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How economic depreciation shapes the relationship of uncertainty with investments' size & timing ☆

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ABSTRACT

This paper identifies and analyzes the effects of the rate of economic depreciation of capital stock on a monopolist's investment option and capacity decision in a dynamic and uncertain market environment, where continuous economic depreciation cannot be fully offset. We find that the firm's capital stock is increasing in the rate of depreciation for low rates and decreasing for higher rates. Further, when considering the timing of investment, we show that the effect of uncertainty on investment is level-dependent on the rate of depreciation: only for sufficiently high rates of depreciation there is a positive relationship between capital investment and uncertainty, and the impact of uncertainty on the present value of the firm is mitigated. The fact that the impact of economic depreciation on the firm's investment problem is level dependent demonstrates that its consideration by investors and managers is not trivial.

1. Introduction

The inevitable physical and productive deterioration of assets, i.e. *economic depreciation*, due to the passage of time or recurrent use, imposes constraints on firms' production capabilities that cannot be overcome in real-life. However, the impact of economic depreciation of capital stock on a firm's investment problem is not trivial to managers and investors when investment is irreversible. This is especially the case in a framework where cash flows are subject to uncertainty, so that the firm may find it best to delay investment, and where the firm can set its production capacity. Indeed, ceteris paribus, in an industry with a higher depreciation rate, the firm could opt to compensate the loss in productive capacity by installing a higher capacity. On the other hand, as a result of a loss in marginal revenue, the firm could also opt to install a lower capacity. Similarly, it is not clear what is the effect of depreciation on the timing of investment, and consequently on the investment–uncertainty relationship, that is, the impact of uncertainty on the expected present value of the acquired capital stock. This paper shows that the impact of economic depreciation (henceforth simply "depreciation") on the firm's decision, considering the timing and scale of investment, is not typically monotonic and leads to valuable insight regarding not only the investment strategy of firms but also how depreciation can change the uncertainty–investment relationship, even in the absence of tax considerations. The choice to leave out tax considerations allows us to abstract from specific accounting and tax regimes, while highlighting to managers and investors the implications that depreciation has, on its own, as a driver of investment size and timing decisions. It allows us to model how investors can optimally consider the merits of the inevitable loss of productive capabilities of capital brought by the passage of time and usage.

The empirical literature has often shown that the impact of uncertainty on (capital) investments can be level dependent (e.g., Lensink and Murinde, 2006, Jeanneret, 2007, Mohn and Misund, 2009, and Henriques and Sadorsky, 2011). This paper corroborates findings in empirical studies (also see e.g., Jeanneret, 2007, Driver et al., 2008, and Samaniego and Sun, 2019), where firms are characterized by, e.g., different technologies, or the same industry in different countries, and where the same uncertainty shocks can lead to either negative or positive impacts to the amount of capital invested depending on those characteristics (also see the review by Sarkar, 2019). What the theoretical literature has not yet shown is how the level of depreciation can shape this relationship. Although there has been some indication of this link in empirical studies (e.g., Samaniego and Sun, 2019), we

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cannot find, to the best of our knowledge, an analytical or intuitive framing of such link in the theoretical literature.

There exist few theoretical studies where depreciation is considered to play some role in investment timing problems. Arkin and Slastnikov (2007) and Jou and Lee (2011) find that depreciation accelerates investment, which is a crucially different finding from the work in this paper. However, their results are driven by the fiscal benefits that arise from the tax regime on depreciation. In the duopoly game by Ruffino and Treussard (2006) with time-to-build and technology adoption, economic depreciation is considered as a necessary requirement for a capital-replacement option to be exercised, and so positive depreciation can trigger further investment. However, the investment–uncertainty relationship is not considered. Finally, Adkins and Paxson (2017) explore the role of depreciation on optimal investment, with stochastic deterioration of the salvage value and operating cost, in a capital replacement model. However, their use of a price-taking assumption limits the application of their set-up to the study of problems where firms can set their capital stock levels, since the direct effect of output on prices is a part of the firm's instantaneous profits, a crucial property in these problems, is lost.

In the spirit of Abel and Eberly (1996), Bertola and Caballero (1994), and Bar-Ilan and Strange (1999), this paper proposes a simple, yet effective, framework where we consider a monopolist that has an American-style perpetual option to undertake a one-off lump-sum irreversible investment. By acquiring capital stock, the firm can immediately start up production in a market with a downward sloping demand curve, where the willingness-to-pay of consumers evolves stochastically over time. Capital stock is assumed to depreciate at a constant geometric rate.

When considering both the scale and the timing dimension of investment, our main results can be summarized as follows.

(i) If the firm decides to invest immediately, under not too strict conditions, we find that the firm *overinvests* for small depreciation rates, relative to a zero-depreciation benchmark, and the firm *underinvests* for large depreciation rates.
We show that an increase in the depreciation rate may have two competing effects on the firm's decisions: on the one hand, the firm may wish to install a higher level of capital stock to hedge against future falls in productivity. On the other hand, the total future aggregate revenue generated by each unit of capital stock decreases as a direct result of depreciation, which makes investment relatively more expensive and gives the firm and incentive to invest less. We find that the first effect, which we call the *buffer effect*, is dominant for small depreciation rates, whereas the second effect, dubbed the *relative cost effect*, is dominant for higher rates.

(ii) Depreciation unambiguously increases the threshold for undertaking investment.
This means that if the consumers' initial willingness-to-pay is not sufficiently high, then depreciation leads to a later exercise of the option, in expectation. Consequently, depreciation also increases the expected size of investment but decreases the present value of capital investment. The increase in the scale of investment found due to a later exercise of the option is a common result in the literature, and in line with, e.g., Manne (1961), Bar-Ilan and Strange (1999), and Dangl (1999), who show that the scale of investment is increasing in the consumers' willingness-to-pay. We show that our result stands, even when the relative cost effect dominates the buffer effect.

(iii) The present value of capital investment is positively related to uncertainty only if depreciation is sufficiently strong.
Uncertainty has a positive effect on both the investment threshold, thereby decreasing the present value of capital investment, and the investment scale, thereby increasing the present value of capital investment. When depreciation is strong, we find that the latter effect dominates whereas the first effect dominates when the rate of depreciation is low.

(iv) Depreciation is able to mitigate the impact of uncertainty on the value of the firm when depreciation is sufficiently strong.
In line with the literature, the value of the firm is measured by the option value, and typically additional uncertainty leads to an increase in the value of waiting. We find that the rate of depreciation does not play a neutral role in that relationship. In fact, this increase in the value of waiting is enhanced for small levels and, conversely, dampened for sufficiently high levels of the depreciation rate.

From a Finance point of view there has been attention for the impact of economic depreciation on the firm's investment problem. However, in the Economics literature there has been little attention dedicated to the role played by depreciation in the absence of tax considerations. Considering the literature on investment under uncertainty in a dynamic framework, only few studies incorporate the effects of economic depreciation on the timing of capital investments. Those studies that do, in turn, do not simultaneously allow for a decision on the size of investment. For some, depreciation is present, but not studied (e.g., Abel, 1983, Bertola and Caballero, 1994, Bloom, 2000, Gryglewicz and Hartman-Glaser, 2019, Mauer and Ott, 1995, Cooper, 2006, Chou et al., 2007, and Lyandres et al., 2018). For others, depreciation plays a more prominent role, as discussed above, but the scale of investment is, as mentioned, fixed (e.g. Arkin and Slastnikov, 2007, Jou and Lee, 2011, Ruffino and Treussard, 2006, and Adkins and Paxson, 2017). To the best of our knowledge, there are currently no other studies that analyze both timing and scale of capital investments, simultaneously, and subject to positive economic depreciation in a dynamic and uncertain environment. We uncover some surprising nontrivial results, which are not yet present in the theoretical literature, in particular that the impact of depreciation on the uncertainty–investment relationship is not unambiguous. Beyond the novelty of the approach, our results allow us to explore the practical relevance of the deterioration in productive capacity to the optimal investment decisions, by investors and managers, both on when (or whether) and on how much to invest, as well as how changes to the volatility of the willingness-to-pay can shape its impact to those same decisions. It should be mentioned that outside this field, there exist other theoretical studies that look at the role of depreciation on investment but in very different set-ups (see, e.g., Schlosser et al., 2021).

The flexibility in investment size is a crucial element in our framework. Conventional dynamic investment set-ups, starting with the seminal work by, e.g., McDonald and Siegel (1986) and Dixit and Pindyck (1994), typically assume that the scale of the investment is fixed and exogenously determined. Dangl (1999) and Bar-Ilan and Strange (1999) were among the first to study capacity choice for single firm set-ups. More recent contributions studying capacity choice in various dynamic lumpy investment monopoly settings are Della Seta et al. (2012), Wen et al. (2017), Azevedo et al. (2021), Sarkar (2019), and Jeon (2021) (also see Huberts et al., 2015, for a general survey for contributions prior to 2015). The work in this paper extends on the single firm case by including and studying economic depreciation.

In the literature, alternative ways of modeling depreciation include assuming finite life-time of capital or complete capital depreciation after production, as studied by, e.g., Gryglewicz et al. (2008), Dixit and Pindyck (1994), and Nakamura (1999, 2002). These "one-hoss shay" models are well suitable to study cases that indeed have this characteristic and therefore apply to different scenarios from ours. It is difficult to assume a one-hoss shay model when capital stock is continuously depreciating and we thus find that this choice of treatment obscures the identification of the opposing and level-dependent effects uncovered in this paper, which are present under more general settings. Nakamura, in a discrete time setting without optimal timing, also considers the relationship between uncertainty and depreciation and finds that market uncertainty has a negative impact on investment. Femminis (2008) and Saltari and Ticchi (2005) challenge his findings: Femminis shows that the negative relationship between investment and uncertainty fully relies on the assumption that capital fully depreciates after production. He also shows that this is not always the case when capital depreciates at a constant geometric rate in a model with risk-aversion. Section 5.1 discusses the implications from using this alternative formulation. We find that, in relation to the above mentioned studies, uncertainty increases investment but it should be noted that in, e.g., Nakamura's discrete time model where capital fully depreciates after two periods, investment represents the injection of new capital stock in each period, whereas in our work uncertainty is compensated for by delaying investment which in turn increases the capital stock upon investment. Thus, our findings are more in line with, e.g., Femminis'.[1]

This paper is organized as follows. First, the model is introduced in Section 2. Section 3 studies the firm's investment strategy and particularly focuses on the impact of depreciation on scale and timing. Section 4 extends the analysis by looking at the uncertainty–investment relationship. We consider two alternative versions of the main model in Section 5. Concluding remarks are given in Section 6, as well as a summary of our results and the managerial implications, followed by suggestions for future research.

2. Model

Consider a monopolist that holds an American style perpetual (real) option to undertake investment and acquire some capital stock. Capital stock is denoted by $K(t)$, where $t \geq 0$ denotes time, and can be obtained by a lump-sum irreversible investment. The firm is assumed to be risk-neutral, rational, financially unconstrained, and value-maximizing. After investment, capital stock is assumed to change over time according to the dynamics

$$\mathrm{d}K(t) = I(t) - \delta \cdot K(t)\mathrm{d}t, \tag{1}$$

where $I(t)$ denotes the instantaneous investment at time t and where $\delta > 0$ denotes the depreciation rate. Although we do not allow for zero-depreciation cases in the problem specification, we will include $\delta = 0$ in our analysis, to better understand the effects of depreciation.

For our main model, we assume that the instantaneous investment takes a positive value at most once.[2] Let $\bar{K}$ denote the initial capital stock at the time of investment, i.e. for some time $s \geq 0$, $I(s) = \bar{K} > 0$ and $I(t \neq s) = 0$. It then follows from (1) that the capital stock held by the firm is given by $K(t) = \bar{K}e^{-\delta(t-s)}, t \geq s$.

To finance this project, the firm faces unit investment cost $\kappa > 0$. Thus, $\kappa\bar{K}$ represents the total financing costs, which is incurred at the moment of investment. Since the investment is irreversible, no portion of these costs can be recovered.

Output $Q(t)$ is determined by the production function

$$Q(t) = \frac{a}{\gamma}K^{\gamma}(t), \tag{2}$$

where $a > 0$ is the production technology parameter and $\gamma \in (0,1)$ denotes an output elasticity that ensures diminishing returns to capital.[3] This type of production function is in line with, e.g., Bertola (1988), Bertola and Caballero (1994), Nakamura (2002) and Lyandres et al. (2018), where labor is assumed to be flexible. Notice that, given a production function that depends positively on $K(t)$, positive depreciation represents the deterioration of the capital's stock productivity. For this reason, we refer to the latter as 'economic depreciation' (simply 'depreciation' henceforth).

