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QUASITRIANGULAR COIDEAL SUBALGEBRAS OF $U_q(\mathfrak{g})$ IN TERMS OF GENERALIZED SATAKE DIAGRAMS

VIDAS REGELSKIS AND BART VLAAR

ABSTRACT. Let \mathfrak{g} be a finite-dimensional semisimple complex Lie algebra and θ an involutive automorphism of \mathfrak{g} . It is well-known from works of Letzter, Kolb and Balagović that the fixed-point subalgebra $\mathfrak{k} = \mathfrak{g}^\theta$ has a quantum counterpart B , a coideal subalgebra of the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ possessing a cylinder-twisted universal K -matrix \mathcal{K} . The objects θ , \mathfrak{k} , B and \mathcal{K} can all be described in terms of a combinatorial datum, a Satake diagram. In the present work we extend this construction to generalized Satake diagrams, objects first considered by Heck. A generalized Satake diagram defines a semisimple automorphism of \mathfrak{g} restricting to the standard Cartan subalgebra \mathfrak{h} as an involution. We show that it naturally leads to a subalgebra $\mathfrak{k} \subset \mathfrak{g}$, not necessarily a fixed-point subalgebra, but still satisfying $\mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}^\theta$. Such a subalgebra \mathfrak{k} can be quantized to a coideal subalgebra of $U_q(\mathfrak{g})$ endowed with a cylinder-twisted universal K -matrix. We conjecture that all such coideal subalgebras of $U_q(\mathfrak{g})$ arise from generalized Satake diagrams in this way.

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1. INTRODUCTION

Given a finite-dimensional semisimple complex Lie algebra \mathfrak{g} and an involutive Lie algebra automorphism $\theta \in \text{Aut}(\mathfrak{g})$, a *symmetric pair* is a pair $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{k} = \mathfrak{g}^\theta$ is the θ -fixed subalgebra of \mathfrak{g} , see [Ara62, Sat71]. *Quantum symmetric pairs* are their quantum analogons. That is to say, the enveloping algebra $U(\mathfrak{g})$ can be quantized to a quasitriangular Hopf algebra, the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ endowed with the universal R -matrix \mathcal{R} , see [Ji85, Dr87]. Similarly, the θ -fixed subalgebra \mathfrak{k} can be quantized to a coideal subalgebra $B \subseteq U_q(\mathfrak{g})$ [Let99, Let02, Ko14] having a compatible quasitriangular structure, the cylinder-twisted universal K -matrix \mathcal{K} [BK16, Ko17].

The involution θ , the corresponding fixed-point subalgebra \mathfrak{k} , the coideal subalgebra B and the universal object \mathcal{K} are all defined in terms of a combinatorial data, the so-called Satake diagram (X, τ) . Here X is a subdiagram of the Dynkin diagram of \mathfrak{g} and τ is an involutive diagram automorphism stabilizing X and satisfying certain compatibility conditions, see [Let02, Ko14].

It is the aim of this paper to extend some of the above work to a more general setting than (quantizations of) fixed-point subalgebras. A direct motivation for this is the fact that the correct quantum group analogue of the fixed-point subalgebra in the Letzter-Kolb approach is not a fixed-point subalgebra itself, but merely tends to one as $q \rightarrow 1$, see [Ko14, Ch. 10]. This suggests that there may be a generalization of this approach that does not require a fixed-point subalgebra as input.

A careful analysis of [Ko14, BK15, BK16] indeed indicates that the compatibility conditions for X and τ can be weakened, leading to the notion of a *generalized Satake diagram*, see Definition 2.2, and the whole theory survives in this setting with very minor adjustments. The resulting Lie subalgebra $\mathfrak{k} = \mathfrak{k}(X, \tau)$ is given in Definition 3.1 and the corresponding coideal subalgebra $B = B(X, \tau)$ in Definition 4.1. Indeed, in [BK15, Rmks. 2.6, 3.14] it is explicitly suggested that some key passages of the theory are amenable for generalizations.

Our proposed generalization of Satake diagrams can be traced back to the work of A. Heck [He84]. In this work Heck provides a classification of involutions of finite root systems such that the corresponding restricted Weyl group is the Weyl group of the restricted root system. We will review this point-of-view and make a connection with a theorem of Lusztig stating that the restricted Weyl group is in fact a Coxeter group.

The characterization in terms of the restricted Weyl group is relevant in the context of the universal R - and K -matrices for quantum symmetric pairs. The universal R -matrix \mathcal{R} has a distinguished factor called quasi R -matrix playing an important role in the theory of canonical bases for $U_q(\mathfrak{g})$ developed by Kashiwara and Lusztig, see [Ka90] and [Lu94, Part IV]. This object possesses a remarkable factorization property expressed in terms of the braid group action on $U_q(\mathfrak{g})$ of the Weyl group associated to \mathfrak{g} , see e.g. [KR90, LS90]. Recently it has become clear that many of these properties extend to the cylinder-twisted universal K -matrix \mathcal{K} . It has a distinguished factor called quasi K -matrix introduced in [BW13] for certain coideal subalgebras of $U_q(\mathfrak{sl}_N)$ and in a more general setting in [BK15], and featuring prominently in the theory of canonical bases for quantum symmetric pairs [BW16]. In [DK18] a factorization property is established for the quasi K -matrix using a braid group action of the aforementioned restricted Weyl group. In the present work we argue that the factorization property extends to quasi K -matrices defined in terms of the generalized Satake diagrams.

A generalization of this approach to the Kac-Moody setting will be addressed in a future work. Another outstanding issue is a Lie-theoretic motivation of the subalgebra \mathfrak{k} , which we define in a rather *ad hoc* manner directly in terms of the combinatorial data (X, τ) , see Definition 3.1.

Therefore let us end the introduction with an additional motivation for the study of the subalgebra \mathfrak{k} and its quantization B by making some observations related to the representation theory of the pair $(U_q(\mathfrak{g}), B)$. Following [BK16, Ko17], there exists a suitable completion \mathcal{U} of $U_q(\mathfrak{g})$ such that the objects $\mathcal{R} \in (\mathcal{U} \otimes \mathcal{U})^\times$ and $\mathcal{K} \in \mathcal{U}^\times$ have well-defined images under any finite-dimensional representation $\rho : U_q(\mathfrak{g}) \rightarrow \text{End}(V)$. Furthermore, there exists an involutive Hopf algebra automorphism ϕ of \mathcal{U} such that $(\phi \otimes \phi)(\mathcal{R}) = \mathcal{R}$ and the following quartic relation is satisfied, known as the (*universal*) ϕ -twisted reflection equation (see [Ko17, Eqs. (3.22-3.23)]):

$$(1.1) \quad \mathcal{R}_{21} \mathcal{K}_2 (\phi \otimes \text{id})(\mathcal{R}) \mathcal{K}_1 = \mathcal{K}_1 (\phi \otimes \text{id})(\mathcal{R}_{21}) \mathcal{K}_2 \mathcal{R} \quad \in \mathcal{U} \otimes \mathcal{U}$$

where $\mathcal{K}_1 = \mathcal{K} \otimes 1$, $\mathcal{K}_2 = 1 \otimes \mathcal{K}$, $\mathcal{R}_{21} = \sigma(\mathcal{R})$ and $\sigma \in \text{Aut}_{\text{alg}}(\mathcal{U} \otimes \mathcal{U})$ is the flip map. Let $R \in \text{GL}(V \otimes V)$ be proportional to $(\rho \otimes \rho)(\mathcal{R})$ and $K \in \text{GL}(V)$ proportional to $\rho(\mathcal{K})$. In the case $\phi = \text{id}$, applying $\rho \otimes \rho$ to (1.1) one obtains the matrix reflection equation

$$(1.2) \quad R_{21} K_2 R K_1 = K_1 R_{21} K_2 R \quad \in \text{End}(V \otimes V)$$

where $K_1 = K \otimes \text{Id}$, $K_2 = \text{Id} \otimes K$ and $R_{21} = PRP$ with $P : V \otimes V \rightarrow V \otimes V$ the permutation operator. When $\phi \neq \text{id}$ one naturally obtains the so-called twisted matrix reflection equation which we omit for simplicity, but this does not significantly affect any of the following remarks. In particular, starting with a Satake diagram, one will recover the solutions of (1.2) used in [NDS95, NS95] to define quantum symmetric pairs.

Treating the matrix R as given, one can of course solve (1.2) for $K \in \text{GL}(V)$. For $U_q(\mathfrak{sl}_N)$ and $V = \mathbb{C}^N$ this was done by A. Mudrov [Mu02]. Based on this result and computations for $U_q(\mathfrak{g})$ with \mathfrak{g} of types B_n , C_n , D_n ($n \leq 4$) and G_2 , and V the vector representation, we formulate the following conjecture.

Conjecture 1.1. *Let $\rho : U_q(\mathfrak{g}) \rightarrow \text{End}(V)$ be the vector representation of $U_q(\mathfrak{g})$. If $K \in \text{GL}(V)$ is a solution of (1.2) then there exists a generalized Satake diagram (X, τ) such that K is proportional to $\rho(\mathcal{K})$ where \mathcal{K} is the universal K -matrix for the coideal subalgebra $B(X, \tau)$, i.e. the quantization of $U(\mathfrak{k}(X, \tau))$.*

Based on the available evidence in terms of solutions to (1.2) known to intertwine restrictions of ρ to coideal subalgebras, we also make the following claim.

Conjecture 1.2. *Let $\rho : U_q(\mathfrak{g}) \rightarrow \text{End}(V)$ be the vector representation of $U_q(\mathfrak{g})$. Then ρ can be used to identify coideal subalgebras, i.e. if the distinct coideal subalgebras $B, B' \subseteq U_q(\mathfrak{g})$ possess the universal K -matrices \mathcal{K} and \mathcal{K}' , respectively, then $\rho(\mathcal{K})$ and $\rho(\mathcal{K}')$ are not scalar multiples of each other.*

If these two conjectures are true, the only coideal subalgebras of $U_q(\mathfrak{g})$ which possess a universal K -matrix in the sense of [Ko17] are those which are quantizations of $U(\mathfrak{k}(X, \tau))$ with (X, τ) a generalized Satake diagram.

We should remark that coideal subalgebras B in the Letzter-Kolb approach carry additional parameters. The generators associated to the nodes $i \in I \setminus X$ depend on scalars $\gamma_i \neq 0$ and s_i , see Definition 4.1. We can thus sharpen Conjecture 1.1. Any invertible matrix solution K of (1.2) is proportional to $\rho(K)$ for some $B(X, \tau)$ with the additional parameters satisfying certain constraints. Most of these constraints were found in [Let03, Ko14] given in terms of the sets Γ_q and \mathcal{S}_q , see (4.3). Always, we must have $(\gamma_i)_{i \in I \setminus X} \in \Gamma_q$. For the conditions on s_i it is helpful to consider the set $I_{\text{ns}} = \{i \in I \setminus X \mid i \text{ does not neighbour } X, \tau(i) = i\}$, see (3.16). The constraints on the s_i are as follows. If $i \notin I_{\text{ns}}$ then $s_i = 0$. For all $(i, j) \in I_{\text{ns}} \times I_{\text{ns}}$ such that $i \neq j$ conjecturally one of three conditions must hold: the Cartan integer a_{ij} is even, $s_j = 0$, or s_i^2/γ_i lies in a particular finite subset of a quadratic completion of $\mathbb{C}(q)$. The defining condition of the set \mathcal{S}_q does not cover the third possibility, which appeared in [BB10].

