**Observation 13.** For any  $A \in \mathcal{S}(Q_1)$ ,  $\Psi[1, A] = \emptyset$ .

From now on assume  $a \geq 2$  and  $A$  be a set in  $\mathcal{S}(Q_a)$ . Let  $H$  be the graph  $G[1, a] - A$  and  $F$  be the graph  $G[1, a - 1] - (A \cap Q_{a-1})$ . Observe that for any two integers  $a, b$ ,  $1 \leq a < b \leq k$  the set  $S_{a-1}^b = S_a^b \cap Q_{a-1}$ . Then, for any  $A \in \mathcal{S}(Q_a)$  we have  $(A \cap Q_{a-1}) \in \mathcal{S}(Q_{a-1})$  and  $\Psi[a - 1, A \cap Q_{a-1}]$  is defined. Note that  $H - F = Y_A^a$ .

In the following lemma we show that  $\Psi[a, A] = \Psi[a - 1, A \cap Q_{a-1}]$  if the frontal component of  $H$  has diameter at most  $s$ .

**Lemma 19.** Let  $H = G[1, a] - A$ , for  $A \in \mathcal{S}(Q_a)$ ,  $1 < a \leq k$ . If the frontal component of  $H$  has diameter at most  $s$ , then  $\Psi[a, A] = \Psi[a - 1, A \cap Q_{a-1}]$ .

**Proof.** Let  $F$  denote the graph  $G[1, a - 1] - (A \cap Q_{a-1})$ . Since  $H - F = Y_A^a$ , if  $Y_A^a = \emptyset$  then  $H = F$  and hence,  $\Psi[a, A] = \Psi[a - 1, A \cap Q_{a-1}]$ . Now assume that  $Y_A^a \neq \emptyset$ . Observe that the connected components of  $H$  and  $F$  are the same except the frontal components. The frontal components of  $H$  and  $F$  differ depending on the set  $S_{a-1}^a$  as follows.

- (i) If  $S_{a-1}^a \cap H = \emptyset$  then the frontal component of  $H$  is  $Y_A^a$ .
- (ii) If  $S_{a-1}^a \cap H \neq \emptyset$  then the frontal component of  $H$  is the union of the frontal component of  $G[1, a - 1] - A$  and  $Y_A^a$ .

If the frontal component of  $H$  is  $Y_A^a$  then  $\Psi[a, A] = \Psi[a - 1, A \cap Q_{a-1}]$  because diameter of  $Y_A^a$  is 1. Hence assume that the frontal component of  $H$  belongs to the case (ii) defined above. Let  $C_f^H$  be the frontal component of  $H$  and  $C_f^F$  be the frontal component of  $F$ . Then  $C_f^H = C_f^F \cup Y_A^a$ . We have the following claim.

**Claim.** Let  $C_f^H = C_f^F \cup Y_A^a$ . If the diameter of  $C_f^H$  is at most  $s$ , then the diameter of  $C_f^F$  is also at most  $s$ .

*Proof of Claim:* Suppose not, then  $C_f^F$  contains two vertices  $u$  and  $v$  such that the distance between  $u$  and  $v$  in  $C_f^F$  is at least  $s + 1$ . Without loss of generality, assume that  $l(u) < l(v)$ . Let  $P$  be a shortest path between  $u$  and  $v$  in  $C_f^F$ . Observe that since  $C_f^F = C_f^H - Y_A^a$ , no vertex  $w \in Y_A^a$  belongs to  $V(P)$ . Moreover, for any vertex  $w \in Y_A^a$  we have  $l(u) < l(v) < l(w)$  in the interval representation. Therefore, any shortest path between  $u$  and  $v$  in  $C_f^H$  does not contain a vertex  $w \in Y_A^a$ . Hence

the shortest path between  $u$  and  $v$  in  $C_f^H$  is also at least  $s + 1$ , which contradicts the assumption that the diameter of  $C_f^H$  is at most  $s$ . ■

Hence by the minimality of  $\Psi[a - 1, A \cap Q_{a-1}]$ , no vertices of  $C_f^F$  are in  $\Psi[a - 1, A \cap Q_{a-1}]$ . Thus it follows that  $\Psi[a, A] = \Psi[a - 1, A \cap Q_{a-1}]$ . □

Now assume that the *frontal* component of  $H = G[1, a] - A$  has diameter at least  $s + 1$ . Recall that if  $Y_A^a = \emptyset$ , we have  $\Psi[a, A] = \Psi[a - 1, A \cap Q_{a-1}]$ . Hence assume that  $Y_A^a \neq \emptyset$ . Due to Lemma 18, any  $s$ -CVD set of  $H$  has to be one of the four types defined in Section 4.2.

