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PDE-based deployment with communicating leaders for a large-scale multi-agent system

Shubham Khansili and Anton Selivanov

***Abstract*— We study the deployment of a first-order multi-agent system (MAS) onto a curve in $\mathbb{R}^n$. The MAS has a chain topology and two types of agents: leaders and followers. The leaders know their positions relative to the target curve. Neighboring leaders can communicate with one another. Each follower is aware of the intended and existing differences between its state and the states of its two nearest neighbors. To solve the formation control problem, we derive a semi-linear parabolic PDE describing the system when the number of agents is sufficiently large. We derive the stability condition in terms of linear matrix inequalities (LMIs). Using numerical simulations, we demonstrate that increased connectivity between the leaders improves the deployment speed of the MAS.**

***Index Terms*— Multi-agent systems, partial differential equations, Lyapunov method, linear matrix inequalities, deployment.**

I. INTRODUCTION

Due to its capacity for resolving intricate computational challenges, distributed artificial intelligence (DAI) has remained a prominent topic of interest for researchers globally. DAI can be broadly divided into three classifications based on the methodology employed for task completion. These categories are Multi-Agent Systems (MAS), parallel AI, and Distributed Problem Solving (DPS). Considerable research effort has been dedicated to the control of MAS, which is the principal subject of this paper, owing to its practical usefulness in diverse fields, such as irrigation, the oil industry, UAVs, smart grids, security, satellites, and many more [1].

Formation control, a highly researched area within the domain of MAS, seeks to steer multiple agents to achieve prescribed constraints on their states. Numerous formation control problems have been examined in the literature, contingent on the agents' sensing ability and interaction topology. When it comes to formation control, the majority of the research carried out thus far to simulate MAS has employed an ordinary differential equation (ODE) methodology. This ODE-based approach becomes intricate and time-consuming when the number of agents is substantial, leading to the scalability problem. When a large-scale MAS is represented using PDEs, whose fundamental structure does not change as it is independent of the number of agents, the scalability problem gets resolved. In addition, when dealing with states of two or more dimensions, PDEs can be useful in describing the physical arrangement of MAS, i.e., by utilizing PDE-based techniques, it is possible to generate more precise formation manifolds that exhibit greater complexity and variety. From the industrial perspective, PDE-based techniques are advantageous due to their efficiency and cost-effectiveness.

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Motivated by the aforementioned ideas, some research has been done to represent MAS using PDEs [2]–[8]. The article [9] employs a PDE-based method to reduce signal transmissions from leaders to other agents by designing a controller using a modal decomposition approach. The design involves using network-based finite-dimensional control of a 1D heat equation under two Neumann actuations and two boundary measurements. In [10], a network of agents is modeled using wave PDEs, and each agent is controlled through the boundary by Neumann-type actuation, assuming only boundary sensing of the agent's state. The article [11] develops a PDE-based control approach to achieve desired formation profiles for a multi-agent system based on a coupled linear, time-varying, parabolic distributed parameter system. Flatness-based motion planning and feedforward control are combined with a backstepping-based boundary controller to stabilize the distributed parameter system of the tracking error. In [12], a new explicit backstepping kernel containing a Poisson kernel is used to stabilize PDEs by boundary control on a disk, based on two reaction-convection-diffusion two-dimensional PDEs for large multi-agent systems deployed in three-dimensional space. In [13], the MAS is modeled using the reaction-advection-diffusion equation, and PDE-backstepping method is employed to design control laws of the agents to stabilize them in the desired formation under a fixed communication topology. The article [14] proposes a boundary control law for a multi-agent system modeled as the heat equation to achieve state consensus. The finite-time deployment of a multi-agent system into a planar formation is studied in [15] using a leader-follower architecture and boundary control via predefined spatial-temporal paths.

