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STEADY-STATE INHOMOGENEOUS DIFFUSION WITH GENERALIZED OBLIQUE BOUNDARY CONDITIONS

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Abstract. We consider the elliptic diffusion (steady-state heat conduction) equation with space-dependent conductivity and inhomogeneous source subject to a generalized oblique boundary condition on a part of the boundary and Dirichlet or Neumann boundary conditions on the remaining part. The oblique boundary condition represents a linear combination between the dependent variable and its normal and tangential derivatives at the boundary. We first prove the well-posedness of the continuous problems. We then develop new finite volume schemes for these problems and prove rigorously the stability and convergence of these schemes. We also address an application to the inverse corrosion problem concerning the reconstruction of the coefficients present in the generalized oblique boundary condition that is prescribed over a portion Γ_0 of the boundary $\partial\Omega$ from Cauchy data on the complementary portion $\Gamma_1 = \partial\Omega \backslash \Gamma_0$.

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1. Introduction

Mathematical models involving oblique derivative boundary conditions appear in various areas of applied sciences, *e.g.* the gravimetric model of geophysics concerned with the determination of the Earth's exterior gravitational field from the magnitude of the gravity gradient [39] and the scattering of long ocean waves by islands [42].

If Ω is a d-dimensional domain ($d \geq 2$) with a smooth boundary $\partial\Omega$ and $\underline{l}$ is a smooth vector field defined on Ω, then an oblique boundary condition can be written as

$$\frac{\partial u}{\partial l} + \overline{\beta} u = g \quad \text{on } \partial\Omega, \tag{1}$$

where $\partial u/\partial l = \nabla u \cdot \underline{l}$ is the directional derivative in the direction $\underline{l}$, $\overline{\beta}$ and g are prescribed functions on $\partial\Omega$ and u is the desired solution that usually satisfies a second-order elliptic partial differential equation, *e.g.* the Laplace's equation in potential theory. Noticing that the oblique derivative is in fact a linear combination of

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the normal $\mathbf{n}$ and tangential $\mathbf{t}$ derivatives, we can generalize (1) to look like (see equation (10))

$$\kappa\frac{\partial u}{\partial \mathrm{n}}+\alpha\frac{\partial u}{\partial \mathrm{t}}+\overline{\beta}u=g \quad \text{on } \partial\Omega, \tag{2}$$

where κ and α are given functions (see also Eq. (7)). For the Laplace's equation in bounded or unbounded domains subject to the oblique boundary condition (1) (or (2)), numerical methods based on the boundary element method (BEM) or the method of fundamental solutions (MFS) were developed in [27,35,38,40], respectively. Moreover, finite volume schemes for the Poisson's and parabolic equations in bounded domains subject to the oblique boundary condition (1) (or (2)) were developed in [9,10]. We refer to [19] for a literature on the subject of numerical methods for oblique boundary conditions.

The present finite volume analyses developed in Sections 6 and 7 for mixed oblique-Dirichlet or oblique-Neumann boundary conditions, respectively, generalize in some sense the results established in [10], where the homogeneous case with oblique condition on the whole boundary was considered. However, the presence of mixed boundary conditions in (7), (8) or (7), (9), the space-dependent conductivity $\kappa(\boldsymbol{x})$ in (6) and the simultaneous presence of the three coefficients κ, α and β in the generalized oblique boundary condition (7) require new changes and new technical tools compared to the analyses of [10], leading to new suitable schemes and results. We justify rigorously the well-posedness of the direct problem given by equation (6) with the mixed boundary conditions (7) and (8) or (7) and (9). We then establish finite volume schemes for these two problems. We employ several technical tools to show both the well-posedness and the convergence of these schemes. In addition, the application of the generalized oblique boundary condition (7) to the inverse problem of corrosion, developed in Section 8, is completely new.

2. Preliminaries

Let Ω be an open bounded polygonal piecewise smooth connected subset of $\mathbb{R}^2$. We assume classical polygons, as in [30], Page 182, in which the boundary is the union of a consecutive finite number of linear segments. Let us denote by $\boldsymbol{x}=(x,y)$ the current point of $\mathbb{R}^2$. For any function $\Psi\in\mathcal{C}^1(\overline{\Omega})$, we denote by $\Psi_{\mathbf{n}}=\nabla\Psi\cdot\mathbf{n}$ and $\Psi_{\mathbf{t}}=\nabla\Psi\cdot\mathbf{t}$, where $\mathbf{n}=(\mathbf{n}_x,\mathbf{n}_y)^{\mathrm{T}}$ (resp. $\mathbf{t}=(-\mathbf{n}_y,\mathbf{n}_x)^{\mathrm{T}}$) is the outward unit normal to the boundary $\partial\Omega$ (resp. $\mathbf{t}$ is the anti-clockwise unit tangent to $\partial\Omega$). The vectors $\mathbf{n}$ and $\mathbf{t}$ are defined everywhere on $\partial\Omega$, except at a finite number of corner points of $\partial\Omega$.

We give now a sense to the operators of normal and tangential derivatives [10], which will allow us to define the weak formulations of subsequent problems:

(i) We define the operator of normal derivative acting on u as an element $u_{\mathbf{n}}$ of $H^{-\frac{1}{2}}(\partial\Omega)$ satisfying the following: if $u\in H^1(\Omega)$ and $f\in L^2(\Omega)$ are such that

$$-\Delta u=f \quad \text{in } \Omega \tag{3}$$

then,

$$\langle u_{\mathbf{n}},v\rangle_{H^{-\frac{1}{2}}(\partial\Omega),H^{\frac{1}{2}}(\partial\Omega)}=\int_\Omega \nabla u(\boldsymbol{x})\cdot\nabla\tilde{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}-\int_\Omega f(\boldsymbol{x})\tilde{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad \forall v\in H^{\frac{1}{2}}(\partial\Omega), \tag{4}$$

where $\tilde{v}$ is an element of $H^1(\Omega)$ such that $\tilde{\gamma}(\tilde{v})=v$ and $\tilde{\gamma}$ is the classical trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$.

(ii) We define the operator of tangential derivative acting on u as an element $u_{\mathbf{t}}$ of $H^{-\frac{1}{2}}(\partial\Omega)$ satisfying the following: if $u\in H^1(\Omega)$,

$$\langle u_{\mathbf{t}},v\rangle_{H^{-\frac{1}{2}}(\partial\Omega),H^{\frac{1}{2}}(\partial\Omega)}=\int_\Omega \tilde{v}_x(\boldsymbol{x})u_y(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}-\int_\Omega u_x(\boldsymbol{x})\tilde{v}_y(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad \forall v\in H^{\frac{1}{2}}(\partial\Omega). \tag{5}$$

It is easy to justify that the operators of normal and tangential derivatives are well-defined.

3. Mathematical formulation

Let Γ_0 and Γ_1 be two open disjoint subsets of $\partial\Omega$ such that $\overline{\Gamma}_0 \cup \overline{\Gamma}_1 = \partial\Omega$ and they are also the union of finite number of linear segments.

Assume also that:

(A_1) $f \in L^2(\Omega)$, $g \in L^2(\Gamma_0)$;
(A_2) $\kappa \in L^\infty(\Omega)$ and for some known positive constant κ_0, $0 < \kappa_0 \le \kappa(\boldsymbol{x})$ *a.e.* $\boldsymbol{x} \in \Omega$;
(A_3) $\beta \in L^\infty(\Gamma_0)$ and for some known positive constant β_0, $0 < \beta_0 \le \beta(\boldsymbol{x})$ *a.e.* $\boldsymbol{x} \in \Gamma_0$;
(A_4) $\alpha \in C^1(\overline{\Omega})$.

We are interested in the following boundary value problem: given functions α, β, f, κ and g satisfying the above conditions solve

$$-\nabla \cdot (\kappa(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{6}$$

with the generalized oblique derivative boundary condition

$$\kappa(\boldsymbol{x})u_{\mathbf{n}}(\boldsymbol{x}) + (\alpha u)_{\mathbf{t}}(\boldsymbol{x}) + \beta(\boldsymbol{x})u(\boldsymbol{x}) = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_0, \tag{7}$$

and the Dirichlet

$$u(\boldsymbol{x}) = h(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_1, \tag{8}$$

or Neumann

$$\kappa(\boldsymbol{x})u_{\mathbf{n}}(\boldsymbol{x}) = q(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_1, \tag{9}$$

boundary conditions, where $h \in H^{1/2}(\partial\Omega)$ and $q \in L^2(\Gamma_1)$ are also given functions.

Remark 3.1. (a) The boundary condition (7) is equivalent to the more common boundary condition

$$\kappa(\boldsymbol{x})u_{\mathbf{n}}(\boldsymbol{x}) + \alpha(\boldsymbol{x})u_{\mathbf{t}}(\boldsymbol{x}) + \overline{\beta}(\boldsymbol{x})u(\boldsymbol{x}) = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_0, \tag{10}$$

where $\overline{\beta} = \beta + \alpha_{\mathbf{t}}$.

(b) Mixed boundary conditions are necessary to be formulated since they arise in the analysis of Cauchy ill-posed problems of the type considered in Section 8.
(c) The above assumptions $(A_1)-(A_4)$ on the input data are needed to get convenient weak formulations for the direct problem (6)–(8) or (6), (7), (9). However, these hypotheses can be weakened. It is worth mentioning that hypothesis (A_4) above or $(A_4)'$ in Section 4.3 below, on α, are needed in order to get a rigorous formulation for the tangential operator involved in equation (7), see, for instance, similar hypotheses in [30], Page 226. Later, we will assume hypotheses which are stronger than those above in order to define suitable finite volume schemes and provide their error estimates. However, the above assumptions can serve us to define convenient finite volume schemes with possible proof for the convergence (without convergence rates) towards the true solution.

4. Existence, uniqueness, and well-posedness of the continuous problems

In this section, we address the existence, uniqueness, and well-posedness of the following two problems:

(1) **First problem.** The elliptic equation (6) with the generalized oblique derivative boundary condition (7) on Γ_0 and the Dirichlet boundary condition (8) on Γ_1.
(2) **Second problem.** The elliptic equation (6) with the generalized oblique derivative boundary condition (7) on Γ_0 and the Neumann boundary condition (9) on Γ_1.

To get a clear and a simple overview, we first consider the existence, uniqueness, and well-posedness of some particular cases of these two problems.

4.1. A simplified version of the problem: Robin convective boundary condition

In case $\Gamma_0 = \emptyset$ then $\Gamma_1 = \partial\Omega$, and therefore equation (6) and (8) or (9) form a classical Dirichlet or Neumann problem, which we do not investigate herein. On the other hand, in the case $\Gamma_1 = \emptyset$, the boundary condition (7) becomes

$$\kappa(\boldsymbol{x})u_{\mathbf{n}}(\boldsymbol{x}) + (\alpha u)_{\mathbf{t}}(\boldsymbol{x}) + \beta(\boldsymbol{x})u(\boldsymbol{x}) = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega. \tag{11}$$

Furthermore, when $\alpha \equiv 0$, the boundary condition (11) becomes the Robin boundary condition

$$\kappa(\boldsymbol{x})u_{\mathbf{n}}(\boldsymbol{x}) + \beta(\boldsymbol{x})u(\boldsymbol{x}) = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega. \tag{12}$$

In such case, the well-posedness of the Robin direct problem for the elliptic equation (6) is given by the following theorem (see [45], (6.1.15) and (6.1.16), Page 162 and [45], (6.1.23), Page 165), whose proof is given for the convenience of the reader.

Theorem 4.1 (Well-posedness of the problem (6) and (12), cf. [45])**.** *Let $\Gamma_1 = \emptyset$ and assumptions (A_1)–(A_3) be held. Then, the elliptic equation (6) with the Robin boundary condition (12) (the particular case of (11) when $\alpha \equiv 0$) has a unique solution in the following weak sense: there exists a unique $u \in H^1(\Omega)$ such that*

$$a(u, v) = b(v), \quad \forall v \in H^1(\Omega), \tag{13}$$

where

$$a(u, v) = \int_\Omega \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \beta(\boldsymbol{x})u(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})$$

and

$$b(v) = \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} g(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}),$$

where $\mathrm{d}\gamma$ denotes the one-dimensional Lebesgue measure on $\partial\Omega$. In addition to this, this weak solution satisfies the following stability estimate:

$$\|u\|_{H^1(\Omega)} \le \frac{C_1}{\min\{\kappa_0, \beta_0\}}\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\big), \tag{14}$$

for some positive constant C_1 that depends only on the geometry of Ω.

Proof. The bilinear form $a(\cdot,\cdot)$ is coercive since

$$\begin{aligned} a(v, v) &= \int_\Omega \kappa(\boldsymbol{x})|\nabla v(\boldsymbol{x})|^2\,\mathrm{d}x + \int_{\partial\Omega} \beta(\boldsymbol{x})v^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) \\ &\ge \min\{\kappa_0, \beta_0\}\Big(|v|_{1,\Omega}^2 + \|\tilde{\gamma}(v)\|_{L^2(\partial\Omega)}^2\Big), \end{aligned} \tag{15}$$

where $|v|_{1,\Omega}$ is the H^1-seminorm defined by $\big(\int_\Omega |\nabla v|^2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\big)^{\frac{1}{2}}$. Using the continuity of the trace (see, for instance, [45], (1.3.1), Page 10 and [22], Lemma 10.4, Page 69) implies that the norm $v \mapsto |v|_{1,\Omega} + \|\tilde{\gamma}(v)\|_{L^2(\partial\Omega)}$ is equivalent to the usual norm $\|\cdot\|_{1,\Omega}$ of $H^1(\Omega)$, that is $\|v\|_{1,\Omega} = (\|v\|_{L^2(\Omega)}^2 + |v|_{1,\Omega}^2)^{\frac{1}{2}}$. Hence, for some constant C_2 which is only depending on Ω, we have, for all $v \in H^1(\Omega)$

$$C_2\big(|v|_{1,\Omega} + \|\tilde{\gamma}(v)\|_{L^2(\partial\Omega)}\big) \le \|v\|_{1,\Omega} \le (C_2)^{-1}\big(|v|_{1,\Omega} + \|\tilde{\gamma}(v)\|_{L^2(\partial\Omega)}\big). \tag{16}$$

This together with (15) imply that

$$a(v, v) \ge \frac{(C_2)^2 \min\{\kappa_0, \beta_0\}}{2}\|v\|_{1,\Omega}^2. \tag{17}$$

Therefore, from the Lax–Milgram lemma there exists a unique solution to the equation (13).

To establish the stability result (14), we take $v = u$ in (13), and use (16) and (17) together with the Cauchy–Schwarz inequality to get

$$\frac{(C_2)^2 \min\{\kappa_0, \beta_0\}}{2} \|u\|_{1,\Omega}^2 \leq |b(u)| \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \|\tilde{\gamma}(u)\|_{L^2(\partial\Omega)}$$
$$\leq \|f\|_{L^2(\Omega)} \|u\|_{1,\Omega} + \frac{1}{C_2} \|g\|_{L^2(\partial\Omega)} \|u\|_{1,\Omega} \leq C_3 \big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\big) \|u\|_{1,\Omega},$$

with $C_3 = \max\{1, C_2^{-1}\}$. Consequently, (14) holds with $C_1 = 2C_3(C_2)^{-2}$. This completes the proof of Theorem 4.1. ☐

4.2. Existence and uniqueness of solution for problem (6) and (7) with Dirichlet boundary condition (8)

Consider the subspace of $H^1(\Omega)$ defined by

$$\mathcal{V} := \Big\{ v \in H^1(\Omega); \tilde{\gamma}(v)\big|_{\Gamma_1} = 0 \Big\},$$

which is a Hilbert space endowed with the usual inner product of $H^1(\Omega)$. Multiplying both sides of (6) by a test function $v \in \mathcal{V}$ and integrating over Ω yield

$$\int_\Omega \kappa(\boldsymbol{x}) \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) \varphi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_0} \kappa(\boldsymbol{x}) u_{\mathbf{n}}(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x}). \tag{18}$$

Using now (7) in (18) imply that

$$\int_\Omega \kappa(\boldsymbol{x}) \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_0} g(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x})$$
$$- \int_{\Gamma_0} (\alpha u)_{\mathbf{t}}(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x}) - \int_{\Gamma_0} \beta(\boldsymbol{x}) \tilde{\gamma}(u)(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x}). \tag{19}$$

Under the assumption (A_4), using the definition of the tangential derivative and that $\tilde{\gamma}(v)\big|_{\Gamma_1} = 0$, we have

$$\int_{\Gamma_0} (\alpha u)_{\mathbf{t}}(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x}) = \int_{\partial\Omega} (\alpha u)_{\mathbf{t}}(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x}) = \int_\Omega \{(\alpha u)_y(\boldsymbol{x}) v_x(\boldsymbol{x}) - (\alpha u)_x(\boldsymbol{x}) v_y(\boldsymbol{x})\}\, \mathrm{d}\boldsymbol{x}. \tag{20}$$

Gathering equations (19) and (20) lead to the following formulation:

$$a(u, v) = b(v), \quad \forall v \in \mathcal{V}, \tag{21}$$

where

$$a(u, v) = \int_\Omega \kappa(\boldsymbol{x}) \nabla u(\boldsymbol{x}) \nabla v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_\Omega \{(\alpha u)_y(\boldsymbol{x}) v_x(\boldsymbol{x}) - (\alpha u)_x(\boldsymbol{x}) v_y(\boldsymbol{x})\}\, \mathrm{d}\boldsymbol{x}$$
$$+ \int_{\Gamma_0} \beta(\boldsymbol{x}) \tilde{\gamma}(u)(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x}) \tag{22}$$

and

$$b(v) = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_0} g(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x}). \tag{23}$$

4.2.1. The particular case $h \equiv 0$ in (8)

In the particular case $h = 0$, the boundary condition (8) becomes

$$u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Gamma_1. \tag{24}$$

Then, the weak formulation of the problem (6), (7) and (24) is: Find $u \in \mathcal{V}$ such that (21) holds.

The assumptions (A_1)–(A_4) are needed to first give a sense to the terms involved in (22) and (23) for all $v \in \mathcal{V}$ and also to prove the existence and uniqueness for problem (21) using the Lax–Milgram lemma. These assumptions, which are similar, for instance, to Assumption 9.1, Pages 32, 33 of [22], and Assumption 11.1, Pages 78, 79 of [22], possibly can be weakened. Let us now check that the hypotheses of the Lax–Milgram lemma are satisfied in order to prove that the problem (21) has a unique solution.

