ON SEMIBOUNDED EXPANSIONS OF ORDERED GROUPS

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ABSTRACT. We explore *semibounded* expansions of arbitrary ordered groups; namely, expansions that do not define a field on the whole universe. We show that if $\mathcal{R} = \langle \mathbb{R}, <, +, \ldots \rangle$ is a semibounded o-minimal structure and $P \subseteq \mathbb{R}$ is a set satisfying certain tameness conditions, then $\langle \mathcal{R}, P \rangle$ remains semibounded. Examples include the cases when $\mathcal{R} = \langle \mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}}, \cdot_{\restriction [0,1]^2} \rangle$, and $P = 2^{\mathbb{Z}}$ or P is an iteration sequence. As an application, we obtain that smooth functions definable in such $\langle \mathcal{R}, P \rangle$ are definable in $\mathcal{R}$.

1. INTRODUCTION

The work of this paper lies at the nexus of two different directions in model theory, both related to o-minimality, which so far have developed independently. The first direction is that of o-minimal *semibounded* structures, which are o-minimal structures that do not interpret a global field, and are obtained, for example, as proper *reducts* of real closed fields. These structures were extensively studied in the 90s by Marker, Peterzil, Pillay [17, 21, 23] and others, they relate to Zilber's dichotomy principle on definable groups and fields, and have continued to develop in recent years [7, 9, 22].

The second direction is that of *expansions* $\widetilde{\mathcal{R}}$ of o-minimal structures $\mathcal{R}$ which are not o-minimal, yet preserve the tame geometric behavior on the class of all definable sets. This area is much richer, originating to A. Robinson [25] and van den Dries [4, 5], it has largely expanded in the last two decades by many authors, and includes broad categories of structures, such as d-minimal expansions of o-minimal structures and expansions with o-minimal open core. Although in general $\mathcal{R}$ is only required to expand a linear order, it is often assumed to expand an ordered group or even a real closed field (and, in fact, the real field $\overline{\mathbb{R}}$).

In recent work [14], Hieronymi-Walsberg considered expansions of ordered groups and explored the dichotomy between defining or not a local field. In this paper, we consider expansions of ordered groups and explore the dichotomy between defining or not a global field (in the latter case, call $\widetilde{\mathcal{R}}$ *semibounded*). As an application, and building on the work from [11], we obtain that for certain semibounded expansions $\widetilde{\mathcal{R}}$ of o-minimal structures $\mathcal{R}$, such as $\widetilde{\mathcal{R}} = \langle \mathcal{R}, 2^{\mathbb{Z}} \rangle$ with $\mathcal{R} = \langle R, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}}, \cdot_{\restriction [0,1]^2} \rangle$, every definable smooth (that is, infinitely differentiable) function is already definable in $\mathcal{R}$.

We now collect some definitions and state our results. We assume familiarity with the basics of o-minimality, as they can be found, for example, in [6]. A standard reference for semibounded o-minimal structures is [7]. The following definition extends the usual notion of a semibounded structure to a general (not necessarily o-minimal) setting.

Definition 1.1. Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be an expansion of an ordered group. We call $\mathcal{M}$ *semibounded* if there is no definable ordered field with domain M whose order agrees with $<$.

There is a number of statements that could be adopted as definitions of a semi-bounded structure $\mathcal{M}$ and which are known to be equivalent in the o-minimal setting (see [7, Fact 1.6]). For example, one could require that there are no definable *poles* (that is, definable bijections between bounded and unbounded sets), or that $\mathcal{M}$ is an expansion of $\langle M, <, + \rangle$ by bounded sets. The latter statement is in fact the key definition in [1]. The equivalence of (suitable versions of) these statements in a general setting appears to be an open question. Our choice of Definition 1.1 in the current setting is due to the fact that it provides a priori the weakest notion (see relevant questions in Section 2.2 below).

The main focus of the current work is to establish in the semibounded setting our results 1.5 - 1.7 below. These results are in the spirit of showing that under certain topological or analytical conditions on objects definable in tame expansions of an o-minimal structure, those objects are actually already definable in the o-minimal reduct. For example, in expansions with *o-minimal open core* ([19]), open definable sets are definable in the o-minimal reduct, which was essential in [10] in showing that in expansions with dense independent sets, every definable group is definable in the o-minimal reduct. In Proposition 1.10 below, we apply our results to obtain that smooth functions definable in certain *d-minimal* expansions of o-minimal structures are again definable in the o-minimal reduct.

Our approach consists in isolating two main properties that hold in our d-minimal examples (Definitions 1.3-1.4) and prove the last statement in Theorem 1.7 under those assumptions only. Previously, in [11], we had introduced special cases of those properties for expansions of real closed fields, where we established the corresponding results, and here we extend them to the setting of expansions of ordered groups. The success of this program might perhaps be an indication that these properties capture indeed the special nature of such expansions and could potentially be a good alternative (say to d-minimality) way to look at them.

Recall that an ordered structure $\mathcal{R}$ is called *definably complete* if every bounded definable subset of its universe has a supremum (in the universe). For any set $X \subseteq R^n$, we define its *dimension* $\dim(X)$ as the maximum k such that some projection of X to k coordinates has non-empty interior, if X is non-empty, and $\dim(\emptyset) = -\infty$.

For the rest of this paper, and unless stated otherwise, we fix an o-minimal expansion $\mathcal{R} = \langle R, <, +, \ldots \rangle$ of an ordered group, and a definably complete expansion $\widetilde{\mathcal{R}} = \langle \mathcal{R}, \ldots \rangle$ of $\mathcal{R}$. By $\mathcal{L}$ we denote the language of $\mathcal{R}$. By 'definable' (respectively, $\mathcal{L}$-definable), we mean definable in $\widetilde{\mathcal{R}}$ (respectively, in $\mathcal{R}$), with parameters. By P we denote a subset of R of dimension 0.

If $\mathcal{R}$ is a real closed field, we call an $\mathcal{L}$-definable set *semialgebraic*.

We fix throughout the paper the following structures over the reals:

- $\overline{\mathbb{R}} = \langle \mathbb{R}, <, +, \cdot \rangle$, the real ordered field.
- $\mathbb{R}_{vec} = \langle \mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}} \rangle$, the real ordered vector space over $\mathbb{R}$, and
- $\mathbb{R}_{sbd} = \langle \mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}}, \cdot_{\restriction [0,1]^2} \rangle$, a semibounded structure.

