Math. Log. Quart. **68,** No. 3, 279–287 (2022) / **DOI** 10.1002/malq.202100028

# Product cones in dense pairs

**Pantelis E. Eleftheriou***

School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom

Received 7 June 2021, revised 16 December 2021, accepted 7 January 2022
Published online 22 January 2022

Let $\mathcal{M} = \langle M, <, +, \dots \rangle$ be an o-minimal expansion of an ordered group, and $P \subseteq M$ a dense set such that certain tameness conditions hold. We introduce the notion of a *product cone* in $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$, and prove: if $\mathcal{M}$ expands a real closed field, then $\widetilde{\mathcal{M}}$ admits a product cone decomposition. If $\mathcal{M}$ is linear, then it does not. In particular, we settle a question from [10].



## 1 Introduction

Tame expansions $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ of an o-minimal structure $\mathcal{M}$ by a set $P \subseteq M$ have received lots of attention in recent literature (cf. [1–4, 6, 7, 12, 14]). One important category is when every definable open set is already definable in $\mathcal{M}$. Dense pairs and expansions of $\mathcal{M}$ by a dense independent set or by a dense multiplicative group with the Mann Property are of this sort. In [10], all these examples were put under a common perspective and a cone decomposition theorem was proved for their definable sets and functions. This theorem provided an analogue of the cell decomposition theorem for o-minimal structures in this context, and was inspired by the cone decomposition theorem established for semi-bounded o-minimal structures (cf. [8, 9, 15]). The central notion is that of a *cone*, and, as its definition in [10] appeared to be quite technical, in [10, Question 5.14], we asked whether it can be simplified in two specific ways. In this paper we refute both ways in general, showing that the definition in [10] is optimal, but prove that if $\mathcal{M}$ expands a real closed field, then a *product cone* decomposition theorem does hold.

In § 2, we provide all necessary background and definitions. For now, let us only point out the difference between product cones and cones, and state our main theorem. Let $\mathcal{M} = \langle M, <, +, \dots \rangle$ be an o-minimal expansion of an ordered group in the language $\mathcal{L}$, and $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ an expansion of $\mathcal{M}$ by a set $P \subseteq M$ such that certain tameness conditions hold (these are listed in § 2). For example, $\widetilde{\mathcal{M}}$ can be a dense pair (cf. [6]), or $P$ can be a dense independent set (cf. [5]) or a multiplicative group with the Mann Property (cf. [7]). By 'definable' we mean 'definable in $\widetilde{\mathcal{M}}$, and by $\mathcal{L}$-definable we mean 'definable in $\mathcal{M}$'. The notion of a *small* set is given in Definition 2.1 below, and it is equivalent to the classical notion of being $P$-internal from geometric stability theory ([10, Lemma 3.11 & Corollary 3.12]). A *supercone* generalizes the notion of being co-small in an interval (Definition 2.2). Now, and roughly speaking, a cone is then defined as a set of the form

$$h\Big(\bigcup_{g\in S}\{g\} \times J_g\Big),$$

where $h$ is an $\mathcal{L}$-definable continuous map with each $h(g, -)$ injective, $S \subseteq M^m$ is a small set, and $\{J_g\}_{g\in S}$ is a definable family of supercones. In Definition 2.4 below, we call a cone a *product cone* if we can replace the above family $\{J_g\}_{g\in S}$ by a product $S \times J$. That is, $C$ has the form

$$h(S \times J),$$

with $h$ and $S$ as above and $J$ a supercone. Let us say that $\widetilde{\mathcal{M}}$ *admits a product cone decomposition* if every definable set is a finite union of product cones. Our main theorem below asserts whether $\widetilde{\mathcal{M}}$ admits a product cone decomposition or not based solely on assumptions on $\mathcal{M}$. Recall that $\mathcal{M}$ is *linear* if it is an expansion of an ordered group and every definable function is piecewise affine (Definition 3.1).

* E-mail: p.eleftheriou@leeds.ac.uk

**Theorem 1.1** *1. If $\mathcal{M}$ is linear, then $\widetilde{\mathcal{M}}$ does not admit a product cone decomposition.*
*2. If $\mathcal{M}$ expands a real closed field, then $\widetilde{\mathcal{M}}$ admits a product cone decomposition.*

The counterexample in (1) is in fact uniform over all linear $\mathcal{M}$: it is a 'triangle' under the diagonal, with small projection (Claim 3.3).

Theorem 1.1(1), in particular, answers [10, Question 5.14(2)] negatively. [10, Question 5.14(1)] further asked whether one can define a supercone as a product of co-small sets in intervals, and still obtain a cone decomposition theorem. In Proposition 4.2 we also answer that question negatively in general, by constructing a counterexample whenever $\mathcal{M}$ expands a real closed field.

