A hidden invariance algebra of Maxwell's equations and the conservation of all Lipkin's zilches from symmetries of the standard electromagnetic action

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In 1964, Lipkin discovered a set of conserved quantities for free electromagnetism without a clear physical interpretation, known as the *zilches*. In 2010, Tang and Cohen realized that one of the zilches, termed as *optical chirality*, provides a measure of the handedness of light, motivating novel investigations into the interactions of light with chiral matter. Although the *zilch symmetries* of Maxwell's equations underlying the conservation of the zilches are known, the question of how to explicitly derive all zilch conservation laws from symmetries of the standard free EM action using Noether's theorem has been answered only in the case of optical chirality. In this Letter, we provide the answer to this question by showing that the zilch symmetries leave invariant the standard free EM action.

In the rest of the article, we provide new insight concerning the conservation of the zilches and their underlying symmetries. First, we show that the zilch symmetries belong to the enveloping algebra of a *"hidden" invariance algebra* of free Maxwell's equations in potential form. The "hidden" algebra closes on $so(6,\mathbb{C})_{\mathbb{R}}$ up to gauge transformations of the four-potential A_μ. The generators of the "hidden" algebra consist of familiar conformal symmetry transformations and certain *"hidden" symmetry transformations* of A_μ. We discuss the generalization of these "hidden" symmetries of Maxwell's equations in the presence of a material four-current, J^μ. The "hidden" symmetries are also discussed for the theory of a complex Abelian gauge field (this is related to the complex formulation of duality-symmetric electromagnetism). Finally, we show that the zilch symmetries of the standard free EM action can be extended to zilch symmetries of the standard interacting action, S', by considering simultaneous transformations of both A_μ and J^μ. This allows us to give a new derivation of the continuity equation for optical chirality in the presence of electric charges and currents, while we also derive new continuity equations for the rest of the zilches.

I. INTRODUCTION

Noether's seminal theorem [1] is the cornerstone in understanding the deep connection between symmetries of physical theories and conservation laws. Starting from continuous symmetries of the action functional of a theory, Noether's theorem can be used to derive conservation laws for the associated Euler-Lagrange equations. In relativistic field theories, such as electromagnetism in Minkowski spacetime, the knowledge of a symmetry leads to a Noether (four-)current, V^μ, which is conserved ($\partial_\mu V^\mu = 0$). This conservation holds for fields satisfying the Euler-Lagrange equations - i.e. for on-shell field configurations - and the corresponding Noether charge, $Q = \int d^3x V^0$, is time-independent.

An example of little-known time-independent quantities in free electromagnetism is given by the ten *zilches* that were discovered by Lipkin in 1964 [2]. One of the zilches, now known as *optical chirality*, started drawing renewed theoretical and experimental interest in 2010, when Tang and Cohen realized that this particular zilch provides a measure of the chirality (or handedness) of light [3]. The optical chirality density for the free electromagnetic (EM) field is [2, 3]

$$C = \frac{1}{2}\left(-\boldsymbol{E}\cdot\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{B}\cdot\frac{\partial \boldsymbol{E}}{\partial t}\right), \tag{1}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, respectively [4]. (Throughout this Letter, we adopt the system of units in which the speed of light and the permittivity of free space are $c = \varepsilon_0 = 1$.) The flux of optical chirality is given by the three-vector

$$\boldsymbol{S} = \frac{1}{2}\boldsymbol{E}\times\frac{\partial \boldsymbol{E}}{\partial t} + \frac{1}{2}\boldsymbol{B}\times\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2}$$

while the differential conservation law for optical chirality [2]

$$\frac{\partial}{\partial t}C + \boldsymbol{\nabla}\cdot\boldsymbol{S} = 0 \tag{3}$$

is satisfied if $\boldsymbol{E}$ and $\boldsymbol{B}$ obey the free Maxwell equations

$$\begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{B} &= \frac{\partial \boldsymbol{E}}{\partial t}, \quad & \boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\ \boldsymbol{\nabla}\cdot\boldsymbol{E} &= 0, & \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0. \end{aligned} \tag{4}$$

Optical chirality is given by the integral of C over the space, $\int d^3x\, C$, and is a constant of motion for free electromagnetism [2].

