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A Tight Lower Bound for Planar Steiner Orientation*

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Abstract. In the STEINER ORIENTATION problem, the input is a mixed graph G (it has both directed and undirected edges) and a set of k terminal pairs $\mathscr{T}$. The question is whether we can orient the undirected edges in a way such that there is a directed $s \rightsquigarrow t$ path for each terminal pair $(s,t) \in \mathscr{T}$. Arkin and Hassin [DAM '02] showed that the STEINER ORIENTATION problem is NP-complete. They also gave a polynomial time algorithm for the special case when $k = 2$.

From the viewpoint of exact algorithms, Cygan, Kortsarz and Nutov [ESA '12, SIDMA '13] designed an XP algorithm running in $n^{O(k)}$ time for all $k \geq 1$. Pilipczuk and Wahlström [SODA '16, TOCT '18] showed that the STEINER ORIENTATION problem is W[1]-hard parameterized by k. As a byproduct of their reduction, they were able to show that under the Exponential Time Hypothesis (ETH) of Impagliazzo, Paturi and Zane [JCSS '01] the STEINER ORIENTATION problem does not admit an $f(k) \cdot n^{o(k/\log k)}$ algorithm for any computable function f.

In this paper, we give a short and easy proof that the $n^{O(k)}$ algorithm of Cygan et al. is asymptotically optimal, even if the input graph is planar. Formally, we show that the PLANAR STEINER ORIENTATION problem is W[1]-hard parameterized by the number k of terminal pairs, and, under ETH, cannot be solved in $f(k) \cdot n^{o(k)}$ time for any computable function f.

Moreover, under a stronger hypothesis called Gap-ETH of Dinur [ECCC '16] and Manurangsi and Raghavendra [ICALP '17], we are able to show that there is no constant $\vartheta > 0$ such that PLANAR STEINER ORIENTATION admits an $(\frac{19}{20} + \vartheta)$-approximation in FPT time, i.e., no $f(k) \cdot n^{o(k)}$ time algorithm can distinguish between the case when all k pairs are satisfiable versus the case when less than $k \cdot (\frac{19}{20} + \vartheta)$ pairs are satisfiable. To the best of our knowledge, this is the first FPT inapproximability result on planar graphs.

* A preliminary version of this article [6] showing the W[1]-hardness and the $f(k) \cdot n^{o(k)}$ running time lower bound under ETH for graphs of genus 1 appeared in CSR '18. This version extends the results to graphs of genus 0 (planar graphs) and also contains an FPT inapproximability result.

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1 Introduction

In the STEINER ORIENTATION problem, the input is a mixed graph $G = (V, E)$ (it has both directed and undirected edges) and a set of terminal pairs $\mathscr{T} \subseteq V \times V$. The question is whether we can orient the undirected edges in a way such that there is a directed $s \rightsquigarrow t$ path for each terminal pair $(s,t) \in \mathscr{T}$. This problem has applications in traffic control (cf. [25] proving Robbins' theorem on graphs with strong orientations) and protein-protein interaction (PPI) networks [22].

STEINER ORIENTATION
Input: A mixed graph $G = (V, E)$, and a set $\mathscr{T} \subseteq V \times V$ of k terminal pairs.
Question: Is there an orientation of the undirected egdes of G such that the resulting graph has an $s \rightsquigarrow t$ path for each $(s,t) \in \mathscr{T}$?
Parameter: k

Hassin and Megiddo [13] showed that STEINER ORIENTATION is polynomial time solvable if the input graph G is completely undirected, i.e., has no directed edges. If the input graph G is actually mixed, then Arkin and Hassin [1] showed that STEINER ORIENTATION is NP-complete. They also gave a polynomial time algorithm for the special case when $k = 2$. Cygan, Kortsarz and Nutov [10] generalized this by giving an $n^{O(k)}$-time algorithm for all $k \geq 1$, i.e., STEINER ORIENTATION is in XP parameterized by k. Although the algorithm of Cygan et al. [10] is polynomial time for fixed k, the degree of the polynomial changes as k changes. This left open the question of whether one could design a much more efficient FPT algorithm for STEINER ORIENTATION parameterized by k, i.e., an algorithm which runs in time $f(k) \cdot n^{O(1)}$ for some computable function f independent of n.

Pilipczuk and Wahlström [23] answered this question negatively by showing that STEINER ORIENTATION is W[1]-hard parameterized by k. As a byproduct of their reduction, they were able to show that under the Exponential Time Hypothesis (ETH) of Impagliazzo, Paturi and Zane [14, 15] the STEINER ORIENTATION problem does not admit an $f(k) \cdot n^{o(k/\log k)}$-time algorithm for any computable function f [24]. That is, the $n^{O(k)}$-time algorithm of Cygan et al. is asymptotically almost optimal. This left open the following two questions:

- Can we close the gap between the $n^{O(k)}$ algorithm and the $f(k) \cdot n^{o(k/\log k)}$ hardness for STEINER ORIENTATION on general graphs?
- Is PLANAR STEINER ORIENTATION (see below) FPT parameterized by k , or can we at least obtain an improved runtime such as $f(k) \cdot n^{O(\sqrt{k})}$?

PLANAR STEINER ORIENTATION, the main problem of interest in this paper, is the restriction of STEINER ORIENTATION to planar mixed graphs. Naturally, these are mixed graphs for which the union of the graph formed by the undirected edges and the underlying undirected graph of the graph formed by the directed edges is planar.

In a preliminary version of this paper [6] the first question was answered by showing a tight lower bound of $f(k) \cdot n^{o(k)}$ for STEINER ORIENTATION under ETH. However, the constructed graph in [6] had genus 1 which left open the complexity for PLANAR

STEINER ORIENTATION since planar graphs have genus 0. In this full version, we completely answer both the above questions. Formally, we show that:

Theorem 1. *The* PLANAR STEINER ORIENTATION *problem is W[1]-hard parameterized by the number k of terminal pairs. Moreover, under ETH,* PLANAR STEINER ORIENTATION *cannot be solved in $f(k) \cdot n^{o(k)}$ time for any computable function f.*

Previously, the only result about the complexity of PLANAR STEINER ORIENTATION was an NP-completeness proof due to Beck et al. [2]. Since our reduction can be performed in polynomial time, it also shows NP-completeness of PLANAR STEINER ORIENTATION.

Furthermore, using the same reduction with a stronger hypothesis we are able to show that it is even hard to approximate the number of satisfiable pairs very precisely.

Theorem 2. *Assuming Gap-ETH, for each $0 < \vartheta \leq \frac{1}{20}$ there exists a constant $\zeta > 0$ such that for any computable function f there is no algorithm for* PLANAR STEINER ORIENTATION*, which given an instance $(G, \mathscr{T})$ with G planar, can distinguish between the following cases in $f(k) \cdot n^{\zeta k}$ time, where $n = |V(G)|$ and $k = |\mathscr{T}|$:*

- *there is an orientation of G that satisfies all pairs of $\mathscr{T}$ and*
- *any orientation of G satisfies less than $(\frac{19}{20} + \vartheta)$-fraction of pairs in $\mathscr{T}$.*

To the best of our knowledge, the only known results on approximability of STEINER ORIENTATION are for the case when the input graph is undirected. In this case, there are several known polynomial time approximation algorithms with ratios $O(\frac{\log|\mathscr{T}|}{\log\log|\mathscr{T}|})$ [10] and $O(\frac{\log n}{\log\log n})$ [12, 22]. We are not aware of any non-trivial approximation algorithms for STEINER ORIENTATION when the input graph is mixed.

