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1 **ORTHOOGONAL DECOMPOSITION OF DEFINABLE GROUPS**

2 ALESSANDRO BERARDUCCI, PANTELIS E. ELEFTHERIOU, AND MARCELLO MAMINO

Abstract. Orthogonality in model theory captures the idea of absence of non-trivial interactions between definable sets. We introduce a somewhat opposite notion of cohesiveness, capturing the idea of interaction among all parts of a given definable set. A cohesive set is indecomposable, in the sense that if it is internal to the product of two orthogonal sets, then it is internal to one of the two. We prove that a definable group in an o-minimal structure is a product of cohesive orthogonal subsets. If the group has dimension one, or it is definably simple, then it is itself cohesive. As an application, we show that an abelian group definable in the disjoint union of finitely many o-minimal structures is a quotient, by a discrete normal subgroup, of a direct product of locally definable groups in the single structures.

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## 1. Introduction

Considering a group  $G$  interpretable in the disjoint union of finitely many structures  $\mathcal{X}_1, \dots, \mathcal{X}_n$  (seen as a multi-sorted structure as in Definition 2.3), one may ask whether  $G$  can be understood in terms of groups definable in the individual structures. One may, for instance, ask whether  $G$  is definably isomorphic to a quotient, modulo a finite normal subgroup  $\Gamma$ , of a direct product  $G_1 \times \dots \times G_n$ , where  $G_i$  is a group definable in  $\mathcal{X}_i$ . This, however, is not true in general: a counterexample is provided by [2, Example 1.2] (a torus obtained from two orthogonal copies of  $\mathbb{R}$  and a lattice generated by two vectors in generic position). After a talk by the first author at the Oberwolfach workshop “Model Theory: Groups, Geometry, and Combinatorics” (2013), Hrushovski suggested that a result of the above kind would require to pass to the locally definable category. So the natural conjecture would be that, if  $G$  is as above, there is a locally definable isomorphism  $G \cong G_1 \times \dots \times G_n / \Gamma$ , where  $G_i$  is a locally definable group in  $\mathcal{X}_i$  and  $\Gamma$  is a compatible *discrete* subgroup (i.e. a subgroup which intersects every definable set at a finite set). Here we establish the conjecture under the additional assumption that the structures  $\mathcal{X}_i$  are o-minimal and  $G$  is abelian. We recall that an o-minimal structure is a structure  $\mathcal{M} = (M, <, \dots)$  expanding a dense linear order without end-points such that every definable subset of  $M$  is a finite union of intervals and points (see [5]). We prove the following result.

**Theorem (9.2).** *Let  $G$  be a definable abelian group in the disjoint union of finitely many o-minimal structures  $\mathcal{X}_1, \dots, \mathcal{X}_n$ . Then there is a locally definable homomorphism*

$$G \cong G_1 \times \dots \times G_n / \Gamma,$$

where  $G_i$  is a locally definable group in  $\mathcal{X}_i$  and  $\Gamma$  is a compatible locally definable discrete subgroup of  $G_1 \times \dots \times G_n$ .

We will deduce Theorem 9.2 from Theorem 8.4 below, which is interesting in itself and holds for non-abelian groups as well. To state the theorem we need to recall the model-theoretic notion of orthogonality. Given definable sets  $X_1, \dots, X_n$  in a structure  $\mathcal{M}$ , we say that  $X_1, \dots, X_n$  are *orthogonal* if, for all  $k_1, \dots, k_n \in \mathbb{N}$ , any definable subset of  $X_1^{k_1} \times \dots \times X_n^{k_n}$  is a finite union of sets of the form  $A_1 \times \dots \times A_n$ , where  $A_i$  is a definable subset of  $X_i^{k_i}$ . Let us also recall that a definable set  $X$  is *internal* to a definable set  $Y$  if there is  $m \in \mathbb{N}$  and a definable surjective map from  $Y^m$  to  $X$ .

**Theorem (8.4).** *Assume  $\mathcal{M}$  is an o-minimal structure. Let  $X_1, \dots, X_n$  be orthogonal sets definable in  $\mathcal{M}$  and  $G$  a group definable in  $\mathcal{M}$ . If  $G$  is internal to the product  $X_1 \times \dots \times X_n$ , then  $G$  is a product*

$$G = A_1 \dots A_n$$

of definable subsets  $A_1, \dots, A_n$ , where  $A_i$  is internal to  $X_i$ , for  $i = 1, \dots, n$  and  $A_1 \dots A_n$  is the set of all products  $a_1 \dots a_n \in G$  with  $a_i \in A_i$ .

To deduce Theorem 9.2 from Theorem 8.4, we take for  $\mathcal{M}$  the o-minimal structure consisting of the concatenation of the structures  $\mathcal{X}_i$  separated by single points. The sets  $X_i$  would then be orthogonal within  $\mathcal{M}$ , so we can apply Theorem 8.4 and take as  $G_i$  an isomorphic copy of the subgroup of  $G$  generated by  $A_i$ . The only delicate point is to show that  $G_i$  is locally definable in  $\mathcal{X}_i$ , but this is not difficult.

1 It is worth stressing that Theorem 8.4 holds for an arbitrary o-minimal structure  $\mathcal{M}$   
 2 and in particular we do not assume that  $\mathcal{M}$  has a group structure. This is important  
 3 for the application to Theorem 9.2 because the concatenation of o-minimal structures  
 4 does not have a group structure even if the single structures do.

5 The proof of Theorem 8.4 uses a number of deep results about groups definable in  
 6 o-minimal structures, such as the solution of Pillay’s Conjecture and Compact Dom-  
 7 ination Conjecture (see [16] and [10] for the definitions). It is, however, conceivable  
 8 that the theorem could be extended far beyond the o-minimal context: indeed we have  
 9 no counterexample even if  $\mathcal{M}$  is allowed to be a completely arbitrary structure. The-  
 10 orem 6.1 provides a partial result in this direction, when  $\mathcal{M}$  is NIP and  $G$  is compactly  
 11 dominated and abelian.

12 Before stating our final result, we notice that the sets  $A_1, \dots, A_n$  in Theorem  
 13 8.4 are orthogonal, so we may ask whether, for a group  $G$  definable in an arbitrary  
 14 o-minimal structure  $\mathcal{M}$ , there is always a natural way to generate it as a product  
 15  $G = A_1 \dots A_n$  of orthogonal definable subsets  $A_i$ . Of course, there is always the  
 16 trivial solution with  $n = 1$  and  $A_1 = G$ , but we would like each  $A_i$  to be in some sense  
 17 “minimal”. To this aim, we introduce the following model-theoretic notions. Let  $Z$   
 18 be a set definable in an arbitrary structure  $\mathcal{M}$ . We say that:

- 19 •  $Z$  is **indecomposable** if whenever  $Z$  is internal to the cartesian product of  
 20 two orthogonal sets, then  $Z$  is internal to one of the two.
- 21 •  $Z$  is **cohesive** if whenever two definable sets are non-orthogonal to  $Z$ , they  
 22 are non-orthogonal to each other.

23 A cohesive set  $Z$  is indecomposable (Proposition 3.2(2)), but it can be shown that  
 24 the converse fails, see Example 11.4. We show that cohesive sets have nice model-  
 25 theoretic properties which we are not able to prove for indecomposable sets. In  
 26 particular, a set internal to a cohesive set is cohesive and the cartesian product of  
 27 two cohesive sets is cohesive. The intuition is that the various parts of a cohesive  
 28 set interact in a non-trivial way, so in particular a cohesive set cannot contain two  
 29 orthogonal infinite definable subsets. Our final result is the following.

30 **Theorem (8.7).** *Let  $G$  be a group interpretable in an o-minimal structure  $\mathcal{M}$ . Then*  
 31 *there are cohesive orthogonal definable sets  $A_1, \dots, A_n$ , such that  $G = A_1 \dots A_n$ .*

32 In the setting of Theorem 8.7 we call the tuple  $A_1, \dots, A_n$  a **cohesive orthog-**  
 33 **onal decomposition** of  $G$ , while in Theorem 8.4 the tuple  $A_1, \dots, A_n$  is called an  
 34 **orthogonal decomposition** of  $G$  with respect to  $X_1, \dots, X_n$ .

35 If  $G$  is infinite, we can choose each  $A_i$  in Theorem 8.7 to be infinite, and in this  
 36 case the number  $n$  is an invariant of  $G$  up to definable isomorphism. Indeed, if  
 37  $G = B_1 \dots B_m$  is another cohesive orthogonal decomposition of  $G$  where  $B_1, \dots, B_m$   
 38 are infinite, then  $m = n$  and each  $B_i$  is bi-internal to a single  $A_j$ . We may call the  
 39 invariant  $n$  the **dimensionality** of  $G$  (not to be confused with the dimension of  $G$ ).  
 40 In this terminology, the unidimensional groups in the sense of [14, Claim 1.26] have  
 41 dimensionality 1.

42 We can show that if  $G$  has dimension one, or it is definably simple, then  $G$  is itself  
 43 cohesive, so these groups have dimensionality one.

44 For the proof of Theorem 8.7, we need both Theorem 8.4 and the results from [8].  
 45 In particular, we need that for every group  $G$  interpretable in an o-minimal structure,

1 there is an injective definable map  $f$  (not necessarily a morphism) from  $G$  to the  
2 cartesian product of finitely many 1-dimensional definable groups ([8, Theorem 3]).

3 **Related work.** Groups definable in the disjoint union of orthogonal structures have  
4 already been considered in [2, 20]. The original motivation comes from the model  
5 theory of group extensions [3, 21]. For instance, the universal cover of a group  
6 definable in an o-minimal expansion of the field  $\mathbb{R}$  is definable in the disjoint union  
7  $\mathbb{R} \sqcup \mathbb{Z}$ , where  $\mathbb{Z}$  has only the additive structure [11]. From a model-theoretic point  
8 of view,  $(\mathbb{Z}, +)$  is an example of a superstable structure of finite and definable Lascar  
9 rank. In [2], it is shown that if a group  $G$  is definable in the disjoint union of an  
10 arbitrary structure  $\mathcal{R}$  and a superstable structure  $\mathcal{Z}$  of finite and definable Lascar  
11 rank, then  $G$  is an extension of a group internal to  $\mathcal{R}$  by a group internal to  $\mathcal{Z}$ .  
12 In [20], Wagner weakened the superstability assumption to assuming only that  $\mathcal{Z}$  is  
13 simple. The simplicity assumption cannot be entirely removed, or replaced by an o-  
14 minimality one: quotients by a lattice of a product of copies of  $\mathbb{R}$  provide examples  
15 of definable groups which may not have infinite definable subgroups internal to any  
16 of the copies ([2, Example 1.2]). However, as Theorem 8.4 shows, the o-minimality  
17 assumption allows for another, in fact more symmetric analysis in terms of generating  
18 subsets instead of subgroups. Theorem 8.7 can be seen as a continuation of the work  
19 in [8].

20 The possibility of extending the results beyond the o-minimal context raises a  
21 number of questions, which are included in Section 10.

