

A PROOF OF THE MATRIX VERSION OF BAKER'S CONJECTURE IN DIOPHANTINE APPROXIMATION

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ABSTRACT. We resolve a question of Beresnevich, Kleinbock, and Margulis ('15) in the affirmative by proving that the matrix analogue of the Veronese curve is strongly extremal in the sense of Diophantine approximation.

1. INTRODUCTION

The origins of the metrical theory of Diophantine approximation on manifolds (also known as Diophantine approximation with dependent quantities) may be traced back to Kurt Mahler's profound conjecture in transcendence theory, [12], that may be translated thus: Lebesgue almost every point on the the Veronese curve $\{(x, x^2, \dots, x^n) : x \in \mathbb{R}\}$ is not very well approximable by rational vectors. Mahler's conjecture was resolved three decades later by Vladimir Sprindžuk [13], who went on to conjecture [14] that any analytic nondegenerate submanifold of \mathbb{R}^n is strongly extremal, viz. that Lebesgue almost every point is not very well multiplicatively approximable by rational vectors. A special case of Sprindžuk's conjecture was conjectured earlier by Alan Baker, [2, p.96], viz. that the Veronese curve is strongly extremal. We refer the reader to [5] for more regarding this history and allied developments.

After a slew of partial results by a number of authors, Sprindžuk's conjecture was finally resolved by Dmitry Kleinbock and Grigory Margulis in their landmark 1998 Annals paper [9], where they translated the problem into dynamical terms and successfully leveraged quantitative non-divergence estimates for unipotent flows on homogenous spaces. There has since been an intense study of Diophantine extremality, both in the case of simultaneous approximation [8, 15, 16] and in the matrix approximation framework [7, 11, 4, 1, 6]. In particular, in a joint work with Junbo Wang, Kleinbock and Margulis generalized their result to the matrix case by proving that any strongly nonplanar manifold in the space of matrices is strongly extremal [11, Theorem 2.1] (cf. Definition 1.2 below). Strong nonplanarity is a stronger requirement than nondegeneracy, but it is necessary since there are known examples of nondegenerate manifolds in the space of matrices which are not extremal [11, Conclusion 2 on p.22].

In joint work with Victor Beresnevich, Kleinbock and Margulis asked [4, §7.1] whether the following matrix version of Baker's conjecture holds: If $m, n \in \mathbb{N}$ are fixed, then for Lebesgue almost every matrix $X \in \mathcal{M}_{m,m}$, the matrix $X \oplus X^2 \oplus \dots \oplus X^n \in \mathcal{M}_{m,mn}$ is not very well multiplicatively approximable.¹ Unlike Baker's original conjecture, it is not obvious that it follows from the matrix analogue of Sprindžuk's conjecture (i.e. the Kleinbock–Margulis–Wang theorem): it is first necessary to show that the manifold $\{X \oplus X^2 \oplus \dots \oplus X^n : X \in \mathcal{M}_{m,m}\}$ is strongly nonplanar. The aim of this note is to fill in this gap by proving the strong nonplanarity of this manifold and thus proving the matrix analogue of Baker's conjecture.

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¹One can also ask about a matrix version of Mahler's conjecture, which is the same as the matrix version of Baker's conjecture except that the word "multiplicatively" is removed from "not very well multiplicatively approximable". The resulting statement was proven by Kleinbock [7, §3.3], but his methods are not powerful enough to yield the matrix version of Baker's conjecture [4, §7.1].

1.1. Diophantine approximation of matrices. In what follows we recall the theorem of Kleinbock–Margulis–Wang regarding strongly extremal manifolds in the space of matrices. We begin by recalling the definition of strong extremality. Fix $m, n \in \mathbb{N}$, let $\mathcal{M} = \mathcal{M}_{m,n}$ denote the set of $m \times n$ matrices, and fix $\mathbf{A} \in \mathcal{M}$. The *exponent of irrationality* of \mathbf{A} is defined as

$$\omega(\mathbf{A}) = \limsup_{\substack{\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\} \\ \mathbf{p} \in \mathbb{Z}^m}} \frac{-\log \|\mathbf{A}\mathbf{q} - \mathbf{p}\|}{\log \|\mathbf{q}\|},$$

and the *exponent of multiplicative irrationality* is the number

$$\omega_{\times}(\mathbf{A}) = \limsup_{\substack{\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\} \\ \mathbf{p} \in \mathbb{Z}^m}} \frac{-\log \prod_{i=1}^m |(\mathbf{A}\mathbf{q} - \mathbf{p})_i|}{\log \prod_{j=1}^n |q_j| \vee 1}.$$

Note that $\omega_{\times}(\mathbf{A}) \geq (m/n)\omega(\mathbf{A})$. The matrix \mathbf{A} is called *very well approximable* if $\omega(\mathbf{A}) > n/m$, and *very well multiplicatively approximable* if $\omega_{\times}(\mathbf{A}) > 1$. Every very well multiplicatively approximable matrix is also very well approximable, and the sets of very well approximable and very well multiplicatively approximable matrices are both Lebesgue nullsets of full Hausdorff dimension. Finally, a measure μ on \mathcal{M} is called *extremal* (resp. *strongly extremal*) if it gives zero measure to the set of very well approximable (resp. very well multiplicatively approximable) vectors.