The paper is organized as follows. Section 2 contains the preliminaries and basic definitions. We define the necessary Lie-theoretic objects surrounding a finite-dimensional semisimple complex Lie algebra \mathfrak{g} and its Cartan subalgebra \mathfrak{h} . We introduce the notion of a generalized Satake diagram as a decoration of the Dynkin diagram of \mathfrak{g} . We explain how the generalized Satake diagrams emerge in the work of A. Heck.

In Section 3 we define the main object of this paper, the subalgebra $\mathfrak{k} = \mathfrak{k}(X, \tau) \subseteq \mathfrak{g}$. Theorem 3.2 is the main result of this section. We show that \mathfrak{k} satisfies the intersection condition $\mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}^\theta$ (which trivially holds when $\mathfrak{k} = \mathfrak{g}^\theta$ with $\theta^2 = \text{id}_{\mathfrak{g}}$) precisely if (X, τ) is a generalized Satake diagram. We then study the derived subalgebra of \mathfrak{k} . When \mathfrak{k} is not a reductive Lie algebra, Propositions 3.5 and 3.6 establish a semidirect product decomposition for \mathfrak{k} in terms of a reductive subalgebra and a nilpotent ideal of class 2. We end this section with some results about the universal enveloping algebra $U(\mathfrak{k})$. (Appendix A contains three technical lemmas in aid of Section 3.)

In Section 4 we briefly review the quasitriangular structure behind the quantum symmetric pairs. We indicate the necessary modifications to the theory of Balagović-Kolb so that it would be applicable to the quantum pair algebras associated to the generalized Satake diagrams.

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2. FINITE-DIMENSIONAL SEMISIMPLE LIE ALGEBRAS AND ROOT SYSTEM INVOLUTIONS

Let I be a finite set and $A = (a_{ij})_{i,j \in I}$ a Cartan matrix. In particular, there exist positive rationals d_i ($i \in I$) such that $d_i a_{ij} = d_j a_{ji}$. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding finite-dimensional semisimple Lie algebra over \mathbb{C} . More precisely, \mathfrak{g} is generated by $\{e_i, f_i, h_i\}_{i \in I}$ subject to

$$(2.1) \quad [h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_i$$

$$(2.2) \quad \text{ad}(e_i)^{1-a_{ij}}(e_j) = \text{ad}(f_i)^{1-a_{ij}}(f_j) = 0 \quad \text{if } i \neq j,$$

for all $i, j \in I$. We denote the standard Cartan subalgebra by $\mathfrak{h} = \langle h_i \mid i \in I \rangle$ and also consider the corresponding nilpotent subalgebras $\mathfrak{n}^+ = \langle e_i \mid i \in I \rangle$, $\mathfrak{n}^- = \langle f_i \mid i \in I \rangle$.

The simple roots $\alpha_i \in \mathfrak{h}^*$ ($i \in I$) satisfy $\alpha_j(h_i) = a_{ij}$ for $i, j \in I$. Let $Q = \sum_{i \in I} \mathbb{Z} \alpha_i$ denote the root lattice. In terms of the root spaces $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}$ ($\alpha \in Q$), \mathfrak{g} is a Q -graded Lie

algebra and we have the following identities for \mathfrak{h} -modules:

$$(2.3) \quad \mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-, \quad \mathfrak{n}^\pm = \bigoplus_{\alpha \in Q^+} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{h} = \mathfrak{g}_0.$$

Hence the root system $\Phi := \{\alpha \in Q \mid \mathfrak{g}_\alpha \neq \{0\}, \alpha \neq 0\}$ satisfies $\Phi = \Phi^+ \cup \Phi^-$ where $\Phi^\pm = \pm(\Phi \cap Q^+)$ and $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$.

The Weyl group W is a finite subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the simple reflections r_i ($i \in I$) acting via $r_i(\alpha) = \alpha - \alpha(h_i)\alpha_i$ for all $i \in I$, $\alpha \in \mathfrak{h}^*$. More precisely, W is a normal subgroup of

$$\mathrm{Aut}(\Phi) := \{g \in \mathrm{GL}(\mathfrak{h}^*) \mid g(\Phi) = \Phi\}.$$

Since W induces a simple transitive action on the set of bases of Φ , one readily obtains that $\mathrm{Aut}(\Phi) = W \rtimes \mathrm{Aut}(A)$, where

$$\mathrm{Aut}(A) = \{\sigma : I \rightarrow I \text{ invertible} \mid a_{\sigma(i)\sigma(j)} = a_{ij} \text{ for all } i, j \in I\}$$

is the group of diagram automorphisms (acting by relabelling).

The following subgroup of $\mathrm{Aut}(\mathfrak{g})$ will be important in what follows:

$$\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) = \{\sigma \in \mathrm{Aut}(\mathfrak{g}) \mid \sigma(\mathfrak{h}) = \mathfrak{h}\} < \mathrm{Aut}(\mathfrak{g}).$$

We briefly review some important subgroups of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$. A braid group action on \mathfrak{g} which extends the dual action of W on \mathfrak{h} is defined by $\mathrm{Ad}(r_i) = \exp(\mathrm{ad}(e_i)) \exp(\mathrm{ad}(-f_i)) \exp(\mathrm{ad}(e_i)) \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ for $i \in I$, yielding $\mathrm{Ad}(W) < \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$. We also have $\mathrm{Aut}(A) < \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ (acting by relabelling). The Chevalley involution $\omega \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ is defined by swapping e_i and $-f_i$ for all $i \in I$; it commutes with $\mathrm{Ad}(W)$ and with $\mathrm{Aut}(A)$. Finally, the group $\tilde{H} := \mathrm{Hom}(Q, \mathbb{C}^\times)$ naturally induces a subgroup $\mathrm{Ad}(\tilde{H}) < \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ via $\mathrm{Ad}(\chi)|_{\mathfrak{g}_\alpha} = \chi(\alpha) \mathrm{id}_{\mathfrak{g}_\alpha}$ for all $\chi \in \tilde{H}$, $\alpha \in Q$.

The elements of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ can be dualized to elements of $\mathrm{Aut}(\Phi)$. Conversely, given $g \in \mathrm{Aut}(\Phi)$ there are $\psi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ whose restriction to \mathfrak{h} dualizes to g . Indeed, from $-\mathrm{id}_{\mathfrak{h}^*} \in \mathrm{Aut}(\Phi)$ and the direct product decomposition $\mathrm{Aut}(\Phi) = W \rtimes \mathrm{Aut}(A)$, there exist unique $(w, \tau) \in W \times \mathrm{Aut}(A)$ such that $g = -w\tau$. Then one easily checks that $\psi = \mathrm{Ad}(w)\omega\tau \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ satisfies $(\psi|_{\mathfrak{h}})^* = g$.

2.1. Compatible decorations and involutions of Φ . Given a subset $X \subseteq I$ denote the corresponding Cartan submatrix by $A_X = (a_{ij})_{i,j \in X}$ and consider the corresponding semisimple Lie algebra $\mathfrak{g}_X := \langle e_i, f_i, h_i \mid i \in X \rangle \subseteq \mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}_X = \mathfrak{h} \cap \mathfrak{g}_X$, dual Weyl vector $\rho_X^\vee \in \mathfrak{h}_X$ and Weyl group $W_X := \langle r_i \mid i \in X \rangle \leq W$. The unique longest element $w_X \in W_X$ is an involution and there exists $\tau_{0,X} \in \mathrm{Aut}(A_X)$ which satisfies

$$(2.4) \quad -w_X(\alpha_i) = \alpha_{\tau_{0,X}(i)} \quad \text{for all } i \in X.$$

Note that $\mathrm{Ad}(w_X)|_{\mathfrak{g}_X} = \tau_{0,X}\omega|_{\mathfrak{g}_X}$ and $\mathrm{Ad}(w_X)^2|_{\mathfrak{g}_\alpha} = \zeta(\alpha) \mathrm{id}_{\mathfrak{g}_\alpha}$ for all $\alpha \in \Phi$, where $\zeta = \zeta(X) \in \tilde{H}$ is defined by

$$\zeta(\alpha_i) := (-1)^{2\alpha_i(\rho_X^\vee)} \quad \text{for } i \in I.$$

We will study

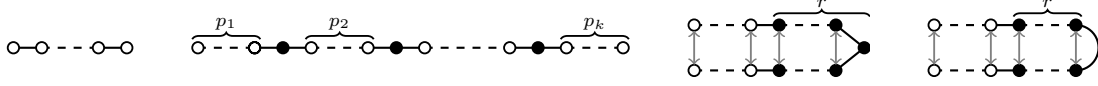
$$\begin{aligned} \mathrm{Aut}^{\mathrm{inv}}(\mathfrak{g}, \mathfrak{h}) &:= \{\psi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) \mid \psi^2|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}\}, \\ \mathrm{Aut}^{\mathrm{inv}}(\Phi) &:= \{g \in \mathrm{Aut}(\Phi) \mid g^2 = \mathrm{id}_{\mathfrak{h}^*}\} \end{aligned}$$

by means of combinatorial data: we define

$$(2.5) \quad \mathrm{CDec}(A) = \{(X, \tau) \mid X \subseteq I, \tau \in \mathrm{Aut}(A), \tau^2 = \mathrm{id}_I, \tau(X) = X, \tau|_X = \tau_{0,X}\}$$

and call its elements *compatible decorations* (of A). In the Dynkin diagram associated to \mathfrak{g} one marks this decoration by filling the nodes corresponding to X and drawing two-sided arrows for the nontrivial orbits of τ .

Example 2.1. Let A be of type A_n , $n \geq 2$. The compatible decorations $\text{CDec}(A)$ are



where $p_1, p_k \in \mathbb{Z}_{\geq 0}$, $p_2, \dots, p_{k-1} \in \mathbb{Z}_{\geq 1}$ for any $k \in \mathbb{Z}_{\geq 2}$ and $0 \leq r \leq \lceil n/2 \rceil$.

