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Strongly complete axiomatization for a logic with probabilistic interventionist counterfactuals

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Abstract. Causal multiteam semantics is a framework where probabilistic notions and causal inference can be studied in a unified setting. We study a logic ($\mathcal{PCO}$) that features marginal probabilities, observations and interventionist counterfactuals, and allows expressing conditional probability statements, *do* expressions and other mixtures of causal and probabilistic reasoning. Our main contribution is a strongly complete infinitary axiomatisation for $\mathcal{PCO}$.

1 Introduction

In the past few decades, the study of causation has transformed from being a topic of mere philosophical speculation to a discipline making use of rigorous mathematical tools. The main two strands of this new discipline, paralleling the division of roles between probability and statistics, are *causal inference* ([16, 28, 29]) and *causal discovery* ([33]). The former studies which causal effects can be inferred from data coupled with causal assumptions about the processes which generated the data. The latter studies which causal connections are compatible with given data (coming from observations or experiments). In both strands new languages, capable of expressing concepts that lie beyond the merely associational or probabilistic properties of data, are needed. A key novel concept that is required is the notion of an *intervention* (modifying a given system). One way of describing interventions is given by expressions called *interventionist counterfactuals*. In their simplest form, these are expressions such as:

If variables $X_1, \ldots, X_n$ were fixed to values $x_1, \ldots, x_n$, then variable Y would take value y

or their (causal-)probabilistic counterparts:

If variables $X_1, \ldots, X_n$ were fixed to values $x_1, \ldots, x_n$, then the probability that variable Y takes value y would be ϵ.

Typically, such expressions are given precise semantics by *causal models* (also known as *structural equation models*). Causal models and interventionist counterfactuals have been reabsorbed as mainstream ideas in the philosophical debate on causation [22,35] but also became widespread tools for the study of causation in disparate applied fields such as epidemiology [21], econometrics [19], social sciences [25] and machine learning [31]. As a recent development, J. Pearl argued that the capability of AI systems to represent and reason about causal knowledge will be the next important leap in the field of artificial intelligence (see, e.g., [30]).

The simple interventionist counterfactuals exhibited above do not exhaust the wide variety of causal-probabilistic expressions that appear in the literature on causal inference (an extended discussion of this issue can be found in [10]). In [28], Pearl emphasizes two kinds of formal notations, the (conditional) *do expressions*, and what we may call, for lack of a better terminology, *Pearl counterfactuals*. These expressions concern probabilities in a post-intervention scenario, but differ in whether one conditions upon events of the pre-intervention or the post-intervention scenario. A conditional *do* expression discusses conditioning over a post-intervention event, as in the statement "The probability that a patient abandons treatment, if he does not quickly improve, is ϵ"; in symbols:

$$Pr(Abandon = 1 \mid do(Treated = 1), Improve = 0) = \epsilon$$

where $Abandon$, $Treated$ and $Improve$ are Boolean variables taking values 1 or 0 depending on whether a certain fact holds or not. On the other hand, a Pearl counterfactual conditions in the pre-intervention system, so that there might even be contradictions between the measured and the conditioning event. E.g., "The probability that a patient who died would have recovered if treated is ϵ":

$$Pr(Dies = 0 \mid do(Treated = 1), Dies = 1) = \epsilon.$$

In [4], Barbero and Sandu propose to tame this wild proliferation of notational devices by decomposing these kinds of expressions in terms of three simpler ingredients: marginal probabilities, interventionist counterfactuals ($\boxright$), and selective implications ($\supset$). The selective implication describes the effect of acquiring new information, whereas the interventionist counterfactual describes the effect of an action. The complex expressions described above become, respectively,

$$Treated = 1 \boxright (Improve = 0 \supset \Pr(Abandon = 1) = \epsilon)$$

$$Dies = 1 \supset (Treated = 1 \boxright \Pr(Dies = 0) = \epsilon)$$

showing that qualitative difference between the two kinds of expressions amounts to an inversion in the order of application of two logical operators.

Interventionist counterfactuals, selective implications, and marginal probability statements can be studied in a shared semantic framework called *causal multiteam semantics*. The framework is meaningful already in a non-probabilistic context, where it generalizes causal models by providing a (qualitative) account of imperfect information,[3] and where it has been studied both from a semantic and a proof-theoretic perspective [3, 5, 9]. The proof-theoretic results rely on a body of earlier work ([12, 14, 17]) on proof systems for (non-probabilistic) counterfactuals evaluated on causal models. In the probabilistic setting, some work in the semantic direction is forthcoming [6, 7].

In this paper, we initiate the proof-theoretic study of logics involving probabilistic counterfactuals in the causal multiteam setting. To the best of our knowledge, there

[3] The idea of modeling imperfect information via *team semantics* was developed by Hodges [23] and Väänänen [34].

has been only one proposal in the literature of a deduction system for probabilistic interventionist counterfactuals ([24]). The language considered in [24] differs in many respects from those we are interested in. It is more expressive in allowing the use of arithmetical operations (sums and products of probabilities and scalars). In contrast, it is also less expressive, since it does not allow for nesting of counterfactuals (iterated interventions), and it has no obvious means for describing complex interactions of interventions and conditioning. For example, it has no obvious way to condition at the same time on *both* a pre-intervention and a post-intervention scenario, or condition on a state of affairs that holds at an intermediate stage between two interventions. Both of these scenarios can be dealt with in relative ease using the framework of Barbero and Sandu [6]: the former by expressions of the form $\alpha \supset (X = x \mathrel{\Box\!\!\rightarrow} (\beta \supset \Pr(\gamma) = \epsilon))$ and the latter by $X = x \mathrel{\Box\!\!\rightarrow} (\alpha \supset (Z = z \mathrel{\Box\!\!\rightarrow} \Pr(\gamma) = \epsilon))$.

Axiomatizing probabilistic logics is a notoriously difficult problem. As soon as a language allows expressing inequalities of the form $\Pr(\alpha) \leq \epsilon$ (ϵ being a rational number), it is not compact, as for example the set of formulas of the form $\Pr(\alpha) \leq \frac{1}{n}$ (n natural number) entails that $\Pr(\alpha) = 0$, but no finite subset yields the same conclusion. Consequently, no usual, finitary deduction system can be strongly complete for such a language. A possible answer to this problem is to settle for a deductive system that is weakly complete, i.e. it captures all the correct inferences from *finite* sets of formulas. This has been achieved for a variety of probabilistic languages with arithmetic operations (e.g. [13]). The result for probabilistic interventionist counterfactuals mentioned above ([24]) is a weak completeness result in this tradition. Proving weak completeness for probabilistic languages *without* arithmetical operations seems to be a more difficult task, and we could find only one such result in the literature ([20])[4]. Unfortunately, the completeness proof of [20] relies on a model-building method that seems not to work for languages where *conditional* probabilities are expressible; thus, it is not adaptable in any straightforward way to our case.