(i) **Coercivity of a given by (22).** In order to prove this (under an additional assumption on $\alpha_{\mathbf{t}}$), we consider $u \in \mathcal{V}$ and use assumptions (A_1)–(A_4) to obtain:

$$\begin{aligned} a(u,u) &= \int_\Omega \kappa(\boldsymbol{x})|\nabla u|^2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_\Omega \{(\alpha u)_y(\boldsymbol{x})u_x(\boldsymbol{x}) - (\alpha u)_x(\boldsymbol{x})u_y(\boldsymbol{x})\}\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_0} \beta(\boldsymbol{x})(\tilde\gamma(u)(\boldsymbol{x}))^2\,\mathrm{d}\gamma(\boldsymbol{x}) \\ &= \int_\Omega \kappa(\boldsymbol{x})|\nabla u|^2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega \{\alpha_y(\boldsymbol{x})(u^2)_x(\boldsymbol{x}) - \alpha_x(\boldsymbol{x})(u^2)_y(\boldsymbol{x})\}\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_0} \beta(\boldsymbol{x})(\tilde\gamma(u)(\boldsymbol{x}))^2\,\mathrm{d}\gamma(\boldsymbol{x}) \\ &= \int_\Omega \kappa(\boldsymbol{x})|\nabla u|^2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\partial\Omega} \alpha_{\mathbf{t}}(\boldsymbol{x})(\tilde\gamma(u)(\boldsymbol{x}))^2\,\mathrm{d}\gamma(\boldsymbol{x}) + \int_{\Gamma_0} \beta(\boldsymbol{x})(\tilde\gamma(u)(\boldsymbol{x}))^2\,\mathrm{d}\gamma(\boldsymbol{x}) \\ &\geq \kappa_0 \int_\Omega |\nabla u|^2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\partial\Omega} \alpha_{\mathbf{t}}(\boldsymbol{x})(\tilde\gamma(u)(\boldsymbol{x}))^2\,\mathrm{d}\gamma(\boldsymbol{x}) + \beta_0\|\tilde\gamma(u)\|^2_{L^2(\Gamma_0)}. \end{aligned} \tag{25}$$

Using this expression together with the continuity of the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$ and denoting $C_\alpha := \min_{\partial\Omega} \alpha_{\mathbf{t}}$ (note that $C_\alpha \leq 0$ since the mean value of $\alpha_{\mathbf{t}}$ on $\partial\Omega$ is 0) imply

$$a(u,u) \geq \kappa_0|u|^2_{1,\Omega} + \frac{C_4 C_\alpha}{2}\|u\|^2_{1,\Omega} + \beta_0\|\tilde\gamma(u)\|^2_{L^2(\Gamma_0)},$$

where, thanks to (16), $C_4 = 1/(C_2)^2$. Further, on using the Poincaré inequality (see, for instance, [45], Thm. 1.3.3, Page 11),

$$\|u\|_{1,\Omega} \leq C_5|u|_{1,\Omega} \leq C_5\big(|u|_{1,\Omega} + \|\tilde\gamma(u)\|_{L^2(\Gamma_0)}\big), \quad \forall u \in \mathcal{V} \tag{26}$$

results in, thanks to the inequality $(a^2+b^2) \geq (a+b)^2/2$,

$$\begin{aligned} a(u,u) &\geq \min\{\kappa_0,\beta_0\}\Big(|u|^2_{1,\Omega} + \|\tilde\gamma(u)\|^2_{L^2(\Gamma_0)}\Big) + \frac{C_4 C_\alpha}{2}\|u\|^2_{1,\Omega} \\ &\geq \left(\frac{\min\{\kappa_0,\beta_0\}}{2(C_5)^2} + \frac{C_4 C_\alpha}{2}\right)\|u\|^2_{1,\Omega}. \end{aligned}$$

Thus, if

$$\min_{\partial\Omega} \alpha_{\mathbf{t}} =: C_\alpha > -\frac{\min\{\kappa_0,\beta_0\}}{(C_5)^2 C_4} \tag{27}$$

holds, then the coercivity of a is satisfied.

(ii) **Continuity of $a(\cdot,\cdot)$, given by (22), on $\mathcal{V}\times\mathcal{V}$.** The continuity of $a(\cdot,\cdot)$ on $\mathcal{V}\times\mathcal{V}$ stems from the Cauchy–Schwarz inequality, the facts that $\kappa \in L^\infty(\Omega)$, $\alpha \in \mathcal{C}^1(\overline{\Omega})$, $\beta \in L^\infty(\Gamma_0)$, and the continuity of the trace operator.

(iii) **Continuity of $b(\cdot)$, given by (23), on $\mathcal{V}$.** The continuity of $b(\cdot)$ on $\mathcal{V}$ stems from the Cauchy–Schwarz inequality, the facts that $f \in L^2(\Omega)$, $g \in L^2(\Gamma_0)$, and the continuity of the trace operator. In addition, we have

$$|b(v)| \leq C_6\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_0)}\big)\|v\|_{1,\Omega}, \tag{28}$$

where, thanks to (16), $C_6 = \max\{1, C_2^{-1}\}$.

At this stage, we have justified that the bi-linear form $a(\cdot,\cdot)$ and the linear form $b(\cdot)$, given by (22) and (23), satisfy the hypotheses of the Lax–Milgram lemma. This implies that the problem (21) has a unique solution. In addition, by taking $v = u$ in (21) and using (28), we obtain the following stability estimate:

$$\|u\|_{1,\Omega} \leq C_7\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_0)}\big). \tag{29}$$

Summarizing the results of this subsection, we have proved the following theorem stating, *via* (21), the well-posedness of the problem (6), (7) and (24).

Theorem 4.2 (Well-posedness for the homogeneous Dirichlet problem)**.** *Let assumptions (A_1)–(A_4) be satisfied. Then, there exists $\delta < 0$, only depending on $\Omega, \Gamma_0, \beta_0$, and κ_0 (see also Remark 4.5 below), such that if α satisfies the condition $C_\alpha := \min_{\partial\Omega} \alpha_{\boldsymbol{t}} \geq \delta$, then there exists a unique solution $u \in \mathcal{V}$ to the problem* (6)*,* (7) *and* (24) *in the sense of the weak formulation* (21)*. In addition to this, the stability estimate* (29) *holds.*

4.2.2. The case $h \not\equiv 0$ in (8)

Let us consider the same problem (6) and (7) but with the non-homogeneous Dirichlet boundary condition (8) with $h \in H^{1/2}(\partial\Omega)$ instead of the homogeneous condition (24). To obtain the weak formulation of the problem (6)–(8), we consider the subset $\mathcal{V}^E$ of $H^1(\Omega)$ defined by

$$\mathcal{V}^E := \Big\{v \in H^1(\Omega); \tilde{\gamma}(v)\big|_{\Gamma_1} = h\Big\},$$

Following the same steps as in (18)–(23), we obtain that any solution to (6)–(8) should satisfy $u \in \mathcal{V}^E$ and also the equation (21). The set $\mathcal{V}^E$ is not a subspace of $H^1(\Omega)$ (when $h \not\equiv 0$) and this does not allow to apply the Lax–Milgram lemma directly. For this reason, we consider a function $\overline{h} \in H^1(\Omega)$ such that $\tilde{\gamma}(\overline{h})(\boldsymbol{x}) = h(\boldsymbol{x})$ a.e. $\boldsymbol{x} \in \Gamma_1$, and define $\omega := u - \overline{h}$ (called the Dirichlet shift). Inserting u by its value $\omega + \overline{h}$ in (21) yields

$$a(\omega, v) = \overline{b}(v), \quad \forall v \in \mathcal{V}, \tag{30}$$

where

$$\overline{b}(v) = b(v) - a(\overline{h}, v). \tag{31}$$

Then, the weak formulation of problem (6)–(8) is as follows: Find $u \in H^1(\Omega)$ such that $u = \omega + \overline{h}$ and $\omega \in \mathcal{V}$ is satisfying (30) and (31). The bilinear form $a(\cdot,\cdot)$ has already been proved to be continuous on $\mathcal{V} \times \mathcal{V}$ and coercive on $\mathcal{V}$ (under the condition (27)). The continuity of the linear form $\overline{b}$, given by (31), follows from the continuity of b and a which have already been proven. Consequently, using the Lax–Milgram lemma, the weak formulation (30) and (31) has a unique solution $\omega \in \mathcal{V}$. To obtain the unique solvability of the original problem (6)–(8) we can choose $\overline{h} \in H^1(\Omega)$ to be the unique solution of the Dirichlet problem

$$\begin{cases} -\nabla \cdot \big(\kappa(\boldsymbol{x})\nabla\overline{h}\big) = 0 & \text{in } \Omega, \\ \overline{h}|_{\partial\Omega} = h. \end{cases}$$

Also, the stability estimate (29) extends to

$$\|u\|_{1,\Omega} \leq C_8\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_0)} + \|h\|_{H^{1/2}(\partial\Omega)}\big), \tag{32}$$

where

$$\|h\|_{H^{1/2}(\partial\Omega)} = \inf_{\overline{h}\in H^1(\Omega):\tilde{\gamma}(\overline{h})=h} \|\overline{h}\|_{1,\Omega}.$$

We summarize now the results of this subsection in the following theorem which extends Theorem 4.2 to the non-homogeneous case.

Theorem 4.3 (Well-posedness for the non-homogeneous Dirichlet problem)**.** *Let assumptions $(A_1)-(A_4)$ be satisfied. Then, there exists $\delta < 0$, only depending on $\Omega, \Gamma_0, \beta_0$, and κ_0 (see also Remark 4.5 below), such that if α satisfies the condition $C_\alpha \geq \delta$, then there exists a unique solution $u \in H^1(\Omega)$ to the problem (6)–(8) in the sense that there exists a unique $\omega \in \mathcal{V}$ satisfying the weak formulation (30)–(31), and $u = \omega + \overline{h}$ with $\overline{h} \in H^1(\Omega)$ such that $\tilde{\gamma}(\overline{h}) = h$. In addition to this, the stability estimate (32) holds.*

4.3. Existence and uniqueness of solution for problem (6) and (7) with Neumann boundary condition (9)

Consider now the problem (6), (7) subject to the Neumann boundary condition (9). In this case we replace the assumption (A_4) by the stronger assumption:

$(A_4)'$ $\alpha \in C^1(\overline{\Omega})$ and $\alpha|_{\Gamma_1} = 0$.

As in the previous subsection, multiplying both sides of (6) by a test function $v \in H^1(\Omega)$ and using (7), (9) and $(A_4)'$, one obtains

$$\begin{aligned}\int_\Omega \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x})\cdot\nabla v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_0}\beta(\boldsymbol{x})\tilde{\gamma}(u)(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})\\ = \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_0} g(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) - \int_{\partial\Omega}(\alpha u)_{\mathbf{t}}(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) + \int_{\Gamma_1} q(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}).\end{aligned} \tag{33}$$

We then have, using the definition of the tangential derivative in equation (5)

$$\int_{\partial\Omega}(\alpha u)_{\mathbf{t}}(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) = \int_\Omega \{(\alpha u)_y(\boldsymbol{x})v_x(\boldsymbol{x}) - (\alpha u)_x(\boldsymbol{x})v_y(\boldsymbol{x})\}\,\mathrm{d}\boldsymbol{x}$$

and equation (33) yields (21), where a is given by (22) and

$$b(v) = \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_0} g(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) + \int_{\Gamma_1} q(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}). \tag{34}$$

The weak formulation of (6), (7) and (9) is: Find $u \in H^1(\Omega)$ such that

$$a(u,v) = b(v), \quad \forall v \in H^1(\Omega), \tag{35}$$

where $a(\cdot,\cdot)$ and $b(\cdot)$ are, respectively, given by (22) and (34). We remark that the new expression of the linear form $b(\cdot)$ defined by (34) contains the additional term $\int_{\Gamma_1} q(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})$ compared to the expression (23).

As before, we check that the conditions of Lax–Milgram lemma are satisfied. The continuity of a and b stems from the Cauchy Schwarz inequality, (A_1)–(A_3) and $(A_4)'$, and the continuity of the trace operator (see (16)). In addition, the following estimate (which extends (28)) on $b(\cdot)$ holds:

$$|b(v)| \leq C_6(\Omega)\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_0)} + \|q\|_{L^2(\Gamma_1)}\big)\|v\|_{1,\Omega}. \tag{36}$$

The coercivity of $a(\cdot,\cdot)$ can be checked by following the same steps used in (25)–(27) (or the idea of Remark 4.5 below) but instead of estimate (26) (which stems from the Poincaré inequality and it cannot be applied since the space of u here is $H^1(\Omega)$), we use the following inequality (cf. [21]):

$$\|u\|_{1,\Omega} \leq C_9\big(|u|_{1,\Omega} + \|\tilde{\gamma}(u)\|_{L^2(\Gamma_0)}\big), \quad \forall u \in H^1(\Omega). \tag{37}$$

Estimate (37) can be deduced from the Extended Poincaré–Steklov inequality ([21], Lemma B.63, Page 490) and using the Cauchy–Schwarz inequality. A discrete version of (16) and (37) will be given later in Lemma 6.3. We have thus justified that the bi-linear form a and the linear form b, given by (22) and (34), satisfy the hypotheses of Lax–Milgram lemma. This implies that the problem (35) has a unique solution. In addition, by taking $v = u$ in (35) and using (36) together with the coercivity of $a(\cdot,\cdot)$, we obtain the following stability estimate:

$$\|u\|_{1,\Omega} \le C_{10}\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_0)} + \|q\|_{L^2(\Gamma_1)}\big). \tag{38}$$

We have thus proved the following theorem.

Theorem 4.4 (Well-posedness for problem (6), (7) and (9))**.** *Let assumptions (A_1)–(A_3) and $(A_4)'$ be satisfied. Then, there exists $\delta < 0$, only depending on $\Omega, \Gamma_0, \beta_0$, and κ_0 (see also Remark 4.5 below) such that if α satisfies the condition $C_\alpha \ge \delta$, then there exists a unique solution to the problem (6), (7) and (9) in the sense of the weak formulation (35), where $a(\cdot,\cdot)$ and $b(\cdot)$ are given, respectively, by (22) and (34). In addition to this, the stability estimate (38) holds.*

Remark 4.5 (Another way of justifying the coercivity and an explicit value for δ)**.** The coercivity proved of $a(\cdot,\cdot)$ can also be proved in another way, as explained below.

(i) The coercivity for the problem (6)–(8) in Section 4.2 (see (25)–(27)) can also be proved using (37) (instead of the use of (26)) since (37) is stronger than (26) and, in particular, it yields (26).

(ii) The coercivity for both problems (6)–(8) (in Sect. 4.2) and (6), (7) and (9) (above) can also be proved in a slightly different way. Indeed, from (25), we can deduce that

$$a(u,u) \ge \kappa_0 |u|^2_{1,\Omega} + \left(\frac{C_\alpha}{2} + \beta_0\right)\|\tilde\gamma(u)\|^2_{L^2(\Gamma_0)}.$$

So, if $C_\alpha \ge -\beta_0$, we have, using inequalities $2(a^2+b^2) \ge (a+b)^2$ and (37) (which generalizes (26)),

$$a(u,u) > \min\{\kappa_0, \beta_0/2\}\Big(|u|^2_{1,\Omega} + \|\tilde\gamma(u)\|^2_{L^2(\Gamma_0)}\Big) \ge \frac{\min\{\kappa_0,\beta_0/2\}}{2(C_9)^2}\|u\|^2_{1,\Omega}.$$

The advantage of this reasoning is that the lower bound δ, stated in Theorems 4.2–4.4, can explicitly be given by $-\beta_0$.

5. Definition of a finite volume mesh: admissible meshes

To establish suitable finite volume schemes for solving the above problems and prove their convergence, we use the following integration rule (see [10]): Let $\mathbf{a}$ and $\mathbf{b}$ be two points in $\mathbb{R}^2$ and $(\mathbf{a},\mathbf{b}) = \{s\mathbf{a} + (1-s)\mathbf{b}, s \in (0,1)\}$. Let $\xi \in \mathcal{C}^1(\mathbb{R}^2)$ and $\mathbf{t} = \frac{\mathbf{b}-\mathbf{a}}{|\mathbf{b}-\mathbf{a}|}$. Let $\xi_{\mathbf{t}} = \nabla\xi\cdot\mathbf{t}$. Then:

$$\int_{(\mathbf{a},\mathbf{b})} \xi_{\mathbf{t}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) = \xi(\mathbf{b}) - \xi(\mathbf{a}). \tag{39}$$

We now describe the assumptions which are needed on the mesh.

Definition 5.1 (Admissible meshes, cf. [23])**.** An admissible finite volume mesh of Ω, denoted by $\mathcal{T}$, is a finite family of open polygonal convex disjoint subsets of Ω (the "control volumes"), with positive measures. To this family we associate a family of disjoint subsets of $\overline{\Omega}$ contained in hyperplanes of $\mathbb{R}^2$, denoted by $\mathcal{E}$ (these are the edges of the control volumes), and a family of points $\mathcal{P} = \{\boldsymbol{x}_K \mid K \in \mathcal{T}\}$ in Ω, satisfying the following properties (as in [23], Definition 9.1):

– $\overline{\Omega} = \cup_{K\in\mathcal{T}}\overline{K}$.

– For all $\sigma \in \mathcal{E}$, there exists a hyperplane $\sigma \subset E \subset \mathbb{R}^2$ and $K \in \mathcal{T}$ such that $\overline{\sigma} = \partial K \cap E$. We denote by $\mathrm{m}(\sigma)$ the one-dimensional measure of σ and assume $\mathrm{m}(\sigma) > 0$. We assume that, for all $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. It then results that, for all $\sigma \in \mathcal{E}$, either $\sigma \subset \partial\Omega$ or there exists $(K, L) \in \mathcal{T} \times \mathcal{T}$ with $K \neq L$ such that $\overline{K} \cap \overline{L} = \overline{\sigma}$; in the latter case we denote $\sigma = K|L$.
– For all $K \in \mathcal{T}$, $\boldsymbol{x}_K \in K$. Furthermore, for all $\sigma \in \mathcal{E}$ such that there exists $(K, L) \in \mathcal{T} \times \mathcal{T}$ with $\sigma = K|L$, it is assumed that the straight line $(\boldsymbol{x}_K, \boldsymbol{x}_L)$ going through $\boldsymbol{x}_K$ and $\boldsymbol{x}_L$ is orthogonal to $K|L$. For $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, let $\mathcal{D}_{K,\sigma}$ be the straight line going through $\boldsymbol{x}_K$ and orthogonal to σ. We assume that $\mathcal{D}_{K,\sigma} \cap \sigma \neq \emptyset$ and we set $\{\boldsymbol{y}_\sigma\} = \mathcal{D}_{K,\sigma} \cap \sigma$.

