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Probability of failure of nonlinear oscillators with fractional derivative elements subject to imprecise Gaussian loads

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Abstract. In this paper, an approach for bounding the first-passage probability of a class of nonlinear oscillators with fractional derivative elements and subjected to imprecise stationary Gaussian loads is presented. Specifically, the statistical linearization and stochastic averaging methodologies are used in conjunction with an operator norm-based solution framework to estimate the bounds of the failure probability in a fully decoupled manner. The proposed technique can treat a wide range of nonlinear and hysteretic behaviors with relatively low computational cost. A numerical example is considered to demonstrate the applicability of the proposed approach. Specifically, the bounds of the first-passage probability of a bilinear hysteretic oscillator with fractional derivative elements are estimated.

1. Introduction

The use of stochastic excitation models constitute a flexible approach to evaluate the effect of uncertain dynamic loads on structural systems [1, 2, 3]. Within this framework, first-passage probabilities enable a suitable measure of structural performance [4]. Nonetheless, it is often challenging to select crisp values for the corresponding excitation model parameters due to, e.g., lack of knowledge, scarce data, or conflicting evidence [5]. In this scenario, interval variables can be adopted to represent the excitation model parameters, whereby first-passage probabilities become interval-valued [6]. Nevertheless, evaluating the corresponding failure probability bounds usually proves a computationally daunting task since, in essence, reliability assessment must be performed for different values of the interval parameters [7]. To address this issue, several frameworks have been developed (e.g., [8, 9]). For linear structural systems under Gaussian excitation, the operator norm-based framework proposed in [10, 11] allows determining failure probability bounds with the solution of two standard optimization problems followed by two evaluations of the failure probability. Recently, this framework has been extended in [12] to address nonlinear systems by resorting to the statistical linearization method [13].

Further, structural dynamical analysis incorporating fractional calculus tools and techniques has gained considerable attention during the recent years [14, 15, 16]. Specifically, several approaches have been proposed to assess the response of systems with fractional derivative elements in a stochastic dynamics framework (e.g., [17, 18, 19]). In this context, a continuing challenge in the field of stochastic engineering dynamics relates to the development of computationally efficient methodologies for assessing the reliability of nonlinear systems with fractional derivative elements (e.g., [20, 21, 22, 23, 24]) and, moreover, for bounding the first-passage probability of such systems under interval-valued stochastic loading. To this end, numerically efficient methodologies such as stochastic averaging [25] and statistical linearization [13] present some attractive features related to their capacity to treat a wide range of nonlinear and hysteretic structural models under diverse types of stochastic excitation (e.g. [26, 27, 28]).

This paper proposes an approach for bounding the first-passage probability of nonlinear oscillators with fractional derivative elements and subject to imprecise stationary Gaussian loads. Specifically, the proposed technique is based on the integration of the statistical linearization and stochastic averaging methodologies with an operator norm-based framework [11], which allows estimating failure probability bounds in a fully decoupled manner with relatively low computational cost. The efficacy of the proposed technique is demonstrated by a numerical example pertaining to a bilinear hysteretic oscillator with fractional derivative elements. The obtained results are compared with a reference solution obtained by a direct double-loop approach.

2. Mathematical formulation

2.1. Equivalent linear oscillator: An imprecise probabilities framework

The class of oscillators of interest satisfies the equation of motion

$$\ddot{x}(t) + \beta D_{0,t}^{\alpha} x(t) + g(t, x, \dot{x}) = q(t), \tag{1}$$

where x denotes the response displacement and a dot over a variable accounts for time differentiation. $g(t, x, \dot{x})$ is an arbitrary nonlinear function, β is a constant coefficient and $D_{0,t}^{\alpha}(\cdot)$ represents the Caputo fractional derivative operator of order α, with $0 < \alpha < 1$, given by [14]

$$\mathrm{D}_{0,t}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau, \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function.