**Remark 1.2** Theorem 1.1 deals with the two main categories of o-minimal structures; namely, $\mathcal{M}$ is linear or it expands a real closed field. In the 'intermediate', semi-bounded case (cf. [9]), where $\mathcal{M}$ defines a field on a bounded interval but not on the whole of $M$, the answer to [10, Question 5.14] is rather unclear. Indeed, in the presence of two different notions of cones in this setting, the semi-bounded cones (from [9]) and the current ones, the methods in §§ 3.1 & 3.2 do not seem to apply and a new approach is needed.

**Notation** The topological closure of a set $X \subseteq M^n$ is denoted by $cl(X)$. Given any subset $X \subseteq M^m \times M^n$ and $a \in M^n$, we write $X_a$ for

$$\{b \in M^m : (b, a) \in X\}.$$

If $m \leq n$, then $\pi_m : M^n \to M^m$ denotes the projection onto the first $m$ coordinates. We write $\pi$ for $\pi_{n-1}$, unless stated otherwise. A family $\mathcal{J} = \{J_g\}_{g \in S}$ of sets is called definable if $\bigcup_{g \in S}\{g\} \times J_g$ is definable. We often identify $\mathcal{J}$ with $\bigcup_{g \in S}\{g\} \times J_g$.

## 2 Preliminaries

In this section we lay out all necessary background and terminology. Most of it is extracted from [10, § 2], where the reader is referred to for an extensive account. We fix an o-minimal theory $T$ expanding the theory of ordered abelian groups with a distinguished positive element 1. We denote by $\mathcal{L}$ the language of $T$ and by $\mathcal{L}(P)$ the language $\mathcal{L}$ augmented by a unary predicate symbol $P$. Let $\widetilde{T}$ be an $\mathcal{L}(P)$-theory extending $T$. If $\mathcal{M} = \langle M, <, +, \dots \rangle \models T$, then $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ denotes an expansion of $\mathcal{M}$ that models $\widetilde{T}$. By '$A$-definable' we mean 'definable in $\widetilde{\mathcal{M}}$ with parameters from $A$'. By '$\mathcal{L}_A$-definable' we mean 'definable in $\mathcal{M}$ with parameters from $A$'. We omit the index $A$ if we do not want to specify the parameters. For a subset $A \subseteq M$, we write $\mathrm{dcl}(A)$ for the definable closure of $A$ in $\mathcal{M}$, and for an $\mathcal{L}$-definable set $X \subseteq M^n$, we write $\dim(X)$ for the corresponding pregeometric dimension. The following definition is taken essentially from [7].

**Definition 2.1** Let $X \subseteq M^n$ be a definable set. We call $X$ *large* if there is some $m$ and an $\mathcal{L}$-definable function $f : M^{nm} \to M$ such that $f(X^m)$ contains an open interval in $M$. We call $X$ *small* if it is not large. We call $X$ *co-small* in a definable set $Y$, if $Y \setminus X$ is small.

Consider the following *Tameness Conditions* (cf. [10]):

(I) $P$ is small.

(II) Every $A$-definable set $X \subseteq M^n$ is a boolean combination of sets of the form

$$\{x \in M^n : \exists z \in P^m \varphi(x, z)\},$$

where $\varphi(x, z)$ is an $\mathcal{L}_A$-formula.

(III) (Open definable sets are $\mathcal{L}$-definable) For every parameter set $A$ such that $A \setminus P$ is dcl-independent over $P$, and for every $A$-definable set $V \subset M^s$, its topological closure $cl(V) \subseteq M^s$ is $\mathcal{L}_A$-definable.

*From now on, we assume that every model $\widetilde{\mathcal{M}} \models \widetilde{T}$ satisfies Conditions (I)-(III) above. We fix a sufficiently saturated model $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle \models \widetilde{T}$.*

We next turn to define the central notions of this paper. As mentioned in the introduction, the notion of a cone is based on that of a supercone, which in turn generalizes the notion of being co-small in an interval. Both notions, supercones and cones, are unions of specific families of sets, which not only are definable, but they are so in a very uniform way.

**Definition 2.2** ([10]) A *supercone* $J \subseteq M^k$, $k \geq 0$, and its *shell* $sh(J)$ are defined recursively as follows:

1. $M^0 = \{0\}$ is a supercone, and $sh(M^0) = M^0$.
2. A definable set $J \subseteq M^{n+1}$ is a supercone if $\pi(J) \subseteq M^n$ is a supercone and there are $\mathcal{L}$-definable continuous maps $h_1, h_2 : sh(\pi(J)) \to M \cup \{\pm\infty\}$ with $h_1 < h_2$, such that for every $a \in \pi(J)$, $J_a$ is contained in $(h_1(a), h_2(a))$ and it is co-small in it. We let $sh(J) = (h_1, h_2)_{sh(\pi(J))}$.

Abusing terminology, we call a supercone *A-definable* if it is an $A$-definable set and its closure is $\mathcal{L}_A$-definable.

