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The Parameterized Hardness of the k-Center Problem in Transportation Networks

Andreas Emil Feldmann*[1] and Dániel Marx†[2]

[1]Department of Applied Mathematics, Charles University, Prague, Czechia, feldmann.a.e@gmail.com
[2]Institute for Computer Science and Control, Hungarian Academy of Sciences, (MTA SZTAKI) dmarx@cs.bme.hu

Abstract

In this paper we study the hardness of the k-Center problem on inputs that model transportation networks. For the problem, a graph $G = (V, E)$ with edge lengths and an integer k are given and a center set $C \subseteq V$ needs to be chosen such that $|C| \leq k$. The aim is to minimize the maximum distance of any vertex in the graph to the closest center. This problem arises in many applications of logistics, and thus it is natural to consider inputs that model transportation networks. Such inputs are often assumed to be planar graphs, low doubling metrics, or bounded highway dimension graphs. For each of these models, parameterized approximation algorithms have been shown to exist. We complement these results by proving that the k-Center problem is W[1]-hard on planar graphs of constant doubling dimension, where the parameter is the combination of the number of centers k, the highway dimension h, and the pathwidth p. Moreover, under the Exponential Time Hypothesis there is no $f(k,p,h) \cdot n^{o(p+\sqrt{k+h})}$ time algorithm for any computable function f. Thus it is unlikely that the optimum solution to k-Center can be found efficiently, even when assuming that the input graph abides to all of the above models for transportation networks at once!

Additionally we give a simple parameterized $(1+\varepsilon)$-approximation algorithm for inputs of doubling dimension d with runtime $(k^k/\varepsilon^{O(kd)}) \cdot n^{O(1)}$. This generalizes a previous result, which considered inputs in D-dimensional L_q metrics.

1 Introduction

Given a graph $G = (V, E)$ with positive edge lengths $\ell : E \to \mathbb{Q}^+$, the k-Center problem asks to find k center vertices such that every vertex of the graph is as close as possible to one of the centers. More formally, a solution to k-Center is a set $C \subseteq V$ of *centers* such that $|C| \leq k$. If $\text{dist}(u, v)$ denotes the length of the shortest path between u and v according to the edge lengths ℓ, the objective is to minimize the *cost* $\rho = \max_{u \in V} \min_{v \in C} \text{dist}(u, v)$ of the solution C. While this is the standard way of defining the problem, throughout this paper we will rather think of it as covering the graph with balls of minimum radius. That is, let $B_v(r) = \{u \in V \mid \text{dist}(u, v) \leq r\}$ be the *ball of radius* r *around* v. The cost of a solution C equivalently is the smallest value ρ for which $\bigcup_{v \in C} B_v(\rho) = V$. The k-Center problem has numerous applications in logistics where easily accessible locations need to be chosen in a network under a budget constraint. For instance, a budget may be available to build k hospitals, shopping malls, or warehouses. These should be placed so that the distance from each point on the map to the closest facility is minimized.

The k-Center problem is NP-hard [28], and so *approximation algorithms* [28, 29] as well as *parameterized algorithms* [9, 12] have been developed for this problem. The former are algorithms that use polynomial time to compute an α-*approximation*, i.e., a solution that is at most α times worse than the optimum. For the latter, a *parameter* q is given as part of the input, and an optimum solution is computed in $f(q) \cdot n^{O(1)}$ time for some computable function f independent of the input size n. The rationale behind such an algorithm is that it solves the problem efficiently in applications where the parameter is small. If such an algorithm exists, the corresponding problem is called *fixed-parameter tractable (FPT)* for q. Another option is to consider *parameterized approximation algorithms* [22, 24], which compute an α-approximation in $f(q) \cdot n^{O(1)}$ time for some parameter q.

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By a result of Hochbaum and Shmoys [19], on general input graphs, a polynomial time 2-approximation algorithm exists, and this approximation factor is also best possible, unless P=NP. A natural parameter for k-Center is the number of centers k, for which however the problem is W[2]-hard [10], and is thus unlikely to be FPT. In fact it is even W[2]-hard [15] to compute a $(2-\varepsilon)$-approximation for any $\varepsilon > 0$, and thus parametrizing by k does not help to overcome the polynomial-time inapproximability. For structural parameters such as the vertex-cover number or the feedback-vertex-set number the problem remains W[1]-hard [21], even when combining with the parameter k. For each of the two more general structural parameters treewidth and cliquewidth, an *efficient parameterized approximation scheme (EPAS)* was shown to exist [21], i.e., a $(1+\varepsilon)$-approximation can be computed in $f(\varepsilon, w) \cdot n^{O(1)}$ time for any $\varepsilon > 0$, if w is either the treewidth or the cliquewidth, and n is the number of vertices.

Arguably however, graphs with low treewidth or cliquewidth do not model transportation networks well, since grid-like structures with large treewidth and cliquewidth can occur in road maps of big cities. As we focus on applications for k-Center in logistics, here we consider more natural models for transportation networks. These include *planar graphs*, *low doubling metrics* such as the Euclidean or Manhattan plane, or the more recently studied *low highway dimension graphs*. Our main result is that k-Center is W[1]-hard on all of these graph classes *combined*, even if adding k and the pathwidth as parameters (note that the pathwidth is a stronger parameter than the treewidth). Before introducing these graph classes, let us formally state our theorem.

Theorem 1. *Even on planar graphs with edge lengths of doubling dimension $O(1)$, the k-Center problem is W[1]-hard for the combined parameter (k, p, h), where p is the pathwidth and h the highway dimension of the input graph. Moreover, under ETH there is no $f(k, p, h) \cdot n^{o(p+\sqrt{k+h})}$ time algorithm[1] for the same restriction on the input graphs, for any computable function f.*

A *planar graph* can be drawn in the plane without crossing edges. Such graphs constitute a realistic model for road networks, since overpasses and tunnels are relatively rare. It is known [27] that also for planar graphs no $(2-\varepsilon)$-approximation can be computed in polynomial time, unless P=NP. On the positive side, k-Center is FPT [10] on unweighted planar graphs for the combined parameter k and the optimum solution cost ρ. However, typically if k is small then ρ is large and vice versa, and thus the applications for this combined parameter are rather limited. If the parameter is only k, then an $n^{O(\sqrt{k})}$ time algorithm exists for planar graphs [25]. By a very recent result [17] the k-Center problem on planar graphs with positive edge lengths admits an *efficient polynomial-time bicriteria approximation scheme*, which for any $\varepsilon > 0$ in $f(\varepsilon) \cdot n^{O(1)}$ time computes a solution that uses at most $(1+\varepsilon)k$ centers and approximates the optimum with at most k centers within a factor of $1+\varepsilon$. This algorithm implies an EPAS for parameter k on planar graphs with edge lengths, since setting $\varepsilon = \min\{\varepsilon', \frac{1}{2k}\}$ forces the algorithm to compute a $(1+\varepsilon')$-approximation in $f(k, \varepsilon') \cdot n^{O(1)}$ time using at most $(1+\varepsilon)k \leq k + \frac{1}{2}$ centers, i.e., at most k centers as k is an integer. This observation is complemented by our hardness result showing that it is necessary to approximate the solution when parametrizing by k in planar graphs with edge lengths.

Definition 2. The *doubling dimension* of a metric (X, dist) is the smallest $d \in \mathbb{R}$ such that for any $r > 0$, every ball of radius $2r$ is contained in the union of at most 2^d balls of radius r. The doubling dimension of a graph is the doubling dimension of its shortest-path metric.