Another path, on which we embark, is to respond to the failure of compactness by aiming for strong completeness using a deduction system with some kind of infinitary resources. The use of infinitary deduction rules (with countably many premises) has proved to be very fruitful and has led to strong completeness theorems for a plethora of probabilistic languages (cf. [26]). Of particular interest to us are [32], where strong completeness is obtained for a language with conditional probabilities, and [27], which obtains strong completeness for "qualitative probabilities" (i.e., for expressions such as $\Pr(\alpha) \leq \Pr(\beta)$, that do not involve numerical constants). We build on these works in order to obtain a strongly complete deduction system (with two infinitary rules) for the probabilistic-causal language $\mathcal{PCO}$ used in [6, 7]. The proof proceeds via a canonical model construction, relying on a Lindenbaum lemma whose proof takes into account the role of infinitary rules. While the proof follows essentially the scheme of [32], it presents peculiar difficulties of its own due to the presence of additional operators (counterfactuals and comparison atoms).

[4] An axiomatization of this kind has also been found for a probabilistic *fuzzy* logic ([15]), which has been proved to be intertranslatable with (classical) probabilistic logic with arithmetical operators ([2]).

2 Preliminaries

Capital letters such as $X, Y, \ldots$ denote **variables** (thought to stand for specific magnitudes such as "temperature", "volume", etc.) which take **values** denoted by small letters (e.g. the values of the variable X will be denoted by $x, x', \ldots$). Sets (and tuples, depending on the context) of variables and values are denoted by boldface letters such as $\mathbf{X}$ and $\mathbf{x}$. We consider probabilities that arise from the counting measures of finite (multi)sets. For finite sets $S \subseteq T$, we define $P_T(S) := \frac{|S|}{|T|}$.

A **signature** is a pair (Dom, Ran), where Dom is a nonempty, finite set of variables and Ran is a function that associates to each variable $X \in \mathrm{Dom}$ a nonempty, finite set $\mathrm{Ran}(X)$ of values (the **range** of X). We consider throughout the paper a fixed ordering of Dom, and write $\mathbf{W}$ for the tuple of all variables of Dom listed in such order. Furthermore, we write $\mathbf{W}_X$ for the variables of $\mathrm{Dom} \setminus \{X\}$ listed according to the fixed order. Given a tuple $\mathbf{X} = (X_1, \ldots, X_n)$ of variables, we denote as $\mathrm{Ran}(\mathbf{X})$ the Cartesian product $\mathrm{Ran}(X_1) \times \cdots \times \mathrm{Ran}(X_n)$. An **assignment** of signature σ is a mapping $s : \mathrm{Dom} \to \bigcup_{X \in \mathrm{Dom}} \mathrm{Ran}(X)$ such that $s(X) \in \mathrm{Ran}(X)$ for each $X \in \mathrm{Dom}$. The set of all assignments of signature σ is denoted by $\mathbb{B}_\sigma$. Given an assignment s that has the variables of $\mathbf{X}$ in its domain, $s(\mathbf{X})$ will denote the tuple $(s(X_1), \ldots, s(X_n))$. For $\mathbf{X} \subseteq \mathrm{Dom}$, $s_{\upharpoonright \mathbf{X}}$ denotes the restriction of s to the variables in $\mathbf{X}$.

A **team** T of signature σ is a subset of $\mathbb{B}_\sigma$. Intuitively, a multiteam is just a multiset analogue of a team. We represent **multiteams** as (finite) sets of assignments with an extra variable *Key* (not belonging to the signature) ranging over $\mathbb{N}$, which takes different values over different assignments of the team, and which is never mentioned in the formal languages. A multiteam can be represented as a table, in which each row represents an assignment. For example, if $Dom = \{X, Y, Z\}$, a multiteam may look like this:

Key	X	Y	Z
0	x	y	z
1	x'	y'	z'
2	x'	y'	z'

The purpose of a multiteam is to encode a probability distribution (over the team obtained by removing the variable *Key*); in this case, that the assigment $s(X) = x, s(Y) = y, s(Z) = z$ has probability $\frac{1}{3}$ while the assignment $t(X) = x', t(Y) = y', t(Z) = z'$ has probability $\frac{2}{3}$. Multiteams by themselves do not encode any solid notion of causation; they do not tell us how a system would be affected by an intervention. We therefore need to enrich multiteams with additional structure.

Definition 1. *A **causal multiteam** T of signature* $(\mathrm{Dom}(T), \mathrm{Ran}(T))$ *with endogenous variables* $\mathbf{V} \subseteq \mathrm{Dom}(T)$ *is a pair* $T = (T^-, \mathcal{F})$ *such that*

1. T^- *is a multiteam of domain* $\mathrm{Dom}(T)$*,*
2. $\mathcal{F}$ *is a function* $\{(V, \mathcal{F}_V) \mid V \in \mathbf{V}\}$ *that assigns to each endogenous variable* V *a non-constant* $|\mathbf{W}_V|$*-ary function* $\mathcal{F}_V : \mathrm{Ran}(\mathbf{W}_V) \to \mathrm{Ran}(V)$*, and*
3. *The **compatibility constraint** holds:* $\mathcal{F}_V(s(\mathbf{W}_V)) = s(V)$ *for all* $s \in T^-$ *and* $V \in \mathbf{V}$.

We will also write End(T) for the set of endogenous variables of T. Due to the compatibility constraint, not all instances for $\mathbf{V}$ and T^- give rise to causal multiteams.

The function $\mathcal{F}$ induces a system of structural equations; an equation

$$V := \mathcal{F}_V(\mathbf{W}_V)$$

for each variable $V \in End(T)$. A structural equation tells how the value of V should be recomputed if the value of some variables in $\mathbf{W}_V$ is modified. Note that that some of the variables in $\mathbf{W}_V$ may not be necessary for evaluating V. For example, if V is given by the structural equation $V := X + 1$, all the variables in $\mathbf{W}_V \setminus \{X\}$ are irrelevant (we call them **dummy arguments** of $\mathcal{F}_V$). The set of non-dummy arguments of $\mathcal{F}_V$ is denoted as PA_V (the set of **parents** of V).