Now we recall the definition of strong nonplanarity, the hypothesis that Kleinbock–Margulis–Wang need in order to prove that a manifold is strongly extremal. To this end, we make the following definitions:

Definition 1.1 (Cf. [4, p.3], [6, p.7]). The *Plücker embedding* is the map which sends a matrix to the list of the determinants of its minors.

Definition 1.2 (Cf. [9, p.341], [4, (2.2)]). A submanifold $M \subseteq \mathbb{R}^d$ is said to be *nondegenerate* if whenever $\phi : U \rightarrow M$ is a coordinate chart, then for all $x \in U$ the components of the derivatives $\phi'(x), \phi''(x), \dots$ span \mathbb{R}^d . A submanifold $M \subseteq \mathcal{M}$ is said to be *strongly nonplanar* if its image under the Plücker embedding is nondegenerate.³

Remark 1.3. A connected real-analytic manifold is nondegenerate if and only if it is not contained in any affine hyperplane, and strongly nonplanar if and only if it is not contained in the preimage of any affine hyperplane under the Plücker embedding.

Remark 1.4. The reason for the adjective “strong” is that there is another notion of nonplanarity, *weak nonplanarity*, which was introduced in [4] and is sufficient to imply strong extremality. Since the matrix analogue of the Veronese curve turns out to be strongly nonplanar, in this paper there is no need to consider weak nonplanarity.

We can now state the matrix analogue of the Kleinbock–Margulis theorem:

Theorem 1.5 ([11], [6, Theorem 1.7]). *Every strongly nonplanar manifold is strongly extremal.*

Remark 1.6. In [11] this theorem was stated under the additional hypothesis that the manifold in question is real-analytic. This hypothesis is necessary if “strongly nonplanar” is defined as in [4], but if its definition is as given above, then the argument of [11] works for non-analytic manifolds as well. Moreover, the validity of Theorem 1.5 for non-analytic manifolds also follows directly from [6, Theorem 1.7].

²This definition agrees with the multiplicative approximation framework considered in [11], but not the one considered in [10]; see comments after [6, Proposition 4.1] for more details.

³This definition agrees with the one given in [4] for all real-analytic manifolds, but the definitions disagree for some manifolds which are not real-analytic. The validity of Theorem 1.5 in the non-analytic category shows that our definition is the “correct” one.

1.2. Main results. Theorem 1.5 shows that the following theorem implies the matrix version of Baker's conjecture:

Theorem 1.7. Fix $m, n \in \mathbb{N}$ and define the map $\psi_{m,n} : \mathcal{M}_{m,m} \rightarrow \mathcal{M}_{m,mn}$ via the formula

$$(1.1) \quad \psi_{m,n}(X) = X \oplus X^2 \oplus \cdots \oplus X^n.^4$$

Then the manifold $\psi_{m,n}(\mathcal{M}_{m,m}) \subseteq \mathcal{M}_{m,mn}$ is strongly nonplanar.

Corollary 1.8 (Matrix version of Baker's conjecture). For Lebesgue almost every $X \in \mathcal{M}_{m,m}$, the matrix $\psi_{m,n}(X)$ defined by (1.1) is not very well multiplicatively approximable.

The techniques we use to prove Theorem 1.7 also allow us to prove a more general result. Returning momentarily to the simultaneous approximation setting, a well-known generalization of the Veronese curve is the *Veronese manifold* of dimension r and order n , which is the image of \mathbb{R}^r under the *Veronese embedding*

$$\psi_{n,r}(x_1, \dots, x_r) := \left(\prod_{k=1}^r x_k^{n_k} \right)_{1 \leq \sum n_k \leq n}.$$

The Veronese curve corresponds to the special case $r = 1$. It is readily verified that all Veronese manifolds are nondegenerate, so by the Kleinbock–Margulis theorem [9], Veronese manifolds are strongly extremal. The following result implies that the same is true for the matrix versions of these manifolds:

Theorem 1.9. Fix $m, n, r \in \mathbb{N}$, and for each $X_1, \dots, X_r \in \mathcal{M}_{m,m}$ let

$$(1.2) \quad \psi_{m,n,r}(X_1, \dots, X_r) := \bigoplus_{1 \leq \sum n_k \leq n} \prod_{k=1}^r X_k^{n_k}.$$

Then $\psi_{m,n,r}(\mathcal{M}_{m,m}^r)$ is strongly nonplanar (and thus strongly extremal).

One aspect of Theorem 1.9 which is somewhat undesirable is that due to the noncommutativity of matrices, the ordering of the product $\prod_{k=1}^r X_k^{n_k}$ matters.⁵ In Theorem 1.9 we understand the ordering to be the default one, that is, $\prod_{k=1}^r X_k^{n_k} = X_1^{n_1} \cdots X_r^{n_r}$. But other orderings are possible, which would lead to different theorems. We now proceed to describe a general theorem that will cover all possible orderings.

