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The partition algebra and the plethysm coefficients I: Stability and Foulkes' conjecture

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This paper is dedicated to the memory of Georgia Benkart

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ABSTRACT

We propose a new approach to study plethysm coefficients by using the Schur-Weyl duality between the symmetric group and the partition algebra. This provides an explanation of the stability properties of plethysm and Kronecker coefficients in a simple and uniform fashion for the first time. We prove the strengthened Foulkes' conjecture for stable plethysm coefficients in an elementary fashion.

Introduction

Understanding the *plethysm coefficients* is a fundamental problem in the representation theories of symmetric and general linear groups and was identified by Richard Stanley as one of the most important open problems in algebraic combinatorics [23]. Perhaps the oldest and most famous question concerning plethysm coefficients is a conjecture of Foulkes from 1950 [8]. To state Foulkes' conjecture, we first require some

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notation. Let $m, n \in \mathbb{N}$ and α be a partition of mn and let $\mathfrak{S}_m \wr \mathfrak{S}_n$ denote the wreath product subgroup of $\mathfrak{S}_{mn}$. The plethysm coefficient $p((n),(m),\alpha)$ is the multiplicity of the irreducible $\mathbb{C}\mathfrak{S}_{mn}$-module $\mathsf{S}(\alpha)$ as a composition factor of the Foulkes module $\mathrm{ind}_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{mn}} \mathbb{C}$. Equivalently, these plethysm coefficients record the decomposition of the $\mathrm{GL}_{mn}(\mathbb{C})$-module $\mathrm{Sym}^n(\mathrm{Sym}^m(\mathbb{C}^{mn}))$ into irreducible summands and also the decomposition of the plethysm of symmetric functions $s_{(n)} \circ s_{(m)}$ as an integral linear combination of Schur functions. Foulkes' conjecture states that, for all $m \leqslant n$ and for all $\alpha \vdash mn$, we have

$$p((m),(n),\alpha) \leqslant p((n),(m),\alpha). \tag{1}$$

A stronger conjecture made in [25] states that for all $m \leqslant n, p, q$ with $mn = pq$ we have

$$p((q),(p),\alpha) \leqslant p((n),(m),\alpha). \tag{2}$$

Plethysm is defined for arbitrary partitions of m and n, but for the purposes of this paper our interest lies in the 'Foulkes case' where both partitions have precisely one row. In this article, we study families of these plethysm coefficients. For an arbitrary partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, set $\lambda_{[mn]} = (mn - |\lambda|, \lambda_1, \lambda_2, \ldots, \lambda_\ell)$. We look at the coefficient $p((n),(m),\lambda_{[mn]})$ as m and n vary. We ask, for a fixed partition λ, whether for all $m < n$ we have that

$$p((m),(n),\lambda_{[mn]}) \leqslant p((n),(m),\lambda_{[mn]}).$$

Our first theorem verifies that this is indeed the case for all except possibly a finite list of values for $m, n \in \mathbb{N}$. Discarding this finite list of values, both Foulkes' conjecture and the strengthened Foulkes' conjecture hold for the partition $\lambda_{[mn]}$. Moreover, we can even drop the assumption in Equation (2) that $mn = pq$, relating plethysm coefficients for $\lambda_{[mn]}$ and $\lambda_{[pq]}$ outside of these values.

Theorem A. *Let λ be an arbitrary partition. For any $m, n, p, q \geqslant |\lambda|$,*

$$p((q),(p),\lambda_{[pq]}) = p((n),(m),\lambda_{[mn]}).$$

In particular, taking $p = n$ and $q = m$, Foulkes' conjecture holds for $\lambda_{[mn]}$ for all but finitely many values of $m, n \in \mathbb{N}$, as does the strengthened Foulkes' conjecture.

The proof of this result constructs a partition algebra isomorphism which "does not see" any difference between m and n providing they are both sufficiently large. This seems to provide the first conceptual explanation for why Foulkes' conjecture "should" be true.

One of the key ideas in our approach is to consider the stable limit of a certain sequence of plethysm coefficients. Brion [3] and Carré–Thibon [5] proved that the following sequences of plethysm coefficients

$$\{p((n),(m),\lambda_{[mn]})\}_{n\in\mathbb{N}} \qquad \{p((n),(m),\lambda_{[mn]})\}_{m\in\mathbb{N}}$$

have stable limits for n (respectively m) sufficiently large with respect to m (respectively n). In fact, Brion's proof of the stability of the former sequence settled a second conjecture from Foulkes' 1950 paper. In this paper we consider the stable limit of the double-sequence

$$\overline{p}_{\infty,\lambda} = \lim_{m,n\to\infty}\{p((n),(m),\lambda_{[mn]})\}. \tag{†}$$

These stable values are achieved whenever $m,n \geqslant |\lambda|$. We study these stable plethysm coefficients through the Schur–Weyl duality between the symmetric group, $\mathfrak{S}_{mn}$, and the partition algebra, $P_r(mn)$, via their actions on the tensor space $(\mathbb{C}^{mn})^{\otimes r}$. This duality results in a functor

$$\mathcal{F}_r : \mathfrak{S}_{mn}\text{-mod} \to \text{mod-}P_r(mn).$$

The key observation is that the module $\mathcal{F}_r(\text{ind}_{\mathfrak{S}_m\wr\mathfrak{S}_n}^{\mathfrak{S}_{mn}}(\mathbb{C}))$ has an elegant diagrammatic description for $m,n \geqslant r$. By considering the module $\mathcal{F}_r(\text{ind}_{\mathfrak{S}_m\wr\mathfrak{S}_n}^{\mathfrak{S}_{mn}}(\mathbb{C}))$ for each $r \geqslant 0$ in turn, we are able to focus solely on the "rth layer" of plethysm constituents $p((n),(m),\lambda_{[mn]})$ for which $|\lambda| = r$. We hence deduce that Foulkes' conjecture holds for $\lambda_{[mn]}$ whenever $m,n \geqslant |\lambda|$ and obtain a simple proof of the stability (†).

Our second main result (which subsumes Theorem A) calculates the value of any stable plethysm coefficient in terms of plethysm coefficients labelled by much smaller partitions and use of the Littlewood–Richardson rule. A geometric proof of this result in the language of jet schemes is given in [14]. Our proof of this result is given by explicitly decomposing the $P_r(mn)$-module $\mathcal{F}_r(\text{ind}_{\mathfrak{S}_m\wr\mathfrak{S}_n}^{\mathfrak{S}_{mn}}(\mathbb{C}))$ using its diagrammatic incarnation.

Theorem B. *Let λ be a partition of $r \in \mathbb{N}$. For all $m,n \geqslant r$ we have that*

$$\overline{p}_{\infty,\lambda} = p((n),(m),\lambda_{[mn]}) = \sum_{\mu\in\mathscr{P}_1(r)} p_\mu(\lambda),$$

where $\mathscr{P}_1(r)$ is the set of all partitions of r whose parts are all strictly greater than 1. The coefficients $p_\mu(\lambda)$ are the generalised plethysm coefficients defined in Section 1.

Our approach brings forward a general tool to study stable and non-stable plethysm coefficients and provides a natural framework for the study of the outstanding problems in the area. In particular, one should notice that our proofs are surprisingly elementary and treat the stabilities of Kronecker and plethysm coefficients uniformly alongside one another for the first time — as the parameters increase, the action of the partition algebra becomes faithful and semisimple exactly as in the case of the Kronecker coefficients [2]. We consider Theorem B to be the natural analogous statement to that for Kronecker

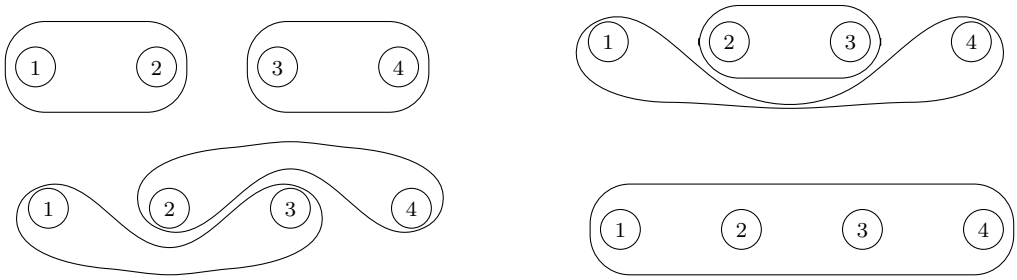
coefficients in [2, Corollary 4.5]. Particular highlights of our approach include easy algebraic proofs of Foulkes' and Weintraub's conjectures for stable plethysm coefficients. The partition algebra approach has proven to be very powerful for understanding Kronecker coefficients: in [1] an algorithm is given for calculating stable Kronecker coefficients in terms of oscillating tableaux. (The partition algebra is essential in the proof and allows one to define a lattice permutation condition on oscillating tableaux.) We hope that the partition algebra is similarly useful in understanding the (stable) plethysm coefficients.

Ramifications

In this paper, we recast the Foulkes' plethysm coefficients in the setting of the partition algebra. In the sequel to this paper we will generalise this to arbitrary plethysm coefficients. The key to this construction will be the *ramified* partition algebra of [18] which we do not discuss here. However, the reader familiar with these constructions is invited to observe that our *stable Foulkes module* for $P_r(mn)$ is equal to the restriction to $P_r(mn)$ of the cell module of the ramified partition algebra denoted by $\Delta_r(\varnothing^{\varnothing})$.

An example

We conclude this introduction with an example, illustrating how to calculate the multiplicities $\overline{p}_{\infty,\lambda} = p((n),(m),\lambda_{[mn]})$ for λ a partition of 4 and $m, n \geqslant 4$. We pass the question from $\mathfrak{S}_{mn}$ to the partition algebra $P_4(mn)$ and take the natural quotient $P_4(mn) \to \mathbb{C}\mathfrak{S}_4$, which dramatically reduces the rank of the problem. We hence obtain an $\mathbb{C}\mathfrak{S}_4$-module with the following basis



with the action given by permuting the diagrams in the obvious fashion. It is easy to see that the first three diagrams span a cyclic module which decomposes as the sum of Specht modules $\mathsf{S}(4) \oplus \mathsf{S}(2,2)$. The fourth diagram provides an indecomposable module isomorphic to $\mathsf{S}(4)$. Hence, for all $m, n \geqslant 4$ we deduce that

$$p((n),(m),(mn-4,4)) = 2 \qquad p((n),(m),(mn-4,2^2)) = 1 \qquad p((n),(m),\lambda_{[mn]}) = 0$$

for $\lambda = (3,1),(2,1^2),(1^4)$.

The structure of the paper

We present sufficient background to make our exposition accessible to a reader familiar with representation theory: Section 1 recalls the definition of the (generalised) plethysm coefficients, their stabilities, and the statement of Foulkes' conjecture and the strengthened Foulkes' conjecture; Section 2 recalls the definition of the partition algebra and the basic facts concerning its representation theory; Section 4 recalls the Schur–Weyl duality between the symmetric groups and partition algebras which underlies the main results of this paper. We apply Schur–Weyl duality to the Foulkes module to obtain a module for the partition algebra. Our fundamental combinatorial object, the fixed depth Foulkes poset, is introduced in Section 3. It is used to construct a diagrammatic module for the partition algebra in Section 5, which we relate to Foulkes modules in Section 6. In Section 7 the construction of a filtration of our diagrammatic module allows us to prove Theorem A, and its decomposition into its irreducible components proves Theorem B.

This paper has been on the arXiv for quite some time, while we were preparing the sequel. Since then, certain plethysm coefficients have been studied by Orellana–Saliola–Schilling–Zabrocki in the context of the *party algebra*, a subalgebra of the partition algebra (see [20]). We do not believe there is any overlap in our results, but the ideas do have a similar diagrammatic flavour.