Note that, for $k > 0$, $sh(J)$ is the unique open cell in $M^k$ such that $cl(sh(J)) = cl(J)$. That is, $sh(J)$ is the interior of $cl(J)$. In particular, if $J$ is $A$-definable, then all defining maps $h_1, h_2$ used in its recursive definition are $\mathcal{L}_A$-definable.

Recall that in our notation we identify a family $\mathcal{J} = \{J_g\}_{g\in S}$ with $\bigcup_{g\in S}\{g\} \times J_g$. In particular, $cl(\mathcal{J})$ and $\pi_n(\mathcal{J})$ denote the closure and a projection of that set, respectively.

**Definition 2.3** (Uniform families of supercones [10]) Let $\mathcal{J} = \bigcup_{g\in S}\{g\} \times J_g \subseteq M^{m+k}$ be a definable family of supercones (so $S \subseteq M^m$, and $J_g \subseteq M^k$, $g \in S$, are supercones). We call $\mathcal{J}$ *uniform* if there is a cell $V \subseteq M^{m+k}$ containing $\mathcal{J}$, such that for every $g \in S$ and $0 < j \leq k$,

$$cl(\pi_{m+j}(\mathcal{J})_g) = cl(\pi_{m+j}(V)_g).$$

We call such a $V$ a *shell* for $\mathcal{J}$. Abusing terminology, we call $\mathcal{J}$ *A-definable*, if it is an $A$-definable family of sets and has an $\mathcal{L}_A$-definable shell.

In case $S$ is a singleton, $\mathcal{J}$ can be identified with a supercone, and its shell with the shell from Definition 2.2 (after projecting on the last $k$ coordinates).

In particular, if $\mathcal{J}$ is uniform, then so is each projection $\pi_{m+j}(\mathcal{J})$. Moreover, if $V$ is a shell for $\mathcal{J}$, then $\pi_{m+j}(V)$ is a shell for $\pi_{m+j}(\mathcal{J})$. Observe also that if $V$ is a shell for $\mathcal{J}$, then for every $x \in \pi_{m+k-1}(\mathcal{J})$, $\mathcal{J}_x$ is co-small in $V_x$.

A shell for $\mathcal{J}$ need not be unique. Whenever we say that $\mathcal{J}$ is a uniform family of supercones with shell $V$, we just mean that $V$ is a shell for $\mathcal{J}$.

**Definition 2.4** (Cones [10] and product cones) A set $C \subseteq M^n$ is a *k-cone*, $k \geq 0$, if there are a definable small $S \subseteq M^m$, a uniform family $\mathcal{J} = \{J_g\}_{g\in S}$ of supercones in $M^k$, and an $\mathcal{L}$-definable continuous function $h : V \subseteq M^{m+k} \to M^n$, where $V$ is a shell for $\mathcal{J}$, such that

1. $C = h(\mathcal{J})$, and
2. for every $g \in S$, $h(g, -) : V_g \subseteq M^k \to M^n$ is injective.

We call $C$ a *k-product cone* if, moreover, $\mathcal{J} = S \times J$, for some supercone $J \subseteq M^k$. A *(product) cone* is a $k$-(product) cone for some $k$. Abusing terminology, we call a (product) cone $h(\mathcal{J})$ *A-definable* if $h$ is $\mathcal{L}_A$-definable and $\mathcal{J}$ is $A$-definable.

The cone decomposition theorem below (Fact 2.6) is a statement about definable sets and functions. The notion of a 'well-behaved' function in this setting is given next.

**Definition 2.5** (Fiber $\mathcal{L}$-definable maps [10]) Let $C = h(\mathcal{J}) \subseteq M^n$ be a $k$-cone with $\mathcal{J} \subseteq M^{m+k}$, and $f : D \to M$ a definable function with $C \subseteq D$. We say that $f$ is *fiber $\mathcal{L}$-definable with respect to* $C$ if there is an $\mathcal{L}$-definable continuous function $F : V \subseteq M^{m+k} \to M$, where $V$ is a shell for $\mathcal{J}$, such that

$$(f \circ h)(x) = F(x), \text{ for all } x \in \mathcal{J}.$$

We call $f$ *fiber $\mathcal{L}_A$-definable with respect to* $C$ if $F$ is $\mathcal{L}_A$-definable.

As remarked in [10, Remark 4.5(4)], the terminology is justified by the fact that, if $f$ is fiber $\mathcal{L}_A$-definable with respect to $C = h(\mathcal{J})$, then for every $g \in \pi(\mathcal{J})$, $f$ agrees on $h(g, J_g)$ with an $\mathcal{L}_{Ag}$-definable map; namely $F \circ h(g, -)^{-1}$. Moreover, the notion of being fiber $\mathcal{L}$-definable with respect to a cone $C = h(\mathcal{J})$, depends on $h$ and $\mathcal{J}$ ([10, Example 4.6]). However, it is immediate from the definition that if $f$ is fiber $\mathcal{L}_A$-definable with respect to a cone $C = h(\mathcal{J})$, and $h(\mathcal{J}') \subseteq h(\mathcal{J})$ is another cone (but with the same $h$), then $f$ is also fiber $\mathcal{L}_A$-definable with respect to it.