Fix $r \in \mathbb{N}$, and let M_r (resp. M_r^*) denote the set of all monomials in the commutative (resp. noncommutative) dummy variables x_1, \dots, x_r . So e.g. x_1x_2 and x_2x_1 denote distinct elements of M_r^* , but they denote the same element of M_r . Let $\pi : M_r^* \rightarrow M_r$ be the natural map, i.e. the one that sends the element of M_r^* denoted by x_1x_2 to the element of M_r denoted by x_1x_2 . Finally, given $p \in M_r^*$ and matrices $X_1, \dots, X_r \in \mathcal{M}_{m,m}$, let $p(X_1, \dots, X_r)$ denote the matrix that results from substituting $x_1 = X_1, \dots, x_r = X_r$ into p . So e.g. $(x_1x_2x_1)(X_1, X_2) = X_1X_2X_1$.

Theorem 1.10. Fix $m, n, r \in \mathbb{N}$, and let $\mathbf{P} = (p_\ell)_{\ell=1}^n$ be a finite sequence in M_r^* such that $\pi(p_1), \dots, \pi(p_n)$ are distinct. For each $X_1, \dots, X_r \in \mathcal{M}_{m,m}$, let

$$(1.3) \quad \psi_{m,n,r}^{(\mathbf{P})}(X_1, \dots, X_r) = \bigoplus_{\ell=1}^n p_\ell(X_1, \dots, X_r).$$

Then $\psi_{m,n,r}^{(\mathbf{P})}(\mathcal{M}_{m,m}^r)$ is strongly nonplanar (and thus strongly extremal).

Obviously, Theorem 1.7 is a special case of Theorem 1.9 and Theorem 1.9 is a special case of Theorem 1.10. So in what follows, we will prove only Theorem 1.10.

⁴In this paper $X \oplus Y$ denotes the direct columnwise sum of two matrices X and Y , i.e. the matrix whose columns are those of X followed by those of Y .

⁵On the other hand, the ordering of the direct sum $\bigoplus_{1 \leq \sum n_k \leq n}$ does not matter, since applying a transformation rearranging the columns of a matrix does not affect the strong nonplanarity of a manifold; it is equivalent to applying a linear transformation to the image of the matrix under the Plücker embedding.

Remark 1.11. In our proof, the requirement that $\pi(p_1), \dots, \pi(p_n)$ be distinct is a necessary one. Indeed, if $\pi(p_1), \dots, \pi(p_n)$ are not distinct, then we may choose $i \neq j$ such that $\pi(p_i) = \pi(p_j)$, and then for all X_1, \dots, X_r we have

$$\det[p_i(X_1, \dots, X_r)] = \det[p_j(X_1, \dots, X_r)]$$

which implies that $\psi_{m,n,r}^{(\mathbf{P})}((\mathcal{M}_{m,m})^r)$ is not strongly nonplanar (its image under the Plücker embedding is contained in a linear subspace of the form $x = y$, where x and y are coordinate functions). However, in this case the manifold $\psi_{m,n,r}^{(\mathbf{P})}((\mathcal{M}_{m,m})^r)$ might still be extremal. For example, if $m = n = r = 2$, $p_1 = x_1x_2$, and $p_2 = x_2x_1$, then it can be checked directly that $\psi_{m,n,r}^{(\mathbf{P})}((\mathcal{M}_{m,m})^r)$ is weakly nonplanar; thus by [4, Corollary 2.4], it is strongly extremal. It is unclear whether all manifolds of the form $\psi_{m,n,r}^{(\mathbf{P})}((\mathcal{M}_{m,m})^r)$ are extremal.

2. MULTISSETS

To make the proof of Theorem 1.10 easier, we introduce some notation for dealing with multisets. If I is a set, then a *multisubset* of I is a map from I to $\mathbb{N} := \{0, 1, \dots\}$; a *multiset* is a multisubset of some set. We use the notation $S \preceq I$ to mean that S is a multisubset of I , and we use the notation $\#(S; i)$ to denote the value of S at a point $i \in I$. (Intuitively, $\#(S; i)$ is the “number of copies of i in S ”.) The *cardinality* of a multiset $S \preceq I$ is $\#(S) := \sum_{i \in I} \#(S; i)$. If $S \preceq I$ and $(a_i)_{i \in I}$ is a collection of real numbers, then

$$\sum_{i \in S} a_i := \sum_{i \in I} \#(S; i) a_i, \quad \prod_{i \in S} a_i := \prod_{i \in I} a_i^{\#(S; i)}.$$

If $S \subseteq I$ is a subset, then S can also be interpreted as a multiset via its characteristic function $\#(S; i) := \#(S \cap \{i\})$, in which case the notations above agree with the usual definition of sums and products over S . Conversely, if $S \preceq I$ is a multiset, then we define the *support* of S to be the unique set $|S| \subseteq I$ such that for all $i \in I$, we have $i \in |S|$ if and only if $\#(S; i) \geq 1$. Note that if $|S|$ is then reinterpreted as a multiset, it is possible that $S \neq |S|$; in fact, this happens whenever $\#(S; i) \geq 2$ for some $i \in I$.