If $\mathcal{T}$ is an admissible mesh, we will also use the following notations:

– The mesh size is defined by $\mathrm{size}(\mathcal{T}) = \sup\{\mathrm{diam}(K) \mid K \in \mathcal{T}\}$, and for any $K \in \mathcal{T}$ denote by $\mathrm{m}(K)$ the two-dimensional Lebesgue measure of K (it is the area of K).
– The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$), $\mathcal{E}_{\mathrm{int}} = \{\sigma \in \mathcal{E} \mid \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\mathrm{ext}} = \{\sigma \in \mathcal{E} \mid \sigma \subset \partial\Omega\}$).
– The set of neighbours of K is denoted by $\mathcal{N}(K) = \{L \in \mathcal{T} \mid \exists\sigma \in \mathcal{E}_K, \overline{\sigma} = \overline{K} \cap \overline{L}\}$.
– If $\sigma = K|L$, we denote by d_σ or $d_{K|L}$ the Euclidean distance between $\boldsymbol{x}_K$ and $\boldsymbol{x}_L$ (which is positive) and $d_{K,\sigma}$ the distance from $\boldsymbol{x}_K$ to σ.
– If $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}$, let d_σ denote the Euclidean distance between $\boldsymbol{x}_K$ and $\boldsymbol{y}_\sigma$ (then, $d_\sigma = d_{K,\sigma}$).
– For any $\sigma \in \mathcal{E}$, the "transmissibility" through σ is defined by $\tau_\sigma = \frac{\mathrm{m}(\sigma)}{d_\sigma}$ (note that $d_\sigma > 0$).

In addition, in order to take care of both the generalized oblique boundary condition (7) and the Dirichlet or Neumann boundary condition (8) or (9), we assume the following assumption on the mesh $\mathcal{T}$.

Assumption 5.2 (Assumption on the mesh $\mathcal{T}$)**.** *Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 5.1. For any $\sigma \in \mathcal{E}_{\mathrm{ext}}$, we assume that either $\sigma \subset \Gamma_0$ or $\sigma \subset \Gamma_1$. We then define the following two subsets of $\mathcal{E}_{\mathrm{ext}}$:*

$$\Gamma_0^{\mathcal{T}} = \{\sigma \in \mathcal{E}_{\mathrm{ext}} \mid \sigma \subset \Gamma_0\} \quad \text{and} \quad \Gamma_1^{\mathcal{T}} = \{\sigma \in \mathcal{E}_{\mathrm{ext}} \mid \sigma \subset \Gamma_1\}. \tag{40}$$

To discretize the governing equation (6) and the oblique boundary condition (7), we need the following definitions.

Definition 5.3. Let $\mathcal{E}_{\mathrm{ext}} \ni \sigma = (\mathbf{a}, \mathbf{b}) = \{s\mathbf{a} + (1 - s)\mathbf{b}, s \in (0, 1)\}$. Denote by σ^- (resp. σ^+) the element of $\mathcal{E}_{\mathrm{ext}}$ such that $\mathbf{a}$ is in the closure of σ^- (resp. $\mathbf{b}$ is in the closure of σ^+) and $\sigma^- \neq \sigma$ (resp. $\sigma^+ \neq \sigma$). We also set $\sigma_e = \mathbf{b}$ and $\sigma_b = \mathbf{a}$ (so that $|\sigma_e - \sigma_b|\mathbf{t} = \sigma_e - \sigma_b$).

Definition 5.4. Let $\alpha \in \mathcal{C}^1(\overline{\Omega})$. For $\sigma \in \mathcal{E}_{\mathrm{ext}}$, the notations σ_e, σ_b, u_{σ^+} and u_{σ^-} are given in Definition 5.3 (in particular, $\sigma = (\sigma_b, \sigma_e)$, with $\mathrm{m}(\sigma)\mathbf{t} = \sigma_e - \sigma_b$). We set:

$$\begin{aligned} u_{\sigma,+} = u_\sigma \text{ and } u_{\sigma,-} = u_{\sigma^+} \quad & \text{if } \alpha(\sigma_e) \geq 0, \\ u_{\sigma,+} = u_{\sigma^+} \text{ and } u_{\sigma,-} = u_\sigma \quad & \text{if } \alpha(\sigma_e) < 0. \end{aligned}$$

Definition 5.5 (The finite volume space)**.** For an admissible mesh $\mathcal{T}$, the space $\mathcal{X}(\mathcal{T})$ is defined by $\mathcal{X}(\mathcal{T}) = \mathcal{Y}(\mathcal{T}) \times \mathcal{Z}(\mathcal{T}) \subset L^2(\Omega) \times L^2(\partial\Omega)$, where $\mathcal{Y}(\mathcal{T})$ is the set of functions from Ω to $\mathbb{R}$, which are constant over each control volume $K \in \mathcal{T}$, and $\mathcal{Z}(\mathcal{T})$ is the set of functions which are constant on each $\sigma \in \mathcal{E}_{\mathrm{ext}}$. In the same manner, we define $\mathcal{Z}_0(\mathcal{T})$ (resp. $\mathcal{Z}_1(\mathcal{T})$) as the space of functions which are constant on each $\sigma \in \Gamma_0^{\mathcal{T}}$ (resp. $\sigma \in \Gamma_1^{\mathcal{T}}$).

Throughout the next sections, the letters C_i denote positive constants, which are independent of the parameters of discretization.

6. Formulation of a finite volume scheme for the problem (6) and (7) with the Dirichlet boundary condition (8) and its convergence analysis

Let us describe the principles of the scheme we want to present in order to approximate the problem (6)–(8). Integrating both sides of (6) over a control volume K, using integration by parts, and summing over the edges σ of the control volume K yield

$$-\sum_{\sigma\in\mathcal{E}_K}\int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x})\cdot\mathbf{n}_{K,\sigma}\,\mathrm{d}\gamma(\boldsymbol{x}) = \mathrm{m}(K)f_K, \tag{41}$$

where $\mathbf{n}_{K,\sigma}$ denotes the normal unit vector to σ outward to K and f_K is the mean value of f on K, *i.e.*

$$f_K = \frac{1}{\mathrm{m}(K)}\int_K f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{42}$$

The discrete set of the unknowns are denoted by $(u_K)_{K\in\mathcal{T}}$ and $(u_\sigma)_{\sigma\in\Gamma_0^{\mathcal{T}}}$. These unknowns are expected to approximate the exact solution u in the control volumes $K\in\mathcal{T}$ and on the edges $\sigma\in\Gamma_0^{\mathcal{T}}$. Note that, due to the Dirichlet boundary condition (8), u is known on Γ_1. We look now for equations satisfied by these discrete unknowns using a convenient approximation to (41) and the boundary conditions (7) and (8).

To this end, we consider the following cases:

(I) $\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{int}}$ with $\sigma = K|L$. In this case, $-\int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x})\cdot\mathbf{n}_{K,\sigma}\,\mathrm{d}\gamma(\boldsymbol{x})$ can be approximated using discrete fluxes which are consistent and conservative, and are given by the classical two–points formula (see [23], Page 818 or also [22], Page 80):

$$F_{K,\sigma} = -\tau_\sigma^\kappa(u_L - u_K), \tag{43}$$

where

$$\tau_\sigma^\kappa = \mathrm{m}(\sigma)\frac{\kappa_K\kappa_L}{d_{K,\sigma}\kappa_L + d_{L,\sigma}\kappa_K},\quad \kappa_K = \frac{1}{\mathrm{m}(K)}\int_K\kappa(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},\quad \kappa_L = \frac{1}{\mathrm{m}(L)}\int_L\kappa(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{44}$$

(II) $\sigma\in\mathcal{E}_K$ and $\sigma\subset\Gamma_1$. In this case, $-\int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x})\cdot\mathbf{n}_{K,\sigma}\,\mathrm{d}\gamma(\boldsymbol{x})$ can be approximated using (43) by taking $u_L = h(\boldsymbol{y}_\sigma)$ (which stems from the Dirichlet boundary conditions (8)), *i.e.*

$$F_{K,\sigma} := -\tau_\sigma^\kappa(h_\sigma - u_K)\quad\text{and}\quad h_\sigma = h(\boldsymbol{y}_\sigma), \tag{45}$$

with

$$\tau_\sigma^\kappa = \mathrm{m}(\sigma)\frac{\kappa_K}{d_{K,\sigma}}. \tag{46}$$

(III) $\sigma\in\mathcal{E}_K$ and $\sigma\subset\Gamma_0$. In this case, $-\int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x})\cdot\mathbf{n}_{K,\sigma}\,\mathrm{d}\gamma(\boldsymbol{x})$ can be expressed using the oblique boundary condition (7) as

$$-\int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x})\cdot\mathbf{n}_{K,\sigma}\,\mathrm{d}\gamma(\boldsymbol{x}) = \int_\sigma(\alpha u)_{\mathbf{t}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) + \int_\sigma\beta(\boldsymbol{x})u(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) - \mathrm{m}(\sigma)g_\sigma, \tag{47}$$

where g_σ is the mean value of g on σ, *i.e.*

$$g_\sigma = \frac{1}{\mathrm{m}(\sigma)}\int_\sigma g(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}). \tag{48}$$

Equation (47) can be approximated as follows [10]:

$$F_{K,\sigma} := -\tau_\sigma^\kappa(u_\sigma - u_K) = \alpha(\sigma_e)u_{\sigma,+} - \alpha(\sigma_b)u_{\sigma^-,+} + \mathrm{m}(\sigma)\beta_\sigma u_\sigma - \mathrm{m}(\sigma)g_\sigma, \tag{49}$$

where τ_σ^κ is defined by (46), β_σ is defined by (48) by replacing g with β, $u_{\sigma,+}$ is given in Definition 5.4, and $u_{\sigma^-,+}$ is obtained by replacing σ with σ^- and σ_e with σ_b in Definition 5.4. To take into account the Dirichlet boundary condition (8), we set in (49), for all $\sigma \in \mathcal{E}_{\text{ext}}$ such that $\sigma \subset \Gamma_0$,

$$u_{\sigma^+} = h(\boldsymbol{y}_\sigma) \quad \text{if} \ \ \sigma^+ \subset \Gamma_1 \quad \text{and} \quad u_{\sigma^-} = h(\boldsymbol{y}_\sigma) \quad \text{if} \ \ \sigma^- \subset \Gamma_1. \tag{50}$$

After having described the principles of the numerical scheme, we give now its formulation in the next subsection.

6.1. Definition of the scheme for the Dirichlet boundary condition (8) on $\boldsymbol{\Gamma_1}$

To summarise our finite volume scheme stated in (43)–(50) we introduce the following definition.

Definition 6.1 (Definition of a finite volume scheme for (6)–(8))**.** Let τ_σ^κ be defined in (44) and (46). Define the discrete fluxes (see (43), (45) and (49)),

$$F_{K,\sigma} = \begin{cases} -\tau_\sigma^\kappa (u_L - u_K), & \forall \sigma = K|L, \\ -\tau_\sigma^\kappa (h(\boldsymbol{y}_\sigma) - u_K), & \forall \sigma \in \mathcal{E}_K \text{ with } \sigma \subset \Gamma_1, \\ -\tau_\sigma^\kappa (u_\sigma - u_K), & \forall \sigma \in \mathcal{E}_K \text{ with } \sigma \subset \Gamma_0. \end{cases} \tag{51}$$

Let $(u_K)_{K\in\mathcal{T}}$ and $(u_\sigma)_{\sigma\in\Gamma_0^{\mathcal{T}}}$ denote the discrete set of unknowns. As a finite volume scheme, we suggest (see (41)):

$$\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} = \mathrm{m}(K) f_K, \quad \forall K \in \mathcal{T}, \tag{52}$$

and, (see (49)), for all $\sigma \in \mathcal{E}_K$ with $\sigma \subset \Gamma_0$,

$$-\tau_\sigma^\kappa (u_\sigma - u_K) = \alpha(\sigma_e) u_{\sigma,+} - \alpha(\sigma_b) u_{\sigma^-,+} + \mathrm{m}(\sigma)\beta_\sigma u_\sigma - \mathrm{m}(\sigma) g_\sigma, \tag{53}$$

where f_K and g_σ are given, respectively, by (42) and (48), and β_σ is given (48) by replacing g with β. Further, in order to take into account the Dirichlet boundary condition (8), we set in (53), for all $\sigma \in \mathcal{E}_{\text{ext}}$ with $\sigma \subset \Gamma_0$,

$$u_{\sigma^+} = h(\boldsymbol{y}_\sigma) \quad \text{if} \ \ \sigma^+ \subset \Gamma_1 \quad \text{and} \quad u_{\sigma^-} = h(\boldsymbol{y}_\sigma) \quad \text{if} \ \ \sigma^- \subset \Gamma_1. \tag{54}$$

6.2. Well-posedness of the discrete problem

Let us extend the definition of u_σ by

$$u_\sigma = \frac{\kappa_K d_{L,\sigma} u_K + \kappa_L d_{K,\sigma} u_L}{d_{K,\sigma}\kappa_L + d_{L,\sigma}\kappa_K} \quad \text{if } \sigma = K|L \in \mathcal{E}_{\text{int}}, \tag{55}$$

$$u_\sigma = h(\boldsymbol{y}_\sigma), \qquad \forall \sigma \in \Gamma_1^{\mathcal{T}}. \tag{56}$$

In this way, the discrete flux (51) can be written in a unified form as

$$F_{K,\sigma} = -\mathrm{m}(\sigma) \frac{\kappa_K}{d_{K,\sigma}} (u_\sigma - u_K) \tag{57}$$

for which we have the conservativity

$$F_{K,\sigma} = -F_{L,\sigma}, \quad \forall \sigma = K|L. \tag{58}$$

We prove the following discrete well-posedness result.

Theorem 6.2 (Well-posedness of scheme (52)–(54))**.** *In addition to the hypotheses of Theorem 4.3, which states the well-posedness of the weak solution of problem (6)–(8), assume that $h \in \mathcal{C}^1(\partial\Omega)$ and that there exists $\overline{h} \in \mathcal{C}^1(\overline{\Omega})$ such that $\tilde{\gamma}(\overline{h}) = h$ (see Remark 6.6 when h is not smooth). Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 5.1 satisfying Assumption 5.2. Then there exists a unique solution $(u_K)_{K\in\mathcal{T}}$ and $(u_\sigma)_{\sigma\in\Gamma_0^{\mathcal{T}}}$ to the finite volume scheme (52)–(54) of Definition 6.1. Corresponding to this unique solution, let $\mathcal{Y}(\mathcal{T}) \ni u_{\mathcal{T}}$ be defined such that its constant value over any $K \in \mathcal{T}$ is u_K (see also Definition 5.5). Also, let $\mathcal{Z}(\mathcal{T}) \ni v_{\mathcal{T}}$ be defined such that its constant value over any $\sigma \in \mathcal{E}_{\rm ext}$ is u_σ if $\sigma \in \Gamma_0^{\mathcal{T}}$ and $h(\boldsymbol{y}_\sigma)$ if $\sigma \in \Gamma_1^{\mathcal{T}}$. Then, the above discrete solution satisfies the following stability estimate:*

$$\begin{aligned}&\Big(|(u_{\mathcal{T}}, v_{\mathcal{T}})|^2_{1,\mathcal{X}(\mathcal{T})} + \|v_{\mathcal{T}}\|^2_{L^2(\Gamma_0)} + |v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}\Big)^{\frac{1}{2}}\\ &\quad\le C_{11}\Big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_0)} + \big(\|\beta\|_{L^\infty(\Gamma_0)} + 1\big)\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}\Big),\end{aligned} \tag{59}$$

where

$$|(u_{\mathcal{T}}, v_{\mathcal{T}})|^2_{1,\mathcal{X}(\mathcal{T})} := |u_{\mathcal{T}}|^2_{1,\mathcal{T}} + \sum_{\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}_K} \tau_\sigma (u_\sigma - u_K)^2 \tag{60}$$

with

$$|u_{\mathcal{T}}|^2_{1,\mathcal{T}} := \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} \tau_\sigma (u_L - u_K)^2, \tag{61}$$

and

$$|v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})} := \sum_{\sigma\in\mathcal{E}_{\rm ext}} |\alpha(\sigma_e)|(u_{\sigma^+} - u_\sigma)^2. \tag{62}$$

To prove Theorem 6.2, we need to use the following technical lemmas. Lemma 6.3 below is not explicitly stated in the existing literature but it can be deduced from Lemmas 10.3 and 10.5 of [22].

Lemma 6.3 (Equivalence of norms, see [22])**.** *Let Ω be a polygonal connected open bounded subset of $\mathbb{R}^d$ and $\mathcal{T}$ be an admissible mesh in the sense of Definition 5.1. Let $I \subset \partial\Omega$ be such that the $(d-1)$-Lebesgue measure of I is positive. Let us define the following norm on $\mathcal{Y}(\mathcal{T})$ (see Definition 5.5):*

$$\|u\|^2_{1,\mathcal{T}} = |u|^2_{1,\mathcal{T}} + \|u\|^2_{L^2(\Omega)}, \tag{63}$$

where $|u|^2_{1,\mathcal{T}}$ is given by (61). Then the norms $\|\cdot\|_{1,\mathcal{T}}$ and $|\cdot|_{1,\mathcal{T}} + \|\overline{\gamma}(\cdot)\|_{L^2(I)}$ are equivalent, with constants only depending on Ω and I, where $\overline{\gamma}(u)(\boldsymbol{x}) = u_K$ a.e. $\boldsymbol{x} \in \sigma$, for all $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\rm ext}$.

Proof. We consider the following steps.