Since a transportation network is embedded on a large sphere (namely the Earth), a reasonable model is to assume that the shortest-path metric abides to the Euclidean L_2-norm. In cities, where blocks of buildings form a grid of streets, it is reasonable to assume that the distances are given by the Manhattan L_1-norm. Every metric for which the distance function is given by the L_q-norm in D-dimensional space $\mathbb{R}^D$ has doubling dimension $O(D)$. Thus a road network, which is embedded into $\mathbb{R}^2$ can reasonably be assumed to have constant doubling dimension. It is known [23] that k-Center is W[1]-hard for parameter k in two-dimensional Manhattan metrics. Also, no polynomial time $(2-\varepsilon)$-approximation algorithm exists for k-Center in two-dimensional Manhattan metrics [14], and no $(1.822-\varepsilon)$-approximation for two-dimensional Euclidean metrics [14]. On the positive side, Agarwal and Procopiuc [4] showed that for any L_q metric in D dimensions, the k-Center problem can be solved optimally in $n^{O(k^{1-1/D})}$ time, and an EPAS exists for the combined parameter (ε, k, D). We generalize the latter to any metric of doubling dimension d, as formalized by the following theorem.

Theorem 3. *Given a metric of doubling dimension d and $\varepsilon > 0$, a $(1+\varepsilon)$-approximation for k-Center can be computed in $(k^k/\varepsilon^{O(kd)}) \cdot n^{O(1)}$ time.*

[1]Here $o(p + \sqrt{k+h})$ means $g(p + \sqrt{k+h})$ for any function g such that $g(x) \in o(x)$.

Theorem 1 complements this result by showing that it is necessary to approximate the cost of the solution if parametrizing by k and d.

Definition 4. The *highway dimension* of a graph G is the smallest $h \in \mathbb{N}$ such that, for some universal constant $c \geq 4$, for every $r \in \mathbb{R}^+$ and every ball $B_{cr}(v)$ of radius cr, there is a set $H \subseteq B_{cr}(v)$ of *hubs* such that $|H| \leq h$ and every shortest path of length more than r lying in $B_{cr}(v)$ contains a hub of H.

The highway dimension was introduced by Abraham et al. [1] as a formalization of the empirical observation by Bast et al. [5, 6] that in a road network, starting from any point A and travelling to a sufficiently far point B along the quickest route, one is bound to pass through some member of a sparse set of "access points", i.e., the hubs. In contrast to planar and low doubling graphs, the highway dimension has the potential to model not only road networks but also more general transportation networks such as those given by air-traffic or public transportation. This is because in such networks longer connections tend to be serviced through larger and sparser stations, which act as hubs. Abraham et al. [1] were able to prove that certain shortest-path heuristics are provably faster in low highway dimension graphs than in general graphs. They specifically chose the constant $c = 4$ in their original definition, but later work by Feldmann et al. [16] showed that when choosing any constant $c > 4$ in the definition, the structure of the resulting graphs can be exploited to obtain quasi-polynomial time approximation schemes for problems such as Travelling Salesman or Facility Location. Note that increasing the constant c in Definition 4 restricts the class of graphs further. Moreover, as shown by Feldmann et al. [16, Section 9], the highway dimension of a graph according to Definition 4 can grow arbitrarily large by just a small change in the constant c: for any c there is a graph of highway dimension 1 when using c in Definition 4, which however has highway dimension $\Omega(n)$ for any constant larger than c.[2] Other definitions of the highway dimension exist as well [1, 2, 3] (see Feldmann et al. [16, Section 9] and Blum [8] for detailed discussions).

Later, Becker et al. [7] used the framework introduced by Feldmann et al. [16] to show that whenever $c > 4$ there is an EPAS for k-Center parameterized by ε, k, and h. Note that the highway dimension is always upper bounded by the vertex-cover number, as every edge of any non-trivial path is incident to a vertex cover. Hence the aforementioned W[1]-hardness result by Katsikarelis et al. [21] for the combined parameter k and the vertex-cover number proves that it is necessary to approximate the optimum when using k and h as the combined parameter. When parametrizing only by the highway dimension but not k, it is not even known if a *parameterized approximation scheme (PAS)* exists, i.e., an $f(\varepsilon, h) \cdot n^{g(\varepsilon)}$ time $(1+\varepsilon)$-approximation algorithm for some computable functions f, g. However, under the *Exponential Time Hypothesis (ETH)* [9], by [15] there is no algorithm with doubly exponential $2^{2^{o(\sqrt{h})}} \cdot n^{O(1)}$ runtime computing a $(2-\varepsilon)$-approximation for any $\varepsilon > 0$. The same paper [15] also presents a 3/2-approximation for k-Center with runtime $2^{O(kh \log h)} \cdot n^{O(1)}$ for a more general definition of the highway dimension than the one given in Definition 4 (based on so-called *shortest path covers*). In contrast to the result of Becker et al. [7], it is not known whether a PAS exists when combining this more general definition of h with k as a parameter. Theorem 1 complements these results by showing that even on planar graphs of constant doubling dimension, for the combined parameter (k, h) no fixed-parameter algorithm exists, unless FPT=W[1]. Therefore approximating the optimum is necessary, regardless of whether h is according to Definition 4 or the more general one from [15], and regardless of how restrictive Definition 4 is made by increasing the constant c.

Definition 5. A *path decomposition* of a graph $G = (V, E)$ is a path P each of whose nodes v is labelled by a bag $K_v \subseteq V$ of vertices of G, and has the following properties:

(a) $\bigcup_{v \in V(P)} K_v = V$,

(b) for every edge $\{u, w\} \in E$ there is a node $v \in V(P)$ such that K_v contains both u and w,

(c) for every $v \in V$ the set $\{u \in V(P) \mid v \in K_u\}$ induces a connected subpath of P.

The *width* of the path decomposition is $\max\{|K_v| - 1 \mid v \in V(P)\}$. The *pathwidth* p of a graph G is the minimum width among all path decompositions for G.

The pathwidth of a graph is always at least as large as its treewidth (for which the path P in the above definition is replaced by a tree). Thus, as mentioned above, arguably, bounded pathwidth graphs are not a good model for transportation networks. Also it is already known that k-Center is W[1]-hard for this parameter, even when combining it with k [21]. We include this well-studied parameter here

[2] We remark that these graphs have unbounded doubling dimension, and that an upper bound of $O(hc^d)$ on the highway dimension of any graph using constant c in Definition 4 can be shown, if the doubling dimension is d and h is the highway dimension using constant 4.

nonetheless, since the reduction of our hardness result in Theorem 1 implies that k-Center is W[1]-hard even for planar graphs with edge lengths when *combining* any of the parameters k, h, d, and p. As noted by Feldmann et al. [16] and Blum [8], these parameters are not bounded in terms of each other, i.e., they are incomparable. Furthermore, the doubling dimension is in fact bounded by a constant in Theorem 1. Hence, even if one were to combine all the models presented above and assume that a transportation network is planar, is embeddable into some metric of constant doubling dimension, has bounded highway dimension, and even has bounded pathwidth, the k-Center problem cannot be solved efficiently, unless FPT=W[1]. Thus it seems unavoidable to approximate the problem in transportation networks, when developing fast algorithms.

1.1 Related work

The above mentioned efficient bicriteria approximation scheme [17] improves on a previous (non-efficient) bicriteria approximation scheme [13], which for any $\varepsilon > 0$ and planar input graph with edge lengths computes a $(1+\varepsilon)$-approximation with at most $(1+\varepsilon)k$ centers in time $n^{f(\varepsilon)}$ for some function f (note that in contrast to above, such an algorithm does not imply a PAS for parameter k). The paper by Demaine et al. [10] on the k-Center problem in unweighted planar graphs also considers the so-called class of *map graphs*, which is a superclass of planar graphs that is not minor-closed. They show that the problem is FPT on unweighted map graphs for the combined parameter (k, ρ). Also for the tree-depth, k-Center is FPT [21]. Another parameter related to transportation networks is the *skeleton dimension*, for which it was recently shown [8] that, under ETH, no $2^{2^{o(\sqrt{s})}} \cdot n^{O(1)}$ time algorithm can compute a $(2-\varepsilon)$-approximation for any $\varepsilon > 0$, if the skeleton dimension is s. It is not known whether this parameter yields any approximation schemes when combined with for instance k, as is the case for the highway dimension.

A closely related problem to k-Center is the ρ-Dominating Set problem, in which ρ is given and the number k of centers covering a given graph with k balls of radius ρ needs to be minimized. As this generalizes the Dominating Set problem, no $(\ln(n) - \varepsilon)$-approximation is possible in polynomial time [11], unless P=NP, and computing an $f(k)$-approximation is W[1]-hard [20] when parametrizing by k, for any computable function f.