First, for each  $j \in \{1, 2, 3, 4\}$ , we find an  $s$ -CVD set of minimum cardinality, which is of type- $j$ . We begin by showing how to construct a minimum cardinality  $s$ -CVD set  $X_1$  of type-1 of  $G[1, a] - A$ . We define  $X_1$  as below.

$$X_1 = Y_A^a \cup \Psi[a - 1, A \cap Q_{a-1}] \tag{21}$$

**Lemma 20.** *The set  $X_1$  is a minimum cardinality  $s$ -CVD set of type-1 of  $G[1, a] - A$ .*

**Proof.** Observe that the graph  $H = G[1, a] - Y_A^a$  is isomorphic to  $G[1, a - 1] - (A \cap Q_{a-1})$ . Hence  $X_1 = Y_A^a \cup \Psi[a - 1, A \cap Q_{a-1}]$  is an  $s$ -CVD set of  $H$ . By definition,  $Y_A^a$  is included in an  $s$ -CVD set of type-1. Hence the minimality of  $\Psi[a - 1, A \cap Q_{a-1}]$  implies that  $X_1$  is a minimum cardinality set of type-1. □

Let  $v$  be some vertex in  $Y_A^a$  and  $b < a$  be the maximum integer such that  $(Q_b \cap L_H(s + 2, v)) \neq \emptyset$ . We construct a minimum cardinality  $s$ -CVD set of type-2 of  $G[1, a] - A$  defined as follows.

$$X_2 = L_H(s + 1, v) \cup \Psi[b, S_b^{b+1}] \tag{22}$$

**Lemma 21.** *The set  $X_2$  is a minimum cardinality  $s$ -CVD set of type-2 of  $G[1, a] - A$ .*

**Proof.** By the maximality of  $b$  we have  $S_b^{b+1} \subseteq L_H(s + 1, v)$ . Moreover, the graph  $(G[b + 1, a] - A) - L_H(s + 1, v)$  is connected: otherwise, if  $L_H(s + 1, v)$  is a separator of  $(G[b + 1, a] - A)$  then  $(S_{b'}^{b'+1} - A) \subseteq L_H(s + 1, v)$  for some  $b' > b$ . Since  $Q_{b'}$  is a maximal clique, there exists at least one vertex  $w \in Q_{b'}$  and  $w \notin Q_{b'+1}$ . Hence the distance between  $w$  and  $v$  is  $s + 2$  and  $(Q_{b'} \cap L_H(s + 2, v)) \neq \emptyset$ . Since  $b' > b$ , this contradicts the maximality of  $b$ .

Since  $(G[b + 1, a] - A) - L_H(s + 1, v)$  is connected we have  $((G[b + 1, a] - A) - L_H(s + 1, v))$  is a frontal component of  $G[1, a] - (A \cup L_H(s + 1, v))$ . Let  $A' = A \cup L_H(s + 1, v)$ . Note that  $Y_{A'}^a = Y_A^a \neq \emptyset$ . Observe that the distance between  $v \in Y_{A'}^a$  and any other vertex in  $(G[b + 1, a] - A) - L_H(s + 1, v)$  is at most  $s$ . Hence by Observation 12,  $(G[b + 1, a] - A) - L_H(s + 1, v)$  has diameter at most  $s$ .

Note that any vertex of  $G[1, b]$  that belongs to  $A$  is also in  $S_b^{b+1}$ . Hence  $G[1, b] - (A \cup S_b^{b+1}) = G[1, b] - S_b^{b+1}$ . Since  $\Psi[b, S_b^{b+1}]$  is a minimum cardinality  $s$ -CVD set of  $G[1, b] - S_b^{b+1}$  the set  $X_2 = L_H(s + 1, v) \cup \Psi[b, S_b^{b+1}]$  is an  $s$ -CVD set of  $H$ . By definition,  $L_H(s + 1, v)$  is included in an  $s$ -CVD set of type-2. Observe that any vertex of  $G[1, b]$  that belongs to  $L_H(s + 1, v)$  is also in  $S_b^{b+1}$  and hence the minimality of  $\Psi[b, S_b^{b+1}]$  implies that  $X_2$  is a minimum cardinality set of type-2. □