In Ref. [3], Tang and Cohen demonstrated that, in the presence of an EM field, the dissymmetry in the excitation rate of two small chiral molecules that are related to

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each other by mirror reflection is determined by the optical chirality. These findings have motivated novel investigations into chiral light-matter interactions [3, 5–16]. Understanding these interactions is very important in various disciplines. For example, it is known that deriving products of a given handedness in chemical reactions can be crucial - because molecules of a given handedness must be used in order to design drugs without negative side-effects [17] - and chiral light has been suggested to serve as a useful tool in order to achieve this [18–20]. Applications of chiral light to the detection and characterization of chiral biomolecules have been also discussed [6]. As for the other nine zilches, recently, Smith and Strange shed light on the mystery of their physical meaning for certain topologically non-trivial vacuum EM fields [21].

Although the *zilch symmetries* - i.e. the symmetries underlying the zilch conservation laws - and their generalization have been discussed in previous works [22–31], there are still certain gaps concerning our mathematical understanding of them. Most importantly, there is a gap in the literature concerning the explicit derivation of all zilch conservation laws from symmetries of the standard free EM action using Noether's theorem. In this Letter, we fill this gap and we also provide new insight concerning the zilch symmetries. Before proceeding to the main part of this article, let us discuss what is already known concerning the zilch symmetries in Subsection I A, as well as review the main findings of the present article in Subsection I B. For later convenience, we present here our notation and conventions.

Conventions.—Greek tensor indices run from 0 to 3 and Latin tensor indices from 1 to 3. We follow the Einstein summation convention, while indices are raised and lowered with the mostly plus Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. A spacetime point in standard Minkowski coordinates is $x^\mu = (x^0, x^1, x^2, x^3) \equiv (t, x^i)$. The totally antisymmetric tensors in 4 and 3 dimensions are $\epsilon^{\mu\nu\rho\sigma}$ and ϵ^{ijk}, respectively ($\epsilon^{0123} = -\epsilon^{123} = -1$).

Let $A_\mu = (-\phi, \boldsymbol{A})$ denote the EM four-potential. The standard free EM action

$$S = \frac{1}{2}\int d^4x\, (\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B}),$$
$$\text{with}\quad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\phi, \quad \boldsymbol{B} = \nabla\times\boldsymbol{A}, \tag{5}$$

is expressed as

$$S = -\frac{1}{4}\int d^4x\, F^{\mu\nu}F_{\mu\nu}, \tag{6}$$

where the antisymmetric EM tensor is defined as $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ (with $F_{0i} = -E_i$ and $F_{ik} = \epsilon_{ikm}B^m$). We denote the dual EM tensor as ${}^\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. The free Maxwell's equations $\partial^\nu F_{\nu\mu} = 0$ are expressed in potential form as

$$\Box A_\mu - \partial_\mu\partial^\nu A_\nu = 0, \tag{7}$$

where $\Box = \partial^\nu\partial_\nu$. Because of the definition of $F_{\mu\nu}$ in terms of the four-potential, the equation

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0 \tag{8}$$

is identically satisfied. Equation (7), as well as the action (6), are invariant under infinitesimal gauge-transformations

$$\delta^{gauge}A_\mu = \partial_\mu a, \tag{9}$$

where a is an arbitrary scalar function.

A. What is known about the zilch symmetries?

The zilch conservation laws can be conveniently described in terms of the zilch tensor [2, 32]

$$Z^\mu{}_{\nu\rho} = -\,{}^\star F^{\mu\lambda}\partial_\rho F_{\lambda\nu} + F^{\mu\lambda}\,\partial_\rho\,{}^\star F_{\lambda\nu}. \tag{10}$$

This is conserved on-shell, $\partial^\rho Z^\mu{}_{\nu\rho} = 0$, and the ten time-independent quantities [2]:

$$\mathscr{Z}^{\mu\nu} = \mathscr{Z}^{\nu\mu} = \int d^3x Z^{\mu\nu 0}$$

are the ten zilches (see Section II for background material concerning the zilches). The optical chirality density (1) is related to the zilch tensor as $Z^{000} = 2C$.