2 The reduction

In this section we give a reduction from the Grid Tiling with Inequality (GT$_\leq$) problem, which was introduced by Marx and Sidiropoulos [26] and is defined as follows. Given κ^2 non-empty sets $S_{i,j} \subseteq [n]^2$ of pairs of integers,[3] where $i, j \in [\kappa]$, the task is to select one pair $s_{i,j} \in S_{i,j}$ for each set such that

- if $s_{i,j} = (a, b)$ and $s_{i+1,j} = (a', b')$ for $i \leq \kappa - 1$ then $a \leq a'$, and
- if $s_{i,j} = (a, b)$ and $s_{i,j+1} = (a', b')$ for $j \leq \kappa - 1$ then $b \leq b'$.

The GT$_\leq$ problem is W[1]-hard [9] for parameter κ, and moreover, under ETH has no $f(\kappa) \cdot n^{o(\kappa)}$ time algorithm for any computable function f.

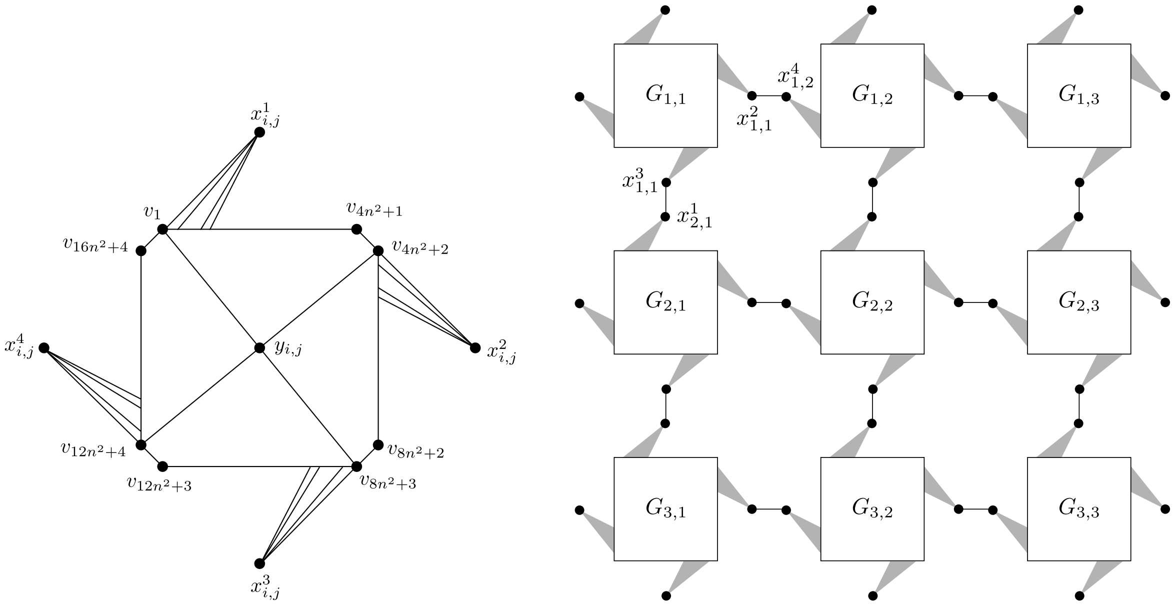
2.1 Construction

Given an instance $\mathcal{I}$ of GT$_\leq$ with κ^2 sets, we construct the following graph $G_\mathcal{I}$. First, for each set $S_{i,j}$, where $1 \leq i, j \leq \kappa$, we fix an arbitrary order on its elements, so that $S_{i,j} = \{s_1, \ldots, s_\sigma\}$, where $\sigma \leq n^2$. We then construct a gadget $G_{i,j}$ for $S_{i,j}$, which contains a cycle $O_{i,j}$ of length $16n^2 + 4$ in which each edge has length 1 (see Fig. 1(a)). Additionally we introduce five vertices $x^1_{i,j}$, $x^2_{i,j}$, $x^3_{i,j}$, $x^4_{i,j}$, and $y_{i,j}$. If $O_{i,j} = (v_1, v_2, \ldots, v_{16n^2+4}, v_1)$ then we connect these five vertices to the cycle as follows. The vertex $y_{i,j}$ is adjacent to the four vertices v_1, v_{4n^2+2}, v_{8n^2+3}, and v_{12n^2+4}, with edges of length $2n^2 + 1$ each. For every $\tau \in [\sigma]$ and $s_\tau \in S_{i,j}$, if $s_\tau = (a, b)$ we add the four edges

- $x^1_{i,j} v_\tau$ of length $\ell'_a = 2n^2 - \frac{a}{n+1}$,
- $x^2_{i,j} v_{\tau+4n^2+1}$ of length $\ell_b = 2n^2 + \frac{b}{n+1} - 1$,
- $x^3_{i,j} v_{\tau+8n^2+2}$ of length $\ell_a = 2n^2 + \frac{a}{n+1} - 1$, and
- $x^4_{i,j} v_{\tau+12n^2+3}$ of length $\ell'_b = 2n^2 - \frac{b}{n+1}$.

We say that the element s_τ corresponds to the four vertices v_τ, $v_{\tau+4n^2+1}$, $v_{\tau+8n^2+2}$, and $v_{\tau+12n^2+3}$. Note that s_1 (which always exists) corresponds to the four vertices adjacent to $y_{i,j}$. Note also that $2n^2 - 1 < \ell_a, \ell'_a, \ell_b, \ell'_b < 2n^2$, since $a, b \in [n]$.

[3] For any positive integer q, throughout this article $[q]$ means $\{1, \ldots, q\}$.



(a) The gadget $G_{i,j}$ in the reduction. **(b)** Connecting the gadgets in a grid-like fashion.

Figure 1: The structure of the graph $G_{\mathcal{I}}$ constructed for the reduction.

The gadgets $G_{i,j}$ are now connected to each other in a grid-like fashion (see Fig. 1(b)). That is, for $j \leq \kappa - 1$ we add a path $P_{i,j}$ between $x^2_{i,j}$ and $x^4_{i,j+1}$ with $n+2$ edges of length $\frac{1}{n+2}$ each. Analogously, for $i \leq \kappa - 1$ we introduce a path $P'_{i,j}$ between $x^3_{i,j}$ and $x^1_{i+1,j}$ that has $n+2$ edges, each of length $\frac{1}{n+2}$. Note that these paths all have length 1.

The resulting graph $G_{\mathcal{I}}$ forms an instance of k-Center with $k = 5\kappa^2$. We claim that the instance $\mathcal{I}$ of $\mathrm{GT}_{\leq}$ has a solution if and only if the optimum solution to k-Center on $G_{\mathcal{I}}$ has cost at most $2n^2$. We note at this point that the reduction would still work when removing the vertices $y_{i,j}$ and decreasing k to $4\kappa^2$. However their existence will greatly simplify analysing the doubling dimension of $G_{\mathcal{I}}$ in Section 3.

2.2 A solution to the $\mathrm{GT}_{\leq}$ instance implies a k-Center instance with cost $2n^2$

Recall that we fixed an order of each set $S_{i,j}$, so that each element $s_\tau \in S_{i,j}$ corresponds to four equidistant vertices on cycle $O_{i,j}$ with distance $4n^2+1$ between consecutive such vertices on the cycle. If $s_\tau \in S_{i,j}$ is in the solution to the $\mathrm{GT}_{\leq}$ instance $\mathcal{I}$, let $C_{i,j} = \{v_\tau, v_{\tau+4n^2+1}, v_{\tau+8n^2+2}, v_{\tau+12n^2+3}, y_{i,j}\}$ contain the vertices of $O_{i,j}$ corresponding to s_τ in addition to $y_{i,j}$. The solution to the k-Center instance $G_{\mathcal{I}}$ is given by the union $\bigcup_{i,j\in[\kappa]} C_{i,j}$, which consists of exactly $5\kappa^2$ centers in total.