We note that, by [17], $\mathbb{R}_{sbd}$ is the unique structure that lies strictly between $\mathbb{R}_{vec}$ and $\overline{\mathbb{R}}$ (in terms of their classes of definable sets).

We let $\Lambda(\mathcal{R})$ be the set of all partial $\emptyset$-definable endomorphisms of $\langle R, <, + \rangle$. Then $\mathcal{R}$ is called *nonlinear* ([16]) if it properly expands $\langle R, <, +, \Lambda(\mathcal{R}) \rangle$ (see also [7, Fact 1.12]). By [24], $\mathcal{R}$ is nonlinear if and only if it defines a real closed field on some bounded interval.

We now extend the tameness properties from [11] to the current setting.

Definition 1.2. Let $Y \subseteq X \subseteq R^n$ be two sets. We say that Y is an $\mathcal{L}$-*chunk of* X if it is an $\mathcal{L}$-definable cell, $\dim Y = \dim X$, and for every $y \in Y$, there is an open box $B \subseteq R^n$ containing y such that $B \cap X \subseteq Y$. Equivalently, Y is a relatively open $\mathcal{L}$-definable cell contained in X with $\dim Y = \dim X$.

Definition 1.3. We say that $\widetilde{\mathcal{R}}$ has the *decomposition property* (DP) if for every definable set $X \subseteq R^n$,

(I) there is an $\mathcal{L}$-definable family $\{Y_t\}_{t \in R^m}$ of subsets of R^n, and a definable set $S \subseteq R^m$ with $\dim S = 0$, such that $X = \bigcup_{t \in S} Y_t$,

(II) X contains an $\mathcal{L}$-chunk.

Definition 1.4. We say that $\widetilde{\mathcal{R}}$ has the *dimension property* (DIM) if for every $\mathcal{L}$-definable family $\{X_t\}_{t \in A}$, and definable set $S \subseteq A$ with $\dim S = 0$, we have

$$\dim \bigcup_{t \in S} X_t = \max_{t \in S} \dim X_t.$$

As mentioned earlier, (DP) and (DIM) extend the corresponding properties from [11] to the current setting.[1] In [11], we showed that if $\mathcal{R}$ is a real closed field and $\widetilde{\mathcal{R}}$ satisfies (DP) and (DIM), then $\widetilde{\mathcal{R}}$ defines no new smooth functions. We extend this theorem to the semibounded setting over the reals (Theorem 1.7 below), in two steps. First, in Section 3, we prove the following result which holds without the assumption of $\widetilde{\mathcal{R}}$ being over the reals. It ensures that $\widetilde{\mathcal{R}}$ defines no new smooth functions that are not semialgebraic.

Theorem 1.5. *Assume $\mathcal{R}$ is a nonlinear reduct of a real closed field, and $\widetilde{\mathcal{R}}$ an expansion of $\mathcal{R}$ satisfying* (DP) *and* (DIM). *Let $f : X \subseteq R^n \to R$ be a definable smooth function, with open semialgebraic domain X. Then f is semialgebraic.*

Second, in Section 4, we restrict $\mathcal{R}$ to be over the reals and let $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$. Using a result from Friedman-Miller [12] (Fact 4.3 below), we prove the following proposition. Note that here $\mathcal{R}$ is any o-minimal semibounded structure over the reals, not necessarily $\mathbb{R}_{sbd}$.

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[1]We take the opportunity to correct two misprints in [11]. First, in [11, Definition 1.1], the phrase '$\mathcal{L}$-definable' should be replaced by 'semialgebraic' (as the notation $\mathcal{L}$ was not defined there). Second, [11, Definition 1.3] should be the same with Definition 1.4 here, with $\mathcal{L}$ being the language of $\overline{\mathbb{R}}$. This is how those definitions are used in the rest of that paper.

Proposition 1.6. *Let $\mathcal{R} = \langle \mathbb{R}, <, +, \ldots \rangle$ be an o-minimal semibounded structure. Suppose $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ has* (DIM). *Then $\widetilde{\mathcal{R}}$ is semibounded.*

We can then conclude our main result.

Theorem 1.7. *Let $\mathcal{R} = \mathbb{R}_{sbd}$, and assume that $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ satisfies* (DP) *and* (DIM). *Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a definable smooth function, with open $\mathcal{L}$-definable domain X. Then f is $\mathcal{L}$-definable.*

Proof. By Theorem 1.5, f is semialgebraic. By Proposition 1.6, $\widetilde{\mathcal{R}}$ is semibounded. In particular, its reduct $\langle \mathcal{R}, f \rangle$ is semibounded. But this reduct is o-minimal, and hence, by [21, Theorem 1.4], f is definable in $\mathbb{R}_{sbd}$. □

Remark 1.8. The following assumptions of Theorem 1.7 are necessary:
(1) $\mathcal{R} = \mathbb{R}_{sbd}$ (and not any semibounded structure over the reals). Indeed, let $\mathcal{R}$ be the expansion of $\langle \mathbb{R}, <, + \rangle$ with all restricted analytic functions, and $\widetilde{\mathcal{R}} = \langle \mathcal{R}, e^{2\pi\mathbb{Z}} \rangle$. Note that, since $\mathcal{R}$ is an o-minimal expansion of $\langle \mathbb{R}, <, + \rangle$ by bounded sets, it is semibounded ([7, Fact 1.6]). Similarly to [11, Example 4.7], let $f : (0, 1) \to \mathbb{R}$ be the function $f(x) = \sin \log(1/x)$. Clearly, f is not definable in the o-minimal $\mathcal{R}$, since its zero set is an infinite discrete set. We show that f is definable in $\widetilde{\mathcal{R}}$. Let $\lambda : (0, 1) \to e^{2\pi\mathbb{Z}}$ be the function sending x to the biggest element of $e^{2\pi\mathbb{Z}}$ lower or equal than x. For every $x \in (0, 1)$, we have

$$f(x) = \sin \log(1/\lambda(x) \cdot \lambda(x)/x) = \sin \log(\lambda(x)/x).$$

But $\lambda(x)/x \in [e^{-2\pi}, 1]$, $\log([e^{-2\pi}, 1]) = [-2\pi, 0]$, and the map

$$(t, x) \to t/x : \bigcup_{t' \in (0,1)} \{t'\} \times [t', e^{2\pi}t'] \to [e^{-2\pi}, 1]$$

is definable in $\mathcal{R}$. Hence, f definable in $\widetilde{\mathcal{R}}$.
(2) f is smooth. If not, we could let $\widetilde{\mathcal{R}} = \langle \mathbb{R}_{sbd}, 2^{\mathbb{Z}} \rangle$ and f_n be the function defined in [11, Remark 4.8(3)]. Then $f_{n \upharpoonright (0,1)}$ is definable, $\mathcal{C}^{n-1}$, not $\mathcal{C}^n$.