The market in which the firm operates is characterized by the following inverse demand function:

$$p(t) = x(t)(1 - Q(t)), \tag{3}$$

so that prices clear markets, where $x(t)$ is an exogenous shock process. The value of x at $t = 0$ is a known parameter but future values are stochastic, thus representing future (unknown) shifts in the demand curve. Thus, a natural interpretation for a higher value of $x(t)$ is that of a higher willingness-to-pay by consumers represented by an upward shift of the inverse demand. Note that this specification results in the firm's investment being associated with uncertainty and therefore risk. This type of inverse demand function follows, e.g., Pindyck (1988), He and Pindyck (1992), Aguerrevere (2003), Wu (2007), and Huisman and Kort (2015). Process $x(t)$ follows a geometric Brownian motion with trend μ and volatility parameter $\sigma > 0$, i.e.

$$\mathrm{d}x(t) = \mu x(t)\mathrm{d}t + \sigma x(t)\mathrm{d}z(t). \tag{4}$$

The first term on the right-hand side represents the trend of the process. The second term on the right-hand side contains the Wiener process $z(t)$ through which exogenous shocks are brought in. The Wiener process has a normal distribution with expected value 0, standard deviation $\sqrt{t}$, and has the property that $(\mathrm{d}z)^2 = \mathrm{d}t$. Let us denote the initial value of the shock process $(x(t))_{t\geq 0}$ by $X = x(0)$. We will assume $2\mu > \sigma^2$ to ensure finite expected hitting times. Discounting is done under a constant risk-free rate ρ, where $\rho > \mu$.

Given the autonomous evolution of K, the choice of the initial capital stock is equivalent to the firm choosing its productive capacity, as given by (2). The firm is assumed to be committed to producing the amount dictated by their capacity allowance. In the literature on capacity constrained firms, this so-called capacity clearing assumption is used on a large scale (e.g. Deneckere et al., 1997, Chod and Rudi, 2005, Anand and Girotra, 2007, Goyal and Netessine, 2007, and Huisman and Kort, 2015). For example, Goyal and Netessine (2007) argue that producing below capacity may be found to be difficult for firms as a result of fixed costs associated with commitments to suppliers, labor, and production ramp-up.

The firm's strategy comprises two decisions: the timing of investment and the size of the initial capital stock ($\bar{K}$). We base ourselves on the work of McDonald and Siegel (1986), Smets (1991), and Dixit and Pindyck (1994) to find the firm's optimal investment (stopping) behavior under uncertainty. To formally write down the firm's optimization problem, denote the filtered probability space of $(x(t))_{t\geq 0}$ by $(\Omega, \mathcal{F}^x, \mathbf{F}, \mathbb{P})$, so that the filtration associated with the process $x(\cdot)$ is denoted by $\mathbf{F} = (\mathcal{F}^x_t)_{t\geq 0}$, with natural filtrations $\mathcal{F}^x_t$, collecting the available information at time $t \geq 0$. Conditional expectation operator $\mathbb{E}_X$ is taken with respect to measure $\mathbb{P}$, i.e. $\mathbb{E}\{\,\cdot \mid \mathcal{F}^x_0\}$, where $X = x(0)$.

Let τ be a stopping time and let $\mathcal{M}$ consist of all finite $\mathcal{F}^x_t$-stopping times. Given inverse demand (3), the firm then faces the following optimization problem, at time $t = 0$, over the initial capital stock $\bar{K}$ and timing τ,[4]

$$V(X) = \sup_{\tau\in\mathcal{M},\bar{K}>0} \mathbb{E}_X \left\{ \int_{\tau}^{\infty} x(t)(1 - Q(t))Q(t)e^{-\rho t}\mathrm{d}t - e^{-\rho\tau}\kappa\bar{K} \right\}, \tag{5}$$

with $Q(t)$ as defined in (2), i.e.,

$$Q(t) = \begin{cases} 0 & \text{if } t < \tau, \\ \frac{a}{\gamma}\left(\bar{K}e^{-\delta(t-\tau)}\right)^{\gamma}, & \text{if } t \geq \tau. \end{cases}$$

[1] However, also Femminis does not take into account that the firm might have an option to delay investment, which we do. In addition, they do not study the relationship between uncertainty and (the rate of) depreciation.

[2] Section 5.2 considers an extension where the firm can replenish its capital stock any number of times and it confirms the results we obtain from our main model.

[3] We assume that $a > \gamma e^{-1}$ to ensure a monotonic relationship between Q and γ. We want to note that our results will still apply for other values of a, however, some of the intuitive properties of the production function are lost in this simple formulation if a is chosen to be too small.

[4] Notice that for all finite stopping times it holds that $\lim_{t\to\infty} K(t) = 0$, which implies that there are no issues with the transversality.

In line with the literature, we will write the optimal stopping moment in terms of X. This means that we will determine a threshold X^* such that, if $x(0) < X^*$ the firm invests when $x(t)$ hits the investment threshold X^* for the first time.[5]

It is important to note that, because the firm's investment is risky, the firm has an incentive to 'delay' investment until the net present value (NPV) is sufficiently positive (McDonald and Siegel, 1986, Dixit and Pindyck, 1994). This leads to a so-called *wedge* between the NPV threshold and our threshold X^*, where the NPV threshold is simply the value of X such that the net present value from investment is zero. Thus, uncertainty creates a *value of waiting*.

If $x(0) \geq X^*$, the firm invests immediately and investment takes place at $t = 0$. It follows that the (stochastic) investment time is given by hitting time $\tau^* = \inf\{t \geq 0 \mid x(t) \geq X^*\}$. We will denote the optimal level of the initial capital stock by $\bar{K}^*(x(0))$.[6] The set of all values of X such that investment takes place immediately is called the *stopping region* and the complementary region is called the *continuation region*. For typical scenarios like ours, the stopping region is given by $\mathcal{S} = \{X \in \mathbb{R}_+ \mid X \geq X^*\}$ and the continuation region equals $\mathcal{C} = \{X \in \mathbb{R}_+ \mid X < X^*\}$. For the latter case, the firm will set, upon investment, $\bar{K}^*(X^*)$, which we will denote as $\bar{K}^{opt}$.

Proposition 1. *Let $\gamma < \frac{\beta-1}{\beta}$, where β is given by*

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}}. \tag{6}$$

Then, $\mathcal{C}$ is empty. The firm's capital stock in the stopping region $\bar{K}^*(X)$ is the solution of

$$\frac{X}{\rho + \gamma\delta - \mu}\left(1 - 2\frac{a}{\gamma}\bar{K}^\gamma \frac{\rho + \gamma\delta - \mu}{\rho + 2\gamma\delta - \mu}\right) = \frac{\kappa}{a}(\bar{K})^{1-\gamma}. \tag{7}$$

Let $\gamma \geq \frac{\beta-1}{\beta}$. Then $\mathcal{C}$ is non-empty. For $X \in \mathcal{S}$, as in the previous case, the firm invests immediately and acquires $\bar{K}^*(X)$, given by the solution of . For $X \in \mathcal{C}$ the firm delays investment and waits until the process $x(t)$ reaches the investment threshold X^* to acquire $\bar{K}^{opt}$, given by

$$X^* = \left(\frac{\gamma}{a}\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu}\right)^{\frac{1-\gamma}{\gamma}} \frac{\beta(2\gamma-1)+1}{\beta-1}\frac{\kappa}{a}(\rho+\delta\gamma-\mu), \tag{8}$$

$$\bar{K}^{opt} = \left(\frac{\gamma}{a}\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu}\right)^{\frac{1}{\gamma}}, \tag{9}$$

respectively. As a result, the firm's value function is given by

$$V(X) = \begin{cases} \left(\frac{X}{X^*}\right)^\beta \frac{\kappa}{\beta-1}\bar{K}^{opt} & \text{if } X < X^*, \\ \frac{X}{\rho+\gamma\delta-\mu}\frac{a}{\gamma}(\bar{K}^*(X))^\gamma \\ \quad \times\left(1 - \frac{a}{\gamma}(\bar{K}^*(X))^\gamma \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) - \kappa\bar{K}^*(X) & \text{if } X \geq X^*. \end{cases} \tag{10}$$

[5] The proof for this result in our set-up is based on Dixit and Pindyck (1994), who show that the state space can be divided into two consecutive regions for standard real options problems without capacity choice, and whose results are extended by Huberts et al. (2019) for models where capacity choice is explicitly modeled. Optimality can be shown using a verification theorem based on, e.g., Gozzi and Russo (2006).

[6] The cost of financing the capital stock, $\kappa\bar{K}$, represents both book and market value at the time of investment, but whereas the number that represents the productive capacity of capital will continue to evolve in time according to (1), its market value will be at most zero immediately after the fact, due to the assumption of irreversibility of investment.

All proofs can be found in Appendix A. Notice that, as a result of the assumption that $\rho > \mu$, we have $\beta > 1$. Additionally, as has been shown extensively in the literature, β has the property that it is decreasing in σ. Also note that the first case of (10) gives the *option value*.

Proposition 1 shows that only for $\gamma > \frac{\beta-1}{\beta}$ we have a non-empty continuation region, i.e., there exists a threshold as given by (8) and it is positive. Bar-Ilan and Strange (1999) also established a relationship between the *marginal productivity of capital* parameter γ and the threshold. As shown by Lyandres et al. (2018), assumptions on γ are required. In order for the value of the waiting option to exceed the value of immediate investment, γ must be sufficiently large. From the formulation of our production function, it follows that capital gets more productive as the output elasticity γ goes down. This means that for each unit of output, less capital is required when γ is smaller, so that investment in each unit of output becomes relatively cheaper, which ultimately accelerates investment. Fig. 1(a) illustrates the regions for different γ.

We would like to note here that we distinguish two ways in which the optimal capital stock is affected by the rate of depreciation. First, assuming the firm invests immediately, i.e. X is fixed, δ has an effect on $\bar{K}^*(X)$ as can be observed from . We will refer to this as the *direct effect* of δ on capital stock. Second, the proof of Proposition 1 shows that $\frac{\partial}{\partial X}K^*(X) > 0$, i.e. the firm's optimal capital stock in the stopping region is increasing in X. Then, in case the firm delays investment, it can be noted from (8) that the direct impact of δ on X^*, in turn, leads to a change in $\bar{K}^{opt} = \bar{K}^*(X^*)$. We will refer to the effect of δ on the acquired capital stock through a change in the threshold of investment as the *indirect effect* of δ on capital stock.

3. Investment and depreciation

With our main model in place, we are equipped to, first, address the question of how economic depreciation affects the firm's optimal investment behavior.

Section 3.1 details how depreciation affects the investment threshold, i.e. the boundary between the continuation region and stopping region, and therefore encapsulates the effects of depreciation on the optimal timing of investment.

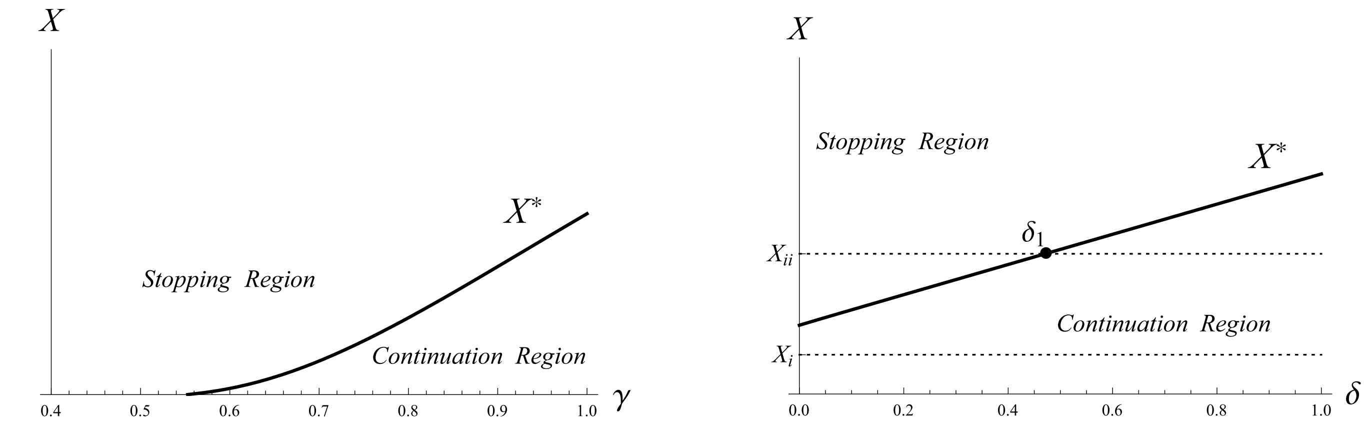
The case where the monopolist remains in the stopping region when studying the effect of different rates of δ, i.e. in the region where the timing of investment is unaffected, offers an opportunity to look at the direct effect of depreciation on the optimal capital stock, which is analyzed in Section 3.2. Then, Section 3.3 extends the analysis to the case where the monopolist remains in the continuation region for different rates of δ, now considering indirect effects due to the simultaneous adjustments of size and timing of investment.

3.1. Timing of investment and δ

Intuition tells us that, since a higher rate of depreciation leads to each unit of capital stock generating less output per stretch of time, total revenue streams are negatively impacted by depreciation. The firm then has, comparatively to a lower rate of depreciation, an incentive to delay investment until the state process reaches a higher level of the consumers' willingness-to-pay. That way, the firm could allow for the gains from the higher prices to compensate the losses in productivity, countering the comparatively lower expected revenue.

One can check directly from Eq. (8) that, since $\frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu}$ is an increasing function of δ, the investment threshold X^* is increasing in δ. Thus, the former effect dominates for all δ and this confirms that, comparatively, depreciation always delays investment in expectation for all states X below the modified threshold.