Given $(X, \tau) \in \text{CDec}(A)$, we define

$$(2.6) \quad \theta = \theta(X, \tau) = -w_X \tau \in \text{Aut}^{\text{inv}}(\Phi).$$

As explained above, the map dual to θ can be extended to an element of $\text{Aut}^{\text{inv}}(\mathfrak{g}, \mathfrak{h})$ which we shall also call θ . It is given by $\theta = \text{Ad}(w_X) \tau \omega$ so that $\theta|_{\mathfrak{h}} = -w_X \tau$. Note that, as a consequence of properties of $\text{Ad}(w_X)$ mentioned earlier, we have

$$(2.7) \quad \theta|_{\mathfrak{g}_X} = \text{id}_{\mathfrak{g}_X},$$

$$(2.8) \quad \theta^2|_{\mathfrak{g}_\alpha} = \zeta(\alpha) \text{id}_{\mathfrak{g}_\alpha} \quad \text{for all } \alpha \in \Phi.$$

2.2. Generalized Satake diagrams and the restricted Weyl group. We choose a subset $I^* \subseteq I \setminus X$ such that it contains precisely one element from each τ -orbit in $I \setminus X$. For $i \in I^*$ denote by $\check{X}(i) \subseteq X$ the union of connected components of X neighbouring $\{i, \tau(i)\}$ and $\check{X}[i] := \check{X}(i) \cup \{i, \tau(i)\}$. By a *minimal subdiagram* of $(X, \tau) \in \text{CDec}(A)$ we mean any subdiagram of the form $\check{X}[i]$ for some $i \in I^*$. By definition $\check{X}[i]$ is a compatible decoration of $A_{\check{X}[i]}$; it is also known as a Satake diagram of (restricted) rank 1.

Definition 2.2. *Generalized Satake diagrams* are elements of the set

$$\text{GSat}(A) := \{(X, \tau) \in \text{CDec}(A) \mid (X, \tau) \text{ contains no minimal subdiagram of the form } \circ \bullet\}. \quad \emptyset$$

The compatible decorations in Example 2.1 are generalized Satake diagrams when $p_1 = p_k = 0$ and $p_2 = \dots = p_{k-1} = 1$.

Remark 2.3. Generalized Satake diagrams were first considered by Heck in [He84] where they are shown to classify involutions of root systems such that the restricted Weyl group is the Weyl group of the restricted root system. Heck uses the symbol σ to denote the negative of our map θ . He also uses the term Satake diagram for any (X, τ) such that $X \subseteq I$, $\tau \in \text{Aut}(A)$, $\tau^2 = \text{id}_I$ and $\tau(X) = X$ (this properly contains the set $\text{CDec}(A)$) and the elements of $\text{GSat}(A)$ are called admissible Satake diagrams. However, the term Satake diagram has become reserved for those combinatorial data which classify involutions of \mathfrak{g} up to conjugacy (and their fixed-point subalgebras), which is the reason for our nomenclature “compatible decoration” and “generalized Satake diagram”. \emptyset

Note that (X, τ) is a generalized Satake diagrams precisely if

$$(2.9) \quad \forall (i, j) \in I \setminus X \times X : \tau(i) = i, w_X(\alpha_i) = \alpha_i + \alpha_j \implies a_{ij} \neq -1,$$

which is precisely the condition needed in [Ko14, Proof of Lemma 5.11, Step 1] and [BK16, Proof of Lemma 6.4]. One can show that (2.9) is equivalent to either of the following more compact conditions:

$$\begin{aligned} \forall i, j \in I : \theta(\alpha_i) = -(\alpha_i + \alpha_j) &\implies a_{ij} \neq -1, \\ \forall i \in I : (\theta(\alpha_i))(h_i) &\neq -1. \end{aligned}$$

Satake diagrams can be defined as the following subset of compatible decorations of A :

$$(2.10) \quad \text{Sat}(A) = \{(X, \tau) \in \text{CDec}(A) \mid \forall i \in I \setminus X : i = \tau(i) \implies \zeta(\alpha_i) = 1\}.$$

It is well-known that Satake diagrams classify involutive Lie algebra automorphisms up to conjugacy, see e.g. [Ara62]. More precisely, in the current setup, for $(X, \tau) \in \text{Sat}(A)$ and $\gamma \in (\mathbb{C}^\times)^{I^*}$ define $s_\gamma \in \check{H}$ by

means of

$$s_\gamma(\alpha_i) = \begin{cases} 1 & \text{if } i \in X, \\ \gamma_i & \text{if } i \in I^*, \\ \gamma_{\tau(i)}\zeta(\alpha_i) & \text{if } i \in (I \setminus X) \setminus I^*, \end{cases}$$

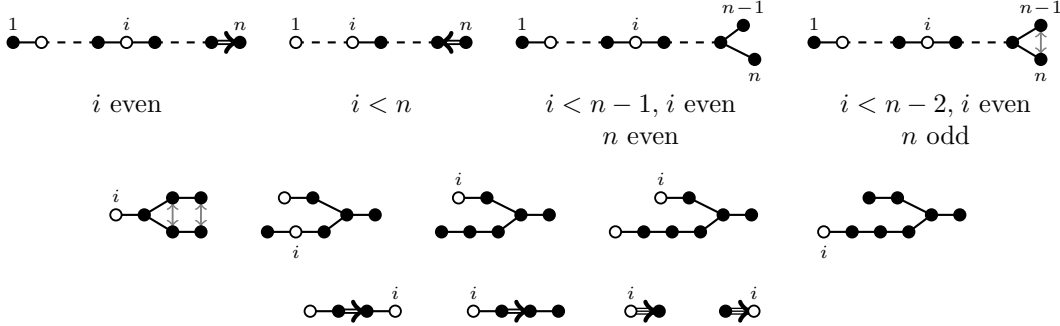
cf. [BK16, Eqs. (5.1-5.2)]. Then it follows from (2.8) that

$$(2.11) \quad \theta_\gamma := \text{Ad}(s_\gamma)\theta$$

satisfies $(\theta_\gamma)^2 = \text{id}_{\mathfrak{g}}$.

If $(X, \tau) \in \text{CDec}(A) \setminus \text{GSat}(A)$ then there exists a pair $(i, j) \in I \setminus X \times X$ such that the union of connected components of X neighbouring i is simply $\{j\}$ and $a_{ji} = -1$. Hence $\rho_X^\vee = \frac{1}{2}h_j$ so that $\zeta(\alpha_i) = (-1)^{a_{ji}} = -1$ implying $(X, \tau) \notin \text{Sat}(A)$. Consequently $\text{Sat}(A) \subseteq \text{GSat}(A)$. The complement $\text{GSat}(A) \setminus \text{Sat}(A)$ is empty if and only if A is of type A_n . We refer the reader to the classification in [He84, Table I], which does not explicitly distinguish between elements of $\text{Sat}(A)$ and $\text{GSat}(A) \setminus \text{Sat}(A)$. It is convenient for our purposes to list the elements of $\text{GSat}(A) \setminus \text{Sat}(A)$, which we do in Table 1.

TABLE 1. All elements of $\text{GSat}(A) \setminus \text{Sat}(A)$ for indecomposable Cartan matrices A . By a case-by-case analysis there is a unique $i \in I$ such that $\zeta(\alpha_i) = -1$; we have indicated the corresponding node in the diagrams. The classical diagrams are labelled in the usual way. For types C_n and D_n upper bounds on i are imposed to avoid the cases when θ is an involution whose fixed-point subalgebra is isomorphic to \mathfrak{gl}_n .



Consider the real vector space $V = \mathbb{R}\Phi$. For a fixed $\theta \in \text{Aut}^{\text{inv}}(\Phi)$ we can decompose V into the positive and negative θ -eigenspaces, $V = V^\theta \oplus V^{-\theta}$. Denote by $\bar{\cdot} : V \rightarrow V$ the corresponding projection onto $V^{-\theta}$. The *restricted roots* are the elements of

$$\overline{\Phi} = \{\bar{\alpha} \mid \alpha \in \Phi\} \setminus \{0\}.$$

Given an arbitrary $\theta \in \text{Aut}^{\text{inv}}(\Phi)$, $\overline{\Phi}$ is not necessarily a root system in its own right. According to [He84, Thm. 6.1], $\overline{\Phi}$ is a (possibly non-reduced or empty) root system precisely if $\theta = \theta(X, \tau) = -w_X\tau$, where $(X, \tau) \in \text{GSat}(A)$ or (X, τ) is the diagram $\circ \bullet$.

Now consider the following groups:

$$\begin{aligned} W^\theta &= \{w \in W \mid w = \theta w \theta\} = \{w \in W \mid w = w_X \tau(w) w_X\}, \\ \overline{W} &= \{w|_{V^{-\theta}} \mid w \in W, w(V^{-\theta}) \subseteq V^{-\theta}\}. \end{aligned}$$

If $\theta = \theta(X, \tau)$ it follows straightforwardly that W_X is a subgroup of W^θ . Moreover, [He84, Prop. 3.1] implies that \overline{W} is isomorphic to W^θ/W_X . For $i \in I^*$ we define $\tilde{r}_i := w_X w_{X[i]} \in W$ where $X[i] = X \cup \{i, \tau(i)\}$ and set $s_i \in \text{GL}(V^{-\theta})$ to be the unique element satisfying $s_i(\bar{\alpha}_i) = -\bar{\alpha}_i$ and $s_i(\beta) = \beta$ for all $\beta \in V^{-\theta}$ such that $\beta(h_i) = 0$. In [He84, Lemma 3.2, Thm. 3.3, Thm. 4.4] the following result is proved.

Theorem 2.4. *Let $(X, \tau) \in \text{CDec}(A)$. The following conditions are equivalent:*

- (i) $(X, \tau) \in \text{GSat}(A)$.
- (ii) For all $i \in I^*$, $s_i \in \overline{W}$.
- (iii) For all $i \in I^*$, \tilde{r}_i lies in W^θ and satisfies $\tilde{r}_i|_{V^{-\theta}} = s_i$.
- (iv) For all $i \in I^*$, $\tau_{0, X[i]}$ preserves X .
- (v) $\overline{W} = W(\overline{\Phi})$.

In [Lu76, 5.9 (i)] it is shown that $(\widetilde{W}, \{\tilde{r}_i\}_{i \in I^*})$ with $\widetilde{W} = \langle \tilde{r}_i \rangle_{i \in I^*}$ is a Coxeter system if condition (iv) in Theorem 2.4 holds (also see [Lu02, 25.1]). If condition (iv) fails then for some $i \in I^*$, $w_{X[i]}$ and w_X do not commute so that $\tilde{r}_i^2 \neq \text{id}_V$. Hence we obtain the following result.

Corollary 2.5. *Let $(X, \tau) \in \text{CDec}(A)$. Then $(\widetilde{W}, \{\tilde{r}_i\}_{i \in I^*})$ is a Coxeter system if and only if $(X, \tau) \in \text{GSat}(A)$.*

3. THE SUBALGEBRA \mathfrak{k}

For $(X, \tau) \in \text{Sat}(A)$ and a suitable choice of $\gamma \in (\mathbb{C}^\times)^{I^*}$ the θ_γ -fixed subalgebra \mathfrak{k} of \mathfrak{g} can be presented in terms of generators, see e.g. [Ko14, Lemma 2.8]. This motivates the following seemingly *ad hoc* definition, where we permit a more general γ .