In Section 5, we turn to examples $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ (Proposition 1.10 below) to which we can apply Theorem 1.7. The archetypical example is that of $\langle \mathbb{R}_{sbd}, 2^{\mathbb{Z}} \rangle$, but our work yields more examples. Let us recall a definition.

Definition 1.9 ([20])**.** Let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathcal{L}$-definable bijection, and f^n the n-th compositional iterate of f. We say that $\mathcal{R}$ is *f-bounded* if for every $\mathcal{L}$-definable function $g : \mathbb{R} \to \mathbb{R}$, there is $n \in \mathbb{N}$ such that ultimately $g < f^n$.

Let $c \in \mathbb{R}$ and f an $\mathcal{L}$-definable function such that $\mathcal{R}$ is f-bounded, and such that $(f^n(c))_n$ is growing and unbounded. We call such $(f^n(c))_n$ an *iteration sequence*.

Proposition 1.10. *Let $\mathcal{R} = \mathbb{R}_{sbd}$, and $\widetilde{\mathcal{R}}$ be any of the following structures:*
(1) $\langle \mathcal{R}, P \rangle$, *where P is an iteration sequence.*
(2) $\langle \mathcal{R}, \alpha^{\mathbb{Z}} \rangle$, *where $1 < \alpha \in \mathbb{R}$.*
Then every smooth definable map with open $\mathcal{L}$-definable domain is $\mathcal{L}$-definable.

We note that if we replaced $\mathbb{R}_{sbd}$ by $\mathbb{R}_{vec}$ in the above examples, the conclusion of Theorem 1.7 also holds, by Hieronymi-Walsberg [14].

Structure of the paper. In Section 2, we fix some notation and establish basic

properties for semibounded structures. In Section 3, we prove Theorem 1.5. In Section 4, we prove Proposition 1.6. In Section 5, we prove Proposition 1.10 and conclude with various open questions about extending our results further.

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2. PRELIMINARIES

In this section, we fix some notation and prove basic facts about semibounded structures. If $A, B \subseteq R$, we denote $\frac{A}{B} = \{a/b : a \in A, b \in B\}$. If $t \in R$, we write $\frac{A}{t}$ for $\frac{A}{\{t\}}$. By a k-cell, we mean a cell of dimension k. If $S \subseteq R^n$ is a set, its closure is denoted by $\overline{S}$, with sole exception $\overline{\mathbb{R}}$, which denotes the real field. By an open box $B \subseteq R^n$, we mean a set of the form

$$B = (a_1, b_1) \times \ldots \times (a_n, b_n),$$

for some $a_i < b_i \in R \cup \{\pm\infty\}$. By an open set we always mean a non-empty open set. For a set $X \subseteq \mathbb{R}$, we define the *convex hull* of X, denoted by $conv(X)$, as the set

$$conv(X) = \bigcup_{x < y \in X} [x, y].$$

We prove a useful lemma about our properties.

Lemma 2.1. *Assume $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ has* (DP)(II). *Then for every definable set X, $\dim(\overline{X}) = \dim(X)$.*

Proof. Let $X \subseteq R^n$ be a definable set. We may assume that $X \neq \emptyset$. Towards a contradiction, suppose that $\dim(X) < \dim(\overline{X})$.

Case I. Suppose first that $\dim(\overline{X}) = n$. Let $B \subseteq \overline{X}$ be an open box. By DP(II), there is a non-empty $\mathcal{L}$-chunk Y of $X \cap B$. Thus $\dim Y = \dim X \cap B < n$. Take any $y \in Y$. By definition of an $\mathcal{L}$-chunk, there is an open box $B' \subseteq B$ that contains y and such that $B' \cap X \subseteq Y$. Since Y is $\mathcal{L}$-definable and has dimension n, there is an open box $B'' \subseteq B'$ that does not intersect Y, and hence not X. That is, X is not dense in B' and thus neither in $B \subseteq \overline{X}$, a contradiction.

Case II. In general, let $\dim(X) < \dim(\overline{X}) = m$. Let π be some projection onto m coordinates, such that $\pi(\overline{X})$ contains an open box B. Clearly, $\dim \pi(X) < m = \dim \pi(\overline{X}) \leq \dim \overline{\pi(X)}$. Hence by Case I, we get a contradiction. □

2.1. **Semibounded o-minimal structures.** In this subsection, we assume that $\mathcal{R}$ is a semibounded structure. Following [22, 9], we say that an interval is *short* if there is an $\mathcal{L}$-definable ordered field on it whose order agrees with $<$. We say that a set is *short* if it is in definable bijection with a subset of a product of short intervals. We will tacitly use the fact that any two short open intervals are in $\mathcal{L}$-definable bijection ([22, Corollary 3.3]).

Definition 2.2. Let $\mathcal{Y} = \{X_t : t \in A\}$ be an $\mathcal{L}$-definable family. We define the equivalence relation $\sim_\mathcal{Y}$ as follows:

$$t \sim_\mathcal{Y} t' \iff X_t = X_{t'}.$$

Lemma 2.3. *Let $I \subseteq R^n$ be a short set, and $\mathcal{Y} = \{X_t\}_{t \in A}$ an $\mathcal{L}$-definable family of subsets of I. Then there is a short set $A' \subseteq A$ of representatives for $\sim_{\mathcal{Y}}$.*

Proof. We prove the lemma by induction on n. Let $n = 1$. By cell decomposition in o-minimal structures, it is easy to see that we may assume that either every X_t is a singleton or every X_t is an open interval. Suppose every X_t is a singleton, $X_t = \{x_t\}$. Define the map $f : A \to I$, with $t \mapsto x_t$. By definable choice in $\mathcal{R}$, there is an $\mathcal{L}$-definable $A' \subseteq A$, such that $f_{\restriction A'} : A' \to I$ is a bijection. By [9, Corollary 3.11], A' is short, as needed. Suppose now that every X_t is an open interval, $X_t = (a_t, b_t)$. Define $f : A \to I^2$, with $t \mapsto (a_t, b_t)$. Again, there is a short $A' \subseteq A$, such that $f(A') = f(A)$, as needed.