At the level of free Maxwell's equations expressed in terms of the EM tensor, the zilch symmetries are known [25, 30]. More specifically, the *zilch symmetry transformations* of the EM tensor are [25, 30]

$$\Delta F_{\mu\nu} = \tilde{n}^\alpha n^\rho\, \partial_\alpha\partial_\rho{}^\star F_{\mu\nu}, \tag{11}$$

where $\tilde{n}^\alpha$ and n^ρ are two arbitrary constant four-vectors. These transformations are symmetries of free Maxwell's equations, i.e. if $F_{\mu\nu}$ is a solution, then so is $\Delta F_{\mu\nu}$. In Ref. [30], a complete classification of all independent local conservation laws of Maxwell's equations was given by using the methods described in Refs. [33, 34]. Using these methods, it was shown that the zilch symmetries (11) of free Maxwell's equations give rise to the conservation of the zilch tensor (10). However, in Ref. [30] the invariance of the standard EM action (6) was not discussed.

The zilch symmetries have also been studied in the case of duality-symmetric electromagnetism [24]. The duality-symmetric EM action is [23]

$$\tilde{S} = -\frac{1}{8}\int d^4x\, (F^{\mu\nu}F_{\mu\nu} + G^{\mu\nu}G_{\mu\nu}). \tag{12}$$

This theory is an extension of the standard EM theory as it has two four-potentials, A_μ and C_μ, and two EM tensors $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. The duality-symmetric theory coincides with the standard EM theory only after we impose the duality constraint $G_{\mu\nu} = {}^*F_{\mu\nu}$. In Ref. [24], following the reverse

Noether procedure, it was shown that the 'generalized' version of the zilch tensor:

$$\begin{aligned}\mathscr{Z}^{\mu}{}_{\nu\rho} = & -\frac{1}{2}G^{\mu\lambda}\partial_\rho F_{\lambda\nu} + \frac{1}{2}F^{\mu\lambda}\,\partial_\rho G_{\lambda\nu} \\ & -\frac{1}{2}G_\nu{}^\lambda\,\partial_\rho F_\lambda{}^\mu + \frac{1}{2}F_\nu{}^\lambda\,\partial_\rho G_\lambda{}^\mu \end{aligned} \tag{13}$$

is the Noether current corresponding to the following zilch symmetry transformations [24]:

$$\begin{aligned}\tilde{\Delta}A_\nu = & \; n^\rho \tilde{n}^\mu\,\partial_\rho G_{\mu\nu} \\ \tilde{\Delta}C_\nu = & -n^\rho \tilde{n}^\mu\,\partial_\rho F_{\mu\nu}. \end{aligned} \tag{14}$$

It has been shown that these transformations leave invariant the duality-symmetric action (12) [24]. Then, the conservation of the zilches follows from the fact that the tensor (13) coincides with the zilch tensor (20) of the standard EM theory if we apply the duality constraint.

The derivation of the zilch conservation laws from symmetries of alternative actions has been studied in Refs. [25, 28].