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1. Foulkes' conjecture and plethysm coefficients

We let $\mathfrak{S}_n$ denote the symmetric group on n letters. The combinatorics underlying the representation theory of the partition algebras and symmetric groups is based on (integer) compositions and partitions. A *composition* λ of n, denoted $\lambda \vDash n$, is defined to be a sequence of non-negative integers which sum to n. If the sequence is weakly decreasing, we write $\lambda \vdash n$ and refer to λ as a *partition* of n. We let $\mathscr{P}(n)$ denote the set of all partitions of n and we let $\mathscr{P}_1(n)$ denote the subset of those partitions whose (non-zero) parts are all strictly greater than 1. We suppress the zero parts of $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell, 0, 0, \dots)$ and write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ where λ_ℓ is non-zero, and we write $\emptyset$ for the unique partition of zero. We say that a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_\ell)$ has *depth* equal to $|\lambda| - \lambda_1 = \lambda_2 + \lambda_3 + \cdots + \lambda_\ell$.

Associated to each partition λ of n, we have a simple right $\mathbb{C}\mathfrak{S}_n$-module, often referred to as a *Specht* module. An explicit construction of these modules is given in [11, §4], where it is shown that the Specht modules provide a complete set of irreducible $\mathbb{C}\mathfrak{S}_n$-modules indexed by the partitions of n.

Now let $m, n \in \mathbb{N}$ and consider the symmetric group $\mathfrak{S}_{mn}$. A copy of the wreath product $\mathfrak{S}_m \wr \mathfrak{S}_n$ occurs as a subgroup of $\mathfrak{S}_{mn}$. Following [12, Section 4.1], we define

$$\mathfrak{S}_m \wr \mathfrak{S}_n = \{(\sigma_1, \sigma_2, \dots, \sigma_n; \pi) : \sigma_i \in \mathfrak{S}_m, i = 1, \dots, n, \pi \in \mathfrak{S}_n\},$$

which we identify with a subgroup of $\mathfrak{S}_{mn}$ via the embedding

$$(\sigma_1, \sigma_2, \dots, \sigma_n; \pi) \mapsto \begin{pmatrix} (j-1)m + i \\ (\pi(j)-1)m + \sigma_{\pi(j)}(i) \end{pmatrix}_{i=1,\dots,m, j=1\dots,n}, \tag{1.1}$$

where on the right we have used the usual two-line notation for permutations. In addition, there are other direct products of wreath product subgroups contained in $\mathfrak{S}_{mn}$ corresponding to various partitions of mn. Suppose $\mu = (m_1^{n_1}, m_2^{n_2}, \dots, m_\ell^{n_\ell}) \vdash mn$ with $m_1 > m_2 > \cdots > m_\ell$, where, here, repeated parts of the partition are denoted by superscripts in the standard fashion. Associated to μ we have a subgroup

$$\prod_i \mathfrak{S}_{m_i} \wr \mathfrak{S}_{n_i} = \mathfrak{S}_{m_1} \wr \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{m_\ell} \wr \mathfrak{S}_{n_\ell} \leqslant \mathfrak{S}_{m_1 n_1} \times \cdots \times \mathfrak{S}_{m_\ell n_\ell} \leqslant \mathfrak{S}_{mn}. \tag{1.2}$$

For $\lambda \vdash mn$, we define the *generalised plethysm coefficient* $p_\mu(\lambda)$ to be the multiplicity of $\mathsf{S}(\lambda)$ as a composition factor of the permutation module $\operatorname{ind}_{\prod_i \mathfrak{S}_{m_i} \wr \mathfrak{S}_{n_i}}^{\mathfrak{S}_{mn}}(\mathbb{C})$,

$$p_\mu(\lambda) = \left[\operatorname{ind}_{\prod_i \mathfrak{S}_{m_i} \wr \mathfrak{S}_{n_i}}^{\mathfrak{S}_{mn}}(\mathbb{C}) : \mathsf{S}(\lambda)\right]_{\mathbb{C}\mathfrak{S}_{mn}}. \tag{1.3}$$

In the special case of a rectangular partition, $\mu = (m^n)$, the subgroup above specialises to be $\mathfrak{S}_m \wr \mathfrak{S}_n$ and we hence obtain the classical plethysm coefficients $p_{(m^n)}(\lambda) = p((n), (m), \lambda)$ defined explicitly in the introduction.

Given a fixed integer $mn \in \mathbb{N}$, the plethysm coefficients (which are associated to a rectangular $\mu = (m^n)$) are the most difficult examples of the coefficients in Equation (1.3) in terms of computation. By which we mean that all other examples of coefficients in Equation (1.3) can be obtained from an understanding of the smaller rank plethysm coefficients and applications of the Littlewood–Richardson rule (for the statement of which, see [11, §16]). To see this, simply note that we are inducing from a product (hence the Littlewood–Richardson coefficients) of wreath product subgroups (hence the plethysm coefficients).

Recall from the introduction that if λ is a partition then $\lambda_{[mn]}$ denotes the partition of mn whose Young diagram is obtained by appending an additional row above those of λ. Note that all partitions of mn are of the form $\lambda_{[mn]}$ for some λ. Brion [3] showed that

if we allow both the value of m and hence the length of the first row of $\lambda_{[mn]}$ to increase, then we obtain a limiting behaviour as follows. For m sufficiently large with respect to n, Brion proved that

$$p((n+k),(m),\lambda_{[m(n+k)]}) = p((n),(m),\lambda_{[mn]})$$

for all $k \geqslant 1$. This stability was conjectured by Foulkes in [8]. In the other direction, Carré–Thibon [5] showed for n sufficiently large with respect to m, we have that

$$p((n),(m+k),\lambda_{[(m+k)n]}) = p((n),(m),\lambda_{[mn]})$$

for all $k \geqslant 1$. Notice that in each case we require that m (respectively n) is sufficiently large with respect to n (respectively m). Therefore one cannot, a priori, consider the limit as n and m both tend to infinity. In this paper we shall consider the stability of the double-sequence

$$\overline{p}_{\infty}(\lambda) = \lim_{m,n\to\infty}\{p((n),(m),\lambda_{[mn]})\},$$

and we shall see that n and m are only required to be greater than $|\lambda|$ (and can be chosen freely with respect to each other) for this stability to occur.

We now recall Foulkes' original conjecture from [8]. This conjecture has been settled for $m \ll n$ by Brion [3]. All other results in the area concern small values of m; namely, $m = 2$ [24], $m = 3$ [7], $m = 4$ [19], $m = 5$ [6].

Conjecture 1.1. ***(Foulkes' conjecture)*** *Let $m, n \in \mathbb{N}$ with $m \leqslant n$. Then*

$$p((m),(n),\lambda_{[mn]}) \leqslant p((n),(m),\lambda_{[mn]})$$

for all partitions λ.

A strengthened version of this conjecture was proposed in [25]. It is currently only known to hold for $m = 2$ and $n = 3$ (see [25]).

Conjecture 1.2. ***(Strengthened Foulkes' conjecture)*** *Let $mn = pq \in \mathbb{N}$ with $m \leqslant n$ and suppose that $p, q \geqslant m$. Then*

$$p((m),(n),\lambda_{[mn]}) \leqslant p((q),(p),\lambda_{[pq]})$$

for all partitions λ.

2. The partition algebra

The partition algebra was originally defined by Martin in [16]. All the results in this section are due to Martin and his collaborators: see [17] and references therein.

2.1. Definitions

For $r \in \mathbb{N}$, we consider the set $\{1, 2, \dots, r, \overline{1}, \overline{2}, \dots, \overline{r}\}$ with the total ordering

$$1 < 2 < \cdots < r < \overline{1} < \overline{2} < \cdots < \overline{r}.$$

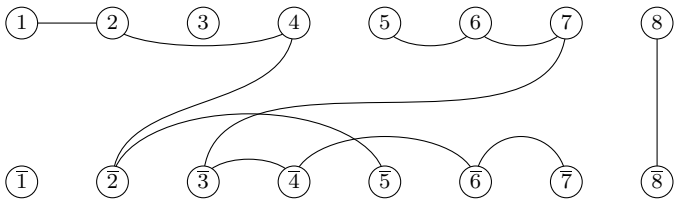
We consider set-partitions of $\{1, 2, \dots, r, \overline{1}, \overline{2}, \dots, \overline{r}\}$. A subset occurring in a set-partition is called a *block*. For example,

$$x = \{\{1, 2, 4, \overline{2}, \overline{5}\}, \{3\}, \{5, 6, 7, \overline{3}, \overline{4}, \overline{6}, \overline{7}\}, \{8, \overline{8}\}, \{\overline{1}\}\},$$

is a set-partition of $\{1, 2, \dots, 8, \overline{1}, \overline{2}, \dots, \overline{8}\}$ with five blocks. By convention, we order the blocks in $x = \{X_1, \dots, X_l\}$ by *increasing minima*, so that

$$1 = \min X_1 < \min X_2 < \cdots < \min X_{l-1} < \min X_l \leqslant \overline{r}.$$

A set-partition, x, can be represented by an (r, r)-*partition diagram*, consisting of r distinguished northern and southern points, which we call vertices. We number the northern vertices from left to right by $1, 2, \dots, r$ and the southern vertices similarly by $\overline{1}, \overline{2}, \dots, \overline{r}$ and connect two vertices by an edge if they belong to the same block and are adjacent in the total ordering given by restriction of the above ordering to the given block. The second condition is imposed to pick a unique representative from the equivalence class of all diagrams having the same connected components. We shall move between set-partitions and their diagrams without further comment. For example, the diagram of the set-partition x given above is as follows:



We will use a generalisation of the definition of (r, r)-partition diagrams. An (m, r)-partition diagram is a diagram representing a set partition of $\{1, \dots, m, \overline{1}, \dots, \overline{r}\}$ with the same conventions.

We now consider a parameter $\delta \in \mathbb{C}$. We define the product xy of two (r, r)-partition diagrams x and y using the concatenation of x above y, where we identify the southern vertices of x with the northern vertices of y. If there are t connected components consisting only of middle vertices, then the product is set equal to δ^t times the (r, r)-partition diagram equivalent to the diagram with the middle components removed. We let $P_r(\delta)$ denote the complex vector space with basis given by all set-partitions of

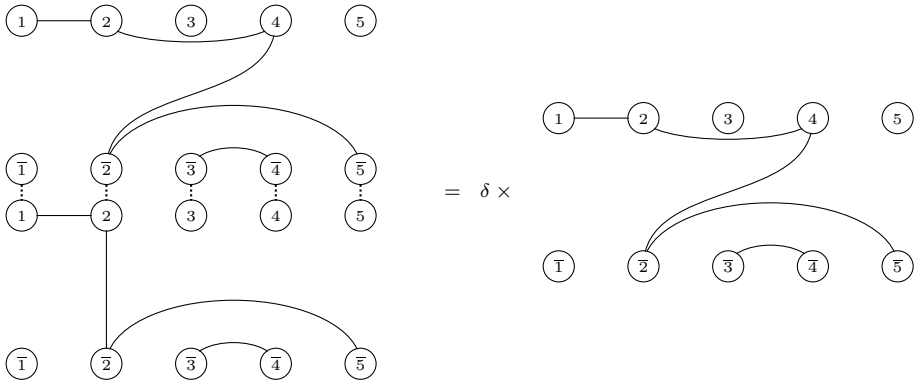


Fig. 1. An example of the multiplication in $P_5(\delta)$.

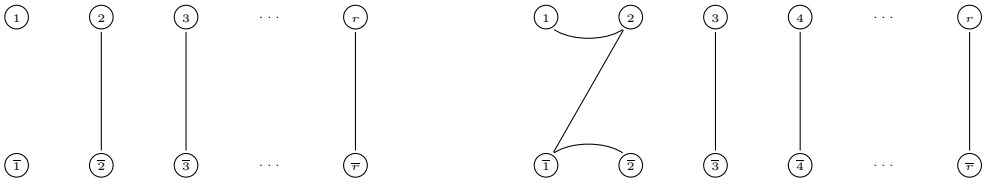


Fig. 2. The non-Coxeter generators, p_1 and $p_{1,2}$, of $P_r(\delta)$, respectively.