First step: first side inequality. Using ([22], Lemmas 10.3 and 10.5) yields that for all $u \in \mathcal{Y}(\mathcal{T})$,

$$\|u - m_I(u)\|_{L^2(\Omega)} \le C_{12}|u|_{1,\mathcal{T}}, \tag{64}$$

where $m_I(u)$ is the mean value of u over I, that is, $m_I(u) = \frac{1}{\mathrm{m}(I)}\int_I u(\boldsymbol{x})\,\mathrm{d}\gamma(\mathbf{x})$. Using the inequality $|a| - |b| \le |a - b|$, inequality (64) implies that

$$\|u\|_{L^2(\Omega)} \le C_{12}|u|_{1,\mathcal{T}} + \sqrt{\mathrm{m}(\Omega)}|m_I(u)|. \tag{65}$$

Using the Cauchy–Schwarz inequality yields

$$|m_I(u)| \le \frac{\sqrt{\mathrm{m}(I)}}{\mathrm{m}(I)}\|\overline{\gamma}(u)\|_{L^2(I)} = \frac{1}{\sqrt{\mathrm{m}(I)}}\|\overline{\gamma}(u)\|_{L^2(I)}.$$

This with (65) imply that

$$\|u\|_{L^2(\Omega)} \le C_{13}\big(|u|_{1,\mathcal{T}} + \|\overline{\gamma}(u)\|_{L^2(I)}\big). \tag{66}$$

Using the definition (63) of the norm $\|\cdot\|_{1,\mathcal{T}}$, inequality (66) implies that

$$\|u\|_{1,\mathcal{T}} \le C_{14}\big(|u|_{1,\mathcal{T}} + \|\overline{\gamma}(u)\|_{L^2(I)}\big).$$

Second step. second side inequality. Using ([22], Lemma 10.5, Page 72) yields

$$\begin{aligned} \|u\|^2_{1,\mathcal{T}} &= |u|^2_{1,\mathcal{T}} + \|u\|^2_{L^2(\Omega)} \ge \frac{1}{2}|u|^2_{1,\mathcal{T}} + \frac{1}{2}|u|^2_{1,\mathcal{T}} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} \\ &\ge \frac{1}{2}|u|^2_{1,\mathcal{T}} + \frac{C_{15}}{2}\|\overline{\gamma}(u)\|^2_{L^2(\partial\Omega)} \ge C_{16}\Big(|u|^2_{1,\mathcal{T}} + \|\overline{\gamma}(u)\|^2_{L^2(I)}\Big). \end{aligned} \tag{67}$$

Inequality (67) together with inequality $\sqrt{a^2+b^2} \ge \frac{1}{\sqrt{2}}(a+b)$ imply that

$$\|u\|_{1,\mathcal{T}} \ge C_{17}\big(|u|_{1,\mathcal{T}} + \|\overline{\gamma}(u)\|_{L^2(I)}\big).$$

This completes the proof of Lemma 6.3.

□

Lemma 6.4. *Let Ω be a polygonal connected open bounded subset of $\mathbb{R}^d$ and $\mathcal{T}$ be an admissible mesh in the sense of Definition 5.1 satisfying Assumption 5.2. Then, for some positive constant C_{18}, which only depends on Ω and Γ_1, for all $u_\mathcal{T} \in \mathcal{Y}(\mathcal{T})$ (see Definition 5.5), we have*

$$\|\overline{\gamma}(u_\mathcal{T})\|_{L^2(\Gamma_0)} \le C_{18}\left(|u_\mathcal{T}|_{1,\mathcal{T}} + \left(\sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^\mathcal{T}} \tau_\sigma (u_K)^2\right)^{\frac{1}{2}}\right), \tag{68}$$

where u_K is the value of $u_\mathcal{T}$ over the control volume K and $\overline{\gamma}$ is the discrete trace operator defined in Lemma 6.3.

Proof. Using Lemma 10.5, Page 72 of [22] (or Lemma 6.3, see (67)) yields

$$\|\overline{\gamma}(u_\mathcal{T})\|_{L^2(\Gamma_0)} \le \frac{1}{\sqrt{C_{15}}}\big(|u_\mathcal{T}|_{1,\mathcal{T}} + \|u_\mathcal{T}\|_{L^2(\Omega)}\big).$$

This with (66) (by choosing $I = \Gamma_1$) and the fact that $d_{K,\sigma} \le \operatorname{diam}(\Omega)$ imply the desired estimate (68). This completes the proof of Lemma 6.4. □

Lemma 6.5. *Let Ω be a polygonal connected open bounded subset of $\mathbb{R}^d$ and $\mathcal{T}$ be an admissible mesh in the sense of Definition 5.1. Let τ_σ^κ be defined in (44) and (46). Then, under Assumption (A_2) (recall that $\tau_\sigma = \mathrm{m}(\sigma)/d_\sigma$ for $\sigma \in \mathcal{E}_{\rm int}$ and $\tau_\sigma = \mathrm{m}(\sigma)/d_{K,\sigma}$ for $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}_K$), we have*

$$\begin{aligned} \frac{\kappa_0^2}{\|\kappa\|_{L^\infty(\Omega)}}\tau_\sigma \le \tau_\sigma^\kappa \le \frac{\|\kappa\|^2_{L^\infty(\Omega)}}{\kappa_0}\tau_\sigma, \quad & \forall \sigma \in \mathcal{E}_{\rm int}, \\ \kappa_0\tau_\sigma \le \tau_\sigma^\kappa \le \|\kappa\|_{L^\infty(\Omega)}\tau_\sigma, \quad & \forall \sigma \in \mathcal{E}_{\rm ext}, \end{aligned}$$

and the following equivalence of semi-norms holds, for all $u_\mathcal{T} \in \mathcal{Y}(\mathcal{T})$:

$$\frac{\kappa_0}{\sqrt{\|\kappa\|_{L^\infty(\Omega)}}}|u_\mathcal{T}|_{1,\mathcal{T}} \le |u_\mathcal{T}|_{1,\kappa,\mathcal{T}} \le \frac{\|\kappa\|_{L^\infty(\Omega)}}{\sqrt{\kappa_0}}|u_\mathcal{T}|_{1,\mathcal{T}},$$

where

$$|u_\mathcal{T}|^2_{1,\kappa,\mathcal{T}} = \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} \tau_\sigma^\kappa (u_L - u_K)^2. \tag{69}$$

Proof. The lemma can be checked using the expression for τ_σ^κ and assumption (A_2). □

Proof of Theorem 6.2. (i) **Proof of (59).** We follow the techniques of [22], Page 51. We introduce $(\overline{u}_K)_{K\in\mathcal{T}} = (u_K - \overline{h}_K)_{K\in\mathcal{T}}$ and $(\overline{u}_\sigma)_{\sigma\in\mathcal{E}_{\rm ext}} = (u_\sigma - h(\boldsymbol{y}_\sigma))_{\sigma\in\mathcal{E}_{\rm ext}}$ where

$$\overline{h}_K = \overline{h}(\boldsymbol{x}_K), \quad \forall K\in\mathcal{T} \quad \text{and} \quad u_\sigma = h(\boldsymbol{y}_\sigma), \quad \forall\sigma\in\Gamma_1^{\mathcal{T}}.$$

The scheme (52) becomes

$$\sum_{\sigma\in\mathcal{E}_K} \overline{F}_{K,\sigma} = \mathrm{m}(K) f_K - \sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}, \quad \forall K\in\mathcal{T}, \tag{70}$$

where

$$\overline{F}_{K,\sigma} = \begin{cases} -\tau_\sigma^\kappa(\overline{u}_L - \overline{u}_K), & \forall\sigma = K|L, \\ -\tau_\sigma^\kappa(0 - \overline{u}_K), & \forall\sigma\in\mathcal{E}_K \text{ with } \sigma\subset\Gamma_1, \\ -\tau_\sigma^\kappa(\overline{u}_\sigma - \overline{u}_K), & \forall\sigma\in\mathcal{E}_K \text{ with } \sigma\subset\Gamma_0. \end{cases}$$

and

$$G_{K,\sigma} = \begin{cases} -\tau_\sigma^\kappa\big(\overline{h}_L - \overline{h}_K\big), & \forall\sigma = K|L, \\ -\tau_\sigma^\kappa\big(h(\boldsymbol{y}_\sigma) - \overline{h}_K\big), & \forall\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm ext}. \end{cases}$$

The scheme (53) then states that for all $\sigma\in\mathcal{E}_K$ with $\sigma\subset\Gamma_0$,

$$\begin{aligned} -\tau_\sigma^\kappa(\overline{u}_\sigma - \overline{u}_K) &= \alpha(\sigma_e)\overline{u}_{\sigma,+} - \alpha(\sigma_b)\overline{u}_{\sigma^-,+} + \mathrm{m}(\sigma)\beta_\sigma\overline{u}_\sigma - \mathrm{m}(\sigma)g_\sigma \\ &\quad + \tau_\sigma^\kappa\big(h_\sigma - \overline{h}_K\big) + \alpha(\sigma_e)h_{\sigma,+} - \alpha(\sigma_b)h_{\sigma^-,+} + \mathrm{m}(\sigma)\beta_\sigma h_\sigma. \end{aligned} \tag{71}$$

We multiply both sides of (70) by $\overline{u}_K$ and sum over $K\in\mathcal{T}$ to obtain

$$\sum_{K\in\mathcal{T}} \overline{u}_K \sum_{\sigma\in\mathcal{E}_K} \overline{F}_{K,\sigma} = \int_\Omega f_{\mathcal{T}}(\boldsymbol{x})\overline{u}_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}\overline{u}_K, \tag{72}$$

where $f_{\mathcal{T}}$, $\overline{u}_{\mathcal{T}}\in\mathcal{Y}(\mathcal{T})$ whose values over K are, respectively, f_K and $\overline{u}_K$. Using (58), (51), (56) (recall that $\Gamma_0^{\mathcal{T}}$ and $\Gamma_1^{\mathcal{T}}$ are given in (40)), and the conservativity property $\overline{F}_{K,\sigma} = -\overline{F}_{L,\sigma}$ for all $\sigma = K|L$, the left hand side of (72) can be written as

$$\begin{aligned} \sum_{K\in\mathcal{T}} \overline{u}_K \sum_{\sigma\in\mathcal{E}_K} \overline{F}_{K,\sigma} &= \sum_{\sigma=K|L} \overline{F}_{K,\sigma}(\overline{u}_K - \overline{u}_L) + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm ext}} \overline{u}_K\overline{F}_{K,\sigma} \\ &= |\overline{u}_{\mathcal{T}}|^2_{1,\kappa,\mathcal{T}} - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa(\overline{u}_\sigma - \overline{u}_K)\overline{u}_K + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^{\mathcal{T}}} \tau_\sigma^\kappa(\overline{u}_K)^2 \\ &= \mathcal{N}^2(\overline{u}_{\mathcal{T}}) - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa(\overline{u}_\sigma - \overline{u}_K)\overline{u}_\sigma, \end{aligned} \tag{73}$$

where

$$\mathcal{N}^2(\overline{u}_{\mathcal{T}}) = |\overline{u}_{\mathcal{T}}|^2_{1,\kappa,\mathcal{T}} + \sum_{\sigma\in\cap\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa(\overline{u}_\sigma - \overline{u}_K)^2 + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^{\mathcal{T}}} \tau_\sigma^\kappa(\overline{u}_K)^2.$$

Let us now express the second term on the RHS of (73) using (71) as

$$-\sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa(\overline{u}_\sigma - \overline{u}_K)\overline{u}_\sigma = \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \big(\alpha(\sigma_e)\overline{u}_{\sigma,+} - \alpha(\sigma_b)\overline{u}_{\sigma^-,+}\big)\overline{u}_\sigma$$

$$
\begin{aligned}
&+ \int_{\Gamma_0} \beta_{\mathcal{T}}(\boldsymbol{x})(\overline{v}_{\mathcal{T}})^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) - \int_{\Gamma_0} g_{\mathcal{T}}(\boldsymbol{x})\overline{v}_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})\\
&+ \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa\big(h_\sigma - \overline{h}_K\big)\overline{u}_\sigma + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \big(\alpha(\sigma_e)h_{\sigma,+} - \alpha(\sigma_b)h_{\sigma^-,+}\big)\overline{u}_\sigma\\
&+ \int_{\Gamma_0} \beta_{\mathcal{T}}(\boldsymbol{x})h_{\mathcal{T}}(\boldsymbol{x})\overline{v}_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}), \qquad (74)
\end{aligned}
$$

where $g_{\mathcal{T}} \in \mathcal{Z}_0(\mathcal{T})$, $h_{\mathcal{T}} \in \mathcal{Z}(\mathcal{T})$, $\beta_{\mathcal{T}} \in \mathcal{Z}_0(\mathcal{T})$, $\overline{v}_{\mathcal{T}} \in \mathcal{Z}(\mathcal{T})$ whose values over σ are, respectively, g_σ, h_σ, β_σ and $\overline{u}_\sigma$. It is useful to note that, since $u_\sigma = h(\boldsymbol{y}_\sigma)$ for all $\sigma \in \Gamma_1^{\mathcal{T}}$, then $\overline{v}_{\mathcal{T}}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Gamma_1$. Since $\overline{u}_\sigma = 0$ for all $\sigma \in \Gamma_1^{\mathcal{T}}$ and $\{\overline{u}_{\sigma,+}, \overline{u}_{\sigma,-}\} = \{\overline{u}_{\sigma^+}, \overline{u}_\sigma\}$, the first term in the RHS of (74) can be written as, see [10], Page 19 (see also [23], Pages 768, 769 and [22], Pages 42, 43),

$$
\begin{aligned}
\sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0} \big(\alpha(\sigma_e)\overline{u}_{\sigma,+} - \alpha(\sigma_b)\overline{u}_{\sigma^-,+}\big)\overline{u}_\sigma &= \sum_{\sigma\in\mathcal{E}_{\text{ext}}} \big(\alpha(\sigma_e)\overline{u}_{\sigma,+} - \alpha(\sigma_b)\overline{u}_{\sigma^-,+}\big)\overline{u}_\sigma\\
&= \sum_{\sigma\in\mathcal{E}_{\text{ext}}} |\alpha(\sigma_e)|(\overline{u}_{\sigma,+} - \overline{u}_{\sigma,-})\overline{u}_{\sigma,+}\\
&= \frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\text{ext}}} |\alpha(\sigma_e)|\Big(\big(\overline{u}_{\sigma,+} - \overline{u}_{\sigma,-}\big)^2 + \overline{u}_{\sigma,+}^2 - \overline{u}_{\sigma,-}^2\Big)\\
&= \frac{1}{2}|\overline{v}_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})} + \frac{1}{2}\int_{\partial\Omega} \alpha_{\mathbf{t}}(\boldsymbol{x})(\overline{v}_{\mathcal{T}})^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})\\
&= \frac{1}{2}|\overline{v}_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})} + \frac{1}{2}\int_{\Gamma_0} \alpha_{\mathbf{t}}(\boldsymbol{x})(\overline{v}_{\mathcal{T}})^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}). \qquad (75)
\end{aligned}
$$

Using the facts that $|h_\sigma - \overline{h}_K| \le C_{19} d_{K,\sigma}\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}$ and $\sum_{\sigma\in\mathcal{E}} \mathrm{m}(\sigma)d_\sigma = 2\mathrm{m}(\Omega)$ (see [25], equation (4.3), Page 1025) together with Lemma 6.5, the fourth term on the RHS of (74) can be estimated as

$$
\left|\sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa\big(h_\sigma - \overline{h}_K\big)\overline{u}_\sigma\right| \le C_{20}\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}\left(\left(\sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa(\overline{u}_\sigma - \overline{u}_K)^2\right)^{\frac{1}{2}} + \|\overline{\gamma}(\overline{u}_{\mathcal{T}})\|_{L^2(\Gamma_0)}\right).
$$

Gathering this estimate together with Lemmas 6.4 and 6.5 implies that

$$
\left|\sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa(h_\sigma - \overline{h}_K)\overline{u}_\sigma\right| \le C_{21}\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}\mathcal{N}(\overline{u}_{\mathcal{T}}). \qquad (76)
$$

Thanks to the facts that $|\alpha(\sigma_e)h_{\sigma,+} - \alpha(\sigma_b)h_{\sigma^-,+}| \le C_{22}(\alpha)\mathrm{m}(\sigma)\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}$ and $d_{K,\sigma} \le \mathrm{diam}(\Omega)$ and using the triangle and the Cauchy–Schwarz inequalities together with the hypothesis (A_2) and Lemmas 6.4 and 6.5, we get the following estimate on the fifth term on the RHS of (74):

$$
\begin{aligned}
&\left|\sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \big(\alpha(\sigma_e)h_{\sigma,+} - \alpha(\sigma_b)h_{\sigma^-,+}\big)\overline{u}_\sigma\right| \le C_{23}(\alpha)\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}\sum_{\sigma\in\Gamma_0^{\mathcal{T}}} \mathrm{m}(\sigma)|\overline{u}_\sigma|\\
&\quad\le C_{24}(\alpha,\kappa,\Omega)\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}\left(\left(\sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa(\overline{u}_\sigma - \overline{u}_K)^2\right)^{\frac{1}{2}} + \|\overline{\gamma}(\overline{u}_{\mathcal{T}})\|_{L^2(\Gamma_0)}\right)\\
&\quad\le C_{25}(\alpha,\kappa,\Omega)\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}\mathcal{N}(\overline{u}_{\mathcal{T}}). \qquad (77)
\end{aligned}
$$

On the other hand, the second term on the RHS of (72) can be estimated using the following equivalent expression, thanks to the property $G_{K,\sigma} = -G_{L,\sigma}$, for all $\sigma = K|L$:

$$\sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}\overline{u}_K = \sum_{\sigma=K|L} G_{K,\sigma}(\overline{u}_K - \overline{u}_L) + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}} G_{K,\sigma}\overline{u}_K.$$

Using this expression, the Cauchy–Schwarz inequality, Lemmas 6.4 and 6.5, and gathering this with the facts that $|G_{K,\sigma}| \le C_{26} d_\sigma \tau_\sigma^\kappa \|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}$ (recall that $d_\sigma = d_{K,\sigma}$ if $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}$) and $\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) d_{K,\sigma} = 2\mathrm{m}(K)$ imply that

$$\begin{aligned}\left|\sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}\overline{u}_K\right| &\le C_{27}\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}\left(|\overline{u}_{\mathcal{T}}|_{1,\kappa,\mathcal{T}} + \|\overline{\gamma}(\overline{u}_{\mathcal{T}})\|_{L^2(\Gamma_0)} + \left(\sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^{\mathcal{T}}} \tau_\sigma^\kappa (\overline{u}_K)^2\right)^{\frac{1}{2}}\right)\\ &\le C_{28}\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}\mathcal{N}(\overline{u}_{\mathcal{T}}).\end{aligned}\tag{78}$$