We associate to each causal multiteam T a **causal graph** G_T, whose vertices are the variables in Dom and where an arrow is drawn from each variable in PA_V to V, whenever $V \in \mathrm{End}(T)$. The variables in $\mathrm{Dom}(T) \setminus \mathrm{End}(T)$ are called **exogenous**. In this paper, we will always assume that causal graphs are acyclic; a causal multiteam with an acyclic causal graph is said to be **recursive**.

Definition 2. *A causal multiteam $S = (S^-, \mathcal{F}_S)$ is a **causal sub-multiteam** of $T = (T^-, \mathcal{F}_T)$, if they have same signature, $S^- \subseteq T^-$, and $\mathcal{F}_S = \mathcal{F}_T$. We then write $S \leq T$.*

We consider causal multiteams as dynamic models, that can be affected by various kinds of operations – specifically, by observations and interventions. Given a causal multiteam $T = (T^-, \mathcal{F})$ and a formula α of some formal language (evaluated over assignments according to some semantic relation $\models$), "observing α" produces the causal sub-multiteam $T^\alpha = ((T^\alpha)^-, \mathcal{F})$ of T, where

$$(T^\alpha)^- := \{s \in T^- \mid (\{s\}, \mathcal{F}) \models \alpha\}.^5$$

An intervention on T will *not*, in general, produce a sub-multiteam of T. It will instead modify the values that appear in some of the columns of T. We consider interventions that are described by formulas of the form $X_1 = x_1 \land \cdots \land X_n = x_n$ (or, shortly, $\mathbf{X} = \mathbf{x}$). Such a formula is **inconsistent** if there are two indexes i, j such that X_i and X_j denote the same variable, while x_i and x_j denote distinct values; it is **consistent** otherwise.

Applying an intervention $do(\mathbf{X} = \mathbf{x})$, where $\mathbf{X} = \mathbf{x}$ is consistent, to a causal multiteam $T = (T^-, \mathcal{F})$ will produce a causal multiteam $T_{\mathbf{X}=\mathbf{x}} = (T^-_{\mathbf{X}=\mathbf{x}}, \mathcal{F}_{\mathbf{X}=\mathbf{x}})$, where the function component is $\mathcal{F}_{\mathbf{X}=\mathbf{x}} := \mathcal{F}_{\upharpoonright(\mathbf{V}\setminus\mathbf{X})}$ (the restriction of $\mathcal{F}$ to the set of variables $\mathbf{V} \setminus \mathbf{X}$) and the multiteam component is $T^-_{\mathbf{X}=\mathbf{x}} := \{s^{\mathcal{F}}_{\mathbf{X}=\mathbf{x}} \mid s \in T^-\}$, where each $s^{\mathcal{F}}_{\mathbf{X}=\mathbf{x}}$ is the unique assignment compatible with $\mathcal{F}_{\mathbf{X}=\mathbf{x}}$ defined (recursively) as

$$s^{\mathcal{F}}_{\mathbf{X}=\mathbf{x}}(V) = \begin{cases} x_i & \text{if } V = X_i \in \mathbf{X} \\ s(V) & \text{if } V \in Exo(T) \setminus \mathbf{X} \\ \mathcal{F}_V(s^{\mathcal{F}}_{\mathbf{X}=\mathbf{x}}(\mathrm{PA}_V)) & \text{if } V \in End(T) \setminus \mathbf{X}. \end{cases}$$

[5] Throughout the paper, the semantic relation in terms of which T^α is defined will be the semantic relation for language $\mathcal{CO}$, which will soon be defined.

Example 3. Consider the following table:

T^-:

Key	$X \rightarrow$	$Y \rightarrow$	Z
0	0	1	0
1	1	2	2
2	1	2	2
3	2	3	6

where each row represents an assignment (e.g., the fourth row represents an assignment s with $s(Key) = 3$, $s(X) = 2$, $s(Y) = 3$, $s(Z) = 6$). Assume further that the variable Z is generated by the function $\mathcal{F}_Z(X, Y) = X \times Y$, Y is generated by $\mathcal{F}_Y(X) = X + 1$, and X is exogenous. The rows of the table are compatible with these laws, so this is a causal multiteam (call it T). It encodes many probabilities; for example, $P_T(Z = 2) = \frac{1}{2}$. Suppose we have a way to enforce the variable Y to take the value 1. We represent the effect of such an intervention ($do(Y = 1)$) by recomputing the Y and then the Z column:

Key	X	$Y \longrightarrow$	Z
0	0	**1**	...
1	1	**1**	...
2	1	**1**	...
3	2	**1**	...

$\rightsquigarrow$ $T^-_{Y=1}$:

Key	X	$Y \longrightarrow$	Z
0	0	1	**0**
1	1	1	**1**
2	1	1	**1**
3	2	1	**2**

where the new value of Z is computed, in each row, as the product of the value for X and the (new) value for Y. The probability distribution has changed: now $P_{T_{Y=1}}(Z = 2)$ is $\frac{1}{4}$. Furthermore, the function $\mathcal{F}_Y$ is now omitted from $T_{Y=1}$ (otherwise the assignments would not be compatible anymore with the laws). Correspondingly, the arrow from X to Y has been omitted from the causal graph.

3 Languages for events and probabilities

The language $\mathcal{CO}$ ("causation and observations") is for the description of events; later we incorporate it in a language for the discussion of probabilities of $\mathcal{CO}$ formulas. For any fixed signature, the formulas of $\mathcal{CO}$ are defined by the following BNF grammar:

$$\alpha ::= Y = y \mid Y \neq y \mid \alpha \land \alpha \mid \alpha \supset \alpha \mid \mathbf{X} = \mathbf{x} \;\Box\!\!\rightarrow \alpha,$$

where $\mathbf{X} \cup \{Y\} \subseteq \mathrm{Dom}$, $y \in \mathrm{Ran}(Y)$, and $\mathbf{x} \in \mathrm{Ran}(\mathbf{X})$. Formulae of the forms $Y = y$ and $Y \neq y$ are called **literals**. The semantics for $\mathcal{CO}$ is given by the following clauses:

$T \models Y = y$	iff	$s(Y) = y$ for all $s \in T^-$.
$T \models Y \neq y$	iff	$s(Y) \neq y$ for all $s \in T^-$.
$T \models \alpha \land \beta$	iff	$T \models \alpha$ and $T \models \beta$.
$T \models \alpha \supset \beta$	iff	$T^\alpha \models \beta$.
$T \models \mathbf{X} = \mathbf{x} \;\Box\!\!\rightarrow \psi$	iff	$T_{\mathbf{X}=\mathbf{x}} \models \psi$ or $\mathbf{X} = \mathbf{x}$ is inconsistent.

where T^α is defined simultaneously with the semantic clauses. We will reserve the letters α, β to denote $\mathcal{CO}$ formulas.