Fig. 1(b) illustrates the investment threshold X^* for different values of δ and distinguishes the regions where the firm delays investment (continuation) and where the firm invests immediately (stopping). As illustrated, for a given parametrization, there exist initial states, like,



(a) Stopping region and continuation region for different γ with $\delta = 0.1$.

(b) Stopping region and continuation region for different δ with $\gamma = 0.8$.

Fig. 1. Stopping region and continuation region for different parametrizations. $\mu = 0.02$, $\rho = 0.1$, $\sigma = 0.2$, $a = 0.6$, and $\kappa = 0.3$.

e.g., X_i, such that they are in the continuation region for any δ. Others, like X_{ii}, can fall in either region depending on the range of δ values being analyzed.

By first restricting our analysis to the range where initial states remain in the stopping region, like X_{ii} for $\delta \in [0, \delta_1)$, we can provide a characterization of the firm's optimal investment scale even as the threshold increases in δ, because it remains optimal to invest immediately.[7]

For our analysis on the continuation region we, equivalently, first implicitly assume that we restrict our analysis to the range where initial states remain in the stopping region. For a characterization of the firm's optimal strategy when initial states do not belong to the same region, we refer the reader to Appendix B.

3.2. Investment in the stopping region and δ

Let $X \in \mathcal{S}$, i.e. the firm undertakes immediate investment. The parameter δ appears in two terms in , each having a different effect on capital stock $\bar{K}^*(X)$. Restructuring equation gives

$$\underbrace{\overset{\text{"Relative Cost Effect"}}{\frac{\kappa}{a}(\bar{K})^{1-\gamma}\frac{\rho+\gamma\delta-\mu}{X}}}_{\uparrow \text{ as } \delta\uparrow \ \Rightarrow \bar{K}\downarrow} = \underbrace{\overset{\text{"Buffer Effect"}}{\left(1 - 2\frac{a}{\gamma}\bar{K}^{\gamma}\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right)}}_{\uparrow \text{ as } \delta\uparrow \ \Rightarrow \bar{K}\uparrow}. \tag{11}$$

Depending on which of the two effects is dominant, $\bar{K}^*(X)$ is either pushed downward or upward as δ increases. To understand why this happens, for the first effect, notice that when depreciation is stronger, each unit of capital stock will produce less units of output in the future, which means that the expected marginal revenue of capital is negatively affected by depreciation. Although the cost of investment, i.e., the cost of acquiring capital, is unaffected by a change in δ, it becomes relatively less rewarding, or more expensive, to invest when δ is higher. As a result, the optimal quantity goes down. We will call this the *relative cost effect*.

The second effect, where depreciation pushes the quantity up, follows from anticipating the changes in capacity that will restrict output. This gives the firm the incentive to set an initially higher output and to therefore acquire a higher level of capital stock upon investment. We will call this the *buffer effect*.

[7] In other words, the timing decision is the same for $X > X^*$ and thus plays no direct role in the choice of $\bar{K}$.

Nevertheless, we do not explicitly exclude X^* from the analysis in the stopping region, but simply note that the sensitivity of the investment scale will differ on this point if taken from the left and from the right of δ.

To illustrate the buffer effect, consider the situation of a firm with a capital stock that does not depreciate, represented by the solid lines in Fig. 2, where Panel (a) represents the capital stock and Panel (b) represents the instantaneous cash flows $\pi(t) = X(t)(1 - Q(t))Q(t)$ in expectation. If the firm now faces a scenario where its capital depreciates, its capital stock and the corresponding instantaneous cash-inflows will erode over time as illustrated by the dotted curves, assuming the firm has the same capital stock at $t = 0$. This gives the firm an incentive to invest in a higher initial capital stock (and move from the dotted to the dashed curve) and thus increase the area under $\mathbb{E}_X \pi(t)$ (from dotted to dashed) to recapture some of the lost total cash-inflows, which, in essence, is the buffer effect.

The degree to which it is optimal to increase the capital stock, however, will depend on the cost associated with acquiring additional capital stock. Note that the dashed line in Panel (b) was built from the optimal level of capital stock, taking the cost of acquisition into consideration.

The next proposition formally shows that the buffer effect is dominant for small values of δ whereas the relative cost effect is dominant for larger values. Fig. 3(a) illustrates the typical shape of $\bar{K}^*(X)$ as a function of depreciation parameter δ. Fig. 3(b) illustrates how $V(X)$ is overall affected by the depreciation rate δ.

Proposition 2. *Let $\hat{\delta}(X)$ be the (unique) solution to*

$$\frac{X(\rho-\mu)}{2(\rho+\gamma\delta-\mu)^2} = \frac{\kappa}{a}\left(\frac{\gamma}{4a}\left(\frac{\rho+2\gamma\delta-\mu}{\rho+\gamma\delta-\mu}\right)^2\right)^{\frac{1-\gamma}{\gamma}}. \tag{12}$$

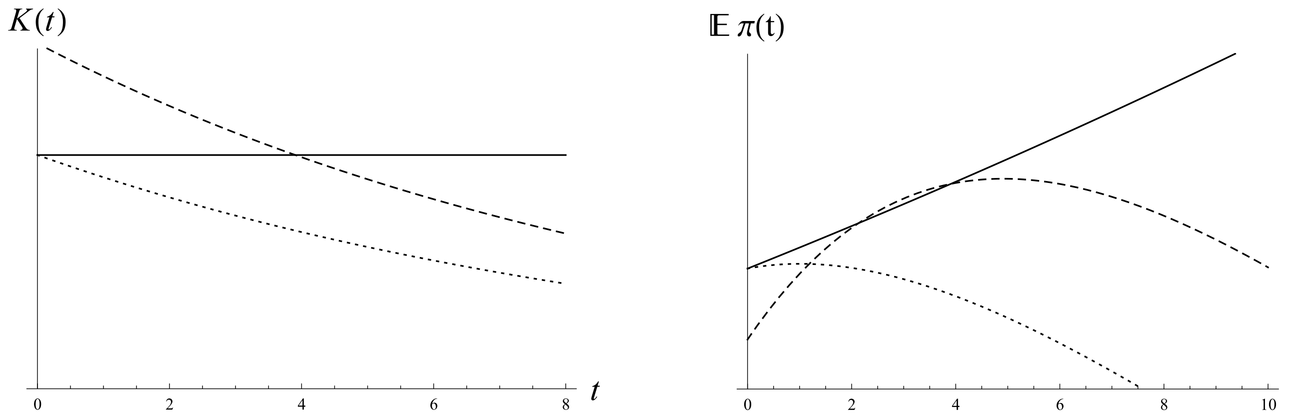
Let $X \in \mathcal{S}$.

(i) If $X > (\rho-\mu)\frac{8\kappa}{\gamma}\left(\frac{\gamma}{4a}\right)^{\frac{1}{\gamma}}$, then $\hat{\delta}(X) > 0$ and
- *for $\delta \in (0, \hat{\delta}(X))$ the capital stock $\bar{K}^*(X)$ is increasing in δ;*
- *for $\delta > \hat{\delta}(X)$ the capital stock $\bar{K}^*(X)$ is decreasing in δ.*

(ii) If $X \leq (\rho-\mu)\frac{8\kappa}{\gamma}\left(\frac{\gamma}{4a}\right)^{\frac{1}{\gamma}}$, then $\hat{\delta}(X) \leq 0$ and
- *for all $\delta > 0$ the capital stock $\bar{K}^*(X)$ is decreasing in δ.*

Appendix B shows that the condition in Case (ii) of Proposition 2 is not typically met, since these values of X are often part of the continuation region.

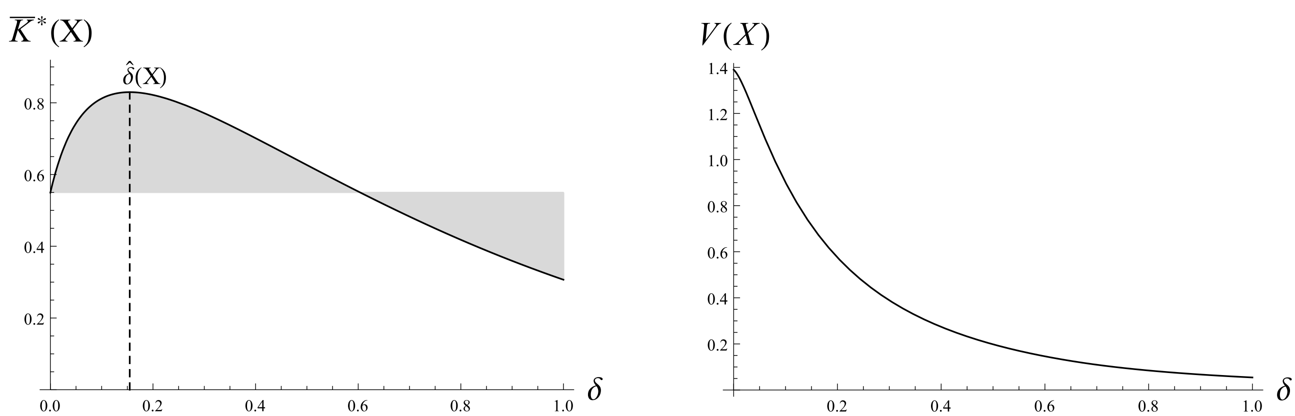
Case (i) of Proposition 2 shows that, in the stopping region, the buffer effect dominates for low rates of depreciation, i.e. the capital stock is increasing in δ. The firm adjusts for the erosion in future productive capacity by *overinvesting*, that is acquiring a higher level of



(a) Level of capital stock as a function of t (stopping region).

(b) Instantaneous cash-inflows (stopping region).

Fig. 2. Illustration of buffer effect. Investment when $\delta = 0$ (solid), when $\delta = 0.1$ but setting capital stock as if $\delta = 0$ (dotted), and optimal investment for $\delta = 0.1$ (dashed). $\mu = 0.02,\ \rho = 0.1,\ \gamma = 0.8,\ X = 0.5$, and $a = 0.6$.



(a) Optimal level of $\bar{K}^*(X)$ as a function of δ.

(b) Firm's value function $V(X)$ as a function of δ.

Fig. 3. The optimal investment size $\bar{K}^*(X)$ and the resulting value to the firm $V(X)$ as a function of depreciation parameter δ. $\mu = 0.02,\ \rho = 0.1,\ \sigma = 0.2,\ \gamma = 0.8,\ X = 0.5,\ a = 0.6$, and $\kappa = 0.3$.

capital stock, relative to the zero-depreciation case, and thus minimizing the loss in total revenue.[8] As illustrated by Fig. 3(a), for higher rates of depreciation, investing in each unit of capital becomes relatively too expensive, since at these rates any additional capital translates very poorly into extra production, while remaining just as costly, and the *net* effect on the size of investment reverses for some $\delta > \hat{\delta}(X)$, which we refer to as *underinvestment* relative to the zero-depreciation case.

Intuitively, economic depreciation creates a need to preemptively replace productive capacity. However, whereas the costs of acquiring more capital scale proportionally in the investment size, the benefits from such investment have diminishing returns on the amount of capital acquired. Therefore, even if the benefits of acquiring capital stock may be large, they can still be dominated by the relative costs of investing at sufficiently high depreciation rates, i.e. when the need for capital replenishment is greater.

For Case (ii) of Proposition 2, consumers' willingness-to-pay is too low for the firm to be able to invest in a large stock and therefore the buffer effect cannot dominate for any δ. This case arises in scenarios where, relatively, κ is high, a is small, and/or X is low relative to $(\rho - \mu)$. In other words, when each unit of capital stock is relatively too expensive.

The overall effect of δ on $V(X)$ in the stopping region, as illustrated in Fig. 3(b), is unambiguously negative. Depreciation leads to lower expected total cash flows which, in turn, translates into a lower value for the firm.

Lemma 1. *Let $X \in \mathcal{S}$. Then $\frac{\partial}{\partial \delta} V(X) < 0$.*

3.3. Investment in the continuation region and δ

Assume that $\mathcal{C}$ is nonempty and let $X \in \mathcal{C}$, i.e. the firm always delays investment. Capital stock $\bar{K}^{opt} = \bar{K}^*(X^*)$ is affected by δ in two ways: indirectly through X^* as studied in Section 3.1 and directly as studied in Section 3.2. The indirect effect is positive: the scale of investment, as established in the real options literature, increases when investment is undertaken at a higher level of the state process, which we find as well. This follows from $\bar{K}^*(X)$ being an increasing function of X (see proof of Proposition 1). The direct effect can be mixed as established in Section 3.2. Similar to what we did to investigate the effect of depreciation on the investment threshold, one can easily verify that, since $\frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu}$ is an increasing function of δ, $\bar{K}^{opt}$ must also be increasing in δ, which follows directly from Eq. (9). Thus, the overall effect of δ on the scale of investment $\bar{K}^{opt}$ is unambiguously positive

[8] It is worth noting that this result is not driven by the capacity clearing constraint. In fact, relaxing this assumption will amplify the buffer effect, and therefore the non-monotonicity observed in this case is preserved.

as depreciation gets stronger and the indirect effect always dominates any negative direct effects.

Consequently, it can be surprising that even though a higher rate of depreciation brings about a larger scale of investment, this does not translate into a higher firm value. In fact, the result that is valid for the stopping region (Lemma 1) can also be observed in the continuation region, as shown in the following lemma.