B. Filling a gap in the literature, main results of this article and outline

In order to derive all zilch conservation laws using Noether's theorem in the case of standard electromagnetism, one needs to find the zilch symmetry transformations of the four-potential that leave the standard action (6) invariant. It is easy to observe that the zilch symmetry transformations $\Delta F_{\mu\nu}$ [Eq. (11)] of free Maxwell's equations are induced by the following zilch transformations of the four-potential:

$$\Delta A_\nu = \; n^\rho \tilde{n}^\mu\,\epsilon_{\mu\nu\sigma\lambda}\,\partial^\sigma \partial_\rho A^\lambda = n^\rho \tilde{n}^\mu\;\partial_\rho\,{}^*F_{\mu\nu}, \tag{15}$$

with $\Delta F_{\mu\nu} \equiv \partial_\mu \Delta A_\nu - \partial_\nu \Delta A_\mu$ for on-shell field configurations. (The transformations (15) coincide with $\tilde{\Delta}A_\nu$ in Eq. (14) if we apply the duality constraint.) Interestingly, the study of the variation of the standard action (6) under the zilch transformations (15) has not been studied in the literature. This means that the following question is still open:

How can we derive all zilch conservation laws from symmetries of the standard free EM action using Noether's theorem?

In this Letter, we give the full answer to this question by showing that the zilch transformations (15) leave the standard EM action (6) invariant, and, then, we derive all zilch conservation laws using the standard Noether procedure (see, e.g. Ref. [35]).

Note that the only zilch conservation law that has hitherto been derived from symmetries of the standard action (5) is the one concerning the conservation of optical chirality [36]. In particular, Philbin showed that optical chirality is the Noether charge corresponding to the following symmetry transformations [36]:

$$\Delta\phi = 0, \qquad \Delta\boldsymbol{A} = \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{A}}{\partial t}. \tag{16}$$

This equation corresponds to a special case of the zilch symmetry transformation (15) with $\tilde{n}^\mu = n^\mu = \delta^\mu_0$. In this article we provide an alternative (and covariant) derivation of Philbin's [36] result for optical chirality.

Outline and main results. The basics concerning the zilch tensor and the zilches are reviewed in Section II. The derivation of all zilch conservation laws using the invariance of the standard action (6) under the zilch symmetries (15) is presented in Section III. Then, we proceed by providing new insight concerning the conservation of the zilches and their underlying symmetries. More specifically, the rest of the investigations and findings of this article are summarized as follows:

- **A hidden invariance algebra of free Maxwell's equations and the zilch symmetries (Subsection IV A).**—We show that the zilch symmetry transformations (15) of the four-potential belong to the enveloping algebra of a "hidden" invariance algebra of free Maxwell's equations in potential form. This "hidden" algebra closes on the 30-dimensional real Lie algebra $so(6,\mathbb{C})_\mathbb{R}$ - i.e. the 'realification' of the complex Lie algebra $so(6,\mathbb{C})$ - up to gauge transformations of the four-potential. (The $so(6,\mathbb{C})_\mathbb{R}$ invariance of free Maxwell's equations in terms of the electric and magnetic fields was uncovered in Ref. [37], but the potential form of Maxwell's equations was not discussed.) The 30 generators of the "hidden" algebra correspond to the 15 well-known infinitesimal conformal transformations [Eq. (40)] and to 15 little-known ("hidden") infinitesimal transformations [Eq. (41)]. The "hidden" transformations (41) take a simpler form when acting on the EM tensor; that is a product of a duality transformation with an infinitesimal conformal transformation [25, 26] (see Eq. (42)) [38].

- **Hidden symmetries in the case of Maxwell's equations in the presence of matter (Subsection IV B) and in the theory of a complex gauge field (Subsection IV C).**—We show that the "hidden" symmetries [Eq. (41)] of free Maxwell's equations persist in the presence of a material four-current [see Eq. (50)]. However, unlike the free case, the invariance algebra does not close on $so(6,\mathbb{C})_\mathbb{R}$. Then, we observe that the "hidden" symmetries of the real potential A_μ also exist for the free field equations (56) of a complex Abelian gauge field $\mathscr{A}_\mu$ - this is related to the complex formulation of duality-symmetric electromagnetism with the complex potential given by $\mathscr{A}_\mu = A_\mu + iC_\mu$ [24].

We show that if we redefine the "hidden" transformations of $\mathscr{A}_\mu$ by multiplying with $i = \sqrt{-1}$, the 30-dimensional algebra becomes $so(4,2) \bigoplus so(4,2)$ (it closes again up to gauge transformations of the complex potential).

