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The Geometry of the Space of Vortices on a Two-Sphere in the Bradlow Limit

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Abstract: It is proved that the normalized L^2 metric on the moduli space of n-vortices on a two-sphere, endowed with any Riemannian metric, converges uniformly in the Bradlow limit to the Fubini–Study metric. This establishes, in a rigorous setting, a longstanding informal conjecture of Baptista and Manton.

1. Introduction

Vortices are the simplest class of topological solitons arising in gauge theory, making them interesting objects of mathematical study, independent of their phenomenological applications in condensed matter physics and cosmology. At critical coupling, they satisfy a self-duality (or Bogomol'nyi) type condition which implies that static vortices exert no net force on one another. The moduli space M_n of static n-vortex solutions is thus exceptionally large, forming a complex n-manifold. There is a well-developed programme for studying their low energy dynamics in this regime, comprehending classical, quantum and statistical mechanics, originally proposed by Manton. The key object underpinning this programme is a natural Riemannian metric g on M_n, called the L^2 metric, obtained by restricting the model's kinetic energy functional to TM_n. Vortex dynamics is modelled by geodesic motion in (M_n, g). For a comprehensive review of the geodesic approximation to vortex dynamics, see [15, ch. 3].

The L^2 metric on M_n is, then, the object of strong and sustained mathematical interest. There are very few situations in which g can be computed exactly, however. Other than the rather trivial case of M_1 when physical space is homogeneous, Strachan has exactly computed the metric on M_2 on the hyperbolic plane [17]. Samols [16] adapted this calculation to obtain a semi-explicit localization formula for the metric on an arbitrary surface (for example the Euclidean plane) which is useful for numerics, and can be used to prove many interesting global and qualitative properties of g, but it does not yield explicit formulae. Even the metric on M_1 is nontrivial if physical space is not homogeneous [12].

This paper examines one case where explicit progress is possible, at least in a certain parametric limit: vortices on a two-sphere. Bradlow [7] observed that, on a compact Riemann surface Σ, there is a lower bound, proportional to n, on the area $|\Sigma|$ required for the surface to accommodate n-vortices. Above this bound, M_n is biholomorphic to $\mathrm{Sym}_n\,\Sigma$, the n-fold symmetric product of Σ, since vortices are uniquely determined by the collection of n points on Σ at which their Higgs field vanishes. As $|\Sigma|$ approaches the bound from above, the vortices spread out and lose their spatial localization. Precisely at the bound, they delocalize entirely: the Higgs field vanishes identically and the magnetic field is uniform. Such vortex solutions are sometimes called "dissolved" vortices, and the limit in which they appear called the "dissolving" limit. The moduli space of dissolved vortices is precisely the moduli space of constant curvature connexions on the line bundle of which the Higgs field is a section. If the genus r of Σ is positive this is a torus of dimension $2r$ which may naturally be identified with the Jacobian variety of Σ. It also has an L^2 metric, to which the L^2 metric g on M_n conjecturally degenerates (more precisely, g conjecturally degenerates to the pullback of the metric on the torus by the Abel-Jacobi map $\mathrm{Sym}_n\,\Sigma \to \mathrm{Jac}(\Sigma)$), a scenario studied in detail in [14]. If $r = 0$, that is, $\Sigma = S^2$, the moduli space of dissolved vortices is a point, while $M_n \equiv \mathbb{C}P^n$. To understand the metric g in the dissolving limit we must rescale it by (for example) demanding that (M_n, g) has some normalized volume. In this context, Baptista and Manton [4] made the remarkable conjecture (on the *round* two-sphere), that g converges in the dissolving limit to the Fubini–Study metric on $\mathbb{C}P^n$, implying an enormous gain in symmetry in this limit.

The purpose of this paper is to give a precise formulation and rigorous proof of a slight generalization of Baptista and Manton's conjecture: we will show that, in the dissolving limit, the normalized L^2 metric on $M_n(S^2)$ converges uniformly to the Fubini–Study metric, regardless of the metric on S^2 (that is, we remove the assumption that the domain sphere is round). The L^2 geometry in the dissolving limit is thus unreasonably simple: a family of metrics with (generically) no nontrivial isometries at all converges uniformly to a homogeneous metric of constant holomorphic sectional curvature. The convergence established implies [2] that the spectrum of the Laplacian on M_n converges uniformly to the spectrum of the Fubini–Study Laplacian, which is easily computed [5]. The theorem thus has immediate consequences for the *quantum* dynamics of vortices on S^2 in the dissolving limit.

The paper is structured as follows. In Sect. 2 we formulate the model and give a precise statement of the theorem. In Sect. 3 we review the notion of *pseudo-vortices*, introduced by Baptista and Manton, and prove that vortices converge uniformly to pseudo-vortices in the dissolving limit. In Sect. 4, we establish convergence of the metric, finishing the proof of the main theorem, while in Sect. 5 we study the convergence of the spectrum of the Laplacian.

2. Statement of the Theorem

Let (Σ, g_Σ) be an oriented Riemannian manifold diffeomorphic to S^2 and (L, h) be a hermitian line bundle over Σ of degree $n \geq 0$. To any section φ of L and unitary connexion A on L we associate the energy

$$E(\varphi, A) = \int_\Sigma \left(\frac{1}{2}|\mathrm{d}_A\varphi|^2 + \frac{1}{2}|F_A|^2 + \frac{1}{8}(\tau - |\varphi|^2)^2\right) \tag{2.1}$$

where $\tau > 0$ is a constant parameter and F_A is the curvature of A. If we choose a local section η of L with $|\eta|^2 = h(\eta, \eta) = 1$, we may identify the connexion A locally with the real one-form $h(i\mathrm{d}_A\eta, \eta)$ which, in a slight abuse of notation, we also denote A. Then for a general local section $f\eta$, where f is smooth and complex valued,

$$\mathrm{d}_A(f\eta) = (\mathrm{d}f - iAf)\eta. \tag{2.2}$$

We adopt the convention that the curvature F_A is real, coinciding (globally) with $\mathrm{d}A$. The energy functional is invariant under gauge transformations,

$$(\varphi, A) \mapsto (e^{i\chi}\varphi, A + \mathrm{d}\chi), \tag{2.3}$$

where $\chi : \Sigma \to \mathbb{R}$ is smooth.

It is well known [6,7] that $E(\varphi, A) \geq \tau\pi n$ with equality if and only if

$$\overline{\partial}_A\varphi = 0 \tag{2.4}$$

$$*F_A = \frac{1}{2}(\tau - |\varphi|^2), \tag{2.5}$$

where $\overline{\partial}_A$ denotes the 0, 1 part of d_A (with respect to the complex structure on Σ defined by its orientation and metric) and $*$ is the Hodge isomorphism defined by g_Σ. Solutions of this system of equations are called vortices. Since they attain a topological lower bound on E, they are automatically global minima of E.