We can introduce more logical operators as useful abbreviations. $\top$ stands for $X = x \mathrel{\Box\!\!\rightarrow} X = x$, and $\bot$ stands for $X = x \mathrel{\Box\!\!\rightarrow} X \neq x$. $\neg\alpha$ (*dual negation*) stands for $\alpha \supset \bot$. This is not a classical (contradictory) negation; it is easy to see that its semantics is:

- $(T^-, \mathcal{F}) \models \neg\alpha$ iff, for every $s \in T^-$, $(\{s\}, \mathcal{F}) \not\models \alpha$.

Thus, it is not the case, in general, that $T \models \alpha$ or $T \models \neg\alpha$. Note that $X \neq x$ is semantically equivalent to $\neg(X = x)$, and $X = x$ is semantically equivalent to $\neg(X \neq x)$. In previous works $\vee$ (*tensor disjunction*) was taken as a primitive operator, but here we define $\alpha \vee \beta$ as $\neg(\neg\alpha \wedge \neg\beta)$. Its semantic clause can be described as follows:

- $T \models \alpha \vee \beta$ iff there are $T_1, T_2 \leq T$ s.t. $T_1^- \cup T_2^- = T^-$, $T_1 \models \alpha$ and $T_2 \models \beta$.

In contrast with the statement above, the formula $\alpha \vee \neg\alpha$ *is* valid. Furthermore, $\alpha \equiv \beta$ abbreviates $(\alpha \supset \beta) \wedge (\beta \supset \alpha)$. Notice that this formula does not state that α and β are logically equivalent, but only that they are satisfied by the same assignments in the specific causal multiteam at hand.

All the operators discussed here (primitive and defined) behave classically over causal multiteams containing exactly one assignment.

A causal multiteam $(T^-, \mathcal{F})$ is **empty** (resp. **nonempty**) if the multiteam T^- is. All the logics $\mathcal{L}$ considered in the paper have the **empty team property**: if T is empty, then $T \models \alpha$ for any $\alpha \in \mathcal{L}$ (and any $\mathcal{F}$ of the same signature).

Our main object of study is the probabilistic language $\mathcal{PCO}$. Besides literals, it allows for **probabilistic atoms**:

$$\Pr(\alpha) \geq \epsilon \mid \Pr(\alpha) > \epsilon \mid \Pr(\alpha) \geq \Pr(\beta) \mid \Pr(\alpha) > \Pr(\beta)$$

where $\alpha, \beta \in \mathcal{CO}$ and $\epsilon \in [0,1] \cap \mathbb{Q}$. The first two are called **evaluation atoms**, and the latter two **comparison atoms**. Probabilistic atoms together with literals of $\mathcal{CO}$ are called **atomic formulas**. The probabilistic language $\mathcal{PCO}$ is then given by the following grammar:

$$\varphi ::= \eta \mid \varphi \wedge \varphi \mid \varphi \sqcup \varphi \mid \alpha \supset \varphi \mid \mathbf{X} = \mathbf{x} \mathrel{\Box\!\!\rightarrow} \varphi,$$

where $\mathbf{X} \subseteq \mathrm{Dom}$, $\mathbf{x} \in \mathrm{Ran}(\mathbf{X})$, η is an atomic formula, and α is a $\mathcal{CO}$ formula. Note that the antecedents of $\supset$ and the arguments of probability operators are $\mathcal{CO}$ formulas. Semantics for the additional operators are given below:

$T \models \psi \sqcup \chi$	iff	$T \models \psi$ or $T \models \chi$
$T \models \Pr(\alpha) \rhd \epsilon$	iff	$T^- = \emptyset$ or $P_T(\alpha) \rhd \epsilon$
$T \models \Pr(\alpha) \rhd \Pr(\beta)$	iff	$T^- = \emptyset$ or $P_T(\alpha) \rhd P_T(\beta)$

where $\rhd \in \{\geq, >\}$ and $P_T(\alpha)$ is a shorthand for $P_{T^-}((T^\alpha)^-)$.

As usual, for a set of formulas Γ, we write $T \models \Gamma$ if T satisfies each of the formulas in Γ. For $\Gamma \cup \{\varphi\} \subseteq \mathcal{PCO}$, we write $\Gamma \models_\sigma \varphi$ if $T \models \Gamma$ implies $T \models \varphi$, for all causal teams T of signature σ. $\models_\sigma \varphi$ abbreviates $\emptyset \models_\sigma \varphi$. We will always assume that some signature is fixed, and omit the subscripts.

The abbreviations $\top, \bot$ can be used freely in $\mathcal{PCO}$, while $\neg$, $\vee$ and $\equiv$ can be applied only to $\mathcal{CO}$ arguments. The definability of the dual negation in $\mathcal{CO}$ allows us to introduce more useful abbreviations:

$$\Pr(\alpha) \leq \epsilon := \Pr(\neg\alpha) \geq 1 - \epsilon \qquad \Pr(\alpha) = \epsilon := \Pr(\alpha) \geq \epsilon \wedge \Pr(\alpha) \leq \epsilon$$
$$\Pr(\alpha) < \epsilon := \Pr(\neg\alpha) > 1 - \epsilon \qquad \Pr(\alpha) \neq \epsilon := \Pr(\alpha) > \epsilon \sqcup \Pr(\alpha) < \epsilon.$$

Furthermore, the $\supset$ operator enables us to express some statements involving conditional probabilities. Writing, as usual, $\Pr(\alpha \mid \gamma)$ for the probability of α conditional on γ, we can define corresponding atoms as follows (where $\rhd \in \{\geq, >\}$):

$$T \models \Pr(\alpha \mid \gamma) \rhd \epsilon \quad \text{iff} \quad (T^{\gamma})^{-} = \emptyset \text{ or } P_{T^{\gamma}}(\alpha) \rhd \epsilon.$$
$$T \models \Pr(\alpha \mid \gamma) \rhd \Pr(\beta \mid \gamma) \quad \text{iff} \quad (T^{\gamma})^{-} = \emptyset \text{ or } P_{T^{\gamma}}(\alpha) \rhd P_{T^{\gamma}}(\beta).$$

It was observed in [6] that $\Pr(\alpha \mid \gamma) \rhd \epsilon$ and $\Pr(\alpha \mid \gamma) \rhd \Pr(\beta \mid \gamma)$ can be defined by $\gamma \supset \Pr(\alpha) \rhd \epsilon$ and $\gamma \supset \Pr(\alpha) \rhd \Pr(\beta)$, respectively.

