Signatures of Supersymmetry in the $\nu = 5/2$ Fractional Quantum Hall Effect

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The Moore-Read state, one of the leading candidates for describing the fractional quantum Hall effect at filling factor $\nu = 5/2$, is a paradigmatic p-wave superconductor with non-Abelian topological order. Among its many exotic properties, the state hosts two collective modes: a bosonic density wave and a neutral fermion mode that arises from an unpaired electron in the condensate. It has recently been proposed that the descriptions of the two modes can be unified by postulating supersymmetry (SUSY) that relates them in the long-wavelength limit. Here we extend the SUSY description to construct wave functions of the two modes on closed surfaces, such as the sphere and torus, and we test the resulting states in large-scale numerical simulations. We demonstrate the equivalence in the long-wavelength limit between SUSY wave functions and previous descriptions of collective modes based on the Girvin-MacDonald-Platzman ansatz, Jack polynomials, and bipartite composite fermions. Leveraging the first-quantized form of the SUSY wave functions, we study their energies using the Monte Carlo method and show that realistic $\nu = 5/2$ systems are close to the putative SUSY point, where the two collective modes become degenerate in energy.

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Introduction.—Exotic topological properties of fractional quantum Hall (FQH) fluids, such as quantized Hall resistance [1], and excitations with fractional charge [2,3] and fractional statistics [4–8], have been the subject of major research efforts over the past decades. More recently, FQH states have come into renewed focus due to their unique *geometric* properties such as the Hall viscosity [9–12] and the Girvin-MacDonald-Platzman (GMP) magnetoroton mode [13,14]. The latter is a low-lying collective mode of any gapped FQH fluid and corresponds to a bosonic excitation that can be viewed as a density wave, analogous to the roton in superfluid ^{4}He [15]. The GMP mode has been observed in several experiments using inelastic light scattering and surface acoustic waves [16–18].

However, certain FQH states, such as the one observed at filling fraction $\nu = 5/2$ [19], can possess an *additional* collective mode, suggested by the numerical simulations [20–22] [see Fig. 1(a) for a schematic illustration]. The additional collective mode in the $\nu = 5/2$ state is naturally accounted for by the Moore-Read (MR) wave function [23] and its particle-hole conjugate the anti-Pfaffian state [24,25], two of the leading candidates for understanding the incompressible state at half filling of the second Landau level (LL). The MR state represents a p-wave superconductor [26,27]

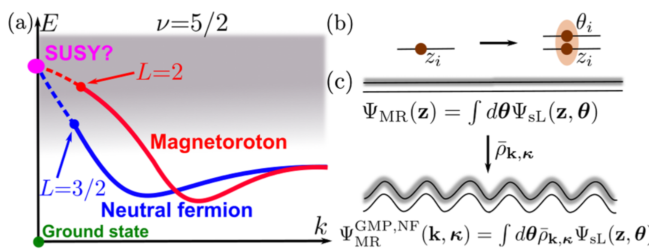

FIG. 1. (a) Sketch of the energy spectrum of a gapped $\nu = 5/2$ FQH state with two neutral collective modes, the magnetoroton and neutral fermion. In the long-wavelength limit $k\ell \to 0$, the two modes carry angular momenta $L = 2$ and $L = 3/2$, respectively, and their degeneracy would give rise to an emergent SUSY. (b) Both modes can be compactly described by introducing, for each particle's coordinate z_i, its SUSY partner with a Grassmann coordinate θ_i [33]. (c) When the total number of particles is even, the Moore-Read state is recovered by integrating out $\boldsymbol{\theta}$ variables (depicted by shading) from Ψ_{sL} the Laughlin wave function at $\nu = 1/2$ in the superplane [34]. Similarly, the wave functions for the GMP and NF modes are obtained by acting with the projected density operator $\bar{\rho}_{\mathbf{k},\boldsymbol{\kappa}}$, which generates a density wave in superspace.

of composite fermions, i.e., the bound states of electrons and vortices [28]. Hence, via analogy with Bardeen-Cooper-Schrieffer (BCS) superconductors [29], the MR state possesses a fermionic collective excitation—the "neutral fermion" (NF)—which is the analog of the Bogoliubov–de Gennes quasiparticle [27,30]. The applications of the MR state in topological quantum computation [31] rest critically on the gap of the NF mode, as the latter can be excited in the process of "fusion" of two elementary excitations of the MR state that simulate the action of quantum gates [32].