A necessary condition for existence of vortices was obtained by Bradlow [7] by integrating (2.5) over Σ:

$$2\pi n = \frac{1}{2}\tau|\Sigma| - \frac{1}{2}\int_\Sigma |\varphi|^2 \leq \frac{1}{2}\tau|\Sigma| \tag{2.6}$$

where $|\Sigma|$ denotes the area of Σ. Hence, if vortices exist,

$$\varepsilon := \tau|\Sigma| - 4\pi n \geq 0. \tag{2.7}$$

We refer to $\varepsilon \searrow 0$ as the Bradlow limit. More precisely, we shall consider the limit where the metric g_Σ (and hence $|\Sigma|$) is fixed, and $\tau \searrow 4\pi n/|\Sigma|$. This is more convenient for our purposes than the limit considered by Baptista and Manton ($\tau \equiv 1$ and g_Σ varying through round metrics, with $|\Sigma|$ approaching $4\pi n$ from above) because it allows us to define Sobolev spaces with fixed norms determined by g_Σ. That our limit includes theirs follows from the observation that, if (φ, A) solves the vortex equations on (Σ, g_Σ) at coupling $\tau = \tau_0$, then $(\tau_0^{-1/2}\varphi, A)$ satisfies the vortex equations on $(\Sigma, \tau_0 g_\Sigma)$ at coupling $\tau = 1$. Note that at $\varepsilon = 0$ any vortex must have $\varphi = 0$ and $*F_A$ constant, so that vortices completely lose their spatial localization, which is why this is often called the *dissolving* limit. For $\varepsilon > 0$, every vortex has $\|\varphi\|^2_{L^2} = \varepsilon$.

The moduli space of vortices M_n is the space of gauge equivalence classes of solutions of (2.4), (2.5). This is a complex n-manifold canonically diffeomorphic to $\mathrm{Sym}_n\Sigma$, the n-fold symmetric product of Σ, the canonical diffeomorphism $F : M_n \to \mathrm{Sym}_n\Sigma$ being the map which sends the (gauge equivalence class of a) vortex solution (φ, A) to the set of points at which φ vanishes, counted with multiplicity. That is, we map the gauge equivalence class of solutions $[(\varphi, A)]$ to the degree n effective divisor $F([(\varphi, A)]) = (\varphi)$ in Σ. In this way we identify the ε-dependent family of manifolds M_n with the single fixed manifold $\mathrm{Sym}_n\Sigma$ which, in a slight abuse of notation, we will also denote M_n.

Since $\Sigma = S^2$, we may specify a degree n effective divisor D by giving a polynomial

$$a_0 + a_1 z + \cdots + a_n z^n \tag{2.8}$$

vanishing on D, where z is a stereographic coordinate. Two such polynomials define the same divisor if and only if one is a constant multiple of the other, so we identify (φ, A), up to gauge, with $[a_0, a_1, \ldots, a_n] \in \mathbb{C}P^n$. Hence $M_n \equiv \mathbb{C}P^n$.

For our purposes, an alternative identification $M_n \equiv \mathbb{C}P^n$ is more convenient, however. Let us choose and fix, once and for all, a unitary connexion $\widehat{A}$ on L of constant curvature. This is unique up to gauge. We equip L with the holomorphic structure whose Dolbeault operator $\overline{\partial}_L = \overline{\partial}_{\widehat{A}}$. Then $H^0(L)$, the space of holomorphic sections of L, is a complex vector space of dimension $n + 1$. Each nonzero section in $H^0(L)$ defines an effective divisor of degree n (on which it vanishes), and two sections define the same divisor if and only if one is a constant nonzero multiple of the other. Hence we have an identification of M_n with the complex projective space of lines through 0 in $H^0(L)$. In other words, we identify M_n with $\mathbb{C}P^n$ via the bijective map

$$f : M_n \to \mathbb{P}(H^0(L)), \qquad [(\varphi, A)] \mapsto \{\psi \in H^0(L) : (\psi) = (\varphi)\} \tag{2.9}$$

which identifies $[(\varphi, A)]$ with the line in $H^0(L)$ consisting of holomorphic sections vanishing on the same divisor as φ. This allows us to equip M_n with a Fubini–Study metric as follows. The L^2 inner product on $\Gamma(L)$ defines a hermitian inner product on $H^0(L)$. Denote by S the unit sphere in $(H^0(L), \|\cdot\|_{L^2})$, and by $\pi : S \to \mathbb{P}(H^0(L))$, $\pi(\psi) = [\psi]$, the Hopf fibration. Then the Fubini–Study metric of constant holomorphic sectional curvature 2 is precisely the metric g_{FS} on $\mathbb{P}(H^0(L))$ with respect to which π is a Riemannian submersion. By the *Fubini–Study metric* on M_n we will always mean

$$g_0 := f^* g_{FS}. \tag{2.10}$$

A priori, there is no reason to expect g_0 to have any relevance to vortex dynamics, which is controlled [15, ch. 3] by a natural Riemannian metric g on M_n called the L^2 metric. This is defined as follows. Any smooth curve $(\varphi(t), A(t))$ of vortex solutions through $(\varphi(0), A(0))$ tautologically defines a tangent vector to M_n at $[(\varphi(0), A(0))]$, to which we associate the squared length

$$\int_\Sigma \left(|P(\dot{\varphi}(0), \dot{A}(0))|^2\right) \tag{2.11}$$

where P denotes projection L^2 orthogonal to the gauge orbit through $(\varphi(0), A(0))$. No explicit formula for the L^2 metric is known (except in the trivial case that $n = 1$ and Σ is round), but it is known to be kähler, and a formula for its kähler class was obtained by Baptista [3]. From this, it follows that the volume of M_n is

$$|M_n| = \frac{\pi^n}{n!}(\tau|\Sigma| - 4\pi n)^n = \frac{\pi^n \varepsilon^n}{n!}, \tag{2.12}$$

which vanishes in the Bradlow limit. Hence g degenerates in this limit so, to discuss its convergence, we must rescale. We define the normalized L^2 metric on M_n to be

$$g_\varepsilon := \frac{1}{\varepsilon} g. \tag{2.13}$$

This is a one-parameter family of metrics on M_n of fixed volume $\pi^n/n!$. The purpose of this paper is to establish

Theorem 1. *In the limit $\varepsilon \searrow 0$, the normalized L^2 metric g_ε converges in C^0 to g_0, the Fubini–Study metric on M_n.*

A result of this kind was conjectured by Baptista and Manton in the case that Σ is round, motivated by the approximation of vortices by what they called *pseudo-vortices* [4]. These are pairs $(\varphi, \widehat{A})$ consisting of a connexion of constant curvature, and a section holomorphic with respect to $\overline{\partial}_{\widehat{A}}$ satisfying a normalization condition. Our proof proceeds by generalizing this approximation to arbitrary spheres, and rigorously controlling its errors.