This paper adopts the leader-follower approach. The only agents who are aware of their position relative to the target curve are the leaders. Each follower is aware of the desired and present differences between their own state and the states of their two closest neighbors. Most existing works on the deployment of agents consider only the local connections between agents. In this research paper, we introduce the long-distance connection between leaders and show how this can be used to improve system performance. Namely, the increased connectivity improves the state approximation, which results in a more accurate control strategy. We derive sufficient conditions for stability in terms of linear matrix inequalities (LMIs). With the help of numerical examples, we show how increased connectivity between leaders improves the overall system performance.

Notations: In this paper, the notation $P < 0, P \in \mathbb{R}^{n\times n}$ indicates that P is a negative definite symmetric matrix. The asterisk symbol $(*)$ is used to denote the symmetric elements of a symmetric matrix. The Euclidean norm is denoted by $|\cdot|$. The norm on the Hilbert space $L^2(a,b)$ is denoted by $||\cdot||$ or $||\cdot||_{[a,b]}$, and the norm on the Sobolev space $\mathcal{H}^1(a,b)$ is denoted by $||\cdot||_{\mathcal{H}^1}$. Partial derivatives are represented using indices, e.g., $\partial z/\partial x = z_x$.

II. Preliminaries

Lemma 1 (Wirtinger's Inequality [16])**.** *Let $z \in \mathcal{H}^1(0,l)$ be a scalar function with $z(0) = 0$ or $z(l) = 0$. Then, $||z|| \leq \frac{2l}{\pi}||z'||$. Moreover, if $z(0) = z(l) = 0$, then $||z|| \leq \frac{l}{\pi}||z'||$.*

Lemma 2. *Let $f \in \mathcal{H}^2(a,b)$ and $\tilde{f}$ denote the linear approximation of f defined as $\tilde{f}(x) = f(a)(\frac{b-x}{b-a}) + (\frac{x-a}{b-a})f(b)$. Then*

$$||f - \tilde{f}|| \leq \frac{2(b-a)^2}{\pi^2}||f''||. \quad (1)$$

Proof: Let $g = f - \tilde{f}$, and note that $g(x)$ vanishes at two distinct points a and b. By Rolle's theorem [17], there exists $c \in (a,b)$ such that $g'(c) = 0$. We then apply Wirtinger's inequality (Lemma 1) to g and g', and obtain

$$\begin{aligned}
||g||^2_{[a,b]} &\leq \frac{(b-a)^2}{\pi^2}\Big(||g'||^2_{[a,c]} + ||g'||^2_{[c,b]}\Big) \\
&\leq \frac{(b-a)^2}{\pi^2}\frac{4(c-a)^2}{\pi^2}||g''||^2_{[a,c]} \\
&+ \frac{(b-a)^2}{\pi^2}\frac{4(b-c)^2}{\pi^2}||g''||^2_{[c,b]} \\
&\leq \frac{4(b-a)^4}{\pi^4}||g''||^2_{[a,b]}.
\end{aligned}$$

In the above, we used $\tilde{f}'' = 0$, which holds since $\tilde{f}$ is a linear function.

III. PDE-based Model of a MAS

Consider a system of ODEs that govern the dynamics of $N+1$ agents in $\mathbb{R}^n$:

$$\dot{z}_i(t) = f(t, z_i(t)) + u_i(t) + v_i(t), \quad i \in \mathcal{I} = \{0, ..., N\}, \quad (2)$$

where $z_i(t) : [0,\infty) \to \mathbb{R}^n$ describes each agent's state, $f : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ describes the local dynamics, and the control inputs are denoted by $u_i(t), v_i(t) : [0,\infty] \to \mathbb{R}^n$. We will first design local controllers, u_i, to make the system amenable to the PDE-based modeling (see Section III-A). Then, we use the PDE model to design global controllers, v_i (see Section III-B). The goal is to deploy agents onto a curve $\gamma \in C^2([0,1],\mathbb{R}^n)$. In particular, the problem is to find the controllers $u_i(t)$ and $v_i(t)$ such that the agents' states converge to the points on the curve γ given by $\gamma_i := \gamma(i/N)$ as t goes to infinity. We assume the following:

1) The function f satisfies the Lipschitz condition in its second argument, i.e., there exists a positive constant L such that for all $t \geq 0$ and all $x, y \in \mathbb{R}^n$, the inequality $|f(t,x) - f(t,y)| \leq L|x-y|$ holds.
2) Each follower has the ability to measure the differences between their state and those of their two closest neighbors. That is, agent i can measure $z_i(t) - z_{i-1}(t)$ and $z_{i+1}(t) - z_i(t)$. They also know the desired relative positions, namely $\gamma_i - \gamma_{i-1}$ and $\gamma_{i+1} - \gamma_i$, as well as $f(t,\gamma_i)$ describing the local dynamics.
3) All leaders are capable of measuring the difference between their state and their target position on the curve, specifically $z_i(t) - \gamma_i$, and they also know the local dynamic function $f(t,\gamma_i)$.

The agents who possess knowledge of its location relative to the target curve are the leaders. However, in situations where absolute positions are irrelevant, such as formation control, this requirement is unnecessary. Each follower is aware of the current and desired differences between its state and that of a neighboring agent. Therefore, if $z_{i-1}(t) = \gamma_{i-1}$ and $z_{i+1}(t) = \gamma_{i+1}$, then agent i can get to γ_i without knowing where it is. Lastly, to remain in position on the target curve, every agent needs to know $f(t,\gamma_i)$.

A. Construction of the local controllers $u_i(t)$

The above assumptions allow us to have displacement-based formation control for all the agents and, therefore, we choose the local controllers for agents $i \in \mathcal{I}\setminus\{0,N\}$ with the local control gain $a > 0$ as

$$\begin{aligned}
u_i(t) = a&\Big(\frac{z_{i+1}(t) - z_i(t)}{h^2} - \frac{z_i(t) - z_{i-1}(t)}{h^2}\Big) \\
&- a\Big(\frac{\gamma_{i+1} - \gamma_i}{h^2} - \frac{\gamma_i - \gamma_{i-1}}{h^2}\Big) \\
&- f(t,\gamma_i), \quad h = \frac{1}{N}.
\end{aligned} \quad (3)$$

We assume that the boundary agents are leader agents. Then the choice of the local controller with control gain $\kappa > 0$ is

$$u_0(t) = -\kappa(z_0(t) - \gamma_0) - f(t,\gamma_0), \quad (4)$$

$$u_N(t) = -\kappa(z_N(t) - \gamma_N) - f(t,\gamma_N). \quad (5)$$

Using (3), we get

$$\begin{aligned}
\dot{z}_i(t) = f(t,z_i(t)) &+ a\Big(\frac{z_{i+1}(t) - z_i(t)}{h^2} - \frac{z_i(t) - z_{i-1}(t)}{h^2}\Big) \\
&- a\Big(\frac{\gamma_{i+1} - \gamma_i}{h^2} - \frac{\gamma_i - \gamma_{i-1}}{h^2}\Big) - f(t,\gamma_i) + v_i(t), \\
&i \in \mathcal{I}\setminus\{0,N\}.
\end{aligned} \quad (6)$$

As suggested in [18], when the number of agents is large enough, i.e., $|\mathcal{I}| = N+1 \to \infty$, the model (6) is an approximation of

$$\begin{aligned}
z_t(t,x) &= f(t,x,z(t,x)) - f(t,x,\gamma(x)) \\
&+ a(z_{xx}(t,x) - \gamma_{xx}(x)) + v(t,x).
\end{aligned} \quad (7)$$

By denoting $e(t,x) = z(t,x) - \gamma(x)$, the error dynamics of (7) is given as the semi-linear diffusion equation

$$e_t(t,x) = ae_{xx}(t,x) + F(t,e) + v(t,x), \quad (8)$$

where $F(t,e) := f(t,\gamma(x)+e) - f(t,\gamma(x))$ and from the *Lipschitz continuity* of f, $\exists\ L > 0$ such that

$$\begin{aligned}
|F(t,e)| = |f(t,\gamma(x)+e) - f(t,\gamma(x))| \leq L|e(t,x)|, \\
\forall t \in [0,\infty),\ x \in [0,1],\ e \in \mathbb{R}^n.
\end{aligned} \quad (9)$$