$\{1, 2, \dots, r, \overline{1}, \overline{2}, \dots, \overline{r}\}$ and with multiplication given by linearly extending the multiplication of diagrams. An example of the multiplication is given in Fig. 1.

We set

$$p_1 = \{\{1\}, \{2, \overline{2}\}, \dots \{r, \overline{r}\}, \{\overline{1}\}\}, \quad p_{1,2} = \{\{1, 2, \overline{1}, \overline{2}\}, \{3, \overline{3}\}, \dots \{r, \overline{r}\}\}$$

and we recall that the Coxeter generators for the symmetric group $s_{i,i+1}$ for $1 \leqslant i < r$ can be thought of as the set-partitions

$$s_{i,i+1} = \{\{1, \overline{1}\}, \dots, \{i-1, \overline{i-1}\}, \{i, \overline{i+1}\}, \{i+1, \overline{i}\}, \{i+2, \overline{i+2}\}, \dots, \{r, \overline{r}\}\}.$$

We set $p_k = s_{k,k-1} \dots s_{1,2} p_1 s_{1,2} \dots s_{k,k-1}$. Some of these diagrams are pictured in Fig. 2.

Proposition 2.1 *([17, Proposition 1]). The algebra $P_r(\delta)$ is generated by the set-partitions $p_1, p_{1,2}$ and $s_{i,i+1}$ for $1 \leqslant i < r$.*

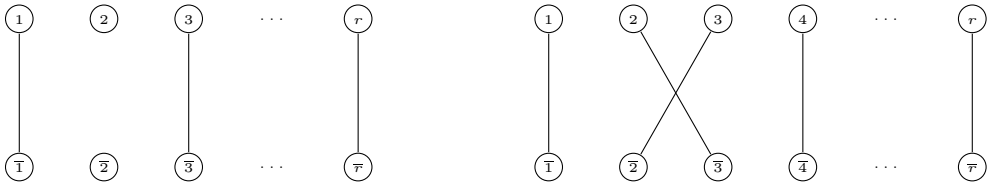


Fig. 3. The generators, p_2 and $s_{2,3}$, of $P_r(\delta)$, respectively.

2.2. Filtration by propagating blocks and standard modules

A block of a set-partition of $\{1, 2, \dots, r, \bar{1}, \bar{2}, \dots, \bar{r}\}$ is called *propagating* if the block contains both northern and southern vertices. In the example from the previous subsection, x has three propagating blocks. Note that the multiplication in $P_r(\delta)$ cannot increase the number of propagating blocks. More precisely, if x, respectively y, is a partition diagram with p_x, respectively p_y, propagating blocks then xy is equal to $\delta^t z$ for some $t \geqslant 0$ and some partition diagram z with p_z propagating blocks, where $p_z \leqslant \min\{p_x, p_y\}$. This gives a filtration of the algebra $P_r(\delta)$ by the number of propagating blocks.

We suppose for the remainder of the paper that $\delta \neq 0$ in order that we can define idempotents $e_k = \delta^{-k} p_1 p_2 \dots p_k$ for $1 \leqslant k \leqslant r$. Notice that the diagram $p_1 p_2 \dots p_k$ has precisely $r-k$ propagating lines (consider, for example, the product $p_1 p_2$ of the diagrams p_1 and p_2 depicted in Figs. 2 and 3). In particular, the aforementioned filtration of $P_r(\delta)$ by the number of propagating blocks can be realised as a chain of idempotent ideals as follows. We have

$$\{0\} \subset P_r(\delta) e_r P_r(\delta) \subset P_r(\delta) e_{r-1} P_r(\delta) \subset \dots \subset P_r(\delta) e_1 P_r(\delta) \subset P_r(\delta).$$

It is easy to see that

$$e_1 P_r(\delta) e_1 \cong P_{r-1}(\delta), \tag{2.1}$$

and that this generalises to $P_{r-k}(\delta) \cong e_k P_r(\delta) e_k$ for $1 \leqslant k \leqslant r$. Moreover, $P_r(\delta) e_1 P_r(\delta)$ is the span of all (r, r)-partition diagrams with at most $r-1$ propagating blocks and hence we have

$$P_r(\delta) / (P_r(\delta) e_1 P_r(\delta)) \cong \mathbb{C}\mathfrak{S}_r. \tag{2.2}$$

We see that any $\mathbb{C}\mathfrak{S}_r$-module can be *inflated* to a $P_r(\delta)$-module: the underlying vector space is the same and the action of $P_r(\delta)$ is via the quotient map provided by Equation (2.2). We also obtain from Equations (2.1) and (2.2), by induction, that the simple $P_r(\delta)$-modules are indexed by the set

$$\mathscr{P}(\leqslant r) = \bigcup_{0\leqslant i\leqslant r} \mathscr{P}(i).$$

For any $\nu \in \mathscr{P}(\leqslant r)$ with $\nu \vdash r-k$, we define the *standard* (right) $P_r(\delta)$-module, $\Delta_{r,\delta}(\nu)$, by

$$\Delta_{r,\delta}(\nu) = \mathsf{S}(\nu) \otimes_{P_{r-k}(\delta)} e_k P_r(\delta), \tag{2.3}$$

where the (right) Specht module $\mathsf{S}(\nu)$ for $\mathbb{C}\mathfrak{S}_{r-k}$ becomes a $P_{r-k}(\delta)$-module by inflation, and we have identified $P_{r-k}(\delta)$ with $e_k P_r(\delta) e_k$ using the isomorphism given in Equation (2.1) providing the left action on $e_k P_r(\delta)$. The action of $P_r(\delta)$ on the standard module $\Delta_{r,\delta}(\nu)$ is given by right multiplication. As $P_{r-k}(\delta)$-modules, we have that

$$\Delta_{r,\delta}(\nu) e_k \cong \Delta_{r-k,\delta}(\nu)$$

if $|\nu| \leqslant r-k$ and is zero otherwise. The following is an immediate consequence of [17, Proposition 23] and is also recorded explicitly in [10, Theorem 2.26].

Theorem 2.2. *The algebra $P_r(\delta)$ is semisimple if*

$$\delta \notin \{0, 1, \dots, 2r-2\} \tag{2.4}$$

and in this case the set $\{\Delta_{r,\delta}(\nu) : \nu \in \mathscr{P}(\leqslant r)\}$ forms a complete set of non-isomorphic simple $P_r(\delta)$-modules.

More generally, we have the following:

Theorem 2.3 *([17, Corollary 5.1]). Recall that $\delta \neq 0$ by assumption. The standard module $\Delta_{r,\delta}(\nu)$ has a unique maximal submodule. The simple module obtained as the quotient by this maximal submodule is denoted by $L_{r,\delta}(\nu)$, and moreover*

$$\{L_{r,\delta}(\nu) : \nu \in \mathscr{P}(\leqslant r)\}$$

forms a complete set of non-isomorphic simple $P_r(\delta)$-modules

We now give an explicit description of the standard modules which follows directly from (2.3). We set $V(r-k, r)$ to be the span of all $(r-k, r)$-partition diagrams (that is, having $r-k$ northern and r southern vertices) having precisely $(r-k)$ propagating blocks. This has the natural structure of a $(\mathfrak{S}_{r-k}, P_r(\delta))$-bimodule. It is easy to see that, as vector spaces, we have

$$\Delta_{r,\delta}(\nu) \cong \mathsf{S}(\nu) \otimes_{\mathfrak{S}_{r-k}} V(r-k, r).$$

The right action of $P_r(\delta)$ is given as follows. Let v be an $(r-k, r)$-partition diagram in $V(r-k, r)$, $x \in \mathsf{S}(\nu)$ and d be an (r, r)-partition diagram. Concatenate v above d to

get $(\delta)^t v'$ for some $(r-k,r)$-partition diagram v' and some non-negative integer t. If v' has fewer than $(r-k)$ propagating blocks then we set $(x \otimes v)d = 0$. Otherwise we set $(x \otimes v)d = \delta^t x \otimes v'$. Note that if $\nu \vdash r$, then we have

$$\Delta_{r,\delta}(\nu) \cong \mathsf{S}(\nu) \otimes_{\mathfrak{S}_r} V(r,r) = \mathsf{S}(\nu), \tag{2.5}$$

viewed as a $P_r(\delta)$-module via Equation (2.2). A special case which will be important later is $\Delta_{r,\delta}(\emptyset)$ which can be viewed as the span of all set-partitions of $\{1,2,\dots,r\}$ with the natural concatenation action.

Remark 2.4. We refer the interested reader to [2] for explicit diagrammatic calculations using the modules constructed in Equation (2.5).

3. The fixed depth Foulkes poset

For $r \in \mathbb{N}$, we consider the set $\{1,2,\dots,r\}$ with the usual total ordering. Given Λ a set-partition of $\{1,2,\dots,r\}$ and $a,b \in \{1,\dots,r\}$, we write $a \sim_\Lambda b$ if a and b belong to the same block of Λ, and we denote the number of blocks of Λ by $\ell(\Lambda)$. For example, if

$$\Lambda = \{\{1,2,4\},\{3\},\{5,7,8\},\{6,9\}\}$$

then $1 \sim_\Lambda 4$, $6 \sim_\Lambda 9$ and $\ell(\Lambda) = 4$. We represent this diagrammatically as follows.

Remark 3.1. Continuing with the convention of Section 2, we order the subsets in $\Lambda = \{\Lambda_1,\dots,\Lambda_l\}$ by *increasing minima*, so that

$$1 = \min\Lambda_1 < \min\Lambda_2 < \cdots < \min\Lambda_{l-1} < \min\Lambda_l \leqslant r.$$

Given Λ, Λ' two set-partitions of $\{1,2,\dots,r\}$, we write $\Lambda \subseteq \Lambda'$ if $a \sim_\Lambda b$ implies $a \sim_{\Lambda'} b$ for any $a,b \in \{1,\dots,r\}$, that is, if Λ' is obtained from Λ by merging blocks. We let F^r denote the set consisting of all pairs (Λ,Λ') of set-partitions of $\{1,\dots,r\}$ such that $\Lambda \subseteq \Lambda'$.

We equip F^r with a partial ordering $\subseteq$ as follows: given $(\Lambda,\Lambda'),(\Gamma,\Gamma') \in \mathsf{F}^r$, we write $(\Lambda,\Lambda') \subseteq (\Gamma,\Gamma')$ if $\Lambda \subseteq \Gamma$ and $\Lambda' \subseteq \Gamma'$. In this case we say that (Λ,Λ') is a refinement of (Γ,Γ') or that (Γ,Γ') is a coarsening of (Λ,Λ'). Given $(\Lambda,\Lambda') \in \mathsf{F}^r$ we associate a diagram, $[\Lambda,\Lambda']$, in the obvious fashion, recording Λ as above then grouping together in a 'bubble' those parts of Λ which together form a part of Λ'. For example if

$$(\Lambda,\Lambda') = \{\{1,2,4\},\{3\},\{5,7,8\},\{6,9\}\},\{\{1,2,3,4\},\{5,6,7,8,9\}\}$$

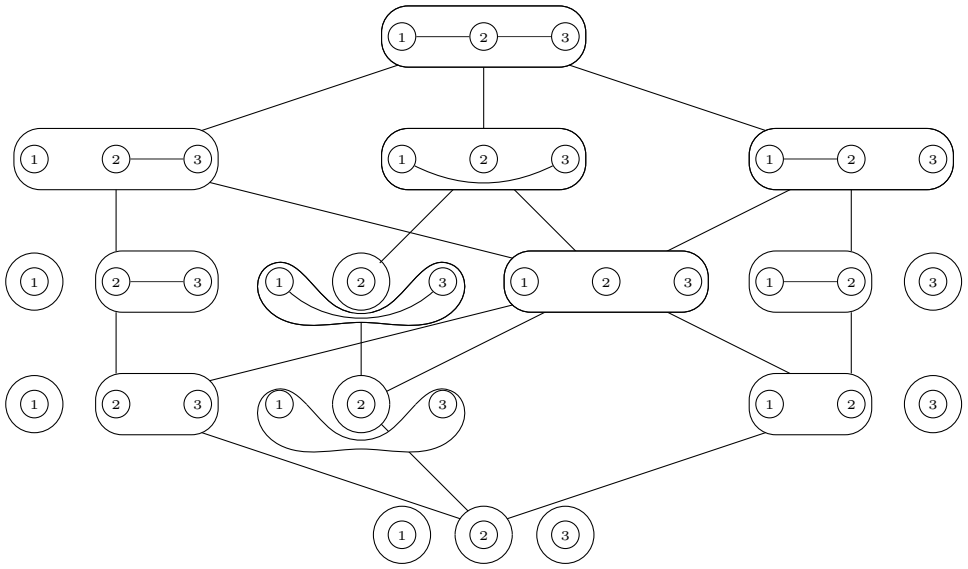
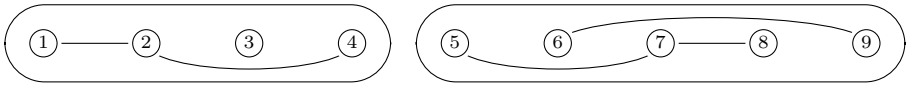


Fig. 4. The poset F^3.