22 **Structure of the paper.** In Sections 2 – 4, we introduce and study the key notions  
23 of this paper. In particular we prove, for definable sets  $X_1, \dots, X_n$  in an arbitrary  
24 structure, that  $X_1, \dots, X_n$  are orthogonal if and only if they are pairwise orthogonal.  
25 We then study indecomposable and cohesive sets and establish that 1-dimensional  
26 groups definable in o-minimal structures are cohesive (Theorem 3.4). The proofs  
27 of Theorems 8.4 and 8.7 then proceed in several steps. In Section 5, we recall the  
28 basics of compact domination for NIP structures, which we employ in Section 6 to  
29 prove Theorem 8.4 for compactly dominated abelian NIP groups that are *contained* in  
30  $X_1 \times \dots \times X_n$  (Theorem 6.1). In Section 7, we specialize in o-minimal structures and  
31 prove Theorem 8.4 for definably compact abelian groups *internal* to  $X_1 \times \dots \times X_n$ . In  
32 Section 8, we employ the rich machinery available for groups definable in o-minimal  
33 structures, and conclude the full Theorem 8.4. Together with the results from [8],  
34 we then establish Theorem 8.7. In the final part of the paper we prove Theorem 9.2.

35 **Notation.** Throughout this paper, we work in a first-order structure  $\mathcal{M}$ . By “defin-  
36 able” we mean definable in  $\mathcal{M}$ , with parameters. Unless stated otherwise,  $X, Y, Z$   
37 denote definable sets. By convention,  $X \times Y^0 = X$ .

38 We assume familiarity with the basics of o-minimality, as in [5]. We also assume  
39 familiarity with the definable manifold topology of definable groups [15, Proposition  
40 2.5]. All topological notions for definable groups, such as connectedness and definable  
41 compactness, are taken with respect to this group topology.

## 42 2. Orthogonality

43 In this section we work in an arbitrary structure  $\mathcal{M}$ . We recall the notions of  
44 orthogonality and internality, and prove some basic facts.

1 **Definition 2.1.** Given definable sets  $X_1, \dots, X_n$ , a  $(X_1, \dots, X_n)$ -**box** is a definable  
 2 set of the form  $U_1 \times \dots \times U_n$  where  $U_i \subseteq X_i$  for every  $i = 1, \dots, n$ . When clear from  
 3 the context, we omit the prefix  $(X_1, \dots, X_n)$ - in front of “box”.

4 **Definition 2.2.** Let  $X_1, \dots, X_n$  be definable sets. We say that  $X_1, \dots, X_n$  are **or-**  
 5 **thogonal** if, for every  $k_1, \dots, k_n \in \mathbb{N}$ , every definable subset  $S$  of  $X_1^{k_1} \times \dots \times X_n^{k_n}$  is  
 6 the union of finitely many  $(X_1^{k_1}, \dots, X_n^{k_n})$ -boxes.

7 An example of the notion of orthogonal sets is provided by the following definition.

8 **Definition 2.3.** Given finitely many structures  $\mathcal{X}_1, \dots, \mathcal{X}_n$ , their **disjoint union**  $\bigsqcup_i \mathcal{X}_i$   
 9 is the multi-sorted structure with a sort for each  $\mathcal{X}_i$  and whose basic relations are the  
 10 definable sets in the single structures  $\mathcal{X}_i$ . Notice that if  $X_i$  is the domain of  $\mathcal{X}_i$ , then  
 11  $X_1, \dots, X_n$  are orthogonal in  $\bigsqcup_i \mathcal{X}_i$ .

12 The definition of orthogonality can be rephrased in terms of types using the fol-  
 13 lowing remark of Wagner. We include a proof to facilitate comparison with similar  
 14 notions of orthogonality in the model-theoretic literature (see [19]), but we will not  
 15 need this fact.

16 **Remark 2.4** ([20, Remark 3.4]). Let  $X_1, \dots, X_n$  be definable sets. The following  
 17 conditions are equivalent:

- 18 (1) Every definable subset of  $X_1 \times \dots \times X_n$  is a finite union of  $(X_1, \dots, X_n)$ -boxes.  
 19 (2) For every type  $p(x_1, \dots, x_n)$  over  $\mathcal{M}$  with  $p(x_1, \dots, x_n) \vdash X_1 \times \dots \times X_n$ , we  
 20 have that  $p_1(x_1) \cup \dots \cup p_n(x_n) \vdash p(x_1, \dots, x_n)$  where  $p_i$  is the  $i$ -th projection  
 21 of  $p$ .

22 *Proof.* To simplify notation we assume  $n = 2$  and write  $X, Y$  for  $X_1, X_2$ .

23 Assume (1). Let  $p(x, y)$  be a type over  $\mathcal{M}$  concentrated on  $X \times Y$ . Let  $\varphi(x, y)$   
 24 be a formula over  $\mathcal{M}$  defining a subset of  $X \times Y$ . The set  $Z$  defined by  $\varphi$  is a finite  
 25 union of boxes  $U_i \times V_i$ . It follows that  $p_1(x) \cup p_2(y) \vdash p(x, y)$ .

26 Assume (2). Let  $Z \subseteq X \times Y$  be definable. We must show that  $Z$  is a finite  
 27 union of boxes  $U_i \times V_i$ . For any type  $p(x, y)$  containing the defining formula of  $Z$ ,  
 28 we have that  $p_1(x) \cup p_2(y) \vdash (x, y) \in Z$ . By compactness, there are  $\varphi_1^p(x) \in p_1(x)$   
 29 and  $\varphi_2^p(y) \in p_2(y)$ , such that  $\varphi_1^p(x) \wedge \varphi_2^p(y) \vdash (x, y) \in Z$ . Again by compactness,  
 30  $(x, y) \in Z$  is equivalent to a finite disjunction of formulas of the form  $\varphi_1^p(x) \wedge \varphi_2^p(y)$   
 31 (if not we reach a contradiction considering a type containing the defining formula of  
 32  $Z$  and the negation of all the formulas  $\varphi_1^p(x) \wedge \varphi_2^p(y)$ ). It follows that  $Z$  is a finite  
 33 union of boxes  $U_i \times V_i$  as desired.  $\square$

34 The following fact follows at once from the definition.

35 **Proposition 2.5.** Let  $X, Y, Z$  be definable sets. Suppose that, for every positive  
 36 integer  $n$ , all definable subsets of  $X^n \times Y$  are a finite union of  $(X^n, Y)$ -boxes, and all  
 37 definable subsets of  $X^n \times Z$  are finite unions of  $(X^n, Z)$ -boxes. Then, for all  $n$ , every  
 38 definable subset of  $X^n \times Y \times Z$  is a finite union of  $(X^n, Y \times Z)$ -boxes.

39 *Proof.* Let  $S$  be a definable subset of  $X^n \times Y \times Z$ . Given  $z \in Z$ , consider the fiber  
 40  $S_z \subseteq X^n \times Y$  consisting of the pairs  $(x, y)$  such that  $(x, y, z) \in S$ . Let  $E_z \subseteq X^n \times X^n$   
 41 be the following equivalence relation:

$$a E_z b \iff \forall y \in Y (a, y) \in S_z \iff (b, y) \in S_z .$$

1 Then  $\{E_z \mid z \in Z\}$  is a family of subsets of  $X^n \times X^n$  indexed by  $Z$ , so by the  
 2 hypothesis applied to  $X^{2n} \times Z$ , it is finite. By the hypothesis on  $X^n \times Y$ , each  
 3 equivalence relation  $E_z$  has finitely many equivalence classes; in fact,  $S_z$  is a finite  
 4 union of  $(X^n, Y)$ -boxes and the equivalence classes of  $E_z$  are the atoms of the boolean  
 5 algebra generated by the projections of these boxes on the component  $X^n$ . It follows  
 6 that  $E = \bigcap_z E_z \subseteq X^n \times X^n$  is again an equivalence relation with finitely many classes.  
 7 On the other hand, for  $a, b \in X^n$ ,

$$a E b \iff \forall y \in Y, z \in Z (a, y, z) \in S \iff (b, y, z) \in S.$$

8 We have thus proved that there are finitely many subsets of the form  $\pi_{Y \times Z}(\pi_{X^n}^{-1}(x) \cap S)$   
 9 with  $x \in X^n$ , which is desired result.  $\square$

10 **Corollary 2.6.** *Let  $X, Y, Z$  be definable sets. If  $X$  is orthogonal to both  $Y$  and  $Z$ ,*  
 11 *then  $X$  is orthogonal to  $Y \times Z$ .*

12 **Corollary 2.7.** *Suppose  $X_1, \dots, X_n$  are pairwise orthogonal definable sets. Then*  
 13  *$X_1, \dots, X_n$  are orthogonal.*

14 *Proof.* It suffices to show that each  $X_i$  is orthogonal to the product of the other sets  
 15  $X_j$ . This follows by 2.6 and induction on  $n$ .  $\square$

16 We shall need the following result.

17 **Corollary 2.8.** *Let  $X$  and  $Y$  be definable sets. Then  $X$  and  $Y$  are orthogonal if and*  
 18 *only if for every positive integer  $n$ , all definable subsets of  $X^n \times Y$  are finite unions of*  
 19  *$(X^n, Y)$ -boxes.*

20 For the main results of this paper we make no saturation assumptions on the ambi-  
 21 ent structure  $\mathcal{M}$ . It is however worth mentioning that under a saturation assumption  
 22 we can strengthen Corollary 2.8 as follows.

23 **Proposition 2.9.** *If  $\mathcal{M}$  is  $\aleph_0$ -saturated, then  $X$  and  $Y$  are orthogonal if and only if*  
 24 *all definable subsets of  $X \times Y$  are finite unions of  $(X, Y)$ -boxes. The same conclusion*  
 25 *holds without saturation provided  $\mathcal{M}$  is o-minimal, or more generally if  $\mathcal{M}$  eliminates*  
 26 *the quantifier  $\exists^\infty$ .*

*Proof.* Suppose that all definable subsets of  $X \times Y$  are finite unions of  $(X, Y)$ -boxes.  
 Let  $S$  be a definable subset of  $X^n \times Y$ . It suffices to show that  $S$  is a finite union  
 of  $(X^n, Y)$ -boxes (by Corollary 2.8). We reason by induction on  $n$ . The case  $n = 1$   
 holds by the assumptions. Assume  $n > 1$ . For  $t \in X$ , consider the set

$$S_t = \{(x, y) \mid (t, x, y) \in S\} \subseteq X^{n-1} \times Y.$$

27 By induction,  $S_t$  is a finite union of  $(X^{n-1}, Y)$ -boxes. Let  $R_t \subseteq Y \times Y$  be the  
 28 equivalence relation defined by  $u R_t v \iff (\forall x \in X^{n-1}) (x, u) \in S_t \leftrightarrow (x, v) \in S_t$ .  
 29 Note that  $R_t$  has finitely many equivalence classes. By saturation (or elimination of  
 30  $\exists^\infty$ ), there is a uniform bound  $k \in \mathbb{N}$ , such that for all  $t \in X$ , there are at most  $k$   
 31 equivalence classes modulo  $R_t$ .