Furthermore,

$$\begin{aligned} |F_e| &= \Big| \lim_{h\to 0} \frac{F(t,e+h)-F(t,e)}{h} \Big| \\ &= \Big| \lim_{h\to 0} \frac{f(t,\gamma(x)+e+h)-f(t,\gamma(x)+e)}{h} \Big| \\ &\le \lim_{h\to 0} \frac{L|h|}{|h|} = L. \end{aligned} \tag{10}$$

For the leader agent $i = 0$, we have $e(t,0) = z(t,0) - \gamma_0$ and therefore,

$$\begin{aligned} e_t(t,0) &= z_t(t,0) = f(t,z(t,0)) + u(t,0) \\ &= f(t,e(t,0)+\gamma_0) - \kappa e(t,0) - f(t,e(t,0)) \\ &= -\kappa e(t,0) + F(t,e(t,0)), \quad t > 0. \end{aligned} \tag{11}$$

Similarly,

$$e_t(t,1) = -\kappa e(t,1) + F(t,e(t,1)), \quad t > 0. \tag{12}$$

Therefore, the PDE given in (8) with the boundary conditions (11), (12) is an approximation of the MAS (6).

B. Construction of the global controller $v(t,x)$.

Since the followers cannot measure the difference between their state and the desired position on the target curve, we design a global controller that lets the followers use the information available from the leaders to approximate their state. For this, we discretize the spatial domain $[0,1]$ into M sampling intervals as $0 = x_0 < x_1 < ... < x_j ... < x_M = 1$. We assume that the $M-1$ leaders are placed at these points (excluding the boundary points). The sampling intervals in space may be variable but bounded, i.e., $x_{j+1} - x_j = \Delta_j \le \Delta$. Since the leaders know the distance to the target curve, the value

$$y_j(t) := e(t,x_j) = z(t,x_j) - \gamma(j/N)$$

is known. The followers use the information from the two nearby agents, i.e., $y_j(t)$ and $y_{j+1}(t)$, where $j = 1, ..., M-1$. Thus, for $x \in [x_j, x_{j+1}]$ we have

$$\begin{aligned} e(t,x) &\approx y_j(t) + \frac{y_{j+1}(t) - y_j(t)}{\Delta}(x - x_j) \\ &= y_j(t)\left(1 - \frac{x - x_j}{\Delta}\right) + y_{j+1}(t)\frac{x - x_j}{\Delta} \end{aligned}$$

Define,

$$b_j(x) = \begin{cases} \frac{x - x_{j-1}}{\Delta}, & x \in [x_{j-1}, x_j), \\ 1 - \frac{x - x_j}{\Delta}, & x \in [x_j, x_{j+1}], \\ 0, & x \notin [x_{j-1}, x_{j+1}]. \end{cases}$$

Then, $e(t,x) \approx \sum_{j=0}^{M-1} b_j(x) y_j(t)$ for $x \in (0,1)$. Therefore, to stabilize the error system (8), we choose the global controller as

$$\begin{aligned} v(t,x) &= -K \sum_{j=0}^{M-1} b_j(x) y_j(t) \\ &= -Ke(t,x) + K\nu(t,x) \end{aligned} \tag{13}$$

with the global controller gain $K > 0$ and $\nu(t,x) = e(t,x) - \sum_{j=0}^{M-1} b_j(x) y_j(t)$.