Let us denote the set containing the four vertices of $C_{i,j} \cap V(O_{i,j})$ by $C^O_{i,j}$, and note that each of these four vertices covers $4n^2+1$ vertices of $O_{i,j}$ with balls of radius $2n^2$, as each edge of $O_{i,j}$ has length 1. Since the distance between any pair of centers in $C^O_{i,j}$ is at least $4n^2+1$, these four sets of covered vertices are pairwise disjoint. Thus the total number of vertices covered by $C^O_{i,j}$ on $O_{i,j}$ is $16n^2+4$, i.e., all vertices of the cycle $O_{i,j}$ are covered. Recall that the lengths of the edges between the vertices $x^1_{i,j}$, $x^2_{i,j}$, $x^3_{i,j}$, and $x^4_{i,j}$ and the cycle $O_{i,j}$ are $\ell_a, \ell'_a, \ell_b, \ell'_b < 2n^2$. Hence the centers in $C^O_{i,j}$ also cover $x^1_{i,j}$, $x^2_{i,j}$, $x^3_{i,j}$, and $x^4_{i,j}$ by balls of radius $2n^2$.

Now consider a path connecting two neighbouring gadgets, e.g., $P_{i,j}$ connecting $x^2_{i,j}$ and $x^4_{i,j+1}$. The center sets $C^O_{i,j}$ and $C^O_{i,j+1}$ contain vertices corresponding to the respective elements $s \in S_{i,j}$ and $s' \in S_{i,j+1}$ of the solution to the $\mathrm{GT}_{\leq}$ instance. This means that if $s = (a,b)$ and $s' = (a',b')$ then $b \leq b'$. Thus the closest centers of $C^O_{i,j}$ and $C^O_{i,j+1}$ are at distance $\ell_b + 1 + \ell'_{b'}$ from each other, as $P_{i,j}$ has length 1. From $b \leq b'$ we get

$$\ell_b + 1 + \ell'_{b'} = 2n^2 + \frac{b}{n+1} - 1 + 1 + 2n^2 - \frac{b'}{n+1} \leq 4n^2.$$

Therefore all vertices of $P_{i,j}$ are covered by the balls of radius $2n^2$ around the two closest centers of $C^O_{i,j}$ and $C^O_{i,j+1}$. Analogously, we can also conclude that any path $P'_{i,j}$ connecting some vertices $x^1_{i,j}$ and $x^3_{i+1,j}$

is covered, using the fact that if $(a,b) \in S_{i,j}$ and $(a',b') \in S_{i+1,j}$ are in the solution to the $\mathrm{GT}_{\leq}$ instance then $a \leq a'$.

Finally, the remaining center vertices in $\bigcup_{i,j\in[\kappa]} C_{i,j} \setminus C^O_{i,j}$ cover the additional vertex $y_{i,j}$ in each gadget $G_{i,j}$.

2.3 A k-Center instance with cost $2n^2$ implies a solution to the $\mathrm{GT}_{\leq}$ instance

Each vertex $y_{i,j}$ must be contained in any solution of cost at most $2n^2$, since the distance from $y_{i,j}$ to any other vertex is more than $2n^2$. This already uses κ^2 of the available $5\kappa^2$ centers.

We now prove that in any solution to the k-Center instance $G_{\mathcal{I}}$ of cost at most $2n^2$, each cycle $O_{i,j}$ must contain exactly four centers. Recall that $\ell_a, \ell'_a, \ell_b, \ell'_b > 2n^2 - 1$, that $y_{i,j}$ is incident to four edges of length $2n^2+1$ each, and that each edge of $O_{i,j}$ has length 1. Now consider the vertices v_{4n^2+1}, v_{8n^2+2}, v_{12n^2+3}, and v_{16n^2+4}, each of which is not connected by an edge to any vertex $x^q_{i,j}$, where $q \in [4]$, nor to $y_{i,j}$. Thus each of these four vertices must be covered by centers on the cycle $O_{i,j}$ if the radius of each ball is at most $2n^2$. Furthermore, the distance between each pair of these four vertices is at least $4n^2+1$, which means that any solution of cost at most $2n^2$ needs at least four centers on $O_{i,j}$ to cover these four vertices. Since there are κ^2 cycles and only $4\kappa^2$ remaining available centers, we proved that each cycle $O_{i,j}$ contains exactly four centers, and apart from the $y_{i,j}$ vertices no other centers exist in the graph $G_{\mathcal{I}}$.

Let $C^O_{i,j}$ be the set of four centers contained in $O_{i,j}$. As each center of $C^O_{i,j}$ covers at most $4n^2+1$ vertices of $O_{i,j}$ by balls of radius at most $2n^2$, to cover all $16n^2+4$ vertices of $O_{i,j}$ these four centers must be equidistant with distance exactly $4n^2+1$ between consecutive centers on $O_{i,j}$. Furthermore, since $\ell_a, \ell'_a, \ell_b, \ell'_b > 2n^2-1$ and each edge of $O_{i,j}$ has length 1, to cover $x^q_{i,j}$ for any $q \in [4]$ some center of $C^O_{i,j}$ must lie on a vertex of $O_{i,j}$ adjacent to $x^q_{i,j}$. This means that the four centers of $C^O_{i,j}$ are exactly those vertices $v_{\tau+(q-1)(4n^2+1)}$ corresponding to element s_τ of $S_{i,j}$.

It remains to show that the elements corresponding to the centers in $\bigcup_{i,j\in[\kappa]} C^O_{i,j}$ form a solution to the $\mathrm{GT}_{\leq}$ instance $\mathcal{I}$. For this, consider two neighbouring gadgets $G_{i,j}$ and $G_{i,j+1}$, and let $(a,b) \in S_{i,j}$ and $(a',b') \in S_{i,j+1}$ be the respective elements corresponding to the center sets $C^O_{i,j}$ and $C^O_{i,j+1}$. Note that for any $\hat{b} \in [n]$ we have $\ell_b \leq \ell_{\hat{b}} + 1$ and $\ell'_{b'} \leq \ell'_{\hat{b}} + 1$. Since every edge of the cycles $O_{i,j}$ and $O_{i,j+1}$ has length 1, this means that the distance from the closest centers $v \in C^O_{i,j}$ and $v' \in C^O_{i,j+1}$ to $x^2_{i,j}$ and $x^4_{i,j+1}$, respectively, is determined by the edges of length ℓ_b and $\ell'_{b'}$ incident to v and v', respectively. In particular, the distance between v and v' is $\ell_b + 1 + \ell'_{b'}$, as the path $P_{i,j}$ connecting $x^2_{i,j}$ and $x^4_{i,j+1}$ has length 1. Assume now that $b > b'$, which means that $b \geq b'+1$ since b and b' are integer. Hence this distance is

$$\ell_b + 1 + \ell'_{b'} = 2n^2 + \frac{b}{n+1} - 1 + 1 + 2n^2 - \frac{b'}{n+1} \geq 4n^2 + \frac{1}{n+1}.$$

As the centers v and v' only cover vertices at distance at most $2n^2$ each, while the edges of the path $P_{i,j}$ have length $\frac{1}{n+2} < \frac{1}{n+1}$, there must be some vertex of $P_{i,j}$ that is not covered by the center set. However this contradicts the fact that the centers form a feasible solution with cost at most $2n^2$, and so $b \leq b'$.

An analogous argument can be made for neighbouring gadgets $G_{i,j}$ and $G_{i+1,j}$, so that $a \leq a'$ for the elements $(a,b) \in S_{i,j}$ and $(a',b') \in S_{i+1,j}$ corresponding to the centers in $C^O_{i,j}$ and $C^O_{i+1,j}$, respectively. Thus a solution to $G_{\mathcal{I}}$ of cost at most $2n^2$ implies a solution to $\mathcal{I}$.