Now let $n > 1$. Denote by π be the projection onto the first $n - 1$-coordinates. By inductive hypothesis for $\mathcal{C} = \{\pi(X_t)\}_{t \in A}$, there is a short set of representatives $C \subseteq A$ for $\sim_{\mathcal{C}}$. For every $s \in C$, consider the set

$$Y_s = \{t \in A : \pi(X_t) = \pi(X_s)\}$$

and the family $\mathcal{D}_s = \{X_t\}_{t \in Y_s}$. It is enough to show that for every $s \in C$, the statement holds for $\mathcal{D}_s$. Namely, it is enough to find a short set of representatives $D_s \subseteq Y_s$ for $\sim_{\mathcal{D}_s}$. Indeed, in that case, $\bigcup_{s \in C} D_s$ will be a set of representatives for $\sim$, and, moreover, by [9, Lemma 4.2], it will be short.

So fix $s \in C$. The family $\mathcal{D}_s$ consists of all sets X_t, $t \in A$, with $\pi(X_t) = \pi(X_s)$. For every $x \in \pi(X_s)$, consider the set of fibers $\mathcal{F}_x = \{(X_t)_x\}_{t \in A}$. By the case of $n = 1$, there is a short set of representatives $F_x \subseteq A$ for $\sim_{\mathcal{F}_x}$. Then the set $D_s = \bigcup_{x \in \pi(X_s)} F_x$ is a set of representatives for $\sim_{\mathcal{D}_s}$, again short by [9, Lemma 4.2], as needed. $\square$

In Section 4, we will use the following fact.

Fact 2.4. *Let $\mathcal{R} = \mathbb{R}_{sbd}$, and $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be an $\mathcal{L}$-definable function. Then there is a bounded interval $B \subseteq \mathbb{R}$ and an affine function $\lambda : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \sum_i \lambda_i x_i + b$, such that for every $x \in X$, $f(x) \in \lambda(x) + B$.*

Proof. Easy to see, using [7, Fact 1.6]. $\square$

2.2. **Open questions.** We conclude this section with some open questions.

Question 2.5. *Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be an expansion of an ordered group. Are the following equivalent?*

- $\mathcal{M}$ *is semibounded,*
- $\mathcal{M}$ *has no definable poles.*

A potential counterexample to the above question could be given by the following structure. For $t \in 2^{-\mathbb{N}}$, let $f_t : [t, 2t) \to (1/2t, 1/t)$ be a linear homeomorphism, and define

$$\mathcal{M} = \langle \mathbb{R}, <, +, \{f_t : t \in 2^{-\mathbb{N}}\} \rangle.$$

It is easy to see that $\mathcal{M}$ defines a pole, but we do not know if it is semibounded.

Question 2.6. *Let $\mathcal{B}$ be the collection of all bounded sets definable in $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$. Do*

$$\langle \mathbb{R}, <, +, \cdot_{\restriction [0,1]^2}, 2^{\mathbb{Z}} \rangle \text{ and } \langle \mathbb{R}, <, +, \{B\}_{B \in \mathcal{B}}, 2^{\mathbb{Z}} \rangle$$

have the same definable sets?

As mentioned in the introduction, $\mathbb{R}_{sbd}$ is the unique structure strictly between $\mathbb{R}_{vec}$ and $\overline{\mathbb{R}}$.

Question 2.7. *What are the possible structures between $\langle \mathbb{R}_{vec}, 2^{\mathbb{Z}} \rangle$ and $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$?*

Unlike in the o-minimal case, there are more than one such structures: besides $\langle \mathbb{R}_{sbd}, 2^{\mathbb{Z}} \rangle$, one can consider, for example, $\langle \mathbb{R}_{vec}, \cdot_{\restriction 2^{\mathbb{Z}} \times \mathbb{R}} \rangle$. Similar examples were studied by Delon in [3].

3. No new non-semialgebraic smooth functions

In this section, $\mathcal{R} = \langle R, <, +, \ldots \rangle$ denotes a nonlinear reduct of a real closed field $\mathcal{R}'$, and $\widetilde{\mathcal{R}}$ an expansion of $\mathcal{R}$, as fixed in the introduction. The goal of this section is to show that if $\widetilde{\mathcal{R}}$ has (DP) and (DIM), then every definable smooth function $f : X \subseteq R^n \to R$ with open semialgebraic domain X is semialgebraic (Theorem 3.8 below). The proof is done in two steps, the first being when X is short. This case is handled by reduction to the semialgebraic case, namely to [11, Theorem 1.4]. In order to do this reduction, we first prove some additional lemmas for semibounded structures in Section 3.1 below. The general case is done by reduction to the short case, using some basic facts from real algebraic geometry, which we recall in Section 3.2.

3.1. **More on semibounded structures.** For the rest of Section 3, we fix a short interval $I = (-a, a) \subseteq R$ and the order-preserving semialgebraic diffeomorphism $\tau(x) = \frac{ax}{\sqrt{x^2+1}} : R \to I$. We let $\mathcal{I} = \langle I, <, \oplus, \odot \rangle$ be the field structure induced on I from $\mathcal{R}'$ via τ. Namely, for every $x, y \in I$,

$$x \oplus y = \tau(\tau^{-1}(x) + \tau^{-1}(y))$$

and

$$x \odot y = \tau(\tau^{-1}(x) \cdot \tau^{-1}(y)).$$

Denote by $\mathcal{L}_{\mathcal{I}}$ the language of $\mathcal{I}$. Clearly, $\mathcal{I}$ is a real closed field. It is in fact pure.