Lemma 2. *Let $X \in C$. Then* $\frac{\partial}{\partial \delta} V(X) < 0$.

One must keep in mind that $\bar{K}^{opt}$ represents the size of investment (at the time of investment) and, although more rapid depreciation leads to an increase in the 'size of initial capital stock', it also leads to an expected delay in the timing of investment, and the delay dominates the effect on the firm's value even when the firm is free to adjust its capacity (δ hurts the cash flow per capital). So, to capture the compounded effect of δ on the size of investment, in line with the literature (e.g., Sarkar, 2019), we resort to studying the expected present value of capital investment, expressed as $EPVI = \mathbb{E}_X \left\{ e^{-\rho \tau^*} \kappa I(\tau^*) \right\} = \left(\frac{X}{X^*} \right)^\beta \kappa \bar{K}^{opt}$. The opposing effects that δ has on the investment size become evident in the formulation of $EPVI$, given by

$$EPVI = \kappa \underbrace{\left(\frac{\gamma}{a} \frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1} \frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu} \right)^{\frac{1-\beta(1-\gamma)}{\gamma}}}_{\uparrow \text{ as } \delta\uparrow} \underbrace{\left(\frac{\beta-1}{\beta(2\gamma-1)+1} \frac{a}{\kappa} \frac{X}{\rho+\delta\gamma-\mu} \right)^{\beta}}_{\downarrow \text{ as } \delta\uparrow}, \quad (13)$$

for all $X \in C$. Notice that $\beta(1-\gamma) < 1$, which follows from the condition in Proposition 1 that allows for non-empty C.

The following lemma shows that the negative effect is always dominant for the present value of capital investment, even when the level of capital stock upon investment is increasing in δ. Hence, the decrease in present value of capital investment due to the delay outweighs the increase in the optimal level of capital stock at the time of investment, in expectation.

Lemma 3. *Let $X \in C$. Then* $\frac{\partial}{\partial \delta} EPVI < 0$.

4. The effect of uncertainty on investment

Section 3.3 showed that X^* and $\bar{K}^{opt}$ are increasing in δ, but the present value of capital (the $EPVI$) is decreasing in δ. This section discusses the effect of uncertainty on the firm's investment and in particular it discusses the interplay between depreciation and uncertainty.

In the early real options literature, studies with fixed capacity (see, e.g., Bernanke, 1983, Bertola, 1988, Pindyck, 1988, and Dixit, 1989) established a direct relationship between the volatility of the underlying process and the firm's real option.[9] Uncertainty pushes the associated investment threshold up and leads to a "late" exercise of the options so that uncertainty is bad for investments. However, since the increased uncertainty also directly affects the distribution of the underlying stochastic process and, therefore, the probability of reaching a higher threshold within a certain amount of time (see, e.g., Lund, 2005, and Sarkar, 2019), an appropriate way to measure the effect of uncertainty on the investment option is to look at the present value of the option. The option value not only incorporates expectations about future cash-flows, it also provides a way of measuring how the firm values its *flexibility*. For these models, the relationship is unambiguous: more uncertainty increases the value of the option. When

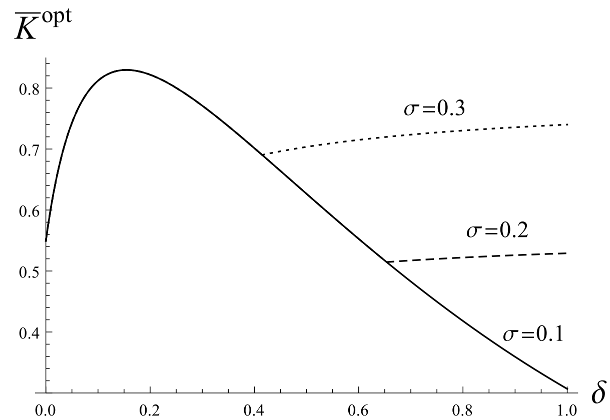


Fig. 4. Optimal level of $\bar{K}$ as a function of δ for different values of σ: $\sigma = 0.1$ (solid), $\sigma = 0.2$ (dashed), and $\sigma = 0.3$ (dotted).

including capacity choice, the same relationship can be established (see, e.g., Bar-Ilan and Strange, 1999, and Huberts et al., 2015).

On the one hand, ceteris paribus, a higher rate of depreciation has a negative impact on cash-flows, as shown by Lemma 1, and when capacity choice is allowed, this translates into a larger capital investment size (i.e., $\bar{K}^{opt}$ goes up), but a lower $EPVI$ (see Lemma 3) due to the expected further delay in the exercise of the option. On the other hand, this expected further delay reflects an increase in the option value. Also the empirical literature has often shown that the impact of uncertainty on investments can be level dependent (e.g., Lensink and Murinde, 2006, Jeanneret, 2007, Mohn and Misund, 2009, and Henriques and Sadorsky, 2011). It is, thus, not immediately clear how depreciation impacts the relationship between uncertainty and investment.

Therefore, similar to the previous section's structure, let us first address the effects of uncertainty on the optimal investment timing, through the effect of σ on the investment threshold X^* in Section 4.1, before addressing the effect of uncertainty on the scale and present value of capital investment in Section 4.2.

4.1. Timing of investment and σ

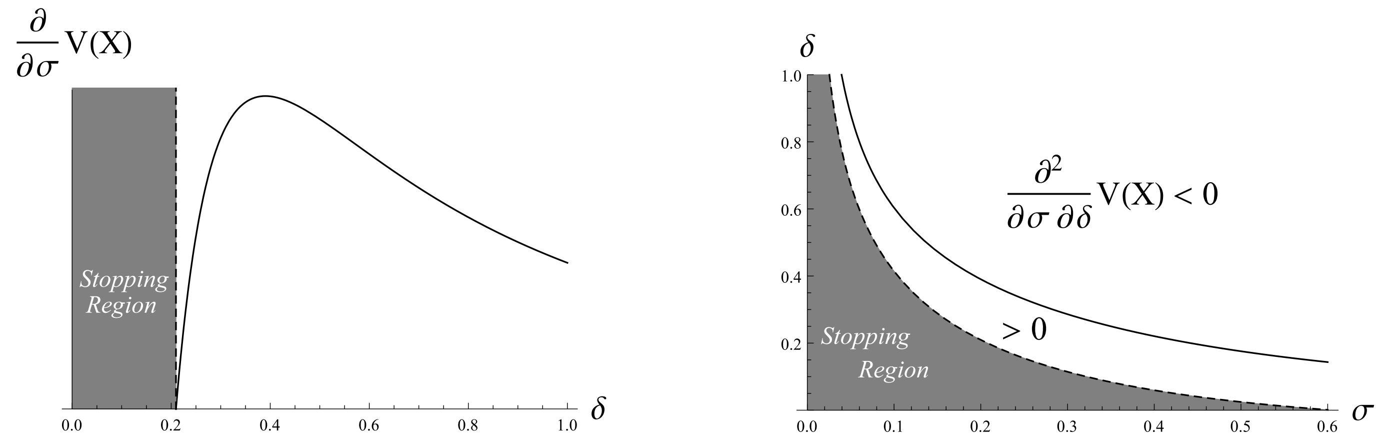
One can check directly from (8) that higher uncertainty increases the investment threshold. In addition, it follows from Eq. (9) that the investment size is decreasing in β and therefore increasing in σ. This means that as a higher level of uncertainty pushes the investment threshold up, the firm acquires a higher capital stock upon investment. Fig. 4 illustrates these findings. The solid line represents the value in the stopping region, i.e., when the firm invests immediately. The dashed and dotted line represent the values if the firm were to delay investment and undertake investment at $X^* > x(0)$, which applies to cases where δ is sufficiently large. Notice that because the investment threshold goes up when σ is higher, the rate of depreciation required to have that $x(0) = X^*$ goes down. Lemma 4 will show that the increase in the threshold, when σ goes up, is amplified by depreciation. In other words, depreciation compounds on the delay associated with more uncertainty. This may suggest that the value the firm attaches to flexibility is larger when depreciation is stronger.

As argued above, an adequate way to measure the impact of uncertainty on timing of investment is to look at the option value V (in (10)). Lemma 4 first shows that, for a fixed rate of depreciation, σ has a positive impact on the option value, i.e., the firm always attaches more value to flexibility when there is more uncertainty.

Lemma 4. *Let $X \in C$. Then*

(i) $\frac{\partial}{\partial \sigma} V(X) > 0$, *and*

[9] See, e.g., Gryglewicz et al. (2008) and Sarkar (2019) for a more contemporary discussion on the uncertainty–investment relationship in the theoretical and empirical literature.



(a) Illustration of derivative $\frac{\partial}{\partial\sigma}V(X)$ for different δ with $\sigma = 0.2$.

(b) Regions distinguishing sign of cross derivative.

Fig. 5. Compound effect of uncertainty and depreciation. $\mu = 0.02$, $\rho = 0.1$, $\gamma = 0.8$, $X = 0.2$, $a = 0.6$, and $\kappa = 0.3$.

(ii) $\frac{\partial^2}{\partial\delta\partial\sigma}X^* > 0.$

To understand the first result, first note that uncertainty does not have an impact on the expected payoffs after investment (cf. the value in the stopping region). However, since σ is positive related to the threshold, the expected firm value upon investment is increasing in σ. In addition, the value of waiting, which stems from uncertainty, is also increasing in σ, as argued above.

To evaluate the role of depreciation further, Fig. 5(a) illustrates the derivative in (i) for different levels of δ. Note that for small δ or σ it holds that $X^* < X$ so that for these parameter values X falls in the stopping region. Although σ always increases V, the role δ plays is level-dependent: the impact of uncertainty is amplified when the δ is low but for high rates the figure shows that depreciation mitigates the impact of uncertainty on option value. The following proposition looks at the relationship between depreciation and uncertainty by studying the sensitivity of the option to uncertainty for changes in δ and provides a condition such that *mitigation* occurs.

Proposition 3. *Let $X \in C$. Then $\frac{\partial^2}{\partial\delta\partial\sigma}V(X) < 0$ if and only if*

$$\ln\left(X^*\right) - \ln(X) > \left(\beta - \frac{\rho-\mu}{\rho + 2\gamma^2\delta - \mu}\right)^{-1}. \tag{14}$$

Since X^* on the left-hand side of (14) is increasing in δ and since the right-hand side is decreasing in δ, the inequality in (14) does not hold for sufficiently high values of δ. As a result, one can indeed distinguish two regions with respect to δ and σ, as illustrated by Fig. 5(b). To interpret this condition, notice that (14) does not hold if X^* and X are sufficiently close. Then, investment is expected to be undertaken soon, which means that the firm values the option, and therefore flexibility, more when the demand is more uncertain. When investment is not expected to be undertaken soon, depreciation mitigates the (positive) impact of uncertainty on the option value, which is reflected by $\frac{\partial^2}{\partial\delta\partial\sigma}V(X) < 0$.

Hence, an increase in the rate of depreciation can mitigate the impact of uncertainty on V but only if depreciation is sufficiently strong.

4.2. Capital investment

Having understood the impact of uncertainty on optimal investment timing (through its impact on V) and how the magnitude of that impact depends on the rate of depreciation, we will now discuss the impact that uncertainty has on the scale of capital investment (through its impact on the $EPVI$) and how this impact's direction and magnitude, both, depend on the rate of depreciation.

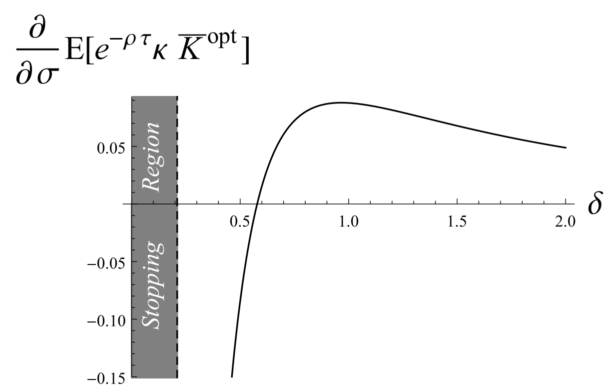


Fig. 6. Illustration of derivative $\frac{\partial}{\partial\sigma}EPVI$ for different δ.