Definition 3.1. For $(X, \tau) \in \text{CDec}(A)$ and $\gamma \in (\mathbb{C}^\times)^{I \setminus X}$ define $\mathfrak{k}_\gamma = \mathfrak{k}_\gamma(X, \tau)$ to be the subalgebra of \mathfrak{g} generated by \mathfrak{g}_X , \mathfrak{h}^θ and

$$(3.1) \quad b_{i, \gamma} = f_i + \gamma_i \theta(f_i) \quad \text{for all } i \in I \setminus X. \quad \emptyset$$

It is convenient to suppress the dependence on γ and simply write b_i and \mathfrak{k} if there is no cause for confusion. We denote $b_i = f_i$ if $i \in X$. Since $\mathfrak{h}_X \subseteq \mathfrak{h}^\theta$ it follows that \mathfrak{k} is generated by $\mathfrak{n}_X^+ := \{e_i | i \in X\}$, \mathfrak{h}^θ and b_i for $i \in I$. Owing to (2.1-2.2), these satisfy

$$(3.2) \quad [e_i, b_j] = \delta_{ij} h_i \in \mathfrak{h}^\theta \quad \text{for all } i \in X, j \in I,$$

$$(3.3) \quad [h, b_j] = -\alpha_j(h) b_j \quad \text{for all } h \in \mathfrak{h}^\theta, j \in I,$$

$$(3.4) \quad [h, e_j] = \alpha_j(h) e_j \quad \text{for all } h \in \mathfrak{h}^\theta, j \in X,$$

$$(3.5) \quad [h, h'] = 0 \quad \text{for all } h, h' \in \mathfrak{h}^\theta,$$

$$(3.6) \quad \text{ad}(e_i)^{1-a_{ij}}(e_j) = 0 \quad \text{for all } i, j \in X, i \neq j.$$

By setting $m = 1 - a_{ij}$ in Lemmas (A.1-A.3) one also obtains analogues of Serre relations among the generators b_i . Namely, for $i, j \in I$ such that $i \neq j$,

$$(3.7) \quad \text{ad}(b_i)^{1-a_{ij}}(b_j) = \begin{cases} (1 + \zeta(\alpha_i)) \gamma_i [\theta(f_i), [f_i, f_j]] \in \mathfrak{n}_X^+ & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j \in \Phi^-, a_{ij} = -1, \\ -18\gamma_i^2 e_j & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j = 0, a_{ij} = -3, \\ -\gamma_i (2h_i + h_j) & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j = 0, a_{ij} = -1, \\ (\gamma_i + \zeta(\alpha_i) \gamma_j) [\theta(f_i), f_j] \in \mathfrak{n}_X^+ & \text{if } \theta(\alpha_i) + \alpha_j \in \Phi^-, a_{ij} = 0, \\ \gamma_j h_i - \gamma_i h_j & \text{if } \theta(\alpha_i) + \alpha_j = 0, a_{ij} = 0, \\ 2(\gamma_i + \gamma_j) b_i & \text{if } \theta(\alpha_i) + \alpha_j = 0, a_{ij} = -1, \\ -\gamma_i b_j & \text{if } \theta(\alpha_i) + \alpha_i = 0, j \in I \setminus X, a_{ij} = -1, \\ -3\gamma_i [b_i, b_j] & \text{if } \theta(\alpha_i) + \alpha_i = 0, j \in I \setminus X, a_{ij} = -2, \\ -6\gamma_i^2 b_j - 3\gamma_i [b_i, [b_i, b_j]] & \text{if } \theta(\alpha_i) + \alpha_i = 0, j \in I \setminus X, a_{ij} = -3, \\ 0 & \text{otherwise.} \end{cases}$$

In order to state the main result of this section, we need some more notation. Consider the subsets

$$I_{\text{diff}} = \{i \in I^* \mid i \neq \tau(i) \text{ and } (\theta(\alpha_i))(h_i) \neq 0\} = \{i \in I^* \mid i \neq \tau(i) \text{ and } \exists j \in X[i] \text{ s.t. } a_{ij} < 0\}$$

and

$$\Gamma = \Gamma(X, \tau) = \{\gamma \in (\mathbb{C}^\times)^{I \setminus X} \mid \forall i \in I^* : \gamma_i \neq \gamma_{\tau(i)} \implies i \in I_{\text{diff}}\}.$$

For $\mathbf{i} \in I^\ell$ with $\ell \in \mathbb{Z}_{>0}$ we write $\alpha_{\mathbf{i}} = \sum_{r=1}^\ell \alpha_{i_r}$ and

$$b_{\mathbf{i}} = \text{ad}(b_{i_1}) \cdots \text{ad}(b_{i_{\ell-1}})(b_{i_\ell}), \quad e_{\mathbf{i}} = \text{ad}(e_{i_1}) \cdots \text{ad}(e_{i_{\ell-1}})(e_{i_\ell}), \quad f_{\mathbf{i}} = \text{ad}(f_{i_1}) \cdots \text{ad}(f_{i_{\ell-1}})(f_{i_\ell}).$$

Observe that $\mathfrak{n}^- = \text{Sp} \bigcup_{\ell > 0} \{f_{\mathbf{i}}\}_{\mathbf{i} \in I^\ell}$. Hence for all $\ell \in \mathbb{Z}_{>0}$ we can choose $\mathcal{J}_\ell \subseteq I^\ell$ such that $\{f_{\mathbf{i}}\}_{\mathbf{i} \in \mathcal{J}_\ell}$ is a basis for $\text{Sp}\{f_{\mathbf{i}}\}_{\mathbf{i} \in I^\ell}$. Then $\{f_{\mathbf{i}}\}_{\mathbf{i} \in \mathcal{J}}$ with $\mathcal{J} := \bigcup_{\ell \in \mathbb{Z}_{>0}} \mathcal{J}_\ell$ is a basis of \mathfrak{n}^- .

Theorem 3.2. *Let $(X, \tau) \in \text{CDec}(A)$ and $\gamma \in (\mathbb{C}^\times)^{I \setminus X}$. The following statements are equivalent:*

(i) $(X, \tau) \in \text{GSat}(A)$ and $\gamma \in \Gamma$.

(ii) For all $i, j \in I$ such that $i \neq j$ we have the following bounded Serre relations:

$$(3.8) \quad \text{ad}(b_i)^{1-a_{ij}}(b_j) \in \mathfrak{n}_X^+ \oplus \mathfrak{h}^\theta \oplus \bigoplus_{\substack{\mathbf{k} \in I^\ell \\ \alpha_{\mathbf{k}} < \lambda_{ij}}} \mathbb{C}b_{\mathbf{k}}$$

where $\lambda_{ij} := (1 - a_{ij})\alpha_i + \alpha_j \in Q^+ \setminus \Phi^+$.

(iii) We have the following identity for \mathfrak{h}^θ -modules:

$$(3.9) \quad \mathfrak{k} = \mathfrak{n}_X^+ \oplus \mathfrak{h}^\theta \oplus \bigoplus_{\mathbf{i} \in \mathcal{J}} \mathbb{C}b_{\mathbf{i}}.$$

(iv) We have

$$(3.10) \quad \mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}^\theta.$$

Remark 3.3. In the fixed-point case $\mathfrak{k} = \mathfrak{g}^{\theta\gamma}$ (3.10) is trivially satisfied (note that $\mathfrak{h}^\theta = \mathfrak{h}^{\theta\gamma}$). \emptyset

Proof of Theorem 3.2.

(i) \iff (ii): This is a direct consequence of (3.7).

(ii) \implies (iii): Owing to (3.3-3.5) it is sufficient to prove (3.9) as an identity for vector spaces. First we prove that $\mathfrak{k} = \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \text{Sp}\{b_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{J}\}$. From (3.2-3.3) it follows that, as vector spaces,

$$(3.11) \quad \mathfrak{k} = \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \langle b_j \rangle_{j \in I} = \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \sum_{\ell \in \mathbb{Z}_{>0}} \sum_{\mathbf{i} \in I^\ell} \mathbb{C}b_{\mathbf{i}}.$$

As a consequence of this, we see that it suffices to prove that for all $\mathbf{j} \in \bigcup_{\ell} I^\ell$ we have

$$(3.12) \quad b_{\mathbf{j}} \in \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \text{Sp}\{b_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{J}\}.$$

We will prove this by induction with respect to the height ℓ . Since for all $j \in I$ we have $\dim(\mathfrak{g}_{-\alpha_j}) = 1$ and hence $(j) \in \mathcal{J}$, the case $\ell = 1$ is trivial. Now fix $\ell \in \mathbb{Z}_{>1}$ and assume that (3.12) holds true for all smaller positive integers. Fix $\mathbf{j} \in I^\ell$ and repeatedly apply the Serre relations (2.2) to obtain that for all $\mathbf{i} \in \mathcal{J}_\ell$ there exist $a_{\mathbf{i}} \in \mathbb{C}$ such that

$$f_{\mathbf{j}} = \sum_{\mathbf{i} \in \mathcal{J}_\ell} a_{\mathbf{i}} f_{\mathbf{i}}.$$

Hence, by virtue of (ii) and equations (3.2-3.3) it follows that

$$b_{\mathbf{j}} - \sum_{\mathbf{i} \in \mathcal{J}_\ell} a_{\mathbf{i}} b_{\mathbf{i}} \in \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \text{Sp}\left\{b_{\mathbf{i}} \mid \mathbf{i} \in \bigcup_{m=1}^{\ell-1} I^m\right\}.$$

Using the induction hypothesis for the elements $b_{\mathbf{i}}$ in the last summation one obtains (3.12).