If $S, T \preceq I$, then we say that S is a *submultiset*⁶ of T , denoted $S \leq T$, if $\#(S; i) \leq \#(T; i)$ for all $i \in I$. If $S \preceq I$ and $(T_i)_{i \in I}$ is a collection of (multi)subsets of a set J , then we define the multiset $\sum_{i \in S} T_i \preceq J$ via the formula

$$\# \left(\sum_{i \in S} T_i; j \right) := \sum_{i \in S} \#(T_i; j) \quad \forall j \in J.$$

Note that even if S and $(T_i)_{i \in I}$ are all ordinary sets, the multiset $\sum_{i \in S} T_i$ may have points with multiplicity ≥ 2 .

3. PROOF OF THEOREM 1.10

For convenience, in the proof we let $\mathcal{A} = (\mathcal{M}_{m,m})^r$, $\mathcal{B} = \mathcal{M}_{m,mn}$, and $\mathcal{C} = \mathbb{R}^c$, where $c = \sum_{k=1}^m \binom{m}{k} \binom{mn}{k}$ is the number of minors that an $m \times mn$ matrix has. We let $\psi = \psi_{m,n,r}^{(\mathbf{P})} : \mathcal{A} \rightarrow \mathcal{B}$ be defined as in (1.3), and we let $\phi : \mathcal{B} \rightarrow \mathcal{C}$ be the Plücker embedding (cf. Definition 1.1). By definition, to show that the manifold $\psi(\mathcal{A}) \subseteq \mathcal{B}$ is strongly nonplanar, we need to show that the manifold $\phi \circ \psi(\mathcal{A}) \subseteq \mathcal{C}$ is nondegenerate. Since $\phi \circ \psi$ is real-analytic (in fact polynomial), it suffices to show that $\phi \circ \psi(\mathcal{A})$ is not contained in any affine hyperplane, and since $\phi \circ \psi(\mathbf{0}) = \mathbf{0}$, it suffices to consider linear hyperplanes. So we need to show that $\mathcal{V} = \mathcal{C}$, where \mathcal{V} denotes the linear span of $\phi \circ \psi(\mathcal{A})$.

We begin by defining appropriate bases of the vector spaces \mathcal{A} , \mathcal{B} , and \mathcal{C} . Let $V = \{1, \dots, m\}$, $Q = \{1, \dots, r\}$, and $E = V \times V \times Q$, and for each $(i, t, q) \in E$ let $\mathbf{e}_{i,t,q} \in \mathcal{A}$ be the basis vector whose q th coordinate is the elementary matrix $\mathbf{e}_{i,t}$ whose (i, t) th entry is 1. The letters V and E are chosen because later in the proof, we will think of (V, E) as a directed multigraph, where the edge $(i, t, q) \in E$ is considered to have initial vertex i and terminal vertex t . Let $L = \{1, \dots, n\}$ and $W = \{1, \dots, mn\}$, and for each $t \in V$

⁶To warn against confusing the two words, note that a multisubset of I is a submultiset of I if and only if it is a set.

and $\ell \in L$ let $\langle t, \ell \rangle = (\ell - 1)n + t \in W$. For each $(i, j) \in V \times W$ let $\mathbf{e}_{i,j} \in \mathcal{B}$ be the elementary matrix whose (i, j) th entry is 1, i.e.

$$\mathbf{e}_{i, \langle t, \ell \rangle} := \mathbf{0} \oplus \cdots \oplus \underbrace{\mathbf{e}_{i,t}}_{\ell\text{th}} \oplus \cdots \oplus \mathbf{0}.$$

Finally, let

$$D = \{(I, J) : I \subseteq V, J \subseteq W, \#(I) = \#(J) \geq 1\},$$

and let $\theta : D \leftrightarrow \{1, \dots, c\}$ be the bijection that describes the order in which the Plücker embedding ϕ lists the determinants of the minors. For each $(I, J) \in D$, let $\mathbf{e}_{I,J} \in \mathbb{R}^c$ be the basis vector whose $\theta(I, J)$ th coordinate is 1.

Given $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $C \in \mathcal{C}$, we will denote by $A_{i,t,q}$, $B_{i,j}$, and $C_{I,J}$ the coordinates of A , B , and C with respect to the bases defined above, so that

$$A = \sum_{(i,t,q) \in E} A_{i,t,q} \mathbf{e}_{i,t,q}, \quad B = \sum_{(i,j) \in V \times W} B_{i,j} \mathbf{e}_{i,j}, \quad C = \sum_{(I,J) \in D} C_{I,J} \mathbf{e}_{I,J}.$$

Now let $B = \psi(A)$ and $C = \phi(B)$. By the definition of the Plücker embedding ϕ , $C_{I,J}$ is the determinant of the matrix $(B_{i,j})_{i \in I, j \in J}$, i.e.

$$(3.1) \quad C_{I,J} = \sum_{\sigma: I \leftrightarrow J} \text{sgn}(\sigma) \prod_{i \in I} B_{i, \sigma(i)}.^7$$

Similarly, by the definition of ψ ,

$$(3.2) \quad B_{i, \langle t, \ell \rangle} = [p_\ell(X_1, \dots, X_r)]_{i,t},$$

where $A = (X_1, \dots, X_r)$.