There is no impact of σ on the $EPVI$ in the stopping region (cf.). In the continuation region, however, for similar reasons as why the $EPVI$ could be ambiguous in δ, it is not immediately clear what the impact is of σ on the $EPVI$. This ambivalence becomes evident if we inspect the impact of σ in (13), and so we would have

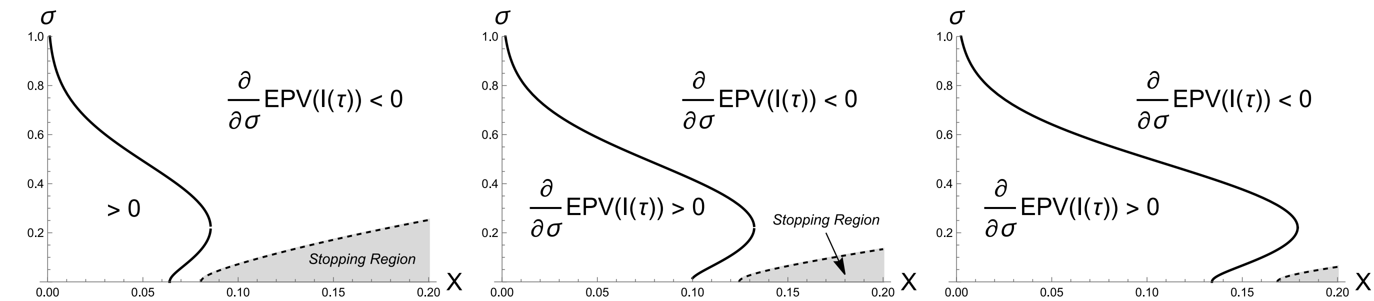
$$EPVI = \kappa \underbrace{\left(\frac{\gamma}{a}\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu}\right)^{\frac{1-\beta(1-\gamma)}{\gamma}}}_{\uparrow\text{ as }\delta,\sigma\uparrow} \underbrace{\left(\frac{\beta-1}{\beta(2\gamma-1)+1}\frac{a}{\kappa}\frac{X}{\rho+\delta\gamma-\mu}\right)^{\beta}}_{\downarrow\text{ as }\delta,\sigma\uparrow}, \tag{15}$$

which shows that σ has mixed effects on the $EPVI$. Fig. 6 illustrates which effect dominates and illustrates that this is dependent on the level of depreciation. The following proposition provides a condition to determine the impact of uncertainty on the $EPVI$, which confirms that the findings in Fig. 6 are, in fact, a general result.

Proposition 4. *Let $X \in C$. Then $\frac{\partial}{\partial\sigma}EPVI > 0$ if and only if*

$$\ln(X^*) - \ln(X) > \frac{(\beta(1-\gamma)-1)^2 + \beta^2\gamma^2}{\beta\gamma(\beta-1)(1+\beta(2\gamma-1))}. \tag{16}$$

Since X^* is increasing in δ, we have that condition (16) is satisfied for sufficiently high δ. To understand this, recall that σ impacts both



(a) $\delta = 0.1$. (b) $\delta = 0.2$. (c) $\delta = 0.3$.

Fig. 7. Regions where the present value of capital is increasing resp. decreasing in σ. $\mu = 0.02$, $\rho = 0.1$, $\gamma = 0.9$, $a = 0.6$, and $\kappa = 0.3$.

the investment timing and size. Since high values of δ already (significantly) delay investment, any further increases in the threshold have a relatively smaller impact on the $EPVI$ when, at the same time, the volatility goes up, which allows the increase in size to dominate. When δ is small, the present value of capital investment is more sensitive to delays in investment so that the $EPVI$ is negatively impacted.

Note that when X is close to zero, the left-hand side of (16) becomes sufficiently large, so that the condition holds for any δ. The regions where (16) holds are illustrated in Fig. 7 for various constellations of X and σ and for different rates of depreciation. It confirms again that the positive effect that σ has on the $EPVI$ is amplified by additional depreciation, and that the negative effect that σ has on the $EPVI$ becomes less dominant in the sense that the area for which the positive effect dominates expands with δ.

Another observation that can be made from Fig. 6 is that for high values of δ, $\frac{\partial}{\partial\sigma}EPVI$ is decreasing in δ. This means that the magnitude of the impact of uncertainty becomes smaller for high values of δ. The following proposition confirms that this applies for sufficiently high rates of depreciation. Thus, we find that uncertainty has a diminished effect on the $EPVI$ when the rate of depreciation is higher.

Proposition 5. *Let $X \in C$ and let $\delta \geq 0$. There exists a $\delta^\circ \geq 0$ such that $\frac{\partial}{\partial\delta}\frac{\partial}{\partial\sigma}EPVI < 0$ if and only if $\delta > \delta^\circ$.*

To understand this result, first note that for high rates of depreciation, investment is taking place at a later point in time than for low rates. Thus, the magnitude of the impact of any increase in the threshold is relatively smaller in terms of present value, when δ is higher. Moreover, as can be observed in Fig. 4, an increase in δ has a relatively small impact on the difference in size of investment when comparing the cases $\sigma = 0.3$ and $\sigma = 0.2$. This observation indicates that $\frac{\partial}{\partial\sigma}\bar{K}^{opt}$ is smaller for higher δ. Therefore, an increase in σ, when simultaneously increasing δ, predominantly impacts timing and not the size of investment. These two effects together lead to the result that the impact of σ diminishes as the rate of depreciation is higher.

5. Robustness

In this section the robustness of our results is tested when considering two alternative versions of the main model. In Section 5.1 an alternative way of modeling economic depreciation is considered: capital is assumed to not depreciate at a constant geometric rate, but instead is assumed to remain fully productive whilst having a finite life-time. This section illustrates that this formulation leads to a qualitatively different investment strategy: some of the intuitive dynamics are lost when eliminating constant depreciation.

In Section 5.2 the main model is extended using a (s,S)-inspired type of (inventory investment) model to offer the firm the possibility to replenish its capital stock.[10] This section shows that our results do not depend, qualitatively, on the assumption that the firm has a one-off opportunity to acquire capital.

Finally, Section 5.3 comments on alternative specifications of the inverse demand function and shows that overinvestment and underinvestment can be found for convex or concave functional forms.

5.1. Full depreciation in finite time

In the literature, there exist studies that consider an alternative approach to depreciation. These papers assume capital stocks to retain full capabilities until a future, and potentially stochastic, moment in time when it fully depreciates.[11] Although such models would describe a different scenario than the scenario we are considering, e.g., Dixit and Pindyck (1994) refer to this *one-hoss shay* model as a case of depreciation as well (Chapter 6). More recent contributions have been made by, e.g., Gryglewicz et al. (2008), Nakamura (2007), Femminis (2008), and Saltari and Ticchi (2005), as discussed in Section 1. To emphasize the relevance of the way this paper models depreciation and to show that the two types of depreciation are not interchangeable/synonymous this section briefly discusses what happens if capital retains its productive capabilities, but fully depreciates after a fixed moment in time.

Denote by $\lambda > 0$ the lifetime of capital. The firm's optimization problem can then be written as

$$V(X) = \sup_{\tau\geq 0, \bar{K}>0} \mathbb{E}_X \left\{ \int_0^\infty p(t)Q(t)e^{-\rho t}\mathrm{d}t - e^{-\rho\tau}\kappa\bar{K} \right\},$$

with

$$Q(t) = \begin{cases} 0 & \text{if } t < \tau \text{ or } t > \tau + \lambda, \\ \frac{a}{\gamma}\left(\bar{K}\right)^\gamma, & \text{if } t \in [\tau, \tau+\lambda]. \end{cases}$$

One can show that the optimal capital stock in the stopping region the solution of

$$\frac{\kappa}{a}(\bar{K})^{1-\gamma}\frac{\rho-\mu}{X} = \underbrace{\overbrace{\left(1 - 2\frac{a}{\gamma}\bar{K}^\gamma\right)}^{\text{"Relative Cost Effect"}}\left(1 - e^{-\rho\lambda}\right)}_{\uparrow \text{ as } \lambda\downarrow \;\Rightarrow\; \bar{K}\downarrow}. \tag{17}$$

[10] This is inspired by the abundant literature on inventory problems, also known as lot sizing problems, emerged after the seminal work by Scarf (1960). This type of models can commonly be found in the Economics literature on irreversible investments with a stochastic state process, see, e.g., Federico et al. (2019). The authors also provide an extensive summary of this literature stream and show optimality for problems very similar to ours.

[11] Classic examples that fit this model include light-bulbs and carriages but in the context of our set-up one can think intuitively of fundamental elements of a machine that once they begin to break, the whole machine needs replacing.

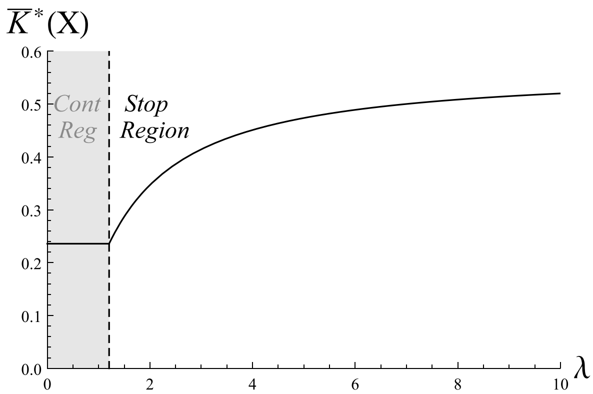


Fig. 8. Optimal capital stock $\bar{K}^*(X)$ as a function of λ.

With λ only appearing in the right-hand side of (17), stronger depreciation is only associated with a lower value of $\bar{K}$, meaning that we lose the buffer effect. A lower value for the lifespan of capital, λ, leads to a smaller period over which revenues are accumulated and, as such, investment is relatively more expensive. Fig. 8 illustrates this effect.

This characterization also illustrates that the buffer effect originates from compensating for losses caused by gradual erosion in productivity. Since this type of depreciation does not impose such losses, the firm does not compensate by increasing the capital stock when depreciation is stronger.

The threshold is given by

$$X^* = \left(\frac{\gamma}{a}\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\right)^{\frac{1-\gamma}{\gamma}} \frac{\beta(2\gamma-1)+1}{\beta-1}\frac{\kappa}{a}\frac{\rho-\mu}{1-e^{-\rho\lambda}}, \tag{18}$$

which is decreasing in λ, i.e. investment is delayed when the lifetime is short, that is, when λ goes down.

It is interesting to note that, in contrast to our main model, the capital stock upon investment, when starting in the continuation region, $\bar{K}^{opt}$ is not affected by λ,

$$\bar{K}^{opt} = \left(\frac{\gamma}{a}\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\right)^{\frac{1}{\gamma}}.$$

This means that, in the absence of the buffer effect, we find that the effect of delaying on the scale of investment is matched by the relative cost effect, with the firm waiting for higher prices to exactly compensate for the loss in expected revenue. This then also leads to the direct result that the expected present value of capital is lower when the lifetime is shorter. In relation to Section 4.2, we therefore also find that when a one-hoss shay model is assumed, depreciation only impacts the uncertainty–investment relationship through timing.

Furthermore, for $X \in C$, we have that

$$\frac{\partial}{\partial\lambda}V(X) = V(X)\frac{\rho}{e^{\rho\lambda}-1}(\beta-1) > 0, \tag{19}$$

i.e. the option value increases when the lifetime is longer. Moreover,

$$\frac{\partial^2}{\partial\sigma\partial\lambda}V(X) < 0 \quad \text{if and only if} \quad \ln(X^*)-\ln(X) < 2(\beta-1)^{-1}. \tag{20}$$

Both (19) and (20) are equivalent results to those found in Lemma 2 and Proposition 3.

5.2. Option to replenish

It could be argued that the effect of depreciation on the scale of investment is partially driven by the assumption that the firm can only acquire capital once. Therefore, in this section, we introduce a simple extension to our model where the firm, in principle, can replenish its capital stock an infinite number of times and we will argue that, qualitatively, this assumption has no impact on what was already established for the main model. We do not aim to fully analyze the outcomes of this extension, but rather to illustrate that both the buffer effect and the relative cost effect are still present, and that the investment behavior in the stopping region is not qualitatively different from our main model.

Consider the scenario where the monopolist, after undertaking investment, can replenish its capital stock. We assume that the firm, upon investment, will choose a $\bar{K}$ and a $\underline{K}$, with $\underline{K} < \bar{K}$, such that when the capital stock reaches the level $\underline{K}$, the firm acquires additional capital to reset its capital stock to a level equal to $\bar{K}$. Purchasing any $\tilde{K} > 0$ units of capital stock is associated with costs $\kappa_0\tilde{K}+\kappa_1$. This formulation introduces a fixed cost, which was not present in our main model. This term ensures that the optimal *replenishment time* T is strictly positive, where T follows from $K(T) = \underline{K}$, i.e., $T(\underline{K},\bar{K}) = \frac{1}{\delta}\ln(\bar{K}/\underline{K})$. When $T \to 0$ the model collapses into a continuous investment model, which contradicts our original assumption that this manner of investment cannot actually take place in real life.

Note that since depreciation is deterministic, the period between replenishing is fixed. This could arise from contractual reasons as, e.g., commitments to the supplier (see, e.g., Dural-Selcuk et al. (2016) for an overview of the literature on (s,S)-type policies with stochastic demand). Thus, the firm places an order every $T(\underline{K},\bar{K})$ periods. Therefore, the firm's optimization problem with respect to $\underline{K}$ and $\bar{K}$ could also be written as the firm choosing $\bar{K}$ and T, so that $\underline{K} = \bar{K}e^{-\delta T}$. Because the latter formulation allows for an easier comparison between the results of this model and the main model, we will continue with the firm choosing $\bar{K}$ and T. In the stopping region, the firm then considers the following optimization problem

$$\sup_{\bar{K},T\geq 0}\mathbb{E}_X\left\{\sum_{i=0}^{\infty}\left(\int_{t=iT}^{(i+1)T} X(t)(1-Q_i(t))Q_i(t)e^{-\rho t}\mathrm{d}t - e^{-\rho iT}\left[\kappa_0\left(\bar{K}-\bar{K}e^{-\delta T}\right)+\kappa_1\right]\right)-\kappa_0\bar{K}e^{-\delta T}\right\}, \tag{21}$$

where $Q_i(t) = \frac{a}{\gamma}(K_i(t))^\gamma$ and $K_i(t) = \bar{K}e^{-\delta(t-iT)}$ are the output and capital stock, respectively, for the $(i+1)$-th cycle, for all $i = 0,1,2,\ldots$, and $iT \leq t < (i+1)T$. Eq. (21) contains of three terms: two terms inside the integral and one term at the end. The first terms represents the firm's discounted instantaneous cash-inflows. The total discounted cost for each replenishment is captured by the second term for $i \geq 1$. The total cost involved with the initial investment is captured by the second term for $i = 0$, $\kappa_0(\bar{K}-\underline{K})+\kappa_1$, plus the final term, which can also be written as $\kappa_0\underline{K}$.