Further, $q(t)$ denotes a zero-mean stationary Gaussian excitation process described by the power spectrum $S_{qq}(\omega)$. It is assumed that a set of parameters $\boldsymbol{\theta} \in \mathbb{R}^{n_\theta}$, which are represented as interval variables, characterize the stochastic excitation model. These are bounded by the hyper-rectangle [5, 6]

$$\boldsymbol{\Theta} = \left\{ \boldsymbol{\theta} \in \mathbb{R}^{n_\theta} : \theta_i^L \le \theta_i \le \theta_i^U, i = 1, \ldots, n_\theta \right\}, \tag{3}$$

where θ_i^L and θ_i^U denote, respectively, the lower and upper bounds between which the true value for the i-th parameter is expected to lie [5]. For reliability assessment purposes, a first-passage failure event is defined as [4]

$$F = \max_{t \in [0,T]} \max_{\ell = 1, \ldots, n_\zeta} \left| \frac{\zeta_\ell(t)}{\zeta_\ell^*} \right| > 1, \tag{4}$$

where $\zeta_\ell(t)$ correspond to the responses of interest with thresholds $\zeta_\ell^* > 0$, $\ell = 1, \ldots, n_\zeta$, and T is the simulation period. Since the parameters $\boldsymbol{\theta}$ are interval-valued, the corresponding first-passage failure probability $P_F(\boldsymbol{\theta}) = P(F|\boldsymbol{\theta})$ is also an interval variable, that is,

$$P_F(\boldsymbol{\theta}) \in \left[P_F^L, P_F^U \right] = \left[\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} P_F(\boldsymbol{\theta}), \max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} P_F(\boldsymbol{\theta}) \right], \tag{5}$$

where P_F^L and P_F^U denote the lower and upper bounds of $P_F(\boldsymbol{\theta})$, respectively. In principle, these bounds can be determined by identifying the extrema of the failure probability function. This leads to the so-called double-loop approaches [7], where reliability analysis is performed in the inner loop and the outer loop comprises an optimization procedure (with respect to the parameters $\boldsymbol{\theta}$).

Next, assuming that the oscillator in equation (1) is lightly damped, its response follows a pseudo-harmonic behavior [13, 29]. In this regard, the response displacement has the form

$$x(t) = A(t)\cos(\omega(A)t + \psi(t)), \tag{6}$$

where $A(t)$ and $\psi(t)$ denote the response amplitude and phase, respectively, and $\omega(A)$ represents the amplitude-dependent natural frequency. Since the system is lightly damped, $A(t)$ and $\psi(t)$ are slowly-varying processes with respect to time. Therefore, they can be considered as constant over one cycle of oscillation [13], i.e., $A(t) = A$ and $\psi(t) = \psi$. Further, equation (1) is written for simplicity as [22]

$$\ddot{x}(t) + \beta_0\dot{x}(t) + z(t, x, D_{0,t}^{\alpha}x, \dot{x}) = q(t), \tag{7}$$

where $z(t, x, D_{0,t}^{\alpha}x, \dot{x}) = \beta D_{0,t}^{\alpha}x + g(t, x, \dot{x}) - \beta_0\dot{x}$, and $\beta_0 = 2\zeta_0\omega_0$ is a damping coefficient with ω_0 and ζ_0 denoting the natural frequency and damping ratio of the corresponding linear oscillator.

Then, the oscillator in equation (7) is approximated by [29, 30]

$$\ddot{x}(t) + (\beta_0 + \beta(A))\,\dot{x}(t) + \omega^2(A)x(t) = q(t), \tag{8}$$

where $\beta(A)$ and $\omega(A)$ denote the amplitude-dependent damping and stiffness, respectively. The determination of the latter is done by forming the difference between equations (7) and (8), and minimizing it in the mean-square sense over one cycle of oscillation [13]. This leads to

$$\beta(A) = \frac{\omega_0^2}{A\omega(A)}S(A) + \frac{\beta}{\omega^{1-\alpha}(A)}\sin\left(\frac{\alpha\pi}{2}\right) - \beta_0, \tag{9}$$

where

$$S(A) = -\frac{1}{\pi}\int_0^{2\pi} g(A\cos\phi, -A\omega(A)\sin\phi)\sin\phi \mathrm{d}\phi \tag{10}$$

and $\phi = \omega(A)t + \psi$, and

$$\omega^2(A) = \frac{\omega_0^2}{A}F(A) + \beta\omega^{\alpha}(A)\cos\left(\frac{\alpha\pi}{2}\right), \tag{11}$$

where

$$F(A) = \frac{1}{\pi}\int_0^{2\pi} g(A\cos\phi, -A\omega(A)\sin\phi)\cos\phi \mathrm{d}\phi. \tag{12}$$

Further, assuming that $p(A)$ denotes the response amplitude probability density function (PDF), taking expectations on equations (9) and (11), the equivalent elements are approximated by

$$\beta_{eq} = \int_0^{\infty} \beta(A)p(A)\mathrm{d}A \tag{13}$$

and

$$\omega_{eq}^2 = \int_0^{\infty} \omega^2(A)p(A)\mathrm{d}A. \tag{14}$$

Therefore, the equivalent linear system in equation (8) takes the form

$$\ddot{x}(t) + (\beta_0 + \beta_{eq})\,\dot{x}(t) + \omega_{eq}^2 x(t) = q(t). \tag{15}$$

The determination of the equivalent linear damping and stiffness elements in equations (13) and (14) rely on the response amplitude PDF, which is done by resorting to the stochastic averaging method.