C. Well-posedness of the PDE

Consider the PDE,

$$e_t(t,x) = a e_{xx}(t,x) + F(t,x,e) + v(t,x) \tag{14}$$

with the boundary conditions

$$e_t(t,0) = -\kappa e(t,0) + F(t,e(t,0)), \quad t > 0, \tag{15}$$

$$e_t(t,1) = -\kappa e(t,1) + F(t,e(t,1)), \quad t > 0. \tag{16}$$

Clearly, (15) and (16) are well-posed ODEs with unique solutions in $C^1[0,\infty)$. Let the solutions be $\psi_1(t)$ and $\psi_2(t)$, respectively. Consider

$$\zeta(t,x) = e(t,x) - (1-x)\psi_1(t) - x\psi_2(t). \tag{17}$$

Then, the system (14)-(16) can be written as

$$\zeta_t = a\zeta_{xx} + G(t,\zeta), \quad \zeta(t,0) = 0 = \zeta(t,1), \tag{18}$$

where the non-linear term is given by

$$\begin{aligned} G(t,\zeta(t,x)) &= F(t,\zeta(t,x)) + v(t,\zeta(t,x)) \\ &\quad - (1-x)\psi_1'(t) - x\psi_2'(t). \end{aligned} \tag{19}$$

The boundary value problem can be formulated as the differential equation

$$\dot{w}(t) = \mathcal{A}w(t) + G(t,w(t)), \ t \ge t_0 \tag{20}$$

defined on the Hilbert space $\mathcal{H}^1[0,1]$, where $w(t) = \zeta(t,\cdot)$. The operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{L}^2(0,1)$ is defined as $\mathcal{A} = a\frac{\partial^2}{\partial x^2}$, and

$$D(\mathcal{A}) = \Big\{ w \in \mathcal{H}^2(0,1) \mid w(0) = 0, \ w(1) = 0 \Big\}. \tag{21}$$

A *strong solution* of (20) on $[0,T]$ is a function

$$w \in \mathcal{L}^2(0,T; D(\mathcal{A})) \cap \mathcal{C}([0,T]; \mathcal{H}^1[0,1]), \tag{22}$$

such that $\dot{w} \in \mathcal{L}^2(0,T;\mathcal{L}^2(0,1))$ and (20) holds almost everywhere on $[0,T]$. Following the steps of [19] and [20], one can show that (20) has a unique strong solution for the initial condition $w(0) = \zeta(0,\cdot) \in \mathcal{H}^1[0,1]$. Therefore, the existence of a unique strong solution of (14) follows from (17).

IV. Stability analysis

In this section, we derived a theorem from which we can draw conclusions about the advantages of having long-distance connections between the leader agents.

Theorem 1. *Consider system* (8) *with boundary conditions* (11) *and* (12). *For a given decay rate δ, global controller gain K, and distance between two consecutive leaders Δ, let us choose the local controller gain $\kappa \ge L + \delta$. If there exist a positive scalar λ_1 such that*

$$\Phi \le 0, \tag{23}$$

where

$$\Phi = \begin{bmatrix} \Phi_{11} & 0 & \frac{(\kappa-K)}{a}K \\ * & \Phi_{22} & -K \\ * & * & -\lambda_1 \end{bmatrix}$$
$$\Phi_{11} = 2\frac{(\kappa-K)}{a}(L-K+\delta),$$
$$\Phi_{22} = \Big(-2a + \lambda_1 \frac{4\Delta^4}{\pi^4}\Big), \tag{24}$$

then the controller (13) *exponentially stabilizes the system* (8), (11), (12) *in the* $\mathcal{H}^1$*-norm with the decay rate* δ*, i.e.,*

$$\exists\ C > 0 : ||e(t,\cdot)||_{\mathcal{H}^1} \leq C \exp(-\delta t)||e(0,\cdot)||_{\mathcal{H}^1}. \tag{25}$$