Our reduction uses some ideas given by Pilipczuk and Wahlström [23], who obtained a lower bound of $f(k) \cdot n^{o(\sqrt{k})}$ via a rather involved reduction from MULTICOLORED CLIQUE. This was later [24] improved by the same authors to $f(k) \cdot n^{o(k/\log k)}$ via the standard trick of reducing from the PARTITIONED SUBGRAPH ISOMORPHISM problem [18] instead. In this paper, we give a parameterized reduction from k-CLIQUE to STEINER ORIENTATION on planar graphs. Typically, W[1]-hardness results for problems on planar graphs are shown via reductions from the GRID TILING problem introduced by Marx [17][4]. Indeed, the GRID TILING problem has turned out to be quite useful in showing W[1]-hardness for several problems on planar graphs [5, 7, 8, 19–21]. However, in this paper we chose to give a parameterized reduction from k-CLIQUE instead so that we can use the same reduction for the FPT inapproximability result of Theorem 2 as well. Note that there is a simple reduction [9, Thm 14.28] from k-CLIQUE to $k \times k$ GRID TILING which is also used implicitly in our reduction.

2 The reduction

We begin by describing the reduction from an instance (G', k) of k-CLIQUE to an instance of PLANAR STEINER ORIENTATION with $O(k)$ terminal pairs. We prove the exact properties of the reduction in subsequent subsections.

[4] In [17] the GRID TILING problem (under the name of MATRIX TILING) was actually used to rule out PTASs for several problems under ETH.

2.1 Construction

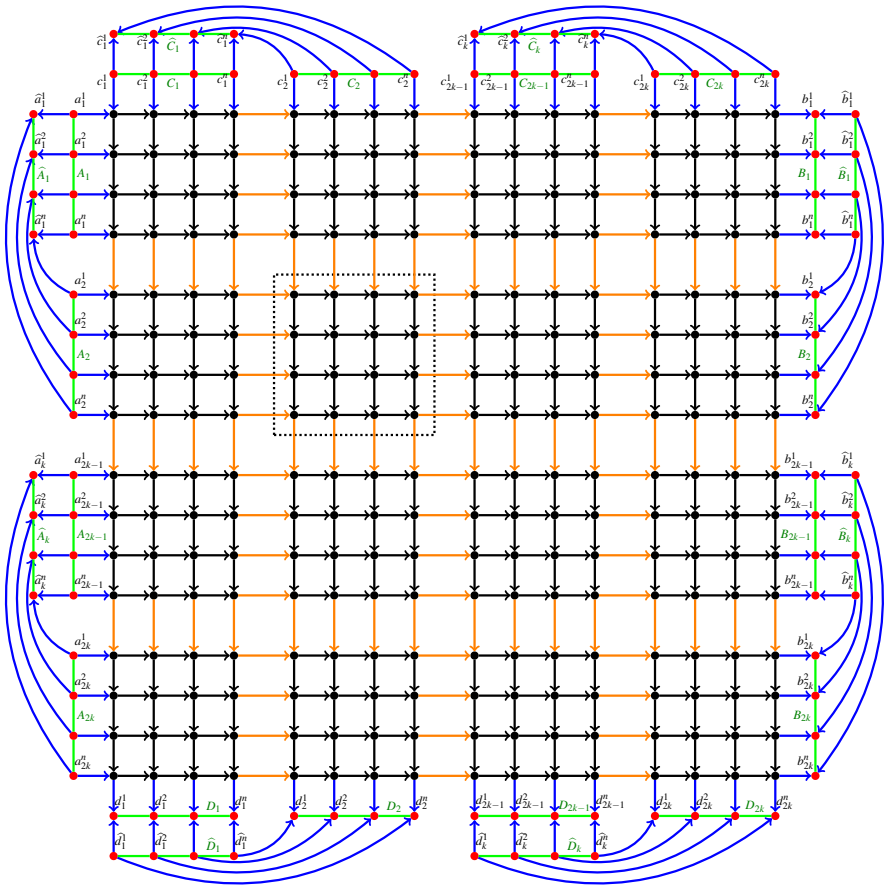

Fig. 1. The instance of STEINER ORIENTATION with $O(k)$ terminal pairs created from an instance of k-Clique (before the splitting operation). At this point, the only undirected edges are the green path edges. A gadget $G_{i,j}$ is highlighted by a dotted rectangle.

Consider an instance (G',k) of k-CLIQUE. We now build an instance $(G,\mathscr{T})$ of PLANAR STEINER ORIENTATION as follows (refer to Figure 1). For simplicity we assume that $|V(G')| = n-1$ and $V(G') = \{u_1,\dots,u_{n-1}\}$ (this allows us to make less case distinctions in some further definitions and descriptions, for example Definition 2 and Definition 3).

– **Black Grid Edges**: For each $1 \le i,j \le 2k$ we introduce the $n \times n$ grid $G_{i,j}$.
  The grids $G_{2i,2j}$ will later be modified to represent adjacencies from the graph G'.

The other grids serve to transfer information. In Figure 1 we highlight a gadget $G_{i,j}$ by a dotted rectangle.

We place the grids $G_{i,j}$ into a big grid of grids left to right according to growing i and from top to bottom according to growing j. That is $G_{1,1}$ is at the top left corner of the construction, the $G_{2k,1}$ at the top right corner, etc.

- In $G_{i,j}$ the column numbers grow from left to right, from 1 to n, and the row numbers grow from top to bottom, from 1 to n.
- For each $1 \leq h, \ell \leq n$ the unique vertex which is the intersection of the hth column and ℓ^{th} row of $G_{i,j}$ is denoted by $v_{i,j}^{h,\ell}$.

Orient each horizontal edge of the grid $G_{i,j}$ to the right, and each vertical edge to the bottom.

– We now define four special sets of vertices for the gadget $G_{i,j}$ given by
  - $\texttt{Left}(G_{i,j}) = \{v_{i,j}^{1,\ell} \mid \ell \in [n]\}$
  - $\texttt{Right}(G_{i,j}) = \{v_{i,j}^{n,\ell} \mid \ell \in [n]\}$
  - $\texttt{Top}(G_{i,j}) = \{v_{i,j}^{\ell,1} \mid \ell \in [n]\}$
  - $\texttt{Bottom}(G_{i,j}) = \{v_{i,j}^{\ell,n} \mid \ell \in [n]\}$

– **Horizontal orange inter-grid edges**:
For each $i \in [2k-1], j \in [2k]$
  - Add the directed perfect matching from vertices of $\texttt{Right}(G_{i,j})$ to $\texttt{Left}(G_{i+1,j})$ given by the set of edges $\{v_{i,j}^{n,\ell} \to v_{i+1,j}^{1,\ell} \mid \ell \in [n]\}$.

– **Vertical orange inter-grid edges**:
For each $i \in [2k], j \in [2k-1]$
  - Add the directed perfect matching from vertices of $\texttt{Bottom}(G_{i,j})$ to $\texttt{Top}(G_{i,j+1})$ given by the set of edges $\{v_{i,j}^{h,n} \to v_{i,j+1}^{h,1} \mid h \in [n]\}$.