In this regard, the stochastic differential equation governing the slowly varying response amplitude process is constructed. The associated Fokker-Planck equation is given by (e.g. [25])

$$\frac{\partial p(A)}{\partial t} = -\frac{\partial}{\partial A}\left\{\left(-\frac{1}{2}(\beta_0+\beta_{eq})A+\frac{\pi S(\omega_{eq})}{2\omega_{eq}^2 A}\right)p(A)\right\} + \frac{1}{4}\frac{\partial}{\partial A}\left\{\frac{\pi S(\omega_{eq})}{\omega_{eq}^2}\frac{\partial p(A)}{\partial A}+\frac{\partial}{\partial A}\left(\frac{\pi S(\omega_{eq})}{\omega_{eq}^2}p(A)\right)\right\}. \quad (16)$$

In general, if the system under consideration is a linear oscillator subject to stationary excitation, the solution of equation (16) is readily available in the form of a Rayleigh distribution (e.g., [31, 32]). Moreover, for the case of a linear oscillator with fractional derivative elements, a closed-form expression for the response amplitude PDF is given by [33]

$$p(A) = \frac{\sin\left(\frac{\alpha\pi}{2}\right)A}{\omega_0^{1-\alpha}\sigma^2}\exp\left(-\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}}\frac{A^2}{2\sigma^2}\right), \quad (17)$$

where σ^2 denotes the stationary response variance of a linear oscillator subject to white noise excitation [13].

2.2. First-passage failure probability bounds estimation

The methodology described in section 2.1 yields a linear relationship between the oscillator response and the stochastic excitation corresponding to a given value of $\boldsymbol{\theta}$. This enables the implementation of an operator norm-based decoupling framework [10, 12] to estimate the bounds of the first-passage failure probability in equation (5). In this regard, the zero-mean discrete Gaussian load in equation (15) is represented via the Karhunen-Loève (K-L) expansion [34]. Specifically, it is assumed that $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ denotes the diagonal $n_\xi \times n_\xi$ matrix containing the n_ξ largest eigenvalues of the stochastic load covariance matrix $\boldsymbol{\Sigma}(\boldsymbol{\theta})$ and $\boldsymbol{\Upsilon}(\boldsymbol{\theta})$ denotes the $n_T \times n_\xi$ matrix of the corresponding eigenvectors. Then, the loading at time $t_k = (k-1)\Delta t$, $k = 1, \dots, n_T$, where Δt represents the time step and $n_T = T/\Delta t + 1$ is the number of time instants, is given by

$$q(t_k, \boldsymbol{\theta}, \boldsymbol{\xi}) = \boldsymbol{\psi}_k^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\xi}. \quad (18)$$

In equation (18), $\boldsymbol{\psi}_k(\boldsymbol{\theta})$ corresponds to the k-th column of the matrix $\boldsymbol{\Psi}(\boldsymbol{\theta}) = \boldsymbol{\Lambda}^{1/2}(\boldsymbol{\theta})\boldsymbol{\Upsilon}^{\mathrm{T}}(\boldsymbol{\theta})$ and $\boldsymbol{\xi} \in \mathbb{R}^{n_\xi}$ comprises n_ξ standard Gaussian random variables.

Further, assuming that the n_T discrete values of the ℓ-th normalized response of interest are contained in the vector

$$\bar{\mathbf{h}}_\ell(\boldsymbol{\theta},\boldsymbol{\xi}) = \frac{1}{h_\ell^*}\left[h_\ell(t_1,\boldsymbol{\theta},\boldsymbol{\xi}) \quad \cdots \quad h_\ell(t_{n_T},\boldsymbol{\theta},\boldsymbol{\xi})\right]^{\mathrm{T}}, \quad (19)$$