It remains to show that the sum in (3.12) is direct. Let $\mathbf{j} \in \mathcal{J}$. Then $f_{\mathbf{j}}$ is nonzero. Because of the explicit formula (3.1) we have

$$(3.13) \quad b_{\mathbf{j}} - f_{\mathbf{j}} \in \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \mathbb{C}\theta(f_{\mathbf{j}}) + \text{Sp}\{b_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{J}, \alpha_{\mathbf{i}} < \alpha_{\mathbf{j}}\}.$$

- Hence $f_j = \pi_{-\alpha_j}(b_j)$ for all $j \in \mathcal{J}$, where π_α is the projection on \mathfrak{g}_α for $\alpha \in \Phi$, see (2.3). Thus the linear independence of $\{f_j\}_{j \in \mathcal{J}}$ together with (2.3) implies that the sum is direct.
- (iii) \implies (iv): By definition, $\mathfrak{h}^\theta \subseteq \mathfrak{k} \cap \mathfrak{h}$ so it suffices to show that $\mathfrak{k} \cap \mathfrak{h} \subseteq \mathfrak{h}^\theta$. Suppose $h \in \mathfrak{k} \cap \mathfrak{h}$. Since $\pi_{-\alpha_j}(b_j) = f_j$ and the triangular decomposition (2.3), part (iii) implies $h \in \mathfrak{n}_X^+ \oplus \mathfrak{h}^\theta \subseteq \mathfrak{g}^\theta$ so $h \in \mathfrak{h}^\theta$.
- (iv) \implies (ii): We prove the contrapositive. If (3.8) fails then (3.14) and (3.7) imply that either $\gamma_j h_i - \gamma_i h_j \in \mathfrak{k} \cap (\mathfrak{h} \setminus \mathfrak{h}^\theta)$ with $\gamma_i \neq \gamma_j$ or $2h_i + h_j \in \mathfrak{k} \cap (\mathfrak{h} \setminus \mathfrak{h}^\theta)$. In either case (3.10) fails. \square

It is convenient to have an explicit description of \mathfrak{h}^θ . Given $i \in I$, by applying θ to $\theta(h_i) - h_i - \theta(h_{\tau(i)}) + h_{\tau(i)} \in \mathfrak{g}_X \cap \mathfrak{h}$ one obtains $\theta(h_i - h_{\tau(i)}) = h_i - h_{\tau(i)}$. From this we straightforwardly deduce

$$(3.14) \quad \mathfrak{h}^\theta = \bigoplus_{i \in X} \mathbb{C}h_i \oplus \bigoplus_{\substack{i \in I^* \\ i \neq \tau(i)}} \mathbb{C}(h_i - h_{\tau(i)})$$

We denote $\Phi_X = \Phi \cap Q_X$ and note that $|\mathcal{J}| = |\Phi|/2$; from (3.14) we also obtain $\dim(\mathfrak{h}^\theta) = |I| - |I^*|$. Hence, given $(X, \tau) \in \text{GSat}(A)$ and $\gamma \in \Gamma$, Theorem 3.2 (iii) implies

$$(3.15) \quad \dim(\mathfrak{k}) = |\Phi_X|/2 + |I| - |I^*| + |\Phi|/2.$$

Corollary 3.4. *Let $(X, \tau) \in \text{GSat}(A)$ and $\gamma \in \Gamma$. The generating set*

$$\{h_i, e_i\}_{i \in X} \cup \{h_i - h_{\tau(i)}\}_{i \in I^*, i \neq \tau(i)} \cup \{b_i\}_{i \in I},$$

and the relations (3.2-3.6) provide a presentation of \mathfrak{k} .

Proof. There are no relations for the b_i other than (3.2), (3.3) and (3.7): otherwise applying $\pi_{-\alpha}$ with $\alpha \in \Phi^+$ maximal produces a relation for the f_i inequivalent to a relation (2.1), (2.2). \square

3.1. Ideal structure of \mathfrak{k} . In this section we assume that A is indecomposable, so that \mathfrak{g} is simple. In order to describe the derived subalgebra of \mathfrak{k} recall the set $I_{\text{diff}} \in I^*$ and define

$$(3.16) \quad \begin{aligned} I_{\text{ns}} &= \{i \in I \mid (\theta(\alpha_i))(h_i) = -2\} = \{i \in I \mid i = \tau(i), \check{X}(i) = \emptyset\}, \\ I_{\text{nsf}} &= \{j \in I_{\text{ns}} \mid \forall i \in I_{\text{ns}} a_{ij} \in 2\mathbb{Z}\}. \end{aligned}$$

Proposition 3.5. *Let $(X, \tau) \in \text{GSat}(A)$ and $\gamma \in \Gamma$. As vector spaces we have*

$$\mathfrak{k} = \mathfrak{k}' \oplus \bigoplus_{i \in I_{\text{diff}}} \mathbb{C}(h_i - h_{\tau(i)}) \oplus \bigoplus_{i \in I_{\text{nsf}}} \mathbb{C}b_i.$$

Proof. Fix $(X, \tau) \in \text{GSat}(A)$. Note that neither $h_i - h_{\tau(i)}$ ($i \in I_{\text{diff}}$) nor b_j ($j \in I_{\text{nsf}}$) is a linear combination of Lie brackets in \mathfrak{k} . This follows from Corollary 3.4 and (3.2-3.7): these elements do not appear as in the expressions for Lie brackets in the defining relations of \mathfrak{k} .

It now suffices to show that the remaining basis elements specified in (3.9) are linear combinations of Lie brackets in \mathfrak{k} , for which we argue as follows.

b_i with $i \in \mathcal{J}_\ell$, $\ell > 1$: This holds by definition.

e_i, f_i, h_i with $i \in X$: This follows from (3.2-3.4).

$h_i - h_{\tau(i)}$ with $i \in I^* \setminus I_{\text{diff}}$ and $i \neq \tau(i)$: The given condition is equivalent to $w_X(\alpha_i) = \alpha_i$ and $a_{i\tau(i)} = 0$.

Hence (3.7) implies that $h_i - h_{\tau(i)} = \gamma_i^{-1}[b_i, b_{\tau(i)}]$.

b_j with $\check{X}(j) \neq \emptyset$: There exists $i \in X$ such that $a_{ij} \neq 0$. By (3.3) we have $b_j = -a_{ij}^{-1}[h_i, b_j]$.

b_j with $j \neq \tau(j)$: Note that $a_{\tau(j)j} \leq 0$. By (3.3) we have $b_j = (a_{\tau(j)j} - 2)^{-1}[h_j - h_{\tau(j)}, b_j]$.

b_j with $j \in I_{\text{ns}} \setminus I_{\text{nsf}}$: By definition of I_{nsf} there exists $i \in I_{\text{ns}}$ such that $a_{ij} \in \{-1, -3\}$. According to (3.7), $b_j = -\gamma_i^{-1}\text{ad}(b_i)^2(b_j)$ if $a_{ij} = -1$ and $b_j = -(2\gamma_i)^{-1}\text{ad}(b_i)^2(b_j) - (6\gamma_i^2)^{-1}\text{ad}(b_i)^4(b_j)$ if $a_{ij} = -3$; in either case $b_j \in \mathfrak{k}'$. \square

It follows that the codimension of \mathfrak{k}' in \mathfrak{k} equals $|I_{\text{diff}}| + |I_{\text{nsf}}|$. For $(X, \tau) \in \text{Sat}(A)$, in [Let02, Sec. 7, Variation 1] it was noted that $|I_{\text{diff}}| \leq 1$ if A is of finite type. In light of the above it is natural to generalize this in two directions: also involve the set I_{nsf} and allow $(X, \tau) \in \text{GSat}(A)$. It turns out the same upper bound holds true and there are generalized Satake diagrams with $|I_{\text{diff}}| + |I_{\text{nsf}}| = 1$ unless A is of type E_8 , F_4 or G_2 . From Table 1 it follows that the only elements of $\text{GSat}(A) \setminus \text{Sat}(A)$ for which $|I_{\text{diff}}| + |I_{\text{nsf}}| = 1$ are of the form $\overset{1}{\circ} - \overset{2}{\circ} - \cdots - \overset{n}{\bullet} \rightleftarrows \bullet$ with $n > 2$ in which case $I_{\text{nsf}} = \{1\}$ and $\zeta(\alpha_2) = -1$.

For the reasons that will become clear a bit later we introduce a further refinement of generalized Satake diagrams. In particular, we define the set of *weak Satake diagrams* by

$$\text{WSat}(A) = \{(X, \tau) \in \text{GSat}(A) \setminus \text{Sat}(A) \mid (X, \tau) \text{ contains no minimal subdiagram of the form } \bullet \rightleftarrows \bullet\}.$$

As mentioned in Table 1, for elements of $\text{GSat}(A) \setminus \text{Sat}(A)$ a case-by-case analysis yields that there can be at most one $i \in I \setminus X$ such that $i = \tau(i)$ and $\zeta(\alpha_i) = -1$. For $(X, \tau) \in \text{WSat}(A)$ we will obtain a semidirect product decomposition in terms of a reductive Lie subalgebra and a nilpotent ideal in which this unique $i \in i \setminus X$ plays an important role.

For any $r \in \mathbb{Z}_{\geq 0}$ and any $i \in I$ denote by $\mathfrak{k}(i)_r$ the span of all b_j such that the coefficient of α_i in α_j is precisely r . We then have the following decomposition

$$\langle b_i \rangle_{i \in I} = \bigoplus_{r=0}^{\infty} \mathfrak{k}(i)_r.$$

Consider the subspace

$$\mathfrak{k}(i) := \bigoplus_{r=1}^{\infty} \mathfrak{k}(i)_r$$

and the subalgebras

$$\mathfrak{k}_i := \langle \mathfrak{n}_X^+, \mathfrak{h}^\theta, \{b_j\}_{j \in I \setminus \{i\}} \rangle \subseteq \mathfrak{k}, \quad \mathfrak{g}_i := \langle \{e_j, f_j, h_j\}_{j \in I \setminus \{i\}} \rangle \subset \mathfrak{g}.$$

Note that $\mathfrak{k} = \mathfrak{k}_i + \mathfrak{k}(i)$ (not necessarily a direct sum, since e.g. b_i may lie in \mathfrak{k}_i).

Proposition 3.6. *Let $(X, \tau) \in \text{WSat}(A)$ and $\gamma \in \Gamma$. Denote by i the unique element of $I \setminus X$ such that $i = \tau(i)$ and $\zeta(\alpha_i) = -1$. Then $\mathfrak{k}(i)_r = \{0\}$ if $r > 2$ and we have the lower central series*

$$\mathfrak{k}(i) = \mathfrak{k}(i)_1 \oplus \mathfrak{k}(i)_2 \supset \mathfrak{k}(i)_2 \supset \{0\}$$

so that $\mathfrak{k}(i)$ is nilpotent of class 2. Furthermore, both $\mathfrak{k}(i)_1$ and $\mathfrak{k}(i)_2$ are \mathfrak{k}_i -modules under the adjoint action, $\mathfrak{k}(i)$ is an ideal of \mathfrak{k} , \mathfrak{k}_i is the fixed-point subalgebra of $\theta|_{\mathfrak{g}_i}$ and we have $\mathfrak{k} = \mathfrak{k}_i \ltimes \mathfrak{k}(i)$.

Proof. Note that (3.7) implies, for all $j \in I \setminus \{i\}$, that

$$(3.17) \quad \text{ad}(b_i)^{1-a_{ij}}(b_j) = 0$$

$$(3.18) \quad \text{ad}(b_j)^{1-a_{ji}}(b_i) \in \sum_{r=1}^{-a_{ij}} \mathbb{F} \text{ad}(b_j)^r(b_i) \subseteq \mathfrak{k}(i)_1.$$

Since (3.3) and (3.18) are the only relations in \mathfrak{k} with b_i appearing on the right-hand side, it follows that $\mathfrak{k}_i = \langle \mathfrak{n}_X^+, \mathfrak{h}^\theta, \mathfrak{k}(i)_0 \rangle$ and $\mathfrak{k} = \mathfrak{k}_i \oplus \mathfrak{k}(i)$ (as vector spaces). Deleting the node i from any diagram in Table 1 one obtains a (possibly disconnected) Satake diagram such that $\theta|_{\mathfrak{g}_i}$ by virtue of (2.8) is an involution. From Table 1 it also follows that $I^* = I \setminus X$ so that \mathfrak{k}_i is the fixed-point subalgebra of \mathfrak{g}_i for the involution θ_γ , see (2.11).