For each $\ell \in L$ let $\|\ell\| \in \mathbb{N}$ and $f_\ell(1), \dots, f_\ell(\|\ell\|) \in Q$ be selected so that

$$p_\ell(X_1, \dots, X_r) = X_{f_\ell(1)} \cdots X_{f_\ell(\|\ell\|)}.$$

Note that these choices are unique because p_ℓ is a monomial in noncommutative variables. We will later need the following fact about the commutative version of p_ℓ : since it can be written in multiset notation as

$$\pi(p_\ell) = \prod_{q \in \sum_{k=1}^{\|\ell\|} \{f_\ell(k)\}} x_q,$$

it follows that $\pi(p_\ell)$ depends only on $\sum_{k=1}^{\|\ell\|} \{f_\ell(k)\}$.

The next step is to write the coordinates $B_{i, \langle t, \ell \rangle} = [X_{f_\ell(1)} \cdots X_{f_\ell(\|\ell\|)}]_{i,t}$ ($i, t \in V$, $\ell \in L$) of B in terms of the coordinates $A_{i,t,q} = [X_q]_{i,t}$ ($(i, t, q) \in E$) of A using the definition of matrix multiplication. To facilitate this, we define a *path* in (V, E) to be a diagram of the form

$$(3.3) \quad v_0 \xrightarrow{f_\ell(1)} v_1 \xrightarrow{f_\ell(2)} \cdots \xrightarrow{f_\ell(\|\ell\|)} v_{\|\ell\|},$$

where $\ell \in L$ and $v_0, \dots, v_{\|\ell\|} \in V$. Let P denote the path represented by the diagram (3.3). Then the vertices $i(P) := v_0$ and $t(P) := v_{\|\ell\|}$ are called the *initial* and *terminal vertices* of P , and $\ell(P) := \ell$ is called the *label* of P . The triples $e_k(P) := (v_{k-1}, v_k, f_\ell(k)) \in E$ ($k = 1, \dots, \|\ell\|$) are called the *edges* of P , and the multiset $F(P) := \sum_{k=1}^{\|\ell\|} \{e_k(P)\}$ is called the *edge multiset* of P .

Using this notation, we can rewrite (3.2) as

$$B_{i, \langle t, \ell \rangle} = [p_\ell(X_1, \dots, X_r)]_{i,t} = \sum_P \prod_{k=1}^{\|\ell\|} [X_{f_\ell(k)}]_{v_{k-1}, v_k} = \sum_P \prod_{e \in F(P)} A_e,$$

⁷If I, J are ordered sets, the *sign* of a bijection $\sigma : I \leftrightarrow J$ is defined as $\text{sgn}(\sigma) := (-1)^{\#\{(i_1, i_2) \in I^2 : i_1 < i_2, \sigma(i_1) > \sigma(i_2)\}}$.

where the sums are taken over all paths P such that $i(P) = i$, $t(P) = t$, and $\ell(P) = \ell$. Plugging into (3.1) and using the notation

$$\begin{aligned} j(P) &:= \langle t(P), \ell(P) \rangle \in W, & I(\mathcal{P}) &:= \sum_{P \in \mathcal{P}} \{i(P)\} \preceq V, \\ J(\mathcal{P}) &:= \sum_{P \in \mathcal{P}} \{j(P)\} \preceq W, & F(\mathcal{P}) &:= \sum_{P \in \mathcal{P}} F(P) \preceq E, \\ d(\mathcal{P}) &:= (I(\mathcal{P}), J(\mathcal{P})). \end{aligned}$$

gives

$$\begin{aligned} C_{I,J} &= \sum_{\sigma: I \leftrightarrow J} \text{sgn}(\sigma) \prod_{i \in I} \sum_{\substack{P \\ i(P)=i \\ j(P)=\sigma(i)}} \prod_{e \in F(P)} A_e \\ &= \sum_{\sigma: I \leftrightarrow J} \text{sgn}(\sigma) \sum_{\substack{P \\ I(\mathcal{P})=I \\ j(P)=\sigma(i(P)) \ \forall P \in |\mathcal{P}|}} \prod_{P \in \mathcal{P}} \prod_{e \in F(P)} A_e \\ &= \sum_{\substack{P \\ d(\mathcal{P})=(I,J)}} (\pm 1) \prod_{e \in F(\mathcal{P})} A_e, \end{aligned}$$

where the last two sums are taken over all multisubsets \mathcal{P} of the collection of paths. Note that every multiset \mathcal{P} included in the summation, i.e. for which $I(\mathcal{P}) = I$ and $J(\mathcal{P}) = J$, is in fact a set (by which we mean that it satisfies $\mathcal{P} = |\mathcal{P}|$). Also note that the inclusion $d(\mathcal{P}) \in D$ is valid for all \mathcal{P} that appear in the summation, but not for every collection of paths \mathcal{P} .