The *weak contradictory negation* φ^C of a formula φ is inductively definable in $\mathcal{PCO}$; this is an operator that behaves exactly as a contradictory negation, except on empty causal multiteams. We list the definitory clauses together with the values produced by the negation of defined formulas.

- $(\Pr(\alpha) \geq \epsilon)^C$ is $\Pr(\alpha) < \epsilon$ (and vice versa)
- $(\Pr(\alpha) > \epsilon)^C$ is $\Pr(\alpha) \leq \epsilon$ (and vice versa)
- $(\Pr(\alpha) = \epsilon)^C$ is $\Pr(\alpha) \neq \epsilon$ (and vice versa)
- $(\mathbf{X} = \mathbf{x} \mathrel{\Box\!\!\rightarrow} \chi)^C$ is $\mathbf{X} = \mathbf{x} \mathrel{\Box\!\!\rightarrow} \chi^C$
- $(\alpha \supset \chi)^C$ is $\Pr(\alpha) > 0 \wedge \alpha \supset \chi^C$
- $(\Pr(\alpha) \geq \Pr(\beta))^C$ is $\Pr(\beta) > \Pr(\alpha)$ (and vice versa)
- $(\psi \wedge \chi)^C$ is $\psi^C \sqcup \chi^C$
- $(\psi \sqcup \chi)^C$ is $\psi^C \wedge \chi^C$
- $(\bot)^C$ is $\top$ (and vice versa)
- $(X = x)^C$ is $\Pr(X = x) < 1$
- $(X \neq x)^C$ is $\Pr(X \neq x) < 1$.

In the clause for $\supset$, the conjunct $\Pr(\alpha) > 0$ (whose intuitive interpretation is "if T is nonempty, then T^α is nonempty") is added to insure that $(\alpha \supset \chi)^C$ is not satisfied by T in case (T is nonempty and) T^α is empty.[6]

We emphasise that, since $\mathcal{CO}$ formulas are $\mathcal{PCO}$ formulas, the weak contradictory negation can also be applied to them; however, the contradictory negation of a $\mathcal{CO}$ formula will typically *not* be itself a $\mathcal{CO}$ formula. The meaning of the weak contradictory negation is as follows.

Theorem 4. *For every $\varphi \in \mathcal{PCO}_\sigma$ and nonempty causal multiteam $T = (T^-, \mathcal{F})$ of signature σ, $T \models \varphi^C \Leftrightarrow T \not\models \varphi$.*

Proof. The proof proceeds by induction on the structure of formulas φ. We show the only non-trivial case of $\supset$.

Suppose $T \models \Pr(\alpha) > 0 \wedge \alpha \supset \chi^C$. Thus $T^\alpha \models \chi^C$. Since T is nonempty and $T \models \Pr(\alpha) > 0$, we conclude that T^α is nonempty as well. Now by applying the induction hypothesis on χ, we obtain $T^\alpha \not\models \chi$. Thus, $T \not\models \alpha \supset \chi$.

[6] Whereas $\Pr(\alpha) > 0$ could be replaced with $(\neg\alpha)^C$, the use of probability atoms in $(X = x)^C$ and $(X \neq x)^C$ seems essential.

For the converse, assume $T \not\models \alpha \supset \chi$. Then $T^\alpha \not\models \chi$, which (by the empty team property) entails that T^α is nonempty, and thus $T \models \Pr(\alpha) > 0$. Moreover, applying the induction hypothesis to χ yields $T^\alpha \models \chi^C$, and thus $T \models \alpha \supset \chi^C$. □

Using the weak contradictory negation, we can define an operator that behaves exactly as the material conditional:

- $\psi \to \chi$ stands for $\psi^C \sqcup \chi$.

Indeed, $T \models \psi \to \chi$ iff T is empty or $T \not\models \psi$ or $T \models \chi$. However, since $\mathcal{PCO}$ has the empty multiteam property, "T is empty" entails $T \models \chi$; thus, for $\mathcal{PCO}$, $\to$ really is the material conditional:

- $\psi \to \chi$ iff $T \not\models \psi$ or $T \models \chi$.

Similarly, we let $\psi \leftrightarrow \chi$ denote $(\psi \to \chi) \land (\chi \to \psi)$.

Note that $\alpha \to \beta$ and $\alpha \supset \beta$ are not in general equivalent even if α, β are $\mathcal{CO}$ formulas. Consider for example a causal multiteam T with two assignments $s = \{(X, 0), (Y, 0)\}$ and $t = \{(X, 1), (Y, 1)\}$. Clearly $T \models X = 0 \to Y = 1$ (since $T \not\models X = 0$), while $T \not\models X = 0 \supset Y = 1$ (since $T^{X=0} \not\models Y = 1$). However, the entailment from $\alpha \supset \psi$ to $\alpha \to \psi$ always holds, provided both formulas are in $\mathcal{PCO}$ (i.e., provided $\alpha \in \mathcal{CO}$). Indeed, suppose $T \models \alpha \supset \psi$ and $T \models \alpha$. From the former we get $T^\alpha \models \psi$. From the latter we get $T = T^\alpha$. Thus, $T \models \psi$. The opposite direction does not preserve truth, but it does preserve validity: if $\models \alpha \to \psi$, then $\models \alpha \supset \psi$. Indeed, the former tells us that any causal multiteam that satisfies α also satisfies ψ. Thus, in particular, for any T, $T^\alpha \models \psi$, and thus $T \models \alpha \supset \psi$.

Similar considerations as above apply to the pair of operators $\equiv$ and $\leftrightarrow$. Futher differences in the proof-theoretical behaviour of these (and other) pairs of operators are illustrated by the axioms T1 and T2 presented in Section 4.2.