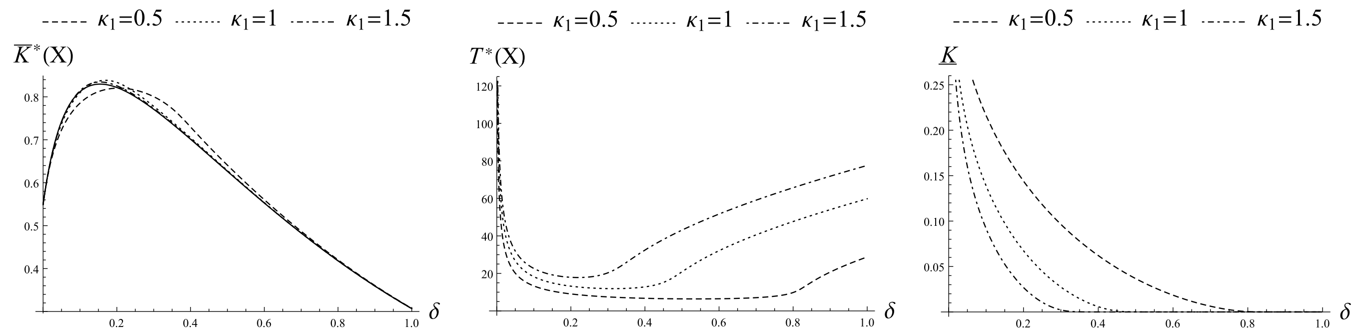
The firm's optimization problem in (21) can be rewritten as

$$\begin{aligned}
&\sup_{\bar{K},T\geq 0}\sum_{i=0}^{\infty} e^{-(\rho-\mu)iT}\left[\frac{X}{\rho+\gamma\delta-\mu}\frac{a}{\gamma}\bar{K}^\gamma\left(1-e^{-(\rho+\gamma\delta-\mu)T}\right)\right.\\
&\qquad\left.-\frac{X}{\rho+2\gamma\delta-\mu}\left(\frac{a}{\gamma}\bar{K}^\gamma\right)^2\left(1-e^{-(\rho+2\gamma\delta-\mu)T}\right)\right]\\
&\qquad-\sum_{i=0}^{\infty}e^{-\rho iT}\left[\kappa_0\left(\bar{K}-\bar{K}e^{-\delta T}\right)+\kappa_1\right]-\kappa_0\bar{K}e^{-\delta T}\\
&=\sup_{\bar{K},T\geq 0}\frac{X}{\rho+\gamma\delta-\mu}\frac{a}{\gamma}\bar{K}^\gamma\frac{1-e^{-(\rho+\gamma\delta-\mu)T}}{1-e^{-(\rho-\mu)T}}\\
&\qquad\left[1-\frac{a}{\gamma}\bar{K}^\gamma\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\frac{1-e^{-(\rho+2\gamma\delta-\mu)T}}{1-e^{-(\rho+\gamma\delta-\mu)T}}\right]\\
&\qquad-\kappa_0\bar{K}\frac{1-e^{-(\rho+\delta)T}}{1-e^{-\rho T}}+\frac{\kappa_1}{1-e^{-\rho T}}.
\end{aligned}$$

First order conditions give the optimal capital stock $\bar{K}^*(X)$ and period $T^*(X)$.[12] The optimal capital stock is given as the solution of

$$\frac{\kappa}{a}(\bar{K})^{1-\gamma}\frac{\rho+\gamma\delta-\mu}{X}\frac{1-e^{-(\rho-\mu)T^*(X)}}{1-e^{-(\rho+\gamma\delta-\mu)T^*(X)}}\frac{1-e^{-(\rho+\delta)T^*(X)}}{1-e^{-\rho T^*(X)}}$$

[12] The Hessian confirms that this is a local maximum. Numerical analysis shows that the maximum is global.



(a) Optimal level of $\bar{K}(X)$ as a function of δ. (b) Optimal $T(X)$ as a function of δ (c) Optimal level of $\underline{K}$ as a function of δ

Fig. 9. Optimal investment strategy for the firm with different κ_1. The solid curve represents the investment strategy for the main model (Section 2). $\mu = 0.02,\ \rho = 0.1,\ \sigma = 0.2,\ \gamma = 0.8,\ a = 0.6$, and $\kappa_0 = 0.3$.

$$= \left(1 - 2\frac{a}{\gamma}\bar{K}^{\gamma}\,\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\,\frac{1-e^{-(\rho+2\gamma\delta-\mu)T^*(X)}}{1-e^{-(\rho+\gamma\delta-\mu)T^*(X)}}\right). \tag{22}$$

Notice that, for sufficiently large T, Eq. (22) gives a (nearly) identical solution to $\bar{K}^*(X)$ as (11), where the buffer effect and relative cost effect can also be observed in a model where the firm can top-up its capital stock. In fact, for $\delta \to 0$, they are identical for any T.

Fig. 9 illustrates the optimal strategy of the firm in the stopping region. Panel (a), equivalent to Fig. 3, illustrates the optimal capital stock $\bar{K}^*(X)$. The solid curve represents the optimal capital stock as found for the main model. The dashed, dotted, and dash–dotted curves correspond to scenarios where the firm is faced with fixed replenishment cost $\kappa_1 = 0.5$, $\kappa_1 = 1$, and $\kappa_1 = 1.5$, respectively. The panel illustrates that the curves are qualitatively equivalent and that for κ_1 sufficiently large, the acquired capital stock is nearly identical to the capital stock the firm would set when it was only allowed to invest once.

Panel (b) and Panel (c) illustrate the optimal replenishment time and the resulting replenishment threshold $\underline{K}$, respectively. For small values for δ, capital depreciates slowly so that the cycle can be long, i.e. $T^*(X)$ is large, whilst not letting capital completely depreciate before replenishment, i.e. $\underline{K} > 0$. For higher values of δ it is optimal to choose a lower $\underline{K}$.

Due to the presence of the fixed cost in this model, investment will be delayed compared to the main model. Nonetheless, we conclude that the direct effects of depreciation on the firm's capital stock, as described in the main model, are preserved when the investment strategy is modeled as a (s,S)-type of policy.

5.3. Alternative inverse demand specifications

One could raise the issue as to whether our results are driven by the linearity of the inverse demand function. Consider a slight generalization of our model by assuming the inverse demand to be

$$p(t) = x(t)(1 - Q^{\eta}(t)),$$

where $\eta > 0$. Notice that for $\eta < 1$ the inverse demand is convex and for $\eta > 1$ the inverse demand is concave. One can verify that for this specification the result as described by Proposition 2, i.e., the buffer effect dominates for small rates δ for sufficiently high X, still applies unless η is sufficiently close to 0. Our other results, in particular the findings in Section 4 are also not qualitatively impacted by a change in η. Therefore, our results are not contingent on the linearity of the inverse demand function.

One could, however, identify some functional forms that lock in the relationship of the two effects we described, such that the relative cost effect always dominates for all δ; incidentally, this drastic characterization of the functional form can be achieved by imposing constant own-price elasticity on the demand function for the product.

6. Conclusions and managerial implications

The framework introduced is admittedly simple, yet sufficiently robust to characterize the effects of eroding productivity on the optimal investment behavior of a monopolist under uncertainty. Without resorting to any tax implications, we reveal competing direct (buffer and relative cost) and indirect (investment threshold) effects on the optimal scale and timing of investment. In addition, the impact of uncertainty is studied in relation to the rate of depreciation. Also here, the role depreciation plays on the present value of capital investment and the option value are level-dependent.

We find that, despite different competing incentives, more depreciation unambiguously increases the investment threshold, i.e. depreciation can lead to a later exercise of the option and, upon a delayed investment, this increases the level of capital stock acquired, while decreasing the present value of capital. Additionally, we find that, under not too strict conditions, an increase in depreciation can lead to overinvestment, when comparing to the standard zero-depreciation case, but always leads to underinvestment for sufficiently high rates.

Furthermore, we find that depreciation can either have a compounding or a mitigating effect on the impact of uncertainty on the firm's option value and the expected present value of capital investment, again depending on the level of depreciation. A higher rate of depreciation makes the firm invest in a larger capital stock albeit delaying the moment of investment. More uncertainty has the same direct effect on timing and scale. When depreciation is strong, the former effect leads to a net increase in the present value of capital investment, whereas the latter is dominant for small rates of depreciation, as hinted by empirical findings in the literature. For the same reason we find that the impact of uncertainty on the option value, representing the present value of the firm, is mitigated only when depreciation is strong. An overview of all our key findings can be found in Table 1.

Our findings illustrate that the treatment of economic depreciation is not trivial when addressing a monopolist's investment problem in a dynamic and uncertain market environment. In fact, alternative modeling choices, as illustrated in Sections 5.1 and 5.3, may actively hide or dismiss the effects we have identified. We are able to verify that our findings are robust to set-ups with multiple sequential replenishment options (Section 5.2) and with non-linear demand specifications.

The results in this paper lead to some important insights for managers and/or investors who have the real option to undertake an irreversible investment and enter a new market. *First*, the analysis in Sections 3 and 4 have shown that the rate of depreciation dictates the

Table 1
Overview of (key) findings.

Main Results
1. Economic depreciation makes the firm delay investment.
2. The rate of economic depreciation dictates whether the firm overinvests or underinvests relative to a zero-depreciation benchmark.
3. Firm value is negatively impacted by economic depreciation.
4. If the firm delays investment, the optimal capital stock level upon investment is increasing in the rate economic depreciation.
5. The present value of capital investment is decreasing in the rate of economic depreciation.
6. Economic depreciation mitigates the impact of uncertainty on firm value only if the rate of economic depreciation is sufficiently high.
If the rate of economic depreciation is sufficiently high, then
7a. the present value of capital investment is increasing in uncertainty; and
7b. uncertainty has a diminished effect on the present value of capital investment.

level of investment when investment is undertaken immediately, impacts the investment timing, and plays a crucial role in the investment–uncertainty relationship (also see Table 1). In fact, efforts to innovate and extend the productive lifetime of a productive asset could be modeled into our framework as a lower δ for the same productive technology. As such, it becomes evident, that it is the productive capacity of the capital stock that drives crucial decisions such as how much and when to invest. Thus, when depreciation cannot be continuously offset by frictionless investment, it is important that managers take care to consider what is the level of depreciation rate of the capital stock they can acquire. *Second*, when evaluating managers' investment plans as measured by the present value of capital investment while optimizing for firm value, the impact of additional uncertainty of the output market needs to be considered in relation to how quickly the productive capacity deteriorates. At low depreciation rates, the impact of extra uncertainty is amplified in discouraging investment (i.e. lower $EPVI$) whereas high rates of depreciation can mitigates the impact of uncertainty shocks and actually encourage investment (i.e. higher $EPVI$).

Third, even if the investor finds themselves in the continuation region, they may wish to be prepared for the eventual triggering condition, and therefore have everything lined up to finance and buy the capital stock $\bar{K}^{opt}$. To prepare diligently and to provision the necessary investment at the right time, they should also update their expectation of how much capital they plan to acquire ahead of time. Should there be any shocks to that lead to a different risk environment, the findings in this paper corroborate with the need to adjust that provision upwards in case there is higher risk, and downwards otherwise, conditional on the depreciation rate in place.