We are now ready to state the cone decomposition theorem from [10].

**Fact 2.6** (Cone decomposition theorem [10, Theorem 5.1])

1. Let $X \subseteq M^n$ be an $A$-definable set. Then $X$ is a finite union of $A$-definable cones.
2. Let $f : X \to M$ be an $A$-definable function. Then there is a finite collection $\mathcal{C}$ of $A$-definable cones, whose union is $X$ and such that $f$ is fiber $\mathcal{L}_A$-definable with respect to each cone in $\mathcal{C}$.

Another important notion from [10] is that of 'large dimension', which we recall next. The proof of Theorem 1.1(2) runs by induction on large dimension.

**Definition 2.7** (Large dimension [10]) Let $X \subseteq M^n$ be definable. If $X \neq \varnothing$, the *large dimension* of $X$ is the maximum $k \in \mathbb{N}$ such that $X$ contains a $k$-cone. The large dimension of the empty set is defined to be $-\infty$. We denote the large dimension of $X$ by $\operatorname{ldim}(X)$.

**Remark 2.8** The tameness conditions that we assume in this paper guarantee that the notion of large dimension is well-defined; namely, the above maximum $k$ always exists ([10, § 4.3]).

## 3 Product cone decompositions

In this section we prove Theorem 1.1.

### 3.1 The linear case

The following definition is taken from [13].

**Definition 3.1** ([13]) Let $\mathcal{N} = \langle N, +, <, 0, \dots \rangle$ be an o-minimal expansion of an ordered group. A function $f : A \subseteq N^n \to N$ is called *affine*, if for every $x, y, x+t, y+t \in A$,

$$f(x+t) - f(x) = f(y+t) - f(y). \tag{1}$$

We call $\mathcal{N}$ *linear* if every definable $f : A \subseteq N^n \to N$ is *piecewise affine*, namely if there is a partition of $A$ into finitely many definable sets $B$, such that each $f_{\restriction B}$ is affine.

The typical example of a linear o-minimal structure is an ordered vector space $\mathcal{V} = \langle V, <, +, 0, \{d\}_{d\in D}\rangle$ over an ordered division ring $D$. In general, if $\mathcal{N}$ is linear, then there exists a reduct $\mathcal{S}$ of such $\mathcal{V}$, such that $\mathcal{S} \equiv \mathcal{N}$ (cf. [13] for details). Using this description, it is not hard to see that every affine function has a continuous extension to the closure of its domain.

Assume now that our fixed structure $\mathcal{M}$ is linear.

**Lemma 3.2** *Let $h : [a, b] \times [c, d] \to M$ be an $\mathcal{L}$-definable continuous function, such that for every $t \in (a, b)$, $h(t, -) : [c, d] \to M$ is strictly increasing. Then*

$$h(b, d) - h(b, c) > 0.$$

P r o o f. Let $\mathcal{W}$ be a cell decomposition of $[a, b] \times [c, d]$ such that for every $W \in \mathcal{W}$, $h_{\restriction W}$ is affine. Since $d - c > 0$, there must be some $W = (f, g)_I \in \mathcal{W}$, where $I$ is an interval with $\sup I = b$, and $r \in I$, such that the map $\delta(t) := g(t) - f(t)$ is increasing on $[r, b)$. We claim that for every $t \in (r, b)$,

$$h(t, g(t)) - h(t, f(t)) \geq h(r, g(r)) - h(r, f(r)).$$

Indeed, there is $k \geq 0$, such that

$$\begin{aligned} h(t, f(t) + \delta(t)) - h(t, f(t)) &= h(t, f(t) + \delta(r) + k) - h(t, f(t)) \\ &= h(t, f(t) + \delta(r) + k) + h(t, f(t) + \delta(r)) \\ &\qquad - h(t, f(t) + \delta(r)) + h(t, f(t)) \end{aligned}$$

$$\geq h(t, f(t) + \delta(r)) - h(t, f(t))$$
$$= h(r, f(r) + \delta(r)) - h(r, f(r)),$$

where the inequality holds because $h(t, -)$ is increasing, and the last equality holds because $h$ is affine on $W$. We conclude that

$$\begin{aligned} h(b, d) - h(b, c) &= \lim_{t \to b} (h(t, d) - h(t, c)) \\ &\geq \lim_{t \to b} (h(t, g(t)) - h(t, f(t))) \\ &\geq h(r, g(r)) - h(r, f(r)) \\ &\leq 0, \end{aligned}$$

where the first and last inequalities hold because $h(t, -)$ and $h(r, -)$ are strictly increasing. □