4 The axiom system

We present a formal deduction system with infinitary rules for $\mathcal{PCO}$ and show it to be strongly complete over recursive causal multiteams. We follow the approach of [32], which proved a similar result for a language with probabilities and conditional probabilities. Our result adds to the picture comparison atoms, counterfactuals, and pre-intervention observations ("Pearl's counterfactuals").

4.1 Further notation

The formulation of some of the axioms – in particular, those involving reasoning with counterfactuals – will involve some additional abbreviations. For example, we will write $\mathbf{X} \neq \mathbf{x}$ for a disjunction $X_1 \neq x_1 \sqcup \cdots \sqcup X_n \neq x_n$.

There will be an axiom (C11) that characterizes recursivity as done in [17]. For it, we need to define the atom $X \rightsquigarrow Y$ ("X causally affects Y") by the formula:

$$\bigvee_{\substack{\mathbf{Z} \subseteq Dom \\ x \neq x' \in \mathrm{Ran}(X) \\ y \neq y' \in \mathrm{Ran}(Y) \\ \mathbf{z} \in \mathrm{Ran}(\mathbf{Z})}} [((\mathbf{Z} = \mathbf{z} \land X = x) \mathrel{\Box\!\!\to} Y = y) \land ((\mathbf{Z} = \mathbf{z} \land X = x') \mathrel{\Box\!\!\to} Y = y')].$$

This formula states that there is some intervention on X that makes a difference for Y; it is the weakest form of causation that is definable in terms of interventionist counterfactuals.

We will also need a formula (from [5]) characterizing the stricter notion of direct cause (X is a direct cause of Y iff $X \in PA_Y$), which is expressible by a $\mathcal{PCO}$ formula $\varphi_{DC(X,Y)}$ defined as:

$$\bigvee_{\substack{x \neq x' \in \mathrm{Ran}(X) \\ y \neq y' \in \mathrm{Ran}(Y) \\ \mathbf{w} \in \mathrm{Ran}(\mathbf{W}_{XY})}} [((\mathbf{W}_{XY} = \mathbf{w} \land X = x) \,\Box\!\!\rightarrow Y = y) \land ((\mathbf{W}_{XY} = \mathbf{w} \land X = x') \,\Box\!\!\rightarrow Y = y')].$$

where $\mathbf{W}_{XY}$ stands for Dom $\setminus \{X, Y\}$. The formula asserts that modifying the value of X may alter the value of Y even when all other variables are held fixed (thus excluding causation via intermediate variables).

Now, some axioms describe specific properties of exogenous or endogenous variables, which can be again characterized in $\mathcal{PCO}$. We can express the fact that a variable Y is endogenous by the following formula (where, as before, $\mathbf{W}_V$ stands for Dom $\setminus \{V\}$):

$$\varphi_{End(Y)} : \bigsqcup_{X \in \mathbf{W}_Y} \varphi_{DC(X,Y)}$$

and its contradictory negation $(\varphi_{End(Y)})^C$ will express that Y is exogenous.

Finally, for each function component $\mathcal{F}$, $\Phi^{\mathcal{F}}$ is a formula that characterizes the fact that a causal team has function component $\mathcal{F}$. In detail,

$$\Phi^{\mathcal{F}} : \bigwedge_{V \in End(\mathcal{F})} \eta_\sigma(V) \land \bigwedge_{V \notin End(\mathcal{F})} \xi_\sigma(V)$$

where

$$\eta_\sigma(V) : \bigwedge_{\mathbf{w} \in \mathrm{Ran}(\mathbf{W}_V)} (\mathbf{W}_V = \mathbf{w} \,\Box\!\!\rightarrow V = \mathcal{F}_V(\mathbf{w}))$$

and

$$\xi_\sigma(V) : \bigwedge_{\substack{\mathbf{w} \in \mathrm{Ran}(\mathbf{W}_V) \\ v \in \mathrm{Ran}(V)}} V = v \supset (\mathbf{W}_V = \mathbf{w} \,\Box\!\!\rightarrow V = v).$$

A nonempty causal multiteam $T = (T^-, \mathcal{G})$ satisfies $\Phi^{\mathcal{F}}$ iff $\mathcal{G} = \mathcal{F}$.[7]

4.2 Axioms and rules

We present a few axiom schemes and rules for $\mathcal{PCO}$, roughly divided in six groups. Each axiom and rule is restricted to formulas of a fixed signature σ, so that actually we have a distinct axiom system for each signature. As usual, α and β are restricted to be $\mathcal{CO}$ formulas.

Tautologies

T1. All instances of classical propositional tautologies in $\land, \sqcup, \rightarrow, ^C, \top, \bot$.

[7] Save for some inessential differences, this is is the content of Theorem 3.4 from [9],

T2. All CO instances of classical propositional tautologies in $\land, \lor, \supset, \neg, \top, \bot$.

Rule MP. $\dfrac{\psi \quad \psi \to \chi}{\chi}$

Rule Rep. $\dfrac{\vdash \varphi \qquad \vdash \theta \leftrightarrow \theta'}{\vdash \varphi[\theta'/\theta]}$ (provided $\varphi[\theta'/\theta]$ is well-formed)

Probabilities

P1. $\alpha \leftrightarrow \Pr(\alpha) = 1$.

P2. $\Pr(\alpha) \geq 0$.

P3. $(\Pr(\alpha) = \delta \land \Pr(\beta) = \epsilon \land \Pr(\alpha \land \beta) = 0) \to \Pr(\alpha \lor \beta) = \delta + \epsilon$
(when $\delta + \epsilon \leq 1$).

P3b. $\Pr(\alpha) \geq \epsilon \land \Pr(\alpha \land \beta) = 0 \to \Pr(\beta) \leq 1 - \epsilon$.

P4. $\Pr(\alpha) \leq \epsilon \to \Pr(\alpha) < \delta$ (if $\delta > \epsilon$).

P5. $\Pr(\alpha) < \epsilon \to \Pr(\alpha) \leq \epsilon$.

P6. $(\alpha \equiv \beta) \to (\Pr(\alpha) = \epsilon \to \Pr(\beta) = \epsilon)$.

P6b. $(\alpha \supset \beta) \to (\Pr(\alpha) = \epsilon \to \Pr(\beta) \geq \epsilon)$.