Fact 3.1 ([17, Corollary 3.6]). *If $X \subseteq I^n$ is semialgebraic (that is, definable in $\mathcal{R}'$), then X is $\mathcal{L}_{\mathcal{I}}$-definable.*

Proof. Let $X \subseteq I^n$ be semialgebraic. Since τ is semialgebraic, so is $\tau^{-1}(X)$. But since τ is also an isomorphism between the structures $\mathcal{R}'$ and $\mathcal{I}$, this means that X is definable in $\mathcal{I}$. $\square$

We write $(\frac{x}{y})_{\mathcal{I}}$ for the division in $\mathcal{I}$. Since the order-topology on I coincides with the subspace topology from R, the dimension of a subset of I^n with respect to either structure is the same. Moreover, if $f : X \subseteq I^n \to I$ is any function, then continuity of f is invariant between the two structures; that is, f is continuous with respect to $\mathcal{I}$ if and only if it is continuous with respect to $\mathcal{R}$. More generally, smoothness of f is also invariant between the two structures. Let us write $f \in \mathcal{C}^\infty(\mathcal{R})$ if f is smooth in the sense of $\mathcal{R}$ (or, rather $\mathcal{R}'$), and $f \in \mathcal{C}^\infty(\mathcal{I})$ if it is smooth in the sense of $\mathcal{I}$.

Fact 3.2. *Let $f : X \subseteq I^n \to I$ be any function with open domain. Then $f \in \mathcal{C}^\infty(\mathcal{R})$ if and only if $f \in \mathcal{C}^\infty(\mathcal{I})$.*

Proof. Immediate from the fact that $\mathcal{R}'$ and $\mathcal{I}$ are isomorphic real closed fields. $\square$

For the rest of this section, we fix $\widetilde{\mathcal{I}}$ to be the structure on I induced from $\widetilde{\mathcal{R}}$. Namely,

$$\widetilde{\mathcal{I}} = \langle I, \{X\}_{X \subseteq I^n \text{ definable}} \rangle.$$

Clearly, $\widetilde{\mathcal{I}}$ expands $\mathcal{I}$. Furthermore, definable completeness of $\widetilde{\mathcal{R}}$ implies that $\widetilde{\mathcal{I}}$ is also definably complete.

Lemma 3.3. *Suppose $\widetilde{\mathcal{R}}$ has (DP) and (DIM). Then so does $\widetilde{\mathcal{I}}$.*

Proof. Let $X \subseteq R^n$ be a set definable in $\widetilde{\mathcal{I}}$.

(DP)(I): Observe that X is also definable in $\widetilde{\mathcal{R}}$. By (DPI) for $\widetilde{\mathcal{R}}$ there is an $\mathcal{L}$-definable family $\{Y_t\}_{t \in R^m}$ of subsets of R^n, and a definable set $S \subseteq R^m$ with $\dim S = 0$, such that $X = \bigcup_{t \in S} Y_t$. By Lemma 2.3, we may assume that $S \subseteq I^m$. By Fact 3.1, the family $\{Y_t\}_{t \in S}$ is $\mathcal{L}_{\mathcal{I}}$-definable, as needed.

(DP)(II): Let X be a set definable in $\widetilde{\mathcal{I}}$ and Y an $\mathcal{L}$-chunk of X. Since the topologies on $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{R}}$ coincide, and, by Fact 3.1, Y is $\mathcal{L}_{\mathcal{I}}$-definable, it follows that Y is also an $\mathcal{L}_{\mathcal{I}}$-chunk of X.

(DIM): Straightforward. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2. **Real algebraic geometry.** Let $\mathcal{R} = \langle R, <, +, \cdot \rangle$ be a real closed field. By an *algebraic set* $A \subseteq R^n$, we mean the zero set of a polynomial in $R[X]$. The *Zariski closure* of a set $V \subseteq R^n$ is the intersection all algebraic sets containing V, denoted by $\overline{V}^{zar}$. Note that $\overline{V}^{zar}$ is algebraic, because $R[X_1, \ldots, X_n]$ is Noetherian.

Let V be an algebraic set. We say that V is *irreducible* if, whenever $V = V_1 \cup V_2$, with each V_i algebraic, we have $V = V_i$, for $i = 1$ or 2.

Fact 3.4. *Let X be a semialgebraic set. Then $\dim X = \dim(\overline{X}^{zar})$.*

Fact 3.5. *Let Y and Y' be two irreducible algebraic sets of dimension n, with $\dim(Y \cap Y') = n$. Then $Y = Y'$.*

Proof. By [11, Lemma 3.4], $Y = Y \cap Y' = Y'$. $\qquad\qquad\qquad\qquad\qquad\square$

Definition 3.6. ([2, Definitions 2.9.3, 2.9.9]) A *Nash function* $f : X \subseteq R^n \to R^m$ is a semialgebraic smooth function with open domain. A *Nash-diffeomorphism* $f : X \to Y$ is a Nash function which is a bijection and whose inverse is also Nash.

A semialgebraic set $V \subseteq R^m$ is a *Nash-submanifold of dimension d* if, for every $x \in V$, there is a Nash-diffeomorphism ϕ from an open semialgebraic neighborhood U of the origin in R^m onto an open semialgebraic neighborhood U' of x in R^m, such that $\phi(0) = x$ and $\phi((R^d \times \{0\}) \cap U) = V \cap U'$.

Note that the graph of a Nash function with connected domain is a connected Nash-submanifold.

Fact 3.7 ([2, Lemma 8.4.1]). *Let $V \subseteq R^m$ be a connected Nash-submanifold. Then $\overline{V}^{zar}$ is irreducible.*

3.3. **Proof of Theorem 1.5.** We are now ready to prove the main result of this section.

Theorem 3.8. *Assume $\widetilde{\mathcal{R}}$ satisfies (DP) and (DIM). Let $f : X \subseteq R^n \to R$ be a definable smooth function, where X is an open semialgebraic set. Then f is semialgebraic.*

Proof. We proceed in two steps:

Step I. Γ_f is short. We handle this case by reduction to the semialgebraic case, [11, Theorem 1.4]. First, we claim that we may assume that $\Gamma_f \subseteq I^{n+1}$. Indeed,

after translating, we may assume that $\Gamma_f \subseteq J^{n+1}$, where J is a short interval. Since any two short intervals are in $\mathcal{L}$-definable bijection, there is an $\mathcal{L}$-definable bijection that embeds Γ_f into I^{n+1}.