32 We claim that there is a finite subset  $A$  of  $Y$  such that for all  $t \in X$  each equivalence  
 33 class of  $R_t$  intersects  $A$ . To this aim we prove, by induction on  $j \leq k$ , that there is a  
 34 finite subset  $A_j$  of  $Y$  such that for all  $t \in Y$  there are at most  $k - j$  equivalence classes  
 35 of  $R_t$  which do not intersect  $A_j$ . Clearly we can take  $A_0$  to be the empty set. Let us

1 construct  $A_{j+1}$ . Consider the definable family  $(Q_t)_{t \in X}$  where  $Q_t \subseteq Y$  is the union of  
 2 the  $R_t$ -equivalence classes which intersect  $A_j$ . By the assumptions the family  $(Q_t)_{t \in X}$   
 3 contains finitely many distinct subsets  $Q_1, \dots, Q_l$  of  $Y$ . We obtain  $A_{j+1}$  adding to  $A_j$   
 4 a point in  $Y \setminus Q_i$  for each  $i = 1, \dots, l$  such that  $Q_i \neq Y$ . The claim is thus proved  
 5 taking  $A = A_k$ .

6 Now we claim that the equivalence relation  $R = \bigcap_{t \in X} R_t$  has finitely many equiva-  
 7 lence classes. Indeed, given  $a \in A$  consider the family  $(P_{t,a})_{t \in X}$  where  $P_{t,a} \subseteq Y$  is the  
 8  $R_t$ -equivalence class of  $a$ . For each  $a \in A$ , by the assumption there are finitely many  
 9 sets of the form  $P_{t,a}$  for  $t \in X$ . Each  $R_t$ -equivalence class is of the form  $P_{t,a}$ . Hence  
 10 each  $R$ -equivalence class belongs to the finite boolean algebra generated by the sets  
 11  $P_{t,a}$ . The claim is thus proved.

12 Now for each  $R$ -equivalence class  $B \subseteq Y$  and for each  $y_1, y_2 \in B$ , we have  $\forall t \in$   
 13  $X, \forall x \in X^{n-1} (t, x, y_1) \in S \iff (t, x, y_2) \in S$ . Thus  $S \cap \pi_Y^{-1}B$  is a box where  
 14  $\pi_Y : X^n \times Y \rightarrow Y$  is the projection. Therefore  $S$  is a finite union of  $(X^n, Y)$ -boxes.  $\square$

15 We now turn to the notion of internality.

16 **Definition 2.10** ([19, Lemma 10.1.4]). Let  $\mathcal{M}$  be a structure. Given two definable  
 17 sets  $X$  and  $V$ , we say that  $X$  is **internal to  $V$** , or  **$V$ -internal**, if there is a definable  
 18 surjection from  $V^n$  to  $X$ , for some  $n$ .

19 **Fact 2.11** ([2, Lemma 2.2]). Let  $X$  and  $Y$  be orthogonal definable sets and let  
 20  $(T_s \mid s \in S)$  be a definable family of subsets of a  $Y$ -internal set  $T$  indexed by an  
 21  $X$ -internal set  $S$ . Then the family contains only finitely many distinct sets  $T_s \subseteq T$ .

22 **Fact 2.12** ([2, Proposition 2.3]). Let  $X$  and  $Y$  be orthogonal definable sets with  
 23  $|X| \geq 2$ , and  $S$  a definable subset of  $X \times Y$ . Then  $S$  is  $X$ -internal if and only if its  
 24 projection onto  $Y$  is finite.

25 It is clear that if  $X$  and  $Y$  are infinite and one of them is internal to the other, then  
 26 the two sets are non-orthogonal. Also, if one of  $X$  and  $Y$  is finite, then the two sets  
 27 are orthogonal.

28 **Proposition 2.13.** Let  $X$  and  $Y$  be orthogonal definable sets. If  $U$  is internal to  $X$   
 29 and  $V$  is internal to  $Y$ , then  $U$  and  $V$  are orthogonal.

30 *Proof.* Let  $S \subseteq U^k \times V^l$  be definable. By the assumption there are definable surjective  
 31 maps  $f : X^m \rightarrow U^k$  and  $g : Y^n \rightarrow V^l$ . Then, by orthogonality,  $(f \times g)^{-1}(S)$  is a finite  
 32 union  $\bigcup_i A_i \times B_i$  of  $(X^m, Y^n)$ -boxes. So  $S = \bigcup_i f(A_i) \times g(B_i)$  is a finite union of  
 33  $(U^k, V^l)$ -boxes.  $\square$

### 34 3. Cohesive sets

35 In this section we work in an arbitrary structure  $\mathcal{M}$ , except for Theorem 3.4 where  
 36 we assume o-minimality. We introduce and study the following two key notions, also  
 37 mentioned in the introduction.

38 **Definition 3.1.** Let  $Z$  be a definable set.

- 39 •  $Z$  is **indecomposable** if for every orthogonal definable sets  $X, Y$ , if  $Z$  is  
 40 internal to  $X \times Y$ , then  $Z$  is either internal to  $X$  or internal to  $Y$ .
- 41 •  $Z$  is **cohesive** if all definable sets  $X$  and  $Y$  not orthogonal to  $Z$  are not  
 42 orthogonal to each other.



1 **Proposition 3.2.**

2 (1) *A definable set internal to a cohesive set is cohesive.*

3 (2) *Any cohesive set is indecomposable.*

4 *Proof.* (1) Let  $X$  be internal to  $Y$ . If  $A$  and  $B$  are non-orthogonal to  $X$ , then by  
5 Proposition 2.13 they are non-orthogonal to  $Y$ . Thus if  $Y$  is cohesive, so is  $X$ .

6 We prove (2). Let  $Z$  be a cohesive set and suppose that  $Z$  is internal to  $X \times Y$  where  
7  $X$  and  $Y$  are orthogonal. By definition there is  $m \in \mathbb{N}$  and a surjective definable map  
8  $f : X^m \times Y^m \rightarrow Z$ . Since  $X^m$  and  $Y^m$  are orthogonal, for the sake of our argument  
9 we can assume  $m = 1$ . So we have a surjective definable map  $f : X \times Y \rightarrow Z$   
10 and we need to show that  $Z$  is  $X$ -internal or  $Y$ -internal. For  $x \in X$  and  $y \in Y$ ,  
11 let  $f_x(y) = f(x, y) = f^y(x)$ . The image of  $f_x$  is  $Y$ -internal and the image of  $f^y$   
12 is  $X$ -internal. These two images are then orthogonal (Proposition 2.13), so they  
13 cannot be both infinite by the hypothesis on  $Z$ . It follows that either  $\text{Im}(f_x)$  is finite  
14 for all  $x \in X$  or  $\text{Im}(f^y)$  is finite for all  $y \in Y$ . By symmetry let us assume that  
15  $\text{Im}(f^y) \subseteq Z$  is finite for all  $y$ . Let  $E_y \subseteq X^2$  be the equivalence relation defined by  
16  $x E_y x' \iff f(x, y) = f(x', y)$ . Then  $E_y$  is a definable equivalence relation on  $X$   
17 of finite index. Since  $(E_y \subseteq X \times X \mid y \in Y)$  is a definable family of subsets of an  
18  $X$ -internal set indexed by a  $Y$ -internal set, by orthogonality there are finitely many sets  
19 of the form  $E_y$  for  $y \in Y$  (Fact 2.11). The intersection  $E = \bigcap_{y \in Y} E_y$  is then again a  
20 definable equivalence relation of finite index on  $X$ . Let  $x_1, \dots, x_k$  be representatives  
21 for the equivalence classes of  $E$ . Then  $Z$  is the image of the restriction of  $f$  to  
22  $\bigcup_{i \leq k} \{x_i\} \times Y$ , and therefore it is  $Y$ -internal.  $\square$

23 **Proposition 3.3.** *If  $X$  and  $Y$  are cohesive and non-orthogonal, then  $X \times Y$  is cohesive.*

24 *Proof.* Let  $A$  and  $B$  be non-orthogonal to  $X \times Y$ . We need to prove that  $A$  and  $B$   
25 are non-orthogonal. By Corollary 2.6 either  $X$  or  $Y$  is non-orthogonal to  $A$ . Similarly,  
26 either  $X$  or  $Y$  is non-orthogonal to  $B$ . There are four cases to consider, but by  
27 symmetry we can consider the following two cases.

28 Case 1.  $X$  is non-orthogonal to both  $A$  and  $B$ .

29 Case 2.  $X$  is non-orthogonal to  $A$ , and  $Y$  is non-orthogonal to  $B$ .

30 In the first case by the cohesiveness of  $X$  the sets  $A$  and  $B$  are non-orthogonal. In  
31 the second case, since  $X$  and  $Y$  are non-orthogonal and  $Y$  is non-orthogonal to  $B$ ,  
32 by the cohesiveness of  $Y$  we conclude that  $X$  is non-orthogonal to  $B$ , so we have a  
33 reduction to the first case.  $\square$

34 We now turn to groups definable in o-minimal structures.

35 **Theorem 3.4.** *Let  $G$  be a definable group of dimension 1 in an o-minimal structure*  
36  *$\mathcal{M}$ . Then  $G$  is cohesive.*

37 *Proof.* Let  $A$  and  $B$  be definable sets non-orthogonal to  $G$ . We need to prove that  
38  $A$  and  $B$  are not orthogonal. Let us first concentrate on  $A$ . By Corollary 2.8 there  
39 are  $n \in \mathbb{N}$  and a definable relation  $R \subseteq A^n \times G$  which is not a finite union of boxes.  
40 Let  $\mathcal{P}(G)$  be the power set of  $G$  and let  $f : A^n \rightarrow \mathcal{P}(G)$  be defined by  $f(a) =$   
41  $\{g \in G \mid (a, g) \in R\}$ . Since  $f(a)$  has dimension  $\leq 1$ , its boundary  $\delta f(a)$ , closure  
42 minus interior, is finite. By the assumption on  $R$ , the image of  $f$  is infinite, thus  
43 also the image of  $\delta f : A^n \rightarrow \mathcal{P}(G)$  is infinite, since in a group of dimension 1 there  
44 can only be finitely many disjoint definable subsets with a given boundary (because,

1 given a cell decomposition compatible with the boundary any definable subset with  
 2 the given boundary is a union of some cells of the decomposition). Now recall that  
 3 in an o-minimal structure a definable family of finite sets is uniformly finite. So  
 4 there is  $k \in \mathbb{N}$  such that  $\delta f(a)$  has at most  $k$  elements for every  $a \in A^n$ . By  
 5 ordering the points of  $\delta f(a)$  lexicographically, we have a map  $h: A^n \rightarrow G^k$  with  
 6 infinite image. It follows that there is  $i \leq k$  such that  $\pi_i \circ h: A^n \rightarrow G$  has infinite  
 7 image, where  $\pi_i: G^k \rightarrow G$  is the projection onto the  $i$ -th component. We have thus  
 8 proved that there is a definable map  $f_A: A^n \rightarrow G$  with infinite image. Similarly, there  
 9 is  $l \in \mathbb{N}$  and a definable map  $f_B: B^l \rightarrow G$  with infinite image. Infinite definable subsets  
 10 of a one-dimensional group have non-empty interior, thus, composing with a group  
 11 translation, we can assume that the two images have an infinite intersection. The  
 12 relation  $xQy : \iff f_A(x) = f_B(y)$  then witnesses the non-orthogonality of  $A$  and  
 13  $B$ .  $\square$

14 **Remark 3.5.** Notice that, by Theorem 3.4, any o-minimal expansion  $\mathcal{M}$  of a group is  
 15 cohesive, hence all sets definable in  $\mathcal{M}$  are cohesive by 3.2. For Theorems 8.4 and 8.7  
 16 it is, therefore, important to work in an arbitrary o-minimal structure.