Now we show how to construct a minimum cardinality  $s$ -CVD set  $X_3$  of type-3 of  $G[1, a] - A$ . Let  $B \subseteq \{1, 2, \dots, a - 1\}$  be the set of integers such that for any  $i \in B$  the graph  $H_i = G[i + 1, a] - (S_i^{i+1} \cup A)$  is connected and has a diameter at most  $s$ . By definition, a type-3  $s$ -CVD set  $X$  of  $H$  contains  $S_c^{c+1}$  for some  $c \in B$ . We call each such type-3  $s$ -CVD set as type-3(c). Now we define minimum type-3(c)  $s$ -CVD set as follows.

$$\text{For each } c \in B, \quad Z_c = (S_c^{c+1} - A) \cup \Psi[c, S_c^{c+1}] \tag{23}$$

**Claim.** *The set  $Z_c$  is a minimum cardinality  $s$ -CVD set of type-3(c) of  $G[1, a] - A$ .*

*Proof of Claim:* Note that any vertex of  $G[1, c]$  that belongs to  $A$  is also in  $S_c^{c+1}$ . By definition,  $S_c^{c+1}$  separates the connected component  $G[c + 1, a] - (S_c^{c+1} \cup A)$  from the rest of the graph namely,  $G[1, c] - (S_c^{c+1})$ . Since the diameter of  $G[c + 1, a] - (S_c^{c+1} \cup A)$  is at most  $s$  and  $\Psi[c, S_c^{c+1}]$  is the minimal cardinality  $s$ -CVD set of  $G[1, c] - S_c^{c+1}$  the set  $Z_c = (S_c^{c+1} - A) \cup \Psi[c, S_c^{c+1}]$  is a minimum cardinality  $s$ -CVD set of  $H$  of type-3(c). ■

We define  $X_3$  as below.

$$X_3 = \min\{Z_c : c \in B\} \tag{24}$$

**Lemma 22.** *The set  $X_3$  is a minimum cardinality  $s$ -CVD set of type-3 of  $G[1, a] - A$ .*

**Proof.** The minimality of each  $Z_c$  implies that the set  $X_3$  is a minimum cardinality type-3  $s$ -CVD set. □

Finally, we show the construction of a minimum cardinality  $s$ -CVD set  $X_4$  of type-4 of  $G[1, a] - A$ . Let  $C \subseteq \{1, 2, \dots, a - 1\}$  be the set of integers such that for any  $i \in C$  the graph  $H_i = G[i + 1, a] - (S_i^{i+1} \cup A)$  is connected and has diameter exactly  $s + 1$ . By definition, a type-4  $s$ -CVD set  $X$  of  $H$  contains  $S_i^{i+1}$  for some  $i \in C$ . We call each such type-4  $s$ -CVD set

as type-4(c). Now we define minimum type-4(c) s-CVD set as follows. Note that  $Y_A^a \neq \emptyset$ . Let  $v$  be some vertex in  $Y_A^a$  and  $Y_i = L_{H_i}(s + 1, v)$ .

$$\text{For each } i \in C, \quad Z_i = (S_i^{i+1} - A) \cup Y_i \cup \Psi[i, S_i^{i+1}] \tag{25}$$

**Claim.** The set  $Z_i$  is a minimum cardinality s-CVD set of type-4(c) of  $G[1, a] - A$ .

*Proof of Claim:* Recall that  $H_i$  is connected and we claim that the graph  $H_i - Y_i$  is also connected: otherwise, if  $Y_i$  is a separator of  $H_i$  then there exists a vertex  $w$  in  $H_i - Y_i$  such that  $w$  does not belong to the component containing  $v$  in  $H_i - Y_i$ . Since any path from  $v$  to  $w$  in  $H_i$  passes through  $Y_i$ , the distance of  $w$  from  $v$  in  $H_i$  is at least  $s + 2$  contradicting the assumption that  $H_i$  has diameter exactly  $s + 1$ .