3. Convergence of Vortices to Pseudo-Vortices

Since we are concerned with the limit $\varepsilon \searrow 0$, we assume henceforth that

$$\frac{4\pi n}{|\Sigma|} < \tau < \frac{1 + 4\pi n}{|\Sigma|}, \tag{3.1}$$

so $\varepsilon \in (0, 1)$. Recall we have fixed a connexion $\widehat{A}$ on L of constant curvature $*F_{\widehat{A}} = 2\pi n/|\Sigma|$, and used this to equip L with a holomorphic structure. Given a divisor $D \in \mathrm{Sym}_n\Sigma$, let us denote by $\widehat{\varphi}_D \in H^0(L)$ a holomorphic section of L with $\|\widehat{\varphi}_D\|_{L^2} = 1$ and $\widehat{\varphi}^{-1}(0) = D$. This is unique up to multiplication by a constant in $U(1)$. Let us define the *pseudo-vortex* corresponding to the divisor D to be the pair $(\sqrt{\varepsilon}\widehat{\varphi}_D, \widehat{A})$, unique up to $\widehat{\varphi}_D \mapsto e^{ic}\widehat{\varphi}_D$ where $c \in \mathbb{R}$ is constant. This pair satisfies the first vortex Eq. (2.4), but not the second (2.5). It does satisfy (2.5) "on average," however, that is, it satisfies the constraint (2.6) obtained by integrating (2.5) over Σ. This observation led Baptista and Manton to conjecture that, for small ε, the actual vortex with divisor D should be well approximated by its corresponding pseudo-vortex [4]. In this section we will prove that this intuition is correct in the following precise sense:

Proposition 2. *There exists a constant $C > 0$, depending only on (L, h) and g_Σ, such that, for all vortex solutions $[(\varphi, A)] \in M_n$,*

$$\|\varepsilon^{-1/2}|\varphi| - |\widehat{\varphi}_D|\|_{C^0} \leq C\varepsilon, \qquad \|F_A - F_{\widehat{A}}\|_{C^0} \leq C\varepsilon,$$

where $D = (\varphi)$ and $\|\cdot\|_{C^0}$ denotes the sup *norm.*

The proof relies on the following observation. Choose and fix D, some effective divisor of degree n on Σ. Then any section φ of L with $(\varphi) = D$ has a unique representative in its gauge equivalence class of the form

$$\varphi = \sqrt{\varepsilon}e^{u/2}\widehat{\varphi}_D \tag{3.2}$$

where $u : \Sigma \to \mathbb{R}$ is a smooth function. If we define the connexion A to be

$$A = \widehat{A} - \frac{1}{2} * \mathrm{d}u, \tag{3.3}$$

then the pair (φ, A) satisfies the first vortex Eq. (2.4) [9]. The connexion A has curvature

$$*F_A = *F_{\widehat{A}} - \frac{1}{2} * \mathrm{d} * \mathrm{d}u = \frac{2\pi n}{|\Sigma|} + \frac{1}{2}\Delta u. \tag{3.4}$$

Hence, (φ, A) satisfies the second vortex Eq. (2.5) if and only if

$$\Delta u - \frac{\varepsilon}{|\Sigma|} + \varepsilon|\widehat{\varphi}_D|^2 e^u = 0. \tag{3.5}$$

It follows from the main theorem of [7], or by direct appeal to [11], that the solution to (3.5) exists and is unique and smooth. In this way we reduce the problem of constructing the vortex with divisor D to a semilinear elliptic PDE for u.

The error in the approximation

$$(\varphi, A) = (\sqrt{\varepsilon}e^{u/2}\widehat{\varphi}_D, \widehat{A} - \frac{1}{2} * \mathrm{d}u) \approx (\sqrt{\varepsilon}\widehat{\varphi}_D, \widehat{A}) \tag{3.6}$$

is controlled by u, and hence we seek bounds on this function. We will use four different norms on $C^\infty(\Sigma)$, namely,

$$\begin{aligned} \|f\|_{L^2}^2 &:= \int_\Sigma |f|^2 \\ \|f\|_{H^1}^2 &:= \int_\Sigma (|f|^2 + |\mathrm{d}f|^2) \\ \|f\|_{H^2}^2 &:= \int_\Sigma (|f|^2 + |\mathrm{d}f|^2 + |\nabla \mathrm{d}f|^2) \\ \|f\|_{C^0} &:= \sup\{|f(x)| : x \in \Sigma\} \end{aligned} \tag{3.7}$$

and will denote by $\langle \cdot, \cdot \rangle_{L^2}$ the L^2 inner product. In the defintion of the H^2 norm, ∇ denotes the Levi-Civita connexion associated to g_Σ. We will also make frequent use of two standard analytic facts which, for convenience, we collect here [1,8]:

Proposition 3. *There exists a constant $c > 0$, depending only on g_Σ, such that*

1. *For all $f \in C^\infty(\Sigma)$, $\|f\|_{C^0} \leq c\|f\|_{H^2}$ (Sobolev embedding);*
2. *For all $f \in C^\infty(\Sigma)$ with $\int_\Sigma f = 0$, $\|f\|_{H^2} \leq c\|\Delta f\|_{L^2}$ (standard elliptic estimate for Δ).*

Proposition 2 will quickly follow once we have shown that $\|u\|_{C^0} \leq C\varepsilon$ uniformly in D. We begin by bounding the average of u in terms of its zero-average component.

Lemma 4. *Let u be a solution of (3.5) and $u_0 = u - \overline{u}$, where*

$$\overline{u} := \frac{1}{|\Sigma|}\int_\Sigma u.$$

Then $|\overline{u}| \leq \|u_0\|_{C^0}$.

Proof. Substituting $u = \overline{u} + u_0$ into (3.5) and integrating over Σ we find (by the Divergence Theorem)

$$0 - \varepsilon + \varepsilon \int_\Sigma |\widehat{\varphi}_D|^2 e^{\overline{u}} e_0^u = 0, \tag{3.8}$$

which may be solved for $\overline{u}$,

$$\overline{u} = -\log\left(\int_\Sigma |\widehat{\varphi}_D|^2 e^{u_0}\right). \tag{3.9}$$

Now $-\|u_0\|_{C^0} \leq u_0 \leq \|u_0\|_{C^0}$ and exp is increasing, so

$$e^{-\|u_0\|_{C^0}} = e^{-\|u_0\|_{C^0}} \int_\Sigma |\widehat{\varphi}_D|^2 \leq \int_\Sigma |\widehat{\varphi}_D|^2 e^{u_0} \leq e^{\|u_0\|_{C^0}} \int_\Sigma |\widehat{\varphi}_D|^2 = e^{\|u\|_{C^0}}, \tag{3.10}$$

where we have used the fact that $\|\widehat{\varphi}_D\|_{L^2} = 1$. But log is also increasing, so

$$-\|u_0\|_{C^0} \leq \log\left(\int_\Sigma |\widehat{\varphi}_D|^2 e^{u_0}\right) \leq \|u_0\|_{C^0} \tag{3.11}$$

whence the claim immediately follows. □

This observation quickly yields a crude bound on $\|u\|_{C^0}$.

Lemma 5. *There exists $C > 0$, depending only on n and g_Σ, such that, for all D, the solution u of (3.5) satisfies $\|u\|_{C^0} \leq C$.*

Proof. As before, let $u = \overline{u} + u_0$ with $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$. Now, by (3.4),

$$\begin{aligned}\frac{1}{4}\|\Delta u_0\|_{L^2}^2 &= \| * F_A - 2\pi n |\Sigma|^{-1}\|_{L^2}^2 \\ &= \|F_A\|_{L^2}^2 - \frac{(2\pi n)^2}{|\Sigma|} \\ &< \|F_A\|_{L^2}^2 \\ &\leq 2E(\varphi, A) \\ &= 2\pi\tau n \leq 2\pi n(1 + 4\pi n)/|\Sigma|. \end{aligned}\tag{3.12}$$

Hence, $\|\Delta u_0\|_{L^2} \leq C_1$, some constant depending only on n and g_Σ. Now $\int_\Sigma u_0 = 0$ so, by the standard elliptic estimate for Δ (Proposition 3),

$$\|u_0\|_{H^2} \leq c\|\Delta u_0\|_{L^2} \leq cC_1, \tag{3.13}$$

and hence, by the Sobolev embedding (Proposition 3),

$$\|u_0\|_{C^0} \leq c^2 C_1. \tag{3.14}$$

Hence, by Lemma 4,

$$\|u\|_{C^0} \leq |\overline{u}| + \|u_0\|_{C^0} \leq 2\|u_0\|_{C^0} \leq 2c^2 C_1. \tag{3.15}$$

□

Equation (3.5) allows us to improve this crude bound.