Note that after adding all inter-grid edges, the graph forms one large grid of size $2kn \times 2kn$ with all edges oriented to the right or to the bottom.

– We introduce $12k \cdot n$ red vertices given by
  - $\widehat{A} := \{\widehat{a}_j^\ell \mid j \in [k], \ell \in [n]\}$
  - $\widehat{B} := \{\widehat{b}_j^\ell \mid j \in [k], \ell \in [n]\}$
  - $\widehat{C} := \{\widehat{c}_i^h \mid i \in [k], h \in [n]\}$
  - $\widehat{D} := \{\widehat{d}_i^h \mid i \in [k], h \in [n]\}$
  - $A := \{a_j^\ell \mid j \in [2k], \ell \in [n]\}$
  - $B := \{b_j^\ell \mid j \in [2k], \ell \in [n]\}$
  - $C := \{c_i^h \mid i \in [2k], h \in [n]\}$
  - $D := \{d_i^h \mid i \in [2k], h \in [n]\}$

– **Blue connector edges**:
  - For each $j \in [2k], \ell \in [n]$
    * add the directed edge $a_j^\ell \to v_{1,j}^{1,\ell}$,
    * add the directed edge $v_{2k,j}^{n,\ell} \to b_j^\ell$.
  - For each $j \in [k], \ell \in [n]$
    * add the directed edge $a_{2j-1}^\ell \to \widehat{a}_j^\ell$,
    * add the directed edge $a_{2j}^\ell \to \widehat{a}_j^{n+1-\ell}$,

    - add the directed edge $\widehat{b}_j^\ell \to b_{2j-1}^\ell$,
    - add the directed edge $\widehat{b}_j^\ell \to b_{2j}^{n+1-\ell}$.
  - For each $i \in [2k], h \in [n]$
    - add the directed edge $c_i^h \to v_{i,1}^{h,1}$,
    - add the directed edge $v_{i,2k}^{h,n} \to d_i^h$.
  - For each $i \in [k], h \in [n]$
    - add the directed edge $c_{2i-1}^h \to \widehat{c}_i^h$,
    - add the directed edge $c_{2i}^h \to \widehat{c}_i^{n+1-h}$,
    - add the directed edge $\widehat{d}_i^h \to d_{2i-1}^h$,
    - add the directed edge $\widehat{d}_i^h \to d_{2i}^{n+1-h}$.
- **Green path edges**:
  For each $j \in [k]$
  - add the undirected path $\widehat{a}_j^1 - \widehat{a}_j^2 - \widehat{a}_j^3 - \ldots\ldots - \widehat{a}_j^{n-1} - \widehat{a}_j^n$, and denote this path[5] by $\widehat{A}_j$,
  - add the undirected path $\widehat{b}_j^1 - \widehat{b}_j^2 - \widehat{b}_j^3 - \ldots\ldots - \widehat{b}_j^{n-1} - \widehat{b}_j^n$, and denote this path by $\widehat{B}_j$.

  For each $j \in [2k]$
  - add the undirected path $a_j^1 - a_j^2 - a_j^3 - \ldots\ldots - a_j^{n-1} - a_j^n$, and denote this path by A_j,
  - add the undirected path $b_j^1 - b_j^2 - b_j^3 - \ldots\ldots - b_j^{n-1} - b_j^n$, and denote this path by B_j.

  For each $i \in [k]$
  - add the undirected path $\widehat{c}_i^1 - \widehat{c}_i^2 - \widehat{c}_i^3 - \ldots\ldots - \widehat{c}_i^{n-1} - \widehat{c}_i^n$, and denote this path by $\widehat{C}_i$,
  - add the undirected path $\widehat{d}_i^1 - \widehat{d}_i^2 - \widehat{d}_i^3 - \ldots\ldots - \widehat{d}_i^{n-1} - \widehat{d}_i^n$, and denote this path by $\widehat{D}_i$.

  For each $i \in [2k]$
  - add the undirected path $c_i^1 - c_i^2 - c_i^3 - \ldots\ldots - c_i^{n-1} - c_i^n$, and denote this path by C_i,
  - add the undirected path $d_i^1 - d_i^2 - d_i^3 - \ldots\ldots - d_i^{n-1} - d_i^n$, and denote this path by D_i.
- **Splitting** operation:
  We first define the sets $S_{i,j} \subseteq [n] \times [n]$ for each $i, j \in [k]$:
  - For each $i \in [k]$ we let $S_{i,i} = \{(x,x) \mid x \in [n-1]\}$.
  - For each $i, j \in [k]$ with $i \neq j$ we let $S_{i,j} = \{(x,y),(y,x) \mid \{u_x, u_y\} \in E(G')\}$. That is, $S_{i,j}$ represents the ones in the adjacency matrix of G' (with one extra row and one extra column of zeroes added).

  Then for each $i, j \in [k]$ and each $x, y \in [n]$ we perform the following operation on the vertex $v_{2i,2j}^{x,y}$:
  - If $(x,y) \in S_{i,j}$ then we keep the vertex $v_{2i,2j}^{x,y}$ as is.

[5] Sometimes we also abuse notation slightly and use $\widehat{A}_j$ to denote this set of vertices

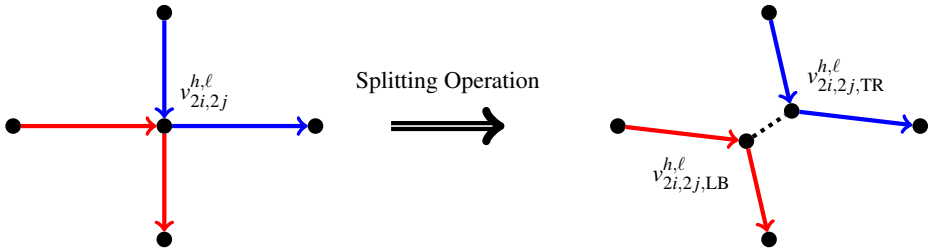

Fig. 2. The splitting operation for vertex $v^{h,\ell}_{2i,2j}$ when $(h,\ell) \notin S_{i,j}$. The idea behind this splitting is that no matter which way we orient the undirected dotted edge we cannot go **both** from left to right and from top to bottom. However, if we just want to go from left to right (top to bottom) then it is possible by orienting the dotted edge to the right (left), respectively.

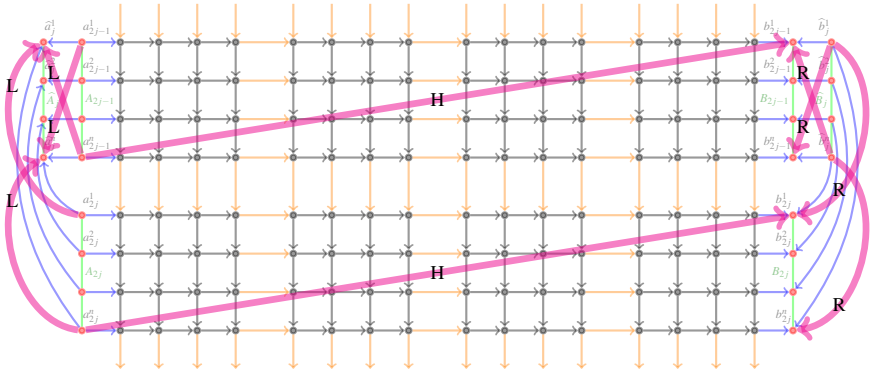

Fig. 3. Illustration of the terminal pairs in two consecutive rows. Each pair is represented by a magenta arrow between the appropriate vertices. The black labels represent the sets the pair belongs to, i.e., L stands for $\mathscr{T}_j^{\mathrm{L}}$, R stands for $\mathscr{T}_j^{\mathrm{R}}$, and H stands for $\mathscr{T}_j^{\mathrm{H}}$.