Since  $H_i - Y_i$  is connected, it is the *frontal* component of  $G[1, a] - A - (S_i^{i+1} \cup Y_i)$ . Let  $A' = A \cup S_i^{i+1} \cup Y_i$ . Note that  $Y_{A'}^a = Y_A^a \neq \emptyset$ . Hence the distance between  $v \in Y_{A'}^a$  and any other vertex in  $H_i - Y_i$  is at most  $s$ . Thus by [Observation 12](#) the graph  $H_i - Y_i$  has diameter at most  $s$ . Note that  $\Psi[i, S_i^{i+1}]$  is the minimal cardinality s-CVD set of  $G[1, i] - S_i^{i+1}$  and any vertex of  $G[1, i] - S_i^{i+1}$  that belongs to  $Y_i$  or  $A$  is also in  $S_i^{i+1}$ . Hence, the set  $Z_i = (S_i^{i+1} - A) \cup Y_i \cup \Psi[i, S_i^{i+1}]$  is a minimum s-CVD set of  $H$  of type-4(c). ■

Now define  $X_4$  as follows.

$$X_4 = \min\{Z_i : i \in C\} \tag{26}$$

**Lemma 23.** The set  $X_4$  is a minimum cardinality s-CVD set of type-4 of  $G[1, a] - A$ .

**Proof.** The minimality of each  $Z_i$  implies that the set  $X_4$  is a minimum cardinality type-4 s-CVD set. □

Now we define a minimum s-CVD set of  $G[1, a] - A$  as the one with minimum cardinality among the sets  $X_i, 1 \leq i \leq 4$ . That is,

$$\Psi[a, A] = \min\{X_1, X_2, X_3, X_4\} \tag{27}$$

A pseudocode of the procedure to find Eq. (27) is given by Procedure 4.

---

**Procedure 4:** Compute\_sCD( $G, a, A$ )

---

- 1 Let  $H = G[1, a] - A$  and  $Y_A^a = (Q_a - Q_{a-1}) - A$
  - 2 Set  $X_1 = Y_A^a \cup \Psi[a - 1, A \cap Q_{a-1}]$
  - 3 For a vertex  $v \in Y_A^a$ , find the maximum integer  $b$  such that  $b < a$  and  $Q_b \cap L_H(s + 2, v) \neq \emptyset$
  - 4 Set  $X_2 = L_H(s + 1, v) \cup \Psi[b, S_b^{b+1}]$
  - 5 Set  $B = C = \emptyset$
  - 6 **for**  $c = 1$  **to**  $a - 1$  **do**
  - 7 **if**  $\text{Diam}[c][a][A] \leq s$  **then**
  - 8  $Z_c = (S_c^{c+1} - A) \cup \Psi[c, S_c^{c+1}] \quad \triangleright H_c = G[c + 1, a] - (S_c^{c+1} \cup A)$
  - 9  $B = B \cup \{c\}$
  - 10 **if**  $\text{Diam}[c][a][A] = s + 1$  **then**
  - 11  $W_c = (S_c^{c+1} - A) \cup L_{H_c}(s + 1, v) \cup \Psi[c, S_c^{c+1}]$
  - 12  $C = C \cup \{c\}$
  - 13 Set  $X_3 = \min\{Z_i : i \in B\}$
  - 14 Set  $X_4 = \min\{W_i : i \in C\}$
  - 15 Set  $\Psi[a, A] = \min\{X_1, X_2, X_3, X_4\}$
  - 16 **Return**  $\Psi[a, A]$
- 

We formally summarise the above discussion in the following lemma.

**Lemma 24.** For  $1 < a \leq k$ , if the diameter of the frontal component of  $G[1, a] - A$  is at least  $s + 1$ , then  $\Psi[a, A] = \min\{X_1, X_2, X_3, X_4\}$ .

**Proof.** The proof follows from [Lemma 18](#) and the above discussion on the minimality of the sets  $X_i, 1 \leq i \leq 4$ , in their respective types. □

The proof of correctness of the algorithm follows from the [Lemmas 13, 19, and 24](#). A pseudocode of the algorithm for finding a minimum s-CVD set of an interval graph is given in [Algorithm 5](#). In the following section, we discuss the time complexity of the algorithm.