$\ell = 1, \dots, n_\zeta$, leads to

$$\bar{\mathbf{h}}(\boldsymbol{\theta},\boldsymbol{\xi}) = \left[\bar{\mathbf{h}}_1^{\mathrm{T}}(\boldsymbol{\theta},\boldsymbol{\xi}) \quad \dots \quad \bar{\mathbf{h}}_{n_\zeta}^{\mathrm{T}}(\boldsymbol{\theta},\boldsymbol{\xi})\right]^{\mathrm{T}}, \quad (20)$$

which comprises all the discrete values of all responses of interest for a given realization of $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$. Then, considering equation (15), the relationship between the discrete responses of interest and the basic random variables is linear and can be written as [35]

$$\bar{\mathbf{h}}(\boldsymbol{\theta},\boldsymbol{\xi}) = \mathbf{M}(\boldsymbol{\theta})\boldsymbol{\xi}. \quad (21)$$

In equation (21), the linear mapping $\mathbf{M}(\boldsymbol{\theta}) \in \mathbb{R}^{n_T n_\zeta \times n_\xi}$ depends on the response thresholds, the adopted integration scheme for equation (15), and the matrix $\boldsymbol{\Psi}(\boldsymbol{\theta})$. Hence, $\mathbf{M}(\boldsymbol{\theta})$ is a function of $\boldsymbol{\theta}$ since these parameters determine the K-L expansion vectors in equation (18) and the properties of the equivalent oscillator in equation (15). The induced (p_1, p_2)-norm of this matrix is defined as

$$\|\mathbf{M}(\boldsymbol{\theta})\|_{p_1,p_2} = \sup_{\boldsymbol{\xi}\neq\mathbf{0}} \frac{\|\mathbf{M}(\boldsymbol{\theta})\boldsymbol{\xi}\|_{p_1}}{\|\boldsymbol{\xi}\|_{p_2}} = \sup_{\boldsymbol{\xi}\neq\mathbf{0}} \frac{\|\bar{\mathbf{h}}(\boldsymbol{\theta},\boldsymbol{\xi})\|_{p_1}}{\|\boldsymbol{\xi}\|_{p_2}}, \tag{22}$$

where $\|\cdot\|_{p_i}$ denotes the p_i-norm of a vector $(i = 1, 2)$. Thus, $\|\mathbf{M}(\boldsymbol{\theta})\|_{p_1,p_2}$ measures the maximum amplification of $\|\bar{\mathbf{h}}(\boldsymbol{\theta},\boldsymbol{\xi})\|_{p_1}$ with respect to the magnitude of the input vector, i.e., $\|\boldsymbol{\xi}\|_{p_2}$. The values $p_1 = \infty$ and $p_2 = 2$ are adopted herein [10, 12]. Then, the main idea of the employed operator norm framework is to use $\|\mathbf{M}(\boldsymbol{\theta})\|_{\infty,2}$ as a numerically efficient proxy of the failure probability function $P_F(\boldsymbol{\theta})$ in order to identify the values of $\boldsymbol{\theta}$ that yield the bounds in equation (5). Specifically, this scheme leads to

$$\left[P_F^L, P_F^U\right] \approx \left[P_F\left(\boldsymbol{\theta}^{*,L}\right), P_F\left(\boldsymbol{\theta}^{*,U}\right)\right], \tag{23}$$

where

$$\boldsymbol{\theta}^{*,L} = \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}}{\operatorname{argmin}} \|\mathbf{M}(\boldsymbol{\theta})\|_{\infty,2} \tag{24}$$

and

$$\boldsymbol{\theta}^{*,U} = \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}}{\operatorname{argmax}} \|\mathbf{M}(\boldsymbol{\theta})\|_{\infty,2}\ . \tag{25}$$

The proposed framework for estimating the bounds in equation (5) is summarized in the following two-step procedure:

I. Determine the parameter values $\boldsymbol{\theta}^{*,L}$ and $\boldsymbol{\theta}^{*,U}$ by solving equations (24) and (25). In this regard, evaluating $\|\mathbf{M}(\boldsymbol{\theta})\|_{\infty,2}$ for a given value of $\boldsymbol{\theta}$ involves determining an equivalent linear oscillator according to section 2.1, and then finding the corresponding matrix $\mathbf{M}(\boldsymbol{\theta})$ in equation (21).
II. Estimate the failure probability bounds $P_F^L \approx P_F(\boldsymbol{\theta}^{*,L})$ and $P_F^U \approx P_F(\boldsymbol{\theta}^{*,U})$ using the nonlinear oscillator in equation (1) and any appropriate reliability analysis technique [4].