3 Properties of the constructed graph

The reduction of Section 2 proves that the k-Center problem is W[1]-hard for parameter k, since the reduction can be done in polynomial time and k is a function of κ. Since this function is quadratic, we can also conclude that, under ETH, there is no $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm for k-Center. We will now show that the graph constructed in the reduction has various additional properties from which we will be able to conclude Theorem 1. First off, it is easy to see that any constructed graph $G_{\mathcal{I}}$ for an instance $\mathcal{I}$ of $\mathrm{GT}_{\leq}$ is planar (cf. Fig. 1). We go on to prove that $G_{\mathcal{I}}$ has constant doubling dimension.

Lemma 6. *The graph $G_{\mathcal{I}}$ has doubling dimension at most $\log_2(324) \approx 8.34$ for $n \geq 2$.*

Proof. planar. To bound the doubling dimension of the graph $G_{\mathcal{I}}$, consider the shortest-path metric on the vertex set $Y = \{y_{i,j} \in V(G_{\mathcal{I}}) \mid i,j \in [\kappa]\}$ given by the distances between these vertices in $G_{\mathcal{I}}$. On an intuitive level, as these vertices are arranged in a grid-like fashion, the shortest-path metric on Y

approximates the L_1-metric. We consider a set of index pairs, for which the corresponding vertices in Y roughly resemble a ball in the shortest-path metric on Y. That is, for any $a \in \mathbb{N}_0$ consider the set of index pairs $A_{i,j}(a) = \{(i',j') \in [\kappa]^2 \mid |i-i'|+|j-j'| \leq a\}$, and let $V_{i,j}(a) \subseteq V(G_{\mathcal{I}})$ contain all vertices of gadgets $G_{i',j'}$ such that $(i',j') \in A_{i,j}(a)$ in addition to the vertices of paths of length 1 connecting these gadgets to each other and to any adjacent gadgets $G_{i'',j''}$ such that $(i'',j'') \notin A_{i,j}(a)$. We call the vertices $x^q_{i'',j''} \in V_{i,j}(a)$ such that $(i'',j'') \notin A_{i,j}(a)$, i.e., the endpoints of the latter paths of length 1, the *boundary vertices of* $V_{i,j}(a)$. We consider $y_{i,j}$ as the center of $V_{i,j}(a)$. We would like to determine the smallest radius of a ball around $y_{i,j}$ that contains all of $V_{i,j}(a)$, and the largest radius of a ball around $y_{i,j}$ that is entirely contained in $V_{i,j}(a)$. For this we need the following claim, which we will also reuse later.

Claim 7. *For any gadget $G_{i,j}$ and $q,q' \in [4]$ with $q \neq q'$, the distance between $x^q_{i,j}$ and $x^{q'}_{i,j}$ in $G_{\mathcal{I}}$ lies between $7n^2-1$ and $8n^2+2$.*

Proof. The distance between $x^q_{i,j}$ and $x^{q'}_{i,j}$ is less than $2(2n^2+2n^2+1) = 8n^2+2$, via the path passing through $y_{i,j}$ and the two vertices of $O_{i,j}$ adjacent to $y_{i,j}$, $x^q_{i,j}$, and $x^{q'}_{i,j}$. Note that the shortest path between $x^q_{i,j}$ and $x^{q'}_{i,j}$ inside the gadget $G_{i,j}$ does not necessarily pass through $y_{i,j}$, but may pass along the cycle $O_{i,j}$ instead. This is because the set $S_{i,j}$ of the $\mathrm{GT}_{\leq}$ instance may contain up to n^2 elements, which would imply a direct edge from $x^q_{i,j}$ to $v_{n^2+(q-1)(4n^2+1)}$ on $O_{i,j}$. Thus we can give a lower bound of $2(2n^2-1)+3n^2+1 = 7n^2-1$ for the distance between $x^q_{i,j}$ and $x^{q'}_{i,j}$ inside of $G_{i,j}$. This is also the shortest path between these vertices in $G_{\mathcal{I}}$, since any other path needs to pass through at least three gadgets. ⌟

We define the *circumradius* of $V_{i,j}(a)$ as the maximum distance from $y_{i,j}$ to any vertex inside of $V_{i,j}(a)$, while the *inradius* of $V_{i,j}(a)$ is the minimum distance from $y_{i,j}$ to any vertex outside of $V_{i,j}(a)$. Note that $V_{i,j}(a) \subseteq B_{y_{i,j}}(r)$ if r is the circumradius, and $B_{y_{i,j}}(r-\varepsilon) \subseteq V_{i,j}(a)$ for any $\varepsilon > 0$ if r is the inradius. Any shortest path from $y_{i,j}$ to a vertex in $V_{i,j}(a)$ passes through the gadget $G_{i,j}$, at most a additional gadgets $G_{i',j'}$ with $(i',j') \in A_{i,j}(a)$, the paths of length 1 connecting these gadgets, and possibly one path of length 1 to reach a boundary vertex of $V_{i,j}(a)$. Hence, by Claim 7, the circumradius of $V_{i,j}(a)$ is less than $(8n^2+2)a+(a+1)+4n^2+1 = (8n^2+3)a+4n^2+2$, since the distance from $y_{i,j}$ to any $x^q_{i,j}$ is less than $4n^2+1$. To reach any vertex outside of $V_{i,j}(a)$ from $y_{i,j}$ it is necessary to first reach $x^q_{i,j}$ for some $q \in [4]$, then pass through a gadgets $G_{i',j'}$ with $(i',j') \in A_{i,j}(a)$, in addition to a paths of length 1 connecting them and $G_{i,j}$, and finally pass through another path of length 1 to reach a boundary vertex of $V_{i,j}(a)$. From the boundary, a vertex not in $V_{i,j}(a)$ can be reached on some cycle $O_{i'',j''}$ with $(i'',j'') \notin A_{i,j}(a)$. The distance from $y_{i,j}$ to $x^q_{i,j}$ is more than $4n^2$ and the distance from a boundary vertex $x^{q'}_{i'',j''}$ to any vertex of $O_{i'',j''}$ is more than $2n^2-1$. Hence, by Claim 7, the inradius of $V_{i,j}(a)$ is more than $(7n^2-1)a+(a+1)+4n^2+(2n^2-1) = 7n^2a+6n^2$.

Now consider any ball $B_v(2r)$ of radius $2r$ around some vertex v in $G_{\mathcal{I}}$ for which we need to bound the number of balls of half the radius with which to cover $B_v(2r)$. Let $y_{i,j}$ be the closest vertex of Y to v. The distance between $y_{i,j}$ and v is at most $2(2n^2+1) = 4n^2+2$, whether v lies on $O_{i,j}$ or on one of the paths of length 1 connecting $G_{i,j}$ with an adjacent gadget. Hence the ball $B_v(2r)$ is contained in a ball of radius $4n^2+2+2r$ around $y_{i,j}$. The latter ball is in turn contained in the set $V_{i,j}(a)$ centered at $y_{i,j}$ if the ball's radius is less than the inradius of $V_{i,j}(a)$. This in particular happens if $4n^2+2+2r \leq 7n^2a+6n^2$, which for instance is true if $a = \lceil \frac{2+2r-2n^2}{7n^2} \rceil$. Assume first that $r \geq 12n^2+5$, which implies that $a > 0$ and so $V_{i,j}(a)$ is well-defined.

At the same time, any set $V_{i',j'}(a')$ is contained in a ball of radius r around $y_{i',j'}$ if its circumradius is at most r, i.e., $(8n^2+3)a'+4n^2+2 \leq r$. This is for instance true if $a' = \lfloor \frac{r-4n^2-2}{8n^2+3} \rfloor$. Note that $r \geq 12n^2+5$ means that $a' \geq 0$ and so $V_{i',j'}(a')$ is well-defined. We may cover all vertices of $A_{i,j}(a)$ with $\lceil \frac{2a+1}{2a'+1} \rceil^2$ sets $A_{i',j'}(a')$, since in Y these sets correspond to "squares rotated by 45 degrees" (i.e., balls in L_1) of diameter $2a+1$ and $2a'+1$, respectively. Thus we can cover $V_{i,j}(a)$ with $\lceil \frac{2a+1}{2a'+1} \rceil^2$ sets $V_{i',j'}(a')$, i.e., we can cover a ball of radius $2r$ in $G_{\mathcal{I}}$ with

$$\left\lceil \frac{2a+1}{2a'+1} \right\rceil^2 \leq \left\lceil \frac{2(\frac{2+2r-2n^2}{7n^2})+3}{2(\frac{r-4n^2-2}{8n^2+3})-1} \right\rceil^2 = \left\lceil \frac{(8n^2+3)(4+4r+17n^2)}{7n^2(2r-7-16n^2)} \right\rceil^2 \leq \left\lceil \frac{9(65r-37)}{7(8r-4)} \right\rceil^2 \leq 121$$

balls of radius r, using that $r \geq 12n^2+5$ implies $n^2 \leq (r-5)/12$.