---

**Algorithm 5:**  $s\text{-CVD}(G, s)$ :  $G$  is an interval graph and  $s$  is a positive integer

---

**Input :** An interval graph  $G$  and a positive integer  $s$

**Output:**  $\Psi[k, \emptyset]$

```

1 Using algorithm in [7] find the ordered set of maximal cliques of  $G$ , say  $Q_1, Q_2, \dots, Q_k$  and  $N_{\text{left}}(v), q_v^+$  and  $q_v^-$  for
  each vertex  $v \in V(G)$ 
2 Find  $S(Q_1)$ 
3 for all  $A \in S(Q_1)$  do
4    $\Psi[1, A] = \emptyset$ 
5 for  $a = 2$  to  $k$  do
6   Find  $S(Q_a)$ 
7   for  $A \in S(Q_a)$  do
8     Set  $Y_A^a = (Q_a - Q_{a-1}) - A$ 
9     if  $Y_A^a = \emptyset$  then
10       $\Psi[a, A] = \Psi[a - 1, A \cap Q_{a-1}]$ 
11    else
12      for  $c = 1$  to  $a - 1$  do
13        Find the diameter of the induced subgraph  $H_c = G[c + 1, a] - (A \cup S_c^{c+1})$  using  $N_{\text{left}}(v), v \in Y_A^a$  and
          store it in  $\text{Diam}[c][a][A]$ .
14      if diameter  $\text{Diam}[1][a][A]$  of the frontal component of  $H_0 = G[1, a] - A \leq s$  then
15         $\Psi[a, A] = \Psi[a - 1, A \cap Q_{a-1}]$ 
16      else
17         $\Psi[a, A] = \text{Compute\_sCD}(G, a, A)$ 
18 return
```

---

4.5. Time complexity

For a given interval graph  $G$  with  $n$  vertices and  $m$  edges, the algorithm first finds the ordered set of maximal cliques of  $G$  as described in Section 4.2. Such an ordered list of the maximal cliques of  $G$  can be produced in linear time as a byproduct of the linear  $(O(n + m))$ -time recognition algorithm for interval graphs due to Booth and Leuker [7]. For each vertex  $v \in G$ , the algorithm gathers the following information during the enumeration of maximal cliques: (i) the values  $q_v^-$  and  $q_v^+$  and (ii)  $N_{\text{left}}(v)$  and are ordered with respect to the left endpoints.

Let  $Q_1, Q_2, \dots, Q_k$  be the ordered set of maximal cliques of  $G$ . From the ordered set of cliques, the algorithm constructs the set  $S(Q_a)$  (steps 2, 6, Algorithm 5) for each  $Q_a, 1 \leq a < k$ . For an integer  $a, 1 \leq a < k$  the set  $S(Q_a)$  can be constructed by adding a vertex  $v \in Q_a$  to each  $S_a^b \in S(Q_a)$  for  $a < b \leq q_v^+$ . For the computation of each  $\Psi[a, A], 1 \leq a \leq k, A \in S(Q_a)$  the algorithm needs to compute the following: (i) the set of vertices,  $Y_A^a$  (step 8, Algorithm 5); (ii) the diameter of the frontal component of the graph  $H = G[1, a] - A$  (step 14, Algorithm 5) and (iii) the diameter of the induced subgraphs  $H_c = G[c + 1, a] - (A \cup S_c^{c+1}), 1 \leq c \leq a - 1$  (steps 12–13, Procedure 4).

The set  $Y_A^a$  can be obtained from the vertex set of  $Q_a$  in linear time by checking the  $q_v^-$  and  $q_v^+$  values of each vertex  $v \in Q_a$ . That is,  $Y_A^a = \{v \in Q_a : q_v^- = a \text{ and } q_v^+ < b, A = S_a^b\}$ . Let  $\text{Diam}[1][a][A]$  be the diameter of the frontal component of  $H = G[1, a] - A$ . By Observation 12, the diameter of the frontal component of  $H$  is equal to the eccentricity of a vertex  $v \in Y_A^a$ . That is, the maximum distance of  $v$  from other vertices in  $H$  which we denote by  $\text{ecc}_H(v)$ . Hence,  $\text{Diam}[1][a][A] = \text{ecc}_H(v)$ . Let  $v_l$  be the leftmost neighbour of  $v$  in  $H$  such that  $q_{v_l}^- = a'$  and  $\text{ecc}_{H'}(v_l)$  be the eccentricity of  $v_l$  in  $H' = G[1, a'] - (Q_{a'} \cap Q_b)$ . Then observe that  $\text{ecc}_H(v) = \text{ecc}_{H'}(v_l) + 1$ . Therefore,  $\text{Diam}[1][a][A] = \text{Diam}[1][a'][Q_{a'} \cap Q_b] + 1$ . Since the leftmost neighbour of  $v$  in  $H$  can be found in linear time from  $N_{\text{left}}(v)$  by checking the  $q_v^-$  and  $q_v^+$  values of each vertex  $u \in N_{\text{left}}(v)$ , the diameter of the frontal component of  $H$  can be found in  $O(n)$  time. Similarly, diameter of the induced subgraphs  $H_c = G[c + 1, a] - (A \cup S_c^{c+1})$  in steps 12–13, Procedure 4 together can be found in  $O(n)$  time by similar arguments as above and the following observation;  $N_{\text{left}}(v) - (A \cup S_c^{c+1}) \supseteq N_{\text{left}}(v) - (A \cup S_{c+1}^{c+2})$ .