Fig. 5. Two diagrams discussed in Example 3.3 which do not belong to certain subposets of F^3 (cross reference with Fig. 4).

then clearly $\Lambda \subseteq \Lambda'$ and we represent this pair diagrammatically as follows.

$[\Lambda, \Lambda'] =$ 

Definition 3.2. For $m, n \in \mathrm{N}$, we let $\mathsf{F}^r_{m,n} \subseteq \mathsf{F}^r$ denote the sub-poset consisting of the elements (Λ, Λ') such that $\ell(\Lambda') \leqslant n$ and such that the number of blocks of Λ contained within any single block of Λ' is at most m.

Example 3.3. The subposet $\mathsf{F}^3_{2,3} \subseteq \mathsf{F}^3$ contains all the elements shown in Fig. 4 except the leftmost diagram in Fig. 5. The subposet $\mathsf{F}^3_{3,2} \subseteq \mathsf{F}^3$ contains all the elements shown in Fig. 4 except the rightmost diagram in Fig. 5.

4. Schur–Weyl duality

Classical Schur–Weyl duality is the relationship between the (left and right) actions of general linear and symmetric groups over tensor space. The symmetric group acts on the right by permuting the factors. The general linear group acts on the left by matrix multiplication on each factor. These two actions commute and each generates the full centraliser of the other. We can restrict the action of the general linear group to the subgroup of permutation matrices and ask what algebra appears on the other side of the duality? The answer is the partition algebra. For the purposes of this paper, we shall be interested in how this allows us to understand certain plethysm coefficients via their stabilities.

The purpose of this paper is to study the decomposition of the Foulkes module $\mathrm{ind}_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{mn}} \mathbb{C}$. We shall attempt to do this via Schur–Weyl duality with the partition algebra. In order to see the Foulkes module within the setting of Schur–Weyl duality, we must consider the left action of $\mathfrak{S}_{mn}$ and the right action of $P_r(mn)$ on $(\mathbb{C}^{mn})^{\otimes r}$ (defined below). Here we have specialised the parameter of the partition algebra to $\delta = mn \in \mathbb{N}$.

Let $I(mn, r) = \{1, \dots, mn\}^r$ be the set of multi-indices. For a given multi-index $i = (i_1, \dots, i_r) \in I(mn, r)$, we put $\mathsf{e}_i = \mathsf{e}_{i_1} \otimes \cdots \otimes \mathsf{e}_{i_r}$. Then $\{\mathsf{e}_i : i \in I(mn, r)\}$ is a basis of tensor space $(\mathbb{C}^{mn})^{\otimes r}$ over $\mathbb{C}$. The action of the Weyl group $\mathfrak{S}_{mn}$ on $(\mathbb{C}^{mn})^{\otimes r}$ is simply the restriction of the diagonal action of $\mathrm{GL}_{mn}(\mathbb{C})$. In particular,

$$\sigma \cdot (\mathsf{e}_{i_1} \otimes \cdots \mathsf{e}_{i_r}) = \mathsf{e}_{\sigma(i_1)} \otimes \cdots \mathsf{e}_{\sigma(i_r)} \tag{4.1}$$

for any $\sigma \in \mathfrak{S}_{mn}$ and $i = (i_1, \dots, i_r) \in I(mn, r)$.

Let $d \in P_r(mn)$ be a partition diagram. Let $\delta_{a,b}$ be the usual Kronecker delta symbol, defined to be 1 if $a = b$ and 0 otherwise. Then the (right) action of d on the basis element e_i of tensor space is given by the matrix $\Psi(d)$ whose entry $[\Psi(d)]_{i_{\overline{1}}, \dots, i_{\overline{r}}}^{i_1, \dots, i_r}$ in row $(i_1, \dots, i_r)$ and column $(i_{\overline{1}}, \dots, i_{\overline{r}})$ is given by

$$[\Psi(d)]_{i_{\overline{1}}, \dots, i_{\overline{r}}}^{i_1, \dots, i_r} = \prod \delta_{i_s, i_t} \tag{4.2}$$

where the product is taken over all pairs s, t in $\{1, \dots, r, \overline{1}, \dots, \overline{r}\}$ which are connected by an edge in d (see, for example, [10, Equation (3.8)]). It is easily checked that the linear extension of the rule $d \mapsto \Psi(d)$ defines a representation $\Psi : P_r(mn) \to \mathrm{End}_{\mathbb{C}}((\mathbb{C}^{mn})^{\otimes r})$. To summarise, writing Φ for the map induced by the action in Equation (4.1), we have actions of the symmetric group and partition algebra on tensor space as follows

$$\mathbb{C}\mathfrak{S}_{mn} \xrightarrow{\Phi} \mathrm{End}_{\mathbb{C}}((\mathbb{C}^{mn})^{\otimes r}) \xleftarrow{\Psi} P_r(mn). \tag{4.3}$$

Theorem 4.1 *([13,16]). In the situation outlined above, the image of each representation is equal to the full centraliser algebra for the other action. More precisely, we have equalities*

$$\Phi(\mathbb{C}\mathfrak{S}_{mn}) = \mathrm{End}_{P_r(mn)}((\mathbb{C}^{mn})^{\otimes r}), \qquad \Psi(P_r(mn)) = \mathrm{End}_{\mathfrak{S}_{mn}}((\mathbb{C}^{mn})^{\otimes r}).$$

As a $(\mathbb{C}\mathfrak{S}_{mn}, P_r(mn))$-bimodule, the tensor space decomposes as

$$(\mathbb{C}^{mn})^{\otimes r} \cong \bigoplus \mathsf{S}(\lambda_{[mn]}) \otimes L_{r,mn}(\lambda)$$

where the sum is over all partitions $\lambda_{[mn]}$ of mn such that $|\lambda| \leqslant r$.

Here, if M is a left $\mathfrak{S}_{mn}$-module then

$$\mathcal{F}_r(M) = \mathrm{Hom}_{\mathfrak{S}_{mn}}(M, (\mathbb{C}^{mn})^{\otimes r})$$

carries the structure of a right $P_r(mn)$-module. Conversely, if N is a right $P_r(mn)$-module, then

$$\mathrm{Hom}_{P_r(mn)}(N, (\mathbb{C}^{mn})^{\otimes r})$$

carries the structure of a left $\mathfrak{S}_{mn}$-module. In particular, the theorem shows that $\mathcal{F}_r(\mathsf{S}(\lambda_{[mn]})) \cong L_{r,mn}(\lambda)$.

Now, $\mathfrak{S}_m \wr \mathfrak{S}_n$ is a subgroup of $\mathfrak{S}_{mn}$ via the embedding of Equation (1.1). The action (4.1) of $\mathfrak{S}_{mn}$ on tensor space restricts to an action of $\mathfrak{S}_m \wr \mathfrak{S}_n$. Having chosen our wreath product subgroup in the fashion above, we let this guide our choice of a new labelling set for the basis of tensor space as follows. For $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, we set

$$v_i^j = \mathsf{e}_{(j-1)m+i},$$

and we note that

$$(\sigma_1, \sigma_2, \dots, \sigma_n; \pi)(v_{i_1}^{j_1} \otimes v_{i_2}^{j_2} \otimes \cdots \otimes v_{i_r}^{j_r}) = v_{\sigma_{\pi(j_1)}(i_1)}^{\pi(j_1)} \otimes v_{\sigma_{\pi(j_2)}(i_2)}^{\pi(j_2)} \otimes \cdots \otimes v_{\sigma_{\pi(j_r)}(i_r)}^{\pi(j_r)}. \quad (4.4)$$

Using Schur–Weyl duality, we now apply $\mathcal{F}_r$ to the Foulkes module to define a $P_r(mn)$-module. This module's decomposition into simple constituents will be governed by the plethysm coefficients. This will allow us to study plethysm coefficients using the tools from the representation theory of partition algebras.

Definition 4.2. We say that the basis vector

$$v = v_{i_1}^{j_1} \otimes v_{i_2}^{j_2} \otimes \cdots \otimes v_{i_r}^{j_r} \in (\mathbb{C}^{mn})^{\otimes r}$$

has *value-type* (Λ, Λ') if $k \sim_{\Lambda'} l$ if and only if $j_k = j_l$ and $k \sim_{\Lambda} l$ if and only if $j_k = j_l$ and $i_k = i_l$. We write $\mathrm{val}(v) = (\Lambda, \Lambda')$. Observe that Λ, Λ' are set-partitions of $\{1, 2, \dots, r\}$ with $\Lambda \subseteq \Lambda'$ and, since there are at most m possible subscripts and n possible superscripts, $\mathrm{val}(v) \in \mathsf{F}_{m,n}^r$. Given $(\Lambda, \Lambda') \in \mathsf{F}_{m,n}^r$, we let $v_{(\Lambda,\Lambda')}$ denote the vector

$$v_{(\Lambda,\Lambda')} = \sum_{\mathrm{val}(v)=(\Lambda,\Lambda')} v,$$

where the sum runs over all basis vectors of $(\mathbb{C}^{mn})^{\otimes r}$ with value-type (Λ, Λ').

Example 4.3. For example, the basis vector $v = v_2^1 \otimes v_1^1 \otimes v_1^1 \otimes v_3^2 \otimes v_2^3 \otimes v_3^2 \otimes v_3^3$ has

$$\mathrm{val}(v) = (\{\{1\},\{2,3\},\{4,6\},\{5\},\{7\}\},\{\{1,2,3\},\{4,6\},\{5,7\}\}).$$

To obtain $\Lambda' = \{\{1,2,3\},\{4,6\},\{5,7\}\}$ note that the superscripts match in positions 1,2,3 and they match in positions 4 and 6 and they also match in positions 5 and 7. Although the subscripts match in positions 1 and 5, the superscripts do not match and so $1 \nsim_\Lambda 5$.

Example 4.4. Let $m = n = r = 2$ and $(\Lambda, \Lambda') = (\{\{1\},\{2\}\},\{\{1,2\}\})$ then

$$v_{(\Lambda,\Lambda')} = v_2^1 \otimes v_1^1 + v_1^1 \otimes v_2^1 + v_2^2 \otimes v_1^2 + v_1^2 \otimes v_2^2.$$

Example 4.5. Let $m = 4$ and $n = r = 5$ and

$$(\Lambda, \Lambda') = (\{\{1,2,4\},\{3\},\{5\}\},\{\{1,2,3,4\},\{5\}\}).$$

Then

$$v_{(\Lambda,\Lambda')} = \sum_{\substack{1\leqslant i_1,i_2,i_3\leqslant 4\\ 1\leqslant j_1,j_2\leqslant 5\\ i_1\neq i_2, j_1\neq j_2}} v_{i_1}^{j_1} \otimes v_{i_1}^{j_1} \otimes v_{i_2}^{j_1} \otimes v_{i_1}^{j_1} \otimes v_{i_3}^{j_2}.$$

Consider the action of the group $\mathfrak{S}_m \wr \mathfrak{S}_n$ on $(\mathbb{C}^{mn})^{\otimes r}$. The following lemma is immediate from Equation (4.4):

Lemma 4.6. *The $\mathfrak{S}_m \wr \mathfrak{S}_n$-orbit of a basis vector v of $(\mathbb{C}^{mn})^{\otimes r}$ consists of precisely those basis vectors having value-type equal to* $\mathrm{val}(v)$.