Rule $\bot^\omega$. $\dfrac{\psi \to \Pr(\alpha) \neq \epsilon, \forall \epsilon \in [0,1] \cap \mathbb{Q}}{\psi \to \bot}$

Comparison

CP1. $(\Pr(\alpha) = \delta \land \Pr(\beta) = \epsilon) \to \Pr(\alpha) \geq \Pr(\beta)$. (if $\delta \geq \epsilon$)

CP2. $(\Pr(\alpha) = \delta \land \Pr(\beta) = \epsilon) \to \Pr(\alpha) > \Pr(\beta)$. (if $\delta > \epsilon$)

Observations

O1. $\Pr(\alpha) = 0 \to (\alpha \supset \psi)$.

O1b. $(\alpha \supset \bot) \to \Pr(\alpha) = 0$.

O2. $(\Pr(\alpha) = \delta \land \Pr(\alpha \land \beta) = \epsilon) \to (\alpha \supset \Pr(\beta) = \frac{\epsilon}{\delta})$. (when $\delta \neq 0$)

O3. $(\alpha \supset \Pr(\beta) = \epsilon) \to (\Pr(\alpha) = \delta \leftrightarrow \Pr(\alpha \land \beta) = \epsilon \cdot \delta)$ (when $\epsilon \neq 0$).

O4. $(\alpha \supset \psi) \to (\alpha \to \psi)$.

O5$_\land$. $\alpha \supset (\psi \land \chi) \leftrightarrow (\alpha \supset \psi) \land (\alpha \supset \chi)$.

O5$_\sqcup$. $\alpha \supset (\psi \sqcup \chi) \leftrightarrow (\alpha \supset \psi) \sqcup (\alpha \supset \chi)$.

O5$_\supset$. $\alpha \supset (\beta \supset \chi) \leftrightarrow (\alpha \land \beta) \supset \chi$.

Rule Mon$_\supset$. $\dfrac{\vdash \psi \to \chi}{\vdash (\alpha \supset \psi) \to (\alpha \supset \chi)}$

Rule $\to$to$\supset$. $\dfrac{\vdash \alpha \to \psi}{\vdash \alpha \supset \psi}$

Rule $\supset^\omega$. $\dfrac{\psi \to (\Pr(\alpha \land \beta) = \delta\epsilon \leftrightarrow \Pr(\alpha) = \epsilon), \forall \epsilon \in (0,1] \cap \mathbb{Q}}{\psi \to (\alpha \supset \Pr(\beta) = \delta)}$

Literals

A1. $\mathbf{Y} = \mathbf{y} \to \mathbf{Y} \neq \mathbf{y}'$. (when $\mathbf{y} \neq \mathbf{y}'$)

A2. $X \neq x \leftrightarrow (X = x \supset \bot)$.

A3. $\bigvee_{\mathbf{y} \in \mathrm{Ran}(\mathbf{Y})} \mathbf{Y} = \mathbf{y}$.

Counterfactuals

C1. $(\mathbf{X} = \mathbf{x} \boxright (\psi \wedge \chi)) \leftrightarrow ((\mathbf{X} = \mathbf{x} \boxright \psi) \wedge (\mathbf{X} = \mathbf{x} \boxright \chi))$.
C2. $(\mathbf{X} = \mathbf{x} \boxright (\psi \sqcup \chi)) \leftrightarrow ((\mathbf{X} = \mathbf{x} \boxright \psi) \sqcup (\mathbf{X} = \mathbf{x} \boxright \chi))$.
C3. $(\mathbf{X} = \mathbf{x} \boxright (\alpha \supset \chi)) \leftrightarrow ((\mathbf{X} = \mathbf{x} \boxright \alpha) \supset (\mathbf{X} = \mathbf{x} \boxright \chi))$.
C4. $(\mathbf{X} = \mathbf{x} \boxright (\mathbf{Y} = \mathbf{y} \boxright \chi)) \to ((\mathbf{X}' = \mathbf{x}' \wedge \mathbf{Y} = \mathbf{y}) \boxright \chi)$
(where $\mathbf{X}' = \mathbf{X} \setminus \mathbf{Y}$ and $\mathbf{x}' = \mathbf{x} \setminus \mathbf{y}$; and provided $\mathbf{X} = \mathbf{x}$ is consistent).
C4b. $((\mathbf{X} = \mathbf{x} \wedge \mathbf{Y} = \mathbf{y}) \boxright \chi) \to (\mathbf{X} = \mathbf{x} \boxright (\mathbf{Y} = \mathbf{y} \boxright \chi))$.
C5. $(\mathbf{X} = \mathbf{x} \boxright \bot) \to \psi$. (when $\mathbf{X} = \mathbf{x}$ is consistent)
C6. $(\mathbf{X} = \mathbf{x} \wedge Y = y) \boxright Y = y$.
C7. $(\mathbf{X} = \mathbf{x} \wedge \gamma) \to (\mathbf{X} = \mathbf{x} \boxright \gamma)$. (where $\gamma \in \mathcal{PCO}$ without occurrences of $\boxright$)
C8. $(\mathbf{X} = \mathbf{x} \boxright \Pr(\alpha) \rhd \epsilon) \leftrightarrow \Pr(\mathbf{X} = \mathbf{x} \boxright \alpha) \rhd \epsilon$. (where $\rhd = \geq$ or $>$)
C8b. $(\mathbf{X} = \mathbf{x} \boxright \Pr(\alpha) \rhd \Pr(\beta)) \leftrightarrow \Pr(\mathbf{X} = \mathbf{x} \boxright \alpha) \rhd \Pr(\mathbf{X} = \mathbf{x} \boxright \beta)$
(where $\rhd = \geq$ or $>$).
C9. $\varphi_{End(Y)} \to (\mathbf{W}_Y = \mathbf{w} \boxright \bigsqcup_{y \in \mathrm{Ran}(Y)} Y = y)$.
C10. $(\varphi_{End(Y)})^C \to (Y = y \supset (\mathbf{W}_V = \mathbf{w} \boxright Y = y))$.
C11. $(X_1 \rightsquigarrow X_2 \wedge \cdots \wedge X_{n-1} \rightsquigarrow X_n) \to (X_n \rightsquigarrow X_1)^C$. (for $n > 1$).

Rule Mon$_{\boxright}$. $\dfrac{\vdash \psi \to \chi}{\vdash (\mathbf{X}=\mathbf{x} \boxright \psi) \to (\mathbf{X}=\mathbf{x} \boxright \chi)}$

We will refer to this list of axioms and rules as the **deduction system**, and write $\Gamma \vdash \varphi$ if there is a countable sequence of $\mathcal{PCO}$ formulas $\varphi_1, \dots, \varphi_\kappa = \varphi$ (enumerated by ordinals $\leq \kappa$) where each formula in the list is either an axiom, a formula from Γ, or it follows from earlier formulas in the list by one of the rules. The sequence itself is called a **proof**.