Combined with (3.2-3.3), (3.18) implies that each summand $\mathfrak{k}(i)_r$ is a \mathfrak{k}_i -module. Hence $\mathfrak{k}(i)$ is a \mathfrak{k}_i -module and by virtue of (3.17) it is a subalgebra of \mathfrak{k} . It follows that $\mathfrak{k}(i)$ is an ideal. Automatically we have that $[\oplus_{r=1}^s \mathfrak{k}(i)_r, \mathfrak{k}(i)_1] \subseteq \oplus_{r=1}^{s+1} \mathfrak{k}(i)_r$ for all $s \in \mathbb{Z}_{\geq 1}$. A case-by-case analysis using Table 1 yields that the coefficient in front of α_i in the highest root of Φ is always 2. This implies $\mathfrak{k}(i)_3 = 0$ so that $\mathfrak{k}(i)_2$ is the centre of $\mathfrak{k}(i)$ and we obtain the indicated lower central series. \square

Regarding the centre \mathfrak{z} of \mathfrak{k} for $(X, \tau) \in \text{WSat}(A)$, recall the notation i for the unique element of $I \setminus X$ such that $i = \tau(i)$ and $\zeta(\alpha_i) = -1$. Since the centre of the ideal $\mathfrak{k}(i)$ is $\mathfrak{k}(i)_2$, we must have $\mathfrak{z} \subseteq \mathfrak{k}(i)_2$. Define

$$\mathcal{J}_{\text{even}} := \{j \in \mathcal{J} \mid \forall k \in I \setminus X \text{ the coefficient of } \alpha_j \text{ in front of } \alpha_k \text{ is even}\}$$

so that

$$\mathfrak{k}(i)_{2,\text{even}} := \bigoplus_{j \in \mathcal{J}_{\text{even}}} \mathbb{C}b_j \subset \mathfrak{k}(i)_2.$$

We claim without proof that \mathfrak{z} is generated by a single element of $\mathfrak{k}(i)_{2,\text{even}}$.

Let us now explain the motivation behind the definition of the set $\text{WSat}(A)$. Consider the excluded generalized Satake diagram $\bullet \rightarrow \circ$. By definition, \mathfrak{k} is the subalgebra of $\mathfrak{g} = \text{Lie}(G_2)$ generated by $e_1, h_1, b_1 = f_1$ and $b_2 = f_2 + \gamma_2 \theta(f_2)$ for some $\gamma_2 \in \mathbb{C}^\times$. The relations (3.2-3.7) specialize to

$$\begin{aligned} [e_1, b_1] &= h_1, & [e_1, b_2] &= 0, & [h_1, b_1] &= -2b_1, & [h_1, b_2] &= b_2, & [h_1, e_1] &= 2e_1, \\ [b_1, [b_1, b_2]] &= 0, & [b_2, [b_2, [b_2, [b_2, b_1]]]] &= -18\gamma_2^2 e_1. \end{aligned}$$

According to (3.15) we have $\dim(\mathfrak{k}) = 8$. A natural basis is given by

$$e_1, \quad b_1, \quad h_1, \quad b_2, \quad b_{(2,1)}, \quad b_{(2,2,1)}, \quad b_{(2,2,2,1)}, \quad b_{(1,2,2,2,1)}.$$

Using the adjoint action of e_1, b_1 and b_2 on \mathfrak{k} it is easy to verify that an ideal of \mathfrak{k} equals \mathfrak{k} if it contains any of the generators listed above. This together with some straightforward computations shows that \mathfrak{k} is in fact a simple Lie algebra. Since $\dim(\mathfrak{k}) = 8$, it must be isomorphic to \mathfrak{sl}_3 . On the other hand, if $(X, \tau) \in \text{WSat}(A)$, since \mathfrak{k} has a nonzero nilpotent ideal by Proposition 3.6, \mathfrak{k} is not a reductive Lie algebra.

Proposition 3.7. *Let $(X, \tau) \in \text{GSat}(A) \setminus \text{Sat}(A)$ and $\gamma \in \Gamma$. Then \mathfrak{k} is not the fixed-point subalgebra of any automorphism of \mathfrak{g} .*

Proof. We first show this for the case when (X, τ) is $\bullet \rightarrow \circ$. Suppose there exists $\phi \in \text{Aut}(\mathfrak{g})$ such that $\mathfrak{k} = \mathfrak{g}^\phi$. From $[h_2, b_1] = 3b_1$ and $[h_2, e_1] = -3e_1$ one establishes straightforwardly that $\phi(h_2) \in \mathfrak{h}$ and hence that $\phi(h_2) = \frac{3}{2}(m-1)h_1 + mh_2$ for some $m \in \mathbb{C}$. Next, from $\theta(f_2) = e_{(2,1)}$ it follows that $[h_2, b_2] = -f_2 - b_2$; hence $\phi(f_2) = mf_2 + \frac{1}{2}(1-m)b_2$. Combining this with $[f_2, b_2] = 3e_1$ one obtains $m = 1$. But this means that h_2 and f_2 are also fixed points of ϕ , contrary to assumption. Hence such ϕ does not exist. Now let $(X, \tau) \in \text{WSat}(A)$. In this case \mathfrak{k} is not a reductive Lie algebra and [Ja62, Thm. 1] implies that \mathfrak{k} cannot be the fixed-point subalgebra of any automorphism of \mathfrak{g} . \square

Nevertheless, in Section 4 we will show that for all $(X, \tau) \in \text{GSat}(A)$, the subalgebra \mathfrak{k} can be quantized resulting in a coideal subalgebra possessing a universal K -matrix.

3.2. The universal enveloping algebra $U(\mathfrak{k})$. Let $(X, \tau) \in \text{GSat}(A)$ and $\gamma \in \Gamma$. We identify \mathfrak{k} with its image in $U(\mathfrak{k})$ under the canonical Lie algebra embedding. The generators of $U(\mathfrak{k})$ corresponding to b_i ($i \in I \setminus X$) can be modified by scalar terms, which is a straightforward generalization of [Ko14, Cor. 2.9].

Proposition 3.8. *For $(X, \tau) \in \text{GSat}(A)$, $\gamma \in \Gamma$ and $\mathbf{s} \in \mathbb{C}^{I \setminus X}$, the universal enveloping algebra $U(\mathfrak{k}_\gamma)_\mathbf{s}$ is generated by e_i, f_i ($i \in X$), $h \in \mathfrak{h}^\theta$ and*

$$(3.19) \quad b_{i;\gamma,\mathbf{s}} = f_i + \gamma_i \theta(f_i) + s_i \quad \text{for all } i \in I \setminus X.$$

Again, if there is no cause for confusion, we will suppress γ and \mathbf{s} from the notation. Because of Corollary 3.4 we immediately obtain the following result, which addresses [Ko14, Rmk. 2.10].

Proposition 3.9. *For $(X, \tau) \in \text{GSat}(A)$, $\gamma \in \Gamma$ and $\mathbf{s} \in \mathbb{C}^{I \setminus X}$, the defining relations of the universal enveloping algebra $U(\mathfrak{k})$ are given by (3.2-3.6), with the Lie bracket interpreted as commutator.*

We may view $U(\mathfrak{k})$ as a Hopf subalgebra of $U(\mathfrak{g})$ so that Lie algebra automorphisms of \mathfrak{g} lift to Hopf algebra automorphisms of $U(\mathfrak{g})$. Call two Hopf subalgebras B, B' of $U(\mathfrak{g})$ equivalent if there exists $\phi \in \text{Aut}_{\text{Hopf}}(U(\mathfrak{g}))$ such that $B' = \phi(B)$. Define

$$(3.20) \quad \begin{aligned} \tilde{\Gamma} &:= \{\gamma \in \Gamma \mid \gamma_i = 1 \text{ unless } i \in I_{\text{diff}}\}, \\ \mathcal{S} &:= \{\mathbf{s} \in \mathbb{C}^{I \setminus X} \mid s_i = 0 \text{ unless } i \in I_{\text{nsf}}\}. \end{aligned}$$

Proposition 3.10. *Let $(X, \tau) \in \text{GSat}(A)$, $\gamma \in \Gamma$ and $\mathbf{s} \in \mathbb{C}^{I \setminus X}$. There exist $\tilde{\gamma} \in \tilde{\Gamma}$ and $\mathbf{s}' \in \mathcal{S}$ such that $U(\mathfrak{k}_{\gamma})_{\mathbf{s}}$ is equivalent to $U(\mathfrak{k}_{\tilde{\gamma}})_{\mathbf{s}'}$.*

Proof. The existence of $\tilde{\gamma}$ can be proven in an argument entirely analogous to the proof of [Ko14, Prop. 9.2 (i)]. It follows that $U(\mathfrak{k}_{\gamma})_{\mathbf{s}}$ is equivalent to $U(\mathfrak{k}_{\tilde{\gamma}})_{\tilde{\mathbf{s}}}$ for some $\tilde{\mathbf{s}} \in \mathbb{C}^{I \setminus X}$.