Lemma 3.1. *There exists a partial order $<$ on D such that for all $d \in D$, there exists a collection of paths \mathcal{P} such that $d(\mathcal{P}) = d$ and such that if \mathcal{P}' is another collection of paths such that $d(\mathcal{P}') \in D$ and $F(\mathcal{P}') = F(\mathcal{P})$, then $d(\mathcal{P}') < d$.*

Proof. For each $d = (I, J) \in D$ let

$$f(d) = \sum_{\substack{i \in I \\ J[i] \neq \emptyset}} \min_{\ell \in J[i]} \|\ell\|,$$

where $J[i] := \{\ell \in L : (i, \ell) \in J\}$ for each $i \in I$. The partial order on D will be defined as follows: we write $d_1 < d_2$ if and only if $f(d_1) < f(d_2)$. Now fix $d = (I, J) \in D$, let $S = \{i \in I : J[i] \neq \emptyset\}$, and for each $i \in S$ let $\ell_i \in J[i]$ be chosen to minimize $\|\ell_i\|$, so that $f(d) = \sum_{i \in S} \|\ell_i\|$. Let $\sigma' : S \rightarrow J$ be defined by the equation $\sigma'(i) = (i, \ell_i)$, and let $\sigma : I \leftrightarrow J$ be an arbitrary bijective extension of σ' . For each $i \in I$ write $\sigma(i) = (t_i, \ell_i)$ (note that this agrees with the previous definition of ℓ_i if $i \in S$) and let P_i be the path represented by the diagram

$$i \xrightarrow{f_{\ell_i(1)}} i \xrightarrow{f_{\ell_i(2)}} \cdots \xrightarrow{f_{\ell_i(\|\ell_i\|-1)}} i \xrightarrow{f_{\ell_i(\|\ell_i\|)}} t_i,$$

so that $i(P_i) = i$ and $j(P_i) = \sigma(i)$. Then, letting $\mathcal{P} = \{P_i : i \in I\}$, we have $I(\mathcal{P}) = I$ and $J(\mathcal{P}) = \sigma(I) = J$, i.e. $d(\mathcal{P}) = d$.

Now let

$$F = F(\mathcal{P}) = \sum_{i \in I} \left[\sum_{k=1}^{\|\ell_i\|-1} \{(i, i, f_{\ell_i}(k))\} + \{(i, t_i, f_{\ell_i}(\|\ell_i\|))\} \right],$$

and let $\mathcal{P}' \neq \mathcal{P}$ be a collection of paths such that $F(\mathcal{P}') = F$ and $d(\mathcal{P}') \in D$. Now the only edges $(i, t, q) \in |F|$ such that $i \neq t$ are those of the form $e_i := (i, t_i, f_{\ell_i}(\|\ell_i\|))$ ($i \in I \setminus S$), which form a directed graph that contains no paths of length greater than one. This implies that if P'_i denotes the unique element of \mathcal{P}' such that $e_i \in |F(P'_i)|$, then:

- for all $i \in I \setminus S$, the diagram representing P'_i is of the form

$$(3.4) \quad i \rightarrow \cdots \rightarrow i \rightarrow t_i \rightarrow \cdots \rightarrow t_i,$$

and in particular $i(P'_i) = i$ and $t(P'_i) = t_i$;

- for all $P' \in \mathcal{Q} := \mathcal{P}' \setminus \{P'_i : i \in I \setminus S\}$, the diagram representing P' is of the form

$$(3.5) \quad i \rightarrow \cdots \rightarrow i,$$

where $i = i(P_i) = t(P_i)$.

Now since $d(\mathcal{P}') \in D$, $I(\mathcal{P}')$ is a set. Thus the paths P'_i ($i \in I \setminus S$) are distinct, and

$$(3.6) \quad i(P') \notin I \setminus S \text{ for all } P' \in \mathcal{Q}.$$

It follows from (3.6) that $t(P') \notin I \setminus S$ for all $P' \in \mathcal{P}'$, so we have $J'[i] = \emptyset$ for all $i \in I \setminus S$, where we write $d(\mathcal{P}') = (I', J')$. So

$$\{i' \in I' : J'[i'] \neq \emptyset\} = \{i(P') : P' \in |\mathcal{Q}|\} \leq \sum_{P' \in \mathcal{Q}} \{i(P')\}$$

and thus

$$f(d(\mathcal{P}')) \leq \sum_{P' \in \mathcal{Q}} \min_{\ell \in J'[i(P')]} \|\ell\| \leq \sum_{P' \in \mathcal{Q}} \|\ell(P')\| = \sum_{P' \in \mathcal{Q}} \#(F(P')) = \#(F) - \sum_{i \in I \setminus S} \#(F(P'_i)).$$