We may thus assume that $\Gamma_f \subseteq I^{n+1}$. In particular, Γ_f is definable in $\widetilde{\mathcal{I}}$. Also, since $f \in \mathcal{C}^\infty(\mathcal{R})$, by Fact 3.2 we obtain $f \in \mathcal{C}^\infty(\mathcal{I})$. Now, by Lemma 3.3, $\widetilde{\mathcal{I}}$ has (DP) and (DIM), and is also definably complete. Hence, by [11, Theorem 1.4], f is $\mathcal{I}$-definable. In particular, it is semialgebraic, as needed.

Step II. General case. By [8], every open semialgebraic set is a finite union of open semialgebraic cells. Hence we may assume X is an open semialgebraic cell. Let π be the projection on the first n-coordinates, B a short open box that intersects Γ_f, and $B' \subseteq \pi(B \cap \Gamma_f)$ an open box. Denote $g = f_{\restriction B'}$. Clearly, Γ_g is contained in a short set and by Step I, g is semialgebraic, and hence Nash. Therefore Γ_g is a connected Nash-submanifold. By Fact 3.7, the set $Y = \overline{\Gamma_g}^{zar}$ is irreducible, and by Fact 3.4 it has dimension n.

Claim. $\Gamma_f \subseteq Y$.

Proof of Claim. Let
$$Z = \{x \in X : (x, f(x)) \in Y\}.$$
It is enough to show $X \subseteq Z$. Note that $B' \subseteq Z$, and hence $\dim Z = n$. Assume towards a contradiction that $X \nsubseteq Z$. Since X is connected and open, there is $z \in \mathrm{fr}(Z) \cap X$ and an open short box $(z, f(z)) \in D_1$, such that for $D := \pi(D_1)$, we have

(1) $\dim(D \cap Z) = n$, and
(2) $D \setminus Z \neq \emptyset$.

Clearly, $\Gamma_{f_{\restriction D}}$ is short, and hence by Step I, we have that $f_{\restriction D}$ is semialgebraic. Since $\dim D = n$, the Zariski closure $Y' = \overline{\Gamma_{f_{\restriction D}}}^{zar}$ has dimension n (Fact 3.4). Moreover, the intersection $Y \cap Y'$ contains $\Gamma_{f_{\restriction D \cap Z}}$ and hence by (2) also has dimension n. By Fact 3.5, $Y = Y'$. It follows that for every $d \in D$,
$$(d, f(d)) \in \Gamma_{f_{\restriction D}} \subseteq Y' = Y.$$
This implies $D \subseteq Z$, which contradicts (2). $\qquad\qquad\square$

Since X is a semialgebraic n-cell, $\Gamma_f \subseteq Y$ and $\dim Y = n$, by [11, Lemma 3.10], we obtain that f is semialgebraic. $\qquad\qquad\square$

4. Staying semibounded

In this section, $\mathcal{R} = \langle \mathbb{R}, <, +, \ldots \rangle$ denotes a semibounded o-minimal expansion of the real ordered group. Besides reducts of the real field, examples include the expansion of $\langle \mathbb{R}, <, + \rangle$ by all restricted analytic functions, and others.

Our goal in this section is to prove Proposition 1.6 (Proposition 4.4 below). We will need some machinery from [12]. For $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$, we denote by $\widetilde{\mathcal{R}}^\#$ the expansion of $\widetilde{\mathcal{R}}$ by all subsets of P^k for all $k \in \mathbb{N}$.

Definition 4.1 ([12])**.** We say that a set $Q \subseteq \mathbb{R}$ is *sparse* if for every $\mathcal{L}$-definable function $f : \mathbb{R}^k \to \mathbb{R}$, $\dim f(Q^k) = 0$.

Lemma 4.2. *If $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ has* (DIM)*, then P is sparse.*

Proof. By (DIM),

$$\dim f(P^k) = \dim \bigcup_{t \in P^k} \{f(t)\} = \max_{t \in P^k} \dim\{f(t)\} = 0,$$

as required.   □

Fact 4.3 ([12, Last claim in the proof of Theorem A]). *Assume $P \subseteq \mathbb{R}$ is sparse. Let $A \subseteq \mathbb{R}^{n+1}$ be definable in $\widetilde{\mathcal{R}}^{\#}$ such that for every $x \in \mathbb{R}^n$, A_x has no interior. Then there is an $\mathcal{L}$-definable function $f : \mathbb{R}^{m+n} \to \mathbb{R}$ such that for every $x \in \mathbb{R}^n$,*

$$A_x \subseteq \overline{f(P^m \times \{x\})}.$$

We are now ready to prove our result.

Proposition 4.4. *If $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ has (DP)(II) and (DIM), then $\widetilde{\mathcal{R}}$ is semibounded.*

Proof. If P is finite, then $\widetilde{\mathcal{R}}$ is semibounded and o-minimal and the result is clear.

Assume towards a contradiction that there is an ordered field $I = \langle \mathbb{R}, <, +_I, \cdot_I, 0_I, 1_I \rangle$ definable in $\widetilde{\mathcal{R}}$. Note that $\cdot_I$ could be different from the standard multiplication. For simplicity, we assume that $0_I = 0$ and use the standard multiplication and division notations (the addition used in the proof being only the standard one). Note that since the order of I is the standard one, for every $x \in \mathbb{R}$, $\lim_{t \to \infty} x/t = 0$. (Indeed, say if $x > 0$, then for every $\varepsilon > 0$, we can take $t > x/\varepsilon$, and have $0 < x/t < \varepsilon$.)

We observe that there is a definable unbounded 0-dimensional set S. Indeed, if P is unbounded we let $S = P$. Suppose P is bounded. By translating, we may assume that P accumulates at 0, say from the right. The bijection $x \mapsto 1/x : \mathbb{R} \to \mathbb{R}$, is definable in I and hence in $\widetilde{\mathcal{R}}$, and moreover its limit at 0^+ is ∞. Hence the image S of P under it is definable, unbounded and 0-dimensional, as needed.