Gathering now (72)–(75) implies that

$$\begin{aligned}&\mathcal{N}^2(\overline{u}_{\mathcal{T}}) + \int_{\Gamma_0} \beta_{\mathcal{T}}(\boldsymbol{x})(\overline{v}_{\mathcal{T}})^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) + \frac{1}{2}|\overline{v}_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})} + \frac{1}{2}\int_{\Gamma_0} \alpha_{\mathbf{t}}(\boldsymbol{x})(\overline{v}_{\mathcal{T}})^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})\\ &\quad= \int_\Omega f_{\mathcal{T}}(\boldsymbol{x})\overline{u}_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_0} g_{\mathcal{T}}(\boldsymbol{x})\overline{v}_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) - \sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}\overline{u}_K - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa(h_\sigma - \overline{h}_K)\overline{u}_\sigma\\ &\quad\quad - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \big(\alpha(\sigma_e)h_{\sigma,+} - \alpha(\sigma_b).h_{\sigma^-,+}\big)\overline{u}_\sigma - \int_{\Gamma_0} \beta_{\mathcal{T}}(\boldsymbol{x})h_{\mathcal{T}}(\boldsymbol{x})\overline{v}_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}).\end{aligned}\tag{79}$$

Using Lemma 6.5 and gathering (79) with (76)–(78) give

$$\begin{aligned}&|(\overline{u}_{\mathcal{T}}, \overline{v}_{\mathcal{T}})|^2_{1,\mathcal{X}(\mathcal{T})} + \int_{\Gamma_0} \beta_{\mathcal{T}}(\boldsymbol{x})\overline{v}_{\mathcal{T}}^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) + |\overline{v}_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})} + \int_{\Gamma_0} \alpha_t(\boldsymbol{x})\overline{v}_{\mathcal{T}}^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})\\ &\quad\le C_{29}(\kappa,\alpha,\Omega)\mathcal{S},\end{aligned}\tag{80}$$

where

$$\mathcal{S} = \int_\Omega f_{\mathcal{T}}(\boldsymbol{x})\overline{u}_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_0} (g_{\mathcal{T}} - \beta_{\mathcal{T}}h_{\mathcal{T}})(\boldsymbol{x})\tilde{v}_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) + \|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}|(\overline{u}_{\mathcal{T}}, \overline{v}_{\mathcal{T}})|_{1,\mathcal{X}(\mathcal{T})}.\tag{81}$$

It is useful to note that, since $\overline{u}_\sigma = 0$ for all $\sigma \in \Gamma_1^{\mathcal{T}}$,

$$|(\overline{u}_{\mathcal{T}}, \overline{v}_{\mathcal{T}})|^2_{1,\mathcal{X}(\mathcal{T})} = |\overline{u}_{\mathcal{T}}|^2_{1,\mathcal{T}} + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma(\overline{u}_\sigma - \overline{u}_K)^2 + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^{\mathcal{T}}} \tau_\sigma \overline{u}_K^2.$$

Using the notation $0 \ge C_\alpha = \min_{\partial\Omega} \alpha_{\mathbf{t}}$ and assumption (A_3), equation (80) implies that

$$|(\overline{u}_{\mathcal{T}}, \overline{v}_{\mathcal{T}})|^2_{1,\mathcal{X}(\mathcal{T})} + (\beta_0 + C_\alpha)\|\overline{v}_{\mathcal{T}}\|^2_{L^2(\Gamma_0)} + |\overline{v}_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})} \le C_{29}(\kappa,\alpha,\Omega)\mathcal{S}.$$

As before, the previous inequality with the hypothesis $C_\alpha \ge -\frac{\beta_0}{2}$ imply that

$$|(\overline{u}_{\mathcal{T}}, \overline{v}_{\mathcal{T}})|^2_{1,\mathcal{X}(\mathcal{T})} + \|\overline{v}_{\mathcal{T}}\|^2_{L^2(\Gamma_0)} + |\overline{v}_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})} \le C_{30}(\kappa,\beta_0,\alpha,\Omega)\mathcal{S}.\tag{82}$$

Using the Cauchy–Schwarz inequality and applying, respectively, Lemmas 6.3 (with $I = \Gamma_0$) and 6.4 yield

$$\left|\int_\Omega f_{\mathcal{T}}(\boldsymbol{x})\overline{u}_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| \le \|f_{\mathcal{T}}\|_{L^2(\Omega)}\|\overline{u}_{\mathcal{T}}\|_{L^2(\Omega)}$$

$$\leq C_{31}(\Omega)\|f\|_{L^2(\Omega)}\Big(|\overline{u}_{\mathcal{T}}|_{1,\mathcal{T}} + \|\overline{\gamma}(\overline{u}_{\mathcal{T}})\|_{L^2(\Gamma_0)}\Big)$$
$$\leq C_{32}(\Omega)\|f\|_{L^2(\Omega)}|(\overline{u}_{\mathcal{T}},\overline{v}_{\mathcal{T}})|_{1,\mathcal{X}(\mathcal{T})}. \tag{83}$$

In addition to this, using the Cauchy–Schwarz inequality, the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, the fact that $\frac{\mathrm{m}(\sigma)}{\tau_\sigma} = d_\sigma < \mathrm{diam}(\Omega)$ and Lemma 6.4 yield

$$\left|\int_{\Gamma_0}(g_{\mathcal{T}} - \beta_{\mathcal{T}} h_{\mathcal{T}})(\boldsymbol{x})\overline{v}_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})\right| \leq \|g_{\mathcal{T}} - \beta_{\mathcal{T}} h_{\mathcal{T}}\|_{L^2(\Gamma_0)}\|\overline{v}_{\mathcal{T}}\|_{L^2(\Gamma_0)}$$
$$\leq C_{33}(\Omega)\|g-\beta h\|_{L^2(\Gamma_0)}\left(\sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}}\tau_\sigma(\overline{u}_\sigma - \overline{u}_K)^2 + \|\overline{\gamma}(\overline{u}_{\mathcal{T}})\|^2_{L^2(\Gamma_0)}\right)^{1/2}$$
$$\leq C_{34}(\Omega)\Big(\|g\|_{L^2(\Gamma_0)} + \|\beta\|_{L^\infty(\Gamma_0)}\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}\Big)|(\overline{u}_{\mathcal{T}},\overline{v}_{\mathcal{T}})|_{1,\mathcal{X}(\mathcal{T})}. \tag{84}$$

Gathering now this inequality with (83) and (81) implies that

$$|\mathcal{S}| \leq C_{35}(\Omega)\Big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_0)} + \Big(\|\beta\|_{L^\infty(\Gamma_0)}+1\Big)\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}\Big)|(\overline{u}_{\mathcal{T}},\overline{v}_{\mathcal{T}})|_{1,\mathcal{X}(\mathcal{T})}.$$

This with (82) imply that

$$\Big(|(\overline{u}_{\mathcal{T}},\overline{v}_{\mathcal{T}})|^2_{1,\mathcal{X}(\mathcal{T})} + \|\overline{v}_{\mathcal{T}}\|^2_{L^2(\Gamma_0)} + |\overline{v}_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}\Big)^{\frac{1}{2}}$$
$$\leq C_{36}(\alpha,\kappa,\Omega,\beta_0)\Big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_0)} + \Big(\|\beta\|_{L^\infty(\Gamma_0)}+1\Big)\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}\Big). \tag{85}$$

Using now the triangle inequality, the definition (60) of $|(\cdot,\cdot)|_{1,\mathcal{X}(\mathcal{T})}$, the fact that $u_K = \overline{u}_K + \overline{h}_K$ and $u_\sigma = \overline{u}_\sigma + h(\boldsymbol{y}_\sigma)$ and also since $|\overline{h}_K - \overline{h}_L|$, $|h_\sigma - \overline{h}_K|$, and $|h(\boldsymbol{y}_{\sigma^+}) - h(\boldsymbol{y}_\sigma)|$ are, respectively, bounded above by, up to multiplicative constants which are independent of the parameters of the discretization, $d_{K|L}\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}$, $d_{K,\sigma}\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}$ and $\mathrm{m}(\sigma)\|\overline{h}\|_{\mathcal{C}^1(\overline{\Omega})}$, estimate (85) yields the desired estimate (59).

(ii) **Proof of the existence and uniqueness.** The finite volume scheme of Definition 6.1 leads to a linear system of equations whose matrix is square since the number of unknowns is $M = M_1 + M_2$, where M_1 is the number of elements of $\mathcal{T}$ and M_2 is the number of elements of $\sigma \in \mathcal{E}_{\mathrm{ext}}$ such that $\sigma \subset \Gamma_0$ (under Assumption 5.2). The number of equations is also M. Therefore, the existence of a solution of such scheme is equivalent to its uniqueness. Such uniqueness can be deduced from the estimate (59) by assuming that $(f,h,g) = (0,0,0)$. This with estimate (59) imply that $|(u_{\mathcal{T}},v_{\mathcal{T}})|_{1,\mathcal{X}(\mathcal{T})} = 0$. On the other hand, the fact that $h=0$ implies that $u_\sigma = 0$ for all $\sigma \in \Gamma_1^{\mathcal{T}}$. From this and the fact that $|(u_{\mathcal{T}},v_{\mathcal{T}})|_{1,\mathcal{X}(\mathcal{T})} = 0$, we deduce $(u_{\mathcal{T}},v_{\mathcal{T}}) = (0,0)$. This completes the proof of Theorem 6.2.

□

Remark 6.6 (Formulation of a scheme when h is not "smooth" and its well-posedness)**.** The formulation of the finite volume scheme of Definition 6.1 requires the regularity assumption $h \in \mathcal{C}^1(\partial\Omega)$ and to get the stability result (59), we have assumed the existence of $\overline{h} \in \mathcal{C}^1(\overline{\Omega})$ such that $\tilde{\gamma}(\overline{h}) = h$. However, these regularity assumptions can be weakened, respectively, to $h \in L^2(\partial\Omega)$ to define a suitable finite volume scheme and to $h \in H^{1/2}(\partial\Omega)$ to get a stability result similar to (59). When $h \in L^2(\partial\Omega)$, there is another possibility to define a finite volume scheme (instead of that given in Definition 6.1) in which $h(\boldsymbol{y}_\sigma)$ involved in Definition 6.1 can be replaced by the mean value on the edge σ, see Remark 9.5, Page 42 of [22], that is,

$$\frac{1}{\mathrm{m}(\sigma)}\int_\sigma h(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}).$$

In order to get a discrete stability similar to the one of (59), it suffices only to assume that $h \in H^{1/2}(\partial\Omega)$. Consequently, there exists a function $\overline{h} \in H^1(\Omega)$ such that $\tilde{\gamma}(\overline{h}) = h$. Using Lemma 9.4, Page 49 of [22], we are able to prove the following discrete well-posedness result, under some hypotheses on the mesh $\mathcal{T}$:

$$\begin{aligned}&\Big(|(u_{\mathcal{T}}, v_{\mathcal{T}})|^2_{1,\mathcal{X}(\mathcal{T})} + \|v_{\mathcal{T}}\|^2_{L^2(\Gamma_0)} + |v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}\Big)^{\frac{1}{2}}\\ &\quad\leq C_{37}(\alpha, \kappa, \Omega, \beta_0)\Big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_0)} + \big(\|\beta\|_{L^\infty(\Gamma_0)} + 1\big)\|\overline{h}\|_{H^1(\Omega)}\Big).\end{aligned}$$

6.3. Convergence rate of scheme (52)–(54)

In this subsection, we prove the following convergence result.

Theorem 6.7 (Convergence rate of scheme (52)–(54))**.** *In addition to the hypotheses of Theorem 6.2, assume that the weak solution ω of (30), (31) and $\overline{h}$ are smooth, namely, ω, $\overline{h} \in \mathcal{C}^2(\overline{\Omega})$ (and consequently $u = \omega + \overline{h} \in \mathcal{C}^2(\overline{\Omega})$), and that $\kappa \in \mathcal{C}^1(\overline{\Omega})$. We define the errors:*

$$e_K = u(\boldsymbol{x}_K) - u_K, \ \ \forall K \in \mathcal{T}, \quad e_\sigma = u(\boldsymbol{y}_\sigma) - u_\sigma, \ \ \forall \sigma \in \Gamma_0^{\mathcal{T}}, \quad \text{and} \quad e_\sigma = 0, \ \ \forall \sigma \in \Gamma_1^{\mathcal{T}},$$

and assume that the $\operatorname{size}(\mathcal{T})$ *is small, namely,* $\operatorname{size}(\mathcal{T}) < 1$*. Then, the following error error estimate holds:*

$$|(e_{\mathcal{T}}, \overline{e}_{\mathcal{T}})|_{1,\mathcal{X}(\mathcal{T})} + \|\overline{e}_{\mathcal{T}}\|_{L^2(\Gamma_0)} + |\overline{e}_{\mathcal{T}}|_{\alpha,\mathcal{Z}(\mathcal{T})} \leq C_{38}\sqrt{\operatorname{size}(\mathcal{T})}, \tag{86}$$

where $(e_{\mathcal{T}}, \overline{e}_{\mathcal{T}}) \in \mathcal{X}(\mathcal{T})$ (see Definition 5.5) whose values over (K,σ) are (e_K, e_σ).

The order $\sqrt{\operatorname{size}(\mathcal{T})}$ is the same as the one obtained in [10] (using the mesh of Definition 5.1) and [9] (for general meshes) for a finite volume scheme approximating the Poisson's equation with oblique boundary conditions on the whole boundary.

Proof. Let us define the following auxiliary errors:

$$R_{K,\sigma} := \begin{cases} \tau_\sigma^\kappa(u(\boldsymbol{x}_L) - u(\boldsymbol{x}_K)) - \int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x}) \cdot \mathbf{n}_{K,\sigma}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}), & \forall \sigma = K|L,\\ \tau_\sigma^\kappa(u(\boldsymbol{y}_\sigma) - u(\boldsymbol{x}_K)) - \int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x}) \cdot \mathbf{n}_{K,\sigma}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}), & \forall \sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}.\end{cases} \tag{87}$$

It is shown in [22], Page 81 that

$$|R_{K,\sigma}| \leq C_{39}\mathrm{m}(\sigma)\operatorname{size}(\mathcal{T}). \tag{88}$$

Since the value of the exact solution u over the boundary Γ_1 is h when the problem considered is (6)–(8), then the errors $R_{K,\sigma}$ in this case are expressed as follows:

$$R_{K,\sigma} := \begin{cases} \tau_\sigma^\kappa(u(\boldsymbol{x}_L) - u(\boldsymbol{x}_K)) - \int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x}) \cdot \mathbf{n}_{K,\sigma}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}), & \forall \sigma = K|L,\\ \tau_\sigma^\kappa(h(\boldsymbol{y}_\sigma) - u(\boldsymbol{x}_K)) - \int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x}) \cdot \mathbf{n}_{K,\sigma}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}), & \forall \sigma \in \mathcal{E}_K \cap \Gamma_1,\\ \tau_\sigma^\kappa(u(\boldsymbol{y}_\sigma) - u(\boldsymbol{x}_K)) - \int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x}) \cdot \mathbf{n}_{K,\sigma}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}), & \forall \sigma \in \mathcal{E}_K \cap \Gamma_0.\end{cases} \tag{89}$$

Using the notations of Definition 5.4, we define the error

$$r_\sigma = \alpha(\sigma_e)u(\sigma_e) - \alpha(\sigma_e)u\big(\boldsymbol{y}_{\sigma,+}\big).$$

Using a Taylor expansion, the following estimate holds:

$$|r_\sigma| \leq C_{40}|\alpha(\sigma_e)|\mathrm{m}(\sigma). \tag{90}$$

Both estimates (88) and (90) hold under the assumption that $u \in \mathcal{C}^2(\overline{\Omega})$. Using (41) and (89) yields that, for all $K \in \mathcal{T}$,

$$-\sum_{\sigma=K|L}\tau_\sigma^\kappa(u(\boldsymbol{x}_L) - u(\boldsymbol{x}_K)) - \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}}\tau_\sigma^\kappa(u(\boldsymbol{y}_\sigma) - u(\boldsymbol{x}_K)) = \mathrm{m}(K)f_K - \sum_{\sigma\in\mathcal{E}}R_{K,\sigma}. \tag{91}$$

Using (7), (89) (third branch) and rule (39) imply that, for all $\sigma \in \mathcal{E}_K \cap \Gamma_0^{\mathcal{T}}$,

$$
\begin{aligned}
-\tau_\sigma^\kappa(u(\boldsymbol{y}_\sigma) - u(\boldsymbol{x}_K)) &= \alpha(\sigma_e)u(\sigma_e) - \alpha(\sigma_b)u(\sigma_b) + \int_\sigma \beta(\boldsymbol{x})u(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) - \mathrm{m}(\sigma)g_\sigma - R_{K,\sigma} \\
&= \alpha(\sigma_e)u\big(\boldsymbol{y}_{\sigma,+}\big) - \alpha(\sigma_b)u\big(\boldsymbol{y}_{\sigma^-,+}\big) + \mathrm{m}(\sigma)\beta_\sigma u(\boldsymbol{y}_\sigma) - \mathrm{m}(\sigma)g_\sigma - R_{K,\sigma} \\
&\quad + r_\sigma - r_{\sigma^-} + l_\sigma,
\end{aligned} \tag{92}
$$

where $l_\sigma = \int_\sigma \beta(\boldsymbol{x})u(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) - \mathrm{m}(\sigma)\beta_\sigma u(\boldsymbol{y}_\sigma)$. Using a Taylor expansion, we obtain

$$
|l_\sigma| \le C_{41}\mathrm{m}(\sigma)\mathrm{size}(\mathcal{T}). \tag{93}
$$

Subtracting (52) and (53) from (91) and (92), respectively, yield that for all $K \in \mathcal{T}$,

$$
-\sum_{\sigma=K|L} \tau_\sigma^\kappa(e_L - e_K) - \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} \tau_\sigma^\kappa(e_\sigma - e_K) = -\sum_{\sigma\in\mathcal{E}_K} R_{K,\sigma} \tag{94}
$$

and, for all $\sigma \in \mathcal{E}_K$ such that $\sigma \subset \Gamma_0$,

$$
-\tau_\sigma^\kappa(e_\sigma - e_K) = \alpha(\sigma_e)e_{\sigma,+} - \alpha(\sigma_b)e_{\sigma^-,+} + \mathrm{m}(\sigma)\beta_\sigma e_\sigma - R_{K,\sigma} + r_\sigma - r_{\sigma^-} + l_\sigma. \tag{95}
$$