Proof: For $\kappa > K$, consider the functional

$$V = \frac{(\kappa-K)}{a}\ ||e(t,x)||^2 + ||e_x(t,x)||^2. \tag{26}$$

Differentiating V and substituting the boundary conditions (11) and (12), we find

$$\begin{aligned}
\dot{V}(t) &= 2\frac{(\kappa-K)}{a}\int_0^1 e^T e_t\ dx + 2\int_0^1 e_x^T e_{xt}\ dx \\
&= 2\frac{(\kappa-K)}{a}\int_0^1 e^T e_t\ dx \\
&+ 2\Big(e_x^T e_t\Big|_0^1 - \int_0^1 e_{xx}^T e_t\ dx\Big) \\
&= 2\frac{(\kappa-K)}{a}\int_0^1 e^T e_t\ dx - 2\kappa e_x^T e\Big|_0^1 \\
&+ 2e_x^T F\Big|_0^1 - 2\int_0^1 (e_{xx}^T)e_t\ dx \\
&= 2\int_0^1 \Big(\frac{(\kappa-K)}{a}e^T - e_{xx}^T\Big)\times \\
&\Big(ae_{xx} + F - K(e-\nu)\Big) - 2\kappa e_x^T e\Big|_0^1 \\
&+ 2e_x^T F\Big|_0^1.
\end{aligned} \tag{27}$$

Integration by parts leads to

$$\begin{aligned}
2(\kappa-K)\int_0^1 e^T e_{xx} dx = 2(\kappa-K)e^T e_x\Big|_0^1 \\
- 2(\kappa-K)||e_x||^2.
\end{aligned} \tag{28}$$

Note that,

$$\begin{aligned}
2\frac{(\kappa-K)}{a}\int_0^1 e^T F\ dx &\leq 2\frac{(\kappa-K)}{a}\int_0^1 |e^T F|\ dx \\
&\leq 2\frac{(\kappa-K)}{a}L||e||^2.
\end{aligned} \tag{29}$$

Using (10), we get

$$\begin{aligned}
-2\int_0^1 e_{xx}^T F\ dx &= -2e_x^T F\Big|_0^1 + 2\int_0^1 e_x^T F_x\ dx \\
&= -2e_x^T F\Big|_0^1 + 2\int_0^1 e_x^T F_e e_x\ dx \\
&\leq -2e_x^T F\Big|_0^1 + 2L||e_x||^2.
\end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned}
\dot{V}(t) \leq\ & 2\frac{(\kappa-K)}{a}(L-K)||e||^2 + 2(L-\kappa)||e_x||^2 \\
& - 2a||e_{xx}||^2 + 2\frac{(\kappa-K)}{a}K\int_0^1 e^T\nu\ dx \\
& - 2K\int_0^1 e_{xx}^T \nu\ dx.
\end{aligned} \tag{31}$$

Since $||e(t,x) - \sum_{j=0}^{M-1} b_j(x)y_j(t)||^2_{[x_0,x_M]} = ||\nu||^2_{[x_0,x_M]}$, Lemma 2 implies

$$\begin{aligned}
||\nu||^2_{[x_0,x_M]} &= ||\nu||^2_{[x_0,x_1]} + \cdots + ||\nu||^2_{[x_{M-1},x_M]} \\
&\leq \frac{4\Delta^4}{\pi^4}\Big(||e_{xx}||^2_{[x_0,x_1]} + \cdots + ||e_{xx}||^2_{[x_{M-1},x_M]}\Big) \\
&= \frac{4\Delta^4}{\pi^4}||e_{xx}||^2_{[x_0,x_M]}.
\end{aligned} \tag{32}$$

Thus, for $\lambda_1 > 0$, we have

$$\lambda_1\Big(\frac{4\Delta^4}{\pi^4}||e_{xx}||^2 - ||\nu||^2\Big) \geq 0. \tag{33}$$

Adding the above inequality to the right-hand side of (31), we get

$$\begin{aligned}
\dot{V}(t) + 2\delta V(t) \leq\ & 2\frac{(\kappa-K)}{a}(L-K+\delta)||e||^2 \\
& + 2(L-\kappa+\delta)||e_x||^2 \\
& + \Big(-2a + \lambda_1\frac{4\Delta^4}{\pi^4}\Big)||e_{xx}||^2 \\
& - \lambda_1||\nu||^2 \\
& + 2\frac{(\kappa-K)}{a}K\int_0^1 e^T\nu\ dx \\
& - 2K\int_0^1 e_{xx}^T\nu\ dx.
\end{aligned} \tag{34}$$

Setting $\eta = [e, e_{xx}, \nu]^T$, we get

$$\dot{V}(t) + 2\delta V(t) \leq \eta^T\Phi\eta + 2(L-\kappa+\delta)||e_x||^2. \tag{35}$$

By the condition of the theorem, $\Phi \leq 0$ and $L - \kappa + \delta \leq 0$, therefore, by the comparison principle, this implies exponential stability with the decay rate δ in the $\mathcal{H}^1$-norm.