The following statement follows by Frobenius reciprocity and Lemma 4.6:

Corollary 4.7. *For each $(\Lambda, \Lambda') \in \mathsf{F}^r_{m,n}$, define an element*

$$\varphi_{(\Lambda,\Lambda')} \in \mathrm{Hom}_{\mathfrak{S}_{mn}}(\mathfrak{S}_{mn} \otimes_{\mathfrak{S}_m \wr \mathfrak{S}_n} \mathbb{C}, (\mathbb{C}^{mn})^{\otimes r})$$

by setting

$$\varphi_{(\Lambda,\Lambda')}(\sigma \otimes 1) = \sigma v_{(\Lambda,\Lambda')}$$

for any $\sigma \in \mathfrak{S}_{mn}$. The set

$$\{\varphi_{(\Lambda,\Lambda')} \mid (\Lambda,\Lambda') \in \mathsf{F}^r_{m,n}\}$$

is a basis for the right $P_r(mn)$*-module* $\mathrm{Hom}_{\mathfrak{S}_{mn}}(\mathfrak{S}_{mn} \otimes_{\mathfrak{S}_m \wr \mathfrak{S}_n} \mathbb{C}, (\mathbb{C}^{mn})^{\otimes r})$.

Definition 4.8. We refer to the (right) $P_r(mn)$-module

$$\mathcal{F}_r(\mathfrak{S}_{mn} \otimes_{\mathfrak{S}_m \wr \mathfrak{S}_n} \mathbb{C}) = \mathrm{Hom}_{\mathfrak{S}_{mn}}(\mathfrak{S}_{mn} \otimes_{\mathfrak{S}_m \wr \mathfrak{S}_n} \mathbb{C}, (\mathbb{C}^{mn})^{\otimes r})$$

as the *Foulkes module* for $P_r(mn)$.

We give a second basis of the Foulkes module for $P_r(mn)$, $\{\overline{\varphi}_{(\Lambda,\Lambda')} : (\Lambda,\Lambda') \in \mathsf{F}^r_{m,n}\}$, by setting

$$\overline{\varphi}_{(\Lambda,\Lambda')} = \sum_{(\Gamma,\Gamma'):(\Lambda,\Lambda') \subseteq (\Gamma,\Gamma')} \varphi_{(\Gamma,\Gamma')}.$$

That this is also a basis is clear, as we have simply applied a uni-triangular summation over the order $\subseteq$.

The element $\overline{\varphi}_{(\Lambda,\Lambda')}$ can be defined for any $(\Lambda,\Lambda') \in \mathsf{F}^r$, sending the generator $1_{\mathfrak{S}_{mn}} \otimes 1$ to

$$\sum_{\substack{1 \leqslant i_S \leqslant m, \forall S \in \Lambda, \\ 1 \leqslant j_\Sigma \leqslant n, \forall \Sigma \in \Lambda', \\ i_a = i_S, j_a = j_\Sigma, \forall a \in S \subseteq \Sigma}} v_{i_1}^{j_1} \otimes v_{i_2}^{j_2} \otimes \cdots v_{i_r}^{j_r}. \tag{4.5}$$

For example, if (Λ,Λ') is as in Example 4.5 then

$$\overline{\varphi}_{(\Lambda,\Lambda')}(1_{\mathfrak{S}_{mn}} \otimes 1) = \sum_{\substack{1 \leqslant i,i',i'' \leqslant 4 \\ 1 \leqslant j,j' \leqslant 5}} v_i^j \otimes v_i^j \otimes v_{i'}^j \otimes v_i^j \otimes v_{i''}^{j'}.$$

5. The stable Foulkes module

In this section we factorise the partition algebra parameter $\delta = \delta_1\delta_2$. In practice, we will often specialise $\delta_1 = m$ and $\delta_2 = n \in \mathbb{Z}_{\geqslant 1}$, however it will be useful to be able to specialise from generic parameters.

Recall from Section 2 that the $P_r(\delta)$-module $\Delta_{r,\delta}(\emptyset)$ has diagrammatic basis given by all set-partitions of $\{1, 2, \ldots, r\}$ with the natural concatenation action. With this in mind, we make the following definition:

Definition 5.1. We define the *stable Foulkes module* to be the complex vector space

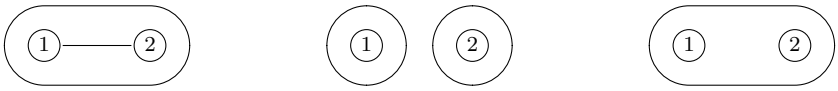
$$\mathbb{F}^r(\delta_1, \delta_2) = \mathrm{Span}_{\mathbb{C}}\{[\Lambda,\Lambda'] \mid (\Lambda,\Lambda') \in \mathsf{F}^r\},$$

equipped with the following right action of $P_r(\delta_1\delta_2)$. For d an (r,r)-partition diagram,

$$[\Lambda, \Lambda']d = \delta_1^{t_1}\delta_2^{t_2}[\Gamma, \Gamma'],$$

if $[\Lambda]d = \delta_1^{t_1}[\Gamma]$ in the $P_r(\delta_1)$-module $\Delta_{r,\delta_1}(\emptyset)$, and $[\Lambda']d = \delta_2^{t_2}[\Gamma']$ in the $P_r(\delta_2)$-module $\Delta_{r,\delta_2}(\emptyset)$.

Example 5.2. Let $r = 2$. The stable Foulkes module $\mathbb{F}^2(\delta_1, \delta_2)$ is 3-dimensional with basis given by the following three diagrams:



The generators of $P_2(\delta_1\delta_2)$ act as follows:

$$p_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & \delta_1\delta_2 & \delta_1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad p_{1,2} \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad s_{1,2} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From this, one observes that the first two diagrams span a 2-dimensional $P_2(\delta_1\delta_2)$-submodule. In fact, we note that this 2-dimensional submodule is isomorphic, as a $P_2(\delta_1\delta_2)$-module, to $\Delta_{2,\delta_1\delta_2}(\varnothing)$ simply by identifying the diagrammatic bases in the obvious manner. The resulting quotient module is 1-dimensional and is isomorphic to $\Delta_{2,\delta_1\delta_2}((2))$ (as there is a unique 1-dimensional simple $P_2(\delta_1\delta_2)$-module that is killed by p_1 and upon which $s_{1,2}$ acts as the scalar 1).

We now describe the action of the generators of the partition algebra $P_r(\delta_1\delta_2)$ on the basis of the stable Foulkes module. We let $\Lambda = \{S_1, S_2, \dots, S_p\}$ and $\Lambda' = \{\Sigma_1, \Sigma_2, \dots, \Sigma_q\}$ for $1 \leqslant q \leqslant p \leqslant r$. Recall the notational convention of Remark 3.1. Observe that

$$[\Lambda, \Lambda']p_{1,2} = \begin{cases} [\{S_1, S_2, S_3, \dots S_p\}, \{\Sigma_1, \Sigma_2, \dots, \Sigma_q\}] & \text{if } 1, 2 \in S_1 \subseteq \Sigma_1 \\ [\{S_1 \cup S_2, S_3, \dots S_p\}, \{\Sigma_1 \cup \Sigma_2, \dots, \Sigma_q\}] & \text{if } 1 \in S_1 \subseteq \Sigma_1, 2 \in S_2 \subseteq \Sigma_2 \\ [\{S_1 \cup S_2, S_3, \dots S_p\}, \{\Sigma_1, \Sigma_2, \dots, \Sigma_q\}] & \text{if } 1 \in S_1 \subseteq \Sigma_1, 2 \in S_2 \subseteq \Sigma_1 \end{cases} \tag{5.1}$$

and also

$$[\Lambda, \Lambda']p_1 = \begin{cases} \delta_1\delta_2 \times [\Lambda, \Lambda'] & \text{if } \{1\} = S_1 = \Sigma_1 \\ [\{\{1\}, S_1 - \{1\}, S_2, S_3, \dots S_p\}, \{\{1\}, \Sigma_1 - \{1\}, \Sigma_2, \dots, \Sigma_q\}] & \text{if } \{1\} \subset S_1 \subseteq \Sigma_1 \\ \delta_1 \times [\{\{1\}, S_2, S_3, \dots S_p\}, \{\{1\}, \Sigma_1 - \{1\}, \Sigma_2, \dots, \Sigma_q\}] & \text{if } \{1\} = S_1 \subset \Sigma_1. \end{cases} \tag{5.2}$$

The generators $s_{i,i+1}$ for $1 \leqslant i < r$ act in the usual fashion by permuting $\{1, 2, \dots, r\}$. For ease of notation, we do not write these actions out explicitly.

6. Relating the Foulkes and stable Foulkes modules for the partition algebra

We again specialise the parameters $\delta_1 = m$ and $\delta_2 = n$ to relate the two right $P_r(mn)$-modules we introduced in the previous sections: the stable Foulkes module $\mathbb{F}^r(m,n)$ and the Foulkes module $\mathrm{Hom}_{\mathfrak{S}_{mn}}(\mathfrak{S}_{mn} \otimes_{\mathfrak{S}_m \wr \mathfrak{S}_n} \mathbb{C}, (\mathbb{C}^{mn})^{\otimes r})$. We shall do this using the elements $\overline{\varphi}_{(\Lambda,\Lambda')}$, which were defined in (4.5).

Theorem 6.1. *Let $m, n, r \in \mathbb{N}$. There is a surjective homomorphism of $P_r(mn)$-modules*

$$\Theta : \mathbb{F}^r(m,n) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{S}_{mn}}(\mathfrak{S}_{mn} \otimes_{\mathfrak{S}_m \wr \mathfrak{S}_n} \mathbb{C}, (\mathbb{C}^{mn})^{\otimes r}),$$

mapping $[\Lambda, \Lambda']$ to $\overline{\varphi}_{(\Lambda,\Lambda')}$. The map Θ is injective if and only if $m, n \geqslant r$.

Proof. We have that $\{\varphi_{(\Lambda,\Lambda')} : (\Lambda,\Lambda') \in \mathsf{F}^r_{m,n}\}$ is a basis of the Foulkes module by Corollary 4.7 and therefore $\{\overline{\varphi}_{(\Lambda,\Lambda')} : (\Lambda,\Lambda') \in \mathsf{F}^r_{m,n}\}$ is a basis by unitriangularity. Thus, surjectivity of Θ is clear. Injectivity follows if and only if $\mathsf{F}^r_{m,n} = \mathsf{F}^r$, that is $m, n \geqslant r$. It remains to check that Θ is a $P_r(mn)$-modules homomorphism, which is simply an exercise in matching-up the action of the partition algebra generators on the two modules. We write $\overline{v}_{(\Lambda,\Lambda')}$ for the image of the generator $1_{\mathfrak{S}_{mn}} \otimes 1$ of the Foulkes module under $\overline{\varphi}_{(\Lambda,\Lambda')}$ from Equation (4.5).

In $\overline{v}_{(\Lambda,\Lambda')}p_{1,2}$, the term $v_{i_1}^{j_1} \otimes v_{i_2}^{j_2} \otimes \cdots v_{i_r}^{j_r}$ is killed if $i_1 \neq i_2$ or $j_1 \neq j_2$. The effect is therefore to merge the blocks of Λ (respectively Λ') containing 1 and 2. Compare this with Equation (5.1).