**Counterexample to product cone decomposition** Let $S \subseteq M$ be a small set such that 0 is in the interior of its closure (by translating $P$ to the origin, such an $S$ exists). Let

$$X = \bigcup_{a \in S^{>0}} \{a\} \times (0, a).$$

**Claim 3.3** *X is not a finite union of product cones.*

Proof. First of all, $X$ cannot contain any $k$-cones for $k > 1$, since $\text{ldim}(X) = 1$, by [10, Lemmas 4.24 & 4.27]. Now let $H(T \times J)$ be an 1-product cone contained in $X$, with $H = (H_1, H_2) : Z \subseteq M^{l+1} \to M^2$, such that the origin is in its closure. Since $H$ is $\mathcal{L}$-definable and continuous, and for each $g \in T$, $H_2(g, -)$ is injective, we may assume that the latter is always strictly increasing. By [10, Lemma 5.10] applied to $J$, $f(-) = \pi_1 H(g, -)$ and $S$, we have

for every $g \in T$, there is $a \in S$, such that $H(g, J) \subseteq \{a\} \times (0, a)$.

By continuity of $H$, it follows that

for every $g \in cl(T) \cap \pi(Z)$, there is $a \in M$, such that $H(g, cl(J)) \subseteq \{a\} \times [0, a]$.

Let $F : \pi(Z) \to M$ be the $\mathcal{L}$-definable map given by

$$F(g) = \pi_1(H(g, cl(J))).$$

Since the origin is in the closure of $H(T \times J)$, there must be an affine $\gamma : (a, b) \to cl(T) \cap \pi(Z)$ with $\lim_{t \to b} F(\gamma(t)) = 0$. Fix any $[c, d] \subseteq cl(J)$. Now the map

$$H_2(\gamma(-), -) : (a, b) \times (c, d) \to M$$

is piecewise affine and hence has a continuous extension $h$ to $[a, b] \times [c, d]$. By definition of $X$,

$$h(b, c) = h(b, d) = 0.$$

But, by Lemma 3.2,

$$h(b, d) - h(b, c) > 0,$$

a contradiction. Since $X$ contains no product cone whose closure contains the origin, $X$ cannot be a finite union of product cones. □

### 3.2 The field case

We now assume that $\mathcal{M}$ expands an ordered field. The main idea behind the proof in this case is as follows. By Fact 2.6, it suffices to write every cone as a finite union of product cones. We illustrate the case of a 1-cone $C = h(\mathcal{J})$, for some $\mathcal{J} = \{J_g\}_{g \in S}$.

*Step I* (Lemma 3.4). Replace $\mathcal{J}$ by a cone $\mathcal{J}' = \{J'_g\}_{g\in S}$, such that for some fixed interval $I$, each $J'_g$ is contained in $I$ and it is co-small in it. Here we use the field structure of $\mathcal{M}$, so this step would fail in the linear case.

*Step II* (Lemma 3.5). By [10, Lemma 4.25], the intersection $J = \bigcap_{g\in S} J'_g$ is co-small in $I$. Moreover, if we let $L = S \times J$, then, by [10, Lemma 4.29], we obtain that the large dimension of $\mathcal{J} \setminus L$ is 0.

*Step III* (Theorem 3.6). Use Steps I and II and induction on large dimension. Here, the inductive hypothesis is only applied to sets of large dimension 0. In general, $\mathrm{ldim}(\mathcal{J} \setminus L) < \mathrm{ldim}(\mathcal{J})$.

To achieve Step I, we first need to make an observation and fix some notation. Using the field operations, one can define an $\mathcal{L}_\varnothing$-definable continuous $f : M^3 \to M$, such that for every $b, c \in M$,

$$f(b, c, -) : (b, c) \to (0, 1)$$

is a bijection. Similarly, there are $\mathcal{L}_\varnothing$-definable continuous maps $f_1, f_2 : M^2 \to M$, such that for every $b, c \in M$, the maps

$$f_1(b, -) : (b, +\infty) \to (0, 1)$$

and

$$f_2(c, -) : (-\infty, c) \to (0, 1)$$

are bijections. To give all these maps a uniform notation, we write $f(b, +\infty, x)$ for $f_1(b, x)$, and $f(-\infty, c, x)$ for $f_2(c, x)$. We fix this $f$ for the next proof. Observe that if $J \subseteq (b, c)$ is co-small in $(b, c)$, for $b, c \in M \cup \{\pm\infty\}$, then $f(b, c, J)$ is co-small in $(0, 1)$.