On the other hand, combining (3.6) with the fact that all edges of P_i have initial vertex i gives that $F(P_i) \leq F(P'_i)$ for all $i \in I \setminus S$. So

$$f(d(\mathcal{P}')) \leq \#(F) - \sum_{i \in I \setminus S} \#(F(P'_i)) \leq \#(F) - \sum_{i \in I \setminus S} \#(F(P_i)) = f(d(\mathcal{P})).$$

If strict inequality holds, we are done, so suppose equality holds. Then $F(P_i) = F(P'_i)$ for all $i \in I \setminus S$. Since $\sum_{P' \in \mathcal{Q}} F(P') = \sum_{i \in S} \sum_{k=1}^{\|\ell_i\|} \{(i, i, f_{\ell_i}(k))\}$, the injectivity of the map $\mathcal{P}' \ni P' \mapsto i(P')$ implies that we can write $\mathcal{Q} = \{P'_i : i \in S\}$ for some paths P'_i ($i \in S$) satisfying $F(P_i) = F(P'_i)$. So $\mathcal{P}' = \{P'_i : i \in I\}$, and $F(P_i) = F(P'_i)$ for all $i \in I$. We claim that $P_i = P'_i$ for all $i \in I$. Indeed, fix $i \in I$. Due to the diagram forms (3.4) and (3.5), the paths P_i and P'_i agree except possibly for their labels $\ell = \ell(P_i)$ and $\ell' = \ell(P'_i)$. But since $F(P_i) = F(P'_i)$, we have $\sum_{k=1}^{\ell} \{f_{\ell}(k)\} = \sum_{k=1}^{\ell'} \{f_{\ell'}(k)\}$. But since the polynomials $\pi(p_1), \dots, \pi(p_n)$ are assumed to be distinct, this implies that $\ell = \ell'$ and thus that $P_i = P'_i$. Since i was arbitrary, we get $\mathcal{P}' = \mathcal{P}$, a contradiction. \triangleleft

To finish the proof, we will show that $\mathbf{e}_d \in \mathcal{V}$ for all $d \in D$ by using strong induction on the partial order $<$. (Recall that \mathcal{V} denotes the linear span of $\phi \circ \psi(\mathcal{A})$, for which we must show that $\mathcal{V} = \mathcal{C}$.) Fix $d \in D$ and suppose that for all $d' < d$, we have $\mathbf{e}_{d'} \in \mathcal{V}$. Let the collection of paths \mathcal{P} be as in Lemma 3.1, and let $F = F(\mathcal{P})$. Let $\mathbf{v}_F \in \mathcal{C}$ be the coefficient of $\prod_{e \in F} A_e$ that appears in $\phi \circ \psi(A)$, thought of as a polynomial function of A . Then we have

$$\mathcal{V} \ni \mathbf{v}_F = \sum_{\substack{\mathcal{P}' \\ F(\mathcal{P}')=F}} (\pm 1) \mathbf{e}_{d(\mathcal{P}')} = (\pm 1) \mathbf{e}_d + \sum_{d' < d} \sum_{\substack{\mathcal{P}' \\ d(\mathcal{P}')=d' \\ F(\mathcal{P}')=F}} (\pm 1) \mathbf{e}_{d'}.$$

Since by induction $\mathbf{e}_{d'} \in \mathcal{V}$ for all $d' < d$, solving for \mathbf{e}_d gives $\mathbf{e}_d \in \mathcal{V}$, completing the proof.

4. CONCLUDING REMARKS AND FURTHER QUESTIONS

4.1. An algorithm for computing weak nonplanarity. In [4, §7.1], the matrix version of Baker's conjecture is motivated by the fact that it is a "special case" of the the question of "how one can in general show that a given [manifold] is weakly nonplanar". This appears to be a broadly philosophical question to which there can be no ultimate precise answer. In addition to motivating the matrix version of Baker's conjecture, this philosophical question also motivates the following mathematical question which can be regarded as a certain way of making the philosophical question precise: is there an algorithm for determining whether a given manifold is weakly nonplanar, based on input parameters which describe the

manifold uniquely in some way (e.g. via a parameterization)? In fact, this mathematical question appears to already have been hinted at in [4, §7.1], where it is stated that “finding an algorithmic way to verify weak nonplanarity seems to be an open problem”.⁸ In the next paragraph, we prove the existence of such an algorithm under the assumption that the manifold is parameterized by an algebraic mapping. However, we warn that this result is not very useful for practical purposes. In particular, it cannot be used to prove the matrix version of Baker’s conjecture, since that conjecture has an infinite list of submanifolds that need to be proven nonplanar, and it is impossible to run the algorithm on all those manifolds simultaneously. Still, the algorithmically checkable nature of weak nonplanarity it is an interesting theoretical result in its own right, complementing the main results of this note.