#### 17 4. Splitting and decomposition

18 In this section we work in an arbitrary structure  $\mathcal{M}$ . Fix definable orthogonal sets  
 19  $X_1, \dots, X_n$ . We introduce the notions of splitting and decomposition for definable  
 20 functions and groups, respectively, contained in products of the sets  $X_i$ , as follows.

21 **Definition 4.1.** We let  $\text{Def}(\prod_i X_i)$  be the collection of all definable sets in  $\mathcal{M}$  that  
 22 are contained in some cartesian product  $\prod_{j=1}^k X_{t(j)}$  with  $t(j) \in \{1, \dots, n\}$ , for all  
 23  $j = 1, \dots, k$ . We say that a function  $f$  is in  $\text{Def}(\prod_i X_i)$  if its graph belongs to  
 24  $\text{Def}(\prod_i X_i)$ .

25 **Notation 4.2.** If  $S \subseteq \prod_{j=1}^k X_{t(j)}$  we define

$$\pi_i : S \rightarrow X_i^{k_i}$$

26 as the projection of  $S$  onto the  $X_i$ -components, where  $k_i$  is the number of indexes  $j$   
 27 with  $t(j) = i$ . For instance, if  $S \subseteq X_1 \times X_2 \times X_1$ , then  $\pi_1 : S \rightarrow X_1^2$  maps  $(a, b, c)$   
 28 to  $(a, c)$ .

29 We now introduce the notion of splitting, which will be a crucial tool for our proofs.

30 **Definition 4.3.** Let  $f : A \rightarrow B$  be a function in  $\text{Def}(\prod_i X_i)$ . We say that  $f$  **splits**  
 (with regard to  $X_1, \dots, X_n$ ) if for every  $x, y \in \text{dom}(f)$  and every  $i \leq n$ ,

$$\pi_i(x) = \pi_i(y) \implies \pi_i(f(x)) = \pi_i(f(y)).$$

31 That is,  $\pi_i(f(x))$  depends only on  $\pi_i(x)$ . Note that if  $f$  splits, then up to a permu-  
 32 tation of the indexes we can write  $f = f_1 \times \dots \times f_n|_A$  where  $f_i : \pi_i(A) \rightarrow \pi_i(B)$ .

33 We will be using the following two facts without specific mentioning.

34 **Fact 4.4.** Let  $f : A \rightarrow B$  be a function in  $\text{Def}(\prod_i X_i)$ . Then there is a finite partition  
 $\mathcal{D}$  of  $\text{dom}(f)$  into definable sets, such that for each  $D \in \mathcal{D}$ , the restriction of  $f$  to  $D$   
 splits.

1 *Proof.* By orthogonality of  $X_1, \dots, X_n$ , up to a permutation of the variables,  $f$  is a  
 2 finite disjoint union  $\bigcup_j f_j$ , where  $f_j = U_{1,j} \times \dots \times U_{n,j}$  and  $U_{i,j} \subseteq \pi_i(A) \times \pi_i(B)$ . We  
 3 conclude observing that each  $f_j$  splits.  $\square$

4 Splitting is preserved under composition in the following sense.

5 **Fact 4.5.** *Let  $h, f_1, \dots, f_m$  be maps in  $\text{Def}(\prod_i X_i)$ , and suppose that the map  $h \circ (f_1 \times$   
 6  $\dots \times f_m)$  is defined (in the sense that the range of  $f_1 \times \dots \times f_m$  is contained in the  
 7 domain of  $h$ ). If  $h, f_1, \dots, f_m$  split, so does  $h \circ (f_1 \times \dots \times f_m)$ .*

8 *Proof.* Straightforward.  $\square$

9 We now turn to definable groups. Recall that  $X_1, \dots, X_n$  are orthogonal.

10 **Definition 4.6.** A definable group  $G$  **admits an orthogonal decomposition** (or  
 11 decomposition for short) with respect to  $X_1, \dots, X_n$ , if there are definable subsets  
 12  $A_1, \dots, A_n$  of  $G$ , such that  $G = A_1 \dots A_n$ , and  $A_i$  is internal to  $X_i$ , for all  $i$ .

13 Note that, since a set internal to a cohesive set is cohesive (Proposition 3.2(1)), a  
 14 definable group admits a (cohesive) orthogonal decomposition as in the introduction  
 15 if and only if there are definable (cohesive) orthogonal sets  $X_1, \dots, X_n$ , such that  $G$   
 16 admits a decomposition with respect to  $X_1, \dots, X_n$ .

17 **Observation 4.7.** Let  $H$  be a finite index subgroup of  $G$ . If  $H$  admits an orthogonal  
 18 decomposition with respect to  $X_1, \dots, X_n$ , then so does  $G$ .

19 Our goal in the next sections is to prove that if  $G$  is definable in an o-minimal  
 20 structure and internal to  $X_1 \times \dots \times X_n$ , then it admits a decomposition in the sense  
 21 above. A relevant case is when  $G$  is contained in – as opposed to being internal to –  
 22  $X_1 \times \dots \times X_n$ . In this situation, by Remark 4.8 below, if the group operation of  $G$  splits,  
 23 then  $G$  admits a decomposition. The converse, however, is not true (Example 4.9).  
 24 On the other hand, if  $G$  is a direct product of groups  $G_1, \dots, G_n$  with  $G_i \subseteq X_i$  for all  
 25  $i$ , then the group operation obviously splits. In the appendix, we will give an example  
 26 in which the group operation splits, yet the group is not even definably isomorphic to  
 27 a direct product.

28 The rest of this section contains some remarks that help to demonstrate the newly  
 29 defined notions.

30 **Remark 4.8.** Let  $(G, \mu, e)$  be a definable group with  $G \subseteq X_1 \times \dots \times X_n$ . Then the  
 31 following are equivalent:

- 32 (1) the group operation  $\mu$  splits (with respect to  $X_1, \dots, X_n$ );  
 33 (2) there are definable groups  $H_1, \dots, H_n$ , contained in  $X_1, \dots, X_n$ , respectively,  
 34 such that  $G < H_1 \times \dots \times H_n$ ;  
 35 (3) there are definable groups  $H_1, \dots, H_n$ , contained in  $X_1, \dots, X_n$ , respectively,  
 36 such that  $G$  is a finite index subgroup of  $H_1 \times \dots \times H_n$ ;

37 and, if either of these holds, then  $G$  admits a decomposition.

38 *Proof.* Assume (1). Consider the projections  $H_1, \dots, H_n$  of  $G$  on  $X_1, \dots, X_n$ , respec-  
 39 tively. The group operation  $\mu$ , by the splitting hypothesis, takes the form

$$\mu((x_1, \dots, x_n), (y_1, \dots, y_n)) = (\mu_1(x_1, y_1), \dots, \mu_n(x_n, y_n))$$

1 It is straightforward to check that  $\mu_i$  is a group operation on  $H_i$ , and (2) is thus  
 2 established.

Now assume (2). Let  $\pi_i : \prod_j X_j \rightarrow X_i$  be the projection onto the  $i$ -th component. Replacing  $H_i$  with  $\pi_i(G)$  we can assume, without loss of generality, that  $\pi_i(G) = H_i$  for each  $i$ . We prove that  $G$  has finite index in  $H_1 \times \dots \times H_n$ . Let

$$p_i : \prod_j X_j \rightarrow \prod_{j \neq i} X_j$$

3 be the projection that omits the  $i$ -th coordinate. Fix an index  $i$  and consider an  
 4 element  $k$  of  $\prod_{j \neq i} H_j$ . Let  $H_i(k) = \pi_i(G \cap p_i^{-1}(k))$ . Observe that  $L_i := H_i(p_i(e))$  is  
 5 a subgroup of  $H_i$ . We claim that  $L := L_1 \times \dots \times L_n$  has finite index in  $H_1 \times \dots \times H_n$ .  
 6 Assuming the claim, since  $L < G$ , also  $G$  must have finite index. It suffices to prove  
 7 that  $L_i$  has finite index in  $H_i$ . The coset  $hL_i$ , for  $h \in H_i$ , coincides with  $H_i(p_i(g))$  for  
 8 any  $g \in G \cap p_i^{-1}(h)$ . Thus, in particular, the cosets of  $L_i$  belong to the family  $H_i(-)$   
 9 indexed over  $\prod_{j \neq i} X_j$ . This family is finite by Fact 2.11, and this concludes the proof  
 10 of (3).

11 It is immediate that (2), hence also (3), implies (1). It remains to show that  
 12  $G$  admits a decomposition. To this aim, observe that  $L$  admits a decomposition:  
 13 it is, in fact, the product of the subgroups  $G \cap p_i^{-1}(p_i(e)) \cong L_i$ . We conclude by  
 14 Observation 4.7.  $\square$

15 **Example 4.9.** We give an example of a group whose operation does not split, even up  
 16 to definable isomorphism, but the group admits a decomposition. Let  $\mathcal{M} = \mathcal{R}_1 \sqcup \mathcal{R}_2$   
 17 where  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are isomorphic copies of  $(\mathbb{R}, <, +)$  and note that their domains  
 18  $R_1$  and  $R_2$  are orthogonal in  $\mathcal{M}$ . Let  $(R_1 \times R_2, +, 0)$  be the product group. Fix  
 19  $a \in [0, 1)$ . Consider the lattice  $\Lambda = \mathbb{Z} \cdot (0, 1) + \mathbb{Z} \cdot (1, a) \subseteq R_1 \times R_2$  and the definable  
 20 set  $[0, 1)^2 \subseteq R_1 \times R_2$ . Define  $G = ([0, 1)^2, \mu, 0)$ , where  $\mu = + \bmod \Lambda$ . Clearly,  
 21  $G = A_1 + A_2$ , where  $A_1 = [0, 1) \times \{0\}$  and  $A_2 = \{0\} \times [0, 1)$ , and hence it admits  
 22 a decomposition with respect to  $R_1, R_2$ . It is easy to see that if  $a \neq 0$ , then  $\mu$  does  
 23 not split. Moreover,  $a \in \mathbb{Q}$ , if and only if  $G$  is definably isomorphic to a group whose  
 24 operation splits, if and only if  $G$  is definably isomorphic to a direct product of groups  
 25 definable in  $R_1$  and  $R_2$ , respectively.