Lemma 6. *There exists $C > 0$, depending only on (L, h) and g_Σ, such that, for all D, the solution u of (3.5) satisfies $\|u\|_{C^0} \leq C\varepsilon$.*

Proof. Once again, by Proposition 3 and Lemma 4, it suffices to show that

$$\|\Delta u_0\|_{L^2} \leq C\varepsilon, \tag{3.16}$$

where u_0 is the zero-average part of u. It is convenient to define

$$\alpha := \sup\{|\widehat{\varphi}_D(p)| : (p, D) \in \Sigma \times \mathrm{Sym}_n \Sigma\}, \tag{3.17}$$

noting that this number certainly exists, by continuity and compactness of Σ, and is, by definition, dependent only on (L, h) and g_Σ. Now, by Eq. (3.5),

$$\|\Delta u_0\|_{L^2} \leq \frac{\varepsilon}{|\Sigma|^{1/2}} + \varepsilon\alpha^2 e^{\|u\|_{C^0}} |\Sigma|^{1/2} \leq C\epsilon, \tag{3.18}$$

by Lemma 5. □

Proof of Proposition 2. By definition of u,

$$\frac{\varphi}{\sqrt{\varepsilon}} - \widehat{\varphi}_D = \widehat{\varphi}_D(1 - e^{u/2}). \tag{3.19}$$

By the mean value theorem, for each $x \in \Sigma$, there exists $q(x)$ between $u(x)$ and 0 such that

$$e^{u(x)/2} - e^{0/2} = \frac{1}{2}e^{q(x)/2}(u(x) - 0). \tag{3.20}$$

Hence

$$\|1 - e^{u/2}\|_{C^0} \leq \frac{1}{2}e^{\|u\|_{C^0}/2}\|u\|_{C^0}, \tag{3.21}$$

and so,

$$\left\|\frac{\varphi}{\sqrt{\varepsilon}} - \widehat{\varphi}_D\right\|_{C^0} \leq \frac{1}{2}\alpha\|u\|_{C^0}e^{\|u\|_{C^0}/2}, \tag{3.22}$$

where α is the constant defined in (3.17). Similarly, $*(F_A - F_{\widehat{A}}) = \Delta u/2$, so by (3.5),

$$\|F_A - F_{\widehat{A}}\|_{C^0} \leq \frac{\varepsilon}{2}\left(\frac{1}{|\Sigma|} + \alpha^2 e^{\|u\|_{C^0}}\right). \tag{3.23}$$

The claimed inequalities now follow immediately from Lemma 6. □

4. Convergence of the Metric

Consider a smooth curve of solutions of the vortex equations, $(\varphi(t), A(t))$, and denote by v the tangent vector to M_n at $[(\varphi(0), A(0))]$ that it generates. Given any such curve, there exists a smooth curve $\widehat{\varphi}(t)$ in the unit sphere $S \subset (H^0(L), \|\cdot\|_{L^2})$ such that

$$\varphi(t) = \sqrt{\varepsilon}\widehat{\varphi}(t)e^{\frac{1}{2}u(t)+i\chi(t)} \tag{4.1}$$

$$A(t) = \widehat{A} + \frac{1}{2} * du(t) + \mathrm{d}\chi(t) \tag{4.2}$$

where $u(t)$, $\chi(t)$ are smooth curves in $C^\infty(\Sigma)$ and, at each time t, $u(t)$ satisfies the PDE (3.5). We may choose $\chi(t)$ freely: changing it merely gauge transforms each solution $(\varphi(t), A(t))$. Note that $\|\widehat{\varphi}(t)\|^2_{L^2} \equiv 1$, so

$$\langle\widehat{\varphi}(t), \dot{\widehat{\varphi}}(t)\rangle_{L^2} \equiv 0, \tag{4.3}$$

where here and henceforth an overdot denotes differentiation with respect to t. Without loss of generality, we may also assume that

$$\langle i\widehat{\varphi}(t), \dot{\widehat{\varphi}}(t)\rangle_{L^2} \equiv 0. \tag{4.4}$$

If not, we gauge transform

$$\widehat{\varphi}(t) \mapsto e^{ic(t)}\widehat{\varphi}(t) \tag{4.5}$$

by a suitable curve of space-independent functions $c(t)$ and redefine $\chi(t) \mapsto \chi(t) - c(t)$. This is convenient because $\widehat{\varphi}(t)$ then moves orthogonal to the fibres of the Hopf fibration, so by definition of the Fubini–Study metric,

$$g_0(v, v) = g_{FS}([\dot{\widehat{\varphi}}(0)], [\dot{\widehat{\varphi}}(0)]) = \|\dot{\widehat{\varphi}}(0)\|^2_{L^2}. \tag{4.6}$$

This, then, is the squared length assigned to v by the Fubini–Study metric g_0. We wish to compare this with the squared length of the same vector with respect to the L^2 metric. For this purpose, it is convenient to choose $\chi(t)$ so that $(\dot{\varphi}(0), \dot{A}(0))$ is L^2 orthogonal to the gauge orbit through $(\varphi(0), A(0))$, and hence $P(\dot{\varphi}(0), \dot{A}(0)) = (\dot{\varphi}(0), \dot{A}(0))$. We may take $\chi(0) = 0$. A general gauge transform is

$$(\varphi, A) \mapsto (e^{is}\varphi, A + \mathrm{d}s), \qquad s \in C^\infty(\Sigma), \tag{4.7}$$

so a general tangent vector to the gauge orbit through (φ, A) takes the form

$$v = (is\varphi, \mathrm{d}s) \in \Gamma(L) \oplus \Omega^1(\Sigma), \qquad s \in C^\infty(\Sigma). \tag{4.8}$$

The tangent vector to our curve is

$$(\dot{\varphi}(0), \dot{A}(0)) = (\sqrt{\varepsilon}(\dot{\widehat{\varphi}}(0) + \widehat{\varphi}(0)(\frac{1}{2}\dot{u}(0) + i\dot{\chi}(0)))e^{u(0)/2}, \frac{1}{2} * \mathrm{d}\dot{u}(0) + \mathrm{d}\dot{\chi}(0)). \tag{4.9}$$

Hence we require that, for all $s \in C^\infty(\Sigma)$,

$$\begin{aligned}
0 &= \langle is\sqrt{\varepsilon}\widehat{\varphi}(0)e^{u(0)/2}, \sqrt{\varepsilon}(\dot{\widehat{\varphi}}(0) + \widehat{\varphi}(0)(\frac{1}{2}\dot{u}(0) + i\dot{\chi}(0))e^{u(0)/2})\rangle_{L^2} \\
&\quad + \langle \mathrm{d}s, \frac{1}{2} * \mathrm{d}\dot{u}(0) + \mathrm{d}\dot{\chi}(0)\rangle_{L^2} \\
&= \int_\Sigma s\varepsilon e^{u(0)} h(i\widehat{\varphi}(0), \dot{\widehat{\varphi}}(0) + \frac{1}{2}\dot{u}(0)\widehat{\varphi}(0) + i\dot{\chi}(0)\widehat{\varphi}(0)) + \langle s, \frac{1}{2}\delta * \mathrm{d}\dot{u}(0) + \delta\mathrm{d}\dot{\chi}(0)\rangle_{L^2} \\
&= \langle s, \Delta\dot{\chi}(0) + \varepsilon|\widehat{\varphi}(0)|^2\dot{\chi}(0) + \varepsilon e^{u(0)} h(i\widehat{\varphi}(0), \dot{\widehat{\varphi}}(0))\rangle_{L^2}.
\end{aligned} \tag{4.10}$$