- **Zilch continuity equations from symmetries in the presence of matter (Section V) and a new question (Section VI).**—We also study the derivation of zilch continuity equations in the presence of electric charges and currents by extending the zilch symmetries of the standard free action (6) to zilch symmetries [Eqs. (60) and (61)] of the standard interacting action (59) (in which A_μ couples to a non-dynamical material four-current J^μ). Taking advantage of the invariance of the interacting action under the zilch symmetries, we present a new way to derive the known continuity equation for optical chirality [39]

$$\frac{\partial}{\partial t} C + \boldsymbol{\nabla} \cdot \boldsymbol{S} = \frac{1}{2}\left(\boldsymbol{j} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \frac{\partial \boldsymbol{j}}{\partial t} \cdot \boldsymbol{B}\right) \tag{17}$$

($\boldsymbol{j}$ is the material electric current density). In Ref. [39], the continuity equation (17) was obtained from the complementary fields formalism, while a similar continuity equation had been first obtained in Ref. [3]. Apart from Eq. (17), in this Letter, we also obtain new continuity equations [Eqs. (65) and (68)] for the rest of the zilches in the presence of electric charges and currents from symmetries of the interacting EM action (59). Then, we pose the interesting open question of whether the aforementioned invariance of the interacting EM action with a non-dynamical material four-current can be extended to the case where the material four-current is dynamical.

II. BACKGROUND MATERIAL CONCERNING THE ZILCH TENSOR AND THE ZILCHES

In this Section we review the basics concerning the zilch tensor and the zilches.

The zilch tensor (10) can be expressed in various forms [2, 32]. For example, using the following identity [32]:

$$\partial_\rho\left({}^\star F_{\lambda\nu} F^{\mu\lambda}\right) = -\frac{1}{4}\delta^\mu_\nu\, \partial_\rho\left({}^\star F^{\lambda\kappa} F_{\lambda\kappa}\right), \tag{18}$$

the zilch tensor (10) can be equivalently expressed as

$$Z^\mu{}_{\nu\rho} = -\,{}^\star F^{\mu\lambda}\partial_\rho F_{\lambda\nu} - {}^\star F_\nu{}^\lambda \partial_\rho F_\lambda{}^\mu - \frac{1}{2}\delta^\mu_\nu\, {}^\star F^{\lambda\kappa}\partial_\rho F_{\lambda\kappa}. \tag{19}$$

This expression makes manifest that the properties $Z^{\mu\nu}{}_\rho = Z^{\nu\mu}{}_\rho$ and $Z^\mu{}_{\mu\rho} = 0$ are identically satisfied. Moreover, by using free Maxwell's equations, it is straightforward to show that the zilch tensor is divergence-free with respect to all of its indices and also satisfies $Z^\rho{}_{\nu\rho} = 0$ [32]. Using the fact that the zilch tensor is symmetric in its first two indices we can rewrite Eq. (10) as

$$\begin{aligned} Z^\mu{}_{\nu\rho} = & -\frac{1}{2}\,{}^\star F^{\mu\lambda}\partial_\rho F_{\lambda\nu} + \frac{1}{2} F^{\mu\lambda}\,\partial_\rho\,{}^\star F_{\lambda\nu} \\ & -\frac{1}{2}\,{}^\star F_\nu{}^\lambda \partial_\rho F_\lambda{}^\mu + \frac{1}{2} F_\nu{}^\lambda\,\partial_\rho\,{}^\star F_\lambda{}^\mu. \end{aligned} \tag{20}$$