Regarding the existence of $\mathbf{s}' \in \mathcal{S}$, note that $b_{i, \tilde{\gamma}} \in (\mathfrak{k}_{\tilde{\gamma}})'$ unless $i \in I_{\text{nsf}}$ owing to Prop. 3.5. Hence $U(\mathfrak{k}_{\tilde{\gamma}})_{\tilde{\mathbf{s}}}$ is already generated by e_i, f_i ($i \in X$), $h \in \mathfrak{h}^{\theta}$, $b_{i, \tilde{\gamma}, 0}$ for $i \in (I \setminus X) \setminus I_{\text{nsf}}$ and $b_{i, \tilde{\gamma}, \tilde{\mathbf{s}}}$ for $i \in I_{\text{nsf}}$. Hence we may take $s'_i = \tilde{s}_i$ if $i \in I_{\text{nsf}}$ and $s'_i = 0$ otherwise. \square

4. THE UNIVERSAL K -MATRIX REVISITED

Assume the d_i are dyadic rationals and let \mathbb{K} be a quadratic closure of $\mathbb{C}(q)$ where q is an indeterminate, so that $q_i := q^{d_i} \in \mathbb{K}$ for all $i \in I$. The Drinfeld-Jimbo quantum group $U_q = U_q(\mathfrak{g})$ is an associative unital algebra over \mathbb{K} which quantizes the universal enveloping algebra $U(\mathfrak{g})$. It is generated by $\{E_i, F_i, t_i^{\pm 1}\}$ where $i \in I$, satisfying the relations given in e.g. [Lu94, 3.1.1]. It is a Hopf algebra whose structure is defined by the choice of the coproduct:

$$\Delta(E_i) = E_i \otimes 1 + t_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes t_i^{-1} + 1 \otimes F_i, \quad \Delta(t_i) = t_i \otimes t_i.$$

For $\alpha = \sum_i n_i \alpha_i \in Q$ with $n_i \in \mathbb{Z}$ we write $t_{\alpha} = \prod_{i \in I} t_i^{n_i}$. The Hopf subalgebra $U_q^0 = U_q(\mathfrak{h})$ is the subalgebra generated by $t_i^{\pm 1}$ for $i \in I$ and spanned by $\{t_{\alpha}\}_{\alpha \in Q}$. In terms of the quantum root spaces

$$(U_q)_{\alpha} = \{u \in U_q \mid \forall i \in I \ t_i u t_i^{-1} = q_i^{\alpha(h_i)} u\}$$

where $\alpha \in Q$, we have the Q -grading

$$(4.1) \quad U_q = \bigoplus_{\alpha \in Q} (U_q)_{\alpha}, \quad (U_q)_{\alpha} (U_q)_{\beta} \subseteq (U_q)_{\alpha + \beta}.$$

According to [Tw92, Thm. 2.1] we have $\text{Aut}_{\text{Hopf}}(U_q) = \text{Ad}(\tilde{H}) \rtimes \text{Aut}(A)$ with $\text{Ad}(\chi)$ for $\chi \in \tilde{H}$ acting on the root space $(U_q)_{\alpha}$ for $\alpha \in Q$ by multiplication by $\chi(\alpha)$, and $\text{Aut}(A)$ acting by relabelling. Other relevant algebra automorphisms are Lusztig's automorphisms T_i for $i \in I$ given as $T_{i,1}''$ in [Lu94, 37.1.3] which define a braid group action on U_q restricting to the Weyl group action on U_q^0 : $T_i(t_{\alpha}) = t_{r_i(\alpha)}$ for $i \in I$ and $\alpha \in Q$. For $X \subseteq I$ with $w_X = r_{i_1} \cdots r_{i_{\ell}}$ a reduced decomposition we define $T_X = T_{i_1} \cdots T_{i_{\ell}}$. Also, we define a quantum analogue of the Chevalley involution by

$$(4.2) \quad \omega_q(E_i) = -t_i^{-1} F_i, \quad \omega_q(F_i) = -E_i t_i, \quad \omega_q(t_i^{\pm 1}) = t_i^{\mp 1}$$

for $i \in I$. Then ω_q commutes with $\text{Aut}(A)$ and with T_i for $i \in I$, see [BK16, Lemma 7.1]. Assuming $\tau(X) = X$, one straightforwardly checks that τ commutes with T_X .

4.1. Quantum pair algebras. We will follow the approach of the papers [Ko14, BK15, BK16] and simply highlight where a definition or formula needs to be changed. The quantum analogon of the map $\theta = \text{Ad}(w_X)\tau\omega$ is the map

$$\theta_q = \theta_q(X, \tau) = T_X \tau \omega_q \in \text{Aut}_{\text{alg}}(U_q).$$

Note the absence of the factor $\text{Ad}(s)$, cf. [Ko14, Def. 4.3] or [BK16, Def. 5.4 and Eqn. (5.4)], which was present in *ibid.* to guarantee that θ_q specializes to the appropriate Lie algebra involution, see [Ko14, Prop. 10.2]. Similar to (3.14) it follows that $U_q(\mathfrak{h})^{\theta_q}$ consists of polynomials in $t_i^{\pm 1}$ ($i \in X$) and $(t_i t_{\tau(i)}^{-1})^{\pm 1}$ ($i \in I^*, i \neq \tau(i)$).

It is equal to the subalgebra denoted $U_{\mathfrak{g}}^{0'}$ in [Ko14].

The quantization of the fixed-point subalgebra in the formalism by [Ko14] relies on the presentation of $\mathfrak{g}^{\theta\gamma}$ in terms of generators given in [Ko14, Lemma 2.8]. Our $\mathfrak{k}(X, \tau)$ with $(X, \tau) \in \text{GSat}(A)$ by definition can be quantized to a right coideal subalgebra in the same way.

Definition 4.1. Let $(X, \tau) \in \text{GSat}(A)$, $\gamma \in (\mathbb{K}^\times)^{I \setminus X}$ and $\mathbf{s} \in \mathbb{K}^{I \setminus X}$. Then $B = B_{\gamma, \mathbf{s}}(X, \tau)$ is the coideal subalgebra generated by $U_q(\mathfrak{g}_X)$, $U_q(\mathfrak{h})^{\theta_q}$ and the elements

$$B_i = B_{i; \gamma, \mathbf{s}} = F_i + \gamma_i \theta_q(F_i t_i) t_i^{-1} + s_i t_i^{-1} \quad \text{for all } i \in I \setminus X. \quad \varnothing$$

To make a direct match between the Kolb-Balogović formalism based on fixed-point subalgebras and our more general approach one should set, for all $i \in I \setminus X$,

$$\gamma_i = s(\alpha_{\tau(i)}) c_i,$$

see also [BK16, Eqn. (7.7)]. If the tuples γ, \mathbf{s} lie in the sets

$$(4.3) \quad \begin{aligned} \Gamma_q &= \{\gamma \in (\mathbb{K}^\times)^{I \setminus X} \mid \forall i \in I^* \gamma_i \neq \gamma_{\tau(i)} \implies i \in I_{\text{diff}}\}, \\ \mathcal{S}_q &= \{\mathbf{s} \in \mathbb{K}^{I \setminus X} \mid s_i = 0 \text{ unless } i \in I_{\text{nsf}}\} \end{aligned}$$

respectively, then according to [Ko14, Sec. 5.3 and Sec. 6] one obtains decompositions of B yielding the quantum analogue of (3.10), namely $B \cap U_q(\mathfrak{h}) = U_q(\mathfrak{h})^{\theta_q}$. The key condition for Satake diagrams, see (2.10), is only used in [Ko14, Proof of Lemma 5.11, Step 1], but it is clear that what is needed is precisely the weaker condition appearing in the definition of a generalized Satake diagram, see Definition 2.2. The rest of [Ko14] is applicable without change in the setting of generalized Satake diagrams; in particular in the specialization ($q \rightarrow 1$) one recovers $U(\mathfrak{k})$, see [Ko14, Sec. 10].

In [BK15] the bar involutions for U_q and B are studied, following earlier work by [ES13] and [BW13] in the case of quantum symmetric pairs of \mathfrak{gl}_N type. The proof of [BK15, Prop. 2.3] relies on a case-by-case analysis of Satake diagrams of finite type from Araki's work [Ara62]. We claim here without proof that a similar analysis using Table 1 yields the same result for all generalized Satake diagrams, in other words that [BK15, Prop. 2.5] holds with $\nu_i = 1$ for all $i \in I \setminus X$ (otherwise $\nu_i = -1$). In the remainder of [BK15] the defining condition of Satake diagrams or a case-by-case analysis is not used so that these results remain valid.

The universal K -matrix for the algebra B is constructed in [BK16] in the case $(X, \tau) \in \text{Sat}(A)$. We restate some key conditions in terms of the parameters γ . Assuming $\nu_i = 1$ for all $i \in I \setminus X$, condition [BK16, Eqn. (5.17)] is equivalent to

$$\gamma_{\tau(i)} = \zeta(\alpha_i) q_i^{\theta(\alpha_i) - 2\rho_X(h_i)} \overline{\gamma_i},$$

where ρ_X is the Weyl vector of \mathfrak{g}_X and $\overline{}$ denotes the bar involution of U_q , which by definition fixes E_i, F_i and inverts $t_i^{\pm 1}$ and q . In [BK16, Proof of Lemma 6.4] the defining condition of Satake diagrams is used, but as before the defining condition of generalized Satake diagrams is what is needed. Then [BK16, Eqn. (7.14)] needs to be replaced by

$$\overline{T_{w_X}(E_{\tau(i)})} = \zeta(\alpha_i) q_i^{-2\rho_X(h_i)} T_{w_X}^{-1}(E_{\tau(i)})$$

so that the scalar ρ_i appearing in [BK16, Lemma 9.3] equals $q_i^{-\theta(\alpha_i)(h_i)} \gamma_{\tau(i)}$ since [BK16, Eqn. (9.8)] is equivalent to

$$\overline{\gamma_i T_{w_X}(E_{\tau(i)})} = q_i^{-\theta(\alpha_i)(h_i)} \gamma_{\tau(i)} T_{w_X}^{-1}(E_{\tau(i)}).$$

Finally, we highlight the paper [DK18] which establishes an elegant factorization property of the quasi K -matrix in terms of the restricted Weyl group of \mathfrak{g} . Sections 2.2 and 2.3 in *ibid.* entail an analysis of the restricted Weyl group and restricted root system following [Lu76]. For completeness, in reference to a comment in [DK18, between Eqs. (2.9) and (2.10)] we remark that also for all $(X, \tau) \in \text{GSat}(A) \setminus \text{Sat}(A)$ the set X is invariant under the diagram automorphism $\tau_0 = \tau_{I,0}$ corresponding to the longest element of W ; this follows from Table 1. The upshot of this in [DK18] is that $\tau_{0, X[i]}$ stabilizes X (for all $i \in I^*$). This is used to derive that the $\tilde{r}_i = w_X w_{X[i]}$ form a Coxeter system for the group they generate. Alternatively, this result follows from Corollary 2.5 for all generalized Satake diagrams.

A. DERIVING MODIFIED SERRE RELATIONS FOR \mathfrak{k}

The following three technical lemmas are used to derive the key equation (3.7). It is convenient to introduce the notation $Q_X = \sum_{i \in X} \mathbb{Z}\alpha_i$ and $Q_X^+ := Q^+ \cap Q_X$.