Thus, the gauge orthogonality condition is

$$\Delta\dot{\chi}(0) + \varepsilon e^{u(0)}|\widehat{\varphi}(0)|^2\dot{\chi}(0) = -\varepsilon e^{u(0)} h(i\widehat{\varphi}(0), \dot{\widehat{\varphi}}(0)). \tag{4.11}$$

Since $u(t)$ satisfies (3.5) at each t, we find, by differentiating with respect to t, that

$$\Delta\dot{u}(0) + \varepsilon e^{u(0)}|\widehat{\varphi}(0)|^2\dot{u}(0) = -2\varepsilon e^{u(0)} h(\widehat{\varphi}(0), \dot{\widehat{\varphi}}(0)). \tag{4.12}$$

The conditions on our curve of solutions imply that $\dot{\chi}(0)$ and $\dot{u}(0)$ satisfy the driven linear PDEs (4.11) and (4.12), and, conversely, given $(\widehat{\varphi}(0), \dot{\widehat{\varphi}}(0))$, these PDEs uniquely determine $\dot{\chi}(0)$ and $\dot{u}(0)$. From now on, we restrict attention to the time $t = 0$ and drop the parameter t from our notation.

The squared length of the velocity vector with respect to the L^2 metric is

$$\begin{aligned}
g(v, v) &= \|\dot{\varphi}\|_{L^2}^2 + \|\dot{A}\|_{L^2}^2 \\
&= \varepsilon \int_\Sigma e^u \left\{ |\dot{\widehat{\varphi}}|^2 + h(\dot{\widehat{\varphi}}, \widehat{\varphi})\dot{u} + 2h(\dot{\widehat{\varphi}}, i\widehat{\varphi})\dot{\chi} + |\widehat{\varphi}|^2 \left(\frac{\dot{u}^2}{4} + \dot{\chi}^2 \right) \right\} \\
&\quad + \frac{1}{4}\|\mathrm{d}\dot{u}\|_{L^2}^2 + \|\mathrm{d}\dot{\chi}\|_{L^2}^2.
\end{aligned} \tag{4.13}$$

Our aim is to show that the normalized L^2 metric $g_\varepsilon = \varepsilon^{-1}g$ satisfies a bound of the form

$$|g_\varepsilon(v, v) - \|\dot{\widehat{\varphi}}\|_{L^2}^2| \leq C\varepsilon\|\dot{\widehat{\varphi}}\|_{L^2}^2, \tag{4.14}$$

where C is independent of v. Comparing with (4.13) it is clear that we must control suitable norms of $\dot{u}$ and $\dot{\chi}$.

Note that both $v = \dot{\chi}$ and $v = \dot{u}/2$ satisfy a PDE of the form

$$\Delta v + av = b \tag{4.15}$$

where $a, b \in C^\infty(\Sigma)$,

$$a = \varepsilon|\widehat{\varphi}|^2 e^u, \qquad b = \begin{cases} -\varepsilon e^u h(i\widehat{\varphi}, \dot{\widehat{\varphi}}), & v = \dot{\chi}, \\ -\varepsilon e^u h(\widehat{\varphi}, \dot{\widehat{\varphi}}), & v = \dot{u}/2. \end{cases} \tag{4.16}$$

An obvious strategy at this point would be to use the standard elliptic estimate for the operator $\Delta + a$ to bound $\|v\|_{H^2}$ in terms of $\|b\|_{L^2}$. Indeed, since $a \geq 0$ and vanishes only on a finite set, it is clear that the kernel of $\Delta + a$ is trivial, so a bound of the form $\|v\|_{H^2} \leq C(a)\|b\|_{L^2}$ is immediately available. The problem with this approach is that the dependence of the coercivity constant $C(a)$ on the function a (hence on $\widehat{\varphi}$) is unknown, so it does not yield a uniform bound as we require.

To proceed further, we appeal to an estimate obtained from the Lax-Milgram Lemma which gives a bound on $\|v\|_{H^1}$ (not $\|v\|_{H^2}$) in terms of a and b. This estimate is likely to be useful in a wide variety of contexts, so we state and prove it in some generality. At first sight, the bound looks (perhaps needlessly) elaborate, and one is tempted to replace the right hand side by a simpler but less sharp expression. Our argument will require the full detail of the bound as stated, however.

Lemma 7. *Let M be a compact Riemannian manifold. Then there exists a constant $C > 0$, depending only on M, such that, for all $a, b, v \in C^\infty(M)$ with $a \geq 0$ and $\int_M a > 0$, if*

$$\Delta v + av = b,$$

then

$$\|v\|_{H^1} \leq C\left\{\left(1 + \frac{\|a\|_{L^2}}{\int_M a}\right)\left(\|b\|_{L^2} + \frac{\|a\|_{L^2}}{\int_M a}\left|\int_M b\right|\right) + \frac{|\int_M b|}{\int_M a}\right\}.$$

Proof. Decompose $v = \overline{v} + v_0$ where

$$\overline{v} = \frac{1}{|M|}\int_M v, \tag{4.17}$$

so $v_0 \in \mathscr{X} = \{f \in H^1 : \int_M f = 0\}$. We note that $\mathscr{X}$ is a Hilbert space with respect to the H^1 inner product. Substituting (4.17) into the PDE for v we find that

$$\Delta v_0 + av_0 = b - a\overline{v}, \tag{4.18}$$

which, on integration over M yields,

$$\overline{v} = \frac{\int_M b - \langle a, v_0\rangle_{L^2}}{\int_M a}. \tag{4.19}$$

Substituting (4.19) back into (4.18), one finds that

$$\Delta v_0 + a\left\{v_0 - \frac{\langle a, v_0\rangle_{L^2}}{\int_M a}\right\} = \left\{b - \frac{\int_M b}{\int_M a}a\right\}. \tag{4.20}$$

It is to (4.20) that we apply the Lax-Milgram Lemma.