In the long-wavelength limit, corresponding to momenta $k \ll \ell^{-1}$, where ℓ is the magnetic length, there is a sharp distinction between the bosonic GMP and fermionic NF modes: the former carries integral angular momentum $L = 2$ on the sphere [35,36], while the latter carries *half odd-integral* $L = 3/2$ [21,37,38]. While the GMP mode has a simple description in terms of acting the lowest LL (LLL) projected density operator on the ground state [13], the microscopic description of the NF mode has required far more elaborate constructions [37–41] that are not amenable to numerical studies in large systems. Recently, Ref. [33] proposed a unified description of the GMP and NF modes in the MR state based on the two modes being supersymmetry (SUSY) partners; see Figs. 1(b) and 1(c) (see also Ref. [42] for the consequences of SUSY for the edge physics of the MR state). While this provides an elegant description of both modes via generalization of the GMP ansatz to superspace, the resulting description has not been tested in numerics. Moreover, it remains unclear if the physical $\nu = 5/2$ system satisfies the assumption of an emergent SUSY.

In this Letter, we extend the construction of Ref. [33] to formulate fully antisymmetric, LLL-projected wave functions for both the GMP and NF modes adapted to closed manifolds, i.e. the sphere and torus, that can be evaluated for large system sizes. We numerically demonstrate the equivalence of these wave functions in the long-wavelength limit to alternative descriptions of the modes based on Jack polynomials [37] and bipartite composite fermions [36,38,39]. We test the existence of SUSY *a posteriori*, by evaluating the energies of the two collective modes in the long-wavelength limit using the Monte Carlo method. While the pure Coulomb interaction at $\nu = 5/2$ gives rise to a weak breaking of SUSY, with the SUSY gap being about 20% of the excitation gap, tuning the interaction by slightly enhancing the V_1 Haldane pseudopotential [43,44] leads to the restoration of SUSY. Importantly, the two modes become degenerate in the long-wavelength limit at approximately the same value of V_1 that maximizes the ground state overlap with the MR wave function [45]. Our results suggest that the SUSY structure is intrinsically present in spectral properties of the $\nu = 5/2$ state, allowing us to probe the conditions for its emergence in large-scale numerics. Moreover, our work provides a foundation for developing an effective field theory of paired FQH states that incorporates SUSY as an emergent symmetry of infrared physics.

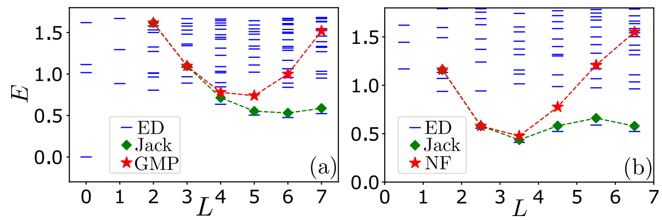

FIG. 2. The energy spectrum of the three-body parent Hamiltonian of the Moore-Read state on the sphere [30] for $N = 14$ (a) and $N = 13$ electrons (b). Blue dashes are energy levels obtained by exact diagonalizations (ED), green diamonds are the energies of collective modes constructed via Jack polynomials [37], while red stars are the energies of the GMP wave function in (a) and the NF wave function in (b). At small L, both collective modes merge with the continuum of the spectrum. The Jack construction captures the entire collective modes, while the GMP and NF wave functions are accurate only in the long-wavelength limit.