As mentioned in the Introduction, the ten zilches are given by the following ten time-independent quantities [2, 32]:

$$\mathscr{Z}^{\mu\nu} = \mathscr{Z}^{\nu\mu} = \int d^3x Z^{\mu\nu 0}, \tag{21}$$

with $\partial \mathscr{Z}^{\mu\nu}/\partial t = 0$. Only nine zilches in Eq. (21) are independent since $Z^\mu{}_{\mu 0} = 0$. The $\mu\nu 0$-component ($Z^{\mu\nu 0}$) of the zilch tensor is the spatial density of the zilch $\mathscr{Z}^{\mu\nu}$, and the $\mu\nu j$-components ($Z^{\mu\nu j}$) are the components of the three-vector describing the corresponding flux [2]. The time-independence of the ten zilches follows from the ten differential conservation laws described by $\partial_\rho Z^{\mu\nu\rho} = 0$. The conservation law (3) for optical chirality corresponds to $\frac{1}{2}\left(\partial_0 Z^{000} + \partial_j Z^{00j}\right) = 0$.

For later convenience, note that the integral in Eq. (21) has the symmetry property

$$\int d^3x Z^{\mu\nu 0} = \int d^3x Z^{\mu 0\nu} \left(= \int d^3x Z^{0\mu\nu}\right) \tag{22}$$

because the difference $Z^{\mu\nu 0} - Z^{\mu 0\nu}$ can always be expressed as a spatial divergence [32]

$$Z^{\mu\nu 0} - Z^{\mu 0\nu} = \partial_j \Lambda^{\mu\nu j},$$

where the explicit expression for the tensor Λ is not needed for the present discussion [40]. It immediately follows that the difference $Z^{\mu 0\nu} - Z^{\nu 0\mu}$ can also be written as a spatial divergence. Hence, the $\mu\nu$-zilch, $\mathscr{Z}^{\mu\nu}$, can be actually interpreted as the time-independent quantity that corresponds to any of the three differential conservation laws: $\partial_\rho Z^{\mu\nu\rho} = 0$ (which is the one used by Lipkin [2]), $\partial_\rho Z^{\mu\rho\nu} = 0$ and $\partial_\rho Z^{\nu\rho\mu} = 0$. These differential conservation laws are not independent of each other. For example, the conservation law $\partial_\rho Z^{\mu\nu\rho} = 0$ can be rewritten as $\partial_\rho Z^{\mu\rho\nu} = 0$ by using the relations

$$\partial_0 Z^{\mu\nu 0} = \partial_0\left(Z^{\mu 0\nu} + \partial_j \Lambda^{\mu\nu j}\right) \tag{23}$$

and

$$\partial_j Z^{\mu\nu j} = \partial_j\left(Z^{\mu j\nu} - \partial_0 \Lambda^{\mu\nu j}\right) \tag{24}$$

for the corresponding spatial densities and fluxes, respectively.

III. INVARIANCE OF THE STANDARD ACTION UNDER THE ZILCH TRANSFORMATIONS AND CONSERVATION LAWS FOR ALL ZILCHES

In this Section we show that the zilch symmetry transformation (15), which is given here again for convenience:

$$\Delta A_\nu = n^\rho \tilde{n}^\mu \,\epsilon_{\mu\nu\sigma\lambda}\, \partial^\sigma \partial_\rho A^\lambda = n^\rho \tilde{n}^\mu \, \partial_\rho {}^*F_{\mu\nu}, \qquad (25)$$

is a symmetry of the standard free EM action (6). Then, we derive all zilch conservation laws using Noether's theorem.

Let us start by examining the way in which the zilch symmetry transformation (25) acts on the EM tensor for off-shell field configurations; that is

$$\begin{aligned}\Delta F_{\mu\nu} &\equiv \partial_\mu \Delta A_\nu - \partial_\nu \Delta A_\mu \\ &= \tilde{n}^\alpha n^\rho \left(\partial_\alpha \partial_\rho {}^\star F_{\mu\nu} - \epsilon_{\alpha\mu\nu\sigma} \partial_\rho \partial_\lambda F^{\lambda\sigma}\right), \qquad (26)\end{aligned}$$

where we have made use of the following important off-shell identity [41]:

$$\partial_\alpha {}^\star F_{\mu\nu} + \partial_\nu {}^\star F_{\alpha\mu} + \partial_\mu {}^\star F_{\nu\alpha} = \epsilon_{\alpha\mu\nu\sigma} \partial^\beta F_\beta{}^\sigma. \qquad (27)$$