Next consider the case when $2n^2+1 \leq r < 12n^2+5$. We know from above that $B_v(2r)$ is contained in $V_{i,j}(a)$ if $a = \lceil \frac{2+2r-2n^2}{7n^2} \rceil$, which is well-defined as $r \geq 2n^2+1$ implies $a \geq 0$. Using $r < 12n^2+5$ and

$n \geq 2$ we get $a \leq 4$. The set $V_{i,j}(4)$ contains at most $(2 \cdot 4 + 1)^2 = 81$ gadgets. On each of the cycles $O_{i',j'}$ with $(i', j') \in A_{i,j}(4)$ we may choose the four vertices v_1, v_{4n^2+2}, v_{8n^2+3}, and v_{12n^2+4} adjacent to $y_{i',j'}$ as centers for balls of radius r. Note that as $r \geq 2n^2 + 1$, the vertex $y_{i',j'}$, every vertex of the cycle $O_{i',j'}$, and also every vertex on a path of length 1 adjacent to gadget $G_{i',j'}$ is at distance at most r to one of these four vertices. Thus at most $4 \cdot 81 = 324$ balls of half the radius are needed to cover all vertices of $B_v(2r)$.

The next case we consider is $n^2 - 1 \leq r < 2n^2 + 1$. Again, $B_v(2r)$ is contained in $V_{i,j}(a)$ if $a = \lceil \frac{2+2r-2n^2}{7n^2} \rceil$, which is well-defined as $r \geq n^2 - 1$ implies $a \geq 0$. Using $r < 2n^2 + 1$ and $n \geq 1$ we obtain $a \leq 1$, which in turn means that the number of gadgets in $V_{i,j}(a)$ now is at most $(2a + 1)^2 \leq 9$. To cover a cycle $O_{i',j'}$ with $(i', j') \in A_{i,j}(1)$, we may choose centers for balls of radius r equidistantly at every $\lfloor 2r \rfloor$-th vertex of $O_{i',j'}$, as all edges of the cycle have length 1. We may lower bound $\lfloor 2r \rfloor \geq 2n^2 - 3$ using $r \geq n^2 - 1$. Since every cycle contains $16n^2 + 4$ vertices, the number of balls to cover a cycle is at most $16n^2 + 4/\lfloor 2r \rfloor \leq 68/5 \leq 14$, using the previous bound and $n \geq 2$. Hence at most $14 \cdot 9 = 126$ balls of half the radius are needed to cover the cycles in $B_v(2r)$. We can then cover the 9 vertices $y_{i',j'}$ where $(i', j') \in A_{i,j}(1)$ and the $2 \cdot 9 + 2 \cdot 3 = 24$ paths of length 1 contained in $V_{i,j}(a)$ with a ball of radius r each, as $r \geq n^2 - 1 \geq 1$ using $n \geq 2$. Hence a total of at most $9 + 24 + 126 = 159$ balls of half the radius suffice to cover $B_v(2r)$.

Finally, if $r < n^2 - 1$, then a ball $B_v(2r)$ contains only a subpath of some cycle $O_{i,j}$, a subpath of a path of length 1 connecting two gadgets, or a single vertex $y_{i,j}$, since any edge connecting a cycle $O_{i,j}$ to $y_{i,j}$ or some $x^q_{i,j}$ has length more than $2n^2 - 1 > 2(n^2 - 1) > 2r$. In this case at most 3 balls of radius r suffice to cover all vertices of $B_v(2r)$. ◻

We next show that we can bound the parameters p and h, i.e., the pathwidth and highway dimension of $G_{\mathcal{I}}$, linearly by κ and $k = \Theta(\kappa^2)$, respectively. Note that the following lemma bounds the highway dimension in terms of k, no matter how restrictive we make Definition 4 by increasing the constant c.

Lemma 8. *For any constant c of Definition 4, the graph $G_{\mathcal{I}}$ has highway dimension at most $O(\kappa^2)$.*

Proof. For any scale $r \in \mathbb{R}^+$ and universal constant $c \geq 4$ we will define a hub set $H_r \subseteq V$ hitting all shortest paths of length more than r in $G_{\mathcal{I}}$, such that $|H_r \cap B_{cr}(v)| = O(\kappa^2)$ for any ball $B_{cr}(v)$ of radius cr in $G_{\mathcal{I}}$. This bounds the highway dimension to $O(\kappa^2)$ according to Definition 4.

Let $X = \{y_{i,j}, x^q_{i,j} \mid q \in [4] \text{ and } i, j \in [\kappa]\}$ so that it contains all vertices connecting gadgets $G_{i,j}$ to each other in addition to the vertices $y_{i,j}$. If $r > 8n^2 + 2$ then $H_r = X$. Any shortest path containing only vertices of a cycle $O_{i,j}$ has length at most $8n^2 + 2$, since the cycle has length $16n^2 + 4$. Any (shortest) path that is a subpath of a path connecting two gadgets has length at most 1. Hence any shortest path of length more than $8n^2 + 2$ must contain some vertex of X. The total size of X is $5\kappa^2$, and so any ball, no matter its radius, also contains at most this many hubs of H_r.

If $1 \leq r \leq 8n^2 + 2$ then any path of length more than r but not containing any vertex of X must lie on some cycle $O_{i,j} = (v_1, v_2, \ldots, v_{16n^2+4}, v_1)$. We define the set $W^{i,j}_r = \{v_{1+\lambda\lfloor r \rfloor} \in V(O_{i,j}) \mid \lambda \in \mathbb{N}_0\}$, i.e., it contains every r-th vertex on the cycle after rounding down. This means that every path on $O_{i,j}$ of length more than r contains a vertex of $W^{i,j}_r$. Thus for these values of r we set $H_r = X \cup \bigcup_{i,j \in [\kappa]} W^{i,j}_r$. Any ball $B_{cr}(v)$ of radius cr contains $O(c)$ hubs of any $W^{i,j}_r$. By Claim 7, the distance between any pair of the four vertices $x^q_{i,j}$, where $q \in [4]$, that connect a gadget $G_{i,j}$ to other gadgets, is more than $7n^2 - 1$. This means that $B_{cr}(v)$ can only intersect $O(c^2)$ gadgets, since $cr \leq c(8n^2 + 2) \leq 2c(7n^2 - 1)$ if $n \geq 1$ and the gadgets are connected in a grid-like fashion. Hence the ball $B_{cr}(v)$ only contains $O(c)$ hubs for each of the $O(c^2)$ sets $W^{i,j}_r$ for which $B_{cr}(v)$ intersect the respective gadget $G_{i,j}$. At the same time each gadget contains only 5 vertices of X. Thus if c is a constant, then the number of hubs of H_r in $B_{cr}(v)$ is constant.