- Otherwise we **split** the vertex $v^{x,y}_{2i,2j}$ into two vertices $v^{x,y}_{2i,2j,\mathrm{LB}}$ and $v^{x,y}_{2i,2j,\mathrm{TR}}$. Note that $v^{x,y}_{2i,2j}$ had 4 incident edges: two incoming (one each from the left and the top) and two outgoing (one each to the right and the bottom). We change the edges as follows (see Figure 2):
  - ∗ Make the left incoming edge and bottom outgoing edge incident on $v^{x,y}_{2i,2j,\mathrm{LB}}$ (denoted by red color in Figure 2).
  - ∗ Make the top incoming edge and right outgoing edge incident on $v^{x,y}_{2i,2j,\mathrm{TR}}$ (denoted by blue color in Figure 2).
  - ∗ Add an **undirected** edge between $v^{x,y}_{2i,2j,\mathrm{LB}}$ and $v^{x,y}_{2i,2j,\mathrm{TR}}$ (denoted by the dotted edge in Figure 2).

By slightly abusing the notation we still refer to the grid as $G_{2i,2j}$, even though some of its vertices were split.

– The set $\mathscr{T}$ of terminal pairs is given by $\bigcup_{i\in[k]} \mathscr{T}_i^{\mathrm{L}} \cup \mathscr{T}_i^{\mathrm{R}} \cup \mathscr{T}_i^{\mathrm{T}} \cup \mathscr{T}_i^{\mathrm{B}} \cup \mathscr{T}_i^{\mathrm{H}} \cup \mathscr{T}_i^{\mathrm{V}}$, where (see also Figure 3)
  - $\mathscr{T}_j^{\mathrm{L}} = \{(a^n_{2j-1}, \widehat{a}^1_j), (a^1_{2j-1}, \widehat{a}^n_j), (a^1_{2j}, \widehat{a}^1_j), (a^n_{2j}, \widehat{a}^n_j)\}$ for each $j \in [k]$,
  - $\mathscr{T}_j^{\mathrm{R}} = \{(\widehat{b}^1_j, b^n_{2j-1}), (\widehat{b}^n_j, b^1_{2j-1}), (\widehat{b}^1_j, b^1_{2j}), (\widehat{b}^n_j, b^n_{2j})\}$ for each $j \in [k]$,

- $\mathscr{T}_i^{\mathrm{T}} = \{(c^n_{2i-1}, \widehat{c}^1_i), (c^1_{2i-1}, \widehat{c}^n_i), (c^1_{2i}, \widehat{c}^1_i), (c^n_{2i}, \widehat{c}^n_i)\}$ for each $i \in [k]$,
- $\mathscr{T}_i^{\mathrm{B}} = \{(\widehat{d}^1_i, d^n_{2i-1}), (\widehat{d}^n_i, d^1_{2i-1}), (\widehat{d}^1_i, d^1_{2i}), (\widehat{d}^n_i, d^n_{2i})\}$ for each $i \in [k]$,
- $\mathscr{T}_j^{\mathrm{H}} = \{(a^n_{2j-1}, b^1_{2j-1}), (a^n_{2j}, b^1_{2j})\}$ for each $j \in [k]$, and
- $\mathscr{T}_i^{\mathrm{V}} = \{(c^n_{2i-1}, d^1_{2i-1}), (c^n_{2i}, d^1_{2i})\}$ for each $i \in [k]$

Note that the total number of terminal pairs is $(4 \cdot 4 + 2 \cdot 2)k = 20k$.

Remark 1. Note that the constructed graph G is planar: Figure 1 gives a planar embedding of the graph before the splitting operation, and the splitting operation (see Figure 2) clearly preserves planarity.

Remark 2. Note that the index n does not appear in either first or second coordinate of any pair in the sets $S_{i,j}$ (for any $i, j \in [k]$) as there is no vertex u_n in G'. This technical condition is not necessary for our reduction, but we added it to help simplify[6] some of the definitions and proofs which appear later in the paper. For example, one can now observe easily that all vertices in $\texttt{Right}(G_{2i,2j})$ and $\texttt{Bottom}(G_{2i,2j})$ are split (for every $i, j \in [k]$).

Before giving an intuition about and proving the properties of the reduction, we first introduce some notation concerning orientations of the green path edges (i.e. potential solutions) of the instance.

Definition 1. *For any $i \in [n]$, a path on n vertices $a_1 - a_2 - \ldots\ldots - a_n$ is said to be oriented towards (away from) i if every edge $a_{j-1} - a_j$ is oriented towards (away from) a_j for every $j \leq i$ and every edge $a_j - a_{j+1}$ is oriented towards (away from) a_j for every $j \geq i$, respectively.*

We now give some intuition on how the individual terminal pairs enforce the desired structure on the solution (see Figure 3 and Figure 4). To satisfy pairs in $\mathscr{T}_j^{\mathrm{L}}$ the path $\widehat{A}_j$ must be oriented away from some index, path A_{2j-1} towards the same index and path A_{2j} towards the opposite index (see Lemma 4). Similarly, to satisfy pairs in $\mathscr{T}_j^{\mathrm{R}}$ the path $\widehat{B}_j$ must be oriented towards some index, the path B_{2j-1} must be oriented away from the same index and the path must be oriented B_{2j} away from the opposite index. Then, the pairs in $\mathscr{T}_j^{\mathrm{H}}$ enforce one inequality on the indexes selected on left and right side each, and to satisfy them, there must be a horizontal path from left to right in each row (see Lemma 5). Pairs in $\mathscr{T}_i^{\mathrm{T}}$, $\mathscr{T}_i^{\mathrm{B}}$, and $\mathscr{T}_i^{\mathrm{V}}$ enforce similar properties on paths $\widehat{C}_i$, C_{2i-1}, C_{2i}, $\widehat{D}_i$, D_{2i-1}, D_{2i} and the existence of a vertical path in each column. The splitting in grids on the diagonal enforces that the indexes selected in rows are the same as those selected in the corresponding columns and thus actually correspond to selecting a vertex of G' (see Lemma 6). The splitting in grids off the diagonal ensures that the selected vertices are adjacent (see Lemma 7).

2.2 Completeness of the reduction

In this subsection we prove the following lemma.

[6] In Definitions 2 and 3, instead of having three conditions (either j is odd or even, and for the case when j is even we would have two further cases depending on whether or not that vertex is split) for the last black vertex of a horizontal canonical path, we now only have two conditions (either j is odd and the vertex is not split, or j is even and the vertex is definitely split).