Now consider p_1. Each term $v_{i_1}^{j_1} \otimes v_{i_2}^{j_2} \otimes \cdots v_{i_r}^{j_r}$ in $\overline{v}_{(\Lambda,\Lambda')}$ is sent to $\sum_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n} v_i^j \otimes v_{i_2}^{j_2} \otimes \cdots v_{i_r}^{j_r}$ in $\overline{v}_{(\Lambda,\Lambda')}p_1$. The effect is to split off a singleton block $\{1\}$ from the blocks of Λ and Λ' containing 1, but there is a multiplicity. If the singleton part $\{1\} \in \Lambda$ and $\{1\} \in \Lambda'$, then there are mn terms in $\overline{v}_{(\Lambda,\Lambda')}$ making this contribution. If $\{1\} \in \Lambda$ but is not a singleton part of Λ' then there are m terms making this contribution. Finally, if $\{1\}$ is not a singleton part of either Λ or Λ' then the single term contributes. Compare this with Equation (5.2).

The symmetric group generators act in the usual way and thus Θ is a $P_r(mn)$-homomorphism. □

By Theorems 4.1 and 6.1 we deduce the following:

Corollary 6.2. *Let $m, n, r \in \mathbb{N}$. We have the following equality of composition multiplicities:*

$$\left[\mathfrak{S}_{mn} \otimes_{\mathfrak{S}_m \wr \mathfrak{S}_n} \mathbb{C} : \mathsf{S}(\lambda_{[mn]})\right]_{\mathbb{C}\mathfrak{S}_{mn}} = \left[\mathbb{F}^r(m,n)/\ker(\Theta) : L_{r,mn}(\lambda)\right]_{P_r(mn)},$$

for $\lambda_{[mn]}$ a partition of mn such that $|\lambda| \leqslant r$. In particular, if $m, n \geqslant r$ then

$$\left[\mathfrak{S}_{mn} \otimes_{\mathfrak{S}_m \wr \mathfrak{S}_n} \mathbb{C} \,:\, \mathsf{S}(\lambda_{[mn]})\right]_{\mathbb{C}\mathfrak{S}_{mn}} = \left[\mathbb{F}^r(m,n) \,:\, L_{r,mn}(\lambda)\right]_{P_r(mn)}.$$

We will demonstrate the power of this corollary in the next two sections, where we use the partition algebra to give elementary algebraic proofs of results about plethysm coefficients.

7. The structure of the stable Foulkes module

We begin this section by giving an elementary filtration on the stable Foulkes module. We show that the layers of this filtration are preserved under swapping the parameters; this gives a simple proof of Theorem A. We then examine these layers of the filtration in more detail; we provide an explicit direct sum decomposition of these layers and hence prove Theorem B.

7.1. A filtration of the stable Foulkes module

In this section we construct a filtration of the stable Foulkes module $\mathbb{F}^r(\delta_1, \delta_2)$ as a $P_r(\delta_1\delta_2)$-module (with arbitrary parameters δ_1, δ_2). This will allow us to deduce that its composition factors depend *only* on the product $\delta_1\delta_2 \in \mathbb{C}$ and do not depend on the distinct parameters δ_1, δ_2. Given Λ a set-partition, recall that $\ell(\Lambda)$ denotes the number of blocks in Λ.

Definition 7.1. Given a pair (Λ, Λ'), we set $\ell([\Lambda, \Lambda']) = \ell(\Lambda) - \ell(\Lambda')$. For $0 \leqslant k < r$, we let $\mathbb{F}^r_k(\delta_1, \delta_2)$ denote the subspace of $\mathbb{F}^r(\delta_1, \delta_2)$ with basis $\{[\Lambda, \Lambda'] \mid \ell([\Lambda, \Lambda']) \leqslant k\}$.

For example for $r = 9$ we see that

$$(\Lambda, \Lambda') = \{\{1,2,4\},\{3\},\{5,7,8\},\{6,9\}\}, \{\{1,2,3,4\},\{5,6,7,8,9\}\} \in \mathbb{F}^9_2(\delta_1, \delta_2).$$

Theorem 7.2. *Let $r \in \mathbb{N}$. Then, as a $P_r(\delta_1\delta_2)$-module, $\mathbb{F}^r(\delta_1, \delta_2)$ has a filtration*

$$0 \subset \mathbb{F}^r_0(\delta_1, \delta_2) \subset \mathbb{F}^r_1(\delta_1, \delta_2) \subset \cdots \subset \mathbb{F}^r_{r-1}(\delta_1, \delta_2) = \mathbb{F}^r(\delta_1, \delta_2).$$

Moreover, all entries in the representing matrices of the generators of $P_r(\delta_1\delta_2)$ on the quotient module $\mathbb{F}^r_k(\delta_1, \delta_2)/\mathbb{F}^r_{k-1}(\delta_1, \delta_2)$ for $1 \leqslant k \leqslant r-1$ consist only of zeroes, ones, and the parameter $\delta_1\delta_2$. In particular, the entries depend only on the product $\delta_1\delta_2$ and are independent of the factors δ_1, δ_2.

Proof. We shall consider the actions of the elements $p_{1,2}$, p_1 and $s_{i,i+1}$ for $1 \leqslant i < r$ in turn. In the three cases of Equation (5.1), we have that

$$\ell([\Lambda,\Lambda']) - \ell([\Lambda,\Lambda']p_{1,2}) = 0,0,1$$

respectively. Each entry in the representing matrix is 0 or 1. Now consider the action of p_1 from Equation (5.2): $[\Lambda,\Lambda']p_1$ is a scalar times a diagram $[\Gamma,\Gamma']$ and

$$\ell([\Lambda,\Lambda']) - \ell([\Gamma,\Gamma']) = 0,0,1$$

respectively. Furthermore, note that the parameter δ_1 appears only in the third case of Equation (5.2), which is precisely the case in which $[\Lambda,\Lambda']p_1$ is zero in the quotient module $\mathbb{F}_k^r(\delta_1,\delta_2)/\mathbb{F}_{k-1}^r(\delta_1,\delta_2)$. Finally, the elements $s_{i,i+1}$ simply permute the numbers $\{1,2,\dots,r\}$ within the blocks of the set-partition and so, firstly, $\ell([\Lambda,\Lambda']) - \ell([\Lambda,\Lambda']s_{i,i+1}) = 0$ for all for $1 \leqslant i < r$ and, secondly, the representing matrices of $s_{i,i+1}$ consist only of entries 0 and 1. The result follows. □

Example 7.3. This filtration is constructed explicitly in Example 5.2 for $r = 2$ and δ_1,δ_2 arbitrary.

Corollary 7.4. *Let $r \in \mathbb{N}$. There exists an isomorphism of $P_r(\delta_1\delta_2)$-modules*

$$\mathbb{F}_k^r(\delta_1,\delta_2)/\mathbb{F}_{k-1}^r(\delta_1,\delta_2) \cong \mathbb{F}_k^r(\delta_2,\delta_1)/\mathbb{F}_{k-1}^r(\delta_2,\delta_1)$$

for $1 \leqslant k \leqslant r-1$. In particular, we have the following equality of composition multiplicities:

$$[\mathbb{F}^r(\delta_1,\delta_2) \,:\, L_{r,\delta_1\delta_2}(\lambda)]_{P_r(\delta_1\delta_2)} = [\mathbb{F}^r(\delta_2,\delta_1) \,:\, L_{r,\delta_1\delta_2}(\lambda)]_{P_r(\delta_1\delta_2)}\,,$$

for $|\lambda| \leqslant r$.

In particular, specialising the parameters $\delta_1\delta_2 = mn \in \mathbb{Z}_{\geqslant 1}$, we obtain Theorem A of the introduction.

7.2. An explicit decomposition of the stable Foulkes module

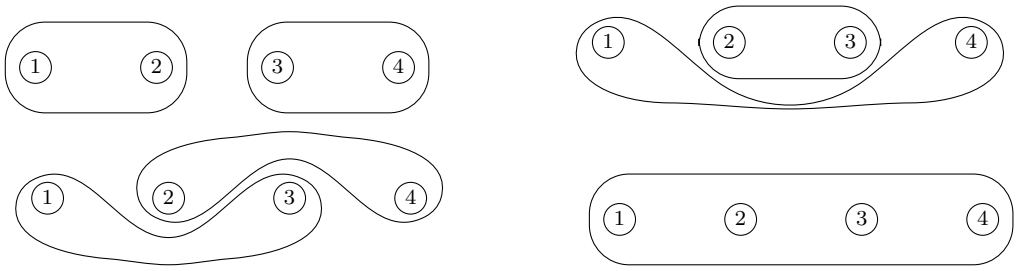
In this section we decompose the stable Foulkes module in the case where $P_r(\delta_1\delta_2)$ is semisimple.

Definition 7.5. Let $r \in \mathbb{N}$. We define the *depth-radical* of $\mathbb{F}^r(\delta_1,\delta_2)$ to be the subspace spanned by the pairs $[\Lambda,\Lambda']$ satisfying either of the following two conditions:

(*i*) The set-partition Λ contains a non-singleton block;
(*ii*) the set-partition Λ' contains a singleton block.

We let $\mathsf{DR}(\mathbb{F}^r(\delta_1,\delta_2))$ denote the depth-radical of $\mathbb{F}^r(\delta_1,\delta_2)$.

Example 7.6. For $r = 4$ the module $\mathbb{F}^4(\delta_1, \delta_2)$ is 60-dimensional and $\mathsf{DR}(\mathbb{F}^4(\delta_1, \delta_2))$ is 56-dimensional. Rather than list the basis elements of $\mathsf{DR}(\mathbb{F}^4(\delta_1, \delta_2))$, we instead list the four pairs (Λ, Λ') which do *not* belong to the depth-radical. These are pictured below.



Proposition 7.7. *Given $r \in \mathbb{N}$, the depth radical $\mathsf{DR}(\mathbb{F}^r(\delta_1, \delta_2))$ is a $P_r(\delta_1\delta_2)$-submodule of $\mathbb{F}^r(\delta_1, \delta_2)$.*

Proof. Clearly the generators $s_{i,i+1}$ for $1 \leqslant i < r$ preserve the space $\mathsf{DR}(\mathbb{F}^r(\delta_1, \delta_2))$ as both conditions of Definition 7.5 are invariant under the permutation action. By Equation (5.2), the generator p_1 acts on a given $[\Lambda, \Lambda']$ either by scalar multiplication, or by removing an edge from Λ *at the expense* of introducing a singleton into Λ'. Therefore the generator p_1 preserves the submodule by (ii) of Definition 7.5. By Equation (5.1) the generator $p_{1,2}$ acts on a given $[\Lambda, \Lambda']$ either trivially or by introducing an edge in Λ. Therefore the generator $p_{1,2}$ preserves the submodule by (i) of Definition 7.5. □

Definition 7.8. Define the *depth quotient* $\mathsf{DQ}(\mathbb{F}^r(\delta_1, \delta_2))$ of $\mathbb{F}^r(\delta_1, \delta_2)$ to be the quotient

$$\mathsf{DQ}(\mathbb{F}^r(\delta_1, \delta_2)) = \mathbb{F}^r(\delta_1, \delta_2)/\mathsf{DR}(\mathbb{F}^r(\delta_1, \delta_2))$$

spanned by the diagrams $[\{\{1\}, \{2\}, \dots, \{r\}\}, \Lambda']$ where Λ' contains no singleton blocks.

Recall that for $\delta_1\delta_2 \neq 0$ the idempotent $e_1 = \frac{1}{\delta_1\delta_2} p_1 \in P_r(\delta_1\delta_2)$. By the general theory of idempotent truncation (see for example [9, Section 6.2]) and Equations (2.1), (2.2) we obtain the following.