If $r < 1$, a path of length more than r may be a subpath of a path connecting two gadgets. Recall that the paths $P_{i,j}$ connecting $x^2_{i,j}$ and $x^4_{i,j+1}$ for $j \leq \kappa - 1$, and the paths $P'_{i,j}$ connecting $x^3_{i,j}$ and $x^1_{i+1,j}$ for $i \leq \kappa - 1$, consist of $n + 2$ edges of length $\frac{1}{n+2}$ each. If $P_{i,j} = (u_0, u_1, \ldots, u_{n+2})$, we define the set $U^{i,j}_r = \{u_{\lambda\lfloor r(n+2) \rfloor} \in V(P_{i,j}) \mid \lambda \in \mathbb{N}_0\}$, and if $P'_{i,j} = (u_0, u_1, \ldots, u_{n+2})$, we define the set $\tilde{U}^{i,j}_r = \{u_{\lambda\lfloor r(n+2) \rfloor} \in V(P'_{i,j}) \mid \lambda \in \mathbb{N}_0\}$, i.e., these sets contain vertices of consecutive distance r on the respective paths, after rounding down. Now let $H_r = \bigcup_{i,j \in [\kappa]} V(G_{i,j}) \cup \bigcup_{i \in [\kappa], j \in [\kappa-1]} U^{i,j}_r \cup \bigcup_{i \in [\kappa-1], j \in [\kappa]} \tilde{U}^{i,j}_r$, so that every path of length more than r contains a hub of H_r. Any ball $B_{cr}(v)$ of radius $cr < c$ intersects only $O(c^2)$ gadgets $G_{i,j}$, as observed above. As the edges of a cycle $O_{i,j}$ have length 1, the ball B contains only $O(c)$ vertices of $O_{i,j}$. Thus $B_{cr}(v)$ contains $O(c)$ hubs of $V(G_{i,j}) \cup U^{i,j}_r \cup \tilde{U}^{i,j}_r$ for each of the $O(c^2)$ gadgets $G_{i,j}$ it intersects. For constant c, this proves the claim. ◻

Lemma 9. *The graph $G_{\mathcal{I}}$ has pathwidth at most $\kappa + O(1)$.*

Proof. We construct a path decomposition of $G_{\mathcal{I}}$ using bags of size $\kappa + O(1)$. For each $i, j \in [\kappa]$ we define the sets $X^2_{i,j} = \{x^2_{i',j} \mid i' \in [i]\}$ and $X^4_{i,j} = \{x^4_{i',j} \mid i' \in [\kappa]\setminus[i-1]\}$ and let $K_{i,j} = \{y_{i,j}, x^1_{i,j}, x^3_{i,j}\} \cup X^2_{i,j} \cup X^4_{i,j}$ be a bag. Intuitively, these bags decompose the graph $G_{\mathcal{I}}$ "from left to right" according to Fig. 1. More precisely, using some additional intermediate bags, the constructed path decomposition will arrange these bags on a path with start vertex $K_{1,1}$, such that traversing the path will consecutively move from $K_{i,j}$ to $K_{i+1,j}$ for each $1 \le i \le \kappa - 1$ and $1 \le j \le \kappa$, and from $K_{\kappa,j}$ to $K_{1,j+1}$ for each $1 \le j \le \kappa - 1$.

To define the intermediate bags, consider a bag $K_{i,j}$ and note that the three connected components left after removing all vertices of $K_{i,j}$ from $G_{\mathcal{I}}$ are (a) the cycle $O_{i,j}$, (b) the subgraph $L_{i,j}$ "to the left of" $K_{i,j}$ induced by all gadgets $G_{i',j'}$ and paths $P_{i',j'}, P'_{i',j'}$ for which $j' \le j - 1$ and $i' \le \kappa$, but also the gadgets $G_{i',j'}$ and paths $P'_{i',j'}$ for which $j' = j$ and $i' \le i - 1$, and finally (c) the subgraph $R_{i,j}$ "to the right of" $K_{i,j}$ induced by all gadgets $G_{i',j'}$ and paths $P_{i',j'}, P'_{i',j'}$ for which either $j' = j$ and $i' \ge i + 1$, or $j' \ge j + 1$ and $i' \le \kappa$, but also the paths $P_{i',j}$ where $i' \le i$ and the path $P'_{i,j}$.

For any $i \le \kappa - 1$, removing the union $K_{i,j} \cup K_{i+1,j}$ from $G_{\mathcal{I}}$ leaves $L_{i,j}$, $R_{i+1,j}$, $O_{i,j}$, $O_{i+1,j}$, and the path $P'_{i,j}$ connecting the gadgets $G_{i,j}$ and $G_{i+1,j}$. The intermediate bags connecting $K_{i,j}$ and $K_{i,j+1}$ for $i \le \kappa - 1$ on the path decomposition will first cover $O_{i,j}$ and then $P'_{i,j}$: if $O_{i,j} = (v_1, v_2, \ldots, v_{16n^2+4}, v_1)$, we define a sequence of bags $K^\tau_{i,j} = K_{i,j} \cup \{v_1, v_\tau, v_{\tau+1}\}$ where $\tau \in [16n^2 + 3]$, and if $P'_{i,j} = (u_0, u_1, \ldots, u_{n+2})$ where $u_0 = x^3_{i,j}$ and $u_{n+2} = x^1_{i+1,j}$ then we define a sequence of bags $J^\tau_{i,j} = K_{i,j} \cup \{u_{\tau-1}, u_\tau\}$ for $\tau \in [n+2]$. Note that for every edge e of $O_{i,j}$ or $P'_{i,j}$ there is a bag containing the vertices of e. Moreover, for every other edge e of gadget $G_{i,j}$ connecting $O_{i,j}$ to $x^q_{i,j}$ for $q \in [4]$ or to $y_{i,j}$ there also is a bag $K^\tau_{i,j}$ containing the vertices of e. Now, the path decomposition contains a subpath between the vertices corresponding to $K_{i,j}$ and $K_{i+1,j}$, which starting from $K_{i,j}$ first traverses vertices for $K^\tau_{i,j}$ with increasing index τ, then connects $K^{16n^2+3}_{i,j}$ to $J^1_{i,j}$, then traverses through $J^\tau_{i,j}$ with increasing τ, and finally connects $J^{n+2}_{i,j}$ to $K_{i+1,j}$. That is, the sequence of bags defined by the subpath is

$$(K_{i,j}, K^1_{i,j}, K^2_{i,j}, \ldots, K^{16n^2+3}_{i,j}, J^1_{i,j}, J^2_{i,j}, \ldots, J^{n+2}_{i,j}, K_{i+1,j}).$$

To connect $K_{\kappa,j}$ to $K_{1,j+1}$ for some $j \le \kappa - 1$, we define additional bags $K'_{i,j} = X^2_{i,j} \cup X^4_{i,j+1}$. Starting from $K_{\kappa,j}$ and using intermediate bags, the path decomposition will traverse the bags $K'_{i,j}$ with decreasing index i until reaching $K_{1,j+1}$.

We first describe the bags of the path decomposition connecting $K_{\kappa,j}$ to the first additional bag $K'_{\kappa,j}$. Defining the intermediate bags is similar to above. For any $i \in [\kappa]$, removing the vertices of $K'_{i,j}$ from $G_{\mathcal{I}}$ leaves three connected components of which one is $P_{i,j}$ connecting the respective gadgets $G_{i,j}$ and $G_{i,j+1}$, one is a component $L'_{i,j}$, which is $L_{1,j+1}$ without the paths $P_{i',j}$ where $i' \le i$, and one is a component $R'_{i,j}$, which is $R_{\kappa,j}$ without the paths $P_{i',j}$ where $i' \ge i$. If $P_{i,j} = (u_0, u_1, \ldots, u_{n+2})$ where $u_0 = x^2_{i,j}$ and $u_{n+2} = x^4_{i,j+1}$, we define a sequence of bags $I^\tau_{i,j} = K'_{i,j} \cup \{u_{\tau-1}, u_\tau\}$ for $\tau \in [n + 2]$. Note that for every edge of $P_{i,j}$ there is a bag $I^\tau_{i,j}$ containing its vertices. Now, the path decomposition contains a subpath connecting vertices corresponding to $K_{\kappa,j}$ and $K'_{\kappa,1}$, which starting from $K_{\kappa,j}$ moves to $K^1_{\kappa,j}$, then through $K^\tau_{\kappa,j}$ with increasing index τ to cover $O_{\kappa,j}$, and then connects $K^{16n^2+3}_{i,j}$ to $I^1_{\kappa,j}$. It then traverses through $I^\tau_{\kappa,j}$ with increasing index τ to cover $P_{\kappa,j}$, after which it moves on to $K'_{\kappa,j}$. That is, the sequence of bags defined by the subpath is

$$(K_{\kappa,j}, K^1_{\kappa,j}, K^2_{\kappa,j}, \ldots, K^{16n^2+3}_{\kappa,j}, I^1_{\kappa,j}, I^2_{\kappa,j}, \ldots, I^{n+2}_{\kappa,j}, \ldots, K'_{\kappa,j}).$$

For any $i \le \kappa - 1$ the path decomposition contains a subpath connecting $K'_{i+1,j}$ to $K'_{i,j}$, covering $P_{i,j}$ via the bags $I^\tau_{i,j}$ with increasing index τ. The sequence defined by this subpath is

$$(K'_{i+1,j}, I^1_{i,j}, I^2_{i,j}, \ldots, I^{n+2}_{i,j}, K'_{i,j}).$$

The last additional bag $K'_{1,j}$ is connected directly to $K_{1,j+1}$ on the path decomposition.