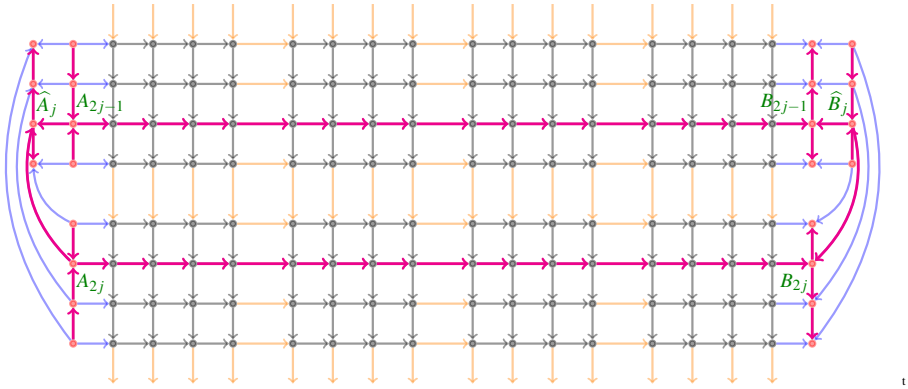

Fig. 4. Illustration of the oriented paths in an intended solution in two consecutive rows of the construction. The arcs used by the paths are drawn in magenta, vertical paths are not drawn. The path $\widehat{A}_j$ is oriented away from $n+1-x_j$ in Lemma 1 and away from $n+1-\lambda_j$ in Lemma 4 and Lemma 6. The path A_{2j-1} is oriented towards $n+1-x_j$ in Lemma 1, towards $n+1-\lambda_j$ in Lemma 4 and Lemma 6, and towards α_{2j-1} in Lemma 5. The path A_{2j} is oriented towards x_j in Lemma 1, towards λ_j in Lemma 4 and Lemma 6, and towards α_{2j} in Lemma 5. The path B_{2j-1} is oriented away from $n+1-x_j$ in Lemma 1, away from $n+1-\mu_j$ in Lemma 4 and Lemma 6, and away from β_{2j-1} in Lemma 5. The path B_{2j} is oriented away from x_j in Lemma 1, away from μ_j in Lemma 4 and Lemma 6, and away from β_{2j} in Lemma 5. The path $\widehat{B}_j$ is oriented towards $n+1-x_j$ in Lemma 1 and towards $n+1-\mu_j$ in Lemma 4 and Lemma 6.

Lemma 1. *On an instance (G',k) of k-*CLIQUE *such that G' contains a clique of size k, the reduction produces an instance $(G,\mathscr{T})$ of* PLANAR STEINER ORIENTATION *which has a solution satisfying all terminal pairs.*

Suppose that G' has a clique X of size k. Let the clique be formed by the vertices $u_{x_1}, u_{x_2}, \ldots, u_{x_k}$. We know that $x_i \in [n]$ for each $i \in [k]$. We now show that the constructed instance $(G,\mathscr{T})$ of PLANAR STEINER ORIENTATION has a solution that satisfies all the terminal pairs. Orient the undirected green path edges as follows (see Figure 4 for an illustration). For each $i, j \in [k]$

- orient $\widehat{A}_j$ and B_{2j-1} away from $n+1-x_j$,
- orient A_{2j-1} and $\widehat{B}_j$ towards $n+1-x_j$,
- orient A_{2j} towards x_j,
- orient B_{2j} away from x_j,
- orient $\widehat{C}_i$ and D_{2i-1} away from $n+1-x_i$,
- orient C_{2i-1} and $\widehat{D}_i$ towards $n+1-x_i$.
- orient C_{2i} towards x_i,
- orient D_{2i} away from x_i,

Note that so far we have only oriented the undirected green path edges, and that the dotted undirected edges arising from the splitting operation (Figure 2) are not yet oriented at this step. It is easy to see that the above orientations ensure that all terminal

pairs of $\bigcup_{i\in[k]} \mathscr{T}_i^{\mathrm{L}} \cup \mathscr{T}_i^{\mathrm{R}} \cup \mathscr{T}_i^{\mathrm{T}} \cup \mathscr{T}_i^{\mathrm{B}}$ are satisfied (see Figure 4). We now show that terminal pairs of $\bigcup_{i\in[k]} \mathscr{T}_i^{\mathrm{H}} \cup \mathscr{T}_i^{\mathrm{V}}$ are also satisfied. First we need some definitions:

Definition 2. ***(horizontal canonical paths)*** *For $j \in [2k]$ and $\ell \in [n]$, we denote by Q_j^ℓ the unique (horizontal) directed $a_j^\ell \to b_j^\ell$ path whose second vertex is $v_{1,j}^{1,\ell}$ and second-last vertex is $v_{2k,j}^{n,\ell}$ if j is odd and $v_{2k,j,TR}^{n,\ell}$ if j is even. This path starts with the blue connector edge from a_j^ℓ to $v_{1,j}^{1,\ell}$ and ends with the blue connector edge from $v_{2k,j}^{n,\ell}$ or $v_{2k,j,TR}^{n,\ell}$ to b_j^ℓ. The intermediate edges are obtained by selecting the paths of black edges given by the ℓth rows of each gadget $G_{i,j}$ for $i \in [2k]$, and connecting these small paths by horizontal orange edges.*

However, we need to address what to do when we encounter a split vertex on this path. Consider the vertex $v_{i,j}^{h,\ell}$ for some $i \in [2k]$ and $h \in [n]$. If $v_{i,j}^{h,\ell}$ is not split, then we do not have to do anything. Otherwise, if $v_{i,j}^{h,\ell}$ is split, then we add the edge $v_{i,j,LB}^{h,\ell} \to v_{i,j,TR}^{h,\ell}$ to Q_j^ℓ.

Note that the orientation of G which orients all dotted edges rightwards, i.e., LB $\to$ TR, contains each of the horizontal canonical paths defined above.

Definition 3. ***(vertical canonical paths)*** *For $i \in [2k]$ and $h \in [n]$, we denote by P_i^h the unique (vertical) $c_i^h \to d_i^h$ path whose second vertex is $v_{i,1}^{h,1}$ and second-last vertex is $v_{i,2k}^{h,n}$ if i is odd and $v_{i,2k,LB}^{h,n}$ if i is even. This path starts with the blue connector edge $(c_i^\ell, v_{i,1}^{h,1})$ and ends with the blue connector edge from $v_{i,2k}^{h,n}$ or $v_{i,2k,LB}^{h,n}$ to d_i^ℓ. The intermediate edges are obtained by selecting the paths of black edges given by the ℓth columns of each gadget $G_{i,j}$ for $j \in [2k]$, and connecting these small paths by vertical orange edges.*

However, we need to address what to do when we encounter a split vertex on this path. Consider the vertex $v_{i,j}^{h,\ell}$ for some $j \in [2k]$ and $\ell \in [n]$. If $v_{i,j}^{h,\ell}$ is not split, then we do not have to do anything. Otherwise, if $v_{i,j}^{h,\ell}$ is split, then we add the edge $v_{i,j,LB}^{h,\ell} \leftarrow v_{i,j,TR}^{h,\ell}$ to P_i^h.

Note that the orientation of G which orients all dotted edges leftwards, i.e., LB $\leftarrow$ TR, contains each of the vertical canonical paths defined above. Observe that both the horizontal canonical paths and vertical canonical paths assign orientations to the dotted edges arising from splitting vertices. Hence, one needs to be careful because the splitting operation (see Figure 2) is designed to ensure that the existence of a horizontal canonical path implies that some vertical canonical path cannot exist (recall that we are allowed to orient each undirected edge in exactly one direction).