We consider the family $\{xS : x \in \mathbb{R}\}$. By Lemma 4.2 and Fact 4.3, there is an $\mathcal{L}$-definable function $f : \mathbb{R}^{k+1} \to \mathbb{R}$ such that

$$(1) \qquad\qquad\qquad\qquad xS \subseteq \overline{f_x(P^k)}.$$

By Fact 2.4, there is a bounded set B and linear functions $\lambda : \mathbb{R}^k \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$, such that

$$(2) \qquad\qquad\qquad\qquad f(x, P^k) \subseteq B + \lambda P^k + b(x).$$

We prove that

$$\mathbb{R} \subseteq \overline{\left(\frac{\lambda P^k}{S} \right)},$$

which will contradict Lemma 2.1 and DP(II). To see this, let $x \in \mathbb{R}$ and $\varepsilon > 0$. We show that there is $p \in P^k$ and $t \in S$, with

$$\left| x - \frac{\lambda p}{t} \right| \leq \varepsilon.$$

Take $t \in S$ with $\frac{B+b(x)}{t} < \varepsilon$. By (1) and (2), there is $p \in P^k$, with

$$x \in \overline{\left(\frac{\lambda p + B + b(x)}{t} \right)} = \left[\frac{\lambda p}{t} - \varepsilon, \frac{\lambda p}{t} + \varepsilon \right],$$

as required.

By (1) and (2),

$$x \in Z = \overline{\{y/t, : \; y \in \lambda(P^k), \, t \in S\}}.$$

Therefore Z has interior. Moreover, by (DIM), we have

$$\dim(Z) = \dim\left(\{y/t \; : \; y \in \lambda(P^k), t \in S\}\right) = \max_{(y,t)\in P^k \times S} \dim\left(\{\lambda(y)/t\}\right) = 0.$$

This is a contradiction. $\qquad\square$

Question 4.5. *Is it true that if $\langle \mathcal{R}, P \rangle$ has (DP) and (DIM), then so does $\langle \mathcal{R}, P \rangle^{\#}$?*

Question 4.6. *Is Theorem 1.7 true for f definable in $\widetilde{\mathcal{R}}^{\#}$?*

5. EXAMPLES

Throughout this section, $\mathcal{R} = \langle \mathbb{R}, <, +, \ldots \rangle$ denotes an o-minimal semibounded structure over the reals. Our goal is to prove Proposition 1.10. For (1), our approach is the following. First, we show that under a certain quantifier elimination result, (DP)(I) holds (Proposition 5.2). Together with d-minimality and the following lemma, we can then conclude its proof. For (2), we reduce the statement to that of $\langle \overline{\mathbb{R}}, \alpha^{\mathbb{Z}} \rangle$ from [11], using Proposition 1.6.

Recall ([18]) that $\widetilde{\mathcal{R}}$ is called *d-minimal* if for every definable family $\{X_t\}_{t \in A}$ of subsets X_t of R, there is $N \in \mathbb{N}$, such that every X_t is the union of an open set and at most N discrete sets.

Lemma 5.1. *Suppose $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ is d-minimal and has (DP)(I). Then it has (DP)(II) and (DIM).*

Proof. By [11, Proposition 4.15], we have DP(II). By [11, Remark 1.5(1)], we have (DIM). (In that reference $\widetilde{\mathcal{R}}$ expanded a field, but the proof of Remark 1.5(1) did not use that assumption.) $\qquad\square$

5.1. **(DP)(I).** In what follows, we assume that P is discrete, closed in its convex hull, has no maximal element, and $0 < P$. We define $\lambda : \mathbb{R} \to P \cup \{0\}$,

$$\lambda(x) = \begin{cases} \max(P \cap (-\infty, x]) & \text{if } x \in conv(P) \\ 0 & \text{otherwise} \end{cases}$$

(which exists since P is a discrete set, closed in its convex hull). We define $s : P \to P$ to be the *successor* function in P; namely,

$$s(x) = \min\{y \in P : x < y\}.$$

By *basic functions* we mean λ, s, s^{-1} and all $\mathcal{L}$-definable functions.

Proposition 5.2. *Assume that every definable set $X \subseteq \mathbb{R}^l$ is a finite union of sets Y, each satisfying the following property:*

(A): there are definable functions $f_1, \ldots, f_n : \mathbb{R}^l \to \mathbb{R}$ and $g_1, \ldots g_m : \mathbb{R}^l \to \mathbb{R}$, which are given by compositional iterates of basic functions, such that

$$Y = \{x \in \mathbb{R}^l \; : \; \forall i, j, \, f_i(x) = 0, \, g_j(x) > 0\}.$$

Then $\langle \mathcal{R}, P \rangle$ has DP(I).

Proof. We begin with a claim.

Claim. *Let h be a composition of basic functions. Then there is an $\mathcal{L}$-definable function f and a definable set $S \subseteq P^k$, such that for all $x \in \mathrm{dom}(h)$,*

(*) $\qquad h(x) = z$ *if and only if there is $y \in S$ such that $f(x, y) = z$.*

Proof of Claim. By induction on the number of iterations of basic functions which compose h.

For $h = \lambda$, let $S = \Gamma_s \subseteq P^2$ and $f(x, y_1, y_2) = y_1$ if $y_1 \leq x < y_2$, and not defined otherwise. We verify $(*)$. If $\lambda(x) = z$ then $f(x, z, s(z)) = z$. By definition of f, if $f(x, y_1, y_2) = y_1$, then $y_1 \leq x < y_2$, and since $(y_1, y_2) \in \Gamma_s$, we have $y_2 = s(y_1)$ and $\lambda(x) = y_1$. Furthermore, we see that if there is $y \in S$ such that $f(x, y)$ is defined then $f(x, y) = h(x)$. The cases $h = s$, s^{-1} are similar and the case where h is $\mathcal{L}$-definable is straightforward.

Now let $h = h_{n+1}(h_1, \ldots, h_k)$ where h_{n+1} is a basic function and assume that for $1 \leq j \leq k$, there are some $\mathcal{L}$-definable functions $h'_j(x, y)$ and definable $S_j \subseteq P^{k_j}$ such that for all $x \in \mathrm{dom}(h)$,

$$h_j(x) = z \text{ if and only if there is } y \in S_j \text{ such that } h_j(x) = h'_j(x, y).$$

For $h_{n+1} = \lambda$ (thus $k = 1$), we define f exactly similarly to the last paragraph, namely $f(x, a_1, a_2, y) = a_1$ if $h'_1(x, y)$ is defined and $a_1 \leq h'_1(x, y) < a_2$, and not defined otherwise. We verify $(*)$. If $h(x) = a_1$ then there are $(a_1, a_2) \in \Gamma_s, y_1 \in S_1$ such that $a_1 \leq h'_1(x, y_1) < a_2$ and $h'_1(x, y_1) = h_1(x)$. Thus $f(x, a_1, a_2, y_1) = a_1$. If there is $y \in S_1$ such that $h'_1(x, y)$ is defined then $h'_1(x, y) = h_1(x)$ and if there are $(a_1, a_2) \in \Gamma_s$ such that $f(x, y, a_1, a_2)$ is defined (that is, $a_1 \leq h'_1(x, y) < a_2$) then $h(x) = a_1 = f(x, y, a_1, a_2)$.