**Lemma 3.4** *Let $\mathcal{J} = \bigcup_{g\in S}\{g\} \times J_g \subseteq M^{m+k}$ be an A-definable uniform family of supercones, with shell $Z \subseteq M^{m+k}$. Then there are*

1. *an A-definable uniform family $\mathcal{J}' = \{J'_g\}_{g\in S}$ of supercones $J'_g \subseteq M^k$, with shell $\pi(Z) \times (0, 1)^k$,*
2. *and an $\mathcal{L}_A$-definable continuous and injective map $F : Z \to M^{m+k}$, such that*

$$F(\mathcal{J}) = \mathcal{J}'.$$

P r o o f . For every $g \in \pi_m(\mathcal{J})$, since $J_g$ is a supercone, it follows that $Z_g$ is an open cell. Hence, for every $0 < j \le k$, there are $\mathcal{L}_A$-definable continuous maps $h_1^j, h_2^j : \pi_{m+j-1}(Z) \to M$ such that

$$\pi_{m+j}(Z) = (h_1^j, h_2^j)_{\pi_{m+j-1}(Z)}.$$

We define

$$F = (F_1, \dots, F_{m+k}) : Z \to M^{m+k},$$

as follows. Let $I = (0, 1)$ and $f$ be the map fixed above. Let $(g, t) \in Z \subseteq M^{m+k}$. If $1 \le i \le m$,

$$F_i(g, t) = g_i$$

(the $i$th coordinate of $g$.) If $i = m + j$, with $0 < j \le k$,

$$F_{m+j}(g, t) = f(h_1^j(g, t_1, \dots, t_{j-1}), h_2^j(g, t_1, \dots, t_{j-1}), t_j).$$

Clearly, $F$ is injective, $\mathcal{L}_A$-definable and continuous. Let

$$\mathcal{J}' = F(\mathcal{J}).$$

That is, $\mathcal{J}' = \{J'_g\}_{g\in S}$, where for every $g \in S$, $J'_g = F(g, J_g)$. It is not hard to check, by induction on $m$, that for every $0 < m \le k$, $\pi_{m+j}(\mathcal{J}')$ is an $A$-definable uniform family of supercones with shell $F(Z) = \pi(Z) \times I^m$. □

**Lemma 3.5** *Let $\mathcal{J} = \bigcup_{g\in S}\{g\} \times J_g \subseteq M^{m+k}$ be an A-definable uniform family of supercones in $M^k$ with shell $Z$, and assume $S \subseteq M^m$ is small. Suppose that $Z = \pi(Z) \times I^k$, where $I = (0, 1)$. Then $\mathcal{J}$ is a disjoint union*

$$(S \times J) \cup Y,$$

*where $S \times J$ is an A-definable uniform family of supercones with shell $Z$, and $Y$ is an A-definable set of large dimension $< k$.*

 

*Proof.* By induction on $k$. For $k = 0$, the statement is trivial. We assume the statement holds for $k$, and prove it for $k + 1$. Let $\pi : M^{m+k+1} \to M^{m+k}$ be the projection onto the first $m + k$ coordinates. Since $\pi(\mathcal{J})$ is also an $A$-definable uniform family of supercones with shell $\pi(Z)$, by inductive hypothesis we can write $\pi(\mathcal{J})$ as a disjoint union

$$\pi(\mathcal{J}) = (S \times T) \cup Y,$$

where $T \subseteq M^k$ is an $A$-definable supercone with $cl(T) = cl(I^k)$, and $Y$ is an $A$-definable set of large dimension $< k$. By [10, Corollary 5.5], the set $\bigcup_{t \in Y}\{t\} \times \mathcal{J}_t \subseteq \mathcal{J}$ has large dimension $< k + 1$, and hence we only need to bring its complement $X$ in $\mathcal{J}$ into the desired form. We have

$$X = \bigcup_{t \in S \times T} \{t\} \times \mathcal{J}_t.$$

Define, for every $a \in T$,

$$K_a = \bigcap_{g \in S} \mathcal{J}_{g,a}.$$

Since each $\mathcal{J}_{g,a}$ is co-small in $I$, by [10, Lemma 4.25] $K_a$ is co-small in $I$. Hence, the set

$$L = \bigcup_{a \in T}\{a\} \times K_a$$

is a supercone in $M^{k+1}$. Since $cl(T) = cl(I^k)$ and for each $a \in T$, $cl(K_a) = cl(I)$, it follows that $cl(L) = cl(I^{k+1})$. In particular, $Z$ is a shell for $S \times L$. Since $S \times L \subseteq X$, it remains to prove that $\mathrm{ldim}(X \setminus (S \times L)) < k + 1$. We have

$$X \setminus (S \times L) = \bigcup_{(g,a) \in S \times T} \{(g, a)\} \times (\mathcal{J}_{g,a} \setminus K_a).$$

But $\mathcal{J}_{g,a} \setminus K_a$ is small, and hence, by [10, Lemma 4.29], the above set has large dimension $= \mathrm{ldim}(S \times T) = k$. □

We can now conclude the main theorem of the paper.