Define the bilinear and linear forms

$$\mathcal{A} : \mathscr{X} \times \mathscr{X} \to \mathbb{R}, \qquad \mathcal{A}(p, q) = \langle \mathrm{d}p, \mathrm{d}q \rangle_{L^2} + \langle ap, q \rangle_{L^2} - \frac{1}{\int_M a} \langle a, p \rangle_{L^2} \langle a, q \rangle_{L^2} \tag{4.21}$$

$$\mathcal{B} : \mathscr{X} \to \mathbb{R}, \qquad \mathcal{B}(q) = \langle b - \frac{\int_M b}{\int_M a} a, q \rangle_{L^2}. \tag{4.22}$$

Then, since v_0 satisfies (4.20), for all $q \in \mathscr{X}$,

$$\mathcal{A}(v_0, q) = \mathcal{B}(q). \tag{4.23}$$

But $\mathcal{A}$ is *coercive* (with respect to the H^1 norm on $\mathscr{X}$) with coercivity constant $\lambda_1/(1+\lambda_1)$, where $\lambda_1 > 0$ is the smallest positive eigenvalue of Δ, that is, for all $q \in \mathscr{X}$,

$$\mathcal{A}(q, q) \geq \frac{\lambda_1}{1+\lambda_1} \|q\|_{H^1}^2. \tag{4.24}$$

To see this, note that

$$\begin{aligned} \mathcal{A}(q, q) &= \|\mathrm{d}q\|_{L^2}^2 + \|\sqrt{a}q\|_{L^2}^2 - \frac{1}{\int_M a} \langle \sqrt{a}, \sqrt{a}q \rangle_{L^2}^2 \\ &\geq \|\mathrm{d}q\|_{L^2}^2 + \|\sqrt{a}q\|_{L^2}^2 - \frac{1}{\int_M a} \|\sqrt{a}\|_{L^2}^2 \|\sqrt{a}q\|_{L^2}^2 \\ &= \|\mathrm{d}q\|_{L^2} \end{aligned} \tag{4.25}$$

by Cauchy-Schwarz. But $\|\mathrm{d}q\|_{L^2}^2 \geq \lambda_1 \|q\|_{L^2}^2$ by the Poincaré inequality, so

$$\|q\|_{H^1}^2 = \|\mathrm{d}q\|_{L^2}^2 + \|q\|_{L^2}^2 \leq (1 + \lambda_1^{-1}) \|\mathrm{d}q\|_{L^2}^2, \tag{4.26}$$

and (4.24) immediately follows.

Hence, by the Lax-Milgram Lemma [10, p. 78],

$$\|v_0\|_{H^1} \leq \frac{1+\lambda_1}{\lambda_1} \|\mathcal{B}\|_{\mathscr{X}^*} \tag{4.27}$$

where

$$\begin{aligned} \|\mathcal{B}\|_{\mathscr{X}^*} &= \sup_{q \in \mathscr{X}, \|q\|_{H^1}=1} |\mathcal{B}(q)| \\ &= \sup_{q \in \mathscr{X}, \|q\|_{H^1}=1} \left| \langle b - \frac{\int_M b}{\int_M a} a, q \rangle_{L^2} \right| \\ &\leq \sup_{q \in \mathscr{X}, \|q\|_{H^1}=1} \|q\|_{L^2} \left\| b - \frac{\int_M b}{\int_M a} a \right\|_{L^2} \\ &\leq \left\| b - \frac{\int_M b}{\int_M a} a \right\|_{L^2} \\ &\leq \|b\|_{L^2} + \frac{\|a\|_{L^2}}{\int_M a} \left| \int_M b \right|. \end{aligned} \tag{4.28}$$

But then, by (4.19),

$$|\overline{v}| \leq \frac{|\int_M b|}{\int_M a} + \frac{\|a\|_{L^2}}{\int_M a}\|v_0\|_{L^2}, \tag{4.29}$$

so

$$\begin{aligned}\|v\|_{H^1} &= \|v_0 + \overline{v}\|_{H^1} \leq \|v_0\|_{H^1} + \sqrt{|M|}|\overline{v}| \\ &\leq \left(1 + \sqrt{|M|}\frac{\|a\|_{L^2}}{\int_M a}\right)\|v_0\|_{H^1} + \sqrt{|M|}\frac{|\int_M b|}{\int_M a}.\end{aligned} \tag{4.30}$$

Defining $C := (1 + \lambda_1^{-1})\max\{1, \sqrt{|M|}\}$, the claimed estimate now follows from inequalities (4.27), (4.28) and (4.30). □

We next apply Lemma 7 to the PDEs for $\dot{\chi}, \dot{u}$, (4.11), (4.12), to obtain order ε bounds on their H^1 norms. The last term in the inequality in Lemma 7, namely $|\int_\Sigma b|/\int_\Sigma a$, looks fatal to this endeavour, since both a and b are of order ε. It will turn out, however, that $\int_\Sigma b$ is actually of order ε^2 due to subtle cancellations in the integral.

Proposition 8. *There exists a constant $C > 0$, depending only on (L, h) and g_Σ, such that*

$$\|\dot{\chi}\|_{H^1},\ \|\dot{u}\|_{H^1} \leq C\varepsilon\|\dot{\widehat{\varphi}}\|_{L^2}.$$

Proof. Throughout this proof, c will denote a positive constant depending (at most) on g_Σ and (L, h). The value of c may vary from line to line. Let $a = \varepsilon|\widehat{\varphi}|^2 e^u \geq 0$, as defined in (4.16). Since $\widehat{\varphi}^{-1}(0)$ is finite, it is clear that $\int_\Sigma a > 0$. By Lemmas 5 and 6 and the Mean Value Theorem,

$$e^u = 1 + U \tag{4.31}$$

where $\|U\|_{C^0} \leq \|u\|_{C^0} e^{\|u\|_{C^0}} \leq c\varepsilon$. Hence

$$\begin{aligned}\int_\Sigma a &= \varepsilon\int_\Sigma |\widehat{\varphi}|^2(1 + U) \\ &= \varepsilon + \varepsilon\int_\Sigma |\widehat{\varphi}|^2 U \\ &\geq \varepsilon - \varepsilon\|U\|_{C^0}\int_\Sigma |\widehat{\varphi}|^2 \\ &\geq \varepsilon - c\varepsilon^2 \geq c\varepsilon.\end{aligned} \tag{4.32}$$

Furthermore,

$$\begin{aligned}\|a\|_{L^2}^2 &= \varepsilon^2\int_\Sigma |\widehat{\varphi}|^4 e^{2u} \\ &\leq 2\varepsilon^2\int_\Sigma |\widehat{\varphi}|^4(1 + U^2) \\ &\leq 2\varepsilon^2\alpha^2(1 + \|U\|_{C^0}^2) \\ &\leq c\varepsilon^2.\end{aligned} \tag{4.33}$$

Consider now the function b defined in (4.16). Note that

$$b = -\varepsilon e^u k, \qquad k := \begin{cases} h(i\widehat{\varphi}, \dot{\widehat{\varphi}}), & v = \dot{\chi}, \\ h(\widehat{\varphi}, \dot{\widehat{\varphi}}), & v = \dot{u}/2, \end{cases} \tag{4.34}$$

and, crucially, in either case $\int_\Sigma k = 0$ by (4.3) or (4.4). Hence

$$\int_\Sigma b = \varepsilon \int_\Sigma (1+U)k = \varepsilon \int_\Sigma Uk, \tag{4.35}$$

and so

$$\begin{aligned} \left| \int_\Sigma b \right| &\le \varepsilon \|U\|_{C^0} \int_\Sigma |k| \\ &\le \varepsilon \|U\|_{C^0} \int_\Sigma |\widehat{\varphi}||\dot{\widehat{\varphi}}| \\ &\le \varepsilon \|U\|_{C^0} \|\widehat{\varphi}\|_{L^2} \|\dot{\widehat{\varphi}}\|_{L^2} \\ &= \varepsilon \|U\|_{C^0} \|\dot{\widehat{\varphi}}\|_{L^2} \\ &\le c\varepsilon^2 \|\dot{\widehat{\varphi}}\|_{L^2}. \end{aligned} \tag{4.36}$$