To compute the overall time complexity of our algorithm, we have the following claims.

**Claim 2.** Total number of subproblems computed by the algorithm, Algorithm 5 is at most  $O(|V| + |E|) = O(n + m)$ .

*Proof of Claim:* Note that with respect to the ordering of maximal cliques of  $G$ , the elements of the set  $S(Q_a)$  have the following relation. For each  $b, a < b \leq k$  we have  $S_a^{b+1} \subseteq S_a^b$ . Hence the number of distinct subproblems computed by the algorithm corresponding to each maximal clique  $Q_a$  is at most  $|S_a^{a+1}| + 1$  (recall that one of the subproblems corresponds to  $\emptyset \in S(Q_a)$ ). Since the number of maximal cliques in  $G$  is at most  $|V| = n$  and  $|S_a^{a+1}| \leq \text{degree}(v), v \in Q_a - Q_{a+1}$ , the total number of subproblems computed by the algorithm is at most  $\sum_{v \in Q_a - Q_{a+1}} \text{degree}(v) + |V| \leq O(|V| + |E|) = O(n + m)$ . ■

**Claim 3.** The procedure  $\text{Compute\_sCD}(G, a, A)$  computes the minimum cardinality  $s\text{-CVD}$  set of  $H = G[1, a] - A$  in  $O(n)$  time.

*Proof of Claim:* Observe that the time complexity of the procedure *Compute\_sCD*( $G, a, A$ ) depends mainly on building the sets  $X_i, 1 \leq i \leq 4$ . Since the set  $Y_A^a, 1 \leq a < k, A \in \mathcal{S}(Q_a)$  is obtained in  $O(n)$  time, the set  $X_1$  can be computed in  $O(n)$  time.

The set  $L_H(s + 1, v)$  can be computed from the leftmost neighbour of  $v$  in  $H$ , say  $v_l$  in linear time by  $s$  iterations: In the first iteration, find the leftmost vertex of  $v_l$  in  $N_{\text{left}}(v_l) - A$ , in the second iteration, find the leftmost vertex in the second neighbourhood and so on. Moreover, the leftmost neighbour of  $v$  in  $H$  can be obtained by a linear search of  $N_{\text{left}}(v)$ .

Since the number of induced subgraphs  $H_c$  is at most  $O(n)$ , the sets  $X_3$  and  $X_4$  can be constructed in  $O(n)$  time. Hence the claim follows. ■

Therefore, by the above claims, the overall time complexity of our algorithm is  $O(n \cdot (n + m))$ , and [Theorem 3](#) follows.

### 5. Hardness for well-partitioned chordal graphs

In this section, we prove [Theorem 2](#). We shall use the following observation.

**Observation 14.** *Let  $H$  be a well-partitioned chordal graph. Let  $H'$  be a graph obtained from  $H$  by adding a vertex of degree 1. Then  $H'$  is a well-partitioned chordal graph.*