SUSY wave functions on the sphere.—We first construct the SUSY wave functions in the spherical geometry [43] (see Supplemental Material [46] for the corresponding construction on the torus). We assume there are N electrons confined to the surface of a sphere, with a Dirac monopole at the center, emanating a radial magnetic flux of strength $2Qhc/e$. The radius of the sphere is $R = \sqrt{Q}\ell$. The total angular momentum L and its z component M are good quantum numbers. The magnitude of the planar wave vector k is given by $k = L/R$. In terms of the spinor coordinates u_j, v_j of the jth electron [43], the MR wave function [23] on the sphere is

$$\Psi_{\rm MR} = {\rm Pf}\left(\frac{1}{u_i v_j - u_j v_i}\right)\Phi_1^2, \tag{1}$$

where Pf(A) stands for the Pfaffian of a skew-symmetric matrix A and $\Phi_1 = \prod_{i<j}(u_i v_j - u_j v_i)$ denotes the Laughlin-Jastrow factor [44].

A neutral density wave excitation of the MR state, carrying angular momentum L, is described by the GMP ansatz [13], $\Psi_{\rm GMP}(L) = \mathcal{P}_{\rm LLL}\hat{\rho}_{L,M}\Psi_{\rm MR}$. Here $\hat{\rho}_{L,M} = \sum_{i=1}^{N} Y_{L,M}(\theta_i, \phi_i)$ is the density operator in the spherical geometry, expressed in terms of spherical harmonics $Y_{L,M}$ [63], and $\mathcal{P}_{\rm LLL}$ is the LLL projection operator. Since the states with different M are degenerate due to rotational symmetry on the sphere, we can set $M = -L$ for simplicity, in which case (up to normalization) we have $Y_{L,-L}(u_j, v_j) \propto v_j^L \bar{u}_j^L$, where the bar denotes complex conjugation. We see that $\Psi_{\rm GMP}(L)$ needs to be explicitly projected to the LLL and this must be done carefully to maintain the efficiency of its evaluation via Monte Carlo simulations. Because of this obstacle, previous approaches [64] could only access the dispersion of the GMP mode via the static structure factor of the ground state.

Remarkably, we find that the LLL projection of the GMP mode of the MR state can be performed *exactly*, with the resulting wave function given by

$$\Psi_{\rm GMP}(L) = \sum_{m=1}^{N} v_m^L \Phi_1^2 {\rm Pf}\left(\frac{\tilde{U}^L_{m,i,j}}{u_i v_j - u_j v_i}\right), \tag{2}$$

where $\tilde{U}^L_{m,i,j}$ is a function of all the electrons' coordinates, defined through a recursion relation (see Supplemental Material [46]). The derivation of Eq. (2) is inspired by the Jain-Kamilla method of projecting composite fermion wave functions [65,66]. However, while the latter method yields a wave function different from that obtained from direct projection, we emphasize that the projection in Eq. (2) results in exactly the same wave function as the one obtained from direct projection. This form allows us to efficiently compute properties of GMP states in large systems, without relying on the static structure factor.

Our exact projection method furthermore allows extending the NF wave function, formulated as a SUSY partner of the GMP mode in the infinite plane [33], to the spherical geometry. The proposed NF wave function from Ref. [33], after stereographic mapping [67], is given by

$$\Psi_{\rm NF}(L) = \mathcal{P}_{\rm LLL}\left[\sum_m (-1)^m \frac{1}{u_m} \widetilde{\rm Pf}_m\left(\frac{1}{u_i v_j - u_j v_i}\right) Y_{L,-L}(u_m, v_m)\Phi_1^2\right], \tag{3}$$

where $\widetilde{\rm Pf}_m$ denotes the Pfaffian of the same matrix as in Eq. (1) but with mth electron unpaired, i.e., $i, j \neq m$, assuming N is odd.