Multiplying both sides of (94) by e_K and summing the result over $K \in \mathcal{T}$, we obtain

$$
-\sum_{K\in\mathcal{T}} e_K \sum_{\sigma\in\mathcal{E}_K} R_{K,\sigma} = |e_{\mathcal{T}}|^2_{1,\kappa,\mathcal{T}} + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa(e_\sigma - e_K)^2 + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^{\mathcal{T}}} \tau_\sigma^\kappa e_K^2 - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa(e_\sigma - e_K)e_\sigma.
$$

This together with (95) imply

$$
\begin{aligned}
-\sum_{K\in\mathcal{T}} e_K \sum_{\sigma\in\mathcal{E}_K} R_{K,\sigma} &= |e_{\mathcal{T}}|^2_{1,\kappa,\mathcal{T}} + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau_\sigma^\kappa(e_\sigma - e_K)^2 + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^{\mathcal{T}}} \tau_\sigma^\kappa e_K^2 \\
&\quad + \sum_{\sigma\in\Gamma_0^{\mathcal{T}}} \big(\alpha(\sigma_e)e_{\sigma,+} - \alpha(\sigma_b)e_{\sigma^-,+}\big)e_\sigma + \int_{\Gamma_0} \beta_{\mathcal{T}}(\boldsymbol{x})(\bar{e}_{\mathcal{T}})^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) \\
&\quad - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} R_{K,\sigma}e_\sigma + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} (r_\sigma - r_{\sigma^-})e_\sigma + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} l_\sigma e_\sigma.
\end{aligned}
$$

This together with the same reasoning used to obtain (82) imply that

$$
|(e_{\mathcal{T}}, \bar{e}_{\mathcal{T}})|^2_{1,\mathcal{X}(\mathcal{T})} + \|\bar{e}_{\mathcal{T}}\|^2_{L^2(\Gamma_0)} + |\bar{e}_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})} \le C_{42}(|\mathbb{T}_1| + |\mathbb{T}_2| + |\mathbb{T}_3|), \tag{96}
$$

where

$$
\mathbb{T}_1 = -\sum_{K\in\mathcal{T}} e_K \sum_{\sigma\in\mathcal{E}_K} R_{K,\sigma} + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} R_{K,\sigma}e_\sigma,\ \mathbb{T}_2 = -\sum_{\sigma\in\Gamma_0^{\mathcal{T}}} (r_\sigma - r_{\sigma^-})e_\sigma,\ \text{and } \mathbb{T}_3 = -\sum_{\sigma\in\Gamma_0^{\mathcal{T}}} l_\sigma e_\sigma.
$$

Let us now estimate each term in the RHS of (96).

(i) **Estimate of the first term in the RHS of (96).** Re-ordering the sum and using the conservativity property $R_{K,\sigma} = -R_{L,\sigma}$ for all $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$ yield

$$
\mathbb{T}_1 = \sum_{\sigma=K|L} R_{K,\sigma}(e_L - e_K) + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} R_{K,\sigma}(e_\sigma - e_K) - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^{\mathcal{T}}} R_{K,\sigma}e_K.
$$

Using this expression together with (88), the fact $\sum_{\sigma\in\mathcal{E}} \mathrm{m}(\sigma)d_\sigma = 2\mathrm{m}(\Omega)$ and the Cauchy–Schwarz inequality imply that

$$
|\mathbb{T}_1| \le C_{43}\mathrm{size}(\mathcal{T})|(e_{\mathcal{T}}, \bar{e}_{\mathcal{T}})|_{1,\mathcal{X}(\mathcal{T})}. \tag{97}
$$

(ii) **Estimate of the second term in the RHS of (96).** Re-ordering the sum and using the Cauchy–Schwarz inequality together with estimate (90) and the fact that $\mathrm{m}(\sigma) \le \mathrm{size}(\mathcal{T})$ imply that

$$\begin{aligned}|\mathbb{T}_2| &= \left|\sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}} r_\sigma(e_{\sigma^+}-e_\sigma)\right| \\ &\le C_{40}\sqrt{\mathrm{size}(\mathcal{T})}\sqrt{\|\alpha\|_{\mathcal{C}(\overline{\Omega})}}\left(\sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}}\mathrm{m}(\sigma)\right)^{\frac{1}{2}}\left(\sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}}|\alpha(\sigma_e)|(e_{\sigma^+}-e_\sigma)^2\right)^{\frac{1}{2}} \\ &\le C_{44}\sqrt{\mathrm{size}(\mathcal{T})}\sqrt{\|\alpha\|_{\mathcal{C}(\overline{\Omega})}}|\overline{e}_{\mathcal{T}}|_{\alpha,\mathcal{Z}(\mathcal{T})} \le C_{45}\sqrt{\mathrm{size}(\mathcal{T})}|\overline{e}_{\mathcal{T}}|_{\alpha,\mathcal{Z}(\mathcal{T})}. \end{aligned} \tag{98}$$

(iii) **Estimate of the third term in the RHS of (96).** Using the Cauchy–Schwarz inequality together with estimate (93) and the reasoning used to obtain (84) yield

$$|\mathbb{T}_3| \le C_{46}\mathrm{size}(\mathcal{T})\|\overline{e}_{\mathcal{T}}\|_{L^2(\Gamma_0)} \le C_{47}\mathrm{size}(\mathcal{T})|(e_{\mathcal{T}},\overline{e}_{\mathcal{T}})|_{1,\mathcal{X}(\mathcal{T})}. \tag{99}$$

Using (96)–(99) and that $\mathrm{size}(\mathcal{T}) < 1$, we deduce the desired estimate (86). This completes the proof of Theorem 6.7.

□

7. Formulation of a finite volume scheme for the problem (6) and (7) with Neumann boundary condition (9) and its convergence analysis

As in Section 6, let us first describe the principles of the finite volume scheme we want to present in order to approximate the problem given by equations (6), (7) and (9). Let $(u_K)_{K\in\mathcal{T}}$ and $(u_\sigma)_{\sigma\in\mathcal{E}_{\mathrm{ext}}}$ denote the discrete unknowns. These unknowns are expected to approximate the exact solution u in the control volumes $K \in \mathcal{T}$ and on the edges $\sigma \in \mathcal{E}_{\mathrm{ext}}$. We look now for equations satisfied by these discrete unknowns using a convenient approximation for (41) and the boundary conditions (7) and (9). To this end, we consider the following cases:

(I) $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{int}}$ with $\sigma = K|L$. In this case, $-\int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x})\cdot\mathbf{n}_{K,\sigma}\,\mathrm{d}\gamma(\boldsymbol{x})$ can be approximated using the same two-points formula (43) and (44).

(II) $\sigma \in \mathcal{E}_K$ and $\sigma \subset \Gamma_1$. In this case, $-\int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x})\cdot\mathbf{n}_{K,\sigma}\,\mathrm{d}\gamma(\boldsymbol{x})$ can be approximated using the Neumann boundary condition (9) on Γ_1 (see [22], equation (10.7), Page 64 and [22], equation (11.8), Page 83), as

$$F_{K,\sigma} = -\tau_\sigma^\kappa(u_\sigma - u_K) = -\mathrm{m}(\sigma)q_\sigma = -\int_\sigma q(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}).$$

(III) $\sigma \in \mathcal{E}_K$ and $\sigma \subset \Gamma_0$. In this case, $-\int_\sigma \kappa(\boldsymbol{x})\nabla u(\boldsymbol{x})\cdot\mathbf{n}_{K,\sigma}\,\mathrm{d}\gamma(\boldsymbol{x})$ can be expressed, using the oblique boundary condition (7), as in (47)–(49).

7.1. Definition of the scheme for the Neumann boundary condition (9) on Γ_1

To summarize our finite volume scheme stated above, we introduce the following definition.

Definition 7.1 (Definition of a finite volume scheme for (6), (7) and (9))**.** Let τ_σ^κ be defined in (44) and (46). Define the discrete fluxes

$$F_{K,\sigma} = \begin{cases} -\tau_\sigma^\kappa(u_L - u_K), & \forall \sigma = K|L, \\ -\tau_\sigma^\kappa(u_\sigma - u_K), & \forall \sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}. \end{cases} \tag{100}$$

Let $(u_K)_{K\in\mathcal{T}}$ and $(u_\sigma)_{\sigma\in\mathcal{E}_{\mathrm{ext}}}$ denote the set of discrete unknowns. As a finite volume scheme, we suggest the scheme given by (52), with

$$F_{K,\sigma} = \begin{cases} -\mathrm{m}(\sigma)q_\sigma, & \forall \sigma \in \mathcal{E}_K \text{ with } \sigma \subset \Gamma_1 \\ \alpha(\sigma_e)u_{\sigma,+} - \alpha(\sigma_b)u_{\sigma^-,+} + \mathrm{m}(\sigma)\beta_\sigma u_\sigma - \mathrm{m}(\sigma)g_\sigma, & \forall \sigma \in \mathcal{E}_K \text{ with } \sigma \subset \Gamma_0. \end{cases} \tag{101}$$

7.2. Well-posedness of the discrete problem

Let us extend the definition of u_σ by (55) such that the discrete fluxes (100) can be written in a unified form as (57). One of the properties we use to analyse the finite volume schemes is conservativity (which can be checked from (100)), that is,

$$F_{K,\sigma} = -F_{L,\sigma}, \quad \forall \sigma = K|L. \tag{102}$$

The results of this subsection are summarized in the following theorem.

Theorem 7.2 (Well-posedness of scheme of Definition 7.1)**.** *Assume that the hypotheses of Theorem 4.4 , which states the well-posedness of the weak solution of problem (6), (7) with the Neumann boundary condition (9), are satisfied. Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 5.1 satisfying Assumption 5.2. Then, there exists a unique solution $(u_K)_{K\in\mathcal{T}}$ and $(u_\sigma)_{\sigma\in\mathcal{E}_{\text{ext}}}$ to the finite volume scheme of Definition 7.1. Corresponding to this unique solution, let $\mathcal{Y}(\mathcal{T}) \ni u_\mathcal{T}$ be defined such that its constant value over any $K \in \mathcal{T}$ is u_K. Also, let $\mathcal{Z}(\mathcal{T}) \ni v_\mathcal{T}$ be defined such that its constant value over any $\sigma \in \mathcal{E}_{\text{ext}}$ is u_σ. Then, the above discrete solution satisfies the following stability estimate:*

$$\|u_\mathcal{T}\|_{1,\mathcal{T}} + \left(\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}} \tau_\sigma (u_\sigma - u_K)^2\right)^{\frac{1}{2}} + |v_\mathcal{T}|_{\alpha,\mathcal{Z}(\mathcal{T})} \le C_{48}\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_0)} + \|q\|_{L^2(\Gamma_1)}\big), \tag{103}$$

where $\|\cdot\|_{1,\mathcal{T}}$ (resp. $|\cdot|_{\alpha,\mathcal{Z}(\mathcal{T})}$) is the discrete norm (resp. semi-norm) defined in (63) (resp. (62)).

Proof. We prove Theorem 7.2 item by item.

(i) **Proof of the discrete stability (103).** To get the discrete well-posedness, we multiply both sides of (52) by u_K and sum over $K \in \mathcal{T}$ to get

$$\sum_{K\in\mathcal{T}} u_K \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} = \int_\Omega f_\mathcal{T}(\boldsymbol{x}) u_\mathcal{T}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{104}$$

where $f_\mathcal{T}$, $u_\mathcal{T} \in \mathcal{Y}(\mathcal{T})$ whose values over K are, respectively, f_K and u_K. Using the conservativity property (102), and equations (100) and (101) (recall that $\Gamma_0^\mathcal{T}$ and $\Gamma_1^\mathcal{T}$ are given in (40)), the LHS of (104) can be written as

$$\begin{aligned}\sum_{K\in\mathcal{T}} u_K \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} &= \sum_{\sigma=K|L} F_{K,\sigma}(u_K - u_L) + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}} u_K F_{K,\sigma}\\ &= |u_\mathcal{T}|^2_{1,\kappa,\mathcal{T}} + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}} \tau^\kappa_\sigma (u_\sigma - u_K)^2 - \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}} \tau^\kappa_\sigma (u_\sigma - u_K) u_\sigma\\ &= |u_\mathcal{T}|^2_{1,\kappa,\mathcal{T}} + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}} \tau^\kappa_\sigma (u_\sigma - u_K)^2\\ &\quad - \int_{\Gamma_1} q_\mathcal{T}(\boldsymbol{x}) v_\mathcal{T}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^\mathcal{T}} \tau^\kappa_\sigma (u_\sigma - u_K) u_\sigma,\end{aligned} \tag{105}$$

where $q_\mathcal{T}$, $v_\mathcal{T}$ are constant over each $\sigma \in \Gamma_1^\mathcal{T}$ whose values over σ are q_σ and u_σ, respectively, and $|\cdot|^2_{1,\kappa,\mathcal{T}}$ is given by (69) in Lemma 6.5. We show that the fourth term on RHS of (105) is positive up to an additive quantity expressed in terms of the function g. Indeed, using the discrete boundary condition (101) on Γ_0, equation (74) holds with $\overline{h} = 0$, $\overline{u}_\mathcal{T} = u_\mathcal{T}$, and $\overline{v}_\mathcal{T} = v_\mathcal{T}$, where $g_\mathcal{T}$, $\beta_\mathcal{T}$ and $v_\mathcal{T}$ are constant over each $\sigma \in \Gamma_0^\mathcal{T}$ whose values over σ are, respectively, g_σ, β_σ and u_σ. Since $\alpha = 0$ over Γ_1, the first term on the RHS of (74) can be written as (see [10], Page 19, [23], Pages 768, 769 and [22], Pages 42, 43),

$$\sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0} \big(\alpha(\sigma_e) u_{\sigma,+} - \alpha(\sigma_b) u_{\sigma^-,+}\big) u_\sigma = \sum_{\sigma\in\mathcal{E}_{\text{ext}}} \big(\alpha(\sigma_e) u_{\sigma,+} - \alpha(\sigma_b) u_{\sigma^-,+}\big) u_\sigma$$

$$
\begin{aligned}
&= \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}} |\alpha(\sigma_e)|(u_{\sigma,+}-u_{\sigma,-})u_{\sigma,+}\\
&= \frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}} |\alpha(\sigma_e)|\big((u_{\sigma,+}-u_{\sigma,-})^2+u_{\sigma,+}^2-u_{\sigma,-}^2\big)\\
&= \frac{1}{2}|v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}+\frac{1}{2}\int_{\partial\Omega}\alpha_{\mathbf{t}}(\boldsymbol{x})v_{\mathcal{T}}^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})\\
&= \frac{1}{2}|v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}+\frac{1}{2}\int_{\Gamma_0}\alpha_{\mathbf{t}}(\boldsymbol{x})v_{\mathcal{T}}^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}),
\end{aligned}\tag{106}
$$

where, since $\alpha(\sigma_e)=0$ for all $\sigma\in\Gamma_1^{\mathcal{T}}$ and $\{u_{\sigma,+},u_{\sigma,-}\}=\{u_{\sigma^+},u_\sigma\}$,

$$
\begin{aligned}
|v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})} &= \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}}|\alpha(\sigma_e)|(u_{\sigma,+}-u_{\sigma,-})^2=\sum_{\sigma\in\Gamma_0^{\mathcal{T}}}|\alpha(\sigma_e)|(u_{\sigma,+}-u_{\sigma,-})^2\\
&= \sum_{\sigma\in\Gamma_0^{\mathcal{T}}}|\alpha(\sigma_e)|(u_{\sigma^+}-u_\sigma)^2.
\end{aligned}\tag{107}
$$

Note that equation (106) has the same form of equation (75). Equation (106) together with (74) (with, as stated before, $(\overline{h},\overline{u}_{\mathcal{T}},\overline{v}_{\mathcal{T}})=(0,u_{\mathcal{T}},v_{\mathcal{T}})$), (104), and (105) imply that

$$
\begin{aligned}
\int_\Omega f_{\mathcal{T}}(\boldsymbol{x})u_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}&+\int_{\Gamma_0}g_{\mathcal{T}}(\boldsymbol{x})v_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})+\int_{\Gamma_1}q_{\mathcal{T}}(\boldsymbol{x})v_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})=|u_{\mathcal{T}}|^2_{1,\kappa,\mathcal{T}}\\
&+\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}}\tau_\sigma^\kappa(u_\sigma-u_K)^2+\int_{\Gamma_0}\beta_{\mathcal{T}}(\boldsymbol{x})v_{\mathcal{T}}^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})+\frac{1}{2}|v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}\\
&+\frac{1}{2}\int_{\Gamma_0}\alpha_{\mathbf{t}}(\boldsymbol{x})v_{\mathcal{T}}^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}).
\end{aligned}
$$

Using this equation, the assumption (A_3), applying Lemma 6.5 and using the notation $0\geq C_\alpha=\min_{\partial\Omega}\alpha_{\mathbf{t}}$, we obtain

$$
\begin{aligned}
|u_{\mathcal{T}}|^2_{1,\mathcal{T}}&+\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}}\tau_\sigma(u_\sigma-u_K)^2+(\beta_0+C_\alpha)\|v_{\mathcal{T}}\|^2_{L^2(\Gamma_0)}+|v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}\\
&\leq C_{49}\Big(\int_\Omega f_{\mathcal{T}}(\boldsymbol{x})u_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}+\int_{\Gamma_0}g_{\mathcal{T}}(\boldsymbol{x})v_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})+\int_{\Gamma_1}q_{\mathcal{T}}(\boldsymbol{x})v_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})\Big).
\end{aligned}\tag{108}
$$