Again, the cases $h_{n+1} = s, s^{-1}$ are similar and the case h_{n+1} $\mathcal{L}$-definable is straightforward. $\square$

Now let X be a definable set. By hypothesis, there are $f_1, \ldots, f_k$ and $g_1, \ldots, g_{k'}$, which are compositional iterates of basic functions, such that

$$X = \{x \in \mathbb{R}^l \ : \ \forall i, j, \ f_i(x) = 0, \ g_j(x) > 0\}$$

Let f'_i, S_i the maps and sets of dimension 0 given by the claim for $h = f_i$, and g'_j, K_j, for $h = g_j$. That is, for every i, j, we have that

$$f_i^{-1}(0) = \bigcup_{t \in S_i} f'_i(-, t)^{-1}(0),$$

$$g_j^{-1}(\mathbb{R}^{>0}) = \bigcup_{t \in K_j} g'_j(-, t)^{-1}(\mathbb{R}^{>0}).$$

Note that X has the form

$$\bigcap_{s \leq m} \bigcup_{t \in S_s} Y_{s,t}$$

where $m = k + k'$, for every $s \leq m$, $t \in S_s$, $Y_{s,t}$ is an $\mathcal{L}$-definable set.

To prove that X has the form $\bigcup_{t \in S} X_t$ where $\{X_t \ : \ t \in S\}$ is a small subfamily of an $\mathcal{L}$-definable family of sets, by an easy induction it is sufficient to prove that the intersection of two sets of the form $\bigcup_{t \in S'} Y_t$ where there is an $\mathcal{L}$-definable family $\{Y_t \ : \ t \in A\}$ and $S' \subseteq A$ has dimension 0 is itself a set of this form. Let $X_1 = \bigcup_{t \in S_1} X_{1,t}$ and $X_2 = \bigcup_{t \in S_2} X_{2,t}$ where there are two $\mathcal{L}$-definable families $\{X_{i,t} \ : \ t \in A_i\}$ and $S_i \subseteq A_i$ being of dimension 0. Then

$$X_1 \cap X_2 = \bigcup_{(t_1, t_2) \in S_1 \times S_2} X_{1,t_1} \cap X_{2,t_2}$$

and the family $\{X_{1,t_1} \cap X_{2,t_2} : t_1 \in S_1, t_2 \in S_2\}$ is a small subfamily of the $\mathcal{L}$-definable family $Z = \{X_{1,t_1} \cap X_{2,t_2} : t_1 \in A_1, t_2 \in A_2\}$. Moreover, by cell decomposition in o-minimal structures, we may assume that Z is a family of cells. This proves the result. $\qquad\square$

We are now ready to conclude the main result of this section.

Proof of Proposition 1.10. For (1), we prove that it has (DP) and (DIM), and hence Theorem 1.7 directly applies. By Proposition 5.1, it suffices to show that it satisfies (DP)(I) and d-minimality. For (DP)(I), by Proposition 5.2, we only need to show Condition (A). Both Condition (A) and d-minimality are shown in [20].

For (2), we cannot directly apply Theorem 1.7, because we do not know if $\widetilde{\mathcal{R}}$ satisfies (DP)(I). However, we can derive the result as follows. Let f be a smooth definable function with open $\mathcal{L}$-definable domain. Observe that f is also definable in $\langle \overline{\mathbb{R}}, \alpha^{\mathbb{Z}} \rangle$. By [11], f is semialgebraic. Also by [11], $\langle \overline{\mathbb{R}}, \alpha^{\mathbb{Z}} \rangle$ satisfies (DIM). Since (DIM) is preserved under taking reducts, $\langle \mathcal{R}, \alpha^{\mathbb{Z}} \rangle$ also satisfies (DIM). Therefore, by Proposition 4.4, $\widetilde{\mathcal{R}}$ is semibounded, and hence so is its reduct $\langle \mathcal{R}, f \rangle$. But this reduct is o-minimal, and hence by [21, Theorem 1.4], f is definable in $\mathbb{R}_{sbd}$. $\qquad\square$

5.2. **Open questions.** We finish with some natural questions and comments that arise from the current work.

Question 5.3. *Does $\langle \mathbb{R}_{sbd}, 2^{\mathbb{Z}} \rangle$ have (DP)?*

The current examples concern semibounded structures that expand $\mathbb{R}_{sbd}$. Suppose that $\mathcal{R}'$ is a structure that lies between $\langle \mathbb{R}, <, + \rangle$ and $\mathbb{R}_{sbd}$, such as

$$\mathcal{R}' = \langle \mathbb{R}, <, +, \cdot_{\restriction [0,1]^2} \rangle.$$

Question 5.4. *For which P does $\langle \mathcal{R}', P \rangle$ satisfy the conclusions of Theorem 1.7 and Proposition 1.10? In particular, does $\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$ do? (The last question was asked by Hieronymi.)*

We note here that $\langle \mathbb{R}_{sbd}, \mathbb{Z} \rangle$, and even $\langle \mathbb{R}, <, +, (x \mapsto \sqrt{2}x), \mathbb{Z} \rangle$, are not d-minimal, as shown in [13].

Question 5.5. *Let $\widetilde{\mathcal{R}}$ be $\langle \mathbb{R}_{vec}, P \rangle$ or $\langle \mathbb{R}_{sbd}, P \rangle$, where P is an iteration sequence or $2^{\mathbb{Z}}$. Is the open core of $\langle \widetilde{\mathcal{R}}, \mathbb{R}^{alg} \rangle$ equal to $\widetilde{\mathcal{R}}$? (extending Khani's relevant result for $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}, \mathbb{R}^{alg} \rangle$ in [15]).*

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