The wave function in Eq. (3) is clearly not analytic due to $1/u_m$ factor. To fix this issue, we replace $1/u_m \to \bar{u}_m$ in Eq. (3), which preserves the total magnetic flux and the shift [68], resulting in a wave function that is an eigenstate of total angular momentum L^2. Following a similar procedure to the GMP mode, we obtain the NF wave function projected to the LLL:

$$\Psi_{\rm NF}(L) = \sum_m (-1)^m v_m^L \Phi_1^2 \widetilde{\rm Pf}_m\left(\frac{1}{u_i v_j - u_j v_i}\right) P_m(L+1), \tag{4}$$

where P_m is defined through a recursion relation in Ref. [46].

Testing SUSY wave functions.—To assess the accuracy of the constructed SUSY-based wave functions, in Fig. 2 we compare their energies with the exact spectrum of the three-body parent Hamiltonian of the MR state [30], as well as against the modes obtained from Jack polynomials [37]. The three-body energies of the GMP and NF wave functions are evaluated in Fock space. The GMP wave function can be directly obtained in Fock space by applying the LLL-projected density operator $\hat{\rho}_{L,M}$ on the MR state. To obtain the NF wave function in Fock space, we use the method of Refs. [36,69] which involves evaluating all the relevant L^2 eigenstates and expanding Eq. (4) in that basis.

First, we note that both the $L = 1$ GMP and $L = 1/2$ NF wave functions vanish with machine precision accuracy, consistent with previous findings [36–39]. Second, the long-wavelength part of the dispersion shows excellent agreement between the GMP (NF) states and the Jack polynomial states from Ref. [37]. In fact, by numerically evaluating the overlap between the Jack states and the SUSY states above, we find the Jack states are *identical* to the GMP states in Eq. (2) at $L = 2$, 3 for even $N \leq 14$, while the Jack states and NF states in Eq. (4) are identical at $L = 3/2, 5/2$ for odd $N \leq 13$. Furthermore, all these states appear to be identical to bipartite composite fermion states from Refs. [36,38], as they give the same overlap with eigenstates of the MR three-body interaction and the Coulomb interaction at $\nu = 5/2$ (up to the second digit quoted in Ref. [36]). At higher values of L, we expect all schemes to produce distinct states. For example, the overlap between the Jack and SUSY-based states decreases at higher values of L [46], and the GMP and NF wave functions no longer capture the exact dispersion at finite momenta $k\ell \gtrsim 1$ (i.e., $L \gtrsim N$). In the following, we focus on the long-wavelength part of the dispersion, where all the constructions result in the same state, as shown above, allowing us to put variational bounds on the conditions for SUSY to emerge.

Emergence of SUSY in realistic systems.—The SUSY construction assumes that the GMP and NF modes have the same energy in the long-wavelength limit. Using the wave functions constructed above, we can probe the SUSY degeneracy in the limit $k\ell \to 0$ by evaluating the energies via the Monte Carlo method on large spheres with $R \gg \ell$ [46]. The dispersions of the GMP and NF modes are shown in Figs. 3(a)–3(c) for the Coulomb interaction projected to the second LL. Furthermore, we explore the neighborhood of the Coulomb interaction by adding a small amount of δV_1 pseudopotential [43], which captures some of the modifications to the interaction potential due to the finite width of the sample and LL mixing.

Figure 3(a) shows that the energy of the GMP mode for the second LL Coulomb interaction is higher than that of the NF mode. However, the GMP dispersion is fairly flat while the energy of the NF mode increases as k decreases. This allows for the possibility of these two modes to meet at $k = 0$. To check this, we performed a finite-size extrapolation of the energies of $L = 2$ GMP and $L = 3/2$ NF states as a function of $1/N$ in Fig. 3(d). The GMP energy is slightly higher than the NF energy in the thermodynamic limit, by $0.005(2)e^2/\epsilon\ell$, which is about 20% of the $\nu = 5/2$ excitation gap [70].