Furthermore

$$\begin{aligned} \|b\|_{L^2} &= \varepsilon \|e^u f\|_{L^2} \\ &\le \varepsilon e^{\|u\|_{C^0}} \|\widehat{\varphi}\|_{C^0} \|\dot{\widehat{\varphi}}\|_{L^2} \\ &\le c\varepsilon \|\dot{\widehat{\varphi}}\|_{L^2} \end{aligned} \tag{4.37}$$

by Lemma 5. Hence

$$\frac{\|a\|_{L^2}}{\int_\Sigma a} \le c, \qquad \text{and} \qquad \frac{|\int_\Sigma b|}{\int_\Sigma a} \le c\varepsilon \|\dot{\widehat{\varphi}}\|_{L^2}. \tag{4.38}$$

Hence, by Lemma 7, both $v = \dot{\chi}$ and $v = \dot{u}/2$ satisfy

$$\begin{aligned} \|v\|_{H^1} &\le c \left(\|b\|_{L^2} + |\int_\Sigma b| + \frac{|\int_\Sigma b|}{\int_\Sigma a} \right) \\ &\le c\varepsilon \|\dot{\widehat{\varphi}}\|_{L^2}, \end{aligned} \tag{4.39}$$

whence the claim follows. □

Proposition 9. *There exists a constant $C > 0$, depending only on (L, h) and g_Σ, such that*

$$\left| \|\dot{\varphi}\|_{L^2}^2 + \|\dot{A}\|_{L^2}^2 - \varepsilon \|\dot{\widehat{\varphi}}\|_{L^2}^2 \right| \le C\varepsilon^2 \|\dot{\widehat{\varphi}}\|_{L^2}^2.$$

Proof. Again, c will denote a positive constant depending (at most) on g_Σ and (L, h), whose value may change from line to line. Differentiating (4.1) and (4.2) with respect to t, we see that

$$\|\dot{\varphi}\|_{L^2}^2 = \varepsilon \|\dot{\widehat{\varphi}}\|_{L^2}^2 + \varepsilon \int_\Sigma U |\dot{\widehat{\varphi}}|^2 + \varepsilon \int_\Sigma \left\{ h(\dot{\widehat{\varphi}}, (\dot{u} + 2i\dot{\chi})\widehat{\varphi}) + \left(\frac{1}{4}\dot{u}^2 + \dot{\chi}^2 \right) |\widehat{\varphi}|^2 \right\} e^u$$

$$\|\dot{A}\|_{L^2}^2 = \frac{1}{4}\|\mathrm{d}\dot{u}\|_{L^2}^2 + \|\mathrm{d}\dot{\chi}\|_{L^2}^2, \tag{4.40}$$

where, as before, $e^u =: 1 + U$ with $\|U\|_{C^0} \leq c\varepsilon$. Hence

$$\begin{aligned}\left|\|\dot{\varphi}\|_{L^2}^2 + \|\dot{A}\|_{L^2}^2 - \varepsilon\|\dot{\widehat{\varphi}}\|_{L^2}^2\right| &\leq \varepsilon\|U\|_{C^0}\|\dot{\widehat{\varphi}}\|_{L^2}^2 + \varepsilon\|\widehat{\varphi}\|_{C^0}e^{\|u\|_{C^0}}\|\dot{\widehat{\varphi}}\|_{L^2}(\|\dot{u}\|_{L^2} + 2\|\dot{\chi}\|_{L^2}) \\ &\quad + \varepsilon e^{\|u\|_{C^0}}\|\widehat{\varphi}\|_{C^0}^2\left(\frac{1}{4}\|\dot{u}\|_{L^2}^2 + \|\dot{\chi}\|_{L^2}^2\right) + \frac{1}{4}\|\dot{u}\|_{H^1}^2 + \|\dot{\chi}\|_{H^1}^2 \\ &\leq c\varepsilon^2 + \varepsilon\alpha c\|\dot{\widehat{\varphi}}\|_{L^2}3c\varepsilon\|\dot{\widehat{\varphi}}\|_{L^2} + \varepsilon c\alpha^2 3C^2\varepsilon^2\|\dot{\widehat{\varphi}}\|_{L^2}^2 \\ &\quad + \frac{3}{4}c^2\varepsilon^2\|\dot{\widehat{\varphi}}\|_{L^2}^2 = c\varepsilon^2\|\dot{\widehat{\varphi}}\|_{L^2}^2,\end{aligned} \tag{4.41}$$

where we have used Lemma 5 to bound $e^{\|u\|_{C^0}}$ and Proposition 8. □

Our main theorem immediately follows.

Proof of Theorem 1. We must show that the symmetric bilinear form $\sigma_\varepsilon := g_\varepsilon - g_0$ converges uniformly to zero, as $\varepsilon \searrow 0$, on the unit tangent bundle of any fixed reference metric for M_n.

Given any tangent vector $X \in T_{[(\varphi,A)]}M_n$ we may choose a representative curve $(\varphi(t), A(t))$ of the form (4.1), (4.2) to compute

$$g_\varepsilon(X, X) = \frac{1}{\varepsilon}(\|\dot{\varphi}\|_{L^2}^2 + \|\dot{A}\|_{L^2}^2), \tag{4.42}$$

and

$$g_0(X, X) = \|\dot{\widehat{\varphi}}\|_{L^2}^2. \tag{4.43}$$

Proposition 9 implies that

$$|g_\varepsilon(X, X) - g_0(X, X)| \leq C\varepsilon g_0(X, X) \tag{4.44}$$

where the constant C is independent of X and ε. Hence, for all unit length vectors X, Y tangent to (M_n, g_0),

$$\begin{aligned}|\sigma_\varepsilon(X, Y)| &= \frac{1}{4}|\sigma_\varepsilon(X + Y, X + Y) - \sigma_\varepsilon(X - Y, X - Y)| \\ &\leq \frac{1}{4}C\varepsilon(g_0(X + Y, X + Y) + g_0(X - Y, X - Y)) \\ &= C\varepsilon.\end{aligned} \tag{4.45}$$

This establishes uniform convergence with respect to the reference metric g_0. □

5. Convergence of the Spectrum

Recall that the *spectrum* of a closed Riemannian manifold (M, g) is the spectrum of its Laplace-Beltrami operator $\Delta_g = \delta\mathrm{d}$, arranged in non-decreasing order,

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \leq \cdots, \tag{5.1}$$

a sequence of numbers that diverges to infinity. In the case of (M_n, g_ε), this spectrum has a direct physical interpretation in terms of the quantum dynamics of n-vortices. The Hamiltonian operator governing this dynamics is (conjecturally) $H = \frac{1}{2}\Delta_g$, where

$g = \varepsilon g_\varepsilon$ is the (non-normalized) L^2 metric. Hence, the energy spectrum of n quantum vortices moving on Σ is

$$E_k(\varepsilon) = \frac{\lambda_k(g_\varepsilon)}{2\varepsilon}, \qquad k = 0, 1, 2, \dots. \tag{5.2}$$