For future research, with this framework at hand, more complex cases can be investigated, such as (i) the decision to replace capital with the same or with superior productivity, (ii) the interplay of the tax benefits and the productivity losses of depreciation, and (iii) optimal investment behavior on Incumbent-Entrant games and/or other competitive setups. In addition, a potentially interesting avenue of exploration could be to bridge our work and the literature that considers risk-adjusted discount rates.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Proofs

Proof of Proposition 1. In the stopping region, i.e. for $X \in \mathcal{S}$, we get

$$\begin{aligned} V(X) &= \mathbb{E}_X \left\{ \int_0^\infty e^{-\rho t} \frac{a}{\gamma} (\bar{K})^\gamma e^{-\delta\gamma t} x(t) \left(1 - \frac{a}{\gamma}(\bar{K})^\gamma e^{-\delta\gamma t}\right) \mathrm{d}t - \kappa\bar{K} \right\} \\ &= \mathbb{E}_X \left\{ \int_0^\infty \frac{a}{\gamma}(\bar{K})^\gamma e^{-t(\rho+\delta\gamma)} x(t)\mathrm{d}t \right. \\ &\quad \left. - \int_0^\infty \left(\frac{a}{\gamma}(\bar{K})^\gamma\right)^2 x(t) e^{-t(\rho+2\delta\gamma)}\mathrm{d}t - \kappa\bar{K} \right\} \\ &= \frac{a}{\gamma}(\bar{K})^\gamma \left(\frac{X}{\rho+\gamma\delta-\mu} - \frac{a}{\gamma}(\bar{K})^\gamma \frac{X}{\rho+2\gamma\delta-\mu}\right) - \kappa\bar{K}. \end{aligned} \tag{23}$$

The optimal capital stock at investment follows from the first order condition where marginal revenue (left-hand side) is equal to marginal cost (right-hand side):

$$\frac{X}{\rho+\gamma\delta-\mu}\left(1 - 2\frac{a}{\gamma}\bar{K}^\gamma \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) = \frac{\kappa}{a}(\bar{K})^{1-\gamma}. \tag{24}$$

The second order condition shows that this is indeed a maximum:

$$\begin{aligned} \frac{\partial^2}{\partial\bar{K}^2}V(X) = -\frac{X}{\rho+\gamma\delta-\mu} a(\bar{K})^{\gamma-2} \Bigg[(1-\gamma)\left(1 - 2\frac{a}{\gamma}\bar{K}^\gamma \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) \\ + a\bar{K}^\gamma \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\Bigg] < 0. \end{aligned}$$

Notice that it follows directly from (24) that (24) only gives solutions where $\bar{K} \geq 0$ so that we always have interior solutions. In addition, we find that $\frac{\partial}{\partial X}\bar{K}^*(X)$ is always positive by applying the implicit function theorem, which gives

$$\begin{aligned} &\frac{\partial\bar{K}^*(X)}{\partial X}\left[\frac{\kappa}{a}(1-\gamma)(\bar{K}^*(X))^\gamma + \frac{X}{\rho+\delta\gamma-\mu}2a\frac{\rho+\delta\gamma-\mu}{\rho+2\delta\gamma-\mu}\right] \\ &= \gamma\frac{1 - 2\frac{\rho+\delta\gamma-\mu}{\rho+2\delta\gamma-\mu}\frac{a}{\gamma}(\bar{K}^*(X))^\gamma}{\rho+\delta\gamma-\mu} > 0. \end{aligned}$$

Following Dixit and Pindyck (1994), the value before investment (i.e. in the continuation region) is $V = \phi$ where ϕ is the solution of $\mathcal{L}\phi = \rho\phi$, where the infinitesimal generator is equal to $\mathcal{L} = \mu X\frac{\partial}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2}{\partial X^2}$. In other words,

$$\frac{1}{2}\sigma^2 X^2 \phi''(X) + \mu X\phi'(X) - \rho\phi(X) = 0.$$

They show that the unique solution to this equation is $\phi(X) = AX^\beta$ where β is the positive root of

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - \rho = 0.$$

The value of $A \in \mathbb{R}$ as well as the investment threshold X^* follow as a solution of the so called *value matching* and *smooth pasting* conditions:

$$AX^\beta = \frac{X}{\rho+\gamma\delta-\mu}\frac{a}{\gamma}(\bar{K})^\gamma\left(1 - \frac{a}{\gamma}(\bar{K})^\gamma\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) - \kappa\bar{K}, \text{ and} \tag{25}$$

$$A\beta X^{\beta-1} = \frac{1}{\rho+\gamma\delta-\mu}\frac{a}{\gamma}(\bar{K})^\gamma\left(1 - \frac{a}{\gamma}(\bar{K})^\gamma\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right), \tag{26}$$

respectively. To find A, $\bar{K}^{opt}$, and X^*, (24), (25), and (26) are solved simultaneously,

$$\begin{aligned} X(\beta-1)\left(1 - \frac{a}{\gamma}(\bar{K})^\gamma\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) &= \beta\kappa(\rho+\delta\gamma-\mu)\frac{\gamma}{a}(\bar{K})^{1-\gamma} \\ &= X\gamma\beta\left(1 - 2\frac{a}{\gamma}(\bar{K})^\gamma\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right). \end{aligned}$$

The first equality follows from the smooth pasting and value matching conditions and the second equality follows from plugging in the first order condition. Rewriting leads to

$$\frac{a}{\gamma}(\bar{K})^{\gamma} = \frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\,\frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu}$$

which leads to (9). The solution to (25) and (26) gives X^* and A. □

Proof of Proposition 2. First notice that, since $\bar{K} = \left(\frac{\gamma}{a}Q(0)\right)^{\frac{1}{\gamma}}$ and $\frac{\partial}{\partial\delta}\bar{K} = \frac{1}{a}\left(\frac{\gamma}{a}Q(0)\right)^{\frac{1-\gamma}{\gamma}}\frac{\partial}{\partial\delta}Q(0)$, the signs of $\frac{\partial}{\partial\delta}\bar{K}$ and $\frac{\partial}{\partial\delta}Q(0)$ are the same. Plugging $\bar{K} = \left(\frac{\gamma}{a}Q(0)\right)^{\frac{1}{\gamma}}$ into gives

$$\frac{X}{\rho+\gamma\delta-\mu}\left(1-2Q(0)\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) = \frac{\kappa}{a}\left(\frac{\gamma}{a}Q(0)\right)^{\frac{1-\gamma}{\gamma}}. \tag{27}$$

Define $\xi = \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}$. Applying the implicit function theorem to (27) leads to

$$\frac{\partial Q(0)}{\partial\delta}\left[\frac{\kappa}{a}\frac{1-\gamma}{a}\left(\frac{\gamma}{a}Q(0)\right)^{\frac{1-2\gamma}{\gamma}} + \frac{2X}{\rho+2\delta\gamma-\mu}\right]$$
$$= -\gamma\frac{X}{(\rho+\delta\gamma-\mu)^2}(1-Q(0)(2\xi)^2). \tag{28}$$

Therefore $\frac{\partial Q(0)}{\partial\delta} > 0 \Leftrightarrow Q(0) > \frac{1}{4\xi^2}$. As the left-hand side of (27) is decreasing in $Q(0)$ and the right-hand side is increasing in $Q(0)$ it is sufficient to evaluate both sides at $Q(0) = \frac{1}{4\xi^2}$ to establish a condition for $\frac{\partial\bar{K}^*(X)}{\partial\delta} > 0$, i.e. plugging $Q(0) = \frac{1}{4\xi^2}$ into

$$\frac{X}{\rho+\gamma\delta-\mu}\left(1-2Q(0)\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) > \frac{\kappa}{a}\left(\frac{\gamma}{a}Q(0)\right)^{\frac{1-\gamma}{\gamma}} \tag{29}$$

gives a sufficient condition.

Next is to show that $\hat{\delta}(X)$ exists and is unique. Notice that the left-hand side of (27) is a function of δ but the right-hand side is not. Therefore, studying the left-hand side of (27) when δ changes is sufficient. As such,

$$\frac{\partial}{\partial\delta}\frac{X}{\rho+\gamma\delta-\mu}\left(1-2Q(0)\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) = \frac{X}{(\rho+\gamma\delta-\mu)^2}(4\xi^2Q(0)-1),$$

which equals 0 for $Q(0) = \frac{1}{4\xi^2}$. As $\frac{1}{\xi^2}$ is increasing in δ we have that there is a unique value of δ, $\hat{\delta}$, such that (29) holds if and only if $\delta < \hat{\delta}(X)$.

For $\delta = 0$ with $Q(0) = \frac{1}{4\xi^2}$, (29) becomes

$$\frac{X}{\rho-\mu} > \frac{2r}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}},$$

which gives a unique value of X such that $\frac{X}{\rho-\mu} = \frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}}$.

Finally, notice that $(\rho-\mu)\frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}} > 0$, so that $(\rho-\mu)\frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}}$ is part of the state space. □

Proof of Lemma 1. Notice that $\frac{\partial}{\partial\bar{K}}V(X) = 0$ as $\bar{K}(X)$ is chosen to maximize $V(X)$. Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}V(X) &= \frac{\partial}{\partial\delta}V(X) + \frac{\partial}{\partial\bar{K}}V(X)\frac{\partial\bar{K}}{\partial\delta}\\ &= -\frac{\gamma X}{(\rho+\gamma\delta-\mu)^2}\left(1-2\frac{a}{\gamma}\bar{K}^{\gamma}(X)\left(\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right)^2\right)+0\\ &< -\frac{\gamma X}{(\rho+\gamma\delta-\mu)^2}\left(1-2\frac{a}{\gamma}\bar{K}^{\gamma}(X)\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) < 0.\end{aligned}$$

For the last inequality, we make use of the fact that the left-hand side of is positive. □

Proof of Lemma 2. Taking the derivative directly gives

$$\begin{aligned}\frac{\partial}{\partial\delta}V(X) &= \frac{\partial}{\partial\delta}\left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{\beta-1}\\ &= \left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{\beta-1}\left(-\frac{\beta}{X^*}\frac{\partial X^*}{\partial\delta}+\frac{1}{\bar{K}^{opt}}\frac{\partial\bar{K}^{opt}}{\partial\delta}\right)\\ &= -\left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{\beta-1}\left(\beta-\frac{\rho-\mu}{\rho+2\gamma^2\delta-\mu}\right)\frac{\partial X^*}{\partial\delta}\frac{1}{X^*} < 0,\end{aligned}$$

so that the statement follows directly. □

Proof of Lemma 3. One can show that the sign of the derivative of (13) with respect to δ is equal to the sign of

$$(\rho-\mu)(1-\beta-\beta(1-\gamma))-2\beta\gamma\delta < 0,$$

which is negative since $1-\beta < 0$ and all other terms are negative. □

Proof of Lemma 4. Taking the derivatives gives

$$\begin{aligned}\frac{\partial}{\partial\sigma}V(X) &= \frac{\partial}{\partial\sigma}\left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{\beta-1}\\ &= \left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{\beta-1}\left(\ln\left(\frac{X}{X^*}\right)-\frac{\beta}{X^*}\frac{\partial X^*}{\partial\beta}+\frac{1}{\bar{K}^{opt}}\frac{\partial\bar{K}^{opt}}{\partial\beta}-\frac{1}{\beta-1}\right)\frac{\partial\beta}{\partial\sigma}\\ &= \left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{\beta-1}\ln\left(\frac{X}{X^*}\right)\frac{\partial\beta}{\partial\sigma} > 0,\end{aligned}$$

so that (i) follows directly.

For (ii), one can directly check that

$$\frac{\partial^2}{\partial\delta\partial\sigma}X^* = -\left(\frac{\gamma}{a}\frac{(\rho+2\gamma\delta-\mu)(1-\beta(1-\gamma))}{(\rho+\gamma\delta-\mu)(1+\beta(1-2\gamma))}\right)^{\frac{1-\gamma}{\gamma}}$$
$$\times\frac{\kappa}{a}\frac{\rho+2\gamma^2\delta-\mu}{\rho+2\gamma\delta-\mu}\frac{\beta(1-\gamma)^2+\gamma(1-\beta(1-\gamma))}{(\beta-1)^2(1-\beta(1-\gamma))}\frac{\partial\beta}{\partial\sigma} > 0.$$

because $\frac{\partial\beta}{\partial\sigma} < 0$. □

Proof of Proposition 3. One can check the following for any function $h:\mathbb{R}^2\to\mathbb{R}$: if $h(x,y)$ can be written as $h(x,y) = f(x)g(y)(l(y))^{k(x)}$ with functions $f,g,k,l:\mathbb{R}\to\mathbb{R}$, then

$$\frac{\partial^2}{\partial x\partial y}h(x,y) = \frac{1}{h(x,y)}\frac{\partial}{\partial y}h(x,y)\frac{\partial}{\partial x}h(x,y)+h(x,y)\frac{1}{l(y)}\frac{\partial l(y)}{\partial y}\frac{\partial k(x)}{\partial x}.$$

This can be used to obtain

$$\begin{aligned}\frac{\partial^2}{\partial\sigma\partial\delta}\left(\frac{X}{X^*}\right)^{\beta}\frac{K^{opt}}{\beta-1} &= \underbrace{\left(\frac{X}{X^*}\right)^{-\beta}\frac{\beta-1}{K^{opt}}}_{>0}\underbrace{\frac{\partial}{\partial\delta}\left(\frac{X}{X^*}\right)^{\beta}\frac{K^{opt}}{\beta-1}}_{<0}\underbrace{\frac{\partial}{\partial\sigma}\left(\frac{X}{X^*}\right)^{\beta}\frac{K^{opt}}{\beta-1}}_{>0}\\ &\quad-\underbrace{\left(\frac{X}{X^*}\right)^{\beta}\frac{K^{opt}}{X^*(\beta-1)}\frac{\partial X^*}{\partial\delta}\frac{\partial\beta}{\partial\sigma}}_{<0}\\ &= \underbrace{\left(\frac{X}{X^*}\right)^{\beta}\frac{\bar{K}^{opt}}{\beta-1}\frac{\partial X^*}{\partial\delta}\frac{1}{X^*}}_{>0}\underbrace{\frac{\partial\beta}{\partial\sigma}}_{<0}\\ &\quad\times\left\{\left(\frac{\rho-\mu}{\rho+2\gamma^2\delta-\mu}-\beta\right)\left(\ln\left(\frac{X}{X^*}\right)\right)-1\right\}.\end{aligned}$$