**Theorem 3.6** (Product cone decomposition in the field case) *Let $X \subseteq M^n$ be an $A$-definable set. Then*

1. *$X$ is a finite union of $A$-definable product cones.*
2. *If $f : X \to M$ is an $A$-definable function, then there is a finite collection $\mathcal{C}$ of $A$-definable product cones, whose union is $X$ and such that $f$ is fiber $\mathcal{L}_A$-definable with respect to each cone in $\mathcal{C}$.*

*Proof.* (1) By induction on the large dimension of $X$. Suppose $\mathrm{ldim}(X) = k$. By Fact 2.6, we may assume that $X$ is a $k$-cone. Every 0-cone is clearly a product cone. Now let $k > 0$. By induction, it suffices to write $X$ as a union of an $A$-definable product cone and an $A$-definable set of large dimension $< k$. Let $X = h(\mathcal{J})$ be as in Definition 2.4, and $Z \subseteq M^{m+k}$ a shell for $\mathcal{J}$.

**Claim** We can write $X$ as a $k$-cone $h'(\mathcal{J}')$, such that for every $g \in \pi(\mathcal{J}')$, $cl(\mathcal{J}')_g = (0, 1)^k$.

*Proof of Claim.* Let $\mathcal{J}'$ and $F : Z \to M^{m+k}$ be as in Lemma 3.4, and define $h' = h \circ F^{-1} : F(Z) \to M^n$. Then

$$h(\mathcal{J}) = hF^{-1}(F(\mathcal{J})) = h'(\mathcal{J}')$$

is as required. □

By the claim, we may assume that for every $g \in S$, $cl(\mathcal{J})_g = (0, 1)^k$. By Lemma 3.5, we have $\mathcal{J} = (S \times J) \cup Y$, where $J \subseteq M^k$ is an $A$-definable supercone, and $\mathrm{ldim}\, Y < k$. Thus $h(\mathcal{J}) = h(S \times J) \cup h(Y)$ has been written in the desired form.

(2) By Fact 2.6, we may assume that $X$ is a $k$-cone and that $f$ is fiber $\mathcal{L}_A$-definable with respect to it. So let again $X = h(\mathcal{J})$ with shell $Z \subseteq M^{m+k}$, and in addition, $\tau : Z \subseteq M^{m+k} \to M$ with $\mathcal{J} \subseteq Z$, be $\mathcal{L}_A$-definable so that for every $x \in \mathcal{J}$,

$$(f \circ h)(x) = \tau(x).$$

By induction on large dimension, it suffices to show that $X$ is the union of a product cone $C$ and a set of large dimension $< k$, such that $f$ is fiber $\mathcal{L}_A$-definable with respect to $C$. Let $X = h'(\mathcal{J}')$ be as in Claim of (1) and $F : Z \to M^{m+k}$ as in its proof. So $h' = h \circ F^{-1} : F(Z) \to M^n$. Define $\tau' : F(Z) \to M^n$ as $\tau' = \tau \circ F^{-1}$. We then have, for every $x' \in \mathcal{J}'$,

$$fh'(x') = fh'F(x) = fh(x) = \tau(x) = \tau F^{-1}(x) = \tau'(x),$$

witnessing that $f$ is fiber $\mathcal{L}_A$-definable with respect to $h'(\mathcal{J}')$.

Therefore, we may replace $h$ by $h$ and $\mathcal{J}$ by $\mathcal{J}'$. Now, as in the proof of (1), we can write $h(\mathcal{J})$ as the union of a product cone $h(S \times J)$ and a set of large dimension $< k$. By the remarks following Definition 2.5, $f$ is also fiber $\mathcal{L}$-definable with respect to $h(S \times J)$. □

**Remark 3.7** From the above proof it follows that in cases where we have disjoint unions in Fact 2.6 (as in [10, Theorem 5.12]), this is also the case in Theorem 3.6.

## 4 Refined supercones

In this section we answer [10, Question 5.14(1)] negatively. The question asked whether the Structure Theorem holds if we strengthen the notion of a supercone as follows.

**Definition 4.1** A supercone $\mathcal{J}$ in $M^k$ is called *refined* if it is of the form

$$\mathcal{J} = J_1 \times \cdots \times J_k,$$

where each $J_i$ is a supercone in $M$. Let us call a ($k$-)cone $C = h(\mathcal{J})$ a ($k$-)*refined cone* if $\mathcal{J}$ is refined.

Our result is the following.[1]

**Proposition 4.2** *Assume $\mathcal{M}$ expands a real closed field. Then there is a supercone in $M^2$ which contains no 2-refined cone. In particular, it is not a finite union of refined cones.*

Proof. The 'in particular' clause follows from [10, Corollaries 4.26 & 4.27]. Now, for every $a \in M$, let

$$J_a = M \setminus (P + aP)$$

and define $\mathcal{J} = \bigcup_{a \in M} \{a\} \times J_a$. Towards a contradiction, assume that $\mathcal{J}$ contains a 2-refined cone. That is, there are supercones $J_1, J_2 \subseteq M$, an open cell $U \subseteq M^2$ with $cl(J_1 \times J_2) = cl(U)$, and an $\mathcal{L}$-definable continuous and injective map $f : U \to M^2$, such that $C = f(J_1 \times J_2) \subseteq \mathcal{J}$. We write $X = f(U)$, and for each $a \in M$, $X_a \subseteq M$ for the fiber of $X$ above $a$. Suppose $C$ is $A$-definable.