As before, if $C_\alpha\geq-\beta_0/2$ then (108) implies that

$$
\begin{aligned}
|u_{\mathcal{T}}|^2_{1,\mathcal{T}}&+\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}}\tau_\sigma(u_\sigma-u_K)^2+\|v_{\mathcal{T}}\|^2_{L^2(\Gamma_0)}+|v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}\\
&\leq C_{50}\Big(\int_\Omega f_{\mathcal{T}}(\boldsymbol{x})u_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}+\int_{\Gamma_0}g_{\mathcal{T}}(\boldsymbol{x})v_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})+\int_{\Gamma_1}q_{\mathcal{T}}(\boldsymbol{x})v_{\mathcal{T}}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})\Big)\\
&\leq C_{51}\big(\|f\|_{L^2(\Omega)}+\|g\|_{L^2(\Gamma_0)}+\|q\|_{L^2(\Gamma_1)}\big)\big(\|u_{\mathcal{T}}\|_{L^2(\Omega)}+\|v_{\mathcal{T}}\|_{L^2(\partial\Omega)}\big).
\end{aligned}\tag{109}
$$

On the other hand, using that d_σ (defined in the notations following Definition 5.1) satisfies $d_\sigma<\mathrm{diam}(\Omega)$, it yields

$$
\frac{1}{2}\|v_{\mathcal{T}}\|^2_{L^2(\partial\Omega)}\leq\sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}}\mathrm{m}(\sigma)(u_\sigma-u_K)^2+\|\overline{\gamma}(u_{\mathcal{T}})\|^2_{L^2(\partial\Omega)}
$$

$$\leq C_{52}\left(\sum_{\sigma\in\mathcal{E}_{\rm ext}}\tau_\sigma(u_\sigma-u_K)^2+\|\overline{\gamma}(u_{\mathcal{T}})\|^2_{L^2(\partial\Omega)}\right).$$

Gathering this with Lemma 6.3 gives the following estimate on the second term on the RHS of (109):

$$\begin{aligned}\|u_{\mathcal{T}}\|^2_{L^2(\Omega)}+\|v_{\mathcal{T}}\|^2_{L^2(\partial\Omega)}&\leq\|u_{\mathcal{T}}\|^2_{L^2(\Omega)}+2C_{52}\left(\sum_{\sigma\in\mathcal{E}_{\rm ext}}\tau_\sigma(u_\sigma-u_K)^2+\|\overline{\gamma}(u_{\mathcal{T}})\|^2_{L^2(\partial\Omega)}\right)\\&\leq C_{53}\left(\sum_{\sigma\in\mathcal{E}_{\rm ext}}\tau_\sigma(u_\sigma-u_K)^2+\|u_{\mathcal{T}}\|^2_{1,\mathcal{T}}\right).\end{aligned}$$

Further, using that $\tau_\sigma\geq\frac{\mathrm{m}(\sigma)}{\mathrm{diam}(\Omega)}$, we get

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm ext}}\tau_\sigma(u_\sigma-u_K)^2+\|v_{\mathcal{T}}\|^2_{L^2(\Gamma_0)}&\geq C_{54}\left(\sum_{\sigma\in\Gamma_0^{\mathcal{T}}}\mathrm{m}(\sigma)(u_\sigma-u_K)^2+\sum_{\sigma\in\Gamma_0^{\mathcal{T}}}\mathrm{m}(\sigma)(u_\sigma)^2\right)\\&\geq\frac{C_{54}}{2}\|\overline{\gamma}(u_{\mathcal{T}})\|^2_{L^2(\Gamma_0)}.\end{aligned}\tag{110}$$

The LHS of (109) is bounded below by

$$|u_{\mathcal{T}}|^2_{1,\mathcal{T}}+\frac{1}{2}\left(\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm ext}}\tau_\sigma(u_\sigma-u_K)^2+\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm ext}}\tau_\sigma(u_\sigma-u_K)^2+\|v_{\mathcal{T}}\|^2_{L^2(\Gamma_0)}\right)+|v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}.$$

Gathering this with (110) and Lemma 6.3 give the following lower bound for the LHS of (109):

$$\begin{aligned}&|u_{\mathcal{T}}|^2_{1,\mathcal{T}}+\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm ext}}\tau_\sigma(u_\sigma-u_K)^2+\|v_{\mathcal{T}}\|^2_{L^2(\Gamma_0)}+|v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}\\&\quad\geq|u_{\mathcal{T}}|^2_{1,\mathcal{T}}+\frac{1}{2}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm ext}}\tau_\sigma(u_\sigma-u_K)^2+C_{55}\|\overline{\gamma}(u_{\mathcal{T}})\|^2_{L^2(\Gamma_0)}+|v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}\\&\quad\geq C_{56}|||(u_{\mathcal{T}},v_{\mathcal{T}})|||^2,\end{aligned}\tag{111}$$

where

$$\begin{aligned}|||(u_{\mathcal{T}},v_{\mathcal{T}})|||^2:=\|u_{\mathcal{T}}\|^2_{1,\mathcal{T}}+\sum_{\sigma\in\mathcal{E}_{\rm ext}}\tau_\sigma(u_\sigma-u_K)^2+|v_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}\\\leq C_{57}\big(\|f\|_{L^2(\Omega)}+\|g\|_{L^2(\Gamma_0)}+\|q\|_{L^2(\Gamma_1)}\big)|||(u_{\mathcal{T}},v_{\mathcal{T}})|||.\end{aligned}$$

This estimate together with the inequality $(a+b+c)^2\leq 3(a^2+b^2+c^2)$ imply the desired estimate (103).

(ii) **Proof of the existence and uniqueness of the solution.** The finite volume scheme of Definition 7.1 leads to a linear system of equations whose matrix is square since the number of unknowns is $M=M_1+M_2$, where M_1 is the number of elements of $\mathcal{T}$ and M_2 is the number of elements of $\sigma\in\mathcal{E}_{\rm ext}$ under Assumption 5.2. The number of equations is also M. Therefore, the existence of the solution for such scheme is equivalent to its uniqueness. Such uniqueness can be deduced from the estimate (103) by assuming that $(f,g,q)=(0,0,0)$ and using the fact that $\|u_{\mathcal{T}}\|_{1,\mathcal{T}}+\big(\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm ext}}\tau_\sigma(u_\sigma-u_K)^2\big)^{1/2}$ is a norm on $\mathcal{X}(\mathcal{T})=\mathcal{Y}(\mathcal{T})\times\mathcal{Z}(\mathcal{T})\subset L^2(\Omega)\times L^2(\partial\Omega)$, which yields $u_{\mathcal{T}}=0$ and $v_{\mathcal{T}}=0$. This completes the proof of Theorem 7.2.

□

7.3. Convergence rate of the scheme for the Neumann boundary condition (9) on Γ_1

The results of this subsection are summarized in the following new theorem.

Theorem 7.3 (Convergence rate of scheme of Definition 7.1)**.** *In addition to the hypotheses of Theorem 7.2, assume that the weak solution stated in Theorem 4.4 for problem (6), (7) and (9) is "smooth", namely $u \in \mathcal{C}^2(\overline{\Omega})$, and that $\kappa \in \mathcal{C}^1(\overline{\Omega})$. For an admissible mesh $\mathcal{T}$ in the sense of Definition 5.1 satisfying Assumption 5.2, let us define the errors:*

$$e_K = u(\boldsymbol{x}_K) - u_K, \ \ \forall K \in \mathcal{T} \quad \text{and} \quad e_\sigma = u(\boldsymbol{y}_\sigma) - u_\sigma, \ \ \forall \sigma \in \mathcal{E}_{\text{ext}},$$

where $(e_\mathcal{T}, \overline{e}_\mathcal{T})$ denotes the element of $\mathcal{X}(\mathcal{T})$ whose values over (K,σ) are (e_K, e_σ). Assume also that the size$(\mathcal{T})$ *is small, namely,* size$(\mathcal{T}) < 1$*. Then the following error estimate holds:*

$$\|e_\mathcal{T}\|_{1,\mathcal{T}} + \left(\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}} \tau_\sigma (e_\sigma - e_K)^2 \right)^{1/2} + |\overline{e}_\mathcal{T}|_{\alpha,\mathcal{Z}(\mathcal{T})} \le C_{58}\sqrt{\text{size}(\mathcal{T})}, \tag{112}$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the discrete norm given by (63).

Proof. From (87) (second branch) and (9), we deduce that

$$R_{K,\sigma} = \tau_\sigma^\kappa(u(\boldsymbol{y}_\sigma) - u(\boldsymbol{x}_K)) - \text{m}(\sigma) q_\sigma, \quad \forall \sigma \in \mathcal{E}_K \cap \Gamma_1^\mathcal{T}. \tag{113}$$

Using (41) and (87) yield that for all $K \in \mathcal{T}$,

$$\begin{aligned} &- \sum_{\sigma=K|L} \tau_\sigma^\kappa(u(\boldsymbol{x}_L) - u(\boldsymbol{x}_K)) - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^\mathcal{T}} \tau_\sigma^\kappa(u(\boldsymbol{y}_\sigma) - u(\boldsymbol{x}_K)) \\ &\quad - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^\mathcal{T}} \tau_\sigma^\kappa(u(\boldsymbol{y}_\sigma) - u(\boldsymbol{x}_K)) = \text{m}(K) f_K - \sum_{\sigma\in\mathcal{E}_K} R_{K,\sigma}. \end{aligned} \tag{114}$$

Using (7), (87) (second branch) and the rule (39) imply that equation (92) holds for all $\sigma \in \mathcal{E}_K \cap \Gamma_0^\mathcal{T}$. Subtracting (52) (see the definition of scheme given in Definition 7.1) and (101) (second branch) from (114) and (92), respectively, yields, that for all $K \in \mathcal{T}$,

$$- \sum_{\sigma=K|L} \tau_\sigma^\kappa(e_L - e_K) - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^\mathcal{T}} \tau_\sigma^\kappa(e_\sigma - e_K) - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^\mathcal{T}} \tau_\sigma^\kappa(e_\sigma - e_K) = - \sum_{\sigma\in\mathcal{E}_K} R_{K,\sigma} \tag{115}$$

and, for all $\sigma \in \mathcal{E}_K \cap \Gamma_0^\mathcal{T}$ equation (95) holds. Multiplying both sides of (115) by e_K, summing the result over $K \in \mathcal{T}$, and using the reasoning of (105) yield

$$\begin{aligned} - \sum_{K\in\mathcal{T}} e_K \sum_{\sigma\in\mathcal{E}_K} R_{K,\sigma} &= |e_\mathcal{T}|_{1,\kappa,\mathcal{T}}^2 + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}} \tau_\sigma^\kappa (e_\sigma - e_K)^2 \\ &\quad - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^\mathcal{T}} \tau_\sigma^\kappa(e_\sigma - e_K)e_\sigma - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^\mathcal{T}} \tau_\sigma^\kappa(e_\sigma - e_K)e_\sigma, \end{aligned} \tag{116}$$

where $|\cdot|_{1,\kappa,\mathcal{T}}$ is the semi-norm given in Lemma 6.5. From (100) (second branch) and (101) (first branch), we deduce that

$$-\tau_\sigma^\kappa(u_\sigma - u_K) = -\text{m}(\sigma) q_\sigma, \quad \forall \sigma \in \mathcal{E}_K \cap \Gamma_1^\mathcal{T}.$$

Subtracting this from (113) yields

$$-\tau_\sigma^\kappa(e_\sigma - e_K) = -R_{K,\sigma}, \quad \forall \sigma \in \mathcal{E}_K \cap \Gamma_1^\mathcal{T}.$$

Gathering this with (116) yields

$$-\sum_{K\in\mathcal{T}} e_K \sum_{\sigma\in\mathcal{E}_K} R_{K,\sigma} + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^{\mathcal{T}}} R_{K,\sigma}e_\sigma = |e_{\mathcal{T}}|^2_{1,\kappa,\mathcal{T}} + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} \tau^\kappa_\sigma (e_\sigma - e_K)^2 - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} \tau^\kappa_\sigma (e_\sigma - e_K)e_\sigma.$$

Re-ordering the sum of the left hand side of the previous expression and using the property $R_{K,\sigma} = -R_{L,\sigma}$ for all $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$ yield

$$\begin{aligned}
&-\sum_{K\in\mathcal{T}} e_K \sum_{\sigma\in\mathcal{E}_K} R_{K,\sigma} + \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_1^{\mathcal{T}}} R_{K,\sigma}e_\sigma \\
&= \sum_{\sigma=K|L} R_{K,\sigma}(e_L - e_K) + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} R_{K,\sigma}(e_\sigma - e_K) - \sum_{\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}} R_{K,\sigma}e_\sigma.
\end{aligned}$$

Gathering the previous two expressions with (95) (which still holds for all $\sigma\in\mathcal{E}_K\cap\Gamma_0^{\mathcal{T}}$, as stated above) imply that

$$\begin{aligned}
&|e_{\mathcal{T}}|^2_{1,\kappa,\mathcal{T}} + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} \tau^\kappa_\sigma (e_\sigma - e_K)^2 + \sum_{\sigma\in\Gamma_0^{\mathcal{T}}} \left(\alpha(\sigma_e)e_{\sigma,+} - \alpha(\sigma_b)e_{\sigma^-,+}\right)e_\sigma + \int_{\Gamma_0} \beta_{\mathcal{T}}(\boldsymbol{x})(\bar{e}_{\mathcal{T}})^2(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) \\
&\quad + \sum_{\sigma\in\Gamma_0^{\mathcal{T}}} (r_\sigma - r_{\sigma^-})e_\sigma + \sum_{\sigma\in\Gamma_0^{\mathcal{T}}} l_\sigma e_\sigma = \sum_{\sigma=K|L} R_{K,\sigma}(e_L - e_K) + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} R_{K,\sigma}(e_\sigma - e_K).
\end{aligned}$$

This with the reasoning followed between (106)–(109) imply that

$$\begin{aligned}
&|e_{\mathcal{T}}|^2_{1,\mathcal{T}} + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} \tau_\sigma (e_\sigma - e_K)^2 + \|\bar{e}_{\mathcal{T}}\|^2_{L^2(\Gamma_0)} + |\bar{e}_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})} \\
&\quad \le C_{59}\left\{\sum_{\sigma=K|L} R_{K,\sigma}(e_L - e_K) + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} R_{K,\sigma}(e_\sigma - e_K) - \sum_{\sigma\in\Gamma_0^{\mathcal{T}}} (r_\sigma - r_{\sigma^-})e_\sigma - \sum_{\sigma\in\Gamma_0^{\mathcal{T}}} l_\sigma e_\sigma\right\}. \quad (117)
\end{aligned}$$

Let us estimate each term on the RHS of (117):

(i) **Estimate of the first and second terms on the RHS of (117).** Using the Cauchy–Schwarz inequality together with (88) and $\sum_{\sigma\in\mathcal{E}} \mathrm{m}(\sigma)d_\sigma = 2\mathrm{m}(\Omega)$ (see [22], Page 62 or [25], equation (4.3), Page 1025),

$$\begin{aligned}
&\left|\sum_{\sigma=K|L} R_{K,\sigma}(e_L - e_K) + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} R_{K,\sigma}(e_\sigma - e_K)\right| \\
&\quad \le C_{60}\mathrm{size}(\mathcal{T})\left(|e_{\mathcal{T}}|_{1,\mathcal{T}} + \left(\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} \tau_\sigma (e_\sigma - e_K)^2\right)^{\frac{1}{2}}\right). \qquad (118)
\end{aligned}$$

(ii) **Estimate of the third term on the RHS of (117).** Re-ordering the sum and using the Cauchy–Schwarz inequality together with estimate (90) and the fact that $\mathrm{m}(\sigma) \le \mathrm{size}(\mathcal{T})$ imply that (recall that $|\cdot|_{\alpha,\mathcal{Z}(\mathcal{T})}$ is the semi-norm on $\mathcal{Z}(\mathcal{T})$ given by (62) and also (107)) we obtain (98).

(iii) **Estimate of the fourth term on the RHS of (117).** Using the Cauchy–Schwarz inequality together with estimate (93) yield (see also (99))

$$\left|\sum_{\sigma\in\Gamma_0^{\mathcal{T}}} l_\sigma e_\sigma\right| \le C_{41}\mathrm{size}(\mathcal{T}) \sum_{\sigma\in\Gamma_0^{\mathcal{T}}} \mathrm{m}(\sigma)|e_\sigma| \le C_{41}\sqrt{\mathrm{m}(\Gamma_0)}\mathrm{size}(\mathcal{T})\|\bar{e}_{\mathcal{T}}\|_{L^2(\Gamma_0)}. \qquad (119)$$

Using (117) along with (98), (118) and (119), and that size($\mathcal{T}$) < 1, we obtain

$$|e_{\mathcal{T}}|^2_{1,\mathcal{T}} + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}} \tau_\sigma(e_\sigma - e_K)^2 + \|\bar{e}_{\mathcal{T}}\|^2_{L^2(\Gamma_0)} + |\bar{e}_{\mathcal{T}}|^2_{\alpha,\mathcal{Z}(\mathcal{T})}$$
$$\leq C_{61}\sqrt{\text{size}(\mathcal{T})}\left(|e_{\mathcal{T}}|_{1,\mathcal{T}} + \left(\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}} \tau_\sigma(e_\sigma - e_K)^2\right)^{1/2} + |\bar{e}_{\mathcal{T}}|_{\alpha,\mathcal{Z}(\mathcal{T})} + \|\bar{e}_{\mathcal{T}}\|_{L^2(\Gamma_0)}\right).$$

This estimate together with the inequality $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$ imply that

$$|e_{\mathcal{T}}|_{1,\mathcal{T}} + \left(\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}} \tau_\sigma(e_\sigma - e_K)^2\right)^{1/2} + \|\bar{e}_{\mathcal{T}}\|_{L^2(\Gamma_0)} + |\bar{e}_{\mathcal{T}}|_{\alpha,\mathcal{Z}(\mathcal{T})} \leq 4C_{61}\sqrt{\text{size}(\mathcal{T})}.$$

This with the inequality (111) (by choosing $(u_{\mathcal{T}}, v_{\mathcal{T}}) = (e_{\mathcal{T}}, \bar{e}_{\mathcal{T}})$) yield the desired estimate (112). This completes the proof of Theorem 7.3.

□

The problems (6)–(8) and (6), (7), (9) investigated both theoretically and numerically so far have been direct and well-posed problems. In the next section, we formulate and investigate inverse and ill-posed problems arising in boundary corrosion in which the coefficients β and/or α in (7) are unknown along with the main dependent variable u.