Let  $s \geq 2$  be an even integer and let  $s = 2t$ . We shall reduce MINIMUM VERTEX COVER (MVC) on general graphs to  $s$ -CVD on well-partitioned chordal graphs. Let  $\langle G, k \rangle$  be an instance of MINIMUM VERTEX COVER such that the maximum degree of  $G$  is at most  $n - 3$ . Let  $\bar{G}$  denote the complement of  $G$ . Now construct a well-partitioned chordal graph  $G_{\text{well}}$  from  $G$  as follows. For each vertex of  $v \in V(G)$ , we introduce a new path  $P_v$  with  $t - 1$  edges and let  $x_v, x'_v$  be the endpoints of  $P_v$ . For each edge  $e \in E(\bar{G})$  we introduce a new vertex  $y_e$  in  $G_{\text{well}}$ . For each pair of edges  $e_1, e_2 \in E(\bar{G})$  we introduce an edge between  $y_{e_1}$  and  $y_{e_2}$  in  $G_{\text{well}}$ . For each edge  $e = uv \in E(\bar{G})$ , we introduce the edges  $x_u y_e$  and  $x_v y_e$  in  $G_{\text{well}}$ . Observe that  $C = \{y_e\}_{e \in E(\bar{G})}$  is a clique,  $I = \{x_v\}_{v \in V(G)}$  is an independent set of  $G_{\text{well}}$ . Therefore  $C \cup I$  induces a split graph, say  $G'$ , in  $G_{\text{well}}$ . Moreover, each vertex of  $I$  has at least two neighbours in  $C$ . (This is due to the fact that minimum degree of  $\bar{G}$  is at least two.) Since  $G_{\text{well}}$  can be obtained from  $G'$  by adding vertices of degree 1, due to [Observation 14](#), we have that  $G_{\text{well}}$  is a well-partitioned chordal graph. We shall show that  $G$  has a vertex cover of size  $k$  if and only if  $G_{\text{well}}$  has a  $s$ -CVD set of size  $k$ .

**Observation 15.** *For each vertex  $v \in C, |N(v) \cap I| = 2$  and for each vertex  $u \in I, |N(u) \cap C| \geq 2$ .*

**Lemma 25.** *Let  $D$  be a subset of  $I$  and let  $T = \{u \in V(G) : x_u \in D\}$ . The set  $D$  is an  $s$ -CVD set of  $G_{\text{well}}$  if and only if  $T$  is a vertex cover of  $G$ .*

**Proof.** Let  $D' = \{x'_v : x_v \in I - D\}$  and  $T' = \{u \in V(G) : x_u \in D'\}$  (note that  $T = V(G) - T'$ ). Note that there is one single component  $G'$  of  $G_{\text{well}} - D$  that contains vertices from  $C$  since there are no isolated vertices by [Observation 15](#). Observe that  $G'$  contains  $I - D$ . Therefore, for any two vertices  $x'_u, x'_v \in D'$ , the distance between  $x'_u, x'_v$  is  $s$  if and only if there is an edge between  $u, v$  in  $\bar{G}$ . Therefore, the distance between any two pairs of vertices in  $D'$  is  $s$  if and only if  $T'$  induces a clique in  $\bar{G}$  and, therefore, an independent set in  $G$ . Since  $T = V(G) - T'$ , we have that distance between any two pairs of vertices in  $D'$  is  $s$  if and only if  $T$  is a vertex cover of  $G$ . Since  $|D'| = |I - D|$ , we have that  $D$  is an  $s$ -CVD set of  $G_{\text{well}}$  if and only if  $T$  is a vertex cover of  $G$ . □

**Lemma 26.** *There is a subset of  $I$ , which is a minimum  $s$ -CVD set of  $G_{\text{well}}$ .*

**Proof.** Let  $S$  be a minimum  $s$ -CVD set of  $G_{\text{well}}$  such that  $|S \cap I|$  is maximum. We claim that  $S \subseteq I$ . Suppose for contradiction this is not true. Let  $I' = \bigcup_{u \in V(G)} V(P_u) - \{x_u\}$ . Then we must have that  $S \cap I' \neq \emptyset$  or  $S \cap C \neq \emptyset$ . Let  $a$  be a vertex of  $S \cap I'$ . Observe that there must be a vertex  $u \in V(G)$  such that  $a \in V(P_u)$  and that  $(S - \{a\}) \cup \{x_u\}$  is an  $s$ -CVD set of  $G_{\text{well}}$ . This contradicts the assumption that  $S$  is a minimum  $s$ -CVD set of  $G_{\text{well}}$  with  $|S \cap I|$  maximum.