The addition of a small amount of $\delta V_1 \approx 0.03$ significantly alters the shape of the modes: it makes the GMP

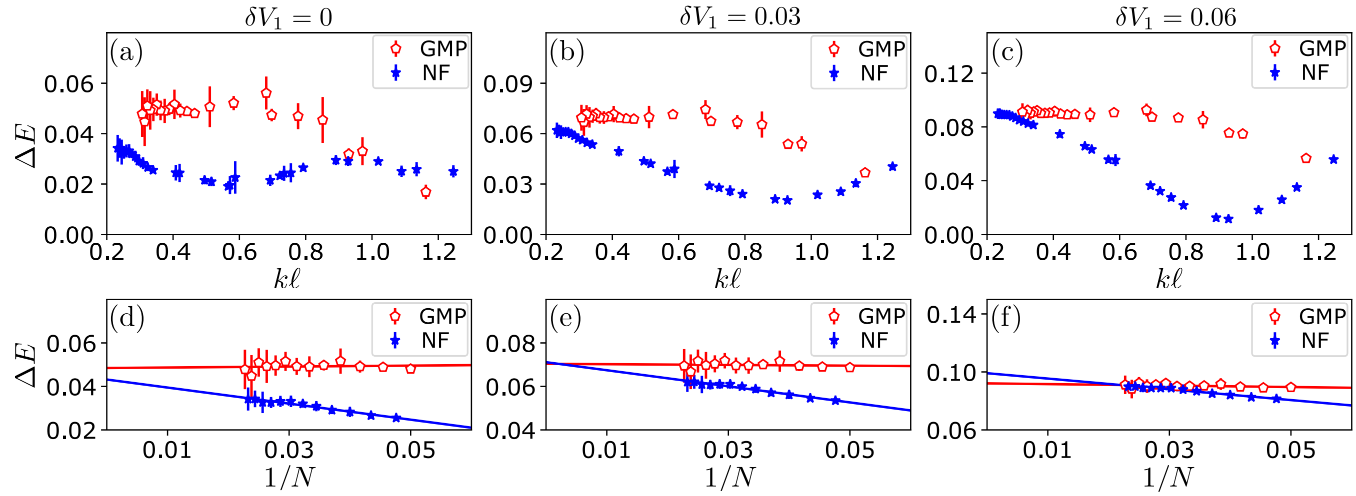

FIG. 3. Dispersion of the GMP and NF modes (a)–(c) and the finite-size scaling of the energies of $L = 2$ and $L = 3/2$ states (d)–(f). The data are for $20 \leq N \leq 44$ electrons for the Coulomb interaction projected to the second LL, with added pseudopotential δV_1 [(a),(d) pure Coulomb interaction ($\delta V_1 = 0$); (b),(e) $\delta V_1 = 0.03$; (c),(f) $\delta V_1 = 0.06$]. The energies are quoted relative to the ground state energy, i.e., $\Delta E = E - E_0$, in units of $e^2/\epsilon\ell$. Here E_0 is the energy of the MR state at the same electron number for the GMP mode, while for the NF mode E_0 is the average value of the MR state energy for $N + 1$ and $N - 1$ electrons.

mode flatter and the NF mode rise faster at small k; see Fig. 3(b). This completely suppresses the gap between the GMP mode and NF mode in the $k\ell \to 0$ limit, as shown in Fig. 3(e). Note that around the same value $\delta V_1 \approx 0.03$, the overlap of the exact ground state with the MR state is maximized [45,71]. Thus, we conclude that adding a small amount of V_1 pseudopotential—approximately 10% of its value in the second LL—enhances the overlap with the MR state while at the same time gives rise to the SUSY degeneracy. Finally, upon an even further increase in δV_1, shown in Figs. 3(c)–3(f), the gap becomes negative, corresponding to the GMP mode being lower in energy than the NF mode. This suggests there is a finite range of δV_1 where SUSY degeneracy can be observed.