Remark 1. *Let* $P = \text{diag}\{\Phi_{11}, \Phi_{22}\}$ *and* $Q = [\frac{(\kappa-K)}{a}K, -K]^T$*. When* δ *and* Δ *tend to zero,* Φ_{11} *and* Φ_{22} *become negative for* $\lambda_1 = \frac{1}{\Delta}$*,* $a > 0$*, and* $\kappa > K > L$*. Therefore,* $P \leq 0$*. Since* $P + \Delta QQ^T < 0$ *for a small enough* Δ*, the Schur Complement Lemma [21] implies* $\Phi \leq 0$*. Therefore, the conditions of Theorem 1 are always true if* δ *and* Δ *are small enough while* κ *and* K *are large enough with* $\kappa > K$*.*

Remark 2. *In the proof of Theorem 1, we relied on inequality* (33)*, which involves* Δ^4*. Note that* Δ^4 *goes to zero faster than* Δ^2*. Therefore, the estimation error decreases more rapidly when the leaders can communicate compared to the isolated leaders considered in [6].*

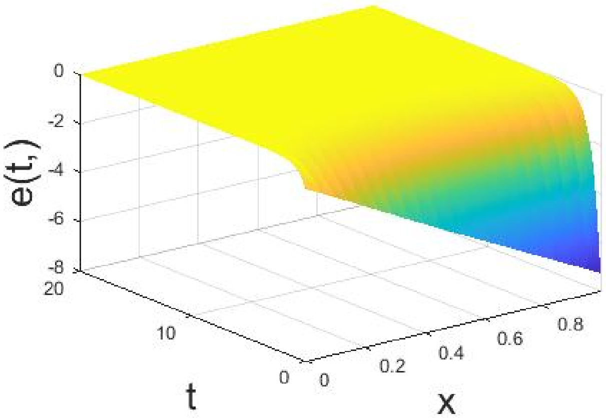

Fig. 1. PDE state error, $e(t,x)$

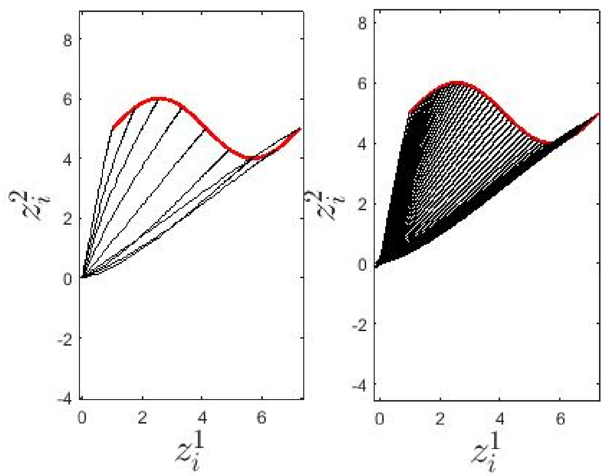

Fig. 2. Phase portrait of MAS with $K=1, \kappa$ = 1.1 for N = 8 (left) and N = 50 (right)