Definition 4. ***(realizable set of paths)*** *A set of directed paths $\mathscr{P}$ in a mixed graph G is* realizable *if there is an orientation G^* of G such that each path $P \in \mathscr{P}$ appears in G^*.*

Lemma 2. *The set of vertical canonical paths $\{P_{2i-1}^{n+1-x_i}, P_{2i}^{x_i} \mid i \in [k]\}$ together with the set of horizontal canonical paths $\{Q_{2j-1}^{n+1-x_j}, Q_{2j}^{x_j} \mid j \in [k]\}$ are realizable in G.*

Proof. Orient each undirected dotted edge which appears on some horizontal path from $\{Q_{2j-1}^{n+1-x_j}, Q_{2j}^{x_j} \mid j \in [k]\}$ in LB $\to$ TR direction and each undirected dotted edge which appears on some vertical path from $\{P_{2i-1}^{n+1-x_i}, P_{2i}^{x_i} \mid i \in [k]\}$ in TR $\to$ LB direction. Since, if $i = j$, then we have $(x_i, x_i) \in S_{i,i}$ by definition and if $i \neq j$, then u_{x_i} and u_{x_j} are adjacent in G' as X is a clique in G', implying that we also have $(x_i, x_j) \in S_{i,j}$ by definition, no black vertex $v_{2i,2j}^{x_i,x_j}$ was split. Thus there is no undirected dotted edge which would appear on both a horizontal and a vertical path from the set. To prove the lemma it remains to arbitrarily orient the edges which do not have a prescribed orientation yet. □

Observe that for each $j \in [k]$, the horizontal path $Q_{2j-1}^{n+1-x_j}$ together with the oriented paths $a_{2j-1}^n \to a_{2j-1}^{n+1-x_j}$ in A_{2j-1} and $b_{2j-1}^{n+1-x_j} \to b_{2j-1}^1$ in B_{2j-1} satisfies the terminal pair (a_{2j-1}^n, b_{2j-1}^1) of $\mathscr{T}_j^{\mathrm{H}}$ for each $j \in [k]$ (see Figure 4). The same holds for the horizontal path $Q_{2j}^{x_j}$, paths in A_{2j} and B_{2j} and the terminal pair (a_{2j}^n, b_{2j}^1) of $\mathscr{T}_j^{\mathrm{H}}$ for each $j \in [k]$.

Similarly, for each $i \in [k]$, the path $P_{2i-1}^{n+1-x_i}$ together with the oriented paths $c_{2i-1}^n \to c_{2i-1}^{n+1-x_i}$ in C_{2i-1} and $d_{2i-1}^{n+1-x_i} \to d_{2i-1}^1$ in D_{2i-1} satisfies the terminal pair (c_{2i-1}^n, d_{2i-1}^1) of $\mathscr{T}_i^{\mathrm{V}}$. Finally, the path $P_{2i}^{x_i}$ together with paths in C_{2i} and D_{2i} satisfies pair (c_{2i}^n, d_{2i}^1) of $\mathscr{T}_i^{\mathrm{V}}$ for each $i \in [k]$.

Lemma 2 guarantees that these families of canonical vertical and horizontal paths can be realized by some orientation of G (note that the canonical paths only orient black edges, and not green path edges whose orientation was already fixed at the start of this subsection). This implies that $(G, \mathscr{T})$ has a solution which satisfies all terminal pairs, concluding the proof Lemma 1.

2.3 Soundness of the reduction

In this subsection we prove the following lemma.

Lemma 3. *Let (G', k) be an input instance of k-*CLIQUE*, $(G, \mathscr{T})$ be instance of* PLANAR STEINER ORIENTATION *produced by the reduction, and $0 < \vartheta \leq \frac{1}{20}$. If there is an orientation G^* of G which satisfies at least $(\frac{19}{20} + \vartheta)$-fraction of the terminal pairs, then there is a clique of size at least $20\vartheta \cdot k$ in G'. In particular, if G^* satisfies all terminal pairs, then G' has a clique of size k.*

Let G^* be the orientation which satisfies at least $(\frac{19}{20} + \vartheta)$-fraction of terminal pairs from $\mathscr{T}$. Note that the set of vertices $A \cup \widehat{B} \cup C \cup \widehat{D}$ has no incoming edges. Similarly, the set of vertices $\widehat{A} \cup B \cup \widehat{C} \cup D$ has no outgoing edges.

The next lemma restricts the orientations of the undirected paths of green path edges.

Lemma 4. *In the orientation G^*, for each $i, j \in [k]$ we have that*

- *if G^* satisfies all pairs of $\mathscr{T}_j^L$, then there exists an integer $\lambda_j \in [n]$ such that the path $\widehat{A}_j$ is oriented away from $n+1-\lambda_j$, the path A_{2j-1} is oriented towards $n+1-\lambda_j$ and the path A_{2j} is oriented towards λ_j.*

- *if G^* satisfies all pairs of $\mathscr{T}_j^R$, then there exists an integer $\mu_j \in [n]$ such that the path $\widehat{B}_j$ is oriented away from $n+1-\mu_j$, the path B_{2j-1} is oriented away from $n+1-\mu_j$ and the path B_{2j} is oriented towards μ_j.*
- *if G^* satisfies all pairs of $\mathscr{T}_i^T$, then there exists an integer $\delta_i \in [n]$ such that the path $\widehat{C}_i$ is oriented away from $n+1-\delta_i$, the path C_{2i-1} is oriented towards $n+1-\delta_i$ and the path C_{2i} is oriented towards δ_i.*
- *if G^* satisfies all pairs of $\mathscr{T}_i^B$, then there exists an integer $\varepsilon_i \in [n]$ such that the path $\widehat{D}_i$ is oriented away from $n+1-\varepsilon_i$, the path D_{2i-1} is oriented away from $n+1-\varepsilon_i$ and the path D_{2i} is oriented towards ε_i.*

Proof. Fix $j \in [k]$. We just prove the lemma for the pairs of $\mathscr{T}_j^{\mathrm{L}}$ and the paths $\widehat{A}_j$, A_{2j-1}, and A_{2j} since the proof for other cases is similar. See again Figure 4 for an illustration. Since the only edges incoming to $\widehat{A}$ are blue connector edges from A (and A has no incoming edges), it follows that the terminal pairs $(a^n_{2j-1}, \widehat{a}^1_j)$ and $(a^1_{2j-1}, \widehat{a}^n_j)$ of $\mathscr{T}_j^{\mathrm{L}}$ are satisfied by edges from the graph $G^*[\widehat{A}_j \cup A_{2j-1}]$. The path satisfying the terminal pair $(a^n_{2j-1}, \widehat{a}^1_j)$ has to travel upwards along A_{2j-1}, use a blue connector edge and then finally travel upwards along $\widehat{A}_j$. Similarly, the path satisfying the terminal pair $(a^1_{2j-1}, \widehat{a}^n_j)$ has to travel downwards along A_{2j-1}, use a blue connector edge and then finally travel downwards along $\widehat{A}_j$. Since we can only orient each green path edge in exactly one direction, it follows that both these paths must use the same blue connector edge, i.e., there exists an integer $\lambda'_i \in [n]$ such that the paths $\widehat{A}_j, A_{2j-1}$ are oriented away from, and towards λ'_i, respectively. A similar argument shows that both the paths satisfying the terminal pairs $(a^1_{2j}, \widehat{a}^1_j)$ and $(a^n_{2j}, \widehat{a}^n_j)$ have to use the blue connector edge $a^{n+1-\lambda'_i}_{2j} \to \widehat{a}^{\lambda'_i}_j$. Hence, the path A_{2j} is oriented towards $n+1-\lambda'_i$. Setting $\lambda_i = n+1-\lambda'_i$, we obtain the desired result. □