Now consider the collection  $\mathcal{C}$  of connected components of  $G_{\text{well}} - S$ . First, observe that there exists at most one connected component in  $\mathcal{C}$  that intersects  $C$  (the clique of  $G_{\text{well}}$ ). We shall call such a component as the *big component* and let  $X$  be the set of vertices of the big component. Note that, by [Observation 15](#), the total number of edges incident on the vertices of  $I$  in the induced subgraph  $I \cup C$  of  $G_{\text{well}}$  is at least  $2 \cdot |I|$  and on the vertices of  $C$  in the induced subgraph  $I \cup C$  is exactly  $2 \cdot |C|$  which implies  $|I| \leq |C|$ . In fact  $I$  itself is a  $s$ -CVD set. Therefore, without loss of generality, we can assume that  $C \not\subseteq S$  and, indeed such a big component exists.

Let  $Y$  denote those vertices of  $G_{\text{well}} - S$  that belong to  $I - X$ . Let  $S_C = S \cap C$  and  $S_I = S \cap I$ . Recall that by assumption,  $S_C \neq \emptyset$ .

If there is a vertex  $v \in S_C$  such that  $|N[v] \cap Y| = 0$ , then  $S - \{v\}$  is a  $s$ -CVD set with  $X \cup \{v\}$  as corresponding big component with diameter less than or equal to  $s$ . This contradicts the minimality of  $S$ . Similarly, if there exists a vertex  $v \in S_C$  such that  $N[v] \cap Y = \{u\}$ , a singleton set then  $S' = S \cup \{u\} - \{v\}$  is a new  $s$ -CVD set with  $X \cup \{v\}$  as corresponding new big component. This contradicts the assumption that  $S$  is a minimum  $s$ -CVD set with  $|S \cap I|$  maximum. Hence together with [Observation 15](#), we infer that  $|N(v) \cap Y| = 2$ , for each  $v \in S_C$ . Note that, by the definition of  $Y$ , all the

neighbours of  $Y$  in  $C$  belong to  $S_C$ . Hence [Observation 15](#) implies that for each vertex  $u \in Y$ ,  $|N(u) \cap S_C| \geq 2$ . Therefore, the total number of edges incident on the vertices of  $Y$  in the induced subgraph  $Y \cup S_C$  is at least  $2 \cdot |Y|$  and on the vertices of  $S_C$  in  $Y \cup S_C$  is exactly  $2 \cdot |S_C|$ . This implies  $|Y| \leq |S_C|$ . Hence,  $S' = (S - S_C) \cup Y$  is a minimum 2-CVD set with  $X \cup S_C$  as the corresponding new big component and  $|S' \cap I| > |S \cap I|$ . This contradicts the assumption for  $S$ .

Hence we conclude that  $S$  is indeed a minimum  $s$ -CVD set such that  $S \subseteq I$ .  $\square$

[Lemmas 25](#) and [26](#) imply that  $G$  has a vertex cover of size  $k$  if and only if  $G_{well}$  has a  $s$ -CVD set of size  $k$ . Now [Theorem 2](#) follows from a result of Khot and Regev [[21](#)], where they showed that unless the Unique Games Conjecture is false, there is no  $(2 - \epsilon)$ -approximation algorithm for MINIMUM VERTEX COVER on general graphs, for any  $\epsilon > 0$ .

## 6. Conclusion

In this paper, we studied the computational complexity of  $s$ -CVD on well-partitioned chordal graphs, a subclass of chordal graphs that generalises split graphs [[18](#)]. We gave a polynomial-time algorithm for  $s = 1$ , and we proved that for any even integer  $s \geq 2$ ,  $s$ -CVD is NP-hard on well-partitioned chordal graphs. We also provide a faster algorithm for  $s$ -CVD on interval graphs for each  $s \geq 1$ . This raises the following questions.

**Question 1.** *What is the time complexity of Cluster Vertex Deletion on chordal graphs?*

**Question 2.** *What is the time complexity of  $s$ -CVD on chordal graphs for odd values of  $s$ ?*

**Question 3.** *Is there a constant factor approximation algorithm for  $s$ -CVD,  $s \geq 2$  on chordal graphs?*

Another generalisation of interval graphs is the class of *cocomparability* graphs. It would be interesting to investigate the following question.

**Question 4.** *What is the time complexity of  $s$ -CVD on cocomparability graphs for each  $s \geq 1$ ?*

## Data availability

No data was used for the research described in the article.

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