V. Numerical Simulations

Example 1. In this example, we will evaluate the distance between two consecutive leader agents and compare the results with [19]. Let $a = 1, K = \pi^2 + 2, \kappa = \pi^2 + 3,$ and $L = \pi^2 + 1$. Solving the LMI of Theorem 1, we determined that the PDE system (8), (11), (12) achieves exponential stability when $\Delta \leq 0.44$. By dividing the spatial domain into five sub-domains with a maximum length of 0.2 between neighboring points x_j, the proposed controller can exponentially stabilize the system. Moreover, if we choose the positions of the four leader agents in the domain as $x_1 = 0.2, x_2 = 0.4, x_3 = 0.6,$ and $x_4 = 0.8$, the system can be exponentially stabilized with a decay rate of $\delta = 0.9$. Compared to [19], our proposed strategy yields a larger value of Δ, as shown in Table I. This is due to the bound on $\nu(t,x) = e(t,x) - \sum_{j=0}^{M-1} b_j(x)y_j(t)$ that depends on Δ^4 in our work, compared to Δ^2 in [19] (see Remark 2). It should be noted that when the value of Δ, which represents the distance between two consecutive leader agents, is increased,

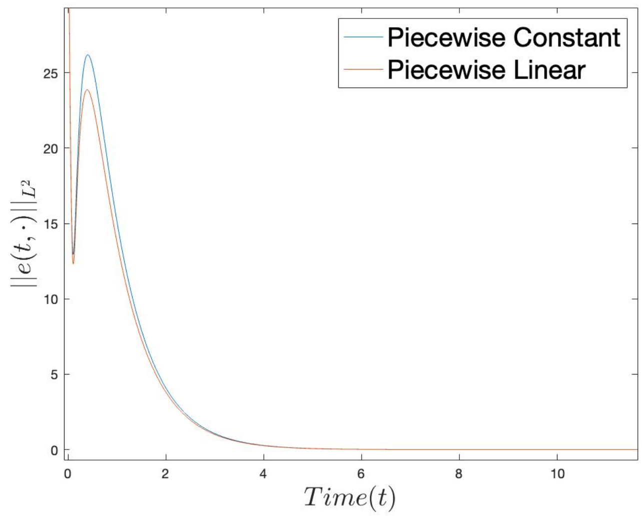

Fig. 3. $||e(t,\cdot)||$ for piecewise linear and constant approximation

TABLE I
Comparison of the results of Theorem (1) with strategy in [19]

Decay rate	δ	0.1	0.3	0.5	0.9
Theorem 1	Δ	0.585	0.575	0.555	0.44
Approach in [19]	Δ	0.505	0.445	0.375	0.17

the required number of leaders will decrease.

Example 2 (Deployment of agents onto the desired curve)**.** Consider the multi-agent system (2) with the dimension of the state vector $n = 2$ and $f(t, z_i) = 0.1(z_i + \sin(z_i))$. Let the desired curve be

$$\gamma(x) = \begin{bmatrix} 1 + 2\pi x \\ 5 + \sin(2\pi x) \end{bmatrix}, \ x \in [0, 1].$$

The LMI of Theorem 1 is feasible for $a = 0.1, \kappa = 1.1, K = 1, \delta = 0.1$ and $\Delta = 0.5$. Therefore, the control law (3),(4) and (5) guarantees that the agents' states converge to the points on the curve γ, given by $\gamma_i := \gamma(i/N)$, as t goes to infinity if the number of agents, N, is sufficiently big. The successful deployment onto the red target curve is shown in Figure 2 for $N = 8$ and $N = 50$ with $\Delta = 0.5$, where

$$z_i(t) = \begin{bmatrix} z_i^1(t) \\ z_i^2(t) \end{bmatrix}, z_i(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ i = 0, 1, \ldots, N.$$

Figure 1 depicts the PDE state error. Figure 3 plots $||e(t,\cdot)||$ versus t for piecewise linear and piecewise constant approximation. It can be seen that the error values are smaller for piecewise linear approximation compared to piecewise constant approximation. This means that the error decreases more rapidly when the leaders can communicate compared to the isolated leaders considered in [6].

VI. CONCLUSIONS

We demonstrated the utility of PDEs in assessing the stability of a large-scale MAS. Specifically, we focused on the problem of formation control for a MAS and derived sufficient stability conditions in the form of Linear Matrix Inequalities (LMIs). Our numerical simulations demonstrated that the performance of the MAS could be improved through the use of leader connections.

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