Lemma 5. *Let G^* be an orientation of G. For each $j \in [2k]$ if A_j is oriented towards $\alpha_j \in [n]$, path B_j is oriented away from $\beta_j \in [n]$, and the pair (a^n_j, b^1_j) is satisfied in G^*, then $\alpha_j \leq \beta_j$. If moreover $\alpha_j = \beta_j$, then G^* contains the horizontal canonical path $Q_j^{\alpha_j}$. For each $i \in [2k]$ if C_i is oriented towards $\gamma_i \in [n]$, path D_i is oriented away from $\zeta_i \in [n]$, and the pair (c^n_i, b^1_i) is satisfied in G^*, then $\gamma_i \leq \zeta_i$. If moreover $\gamma_i = \zeta_i$, then G^* contains the vertical canonical path $P_i^{\gamma_i}$.*

Proof. We only prove the first two claims, the proof for the other two is completely analogous. See again Figure 4 for an illustration. Fix any $j \in [2k]$ and consider the terminal pair (a^n_j, b^1_j). Since G^* satisfies the pair (a^n_j, b^1_j), it contains a path P from a^n_j to b^1_j. Since $\widehat{A}$ has no outgoing edges, P does not contain any vertices of $\widehat{A}$. Similarly, P does not contain any vertices of $\widehat{B}$. Hence, the path P consist of the following five parts: the vertical upwards path $a^n_j \to a^{n-1}_j \to \ldots \to a^\tau_j$ followed by the blue connector edge $a^\tau_j \to v^{1,\tau}_{1,j}$, then a path in the graph $G^*\left[\bigcup_{i=1}^{2k} \bigcup_{j'=1}^{2k} V(G_{i,j'})\right]$ followed by a blue connector edge $v^{n,\upsilon}_{2k,j} \to b^\upsilon_j$ and a vertical upwards path $b^\upsilon_j \to b^{\upsilon-1}_j \ldots \to b^1_j$. If j is even, then the second blue connector edge is actually $v^{n,\upsilon}_{2k,j,\mathrm{TR}} \to b^\upsilon_j$.

Since A_j is oriented towards α_j, for the first part of P to exist we have $\alpha_j \leq \tau$. Similarly, since B_j is oriented away from β_j, we have $\upsilon \leq \beta_j$. Furthermore, all edges in $G^*\left[\bigcup_{i=1}^{2k}\bigcup_{j'=1}^{2k} V(G_{i,j'})\right]$, except for those formed by splitting vertices, are oriented either downward or to the right. Moreover, the splitting does not really change the row/column level. Therefore, the central part of P only goes down or right and $\tau \leq \upsilon$. Together we obtain $\alpha_j \leq \tau \leq \upsilon \leq \beta_j$, and the first claim follows.

Moreover, if $\alpha_j = \beta_j$, then $\tau = \upsilon = \alpha_j$ and the central part of P only goes right and from LB to TR within the split vertices. It follows that it is the horizontal canonical path $Q_j^{\alpha_j}$. □

Lemma 6. *For each $i \in [k]$ and $j \in [k]$ and integers λ_j, μ_j, δ_i, ε_i as given by Lemma 4 we have the following.*

- *If G^* satisfies all pairs in $\mathscr{T}_j^L \cup \mathscr{T}_j^R \cup \mathscr{T}_j^H$, then $\lambda_j = \mu_j$. Moreover, G^* contains the horizontal canonical path $Q_{2j}^{\lambda_j}$.*
- *If G^* satisfies all pairs in $\mathscr{T}_i^T \cup \mathscr{T}_i^B \cup \mathscr{T}_i^V$, then $\delta_i = \varepsilon_i$. Moreover, G^* contains the vertical canonical path $P_{2i}^{\delta_i}$.*
- *If $i = j$ and both previous conditions hold, then moreover $\lambda_i = \delta_i$.*

Proof. Fix $j \in [k]$. See again Figure 4 for an illustration. We first prove that $\lambda_j = \mu_j$, if G^* satisfies the pairs in $\mathscr{T}_j^{\mathrm{L}} \cup \mathscr{T}_j^{\mathrm{R}} \cup \mathscr{T}_j^{\mathrm{H}}$. In this case, by Lemma 4, A_{2j-1} is oriented towards $n+1-\lambda_j$ and B_{2j-1} is oriented away from $n+1-\mu_j$ and from Lemma 5 it follows that $n+1-\lambda_j \leq n+1-\mu_j$. That is, $\lambda_j \geq \mu_j$. Since A_{2j} is oriented towards λ_j and B_{2j} is oriented away from μ_j, from Lemma 5 it follows that $\lambda_j \leq \mu_j$. Thus $\lambda_j = \mu_j$ and applying Lemma 5 again to paths A_{2j} and B_{2j} we get that G^* contains the horizontal canonical path $Q_{2j}^{\lambda_j}$.

The proof for the second part is completely analogous to the proof of the first part. Thus we move to the third part. From the previous two parts we know that G^* contains both the the horizontal canonical path $Q_{2i}^{\lambda_i}$ and the vertical canonical path $P_{2i}^{\delta_i}$. It follows that the vertex $v_{2i,2i}^{\delta_i,\lambda_i}$ was not split, that is $(\delta_i, \lambda_i) \in S_{i,i}$. This implies $\delta_i = \lambda_i$ by the definition of $S_{i,i}$, concluding the proof. □

Definition 5. *An index $i \in [k]$ is called* good *(for G^*) if all terminal pairs from $\mathscr{T}_i^L \cup \mathscr{T}_i^R \cup \mathscr{T}_i^T \cup \mathscr{T}_i^B \cup \mathscr{T}_i^H \cup \mathscr{T}_i^V$ are satisfied in G^*.*

Lemma 7. *If both $i, j \in [k]$, $i \neq j$ are good in G^*, then $\{u_{\delta_i}, u_{\delta_j}\} \in E(G')$. In particular $\delta_i \neq \delta_j$.*

Proof. By Lemma 6, if i, j are good, then G^* contains the horizontal canonical path $Q_{2j}^{\delta_j}$ and the vertical canonical path $P_{2i}^{\delta_i}$. It follows that the vertex $v_{2i,2j}^{\delta_i,\delta_j}$ was not split, that is $(\delta_i, \delta_j) \in S_{i,j}$. This implies $\{u_{\delta_i}, u_{\delta_j}\} \in E(G')$ by the definition of $S_{i,j}$. The second part follows as we do not allow loops. □

Lemma 8. *Let $Y \subseteq [k]$ be the set of good indices. For each $0 \leq \vartheta \leq \frac{1}{20}$, if G^* satisfies at least $(\frac{19}{20} + \vartheta)$-fraction of the terminal pairs, then $|Y| \geq 20\vartheta \cdot k$.*

Proof. If i is good then all 20 pairs from $\mathscr{T}_i^{\mathrm{L}} \cup \mathscr{T}_i^{\mathrm{R}} \cup \mathscr{T}_i^{\mathrm{T}} \cup \mathscr{T}_i^{\mathrm{B}} \cup \mathscr{T}_i^{\mathrm{H}} \cup \mathscr{T}_i^{\mathrm{V}}$ are satisfied in G^*. Otherwise, at most 19 pairs from this set are satisfied. Hence, the total number of satisfied pairs is at most $20|Y| + 19(k - |Y|) = 19k + |Y|$. However, we know that G^* satisfies at least $(\frac{19}{20} + \vartheta)|\mathscr{T}| = (\frac{19}{20} + \vartheta)20k = 19k + 20\vartheta \cdot k$ pairs. Hence, we have $|Y| \geq 20\vartheta \cdot k$. □

Proof (of Lemma 3). If G^* satisfies at least $(\frac{19}{20} + \vartheta)$-fraction of the terminal pairs, then by Lemma 8 there is a set Y of good indices of size at least $20\vartheta \cdot k$. By Lemma 7 the set $X = \{u_{\delta_i} \mid i \in Y\}$ forms a clique in G' of size at least $20\vartheta \cdot k$. □

2.4 Proving the lower bounds

In this subsection we finish the proof of both our theorems.