## 26 5. Compact domination

27 Let again  $\mathcal{M}$  be an arbitrary structure and  $G$  a definable group. We call a de-  
 28 finable set  $S \subseteq G$  *left-generic* (*right-generic*) if finitely many left-translates (respec-  
 29 tively right-translates) of  $S$  cover  $G$ . We call  $S$  *generic* if it both left-generic and  
 30 right-generic (which are equivalent when  $G$  has *finitely satisfiable generics* (*fsg*) [10,  
 31 Proposition 4.2]). We are mainly interested in o-minimal structures, but in this sec-  
 32 tion we consider the larger NIP class. We recall that NIP structures include both  
 33 the o-minimal structures (e.g. the real field) and the stable structures (e.g. the  
 34 complex field). If  $\mathcal{M}$  is a NIP structure and  $G$  is *compactly dominated* (see [17]  
 35 for the definitions), then  $G$  has *fsg* [17, Theorem 8.37], [18, Proposition 3.23], [17,  
 36 Proposition 8.33]. Compact domination is a model-theoretic form of compactness:  
 37 for instance a semialgebraic linear group  $G < GL(n, \mathbb{R})$  is compactly dominated if and  
 38 only if it is compact. The following proposition subsumes all we need about these  
 39 notions.

1 **Proposition 5.1.** *Let  $\mathcal{M}$  be a NIP structure and  $G$  a compactly dominated group*  
 2 *definable in  $\mathcal{M}$ . Then:*

- 3 (1) *If the union of two definable subsets of  $G$  is generic, then one of the two is*  
 4 *generic.*  
 5 (2) *Suppose  $G = HK$ , where  $H$  and  $K$  are definable subgroups and  $K$  is normal.*  
 6 *Then  $S \subseteq G$  is generic if and only if it contains a set of the form  $AB$  where*  
 7  *$A$  is a generic subset of  $H$  and  $B$  is a generic subset of  $K$ .*  
 8 (3) *If  $G$  and  $H$  are compactly dominated, then  $G \times H$  is compactly dominated.*

9 *Proof.* Point (1) holds for all groups with *fsg* [10, Proposition 4.2].

10 To prove point (2) we may assume  $\mathcal{M}$  is  $\kappa$ -saturated for some sufficiently big  
 11 cardinal  $\kappa$ . We make use of the *infinitesimal subgroup*  $G^{00}$  (see [17] for the definition).  
 12 Specifically, we need to observe that in a compactly dominated group  $G$ , a subset  $S$  is  
 13 generic if and only if some translate of  $S$  contains  $G^{00}$  [1, Proposition 2.1]. If  $G = HK$   
 14 with  $H < G$  and  $K \triangleleft G$ , then  $G^{00} = (HK)^{00} = H^{00}K^{00}$  [4, Theorem 4.2.5] (this holds  
 15 without the hypothesis that  $G$  is compactly dominated). Now suppose  $S \subseteq G$  is  
 16 generic. So there is  $g \in G$  such that  $gG^{00} \subseteq S$ . Now,  $K^{00}$  is a normal subgroup of  $G$   
 17 (being a definably characteristic subgroup of  $K$ ), and so, writing  $g = kh$  for  $h \in H$  and  
 18  $k \in K$ , we get  $gG^{00} = khK^{00}H^{00} = (kK^{00})(hH^{00}) \subseteq S$ . By  $\kappa$ -saturation there are  
 19 definable sets  $U, V$  with  $K^{00} \subseteq U \subseteq K$  and  $H^{00} \subseteq V \subseteq H$  such that  $(kU)(hV) \subseteq S$ .  
 20 By construction,  $kU$  and  $hV$  are generic in  $K$  and  $H$ , respectively.

21 Point (3) follows from [18, Corollary 3.17] (the product of smooth measures is  
 22 smooth) and the fact that a group is compactly dominated if and only if it has a  
 23 smooth left-invariant measure [17, Theorem 8.37].  $\square$

24 **Fact 5.2.** *Definably compact groups in an o-minimal structure are compactly domi-*  
 25 *nated.*

26 *Proof.* This was first proved for o-minimal expansions of a field in [9]. We give some  
 27 bibliographical pointers to obtain the result for arbitrary o-minimal structures. First  
 28 one shows that a definably compact group in an o-minimal structures has *fsg* [12,  
 29 Theorem 8.6]. From this one deduces that  $G$  admits a *generically stable left-invariant*  
 30 *measure* [17, Proposition 8.32]. In an o-minimal structure (and more generally in  
 31 a *distal* structure), a generically stable measure is *smooth* [17, Proposition 9.26].  
 32 Finally, a NIP group with a smooth left-invariant measure is compactly dominated  
 33 [17, Theorem 8.37].  $\square$

34 **Proposition 5.3.** *Let  $G$  be a compactly dominated group. Given two generic sets*  
 35  *$A \subseteq G$  and  $B \subseteq G$ , there is  $h \in G$  such that  $A \cap hB$  is generic.*

36 *Proof.* There is a finite subset  $I \subseteq G$  such that  $A \subseteq \bigcup_{h \in I} hB$ . By Proposition 5.1(1),  
 37 there is  $h \in I$  such that  $A \cap hB$  is generic.  $\square$

### 38 6. Decomposition of compactly dominated abelian groups (NIP)

39 In this section  $\mathcal{M}$  is a NIP structure and  $X_1, \dots, X_n$  are orthogonal definable sets.

40 **Theorem 6.1.** *Let  $G$  be a compactly dominated abelian group contained in  $X_1 \times \dots \times$*   
 41  *$X_n$ . Then  $G$  admits a decomposition with respect to  $X_1, \dots, X_n$ .*

*Proof.* Let  $P_n : G^n \rightarrow G$  be the function sending  $(x_1, \dots, x_n)$  to  $\prod_{i=1}^n x_i$ . Then  $P_n \in \text{Def}(\prod_i X_i)$ . Since  $G$  is compactly dominated, so is  $G^n$  (Proposition 5.1(3)). By Fact 4.4 and Proposition 5.1(1),  $P_n$  splits on a generic definable set  $S \subseteq G^n$ . By Proposition 5.1(2) we find definable generic sets  $A_1, \dots, A_n \subseteq G$  such that  $A_1 \times \dots \times A_n \subseteq S$ . By Proposition 5.3 we find  $a_1, \dots, a_n \in G$  such that the set

$$U = \bigcap_{i=1}^n a_i A_i$$

is generic in  $G$ . Again by Fact 4.4, there is a generic set  $D \subseteq U$  such that for every  $i = 1, \dots, n$  the function

$$f_i : x \in D \mapsto a_i^{-1}x$$

splits on  $D$ . Since  $D \subseteq a_i A_i$ , the image of  $f_i$  is contained in  $A_i$ . It follows that the function

$$f_1 \times \dots \times f_n : D^n \rightarrow A_1 \times \dots \times A_n$$

splits. By Fact 4.5, since  $P_n$  splits on  $A_1 \times \dots \times A_n$ , we deduce that the function

$$f = P_n \circ (f_1 \times \dots \times f_n)$$

1 splits on  $D^n$ . Since  $G$  is abelian,

$$f(x_1, \dots, x_n) = a \prod_{i=1}^n x_i \tag{6.1}$$

where  $a = \prod_{i=1}^n a_i^{-1}$ . By the orthogonality assumption,  $D$  is a finite union of sets of the form  $U_1 \times \dots \times U_n$  with  $U_i \subseteq X_i$ . By Proposition 5.1(1) one of these sets is generic, so by replacing  $D$  with a smaller set we can assume that

$$D = U_1 \times \dots \times U_n.$$

Let  $\pi_i : \prod_j X_j \rightarrow X_i$  be the projection onto the  $i$ -th component and let

$$p_i : \prod_j X_j \rightarrow \prod_{j \neq i} X_j$$

be the projection that omits the  $i$ -th coordinate. Now fix  $k \in D$  and let

$$D_i = p_i^{-1} p_i(k) \cap D = \{x \in D \mid \forall j \neq i \pi_j(x) = \pi_j(k)\}.$$

2 Notice that  $D_i$  is  $U_i$ -internal, hence *a fortiori* it is  $X_i$ -internal. We claim that

$$k^{n-1}D \subseteq D_1 \dots D_n \tag{6.2}$$

To prove the claim, let  $g \in D$  and let  $x_i \in G$  be such that

$$\pi_i(x_i) = \pi_i(g) \text{ \& } \pi_j(x_i) = \pi_j(k) \text{ for } j \neq i.$$

Notice that  $x_i \in D_i$ . Since  $f$  splits on  $D^n$ , the value of  $\pi_i f(x_1, \dots, x_n)$  does not change if we replace  $x_i$  with  $g$  and  $x_j$  with  $k$  for  $j \neq i$ . By Equation (6.1) we then obtain

$$\pi_i f(x_1, \dots, x_n) = \pi_i(ak^{n-1}g).$$

Since this holds for every  $i$ , we deduce that

$$f(x_1, \dots, x_n) = ak^{n-1}g$$

3 and since  $f(x_1, \dots, x_n) = a \prod_{i=1}^n x_i$  we obtain Equation 6.2.

1 We have thus shown that a translate of  $D$  is contained in a product of  $X_i$ -internal  
 2 sets. Since  $D$  is generic, the same holds for  $G$ , so  $G$  is a product of  $X_i$ -internal  
 3 sets.  $\square$

#### 4 7. Decomposition of definably compact abelian groups (o-minimal)

5 In this section  $\mathcal{M}$  is an o-minimal structure and  $X_1, \dots, X_n$  are orthogonal definable  
 6 sets. We prove the following variant of Theorem 6.1, where the NIP hypothesis is  
 7 replaced by o-minimality, but the group is only assumed to be internal to  $X_1 \times \dots \times X_n$ .

8 **Theorem 7.1.** *Let  $G$  be a definably compact abelian group internal to  $X_1 \times \dots \times X_n$ .  
 9 Then  $G$  admits a decomposition with respect to  $X_1, \dots, X_n$ .*

10 We need the following lemma.

11 **Lemma 7.2.** *Let  $X$  be an infinite set definable in an o-minimal structure  $\mathcal{M}$ . Then  
 12 there is a definable set  $Y = [X]^{o\text{-min}}$  such that:*

- 13 (1)  $X$  and  $Y$  are internal to each other;
- 14 (2) there is  $k \in \mathbb{N}$  and a definable injective map from  $X$  to  $Y^k$ ;
- 15 (3)  $Y$  has a definable linear order  $\prec$  such that  $(Y, \prec)$  with the induced structure  
 16 from  $\mathcal{M}$  is o-minimal.<sup>1</sup> We call the resulting structure  $\mathcal{Y}$  an **o-minimal**  
 17 **envelope** of  $X$ .

18 Moreover we have:

- 19 (4) if  $X_1, \dots, X_n$  are definable infinite sets, there there are definable sets  $X'_1, \dots, X'_n$   
 20 such that  $X'_i$  is bi-internal to  $X_i$  for each  $i$  and  $[X_1 \times \dots \times X_n]^{o\text{-min}}$  can be  
 21 definably embedded in  $X'_1 \times \dots \times X'_n$ .