Note that, in local coordinates,

$$\Delta_{g_\varepsilon} = -g_\varepsilon^{ij}\left(\frac{\partial}{\partial x_i \partial x_j} - \Gamma_{ij}^{\varepsilon,k}\frac{\partial}{\partial x_k}\right), \tag{5.3}$$

where Γ^ε denotes the Christoffel symbols of the metric g_ε. Since Γ^ε involves derivatives of g_ε, the C^0 convergence proved in Theorem 1 does *not* imply that Δ_{g_ε} converges to Δ_{g_0}. Rather remarkably, we can still conclude, by work of Bando and Urakawa [2], that the *spectrum* of Δ_{g_ε} converges uniformly to the *spectrum* of Δ_{g_0}, in the sense that

$$\frac{\lambda_k(g_\varepsilon)}{\lambda_k(g_0)} \xrightarrow{\varepsilon \to 0} 1 \tag{5.4}$$

uniformly in k. More precisely:

Corollary 10. *Let $C > 0$ be the constant, depending only on (L, h) and g_Σ, whose existence is established in Proposition 9. Then for all $\varepsilon \in [0, 1/C)$ and all $k \geq 1$,*

$$\frac{(1 - C\varepsilon)^n}{(1 + C\varepsilon)^{n+1}} \leq \frac{\lambda_k(g_\varepsilon)}{\lambda_k(g_0)} \leq \frac{(1 + C\varepsilon)^n}{(1 - C\varepsilon)^{n+1}}.$$

Proof. Given a Riemannian metric g on M_n, and a finite dimensional subspace V of $C^\infty(M_n)$, let

$$\Lambda_g(V) := \sup\{\|\mathrm{d}f\|^2_{L^2,g}/\|f\|^2_{L^2,g} \ : \ f \in V\backslash\{0\}\}. \tag{5.5}$$

Then, by Proposition 2.1 of [2],

$$\lambda_k(g) = \inf\{\Lambda_g(V) \ : \ V < C^\infty(M_n), \ \dim V = k + 1\}, \tag{5.6}$$

the infimum of Λ_g over all $(k + 1)$-dimensional subspaces of $C^\infty(M_n)$.

At a given point $p \in M_n$, let $E_1, \dots, E_{2n}$ be an orthonormal frame for (T_pM_n, g_0), G_ε be the matrix with entries $g_\varepsilon(E_i, E_j)$, S_ε the matrix with entries $\sigma_\varepsilon(E_i, E_j)$ (where, as before, $\sigma_\varepsilon = g_\varepsilon - g_0$) and H_ε be the inverse of G_ε. Then the volume form defined by g_ε is

$$\mathrm{vol}_{g_\varepsilon} = v_\varepsilon \mathrm{vol}_{g_0}, \tag{5.7}$$

where $v_\varepsilon(p) = (\det G_\varepsilon)^{1/2}$. Now, by (4.44), every eigenvalue μ of S_ε satisfies $|\mu| \leq C\varepsilon$, so the eigenvalues $1+\mu$ of $G_\varepsilon = \mathbb{I}_{2n} + S_\varepsilon$ lie in $[1 - C\varepsilon, 1 + C\varepsilon]$. Hence, for all $\varepsilon < 1/C$,

$$(1 - C\varepsilon)^n \leq v_\varepsilon(p) \leq (1 + C\varepsilon)^n. \tag{5.8}$$

Furthermore, the eigenvalues $\nu = (1 + \mu)^{-1}$ of H_ε lie in $[(1 + C\varepsilon)^{-1}, (1 - C\varepsilon)^{-1}]$. Now defining, for any given smooth function f, v to be the column matrix with entries $\mathrm{d}f_p(E_i)$,

$$|\mathrm{d}f_p|^2_{g_\varepsilon} = v^T H_\varepsilon v, \qquad |\mathrm{d}f_p|^2_{g_0} = v^T v. \tag{5.9}$$

Hence

$$\frac{|\mathrm{d}f_p|^2_{g_0}}{(1 + C\varepsilon)} \leq \nu_{min}|\mathrm{d}f_p|^2_{g_0} \leq |\mathrm{d}f_p|^2_{g_\varepsilon} \leq \nu_{max}|\mathrm{d}f_p|^2_{g_0} \leq \frac{|\mathrm{d}f_p|^2_{g_0}}{(1 - C\varepsilon)}. \tag{5.10}$$

The bounds (5.8) and (5.10) are independent of $p \in M_n$, so hold globally. Hence, for all $f \in C^\infty(M_n)$,

$$\begin{aligned}(1-C\varepsilon)^n \|f\|^2_{L^2,g_0} \leq \ \|f\|^2_{L^2,g_\varepsilon} \ &\leq (1+C\varepsilon)^n \|f\|^2_{L^2,g_0}, \\ \frac{(1-C\varepsilon)^n}{1+C\varepsilon}\|\mathrm{d}f\|^2_{L^2,g_0} \leq \|\mathrm{d}f\|^2_{L^2,g_\varepsilon} \ &\leq \frac{(1+C\varepsilon)^n}{1-C\varepsilon}\|\mathrm{d}f\|^2_{L^2,g_0},\end{aligned} \tag{5.11}$$

and so, for all $f \neq 0$,

$$\begin{aligned}\frac{\|\mathrm{d}f\|^2_{L^2,g_\varepsilon}}{\|f\|^2_{L^2,g_\varepsilon}} &\leq \frac{(1+C\varepsilon)^n}{(1-C\varepsilon)^{n+1}}\frac{\|\mathrm{d}f\|^2_{L^2,g_0}}{\|f\|^2_{L^2,g_0}} \\ \frac{\|\mathrm{d}f\|^2_{L^2,g_0}}{\|f\|^2_{L^2,g_0}} &\leq \frac{(1+C\varepsilon)^{n+1}}{(1-C\varepsilon)^n}\frac{\|\mathrm{d}f\|^2_{L^2,g_\varepsilon}}{\|f\|^2_{L^2,g_\varepsilon}}.\end{aligned} \tag{5.12}$$

Hence, for any $(k+1)$ dimensional subspace V of $C^\infty(M_n)$,

$$\begin{aligned}\Lambda_{g_\varepsilon}(V) &\leq \frac{(1+C\varepsilon)^n}{(1-C\varepsilon)^{n+1}}\Lambda_{g_0}(V), \\ \Lambda_{g_0}(V) &\leq \frac{(1+C\varepsilon)^{n+1}}{(1-C\varepsilon)^n}\Lambda_{g_\varepsilon}(V),\end{aligned} \tag{5.13}$$

and so

$$\begin{aligned}\lambda_k(g_0) &\geq \frac{(1-C\varepsilon)^{n+1}}{(1+C\varepsilon)^n}\lambda_k(g_\varepsilon), \\ \lambda_k(g_\varepsilon) &\geq \frac{(1-C\varepsilon)^n}{(1+C\varepsilon)^{n+1}}\lambda_k(g_0),\end{aligned} \tag{5.14}$$

which completes the proof. □

The spectrum of g_0 is known explicitly [5],

$$\lambda_k(g_0) = 4k(n+k), \qquad d_k(g_0) = n(n+2k)\binom{n-1+k}{k}^2, \tag{5.15}$$

where we have changed our labelling convention so that $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ and d_k is the degeneracy of λ_k. In order to extract explicit bounds on $\lambda_k(g_\varepsilon)$ from Corollary 10, we need an explicit upper bound on the constant C. Such a bound should be obtainable when Σ is given the round metric, since explicit formulae for $\widehat{\varphi}$ are available in this case [4]. A particularly important issue, if one is to apply Corollary 10 to the study of the quantum statistical mechanics of vortices [13], is the n dependence of C.

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