Finally, when at $K_{\kappa,\kappa}$ the path decomposition only needs to cover $O_{\kappa,\kappa}$ to finish, i.e., to make sure that every vertex of $G_{\mathcal{I}}$ is contained in some bag. This can be done using the sequence $K^\tau_{\kappa,\kappa}$ with increasing index τ, as above. That is, the sequence of bags defined by the final subpath of the path decomposition is

$$(K_{\kappa,\kappa}, K^1_{\kappa,\kappa}, K^2_{\kappa,\kappa}, \ldots, K^{16n^2+3}_{\kappa,\kappa})$$

As argued above, for every edge of $G_{\mathcal{I}}$ there is a bag containing its vertices. To argue that all bags containing some vertex of $G_{\mathcal{I}}$ form a subpath of the path decomposition, note that for intermediate bags $K^{\tau}_{i,j}$, $J^{\tau}_{i,j}$, and $I^{\tau}_{i,j}$ we have $K^{\tau}_{i,j}, J^{\tau}_{i,j} \supseteq K_{i,j}$ and $I^{\tau}_{i,j} \supseteq K'_{i,j}$. Also note that every vertex of $X = \{y_{i,j}, x^q_{i,j} \mid q \in [4] \text{ and } i,j \in [\kappa]\}$ lies in some bag $K_{i,j}$ or $K'_{i,j}$. Let $x \in X$ be a vertex that appears in a bag B but not in bag B', and assume first that B comes before B' in the sequence defined by the path decomposition. Since the path decomposition traverses $G_{\mathcal{I}}$ "from left to right", this means that x lies in the set $L_{i,j}$ of some bag $K_{i,j}$, or the set $L'_{i,j}$ of some bag $K'_{i,j}$, for which $B' \supseteq K_{i,j}$ or $B' \supseteq K'_{i,j}$, respectively. Similarly, if B' comes before B in the sequence, then x lies in the set $R_{i,j}$ of some bag $K_{i,j}$, or the set $R'_{i,j}$ of some bag $K'_{i,j}$, for which $B' \supseteq K_{i,j}$ or $B' \supseteq K'_{i,j}$, respectively. Now observe that $L_{i,j} \subset L_{i',j'}$ and $R_{i,j} \supset R_{i',j'}$ for any $i < i'$ where $j = j'$ but also for any $j < j'$, while for bags $K'_{i,j}$, for any j and $i > i'$ we have $L_{\kappa,j} \subset L'_{i,j} \subset L'_{i',j}$ and $R'_{i,j} \supset R'_{i',j} \supset R_{1,j}$. This means that, by definition of the sequence of bags along the path decomposition, if B comes before B' the vertex x cannot appear in any bag after B' either, while if B' comes before B then x cannot appear in any bag before B' either. As a consequence, the bags containing any $x \in X$ must form a subpath of the path decomposition.

It remains to argue about vertices not in X. Note that these only occur in intermediate bags $K^{\tau}_{i,j}$, $J^{\tau}_{i,j}$, and $I^{\tau}_{i,j}$ on some cycle of a gadget or a path connecting gadgets. Furthermore, the vertices of a cycle $O_{i,j}$ only occur in the bags $K^{\tau}_{i,j}$, vertices of paths $P'_{i,j}$ (except the endpoints which lie in X) only occur in the bags $J^{\tau}_{i,j}$, and vertices of paths $P_{i,j}$ (except the endpoints which lie in X) only occur in the bags $I^{\tau}_{i,j}$. It is thus easy to see from the definition of the sequences of bags above, that any vertex not in X only lies in bags that form a subpath of the path decomposition.

Note that each bag contains $\kappa + O(1)$ vertices, which concludes the proof. ☐

The reduction given in Section 2 together with Lemma 6, Lemma 8, and Lemma 9 imply Theorem 1, since the $\mathrm{GT}_{\leq}$ problem is W[1]-hard [9] for parameter κ, and we may assume w.l.o.g. that $n \geq 2$. Moreover $\kappa = \Theta(\sqrt{k})$ and, under ETH, $\mathrm{GT}_{\leq}$ has no $f(\kappa) \cdot n^{o(\kappa)}$ time algorithm [9] for any computable function f.

4 An algorithm for low doubling metrics

In this section we give a simple algorithm that generalizes one of Agarwal and Procopiuc [4], which for D-dimensional L_q metrics computes a $(1+\varepsilon)$-approximation in time $f(\varepsilon, k, D) \cdot n^{O(1)}$. In particular, any such metric has doubling dimension $O(D)$. Here we assume that the input metric has doubling dimension d. A fundamental observation about metrics of bounded doubling dimension is the following, which can be proved by a simple recursive application of Definition 2. Here the aspect ratio of a set $Y \subseteq X$ is the diameter of Y divided by the minimum distance between any two points of Y.

Lemma 10 ([18])**.** *Let (X, dist) be a metric with doubling dimension d and $Y \subseteq X$ be a subset with aspect ratio α. Then $|Y| \leq 2^{d\lceil \log_2 \alpha \rceil}$.*

To compute a $(1+\varepsilon)$-approximation to k-Center given a graph G with vertex set V, we first compute its shortest-path metric (V, dist). We then compute several *nets* of this metric, which are defined as follows.

Definition 11. For a metric (X, dist), a subset $Y \subseteq X$ is called a *δ-cover* if for every $u \in X$ there is a $v \in Y$ such that $\mathrm{dist}(u,v) \leq \delta$. A *$\delta$-net* is a δ-cover with the additional property that $\mathrm{dist}(u,v) > \delta$ for all distinct points $u, v \in Y$.

Note that a δ-net can be computed greedily in polynomial time. The first step of our algorithm is to guess the optimum cost ρ by trying each of the $\binom{n}{2}$ possible values. For each guess we compute an $\frac{\varepsilon\rho}{2}$-net $Y \subseteq V$. We know that the metric (V, dist) can be covered by k balls of diameter 2ρ each, which means that the aspect ratio of Y inside of each ball is at most $4/\varepsilon$. Thus by Lemma 10, each ball contains $1/\varepsilon^{O(d)}$ vertices of Y, and so $|Y| \leq k/\varepsilon^{O(d)}$.

An optimum k-Center solution $C \subseteq Y$ for (Y, dist) can be computed by brute force in $\binom{|Y|}{k} = k^k/\varepsilon^{O(kd)}$ steps. Since every center of the optimum solution $C^* \subseteq V$ of the input graph has a net point of Y at distance at most $\frac{\varepsilon\rho}{2}$, there exists a k-Center solution in Y of cost at most $(1+\varepsilon/2)\rho$, given that ρ is the optimum cost. The computed center set $C \subseteq Y$ thus also has cost at most $(1+\varepsilon/2)\rho$. Therefore C covers all of V with balls of radius $(1+\varepsilon)\rho$, since every vertex of V is at distance $\frac{\varepsilon\rho}{2}$ from some vertex of Y. Thus C is a $(1+\varepsilon)$-approximation of the input graph. Considering the guessed values of ρ in increasing order, outputting the first computed solution with cost at most $(1+\varepsilon)\rho$ gives the algorithm of Theorem 3.

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