22 *Proof.* Suppose  $X \subseteq M^m$  and let  $\pi_i : M^m \rightarrow M$  be the projection onto the  $i$ -th  
 23 coordinate ( $i = 1, \dots, m$ ). Fix parameters  $a_1 < \dots < a_m$  in  $M$ , let  $Z_i = \{a_i\} \times \pi_i(X)$   
 24 and let  $Z = \bigcup_i Z_i \subseteq M \times M$ . Then  $Z$  has dimension 1 and is bi-internal to  $X$ .  
 25 Each  $\pi_i(X) \subseteq M$  is a finite union of points and intervals with the induced order from  
 26  $\mathcal{M}$ , and we have an induced order on  $Z_i$  via the obvious bijection. We then order  $Z$   
 27 lexicographically by stipulating that all the elements of  $Z_i$  precede all the elements of  
 28  $Z_{i+1}$ . We thus obtain a definable linear order  $<_Z$  on  $Z$ , but notice that  $Z$  need not  
 29 be dense, and even if it is,  $Z_i$  need not be a finite union of points and intervals in the  
 30 order  $(Z, <_Z)$  (e.g.  $Z_i$  may be bounded but with no supremum in  $Z$ ). We can remedy  
 31 this by adding and removing from  $Z$  finitely many points, thus obtaining a definable  
 32 set  $Y$  with a dense linear order  $\prec$  without end-points which agrees with  $<_Z$  on  $Y \cap Z$   
 33 (for example if  $\mathcal{M} = \mathbb{R}$  and  $Z = [1, 2) \cup (3, 4] \cup [5, 6) \subset M$ , we can add the point  
 34 2 and remove 1 and 4). The set  $Y$  can be constructed as a disjoint union  $\bigcup_{i=1}^m Y_i$   
 35 where, for each  $i$ , the symmetric difference  $Y_i \Delta Z_i$  is finite and  $Y_i$  is a finite union of  
 36 intervals and points of  $(Y, \prec)$ , so in particular it is definable in  $(Y, \prec)$ .

37 To prove (3) it suffices to observe that any definable subset  $D$  of  $Y$  is the union  
 38 of its traces  $D \cap Y_i$  on the various  $Y_i$ , so it is a finite union of points and intervals of  
 39  $(Y, \prec)$ .

40 To prove the remaining points we make some preliminary observations (where all  
 41 the relevant sets are assumed to be definable and  $n$  is arbitrary):

<sup>1</sup>The induced structure contains a predicate for each  $\mathcal{M}$ -definable subset of  $Y^n$  for  $n \in \mathbb{N}$ .

- 1        i)  $A_1 \times \dots \times A_n$  can be definably embedded in  $(A_1 \cup \dots \cup A_n)^n$ .  
 2        ii) If  $A$  is infinite and  $F$  is finite,  $A \cup F$  can be embedded in  $A \times A$ .  
 3        iii)  $A_1 \cup \dots \cup A_n$  can be definably embedded in  $A_1 \times \dots \times A_n \times B$  where  $B$  is  
 4            any infinite set. If, for some  $i$ , the set  $A_i$  is infinite, we can take  $B = A_i$  and  
 5            embed  $A_1 \cup \dots \cup A_n$  in  $A_1 \times \dots \times A_n \times A_i$ , hence *a fortiori* in  $A_1^2 \times \dots \times A_n^2$ .

6 By i) and iii)  $A_1 \times \dots \times A_n$  is bi-internal to  $A_1 \cup \dots \cup A_n$  provided at least one of the  
 7 sets  $A_i$  is infinite.

8        The set  $X$  is included in the cartesian product of its projections  $\pi_i(X)$ , so it can be  
 9 embedded in the cartesian product of the sets  $Z_i = \{a_i\} \times \pi_i(X)$ . On the other hand,  
 10 by i),  $Z_1 \times \dots \times Z_n$  can be embedded in  $(Z_1 \cup \dots \cup Z_n)^m$ . Moreover,  $Z_1 \cup \dots \cup Z_n$   
 11 differs from the o-minimal envelope  $Y$  by a finite set, so by ii) it can be embedded  
 12 into  $Y^2$  (note that  $Y$  must be infinite as  $X$  is assumed to be infinite). It follows that  
 13  $X$  can be embedded into  $Y^{2m}$ , thus proving (2).

14        By a similar argument, using ii),  $Y$  can be embedded into  $(Z_1 \cup \dots \cup Z_n)^2$ , which in  
 15 turn can be embedded, by iii), into the cartesian product  $\prod_i \pi_i(X)^4$  of the projections  
 16 of  $X$  to the 4-th power. In particular  $X$  and  $Y$  are bi-internal and we get (1).

17        To prove point (4), let  $X = X_1 \times \dots \times X_n$  where each  $X_i \subseteq M^{n_i}$  is infinite. Let  
 18  $X'_i$  be the product of the projections of  $X_i$  to the 4-th power. Then the o-minimal  
 19 envelope of  $X$  can be embedded in  $X'_1 \times \dots \times X'_n$  and we get (4).  $\square$

20 **Definition 7.3** ([8, Def. 1.1]). Let  $X, Y$  be definable sets,  $E_1, E_2$  two definable  
 21 equivalence relations on  $X$  and  $Y$  respectively. A function  $f : X/E_1 \rightarrow Y/E_2$  is called  
 22 definable if the set  $\{(x, y) \in X \times Y \mid f([x]_{E_1}) = [y]_{E_2}\}$  is definable.

23 **Proposition 7.4.** *Let  $G$  be a definable group in an o-minimal structure  $\mathcal{M}$  and let*  
 24  *$X_1, \dots, X_n$  be definable sets in  $\mathcal{M}$ . If  $G$  is internal to  $X_1 \times \dots \times X_n$ , then there is*  
 25 *an injective definable map  $f : G \rightarrow X'_1 \times \dots \times X'_n$ , where each  $X'_i$  is bi-internal to  $X_i$*   
 26 *( $i = 1, \dots, n$ ).*

27 *Proof.* We can assume that  $G$  is infinite as otherwise the result is clear. We can then  
 28 also assume that each  $X_i$  is infinite, because if  $X_n$  is finite, say, then  $G$  is internal to  
 29  $X_1 \times \dots \times X_{n-1}$ .

30        By Lemma 7.2,  $G$  is internal to  $Y = [X_1 \times \dots \times X_n]^{\text{o-min}}$ , so it can be considered  
 31 as an interpretable group in the o-minimal structure  $\mathcal{Y}$ . By [8] interpretable groups in  
 32 an o-minimal structure are definably isomorphic to definable groups (in the sense of  
 33 Definition 7.3). It follows that there is a definable injective map  $f : G \rightarrow Y^k$  for some  
 34  $k \in \mathbb{N}$ . By the construction of the o-minimal envelope,  $Y$  can be embedded into a  
 35 product of sets  $Y_i$ , where  $Y_i$  is bi-internal to  $X_i$  (Lemma 7.2(4)). Now it suffices to  
 36 take  $X'_i = Y_i^k$ .  $\square$

37        We are now ready to finish the proof of the theorem.

38 *Proof of Theorem 7.1.* By Proposition 7.4 there are definable sets  $X'_1, \dots, X'_n$  with  
 39  $X'_i$  bi-internal to  $X_i$  such that  $G$  is definably isomorphic to a group  $G'$  contained in  
 40  $X'_1 \times \dots \times X'_n$ . Since  $G$  is definably compact,  $G'$  also is. By Theorem 6.1  $G'$  admits  
 41 a decomposition with respect to  $X'_1, \dots, X'_n$ , hence also with respect to  $X_1, \dots, X_n$ .  
 42 Hence so does  $G$ .  $\square$



## 1 8. Decomposition: general case (o-minimal)

2 In this section, we prove our main theorems (8.4 and 8.7). Fix an o-minimal struc-  
 3 ture  $\mathcal{M}$  and definable orthogonal sets  $X_1, \dots, X_n$ . We first prove that the existence  
 4 of decompositions is preserved under taking central extensions.

5 **Lemma 8.1.** *Let  $1 \rightarrow N \rightarrow G \xrightarrow{f} H \rightarrow 1$  be a definable exact sequence of definable  
 6 groups internal to  $X_1 \times \dots \times X_n$  with  $N < Z(G)$ . If  $N$  and  $H$  admit decompositions  
 7 with respect to  $X_1, \dots, X_n$ , then  $G$  too admits a decomposition with respect to the  
 8 same orthogonal sets.*

*Proof.* By assumption, we can write  $H = H_1 \dots H_n$  and  $N = N_1 \dots N_n$ , where  $H_i$  and  
 $N_i$  are  $X_i$ -internal definable sets (not necessarily subgroups). We have

$$G = f^{-1}(H_1) \dots f^{-1}(H_n).$$

By [7, Theorem 2.5], there is a definable section  $\sigma : H \rightarrow G$ . Since  $f^{-1}(H_i) =$   
 $\sigma(H_i)N$ , we have

$$G = \sigma(H_1)N \dots \sigma(H_n)N.$$

9 Since  $N = N_1 \dots N_n$  is contained in the center of  $G$ , it follows that  $G = U_1 \dots U_n$ ,  
 10 where  $U_i$  is the  $X_i$ -internal set  $\sigma(H_i)N_i \dots N_i$  ( $n$  occurrences of  $N_i$ ).  $\square$

11 We can now handle the abelian case.

12 **Proposition 8.2.** *Let  $G$  be a definable group internal to  $X_1 \times \dots \times X_n$ . If  $G$  is abelian,  
 13 then  $G$  admits a decomposition with respect to  $X_1, \dots, X_n$ .*

14 *Proof.* We reason by induction on dimension. For  $\dim(G) = 1$ ,  $G$  is cohesive by  
 15 Theorem 3.4, so it is indecomposable (Proposition 3.2), hence it is internal to one of  
 16 the  $X_j$ . Let  $\dim(G) > 1$ . If  $G$  is definably compact, then we conclude by 7.1. If not,  
 17 then by [13, Theorem 1.2],  $G$  has a 1-dimensional torsion-free definable subgroup  $H <$   
 18  $G$ . Since  $\dim(H) = 1$ ,  $H$  admits a decomposition. By induction on dimension, so  
 19 does  $G/H$ . Therefore, since  $G$  is abelian, we can apply Lemma 8.1.  $\square$

20 The following fact must be well-known, but we include a proof for completeness.

21 **Fact 8.3.** *Let  $G$  be a connected group definable in an o-minimal structure. If  $Z(G)$   
 22 is finite, then  $G/Z(G)$  is centerless.*

23 *Proof.* Let  $a \in G$ , such that  $aZ(G)$  is in the center of  $G/Z(G)$ . We want to prove that  
 24  $a \in Z(G)$ . We have that for all  $b \in G$ ,  $a^{-1}b^{-1}ab \in Z(G)$ . Since  $G$  is connected, the  
 25 image of the map  $f : G \rightarrow Z(G)$  sending  $b \in G$  to  $a^{-1}b^{-1}ab \in Z(G)$  is connected.  
 26 Since  $Z(G)$  is finite,  $f$  must be constant. Since  $f$  maps the identity  $e \in G$  to  $e$ , we  
 27 have  $a^{-1}b^{-1}ab = e$ . Thus  $a \in Z(G)$ , as needed.  $\square$

28 We can now prove our first main result.

29 **Theorem 8.4.** *Let  $G$  be internal to  $X_1 \times \dots \times X_n$  where  $X_1, \dots, X_n$  are orthogonal  
 30 definable sets. Then  $G$  admits a decomposition with respect to  $X_1, \dots, X_n$ .*

31 *Proof.* We observe that, since the connected component of the identity  $G^0$  has finite  
 32 index in  $G$ , by Observation 4.7, it suffices to find a decomposition of  $G^0$ . We may  
 33 thus assume that  $G$  is connected.