We write $\vdash \varphi$ for $\emptyset \vdash \varphi$; if it holds, we say that φ is a **theorem**. Notice that some of the rules (Rep, Mon$_\supset$, Mon$_{\boxright}$, $\to$to$\supset$) can only be applied to theorems, since they preserve validity but not truth.

5 Discussion of the proof system

We have described a family of infinitary axiom systems, one for each finite signature σ. Our main result is that each such axiom system is sound and strongly complete for $\mathcal{PCO}_\sigma$ over the corresponding class of multiteams of signature σ. By saying that a deduction system is **sound** for $\mathcal{PCO}_\sigma$ we mean that, for all formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{PCO}_\sigma$, $\Gamma \vdash \varphi$ entails $\Gamma \models_\sigma \varphi$; and it is **strongly complete** for $\mathcal{PCO}_\sigma$ if $\Gamma \models_\sigma \varphi$ entails $\Gamma \vdash \varphi$. As discussed in the Introduction, a finitary axiom system could at most aspire to be (sound and) *weakly* complete for $\mathcal{PCO}_\sigma$, i.e. to satisfy the equivalence $\Gamma_0 \models \varphi$ iff $\Gamma_0 \vdash \varphi$, for *finite* sets Γ_0.

Theorem 5 (Soundness and strong completeness). *Let σ be a signature and $\Gamma \cup \{\varphi\} \subseteq \mathcal{PCO}_\sigma$. Then $\Gamma \models \varphi$ if and only if $\Gamma \vdash \varphi$.*

The proof of this result (which can be found in the full version of the paper, [8]) uses a Henkin-style canonical model construction, i.e. it proceeds by showing that each maximal consistent set Γ of formulas of $\mathcal{PCO}_\sigma$ provides sufficient information for constructing a *canonical causal multiteam* $\mathbb{T}$ that satisfies Γ. The proof essentially follows the lines of the completeness proof given in [32], but it presents some novel difficulties in dealing with the additional operators $\supset$ and $\Box\!\!\rightarrow$, especially towards obtaining a Truth Lemma, which takes the unusual form:

$$\text{For all } \alpha \in \mathcal{CO} \text{ and } \varphi \in \mathcal{PCO}, \mathbb{T}^\alpha \models \varphi \iff \alpha \supset \varphi \in \Gamma.$$

The choice of axioms and rules is largely built on earlier axiomatizations of simpler languages for probabilistic or causal reasoning; let us briefly illustrate how our system adapts or differs from earlier sources. Rules MP, $\bot^\omega$, $\supset^\omega$ and axioms P1-2-3-4-5 and O1-2-3 are essentially adapted from the paper [32] (the rule $\bot^\omega$ comes from the earlier [1]). Keeping in mind that a formula of the form $\alpha \supset \Pr(\beta) = \epsilon$ is semantically equivalent to a conditional probability statement $\Pr(\beta \mid \alpha) = \epsilon$, axioms O2-3 encode the usual definition of conditional probability in terms of marginal probability. Our Rule $\text{Mon}_\supset$ allows omitting axioms 8, 11 and 12 from [32], which follow from it. Our restriction $\delta + \epsilon \leq 1$ in axiom P3 is imposed by the syntax (we do not allow numbers greater than 1 as symbols). The additional axiom P3b guarantees that, despite this restriction, axiom scheme P3 is always applicable, in the sense that, if an instance of it is not admitted as an axiom, then the premises of said instance are contradictory.[8] Axiom P6 derives from [32], but in our case the correct formulation requires the interaction of the two conditionals $\supset$ (used to define $\equiv$) and $\rightarrow$; notice that the analogous formulation $(\alpha \leftrightarrow \beta) \rightarrow (\Pr(\alpha) = \epsilon \rightarrow \Pr(\beta) = \epsilon)$ is *not* valid. The variant P6b is our addition. These adaptations are due both to differences in the syntax ([32] has an explicit conditional probability operator, while we talk of conditional probabilities only indirectly, by means of the selective implication; and we have distinct logical operators at the level of events vs. the level of probabilities) and in the semantics (in particular, we differ in the treatment of truth over empty models).

Regarding comparison atoms, analogues of CP1-2 appear, for example, in [27], and in earlier literature. An interesting difference from [27] is that in our sytem we do not need an additional infinitary rule to deal with the comparison atoms.

Axioms C6, C7 and C11 take the same roles as the principles of *Effectiveness*, *Composition* and *Recursivity* from [14]. The current, more intuitive form of axiom C7 was introduced in [9]; it captures the intuition that intervening by fixing some variables to values they already possess will not alter the value of any variable (although it may alter the set of causal laws, whence the restriction to γ without occurrences of $\Box\!\!\rightarrow$). Halpern [17] noticed that $\Box\!\!\rightarrow$ distributes over Boolean operators, and formulated analogues of C1 and C2. The validity of C3-4-4b was pointed out in [5] (although an earlier axiom for dealing with nested counterfactuals had already been devised in [12]), and the importance of C5 emerged in [9].

6 Conclusions

We produced a strongly complete axiom system for a language $\mathcal{PCO}$ for probabilistic counterfactual reasoning (without arithmetical operations). As for most analogous results in the literature on interventionist counterfactuals, we have assumed that the signatures are finite; it would be interesting to find out if the recently developed methods of [18] for

[8] It seems to us that an axiom analogous to P3b should be added also to the system in [32].

axiomatizatizing infinite signatures may be extended to our case. Our system features infinitary rules, and it is therefore natural to wonder whether finitary axiomatizations could be obtained. Due to the failure of compactness, such axiomatizations can aspire at most at weak completeness.

There is another closely related axiomatization issue that would be important to settle. In [6], an extension $\mathcal{PCO}^\omega$ of $\mathcal{PCO}$ is considered that features a countably infinite version of the global disjunction $\sqcup$. This uncountable language is much more expressive than $\mathcal{PCO}$ and it can be proved that, in a sense, it encompasses all the expressive resources that a probabilistic language for interventionist counterfactuals should have. Given the special semantic role of this language, it would be important to find out whether an (infinitary) strongly complete axiomatization can be obtained for it. The main obstacle is proving an appropriate Lindenbaum lemma; as shown e.g. in [11], for an uncountable language with an infinitary axiom system the Lindenbaum lemma can even be false.

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