Proposition 7.9. *For $r \geqslant 2$,*

$$\mathsf{DR}(\mathbb{F}^r(\delta_1, \delta_2))e_1 P_r(\delta_1\delta_2) = \mathsf{DR}(\mathbb{F}^r(\delta_1, \delta_2)), \qquad \mathsf{DQ}(\mathbb{F}^r(\delta_1, \delta_2))e_1 = 0,$$

and

$$\mathsf{DR}(\mathbb{F}^r(\delta_1, \delta_2))e_1 \cong \mathbb{F}^{r-1}(\delta_1, \delta_2)$$

as an $e_1 P_r(\delta_1\delta_2)e_1 \cong P_{r-1}(\delta_1\delta_2)$-module. When $r = 1$, $\mathbb{F}^1(\delta_1, \delta_2) \cong \Delta_{1,\delta_1\delta_2}(\emptyset)$.

Proof. We consider the first statement. We let $[\Lambda, \Lambda']$ be an arbitrary basis element of $\mathsf{DR}(\mathbb{F}^r(\delta_1, \delta_2))$. We shall write $[\Lambda, \Lambda']$ in the form

$$[\Lambda, \Lambda'] = [\overline{\Lambda}, \overline{\Lambda'}]e_1 d$$

for some $[\overline{\Lambda}, \overline{\Lambda'}] \in \mathsf{DR}(\mathbb{F}^r(\delta_1, \delta_2))$ and some partition diagram $d \in P_r(\delta_1\delta_2)$ and hence deduce the result. First, suppose that Λ' contains a singleton block $\{i\}$ for $1 \leqslant i \leqslant r$. In this case we set

$$[\overline{\Lambda}, \overline{\Lambda'}] = [\Lambda, \Lambda']s_{1,i},$$

where $s_{1,i} = s_{i-1,i} \cdots s_{2,3}s_{1,2}s_{2,3} \cdots s_{i-1,i}$. We find

$$[\Lambda, \Lambda'] = [\overline{\Lambda}, \overline{\Lambda'}]e_1 s_{1,i}$$

as required. Now suppose that Λ' contains no singleton block; by Definition 7.5 we deduce that Λ contains a non-singleton block. In other words, we suppose that there exist distinct $j, k \in \{1, \dots, r\}$ with $j \sim_\Lambda k$. In this case we set

$$[\overline{\Lambda}, \overline{\Lambda'}] = [\Lambda, \Lambda']s_{1,j}s_{2,k},$$

where $s_{2,k} = s_{1,2}s_{1,k}s_{1,2}$. We observe that

$$[\Lambda, \Lambda'] = [\overline{\Lambda}, \overline{\Lambda'}]e_1(p_{1,2}s_{1,j}s_{2,k})$$

as required. The first statement follows.

We now consider the second and third statements. Let $[\Lambda, \Lambda']$ be a basis element of $\mathbb{F}^r(\delta_1, \delta_2)$ and consider $[\Lambda, \Lambda']e_1$ using Equation (5.2). In all three cases the resulting outer partition contains a singleton block and therefore $[\Lambda, \Lambda']e_1 \in \mathsf{DR}(\mathbb{F}^r(\delta_1, \delta_2))$. Therefore the second statement holds. Considering only $[\Lambda, \Lambda'] \in \mathsf{DR}(\mathbb{F}^r(\delta_1, \delta_2))$, we see that all possible $[\Gamma, \Gamma']$ with a singleton part $\{1\}$ in both Γ and Γ' can occur as $[\Lambda, \Lambda']e_1$, thus the third statement holds.

Finally, it is clear that $\mathbb{F}^1(\delta_1, \delta_2)$ is 1-dimensional and p_1 acts by scalar multiplication by $\delta_1\delta_2$ as in $\Delta_{1,\delta_1\delta_2}(\emptyset)$. □

By Proposition 7.9 and the construction of simple modules of the partition algebra in Subsection 2.2we deduce the following:

Corollary 7.10. *In the case where $P_r(\delta_1\delta_2)$ is semisimple, we have the following equality of composition multiplicities:*

$$[\mathbb{F}^r(\delta_1, \delta_2) \,:\, L_{r,\delta_1\delta_2}(\lambda)]_{P_r(\delta_1\delta_2)} = \begin{cases} [\mathsf{DQ}(\mathbb{F}^r(\delta_1, \delta_2)) \,:\, L_{r,\delta_1\delta_2}(\lambda)]_{P_r(\delta_1\delta_2)} & \text{if } |\lambda| = r, \\ [\mathbb{F}^{r-1}(\delta_1, \delta_2) \,:\, L_{r-1,\delta_1\delta_2}(\lambda)]_{P_{r-1}(\delta_1\delta_2)} & \text{if } |\lambda| < r. \end{cases}$$

We now describe these composition multiplicities in the semisimple case. If λ is a partition of r then, by Equation (2.5), the simple $P_r(\delta_1\delta_2)$-module $L_{r,\delta_1\delta_2}(\lambda)$ is the (inflation of the) Specht module $\mathsf{S}(\lambda)$. Therefore, for $|\lambda| = r$,

$$[\mathsf{DQ}(\mathbb{F}^r(\delta_1,\delta_2)) : L_{r,\delta_1\delta_2}(\lambda)]_{P_r(\delta_1\delta_2)} = [\mathsf{DQ}(\mathbb{F}^r(\delta_1,\delta_2)) : \mathsf{S}(\lambda)]_{\mathbb{C}\mathfrak{S}_r}. \tag{7.1}$$

Now, as a $\mathbb{C}\mathfrak{S}_r$-module, $\mathsf{DQ}(\mathbb{F}^r(\delta_1,\delta_2))$ is a permutation module and its decomposition into transitive permutation modules is readily seen (from Definition 7.8) to be

$$\mathsf{DQ}(\mathbb{F}^r(\delta_1,\delta_2)) = \bigoplus_{\mu\in\mathscr{P}_1(r)} [\Lambda_{(1^r)}, \Lambda_\mu]\mathbb{C}\mathfrak{S}_r,$$

where, recall, $\mathscr{P}_1(r)$ denotes the set of partitions of r which have no part equal to 1, and, for μ a partition of r we define a corresponding set-partition $\Lambda_\mu = \{\{1,2,\dots,\mu_1\},\{\mu_1+1,\dots,\mu_2\},\dots\}$. Therefore, the $\mathbb{C}\mathfrak{S}_r$-module

$$\mathsf{DQ}(\mathbb{F}^r(\delta_1,\delta_2)) = \bigoplus_{\mu\in\mathscr{P}_1(r)} \mathrm{ind}_{\mathrm{Stab}(\Lambda_\mu)}^{\mathfrak{S}_r} \mathbb{C}. \tag{7.2}$$

The groups $\mathrm{Stab}(\Lambda_\mu)$ appearing here are direct products of wreath products of symmetric groups as in Equation (1.2).

Theorem 7.11. *Suppose that $P_r(\delta_1\delta_2)$ is semisimple and λ is a partition of r. Then*

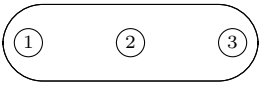
$$[\mathsf{DQ}(\mathbb{F}^r(\delta_1,\delta_2)) : L_{r,\delta_1\delta_2}(\lambda)]_{P_r(\delta_1\delta_2)} = \sum_{\mu\in\mathscr{P}_1(r)} \left[\mathrm{ind}_{\mathrm{Stab}(\Lambda_\mu)}^{\mathfrak{S}_r} \mathbb{C} : \mathsf{S}(\lambda)\right]_{\mathbb{C}\mathfrak{S}_r}.$$

Proof. This follows from Equation (7.2) and Equation (7.1). □

Example 7.12. We continue the example of $\mathbb{F}^4(\delta_1,\delta_2)$ from Example 7.6 in the case where $P_r(\delta_1\delta_2)$ is semisimple. The first three diagrams shown in that example belong to the same orbit; the first is $\Lambda_{(2,2)}$, and the stabiliser of any one of these diagrams is isomorphic to $\mathrm{Stab}(\Lambda_{(2,2)}) = \mathfrak{S}_2 \wr \mathfrak{S}_2$. The final diagram is $\Lambda_{(4)}$ and its stabiliser is $\mathrm{Stab}(\Lambda_{(4)}) = \mathfrak{S}_4$. If $\delta_1\delta_2 \notin \{0,1,2,3,4,5,6\}$ then $P_r(\delta_1\delta_2)$ is semisimple by Equation (2.4). Since $\mathrm{ind}_{\mathfrak{S}_2\wr\mathfrak{S}_2}^{\mathfrak{S}_4}(\mathbb{C}) \cong \mathsf{S}(4) \oplus \mathsf{S}(2^2)$,

$$\mathsf{DQ}(\mathbb{F}^4(\delta_1,\delta_2)) \cong L_{4,\delta_1\delta_2}(4) \oplus L_{4,\delta_1\delta_2}(4) \oplus L_{4,\delta_1\delta_2}(2^2).$$

The rest of the decomposition of $\mathbb{F}^4(\delta_1,\delta_2)$, the decomposition of its depth radical, is obtained by Corollary 7.10. Accordingly, we must decompose the $P_3(\delta_1\delta_2)$-module $\mathbb{F}^3(\delta_1,\delta_2)$. As $\mathsf{DQ}(\mathbb{F}^3(\delta_1,\delta_2)$ is 1-dimensional, with basis vector



it follows that $\mathsf{DQ}(\mathbb{F}^3(\delta_1,\delta_2))$ is isomorphic to $L_{3,\delta_1\delta_2}(3)$ and so it contributes a summand $L_{4,\delta_1\delta_2}(3)$ to $\mathbb{F}^4(\delta_1,\delta_2)$. To decompose $\mathsf{DR}(\mathbb{F}^3(\delta_1,\delta_2))$, we must decompose the $P_2(\delta_1\delta_2)$-module $\mathbb{F}^2(\delta_1,\delta_2)$. By the previous argument, $\mathsf{DQ}(\mathbb{F}^2(\delta_1,\delta_2)$ is isomorphic to $L_{2,\delta_1\delta_2}(2)$, and the decomposition of $\mathsf{DR}(\mathbb{F}^2(\delta_1,\delta_2))$ is governed by that of $\mathbb{F}^1(\delta_1,\delta_2)\cong L_{1,\delta_1\delta_2}(\emptyset)$. Putting all this together shows

$$\mathsf{DR}(\mathbb{F}^4(\delta_1,\delta_2))\cong L_{4,\delta_1\delta_2}(3)\oplus L_{4,\delta_1\delta_2}(2)\oplus L_{4,\delta_1\delta_2}(\emptyset).$$

Combining the decompositions of $\mathsf{DR}(\mathbb{F}^4(\delta_1,\delta_2))$ and $\mathsf{DQR}(\mathbb{F}^4(\delta_1,\delta_2))$ obtained above shows that

$$\mathbb{F}^4(\delta_1,\delta_2)\cong 2L_{4,\delta_1\delta_2}(4)\oplus L_{4,\delta_1\delta_2}(2^2)\oplus L_{4,\delta_1\delta_2}(3)\oplus L_{4,\delta_1\delta_2}(2)\oplus L_{4,\delta_1\delta_2}(\emptyset).$$

Remark 7.13. Alternatively, one can note that we have an injective $\mathbb{C}$-linear map $\varphi:\mathbb{F}^{r-1}(\delta_1,\delta_2)\to\mathbb{F}^r(\delta_1,\delta_2)$ given by $\varphi(\{S_1,S_2,\dots,S_p\},\{\Sigma_1,\Sigma_2,\dots\Sigma_q\})=(\{S_1,S_2,\dots,S_p,\{r\}\},\{\Sigma_1,\Sigma_2,\dots\Sigma_q,\{r\}\})$. The image of this map generates the submodule $\mathsf{DR}(\mathbb{F}^r(\delta_1,\delta_2))$. The form of the map φ is how we first arrived at condition (ii) of Definition 7.5; condition (i) was then deduced by considering the submodule generated by the image.

8. Consequences for plethysm coefficients

For the final section we specialise δ_1 and δ_2 to be m and n respectively. Combining Corollary 7.10 and Theorem 7.11 with Corollary 6.2, we obtain a formula for certain plethysm coefficients in terms of smaller generalised plethysm coefficients (as defined in Equation (1.3)).