Lemma A.1. *Let $(X, \tau) \in \text{CDec}(A)$ and $\gamma \in (\mathbb{C}^\times)^{I \setminus X}$. For all $i \in X$, $j \in I$ and $m \in \mathbb{Z}_{\geq 1}$ we have*

$$\text{ad}(b_i)^m(b_j) = \begin{cases} \text{ad}(f_i)^m(f_j) + \gamma_j \theta(\text{ad}(f_i)^m(f_j)) & \text{if } j \in I \setminus X, \\ \text{ad}(f_i)^m(f_j) & \text{if } j \in X. \end{cases}$$

Proof. This follows immediately from (2.7) and the fact that θ is a Lie algebra automorphism. \square

Lemma A.2. *Let $(X, \tau) \in \text{CDec}(A)$ and $\gamma \in (\mathbb{C}^\times)^{I \setminus X}$. For all $i \in I \setminus X$, $j \in X$ and $m \in \mathbb{Z}_{\geq 1}$ we have*

$$\text{ad}(b_i)^m(b_j) = \text{ad}(f_i)^m(f_j) + \gamma_i^m \theta(\text{ad}(f_i)^m(f_j)) + \text{LO}_{ij}(m)$$

where

$$\text{LO}_{ij}(m) = \begin{cases} (1 + \zeta(\alpha_i))\gamma_i [\theta(f_i), [f_i, f_j]] \in \mathfrak{n}_X^+ & \text{if } \tau(i) = i, w_X(\alpha_i) - \alpha_i - \alpha_j \in \Phi^+, m = 2, \\ -\gamma_i(2h_i - a_{ij}h_j) & \text{if } \tau(i) = i, w_X(\alpha_i) = \alpha_i + \alpha_j, m = 2, \\ -3(2 + a_{ij})\gamma_i(f_i - \theta(f_i)) & \text{if } \tau(i) = i, w_X(\alpha_i) = \alpha_i + \alpha_j, m = 3, \\ -6a_{ij}(2 + a_{ij})\gamma_i^2 e_j & \text{if } \tau(i) = i, w_X(\alpha_i) = \alpha_i + \alpha_j, m = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By induction with respect to m . For $m = 1$, (2.7) implies

$$\text{ad}(b_i)^1(b_j) = [f_i + \gamma_i \theta(f_i), f_j] = \text{ad}(f_i)^1(f_j) + \gamma_i^1 \theta(\text{ad}(f_i)^1(f_j)) + \text{LO}_{ij}(1)$$

with $\text{LO}_{ij}(1) = 0$ as required. Now assume $m \in \mathbb{Z}_{>1}$ and suppose the statement holds for all smaller values. Then, by virtue of the induction hypothesis, the fact that θ is a Lie algebra automorphism and (2.7), we find

$$\begin{aligned} \text{ad}(b_i)^m(b_j) &= [b_i, \text{ad}(b_i)^{m-1}(b_j)] \\ &= [f_i + \gamma_i \theta(f_i), \text{ad}(f_i)^{m-1}(f_j) + \gamma_i^{m-1} \theta(\text{ad}(f_i)^{m-1}(f_j)) + \text{LO}_{ij}(m-1)] \\ &= \text{ad}(f_i)^m(f_j) + \gamma_i^m \theta(\text{ad}(f_i)^m(f_j)) \\ &\quad + \gamma_i [\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] + \gamma_i^{m-1} [f_i, \theta(\text{ad}(f_i)^{m-1}(f_j))] + [b_i, \text{LO}_{ij}(m-1)]. \end{aligned}$$

Using (2.8) we have $\theta^2(f_i) = \zeta(\alpha_i)f_i$ so that

$$(A.1) \quad \text{LO}_{ij}(m) = \gamma_i [\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] + \zeta(\alpha_i)\gamma_i^{m-1} \theta([\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)]) + [b_i, \text{LO}_{ij}(m-1)].$$

Suppose that $[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] \neq 0$. Then $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi \cup \{0\}$. Now $\Phi = \Phi^+ \cup \Phi^-$ implies that $\tau(i) = i$ and $j \in \check{X}(i)$.

If $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi^+$ we must have $\tau(i) = i$ and $m = 2$; because $w_X(\alpha_{\tau(i)}) - \alpha_i - \alpha_j \in Q_X^+$ it follows that $[\theta(f_i), [f_i, f_j]] \in \mathfrak{n}_X^+$. The claimed expression for $\text{LO}_{ij}(2)$ follows immediately from (A.1); those for $\text{LO}_{ij}(m)$ with $m > 2$ from (3.2).

Now suppose $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi^- \cup \{0\}$. Then $\tau(i) = i$ and $w_X(\alpha_i) \leq (m-1)\alpha_i + \alpha_j$ so that $\check{X}(i) = \{j\}$ and hence $a_{ji} < 0$. In this case we readily obtain

$$w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j = (2-m)\alpha_i - (1+a_{ji})\alpha_j.$$

From $\Phi = \Phi^+ \cup \Phi^-$ it follows that $a_{ji} = -1$. Now $\mathbb{Z}\alpha_i \cap \Phi = \{\pm\alpha_i\}$ implies that $m \in \{2, 3\}$. We straightforwardly compute

$$[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] = \begin{cases} a_{ij}h_j - h_i & \text{if } m = 2, \\ -2(1+a_{ij})f_i & \text{if } m = 3, \end{cases}$$

and the claimed expressions for $\text{LO}_{ij}(m)$ readily follow. \square

For $i, j \in I$ and $m, r \in \mathbb{Z}$ such that $0 \leq r \leq \lfloor m/2 \rfloor$ define $p_{ij}^{(r,m)} \in \mathbb{Z}$ by

$$(A.2) \quad p_{ij}^{(0,m)} = -1, \quad p_{ij}^{(\frac{m+1}{2},m)} = 0, \quad p_{ij}^{(r,m+2)} = p_{ij}^{(r,m+1)} - (m+1)(m+a_{ij})p_{ij}^{(r-1,m)}.$$

Lemma A.3. *Let $(X, \tau) \in \text{CDec}(A)$ and $\gamma \in (\mathbb{C}^\times)^{I \setminus X}$. For all $i, j \in I \setminus X$ such that $i \neq j$ and $m \in \mathbb{Z}_{\geq 0}$ we have*

$$\text{ad}(b_i)^m(b_j) = \text{ad}(f_i)^m(f_j) + \theta(\text{ad}(f_i)^m(f_j)) + \text{LO}_{ij}(m)$$

where

$$\text{LO}_{ij}(m) = \begin{cases} (\gamma_i + \zeta(\alpha_i)\gamma_j)[\theta(f_i), f_j] \in \mathfrak{n}_X^+ & \text{if } \tau(i) = j, w_X(\alpha_i) - \alpha_i \in \Phi^+, m = 1, \\ \gamma_j h_i - \gamma_i h_j & \text{if } \tau(i) = j, w_X(\alpha_i) = \alpha_i, m = 1, \\ 2((\gamma_j - a_{ij}\gamma_i)f_i - \gamma_i(\gamma_i - a_{ij}\gamma_j)e_j) & \text{if } \tau(i) = j, w_X(\alpha_i) = \alpha_i, m = 2, \\ \sum_{r=1}^{\lfloor m/2 \rfloor} p_{ij}^{(r,m)} \gamma_i^r \text{ad}(b_i)^{m-2r}(b_j) & \text{if } \tau(i) = i, w_X(\alpha_i) = \alpha_i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As in the proof of Lemma A.2 we apply induction with respect to m . For $m = 0$ we have

$$\text{ad}(b_i)^0(b_j) = b_j = f_j + \gamma_j \theta(f_j) = \text{ad}(f_i)^0(f_j) + \gamma_i^0 \gamma_j \theta(\text{ad}(f_i)^0(f_j)) + \text{LO}_{ij}(0)$$

with $\text{LO}_{ij}(0) = 0$ as required. Now assume $m \in \mathbb{Z}_{>0}$ and suppose the statement holds for all smaller values. Then, by the induction hypothesis,

$$\begin{aligned} \text{ad}(b_i)^m(b_j) &= [b_i, \text{ad}(b_i)^{m-1}(b_j)] \\ &= [f_i + \gamma_i \theta(f_i), \text{ad}(f_i)^{m-1}(f_j) + \gamma_i^{m-1} \gamma_j \theta(\text{ad}(f_i)^{m-1}(f_j)) + \text{LO}_{ij}(m-1)]. \end{aligned}$$

Rearranging terms and using that θ is a Lie algebra automorphism we obtain

$$\begin{aligned} \text{ad}(b_i)^m(b_j) &= \text{ad}(f_i)^m(f_j) + \gamma_i^{m-1} \gamma_j \theta(\text{ad}(f_i)^m(f_j)) + \text{LO}_{ij}(m) \quad \text{where} \\ \text{LO}_{ij}(m) &= \gamma_i [\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] + \gamma_i^{m-1} \gamma_j [f_i, \theta(\text{ad}(f_i)^{m-1}(f_j))] + [b_i, \text{LO}_{ij}(m-1)]. \end{aligned}$$

Using (2.8) we obtain

$$(A.3) \quad \text{LO}_{ij}(m) = \gamma_i [\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] + \zeta(\alpha_i) \gamma_i^{m-1} \gamma_j \theta([\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)]) + [b_i, \text{LO}_{ij}(m-1)].$$

If $[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] \neq 0$ then $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi \cup \{0\}$.

If $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi^+$ we must have $j = \tau(i)$, $\check{X}(i) \neq \emptyset$, $m = 1$; since $w_X(\alpha_{\tau(i)}) - \alpha_j \in Q_X^+$ it follows that $[\theta(f_i), f_j] \in \mathfrak{n}_X^+$. The expression for $\text{LO}_{ij}(1)$ follows from (A.3); $\text{LO}_{ij}(m) = 0$ with $m > 1$ is a consequence of (3.2).

Now suppose $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi^- \cup \{0\}$. It follows that $\check{X}(i) = \emptyset$, so $\zeta(\alpha_i) = 1$, and $\tau(i) \in \{i, j\}$. If $\tau(i) = j$ then $\mathbb{Z}\alpha_i \cap \Phi = \{\pm\alpha_i\}$ implies that $m \in \{1, 2\}$. Furthermore, $\theta(f_i) = -e_j$ and $a_{ij} = a_{ji}$. Now (A.3) implies, as required, $\text{LO}_{ij}(1) = \gamma_j h_i - \gamma_i h_j$,

$$\begin{aligned} \text{LO}_{ij}(2) &= \gamma_i \gamma_j \theta([-e_j, [f_i, f_j]]) + \gamma_i [-e_j, [f_i, f_j]] + [b_i, \text{LO}_{ij}(1)] \\ &= \gamma_i \gamma_j \theta([h_j, f_i]) + \gamma_i [h_j, f_i] + [\gamma_i h_j - \gamma_j h_i, f_i - \gamma_i e_j] \\ &= 2((\gamma_j - a_{ij}\gamma_i)f_i - \gamma_i(\gamma_i - a_{ij}\gamma_j)e_j) \end{aligned}$$

and $\text{LO}_{ij}(m) = 0$ if $m > 2$.

It remains to deal with the case $\check{X}(i) = \emptyset$ and $\tau(i) = i$, in which case $\theta(f_i) = -e_i$. A straightforward computation gives

$$[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] = (m-1)(m-2+a_{ij})\text{ad}(f_i)^{m-2}(f_j).$$

By virtue of the induction hypothesis, (A.3) simplifies to

$$\text{LO}_{ij}(m) = (m-1)(m-2+a_{ij})\gamma_i (\text{ad}(b_i)^{m-2}(b_j) - \text{LO}_{ij}(m-2)) + [b_i, \text{LO}_{ij}(m-1)],$$

from which the recursion (A.2) follows straightforwardly. \square

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