1 We prove, now, the theorem, by induction on  $\dim(G)$ . For  $\dim(G) = 0$ , it is  
 2 obvious. Assume  $\dim(G) > 0$ .

3 Case 1. Suppose that the centre  $Z(G)$  is finite. Then  $Z(G)$  admits an orthogonal  
 4 decomposition, and by Lemma 8.1 it suffices to prove that  $H = G/Z(G)$  has a  
 5 decomposition. Since  $Z(G)$  is finite,  $H$  is centerless (Fact 8.3).

6 By [8],  $H$  is definably isomorphic to a definable group. By [14, Theorems 3.1 and  
 7 3.2], it follows that there are definable real closed fields  $R_1, \dots, R_k$  and definable  
 8 linear groups  $H_i < GL(n, R_i)$  such that  $H$  is definably isomorphic to  $H_1 \times \dots \times H_k$ .  
 9 A definable real closed field in an o-minimal structure has dimension 1 [13, Theorem  
 10 4.1]. We can arrange so that  $R_i$  is internal to  $X_1 \times \dots \times X_n$  because we can apply the  
 11 cited results inside the o-minimal structure  $[X_1 \times \dots \times X_n]^{\text{o-min}}$  (as defined in Lemma  
 12 7.2). By Theorem 3.4  $R_i$  is cohesive, so it is internal to some  $X_j$  by Proposition  
 13 3.2(2). It follows that each  $H_i$  is internal to some  $X_j$ . We have thus proved that  $H$   
 14 admits a decomposition with respect to the orthogonal sets  $X_1, \dots, X_n$ . Therefore  
 15 we can conclude by Lemma 8.1.

16 Case 2. Suppose  $Z(G)$  is infinite. By the abelian case (Proposition 8.2),  $Z(G)$  has  
 17 a decomposition with respect to  $X_1, \dots, X_n$ . By induction on the dimension,  $G/Z(G)$   
 18 has a decomposition too. Therefore we can again conclude by Lemma 8.1.  $\square$

19 As a by-product of the proof we obtain:

20 **Proposition 8.5.** *If  $G$  is definably simple, then  $G$  is cohesive.*

21 *Proof.* Let  $G$  be definably simple. By the proof of Theorem 8.4 there is a definable real  
 22 closed field  $R$  such that  $G$  is definably isomorphic to a definable subgroup of  $GL(n, R)$ .  
 23 Since  $\dim(R) = 1$ , by Theorem 3.4  $R$  is cohesive. But  $GL(n, R)$  is internal to  $R$ ,  
 24 thus all its definable subsets are cohesive.  $\square$

25 We now proceed towards our second main result.

26 **Lemma 8.6.** *Let  $G$  be an interpretable group. Then there are cohesive orthogonal*  
 27 *definable sets  $X_1, \dots, X_n$  and an injective definable map  $h : G \rightarrow \prod_{i=1}^n X_i$ .*

28 *Proof.* By [8, Theorem 3], there is a definable injective map  $f : G \rightarrow \prod_{j=1}^k G_j$  where  
 29 each  $G_j$  is a 1-dimensional definable group. Define a relation  $R$  on  $\{1, \dots, k\}$  by  $iRj$   
 30 if  $G_i$  and  $G_j$  are not orthogonal. By Theorem 3.4, the groups  $G_i$  are cohesive, hence  
 31  $R$  is an equivalence relation. Suppose there are  $n$  equivalence classes. Let  $X_i$  be the  
 32 product of the groups  $G_j$ , with  $j$  in the  $i$ -th class. Using  $f$ , it is easy to define (by a  
 33 permutation of the coordinates on the image) the injective map  $h : G \rightarrow \prod_{i=1}^n X_i$ .

34 The sets  $X_i$  are cohesive by Proposition 3.3 and mutually orthogonal by Corol-  
 35 lary 2.6.  $\square$

36 Recall the notion of cohesive orthogonal decomposition from the introduction.

37 **Theorem 8.7.** *If  $G$  is a group interpretable in an o-minimal structure  $\mathcal{M}$ , then  $G$*   
 38 *admits a cohesive orthogonal decomposition  $G = A_1 \dots A_n$ .*

*Proof.* Let  $X_1, \dots, X_n$  be the sets provided by Lemma 8.6. By Theorem 8.4, there  
 are  $X_i$ -internal sets  $A_i \subseteq G$ ,  $i = 1, \dots, n$ , such that

$$G = A_1 \dots A_n.$$

1 Clearly, the  $A_i$ 's are orthogonal and cohesive, as they inherit those properties from  
2 the  $X_i$ 's.  $\square$

3 **Corollary 8.8.** *If  $G$  is infinite, in Theorem 8.7 we can choose each  $A_i$  to be infinite.  
4 In this case, the number  $n$  is an invariant of  $G$  up to definable isomorphism. Indeed,  
5 if  $G = B_1 \dots B_m$  is another cohesive orthogonal decomposition of  $G$  with  $B_1, \dots, B_m$   
6 infinite, then  $m = n$  and each  $B_i$  is bi-internal to a unique  $A_j$ .*

7 *Proof.* Suppose that  $G$  is infinite and fix a cohesive orthogonal decomposition  $G =$   
8  $A_1 \dots A_n$ . Then at least one  $A_i$  is infinite, say  $A_1$ . If some  $A_i$  is finite we may replace  
9  $A_1$  with  $A_1 A_i$  and omit  $A_i$  obtaining another cohesive orthogonal decomposition.  
10 So we may assume that  $A_1, \dots, A_n$  are all infinite. Now consider another cohesive  
11 orthogonal decomposition  $G = B_1 \dots B_m$  into infinite sets. Fix  $B_i$  and observe that  
12  $B_i$  is internal to  $G$ , which is internal to the cartesian product  $A_1 \times \dots \times A_n$ . Since  
13  $B_i$  is indecomposable, it must be internal to some  $A_j$ . Moreover,  $j$  must be unique,  
14 because if  $B_i$  is internal to both  $A_j$  and  $A_h$ , with  $j \neq h$ , then it is non-orthogonal to  
15 both, so by cohesiveness of  $B_i$ ,  $A_j$  and  $A_h$  are non-orthogonal, a contradiction. The  
16 argument also shows that  $m = n$  and  $B_i$  is in fact bi-internal to the corresponding  
17  $A_j$ .  $\square$

## 18 9. Locally definable groups

19 In this section, we fix again an o-minimal structure  $\mathcal{M}$ , and prove Theorem 9.2.  
20 Let us first recall a few definitions concerning locally definable sets and groups (which  
21 can, in fact, be given for arbitrary structures).

22 **Definition 9.1.** A **locally definable set**  $X$  is a countable union of definable sets  
23 together with a given presentation as such a countable union. A subset of a locally  
24 definable set  $X$  is said to be **definable** if it is definable in  $\mathcal{M}$  and is contained in the  
25 union of finitely many sets of the presentation of  $X$  (this last condition is automatically  
26 satisfied if  $\mathcal{M}$  is  $\aleph_0$ -saturated). A **compatible** subset of a locally definable set  $X$  is a  
27 subset which intersects every definable subset of  $X$  at a definable set. A compatible  
28 subset is **discrete** if it intersects every definable set into a finite set. A **locally**  
29 **definable function** is a function between locally definable sets whose restriction to  
30 each definable set is definable. Similar definitions apply to groups. A locally definable  
31 group is a locally definable set with a locally definable group operation. We can then  
32 speak of compatible and discrete subgroups and locally definable homomorphisms. A  
33 locally definable group is **definably generated** if it is generated by a definable subset.

**Theorem 9.2.** *Let  $G$  be an abelian group definable in the disjoint union  $\bigsqcup_i \mathcal{X}_i$  of finitely  
many o-minimal structures  $\mathcal{X}_1, \dots, \mathcal{X}_n$ . Then there is a locally definable isomorphism*

$$G \cong G_1 \times \dots \times G_n / \Gamma,$$

34 *where  $G_i$  is a locally definable and definably generated group in  $\mathcal{X}_i$ , and  $\Gamma$  is a com-*  
35 *patible locally definable discrete subgroup of  $G_1 \times \dots \times G_n$ .*

36 *Proof.* The structure  $\bigsqcup_i \mathcal{X}_i$  is bi-interpretable with the o-minimal structure  $\mathcal{M}$  ob-  
37 tained by concatenating  $\mathcal{X}_1, \dots, \mathcal{X}_n$  in the given order and adding  $n - 1$  points to  
38 separate  $\mathcal{X}_i$  from  $\mathcal{X}_{i+1}$  for  $i < n$ . We can therefore apply to  $\mathcal{M} = \bigsqcup_i \mathcal{X}_i$  the various

1 results concerning o-minimal structures. By Theorem 8.4, the group  $G$  admits a de-  
 2 composition  $G = A_1 \dots A_n$  with respect to  $X_1, \dots, X_n$ , where  $X_i$  is the domain of  $\mathcal{X}_i$ .  
 3 Let  $\langle A_i \rangle$  be the locally definable subgroup of  $G$  generated by  $A_i$ .

4 We claim that  $\langle A_i \rangle$  is locally definably isomorphic to a definably generated group  $G_i$   
 5 in the structure  $\mathcal{X}_i$ . To this aim, let  $A_i^{(m)} \subseteq G$  consist of the  $m$ -fold products  $a_1 \dots a_m$ ,  
 6 where each  $a_i$  is either an element of  $A_i$  or is the group-inverse of an element of  $A_i$ .  
 7 Without loss of generality, after permuting coordinates, we can choose  $k_1, \dots, k_n \in \mathbb{N}$   
 8 such that  $G$  is included in  $X_1^{k_1} \times \dots \times X_n^{k_n}$ . Since  $A_i^{(m)}$  is  $X_i$ -internal and included  
 9 in  $G$ , by Fact 2.12 it must have a finite projection on the factors different from  
 10  $X_i$ . Thus we can write  $A_i^{(m)} \subseteq L_m \times X_i^{k_i} \times F_m$  where  $L_m$  and  $F_m$  are finite sets.  
 11 It follows that the subgroup  $\langle A_i \rangle$  of  $G$  generated by  $A_i$  is included in  $L \times X_i^{k_i} \times F$   
 12 where  $L = \bigcup_m L_m$  and  $F = \bigcup_m F_m$  are countable sets. Consider a bijection sending  
 13  $L \cup F$  to a countable subset of  $X_i$ . This induces a locally definable bijection between  
 14  $\langle A_i \rangle$  and a locally definable subset  $G_i \subseteq X_i^{k_i+1}$ . We can endow  $G_i$  with a group  
 15 operation via the bijection. The resulting group  $G_i$  will then be locally definable, and  
 16 in fact definably generated, in the structure  $\mathcal{X}_i$ . There is a locally definable group  
 17 homomorphism  $f : G_1 \times \dots \times G_n \rightarrow G$  induced by the composition  $G_1 \times \dots \times G_n \cong$   
 18  $\langle A_1 \rangle \times \dots \times \langle A_n \rangle \rightarrow G$ , where the last map sends  $(x_1, \dots, x_n)$  to their product in  $G$ .  
 19 Note that  $f$  is a homomorphism since  $G$  is abelian.