We now proceed to demonstrate that the zilch symmetry transformation (25) is indeed a symmetry of the action (6) and then apply Noether's theorem. We find that the variation

$$\Delta S = -\frac{1}{2} \int d^4x \, F^{\mu\nu} \, \Delta F_{\mu\nu} \qquad (28)$$

is given by a total divergence (without making use of the equations of motion), as

$$\Delta S = \int d^4x \, \partial_\nu D^\nu \qquad (29)$$

with

$$D^\nu = \frac{1}{2} n^\rho \tilde{n}^\mu \left(2\, {}^\star F^{\lambda\nu} \partial_\rho F_{\mu\lambda} + Z_\mu{}^\nu{}_\rho + \delta^\nu_\rho \, {}^\star F_{\mu\sigma} \, \partial^\beta F_\beta{}^\sigma\right) \qquad (30)$$

- see Appendix A for some details of the calculation. Now, the usual procedure [35] can be followed in order to construct the conserved Noether current, V^ν, associated with the zilch symmetry transformation (25), as

$$V^\nu = \frac{\partial \mathscr{L}}{\partial(\partial_\nu A_\mu)} \Delta A_\mu - D^\nu, \qquad (31)$$

where $\mathscr{L} = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}$ is the free EM Lagrangian density. Substituting the expression for D^ν [Eq. (30)] into Eq. (31) and making use of the identity (18), we find

$$V^\nu = \frac{1}{2} n^\rho \tilde{n}^\mu \left(Z_\mu{}^\nu{}_\rho - \delta^\nu_\rho \, {}^\star F_{\mu\sigma} \, \partial^\beta F_\beta{}^\sigma\right). \qquad (32)$$

The definition of a conserved Noether current is not unique; we are free to add any term that vanishes on-shell and/or any term that is equal to the divergence of any rank-2 antisymmetric tensor to the expression for the Noether current [42]. Thus, we are allowed to express the Noether current in Eq. (32) as

$$V^\nu_{zilch} = \frac{1}{2} n^\rho \tilde{n}^\mu Z_\mu{}^\nu{}_\rho \qquad (33)$$

with $\partial_\nu V^\nu_{zilch} = 0$. Since the constant four-vectors n^ρ and $\tilde{n}^\mu$ in Eq. (33) are arbitrary, we conclude that

$$\partial_\nu Z^{\mu\nu\rho} = 0. \qquad (34)$$

In other words, the zilch tensor is the conserved Noether current corresponding to the zilch symmetries (25) of the standard free action (6), while the corresponding Noether charges are the zilches (21).

IV. "HIDDEN" SYMMETRIES

A. "Hidden" invariance algebra of free Maxwell's equations and the zilch symmetries

Here we investigate the relation of the zilch symmetry transformations (25) to a "hidden" $so(6,\mathbb{C})_\mathbb{R}$ invariance algebra of free Maxwell's equations in potential form (7).

Let ξ^μ denote any of the fifteen conformal Killing vectors of Minkowski spacetime satisfying

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{\partial^\alpha \xi_\alpha}{2} \eta_{\mu\nu}. \qquad (35)$$

The conformal Killing vectors ξ^μ of Minkowski spacetime consist of [43]: the four generators of spacetime translations,

$$P_{(\alpha)} = P^\mu_{(\alpha)} \partial_\mu = \partial_\alpha, \qquad (36)$$

the six generators of the Lorentz algebra $so(3,1)$,

$$M_{(\beta,\gamma)} = M^\mu_{(\beta,\gamma)} \partial_\mu = x_\beta \partial_\gamma - x_\gamma \partial_\beta, \qquad (37)$$

the generator of dilations

$$D = D^\mu \partial_\mu = x^\mu \partial_\mu, \qquad (38)$$

and the four generators of special conformal transformations

$$K_{(\alpha)} = K^\mu_{(\alpha)} \partial_\mu = x^\nu x_\nu \partial_\alpha - 2 x_\alpha x^\mu \partial_\mu. \qquad (39)$$

These fifteen vectors form a basis for the algebra of infinitesimal conformal transformations of Minkowski spacetime which is isomorphic to $so(4,2)$.