Proof (of Theorem 1). It is easy to see that the graph G has $O(n^2k^2)$ vertices and can be constructed in $\mathrm{poly}(n+k)$ time. Recall that (Remark 1) the graph G constructed in the STEINER ORIENTATION instance is planar. Combining this with the two directions from Lemma 1 and Lemma 3, we get a parameterized reduction from k-CLIQUE to PLANAR STEINER ORIENTATION. Hence, the W[1]-hardness of PLANAR STEINER ORIENTATION follows from the W[1]-hardness of k-CLIQUE. Chen et al. [4] showed that, for any function f, the existence of an $f(k) \cdot n^{o(k)}$ algorithm for k-CLIQUE violates ETH. Our reduction transforms an instance of k-CLIQUE into an equivalent instance of PLANAR STEINER ORIENTATION with $O(k)$ demand pairs. We obtain that under ETH there is no $f(k) \cdot n^{o(k)}$ time algorithm for PLANAR STEINER ORIENTATION. This concludes the proof. □

For the inapproximability proof we need some more definitions. We will use the recent result on parameterized inapproximability of k-CLIQUE from [3]. To state the result precisely, let us first state the underlying assumption, the Gap Exponential Time Hypothesis (Gap-ETH).

Hypothesis 1 ((Randomized) Gap-ETH [11,16]) *There exists a constant $\delta > 0$ such that, given a 3CNF formula Φ on n variables, no (possibly randomized) $2^{o(n)}$-time algorithm can distinguish between the following two cases correctly with probability at least 2/3:*

- *Φ is satisfiable.*
- *Every assignment to the variables violates at least a δ-fraction of the clauses of Φ.*

Here we do not attempt to reason why Gap-ETH is a plausible assumption; for more detailed discussions on the topic, please refer to [11] or [3]. For now, let us move on to state the inapproximability result from [3] that we need. Chalermsook et al. [3] showed that the only way to distinguish graphs with large cliques from those with small cliques is essentially to enumerate all vertex subsets of certain size, as stated more formally below.

Theorem 3 ([3, Theorem 18]). *Assuming Gap-ETH, there exist constants $\delta, r_0 > 0$ such that, for any function g and for any positive integers $q \geq r \geq r_0$, there is no algorithm that, given a graph G', can distinguish between the following cases in $g(q,r) \cdot n^{\delta r}$ time, where $n = |V(G')|$:*

– $\textsc{Clique}(G') \geq q$ *and*
– $\textsc{Clique}(G') < r$,

where $\textsc{Clique}(G')$ *denotes the maximum size of a clique in* G'.

While the construction presented in [3] is randomized, the authors claim that it can be derandomized so that the assumption can be limited to the (deterministic) Gap-ETH.

Proof (of Theorem 2). Let δ and r_0 be the constants from Theorem 3, ϑ be a constant such that $0 < \vartheta \leq \frac{1}{20}$ and let $\zeta = \frac{\vartheta\delta}{4}$. Let $r = \max\{3, r_0\}$ and $q = \lceil \frac{r}{20\vartheta} \rceil$. Note that since $r > 2$, we have $q > 2$ and, thus $q \leq 2\frac{r}{20\vartheta}$.

Assume for contradiction that there is a function f and an algorithm $\mathbb{A}$ that can distinguish between the two cases:

– there is an orientation of G that satisfies all pairs of $\mathscr{T}$ and
– any orientation of G satisfies less than $\frac{19}{20} + \vartheta$ fraction of pairs in $\mathscr{T}$

in $f(|\mathscr{T}|)(|V(G)|^{\zeta|\mathscr{T}|})$ time.

We design an algorithm $\mathbb{B}$ that will contradict Theorem 3 for q and r as follows. Given a graph G' we apply our reduction from Subsection 2.1 to the instance (G', q) of $\textsc{Clique}$. Then we apply algorithm $\mathbb{A}$ to the resulting instance $(G, \mathscr{T})$ of $\textsc{Steiner Orientation}$. If G' contains a clique of size q, then, by Lemma 1, there is an orientation of G satisfying all pairs from $\mathscr{T}$ and the algorithm $\mathbb{A}$ recognizes that. If each clique in G' is of size less than $r \leq 20\vartheta q$, then each orientation of G satisfies less than a $\frac{19}{20} + \vartheta$ fraction of pairs in $\mathscr{T}$. Indeed, by Lemma 3, if any orientation satisfied a $\frac{19}{20} + \vartheta$ fraction of pairs in $\mathscr{T}$, then there would be a clique of size $20\vartheta q \geq r$ in G'. Again, the algorithm $\mathbb{A}$ recognizes that.

Now since $|V(G)| = O(q^2|V(G')|^2)$, the reduction can be performed in linear time in the size of the output, and $\mathbb{A}$ runs in time $f(|\mathscr{T}|)(|V(G)|^{\zeta|\mathscr{T}|})$, we obtain that $\mathbb{B}$ runs in time

$$f(|\mathscr{T}|)(|V(G)|^{\zeta|\mathscr{T}|}) = f(20q)((O(q^2|V(G')|^2))^{\zeta\cdot 20q}) =$$
$$f(20q)((O(q^2))^{\zeta\cdot 20q})(|V(G')|^2)^{\zeta\cdot 20q}) = g(q)(|V(G')|^{2\cdot\zeta\cdot 20q}) \leq g(q)(|V(G')|^{\delta\cdot r}),$$

for suitable g. The last inequality holds since $2\zeta\cdot 20q = 2\zeta\cdot 20\cdot\lceil\frac{r}{20\vartheta}\rceil \leq 2\zeta\cdot 20\cdot 2\frac{r}{20\vartheta} = \frac{4\zeta}{\vartheta}r = \frac{4\vartheta\delta}{4\vartheta}r = \delta r$ by the definition of ζ.

Thus the existence of $\mathbb{A}$ would indeed contradict Theorem 3. □

3 Open Problems

This work completely closes the gap between upper and lower bounds for the $\textsc{Steiner Orientation}$ problem parameterized by the number of terminal pairs k, even on planar graphs. However, there are still some interesting open questions to pursue. We list some of them below:

– Does $\textsc{Steiner Orientation}$ belong to the class W[1]?

– Can we show W[2]-hardness for STEINER ORIENTATION parameterized by k?
– Can we obtain O(1)-FPT approximation for PLANAR STEINER ORIENTATION parameterized by k?

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