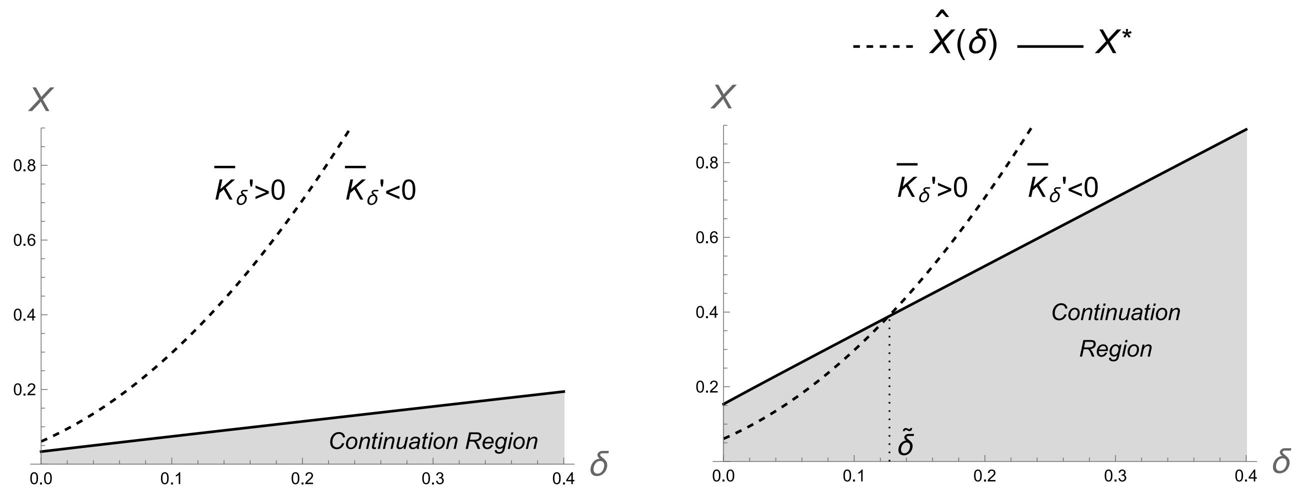
Rewriting the last term gives (14). □

Proof of Proposition 4. Taking the derivative of $EPVI$ with respect to σ gives

$$\begin{aligned}\frac{\partial}{\partial\sigma}\left(\frac{X}{X^*}\right)^{\beta}\kappa\bar{K}^{opt} &= \left(\frac{X}{X^*}\right)^{\beta}\kappa\bar{K}^{opt}\left(\frac{\partial\bar{K}^{opt}/\partial\beta}{\bar{K}^{opt}}-\frac{\beta}{X^*}\frac{\partial X^*}{\partial\beta}+\ln\left(\frac{X}{X^*}\right)\right)\frac{\partial\beta}{\partial\sigma}\\ &= \left(\frac{X}{X^*}\right)^{\beta}\kappa\bar{K}^{opt}\left(\underbrace{\frac{(\beta(1-\gamma)-1)^2+\beta^2\gamma^2}{\beta\gamma(\beta-1)(1+\beta(2\gamma-1))}}_{>0}+\underbrace{\ln\left(\frac{X}{X^*}\right)}_{<0}\right)\frac{\partial\beta}{\partial\sigma}\end{aligned} \tag{30}$$

To show that the first term between brackets is strictly positive, notice that the denominator is positive because $1+\beta(2\gamma-1) \geq \beta-1 > 0$, which follows from our condition that $\gamma \geq \frac{\beta-1}{\beta}$. Also notice that the second term is negative since $X < X^*$ for all $X \in C$.

For the smallest value of σ such that $X^* \geq X$ it holds that $\ln\left(\frac{X}{X^*}\right) = 0$ so that indeed $\frac{\partial}{\partial\sigma}\left(\frac{X}{X^*}\right)^{\beta}\kappa\bar{K}^{opt} < 0$ for sufficiently small



(a) $\sigma = 0.1 < 0.22 = \tilde{\sigma}$. (b) $\sigma = 0.5 > 0.22 = \tilde{\sigma}$.

Fig. 10. Curves corresponding to $\hat{X}$ and X^*. Solid curve separates regions where $\bar{K}^*(X)$ is increasing and decreasing in δ for $X \in \mathcal{S}$ and dashed line separates stopping region and continuation region (dotted area).
$\mu = 0.02$, $\rho = 0.1$, $\gamma = 0.8$, $a = 0.6$, and $\kappa = 0.3$.

σ. In addition, for $X \searrow 0$ it holds that $\ln\left(\frac{X}{X^*}\right) \searrow -\infty$ so that $\frac{\partial}{\partial\sigma}\left(\frac{X}{X^*}\right)^{\beta}\kappa\bar{K}^{opt} > 0$. □

Lemma 5. *Let $X \in \mathcal{C}$. Then $\frac{\partial^2}{\partial\delta\partial\sigma}EPVI > 0$ if and only if*

$$\ln(X^*) - \ln(X) < \frac{(\beta(1-\gamma)-1)^2 + \beta^2\gamma^2}{\beta\gamma(\beta-1)(1+\beta(2\gamma-1))} + \left(\beta - \frac{\rho-\mu}{\rho+2\gamma^2\delta-\mu}\right)^{-1}. \quad (31)$$

Proof of Lemma 5. Using that $\frac{\partial^2}{\partial\delta\partial\sigma}EPVI = \frac{\partial}{\partial\delta}\frac{\partial}{\partial\sigma}EPVI$, differentiating equation (30) gives

$$\begin{aligned}\frac{\partial^2}{\partial\sigma\partial\delta}\left(\frac{X}{X^*}\right)^{\beta}\kappa\bar{K}^{opt} &= \frac{\partial}{\partial\delta}\left\{\left(\frac{X}{X^*}\right)^{\beta}\kappa\bar{K}^{opt}\right\}\\ &\times\left(\frac{(\beta(1-\gamma)-1)^2+\beta^2\gamma^2}{\beta\gamma(\beta-1)(1+\beta(2\gamma-1))} + \ln\left(\frac{X}{X^*}\right)\right)\frac{\partial\beta}{\partial\sigma}\\ &+\left(\frac{X}{X^*}\right)^{\beta}\kappa\bar{K}^{opt}\left(-\frac{1}{X^*}\frac{\partial}{\partial\delta}X^*\right)\frac{\partial\beta}{\partial\sigma}\\ &= -\left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{X^*}\frac{\partial X^*}{\partial\delta}\frac{\partial\beta}{\partial\sigma}\times\\ &\left\{\underbrace{\left(\beta - \frac{\rho-\mu}{\rho+2\gamma^2\delta-\mu}\right)}_{>0}\right.\\ &\left.\times\left(\underbrace{\frac{(\beta(1-\gamma)-1)^2+\beta^2\gamma^2}{\beta\gamma(\beta-1)(1+\beta(2\gamma-1))}}_{>0} + \underbrace{\ln\left(\frac{X}{X^*}\right)}_{<0}\right)+1\right\}\end{aligned}$$

Rewriting the terms between brackets gives (31). □

Proof of Proposition 5. First note that the right-hand side of (31) is decreasing in δ and the left-hand side of (31) is increasing in δ. Since the left-hand side is not bounded, i.e. $\ln(X^*) \to \infty$ as $\delta \to \infty$, and since the right-hand side is always finite, the left-hand side exceeds the right-hand side for sufficiently high rates. It then also follows that $\delta^\circ > 0$ if condition (31) is met for $\delta = 0$, i.e., when X^* is closest to X. Otherwise, $\delta^\circ = 0$. □

Appendix B. Optimal investment across regions

We now bring the analyses on the stopping and continuation regions together, allowing for cases where the monopolist may switch between these regions for a change in δ, and identify the overall effect of depreciation on the firm's investment behavior. In order to establish how the size of capital investment is affected by the depreciation rate for all X, let us denote the inverse of the $\hat{\delta}(X)$ in Proposition 2 by $\hat{X}(\delta)$, i.e.

$$\hat{X}(\delta) = \frac{2(\rho+\gamma\delta-\mu)^2}{\rho-\mu}\frac{\kappa}{a}\left(\frac{\gamma}{4a}\left(\frac{\rho+2\gamma\delta-\mu}{\rho+\gamma\delta-\mu}\right)^2\right)^{\frac{1-\gamma}{\gamma}}. \quad (32)$$

Notice that $\hat{X}(\delta)$ is increasing in δ and, following Proposition 2, it divides the stopping region into a region with net positive and a region with net negative *direct* effects of an increase in the value of δ on $\bar{K}^*(X)$, i.e. for $X < \hat{X}(\delta)$ and $X \geq \hat{X}(\delta)$, respectively.

Section 3.1 established that X^* is increasing in δ, and that X^* divides the space into a region where the firm delays investment (continuation) with the *indirect* effect of an increase in the value of δ on $\bar{K}^{opt}$ dominating positively over all others (for $X < X^*$), and a region where the firm invests immediately (stopping) with the indirect effect of δ on capital stock not playing any part (for $X \geq X^*$).

In summary, we can compare both boundaries to distinguish cases where $\hat{X} < X^*$ and where $\hat{X} > X^*$, which leads to the following two cases.

Case 1:

- For $X \in (0, X^*)$ the firm delays investment until the state process hits X^* for the first time. The scale of investment $\bar{K}^{opt} = \bar{K}^*(X^*)$ is increasing in δ, while the present value of capital is decreasing in δ.
- For $X \in [X^*, \hat{X})$ the firm undertakes investment immediately and $\bar{K}^*(X)$ is decreasing in δ.
- For $X \in (\hat{X}, \infty)$ the firm undertakes investment immediately and $\bar{K}^*(X)$ is increasing in δ.

Case 2:

- For $X \in (0, X^*)$ the firm delays investment until the state process hits X^* for the first time. The scale of investment $\bar{K}^{opt} = \bar{K}^*(X^*)$ is increasing in δ, while the present value of capital is decreasing in δ;
- For $X \in [X^*, \infty)$ the firm undertakes investment immediately and $\bar{K}^*(X)$ is increasing in δ.

Since X^* is increasing in σ and since $\hat{X}(\delta)$ is not affected by σ, the condition for the stopping region to consist only of points where $\bar{K}^*(X)$ is positively affected by depreciation is that σ be sufficiently large. For lower levels on uncertainty, capital is negatively impacted by depreciation for $X \in [X^*, \hat{X}(\delta))$ and positively otherwise.

Proposition 6. *Let $\tilde{\sigma}$ be the (unique) solution to*

$$\left(\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\right)^{1-\gamma}\left(\frac{\beta(2\gamma-1)+1}{\beta-1}\right)^{\gamma}=\frac{(8)^{\gamma}}{4}. \tag{33}$$

(i) If $\sigma < \tilde{\sigma}$, then Case 1 applies, for all δ.
(ii) If $\sigma \geq \tilde{\sigma}$, then Case 1 applies if and only if δ is larger than some $\tilde{\delta}$, which is the solution of

$$\frac{8^{\gamma}}{4}\left(\frac{\rho+\gamma\delta-\mu}{\rho-\mu}\right)^{\gamma}\left(\frac{\rho+2\gamma\delta-\mu}{\rho+\gamma\delta-\mu}\right)^{1-\gamma}=\left(\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\right)^{1-\gamma}\left(\frac{\beta(2\gamma-1)+1}{\beta-1}\right)^{\gamma}. \tag{34}$$

Moreover, Case 2 applies if and only if δ is smaller than some $\tilde{\delta}$, which is the solution of (34).

The proposition is illustrated by Fig. 10 where $\tilde{\sigma} = 0.22$. Notice that (33) only depends on σ, μ, ρ, and γ.

The proposition shows that, for depreciation to have an unambiguously positive effect on (the level of) capital stock for all X, market uncertainty needs to be sufficiently high, which comes a result of the expansion of the continuation region.

Proof of Proposition 6. A large part of the proof follows from the main text.

Rewriting $\hat{X}(\delta) = X^*$ gives (34). Since the left-hand side of (34) is (strictly) increasing in δ and the right-hand side does not depend on δ, the intersection is unique. This can be used to show that $\frac{\partial}{\partial\delta}\hat{X} > \frac{\partial}{\partial\delta}X^*$ for all δ. Substituting the left-hand side of (34) into $\hat{X}(\delta)$ gives

$$\tilde{X}=\left(\frac{\gamma}{a}\right)^{\frac{1-\gamma}{\gamma}}\frac{\kappa}{a}\left(\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\right)^{2\frac{1-\gamma}{\gamma}}\left(\frac{\beta(2\gamma-1)+1}{\beta-1}\right)^{2}(\rho-\mu)\frac{4^{\frac{1}{\gamma}}}{8}.$$

We now need to check conditions such that $\tilde{X} > (\rho-\mu)\frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}}$ where $(\rho-\mu)\frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}}$ follows from Proposition 2. Rewriting $\tilde{X} = (\rho-\mu)\frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}}$ gives (33). Uniqueness of $\tilde{\sigma}$ follows from the fact that $\tilde{X}$ is monotone in σ and that $(\rho-\mu)\frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}}$ does not depend on σ. □

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