Recall (cf. [4, (2.5)]) that a connected real-analytic manifold $M \subseteq \mathcal{M} = \mathcal{M}_{m,n}$ is said to be *weakly nonplanar* if it is not contained in any set of the form

$$\mathcal{H}_{A,B} = \{Y \in \mathcal{M} : \det(AY + B) = 0\},$$

where $A \in \mathcal{M}_{n,m}$ and $B \in \mathcal{M}_{n,n}$ are not both zero. It is shown in [4, Corollary 2.4] that every weakly nonplanar manifold is strongly extremal. Now suppose that the manifold M is given in the form $M = \mathbf{f}(\mathbb{R}^k)$, where $\mathbf{f} : \mathbb{R}^k \rightarrow \mathcal{M}_{m,n}$ is a polynomial map. Then M is weakly nonplanar if and only if there exist A, B , not both zero, such that for all $\mathbf{x} \in \mathbb{R}^k$, $\det(A\mathbf{f}(\mathbf{x}) + B) = 0$. This condition is expressible in first-order quantifier logic over the real numbers. Thus there is an algorithm [3, Algorithm 11.14] to determine whether this condition is true or not.

We remark that if A, B are allowed to be complex matrices rather than real ones, then there is an algorithm for determining the truth of the condition which is somewhat simpler than the one described in [3, Algorithm 11.14]. In practice this is probably the better algorithm to use, since the case where M is weakly nonplanar even though it is contained in a manifold of the form $\mathcal{H}_{A,B}$ with A, B complex matrices is rare.⁹ For this algorithm, first write $\det(A\mathbf{f}(\mathbf{x}) + B) = \mathbf{g}(A, B) \cdot \psi \circ \mathbf{f}(\mathbf{x})$ for some polynomial function \mathbf{g} , where ψ is the Plücker embedding. Then compute the linear span of $\psi \circ \mathbf{f}(\mathbb{R}^k)$; this is possible since $\psi \circ \mathbf{f}$ is a polynomial. Write the linear span in the form $\langle \mathbf{z}_1, \dots, \mathbf{z}_n \rangle$. Then the question is whether there exist complex matrices A, B , not both zero, such that

$$\mathbf{g}(A, B) \cdot \mathbf{z}_i = 0 \quad \forall i = 1, \dots, n.$$

This is a finite number of polynomial equations in the entries of the matrices A and B . Hilbert’s Nullstellensatz implies that the existence of such matrices is equivalent to the condition that there is some entry x of A or B such that no power of x is in the ideal I generated by the polynomials $\mathbf{g}(A, B) \cdot \mathbf{z}_i$ ($i = 1, \dots, n$). Given any entry x , it can be determined whether this condition holds by computing a Gröbner basis for I with respect to some monomial ordering with respect to which the monomials x, x^2, \dots are ordered lower than all other monomials, and then checking whether any power of x is in the Gröbner basis.

4.2. Algorithms for computing extremality and strong extremality within the class of rational submanifolds. Subsequent to the question asked in [4, §7.1], Aka, Breuillard, Rosenzweig, and de Saxcé found a sufficient condition for extremality which is weaker than weak nonplanarity [1, Theorem 3.1]. Their result is optimal in the sense that it completely characterizes extremality within the class of rational submanifolds (i.e. those defined by polynomial equations with rational coefficients), indicating that extremal manifolds which do not satisfy their condition must be extremal for “Diophantine” reasons rather than “geometric” ones. The above argument goes through with minor modifications to show that there is an algorithm for determining whether their condition holds for any given manifold. In particular, there is an algorithm for determining whether any rational submanifold of \mathcal{M} is extremal or not.

⁸In context, it is somewhat unclear precisely what is meant by this statement.

⁹It happens for example when

$$M = \left\{ \begin{bmatrix} x & y \\ -y & x \end{bmatrix} : x, y \in \mathbb{R} \right\}, \quad A = \begin{bmatrix} 1 & \sqrt{-1} \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & \sqrt{-1} \end{bmatrix}.$$

A forthcoming joint work of the authors with Breuillard and de Saxcé will include the analogous version of their result for strong extremality. The above argument would also apply to this setting, producing an algorithm for determining whether any rational submanifold of \mathcal{M} is strongly extremal or not.

4.3. Open question. We end with the following question:

Question 4.1 (Cf. Remark 1.11). Fix $m, n, r \in \mathbb{N}$ and let $(p_\ell)_{\ell=1}^n$ be a finite sequence in M_r^* , with $\pi(p_1), \dots, \pi(p_n)$ not necessarily distinct. Is it true that the manifold $\psi_{m,n,r}^{(\mathbf{P})}((\mathcal{M}_{m,m})^r)$ must be

- (i) weakly nonplanar?
- (ii) strongly extremal?
- (iii) extremal?

(The distinctness of $\pi(p_1), \dots, \pi(p_n)$ is a necessary and sufficient condition for the strong nonplanarity of $\psi_{m,n,r}^{(\mathbf{P})}((\mathcal{M}_{m,m})^r)$, as shown by combining Theorem 1.10 with Remark 1.11.)

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