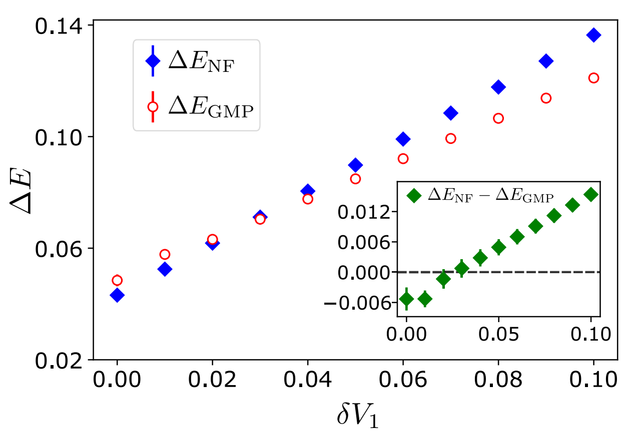

FIG. 4. The extrapolated gaps of the $L = 2$ GMP mode and $L = 3/2$ NF mode in $N \to \infty$ limit, for different values of δV_1 pseudopotential added to the second LL Coulomb interaction. The inset shows the difference between the gaps. The error only includes the regression error from finite-size extrapolations while ignoring the small statistical error of the Monte Carlo method.

A more systematic analysis of the effect of δV_1 is presented in Fig. 4, which shows the extrapolated gaps of the $L = 2$ GMP and the $L = 3/2$ NF states, as a function of δV_1. The gaps vary linearly with δV_1, and the slope of the NF gap is larger than that of the GMP gap. This indicates that the NF mode is more sensitive to the hard-core interaction than the GMP mode. We show the difference between the GMP and NF gaps in the long-wavelength limit in the inset of Fig 4. As a result of the linearity of the GMP and NF energies, their difference is also linear. By varying δV_1, the difference can be tuned to zero and this happens near $\delta V_1 \approx 0.03$. Increasing δV_1 beyond this value breaks the SUSY degeneracy. However, in the limit of large δV_1, we no longer expect our wave functions to provide a good description of the underlying physics, because the effective interaction becomes increasingly LLL-like, which ultimately induces a transition to the gapless composite fermion Fermi liquid [45,72].

Conclusions.—In this Letter, we have constructed the wave functions for the neutral collective modes of the MR state in the spherical and torus geometries based on the SUSY description. We have shown that, in the long-wavelength limit, these wave functions encompass the previous independent constructions, with the advantage of allowing numerical calculations in large systems. The variational energies of these wave functions suggest that the $\nu = 5/2$ FQH state is close to the SUSY point and can be driven to it by adding δV_1. We have confirmed that similar conclusions apply to other modifications of the interaction, e.g., via longer-range pseudopotentials or by increasing the width of the sample, and they also hold for the MR state of bosons at $\nu = 1$ [46].

The LL-projected Coulomb interaction is particle-hole symmetric; hence, our conclusions apply to both the MR and anti-Pfaffian states. One obvious question is the effect of particle-hole symmetry-breaking interactions on SUSY, such as the three-body interaction that distinguishes the MR and anti-Pfaffian states. Unfortunately, this question is challenging to address via the method presented here, due to a lack of efficient ways of evaluating three-body interactions in real space. A natural source of three-body interactions is LL mixing, which we have neglected above. It would be interesting to explore its effect on the collective modes in order to obtain a closer comparison with experiments.

On the theory side, our demonstration of SUSY at the microscopic level calls for a development of an effective field theory [73–75] that incorporates SUSY and treats the two collective modes on the same footing. Beyond the two collective modes discussed here, it would be interesting to understand if SUSY leaves an imprint on the rest of the spectrum of the $\nu = 5/2$ state, e.g., its dynamical response functions. For example, the weak breaking of SUSY is expected to give rise to a gapless Goldstino mode in the bulk [76], which could be probed in numerics. Other paired states such as Haldane-Rezayi [77,78], the Halperin 331 state [79], the $\mathbb{Z}_n$ superconductor of composite bosons [80], and the “permanent state” [23,81] are all expected to carry a neutral excitation in addition to the GMP mode; hence, it would be interesting to understand if SUSY could emerge in them. Finally, on the experimental front, it would be important to develop protocols for detecting the NF mode, which is invisible to conventional optical probes [82].

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