3. Numerical example

A bilinear hysteretic oscillator with fractional derivative elements is considered as numerical example (e.g., [36]). The oscillator response is given by equation (1) with

$$g(t, x, \dot{x}) = \gamma\omega_0^2 x(t) + (1-\gamma)\omega_0^2 x_y z. \tag{26}$$

In equation (26), γ denotes the post- to pre-yield stiffness ratio, x_y is the yielding onset, and z is a state variable such that

$$x_y \dot{z} = \dot{x}\left(1 - H(\dot{x})H(z-1) - H(-\dot{x})H(-z-1)\right), \tag{27}$$

where $H(\cdot)$ is the Heaviside step function. The following parameter values are considered: $\beta = 6.32$, $\alpha = 0.5$, $\gamma = 0.7$, $\omega_0 = 10$, and $x_y = 0.01$.

Further, the zero-mean Gaussian stochastic excitation $q(t)$ in equation (1) is described by the Clough-Penzien spectrum [37]

$$S_{qq}(\omega) = \frac{\omega^4\left(\omega_g^4 + (2\zeta_g\omega_g\omega)^2\right)S_0}{\left((\omega_g^2-\omega^2)^2 + (2\zeta_g\omega_g\omega)^2\right)\left((\omega_f^2-\omega^2)^2 + (2\zeta_f\omega_f\omega)^2\right)}, \tag{28}$$

where S_0 is the intensity of the excitation (in $(m/s^2)^2/Hz$), ω_g and ω_f are the natural circular frequencies of the filter (in rad/s), and ζ_g and ζ_f the corresponding damping ratios. It is assumed that all parameters characterizing $S_{qq}(\omega)$ in equation (28) are interval-valued, and thus, $\boldsymbol{\theta} = [S_0, \omega_g, \omega_f, \zeta_g, \zeta_f]^{\mathrm{T}}$. The corresponding lower and upper bounds of the different parameters are shown in table 1.

Table 1. Lower and upper bounds of the interval parameters.

Parameter	Lower bound	Upper bound
S_0	0.040	0.060
ω_g	9.976	14.964
ω_f	4.344	6.516
ζ_g	0.640	0.960
ζ_f	0.544	0.816

The first-passage failure event is defined in terms of the oscillator displacement $x(t)$ as

$$F = \max_{t \in [0,T]} \frac{|x(t)|}{x^*} > 1 \tag{29}$$

where $x^* = 0.111$ and $T = 20\ s$ is the reference period with time step $\Delta t = 0.01$. Further, considering the entire set of eigenvectors of the covariance matrix are retained to construct the K-L expansion in equation (18), a total of $n_\xi = 2001$ random variables are involved in the discrete representation of $q(t)$.

The implementation of the proposed framework relies on determining the equivalent linear oscillator in equation (15). Therefore, taking into account the hysteretic term defined by equations (26) and (27), equations (10) and (12) yield

$$S(A) = \begin{cases} \frac{4x_y}{\pi}\left(1 - \frac{x_y}{A}\right), & A > x_y \\ 0, & A \le x_y \end{cases} \tag{30}$$

and

$$F(A) = \begin{cases} \frac{A}{\pi}\left(\Lambda - \frac{1}{2}\sin(2\Lambda)\right), & A > x_y \\ A, & A \le x_y \end{cases}, \tag{31}$$

respectively, where $\Lambda = \arccos\left(1 - \frac{2x_y}{A}\right)$. Then, taking into account equations (9), (11), (30) and (31), as well as equation (17), the equivalent elements in equation (15) are determined as functions of the stationary response variance σ^2, i.e., $\beta_{eq} = \beta_{eq}(\sigma^2)$ and $\omega_{eq} = \omega_{eq}(\sigma^2)$. Indicatively,

$$\begin{aligned} \beta_{eq} = & -\beta_0 + \frac{\beta \sin^2\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}\sigma^2} \int_0^\infty \frac{A}{\omega^{1-\alpha}(A)} \exp\left(-\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}} \frac{A^2}{2\sigma^2}\right) \mathrm{d}A \\ & + \frac{4x_y\omega_0^2(1-\gamma)\sin\left(\frac{\alpha\pi}{2}\right)}{\pi\omega_0^{1-\alpha}\sigma^2} \int_{x_y}^\infty \frac{1 - \frac{x_y}{A}}{\omega(A)} \exp\left(-\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}} \frac{A^2}{2\sigma^2}\right) \mathrm{d}A \end{aligned} \tag{32}$$

and a corresponding expression holds for ω_{eq}^2. The stationary response variance corresponding to the equivalent linear oscillator in equation (8) depends, in turn, on the equivalent linear damping and stiffness elements [13], i.e., $\sigma^2 = \sigma^2(\omega_{eq}, \beta_{eq})$. Thus, a coupled set of nonlinear

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algebraic equations is constructed and then solved for computing β_{eq}, ω_{eq} and σ^2. This is done in an iterative manner (e.g., [13, 38]) or by resorting to any pertinent numerical scheme.