20 It remains to prove that the kernel  $\Gamma$  of the above  $f$  is discrete. To this aim fix,  
 21 for  $i = 1, \dots, n$ , a definable subset  $U_i$  of  $\langle A_i \rangle$  and let  $S$  be the set of all tuples  
 22  $(a_1, \dots, a_n) \in U_1 \times \dots \times U_n$  such that  $a_1 \dots a_n = 1_G$ . It suffices to show that  $S$  is  
 23 finite. The sets  $U_1, \dots, U_n$  are orthogonal by Proposition 2.13, since they are internal  
 24 to  $A_1, \dots, A_n$  respectively. It follows that  $S$  is a finite union of sets of the form  
 25  $B_1 \times \dots \times B_n$  with  $B_i \subseteq U_i$ . However, each  $B_i$  can only be a singleton because any  
 26 choice of  $n - 1$  elements from  $a_1, \dots, a_n$  determines the last one via the equation  
 27  $a_1 \dots a_n = 1_G$ .  $\square$

## 28 10. Questions

- 29 (1) Does Theorem 8.4 extend to the case when  $\mathcal{M}$  is an arbitrary structure?  
 30 (2) Can the abelianity hypothesis in Theorem 9.2 be removed?  
 31 (3) Does Proposition 2.9 hold without the saturation hypothesis?

## 32 11. Appendix

33 We construct an example of a definable group whose group operation splits, but  
 34 the group is not a product of two infinite groups, hence in particular two orthogonal  
 35 groups. First we need the following observation.

36 **Example 11.1.** There is a real Lie group  $G$ , definable in the pure real field structure,  
 37 which is not a semidirect product of the connected component of the identity  $G^0$  and  
 38 a finite non-trivial group.

39 *Proof.* Let  $p$  be an odd prime. The Heisenberg group mod  $p$  is the semialgebraic  
 40 group  $H$  of matrices of the form  $\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$  where  $c \in \mathbb{R}/p\mathbb{Z}$  and  $a, b \in \mathbb{Z}/p\mathbb{Z}$ . We  
 41 claim that  $H$  is not Lie isomorphic to a semidirect product of a connected real Lie

1 group and a discrete group. Taking  $a, b = 0$  we obtain the center  $Z(H)$  of  $H$ , which  
 2 coincides with  $H^0$  and it is isomorphic to the circle group  $\mathbb{R}/p\mathbb{Z}$ . The quotient  $H/H^0$   
 3 is isomorphic to  $(\mathbb{Z}/p\mathbb{Z})^2$ . Since  $H$  is not abelian,  $H$  is not isomorphic to the direct  
 4 product  $H^0 \times (\mathbb{Z}/p\mathbb{Z})^2$ . Moreover the direct product is the only possible semidirect  
 5 product in the Lie category because  $(\mathbb{Z}/p\mathbb{Z})^2$  has no non-trivial continuous action on  
 6  $\mathbb{R}/p\mathbb{Z}$  (since the only non-trivial definable automorphism of  $\mathbb{R}/p\mathbb{Z}$  is the inverse, and  
 7  $p$  is odd).  $\square$

**Example 11.2.** Let  $R_1$  and  $R_2$  be two orthogonal copies of the field  $\mathbb{R}$  and work  
 in the o-minimal structure  $M = R_1 \sqcup R_2$  obtained by concatenation of  $R_1$  and  $R_2$   
 with a separating element between them. Let  $H_i$  be the Heisenberg group mod 3  
 over  $R_i$ . Consider the definable group  $H_1 \times H_2$ . Now consider the definable subgroup  
 $G < H_1 \times H_2$  consisting of the pairs of matrices

$$\left\langle \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a & c' \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \right\rangle$$

8 with  $a, b \in \mathbb{Z}/3\mathbb{Z}$ ,  $c, c' \in \mathbb{R}/3\mathbb{Z}$ . Note that  $G$  is definably isomorphic to an extension  
 9 of  $(\mathbb{R}/3\mathbb{Z})^2$  by  $(\mathbb{Z}/3\mathbb{Z})^2$ . In Proposition 11.3 below we prove that  $G$  is not a direct  
 10 product of two infinite definable subgroups. This shows that, although the group  
 11 operation splits with respect to  $R_1$  and  $R_2$  (because it is induced by the direct product  
 12  $H_1 \times H_2$ ),  $G$  is not a direct product of orthogonal subgroups.

13 **Proposition 11.3.** *The group  $G$  in Example 11.2 is not a direct product of two infinite*  
 14 *definable subgroups.*

15 *Proof.* Assume that  $G$  is the direct product of two definable infinite subgroups  $G_1$   
 16 and  $G_2$ . Then  $\dim(G_1) = \dim(G_2) = 1$ . Since definable groups of dimension 1 are  
 17 cohesive (Theorem 3.4), we may assume that  $G_1$  is  $R_1$ -internal and  $G_2$  is  $R_2$ -internal.  
 18 Consider the natural (surjective) projections  $\pi_i : G \rightarrow H_i$ .

19 We claim that  $G_1^0 = \pi_2^{-1}(1_G)$  and  $G_2^0 = \pi_1^{-1}(1_G)$  where  $G_i^0$  is the connected  
 20 component of the identity of  $G_i$ . Consider for instance  $G_1$ . Clearly  $\pi_2(G_1)$  is finite by  
 21 orthogonality. Hence  $\pi_2^{-1}(1_G) \cap G_1$  has finite index in  $G_1$ , so it is infinite. Moreover  
 22  $\pi_2^{-1}(1_G)$  is  $H_1^0 \times \{1_G\}$ , hence it is connected and, since two definably connected one  
 23 dimensional groups having infinite intersection coincide, it must coincide with  $G_1^0$ . The  
 24 claim is thus proved.

25 Observe that  $Z(G) = G^0 = G_1^0 \times G_2^0$ , thus  $[G_1 : G_1^0][G_2 : G_2^0] = [G : G^0] =$   
 26 9. So, there are three cases for the possible values of the indexes of  $G_1^0$  and  $G_2^0$ :  
 27  $(1, 9), (3, 3), (9, 1)$ .

First case:  $[G_1 : G_1^0] = 1$  and  $[G_2 : G_2^0] = 9$  (observe that the third case is  
 symmetric). In this case  $G_1$  is connected, thus  $G_1 = G_1^0$ , and  $|\pi_1(G_2)| = 9$  because  
 $G_2^0 = \pi_1^{-1}(1_G)$  has index 9 in  $G_2$ . On the other hand

$$H_1 = \pi_1(G) = \pi_1(G_1)\pi_1(G_2) = H_1^0\pi_1(G_2) = Z(H_1)\pi_1(G_2)$$

28 and since  $\pi_1(G_2)$  has 9 elements and  $Z(H_1)$  has index 9 in  $H_1$ , it follows that  $H_1$  must  
 29 be the direct product of  $Z(H_1)$  and  $\pi_1(G_2)$ , contradicting the claim in Example 11.1.

30 Second case:  $[G_1 : G_1^0] = 3 = [G_2 : G_2^0]$ . In this case we show that  $G_1$  and  $G_2$  are  
 31 abelian and we reach a contradiction since  $G$  is not abelian. By symmetry it suffices

1 to show that  $G_1$  is abelian. First recall that  $G_1^0 < Z(G)$ , so in particular  $G_1^0$  is central  
 2 in  $G_1$ , and definably isomorphic to  $\mathbb{R}/3\mathbb{Z}$ . It follows that  $G_1$  is definably isomorphic  
 3 to a central extension of  $\mathbb{R}/3\mathbb{Z}$  by  $\mathbb{Z}/3\mathbb{Z}$ . We claim that such a group is necessarily  
 4 abelian. To this aim we show that there is a copy of  $\mathbb{Z}/3\mathbb{Z}$  which is a complement of  
 5  $G_1^0$ . Let  $G_1^0, aG_1^0, bG_1^0$  be the three connected components of  $G_1$ . Note that the map  
 6  $x \mapsto x^3$  has image contained in  $G_1^0$  and its restriction to  $G_1^0$  is onto. Consider the map  
 7 sending  $x \in G_1^0$  to  $(ax)^3 \in G_1^0$ . Since  $G_1^0$  is central,  $(ax)^3 = a^3x^3$ . Now let  $y \in G_1^0$   
 8 be such that  $y^3 = a^3$ . Then  $ay^{-1}$  has order three and generates a complement  $C$  of  
 9  $G_1^0$  in  $G_1$ . Since  $G_1^0$  is central we conclude that  $G_1$  is the direct product of  $C$  and  $G_1^0$ ,  
 10 hence it is abelian.  $\square$

11 **Example 11.4.** Let  $\mathcal{M} = (K, k, \Gamma; \pi, \nu)$  be a three sorted structure consisting of an  
 12 algebraically closed valued field  $K$  with residue field  $k$  of characteristic zero, value  
 13 group  $\Gamma$ , and the two projections of the field on the other sorts. Then  $K$  is indecom-  
 14 posable in  $\mathcal{M}$  but not cohesive.

15 *Proof.* We need the following facts:

- 16 (1)  $k$  and  $\Gamma$  are internal to  $K$ .
- 17 (2)  $k$  and  $\Gamma$  are orthogonal.
- 18 (3)  $K$  is internal to any infinite definable subset of  $K$ .
- 19 (4)  $K$  is not internal to  $k \times \Gamma$ .

20 Point (1) is obvious. The proof of (2) is in [6, Corollary 5.25]. (3) follows from the  
 21 fact that any definable subset of  $K$  is a boolean combination of valuation balls [6,  
 22 Corollary 3.32], so in particular if it is infinite it contains a valuation ball, and  $K$  itself  
 23 is internal to any valuation ball. Point (4) is clear if  $k$  and  $\Gamma$  have cardinality less  
 24 than  $K$  and we can reduce to this case going to an elementary equivalent model (for  
 25 instance, by the theorem of Ax-Kochen and Ershov (see [6, Theorem 5.1]) we can  
 26 take  $K = \mathbb{Q}^{ac}((t^{\mathbb{Q}})), \Gamma = \mathbb{Q}, k = \mathbb{Q}^{ac}$  where  $\mathbb{Q}^{ac}$  is the algebraic closure of  $\mathbb{Q}$ ).

27 Granted (1)–(4) we continue as follows. By (1) and (2)  $K$  is not cohesive. We  
 28 claim that  $K$  is indecomposable. So let  $K$  be internal to a product  $X \times Y$  of orthogonal  
 29 definable sets. We must show that  $K$  is internal to either  $X$  or  $Y$ . If this is not the  
 30 case, then by (3) neither  $X$  nor  $Y$  can have an infinite projection on  $K$ . But then  
 31 both  $X$  and  $Y$  are internal to  $k \times \Gamma$ . So  $K$  itself is internal to  $k \times \Gamma$ , contradicting  
 32 (4).  $\square$

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