8. An application to corrosion

Although material losses resulting from the deterioration of a metal leads to an unknown perturbation of the corroded boundary [37], the linearization of the Butler-Volmer nonlinear boundary condition [47] leads to a simpler problem [31,32] concerned with the determination of the corrosion coefficient β in the Robin boundary condition

$$\kappa(\boldsymbol{x})u_{\mathbf{n}}(\boldsymbol{x}) + \beta(\boldsymbol{x})u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Gamma_0, \tag{120}$$

while assuming that the corroded boundary Γ_0 is known. Related inverse corrosion problems in which Γ_0 is unknown but β is known, or when both Γ_0 and β are unknown can be found *e.g.* in [11,12,33] and [1,4,44,46], respectively.

In order to compensate for the missing information, the potential u is measured on some non-empty open subset K of Γ_1 as

$$u\big|_K = h. \tag{121}$$

The inverse Robin problem concerned with the identification of the impedance corrosion coefficient β in some admissible set

$$\mathcal{B} := \left\{\beta \in \mathcal{C}(\overline{\Gamma_0}) \middle| 0 \not\equiv \beta \geq 0\right\}, \tag{122}$$

along with the potential $u \in H^1(\Omega)$ satifying the Laplace's equation

$$\Delta u = 0 \quad \text{in } \Omega, \tag{123}$$

the Robin boundary condition

$$u_{\mathbf{n}}(\boldsymbol{x}) + \beta(\boldsymbol{x})u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Gamma_0, \tag{124}$$

the Neumann boundary condition (9) with $0 \not\equiv q \in L^2(\Gamma_1)$ and the measured boundary potential (voltage) (121) with $h \in L^2(K)$, was investigated in [15] where the uniqueness and Lipschitz stability of solution were established. In this section, we extend the previous investigations by considering the conductivity equation

$$-\nabla \cdot (\kappa(\boldsymbol{x})\nabla u) = 0 \quad \boldsymbol{x} \in \Omega, \tag{125}$$

in an inhomogeneous material Ω with space-dependent conductivity κ satisfying assumption (A_2), instead of the homogeneous Laplace's equation (123), subject to the generalized boundary condition

$$\kappa(\boldsymbol{x})u_{\mathbf{n}}(\boldsymbol{x}) + (\alpha u)_{\mathbf{t}}(\boldsymbol{x}) + \beta(\boldsymbol{x})u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Gamma_0, \tag{126}$$

instead of the Robin boundary condition (124) (or (120)). Moreover, we investigate the case when the coefficient α satisfying assumption (A_4) (or (A_4')) may also be unknown.

Theorem 8.1 (Identification of β when α is known)**.** *Let $0 \not\equiv q \in L^2(\Gamma_1)$, $h \in L^2(K)$, κ satisfying assumption (A_2) and $\alpha \in \mathcal{C}^1(\overline{\Omega})$ be known. Then the inverse problem (9), (121), (125) and (126) has at most one solution $(\beta, u) \in \mathcal{B} \times H^1(\Omega)$.*

Proof. Following [15], let $(\beta_i, u_i) \in \mathcal{B} \times H^1(\Omega)$ for $i = 1, 2$ be two solutions of the inverse problem (9), (121), (125) and (126). Then the difference $w = u_1 - u_2$ satisfies the Cauchy problem

$$\begin{cases} -\nabla \cdot (\kappa \nabla w) = 0, & \text{in } \Omega, \\ w\big|_K = 0, & w_{\mathbf{n}}\big|_K = 0. \end{cases}$$

Then from the Holmgren's unique continuation theorem, we obtain that $w \equiv 0$ in Ω. This implies that $u_1 = u_2$ in Ω and therefore:

$$\kappa(u_1)_{\mathbf{n}} + (\alpha u_1)_{\mathbf{t}} + \beta_1 u_1 = 0 = \kappa(u_1)_{\mathbf{n}} + (\alpha u_1)_{\mathbf{t}} + \beta_2 u_1 \quad \text{on } \Gamma_0. \tag{127}$$

Thus, $(\beta_1 - \beta_2)u_1 = 0$ on Γ_0. If we assume, by contradiction, that there exists a point in Γ_0 where β_1 and β_2 are not equal, then from the continuity of $\beta_i \in \mathcal{B}$ for $i = 1, 2$, it follows that there exists a neigbourhood V of that point, which is strictly included in Γ_0, where β_1 is distinct of β_2 and hence, it follows that $u_1 = 0$ on V and from (127) also that $(u_1)_{\mathbf{n}} = 0$ on V. This means that $u_1 \in H^1(\Omega)$ satisfies the Cauchy problem

$$\begin{cases} -\nabla \cdot (\kappa \nabla u_1) = 0, & \text{in } \Omega, \\ u_1\big|_V = 0, & (u_1)_{\mathbf{n}}\big|_V = 0. \end{cases}$$

From Holmgren's unique continuation theorem we obtain that $u_1 \equiv 0$ in Ω, which is in contradiction with the fact that $k(u_1)_{\mathbf{n}} = q \not\equiv 0$ on Γ_1. Hence, β_1 and β_2 must coincide on Γ_0 and this concludes the proof of the theorem. □

According to the assumption $(A_4)'$, let us now define

$$\mathcal{A} := \left\{ \alpha \in \mathcal{C}^1(\overline{\Omega}); \alpha\big|_{\Gamma_1} = 0 \right\}.$$

Theorem 8.2 (Identification of α when β is known)**.** *Let $0 \not\equiv q \in L^2(\Gamma_1)$, κ satisfying assumption (A_2) and $\beta \in \mathcal{B}$ be known. For α_1, $\alpha_2 \in \mathcal{A}$, let u_1, $u_2 \in H^1(\Omega)$ be the corresponding solutions of the direct problem (9), (125) and (126). Suppose that $u_1\big|_K = u_2\big|_K$. Then, $u_1 = u_2$ in $\overline{\Omega}$ and $\alpha_1 = \alpha_2$ on Γ_0 (and hence on $\partial\Omega$).*

Proof. As before in the proof of Theorem 8.1, we first obtain that $u_1 = u_2$ in Ω and that

$$\kappa(u_1)_{\mathbf{n}} + (\alpha_1 u_1)_{\mathbf{t}} + \beta u_1 = 0 = \kappa(u_1)_{\mathbf{n}} + (\alpha_2 u_1)_{\mathbf{t}} + \beta u_1 = 0 \quad \text{on } \Gamma_0. \tag{128}$$

Thus, $((\alpha_1-\alpha_2)u_1)_{\mathbf{t}}=0$ on Γ_0. Let us denote $Z:=(\alpha_1-\alpha_2)u_1$ in $\overline{\Omega}$, which satisfies that $\alpha_1,\ \alpha_2\in\mathcal{A}$, $u_1\in H^1(\Omega)$ and $Z_{\mathbf{t}}=0$ on $\partial\Omega$. Then, according to the definition (5) of the tangential derivative we have $\int_\Omega(Z_xv_y-Z_yv_x)\,\mathrm{d}\Omega=0$ for all $v\in H^1(\Omega)$. On sampling with the conjugates of harmonic functions. *i.e.* $w_x=v_y$ and $w_y=-v_x$, we obtain that $0=\int_\Omega\nabla Z\cdot\nabla w\,\mathrm{d}\Omega=\int_{\partial\Omega}Zw_{\mathbf{n}}\,\mathrm{d}\gamma$ for any harmonic function w in Ω. Since $\Omega\subset\mathbb{R}^2$ is simply-connected we can conformally map it to the unit circle and on choosing the trial harmonics $w_k(r,\theta)=r^k\cos(k\theta)$ and $r^k\sin(k\theta)$ for $k\in\mathbb{N}$, we obtain, *via* Fourier analysis, that $Z\big|_{\partial\Omega}=$ constant. Since $\alpha_1=\alpha_2=0$ on Γ_1 it follows that $Z\big|_{\partial\Omega}=0$. Thus, $(\alpha_1-\alpha_2)u_1=0$ on Γ_0. Proceeding as before in the proof of Theorem 8.1, we readily obtain that $\alpha_1=\alpha_2$ on Γ_0 (and hence on $\partial\Omega$). Remark that we only obtain that $\alpha_1=\alpha_2$ on Γ_0 (and hence on $\partial\Omega$), but not inside the domain Ω. ☐

Theorem 8.3 (Simultaneous identification of both α and β)**.** *Let κ satisfying assumption (A_2) be known. Let $0\not\equiv q\in L^2(\Gamma_1)$, $0\not\equiv Q\in L^2(\Gamma_1)$, $h\in L^2(K)$ and $H\in L^2(K)$ be given boundary data. We further assume that q and Q are linearly independent. For $i=1,2$, let $\alpha_i\in\mathcal{A}$, $\beta_i\in\mathcal{B}$ and let $u_i,U_i\in H^1(\Omega)$ be the corresponding solutions of the direct problems*

$$\begin{cases}-\nabla\cdot(\kappa\nabla u_i)=0 & \text{in }\Omega,\\ \kappa(\boldsymbol{x})(u_i)_{\boldsymbol{n}}(\boldsymbol{x})+(\alpha_iu_i)_{\boldsymbol{t}}(\boldsymbol{x})+\beta_i(\boldsymbol{x})u_i(\boldsymbol{x})=0, & \boldsymbol{x}\in\Gamma_0,\\ \kappa(u_i)_{\boldsymbol{n}}=q, & \text{on }\Gamma_1,\\ u_i=h & \text{on }K\end{cases}\tag{129}$$

and

$$\begin{cases}-\nabla\cdot(\kappa\nabla U_i)=0 & \text{in }\Omega,\\ \kappa(\boldsymbol{x})(U_i)_{\boldsymbol{n}}(\boldsymbol{x})+(\alpha_iU_i)_{\boldsymbol{t}}(\boldsymbol{x})+\beta_i(\boldsymbol{x})U_i(\boldsymbol{x})=0, & \boldsymbol{x}\in\Gamma_0,\\ \kappa(U_i)_{\boldsymbol{n}}=Q, & \text{on }\Gamma_1,\\ U_i=H & \text{on }K.\end{cases}\tag{130}$$

Then, $u_1=u_2$, $U_1=U_2$ in $\overline{\Omega}$, $\beta_1=\beta_2$ on Γ_0 and $\alpha_1=\alpha_2$ on Γ_0 (and hence on $\partial\Omega$).

Proof. As before, by Holmgren's unique continuation we obtain that $u_1=u_2$ and $U_1=U_2$ in Ω and

$$\begin{cases}((\alpha_1-\alpha_2)u_1)_{\mathbf{t}}+(\beta_1-\beta_2)u_1=0 & \text{on }\Gamma_0,\\ ((\alpha_1-\alpha_2)U_1)_{\mathbf{t}}+(\beta_1-\beta_2)U_1=0 & \text{on }\Gamma_0.\end{cases}\tag{131}$$

Multiplying in (131) the first equation by U_1 and the second equation by u_1 and subtracting, we obtain

$$(\alpha_1-\alpha_2)((u_1)_{\mathbf{t}}U_1-(U_1)_{\mathbf{t}}u_1)=0\quad\text{on }\Gamma_0.\tag{132}$$

Given that q and Q are non-trivial linearly independent functions on Γ_1 we can infer that the traces $u\big|_{\Gamma_0}$ and $U\big|_{\Gamma_0}$ will also be linearly independent [13]. Indeed, if that is not the case equation (126) implies that $\kappa u_{\mathbf{n}}\big|_{\Gamma_0}$ and $\kappa U_{\mathbf{n}}\big|_{\Gamma_0}$ would also be linearly dependent and, from Holmgren's unique continuation, it would follow that u and U would be linearly dependent in Ω. Then, there would exist $(c_1,c_2)\in\mathbb{R}^2\backslash(0,0)$ such that $c_1u+c_2U\equiv0$ in Ω. Again, by the unique continuation property, it follows that $c_1q+c_2Q\equiv0$ on Γ_1, which contradicts the assumption that q and Q are linearly independent [1]. Therefore, we have proved that $u\big|_{\Gamma_0}$ and $U\big|_{\Gamma_0}$ are linearly independent, which means that their Wronskian $W(u,U):=u_{\mathbf{t}}U-U_{\mathbf{t}}u$ does not vanish on open subsets of Γ_0. From equation (132) and continuity of α_i for $i=1,2$, this implies that $\alpha_1=\alpha_2$ on Γ_0. The remaining statement that $\beta_1=\beta_2$ on Γ_0 follows from one of the equations in (131) along with the arguments used in the proof of Theorem 8.1. ☐

Remark 8.4. Theorems 8.1–8.3 also holds when we consider the Dirichlet data (8) on Γ_1 and the measured flux

$$\kappa u_{\mathbf{n}}\big|_K=q\in L^2(K),\tag{133}$$

on $K\subset\Gamma_1$, instead of the Neumann data (9) on Γ_1 and the measured potential (121) on K.

Remark 8.5. Identifying the boundary Γ_0 along with the corrosion coefficients β and α may also be possible using the techniques of [1, 14], but this investigation is deferred to future work.

Various objective functionals to be minimized have been proposed for the reconstruction of the Robin coefficient β in the inverse problem (121)–(124) and (9). For example, the Tikhonov regularization functional

$$\mathcal{J}_1(\beta) := \|u(\beta) - h\|^2_{L^2(K)} + \lambda\|\beta\|^2_{L^2(\Gamma_0)}, \tag{134}$$

where $\lambda \geq 0$ is the regularization parameter, has been minimized in [34] using the conjugate gradient method (CGM). For ill-posed problems, the functional (134) may not stabilize the solution if $\lambda \geq 0$ is chosen too small while for nonlinear problems it may not be strictly convex and hence the numerical solution will depend on the initial guess. In order to overcome this non-robustness with respect to the initial guess, domain objective functionals (named after R. Kohn and M. Vogelius) have been proposed for solving (121)–(124) and (9), assuming, for simplicity, that $K = \Gamma_1$, *e.g.*:

$$\mathcal{J}_2(\beta) := \|u^D(\beta) - u^N(\beta)\|^2_{L^2(\Omega)} + \lambda\|\beta\|^2_{L^2(\Gamma_0)}, \tag{135}$$

or

$$\mathcal{J}_3(\beta) := \left\|u^D(\beta) - u^N(\beta)\right\|^2_{H^1(\Omega)} + \lambda\|\beta\|^2_{L^2(\Gamma_0)}, \tag{136}$$

see [49], or

$$\mathcal{J}_4(\beta) := \left\|\nabla u^D(\beta) - \nabla u^N(\beta)\right\|^2_{L^2(\Omega)} + \int_{\Gamma_0} \beta|u^D(\beta) - u^N(\beta)|^2, \tag{137}$$

see [15–17], where $u^D(\beta)$, $u^D(\beta) \in H^1(\Omega)$ are the unique solutions of the direct well-posed problems

$$\begin{cases} -\Delta u^D = 0 & \text{in } \Omega, \\ u^D = h & \text{on } \Gamma_1, \\ \partial_{\mathbf{n}} u^D + \beta u^D = 0 & \text{on } \Gamma_0, \end{cases} \tag{138}$$

and

$$\begin{cases} -\Delta u^N = 0 & \text{in } \Omega, \\ \partial_{\mathbf{n}} u^N = q & \text{on } \Gamma_1, \\ \partial_{\mathbf{n}} u^N + \beta u^N = 0 & \text{on } \Gamma_0. \end{cases} \tag{139}$$

As for the set of admissible functions β, one can choose

$$\mathcal{B}_0 := \left\{\beta \in L^\infty(\Gamma_0); 0 < \beta_0 \leq \beta(\boldsymbol{x}) \leq \beta_1 < \infty \; a.e. \; \boldsymbol{x} \in \overline{\Gamma_0}\right\},$$

see [49], such that the assumption (A_3) is satisfied. Compared to the set $\mathcal{B}$, the assumption $\beta > 0$ imposed in the set $\mathcal{B}_0$ is quite natural in atmospheric corrosion [32, 36].

For our inverse problem given by equations (6)–(9) for reconstructing the potential u along with the boundary coefficients β and/or α, similar functionals to $\mathcal{J}_1$–$\mathcal{J}_4$ can be defined, where u^D, $u^N \in H^1(\Omega)$ are the unique solutions of the direct well-posed problems

$$\begin{cases} -\nabla \cdot \left(\kappa \nabla u^D\right) = 0 & \text{in } \Omega, \\ u^D = h & \text{on } \Gamma_1, \\ \kappa\partial_{\mathbf{n}} u^D + \beta u^D + \left(\alpha u^D\right)_{\mathbf{t}} = 0 & \text{on } \Gamma_0, \end{cases} \tag{140}$$

and

$$\begin{cases} -\nabla \cdot \left(\kappa \nabla u^N\right) = 0 & \text{in } \Omega, \\ \kappa \partial_{\mathbf{n}} u^N = q & \text{on } \Gamma_1, \\ \kappa \partial_{\mathbf{n}} u^N + \beta u^N + \left(\alpha u^N\right)_{\mathbf{t}} = 0 & \text{on } \Gamma_0. \end{cases} \tag{141}$$

Finally, remark that the problems (140) and (141) are exactly the direct and well-posed problems analysed and solved using the finite volume method in the previous sections.

9. Conclusion and a perspective

The elliptic diffusion equation with space-dependent conductivity and a generalized oblique boundary condition on a part of the boundary and Dirichlet or Neumann boundary conditions on the remaining part has been considered. The well-posedness of the continuous problems has been established along with new finite volume schemes. The discrete stability and the convergence of these schemes have been shown. The convergence order of these schemes is the same as the one obtained in our previous works [5, 10] which dealt with Poisson's equation with an oblique boundary condition on the whole boundary of the domain. We have also addressed an application to the inverse corrosion problem concerning the reconstruction of coefficients present in the generalized oblique boundary condition that is prescribed over a portion Γ_0 of the boundary $\partial\Omega$ from Cauchy data on the complementary portion $\Gamma_1 = \partial\Omega \backslash \Gamma_0$.

The results obtained in the present paper can be extended, in principle, to the general non-conforming meshes SUSHI of [25] in the following sense: (i) in the case of constant conductivity κ, we can obtain the same discrete well-posedness and error estimates (thanks to [25], Thm. 4.8, Page 1033); (ii) in the case of non-constant conductivity $\kappa(\boldsymbol{x})$, we can obtain the discrete well-posedness and convergence (without rate of convergence) as in Remark 4, Pages 2547, 2548 of [9]. However, this needs a separate study in which the functional tools of [18] may need to be employed. We will address this extension to SUSHI and also to the general framework of the Gradient Discretization Method (GDM), [18, 20], for elliptic and parabolic equations in the future. Finally, as a perspective of this work, we plan to extend the results and applications of this paper to parabolic equations.

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