By saturation, there is $a \in M$ which is dcl-independent over $A \cup P$, and further $g, h \in P$ which are dcl-independent over $a$. So

$$\dim(g, h, a) = 3.$$

By assumption, there are $(p, q) \in U \setminus (J_1 \times J_2)$, such that

$$f(p, q) = (a, g + ha).$$

Observe that $a \in \operatorname{dcl}(p, q)$. Also, one of $p, q$ must be in $\operatorname{dcl}(AP)$. Indeed, we have $p \notin J_1$ or $q \notin J_2$. If, say, the former holds, then $p \in \pi(U) \setminus J_1$. Since the last set is $A$-definable and small, we obtain by [10, Lemma 3.11], that $p \in \operatorname{dcl}(AP)$.

We may now assume that $p \in \operatorname{dcl}(AP)$. If we write $f = (f_1, f_2)$, we obtain

$$f_2(p, q) = g + hf_1(p, q). \tag{2}$$

Since $a$ is dcl-independent over $A \cup P$, there must be an open interval $I \subseteq M$ of $p$, such that for every $x \in I$,

$$f_2(x, q) = g + hf_1(x, q).$$

[1] The proof is based on an idea suggested by Hieronymi.

 

Viewing both sides of the equation as functions in the variable $f_1(x, q)$, and taking their derivatives with respect to it, we obtain:

$$\frac{\partial f_2(x, q)}{\partial f_1(x, q)} = f_1(x, q) + h.$$

Evaluated at $p$, the last equality gives $h \in \mathrm{dcl}(p, q)$. By (2), also $g \in \mathrm{dcl}(p, q)$. All together, we have proved that $g, h, a \in \mathrm{dcl}(p, q)$. It follows that

$$\dim(g, h, a) \leq \dim(p, q) \leq 2,$$

a contradiction. □

**Acknowledgements** I thank Philipp Hieronymi for several discussions on this topic, and the referee for a very careful reading of the paper. This research was partially supported by an EPSRC Early Career Fellowship (EP/V003291/1) and a Zukunftskolleg Research Fellowship (Konstanz).

## References

[1] O. Belegradek and B. Zilber, The model theory of the field of reals with a subgroup of the unit circle, J. Lond. Math. Soc. **78**, 563–579 (2008).
[2] A. Berenstein, C. Ealy, and A. Günaydin, Thorn independence in the field of real numbers with a small multiplicative group, Ann. Pure Appl. Log. **150**, 1–18 (2007).
[3] G. Boxall, P. Hieronymi, Expansions which introduce no new open sets, J. Symb. Log. **77**, 111–121 (2012).
[4] A. Dolich, C. Miller, and C. Steinhorn, Structures having o-minimal open core, Trans. Amer. Math. Soc. **362**, 1371–1411 (2010).
[5] A. Dolich, C. Miller, and C. Steinhorn, Expansions of o-minimal structures by dense independent sets, Ann. Pure Appl. Log. **167**, 684–706 (2016).
[6] L. van den Dries, Dense pairs of o-minimal structures, Fund. Math. **157**, 61–78 (1988).
[7] L. van den Dries and A. Günaydın, The fields of real and complex numbers with a small multiplicative group, Proc. Lond. Math. Soc. **93**, 43–81 (2006).
[8] M. Edmundo, Structure theorems for o-minimal expansions of groups, Ann. Pure Appl. Log. **102**, 159–181 (2000).
[9] P. Eleftheriou, Local analysis for semi-bounded groups, Fund. Math. **216**, 223–258 (2012).
[10] P. Eleftheriou, A. Günaydin, and P. Hieronymi, Structure theorems in tame expansions of o-minimal structures by dense sets, Israel J. Math. **239**, 435–500 (2020).
[11] P. Eleftheriou and S. Starchenko, Groups definable in ordered vector spaces over ordered division rings, J. Symb. Log. **72**, 1108–1140 (2007).
[12] A. Günaydın and P. Hieronymi, The real field with the rational points of an elliptic curve, Fund. Math. **215**, 167–175 (2011).
[13] J. Loveys and Y. Peterzil, Linear o-minimal structures, Israel J. Math. **81**, 1–30 (1993).
[14] C. Miller and P. Speissegger, Expansions of the real line by open sets: o-minimality and open cores, Fund. Math. **162**, 193–208 (1999).
[15] Y. Peterzil, A structure theorem for semibounded sets in the reals, J. Symb. Log. **57**, 779–794 (1992).

 