Theorem 8.1. *Let $m,n\in\mathbb{N}$ and let $\lambda_{[mn]}$ be a partition of mn with $m,n\geqslant|\lambda|$. Then*

$$p((n),(m),\lambda_{[mn]})=\sum_{\mu\in\mathscr{P}_1(|\lambda|)}p_\mu(\lambda).$$

Proof. Let $r=|\lambda|$. Under the hypotheses, $mn\geqslant r^2>2r-2$ and so $P_r(mn)$ is semisimple. Schur–Weyl duality and Theorem 6.1 show that

$$p((n),(m),\lambda_{[mn]})=\left[\mathbb{F}^r(m,n)\,:\,L_{r,mn}(\lambda)\right]_{P_r(mn)}.$$

Applying Theorem 7.11 then gives

$$p((n),(m),\lambda_{[mn]})=\sum_{\mu\in\mathscr{P}_1(|\lambda|)}\left[\operatorname{ind}_{\operatorname{Stab}(\Lambda_\mu)}^{\mathfrak{S}_{|\lambda|}}\mathbb{C}\,:\,\mathsf{S}(\lambda)\right]_{\mathbb{C}\mathfrak{S}_{|\lambda|}}=\sum_{\mu\in\mathscr{P}_1(|\lambda|)}p_\mu(\lambda).\quad\square$$

Example 8.2. Continuing Example 7.6 and Example 7.12, when $r=4$ and $m,n\geqslant 4$ we obtain the following plethysm coefficients:

$$p((n),(m),(mn-4,4)) = 2$$
$$p((n),(m),(mn-4,2^2)) = 1$$
$$p((n),(m),(mn-3,3)) = 1$$
$$p((n),(m),(mn-2,2)) = 1$$
$$p((n),(m),(mn)) = 1$$

and the coefficients $p((n),(m),\alpha) = 0$ for all other partitions α of depth at most 4.

The following result is immediate from Theorem 8.1.

Corollary 8.3. *Let λ be an arbitrary partition. The double sequence*

$$\{p((n),(m),\lambda_{[mn]})\}_{m,n\in\mathbb{N}}$$

stabilises for all $m,n \geqslant |\lambda|$; we denote the stable values by $\overline{p}_{\infty,\lambda}$. In other words

$$\overline{p}_{\infty,\lambda} = p((n),(m),\lambda_{[mn]})$$

for all $m,n \geqslant |\lambda|$.

Theorem 8.1 also allows us to deduce the following result, with a little more work.

Proposition 8.4. *For $n,m \geqslant r$, we have that*

$$\overline{p}_{\infty,(r)} = p((n),(m),(mn-r,r)) = p((r),(r),(r^2-r,r)) = |\mathscr{P}_1(r)|.$$

Moreover, for $r > 2$,

$$p((r-1),(r),(r^2-2r,r)) = p((r),(r-1),(r^2-2r,r)) = |\mathscr{P}_1(r)| - 1.$$

In particular, the bound on stability in Corollary 8.3 is sharp for λ an arbitrary partition of r.

Proof. The two statements together prove that the bounds on n,m for stability in Corollary 8.3 are sharp using partition $\lambda = (r)$. The first statement follows immediately from Theorem 8.1 and the fact that $\mathsf{S}((r))$ occurs as a summand once in each transitive $\mathbb{C}\mathfrak{S}_r$-permutation module $\mathrm{ind}_{\mathrm{Stab}(\Lambda_\mu)}^{\mathfrak{S}_r}\mathbb{C}$. For the second part, recall that the Cayley–Sylvester formula (see, for example [15]) provides the plethysm coefficients for two-part partitions:

$$p((n),(m),(mn-r,r)) = |\mathscr{P}_{m\times n}(r)| - |\mathscr{P}_{m\times n}(r-1)|,$$

where $\mathscr{P}_{m\times n}(r)$ equals the set of all those partitions of r whose Young diagrams fit inside the $m \times n$ rectangle. Taking $m = r$ and $n = r-1$ (and the calculation is identical for $m = r-1$, $n = r$),

$$\begin{aligned} p((r-1),(r),(r^2-2r,r)) &= |\mathscr{P}_{r\times(r-1)}(r)| - |\mathscr{P}_{r\times(r-1)}(r-1)| \\ &= |\mathscr{P}(r)| - 1 - |\mathscr{P}(r-1)| \\ &= |\mathscr{P}_1(r)| - 1, \end{aligned}$$

since adding a part of size 1 provides a bijection $\mathscr{P}(r-1) \to \mathscr{P}(r) \setminus \mathscr{P}_1(r)$. □

Example 8.5. The non-zero stable plethysm coefficients $\overline{p}_{\infty,\lambda}$ for $\lambda \vdash 8$ are as follows:

$$\overline{p}_{\infty,(8)} = 7, \quad \overline{p}_{\infty,(7,1)} = 4, \quad \overline{p}_{\infty,(6,2)} = 8, \quad \overline{p}_{\infty,(5,3)} = 3, \quad \overline{p}_{\infty,(5,2,1)} = 2,$$
$$\overline{p}_{\infty,(4^2)} = 4, \quad \overline{p}_{\infty,(4,3,1)} = 1, \quad \overline{p}_{\infty,(4,2^2)} = 3, \quad \overline{p}_{\infty,(2^4)} = 1.$$

To see this, we note that the elements of $\mathscr{P}_1(8)$ are (8), $(6,2)$, $(5,3)$, $(4,4)$, $(4,2^2)$, $(3^2,2)$ and (2^4). The required products of (smaller) plethysm and Littlewood-Richardson coefficients can be calculated by hand or using SAGE [22] (whereas the coefficients $p((8),(8),\lambda)$ seem to be beyond SAGE). The decompositions of the relevant transitive permutation modules are as follows:

$$\operatorname{ind}_{\mathfrak{S}_8}^{\mathfrak{S}_8}(\mathbb{C}) = \mathsf{S}(8), \qquad \operatorname{ind}_{\mathfrak{S}_6\times\mathfrak{S}_2}^{\mathfrak{S}_8}(\mathbb{C}) = \mathsf{S}(6,2) \oplus \mathsf{S}(7,1) \oplus \mathsf{S}(8),$$
$$\operatorname{ind}_{\mathfrak{S}_5\times\mathfrak{S}_3}^{\mathfrak{S}_8}(\mathbb{C}) = \mathsf{S}(5,3)\oplus\mathsf{S}(6,2)\oplus\mathsf{S}(7,1)\oplus\mathsf{S}(8), \qquad \operatorname{ind}_{\mathfrak{S}_4\wr\mathfrak{S}_2}^{\mathfrak{S}_8}(\mathbb{C}) = \mathsf{S}(4^2)\oplus\mathsf{S}(6,2)\oplus\mathsf{S}(8),$$
$$\operatorname{ind}_{\mathfrak{S}_4\times\mathfrak{S}_2\wr\mathfrak{S}_2}^{\mathfrak{S}_8}(\mathbb{C}) = \mathsf{S}(4,2^2) \oplus \mathsf{S}(4^2) \oplus \mathsf{S}(5,2,1) \oplus \mathsf{S}(5,3) \oplus 2\mathsf{S}(6,2) \oplus \mathsf{S}(7,1) \oplus \mathsf{S}(8),$$
$$\operatorname{ind}_{\mathfrak{S}_3\wr\mathfrak{S}_2\times\mathfrak{S}_2}^{\mathfrak{S}_8}(\mathbb{C}) = \mathsf{S}(4,2^2)\oplus\mathsf{S}(4,3,1)\oplus\mathsf{S}(4^2)\oplus\mathsf{S}(5,2,1)\oplus\mathsf{S}(5,3)\oplus2\mathsf{S}(6,2)\oplus\mathsf{S}(7,1)\oplus\mathsf{S}(8),$$
$$\operatorname{ind}_{\mathfrak{S}_2\wr\mathfrak{S}_4}^{\mathfrak{S}_8}(\mathbb{C}) = \mathsf{S}(2^4) \oplus \mathsf{S}(4,2^2) \oplus \mathsf{S}(4^2) \oplus \mathsf{S}(6,2) \oplus \mathsf{S}(8).$$

One can deduce that Foulkes' conjecture is satisfied at the partition $\lambda_{[mn]}$ for all pairs m, n of integers which are both greater than or equal to 8. For example,

$$\overline{p}_{\infty,(6,2)} = p((8),(8),(56,6,2)) = p((9),(8),(64,6,2)) = p((8),(9),(64,6,2)) = 8.$$

We also deduce the strengthened Foulkes' conjecture is satisfied for all quadruples of integers which are each greater than or equal to 8. For example,

$$p((240),(8),(1912,6,2)) = p((40),(48),(1912,6,2)) = p((16),(120),(1912,6,2)) = 8.$$

Example 8.6. We find $p((10),(10),(90,4^2,2)) = 6$. This can be calculated as follows:

$$[\operatorname{ind}_{\mathfrak{S}_4\wr\mathfrak{S}_2\times\mathfrak{S}_2}^{\mathfrak{S}_{10}}(\mathbb{C}) : \mathsf{S}(4^2,2)] = 1, \quad [\operatorname{ind}_{\mathfrak{S}_4\times\mathfrak{S}_3\wr\mathfrak{S}_2}^{\mathfrak{S}_{10}}(\mathbb{C}) : \mathsf{S}(4^2,2)] = 1,$$
$$[\operatorname{ind}_{\mathfrak{S}_4\times\mathfrak{S}_2\wr\mathfrak{S}_3}^{\mathfrak{S}_{10}}(\mathbb{C}) : \mathsf{S}(4^2,2)] = 1,$$

$$[\mathrm{ind}_{\mathfrak{S}_3\wr\mathfrak{S}_2\times\mathfrak{S}_2\wr\mathfrak{S}_2}^{\mathfrak{S}_{10}}(\mathbb{C}) : \mathsf{S}(4^2,2)] = 2, \quad [\mathrm{ind}_{\mathfrak{S}_2\wr\mathfrak{S}_5}^{\mathfrak{S}_{10}}(\mathbb{C}) : \mathsf{S}(4^2,2)] = 1$$

with

$$[\mathrm{ind}_{\mathrm{Stab}(\Lambda_\mu)}^{\mathfrak{S}_{10}}(\mathbb{C}) : \mathsf{S}(4^2,2)] = 0$$

for all other $\mu \in \mathscr{P}_1(10)$.

Our results provide an elementary new proof of Weintraub's conjecture [26] for stable plethysm coefficients. Recall that the conjecture, recently proven in [4], states that if m is even and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is an even partition (that is $\lambda_1, \lambda_2, \ldots, \lambda_\ell \in 2\mathbb{Z}_{\geqslant 1}$) then the plethysm coefficient $p((n),(m),\lambda)$ is non-zero.

Corollary 8.7 *(Stable Weintraub's conjecture). For λ an even partition, we have that*

$$\overline{p}_{\infty,\lambda} > 0.$$

Proof. Let $\lambda = (a_1^{b_1}, a_2^{b_2}, \ldots, a_\ell^{b_\ell})$ be an even partition, and pick $m, n \geqslant |\lambda|$. We use the formula for $p((n),(m),\lambda_{[mn]})$ in Theorem 8.1. Since λ is even, $\lambda \in \mathscr{P}_1(|\lambda|)$. The contribution to the sum from taking $\mu = \lambda$ is 1 since $p((b_i),(a_i),(a_i{}^{b_i})) = 1$ for even a_i by [21, Theorem 2.6] and, by the Littlewood–Richardson rule, $p_\mu(\lambda) = 1$ for $\mu = (a_1^{b_1}, \ldots, a_\ell^{b_\ell}) = \lambda$. Therefore the stable plethysm coefficient $\overline{p}_{\infty,\lambda} = p((n),(m),\lambda_{[mn]})$ is strictly positive. □

Data availability

No data was used for the research described in the article.

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