The optimization problems defined in equations (24) and (25) are solved to identify the parameter values that bound the failure probability and, subsequently, the failure probability is evaluated at these values by considering the nonlinear oscillator in equations (1), (26) and (27). A pattern search algorithm (e.g., [39]) is implemented to solve the optimization problems, while direct Monte Carlo simulation with 2.5×10^4 generated samples is employed for reliability assessment. In passing, note that alternative optimization techniques and reliability assessment methods can be also considered for the practical implementation of the proposed approach.

The results obtained by the proposed approach are presented in table 2. Specifically, the optimal values $\boldsymbol{\theta}^* = [S_0^*, \omega_g^*, \omega_f^*, \zeta_g^*, \zeta_f^*]^{\mathrm{T}}$ corresponding to the failure probability bounds P_F^L and P_F^U, as well as the associated values of the failure probability, $P_F(\boldsymbol{\theta}^*)$, and operator norm, $\|\mathbf{M}(\boldsymbol{\theta}^*)\|_{\infty,2}$, are shown in the table. Further, the failure probability bounds obtained by a double-loop approach are also reported. These results, which can be regarded as reference values, are obtained by solving the optimization problems in equation (5) directly. To this end, a pattern search algorithm is considered and direct Monte Carlo simulation with a sample size of 2.5×10^4 is used for estimating $P_F(\boldsymbol{\theta})$. It is readily seen that the reference values for the failure probability bounds agree with those estimated by the proposed approach. Furthermore, the optimal parameter values identified by both methods are relatively similar between each other. Notably, the proposed approach requires only two reliability analyses, which leads to significant computational savings in this case. The obtained results highlight the validity of the adopted solution strategy for this example. That is, the use of the statistical linearization and stochastic averaging methodologies in conjunction with the operator norm defined in equation (22) enables a numerically efficient proxy that allows bounding the failure probability function in a fully decoupled manner.

Table 2. Results obtained by the proposed approach and a direct solution scheme.

	Proposed approach		Double-loop approach	
	P_F^L	P_F^U	P_F^L	P_F^U
S_0^*	0.040	0.060	0.040	0.060
ω_g^*	9.976	11.167	10.470	12.275
ω_f^*	6.516	4.344	6.491	4.434
ζ_g^*	0.960	0.640	0.925	0.644
ζ_f^*	0.816	0.544	0.805	0.548
$P_F(\boldsymbol{\theta}^*)$	6.8×10^{-4}	7.96×10^{-1}	7.2×10^{-4}	7.99×10^{-1}
$\|\mathbf{M}(\boldsymbol{\theta}^*)\|_{\infty,2}$	3.7×10^{-3}	1.92×10^{-2}	4.1×10^{-3}	1.86×10^{-2}
Time (s)	8.09×10^2	7.10×10^2	3.56×10^4	6.79×10^4

4. Conclusion

In this paper an approach for bounding the first-passage probability of a class of nonlinear oscillators with fractional derivative elements and subject to imprecise stationary Gaussian loads has been developed. This has been done by combining the statistical linearization and stochastic averaging methodologies with a recently proposed operator norm-based solution framework, which allows bounding the first-passage probability in a fully decoupled manner. Specifically, an operator norm related to the equivalent linear oscillator obtained at any given value of the excitation model parameters is employed as a numerically efficient proxy of the failure probability in order to identify the parameter values that yield the extrema of the failure probability. Ultimately, the solution of two standard optimization problems and two corresponding reliability analyses are required to estimate the bounds of the first-passage probability. The efficacy of the proposed approach has been demonstrated by a numerical example consisting of a bilinear hysteretic oscillator with fractional derivative elements subject to a stochastic excitation modeled by filtered Gaussian noise, while reference results obtained from a standard double-loop approach have been also provided for comparison.

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