The "hidden" invariance algebra of free Maxwell's equations (7) is generated by two types of infinitesimal symmetry transformations of the four-potential. The first type corresponds to the well-known infinitesimal conformal transformations, conveniently described by the Lie derivative

$$L_\xi A_\mu = \xi^\lambda \partial_\lambda A_\mu + A_\lambda \partial_\mu \xi^\lambda, \qquad \xi \in so(4,2). \qquad (40)$$

These transformations generate a representation of $so(4,2)$ on the solution space of Maxwell's equations (7). The second type of transformations corresponds to the little-known ("hidden") transformations [44]

$$T_\xi A_\mu = \xi^\rho \epsilon_{\rho\mu\sigma\lambda} \partial^\sigma A^\lambda, \quad \xi \in so(4,2). \tag{41}$$

If A_μ is a solution of Maxwell's equations, then so are $L_\xi A_\mu$ and $T_\xi A_\mu$ for all $\xi \in so(4,2)$ [44]. The effect of the "hidden" transformation (41) on $F_{\mu\nu}$ corresponds to the product of a duality transformation with an infinitesimal conformal transformation as

$$T_\xi F_{\mu\nu} \equiv \partial_\mu T_\xi A_\nu - \partial_\nu T_\xi A_\mu = L_\xi {}^\star F_{\mu\nu}, \tag{42}$$

where

$$L_\xi {}^\star F_{\mu\nu} = \xi^\rho \partial_\rho {}^\star F_{\mu\nu} + {}^\star F_{\rho\nu}\, \partial_\mu \xi^\rho + {}^\star F_{\mu\rho}\, \partial_\nu \xi^\rho. \tag{43}$$

This symmetry transformation of the EM tensor was first found in Refs. [25, 26].

The structure of the "hidden" invariance algebra of Maxwell's equations in potential form is determined by the Lie brackets:

$$[L_{\xi'}, L_\xi] A_\mu = L_{[\xi',\xi]} A_\mu \tag{44}$$

$$[L_{\xi'}, T_\xi] A_\mu = T_{[\xi',\xi]} A_\mu \tag{45}$$

and

$$[T_{\xi'}, T_\xi] A_\mu = -L_{[\xi',\xi]} A_\mu + \partial_\mu \left([\xi',\xi]^\sigma A_\sigma - \xi'_\sigma \xi_\rho F^{\sigma\rho} \right), \tag{46}$$

where, e.g., $[L_{\xi'}, L_\xi] = L_{\xi'} L_\xi - L_\xi L_{\xi'}$, while ξ and ξ' are any two basis elements of $so(4,2)$ with $[\xi',\xi]^\rho = L_{\xi'}\xi^\rho$. We observe the appearance of a gauge transformation of the form (9) in the last term of Eq. (46). To the best of our knowledge, the explicit expressions for the commutators (45) and (46) appear here for the first time. The commutation relations in Eqs. (44)-(46) coincide with the commutation relations of the 30-dimensional real Lie algebra $so(6,\mathbb{C})_\mathbb{R}$ [37] (up to the gauge transformation in Eq. (46)).

Now, let us denote the zilch symmetry transformation (25) with associated Noether current corresponding to $Z_\alpha{}^\nu{}_\beta$ (α and β have fixed values) as $\Delta_{(\beta,\alpha)} A_\mu$. The latter is readily expressed as [see Eq. (25)]

$$\Delta_{(\beta,\alpha)} A_\mu = \partial_\beta \left(\epsilon_{\alpha\mu\sigma\lambda}\, \partial^\sigma A^\lambda \right) = L_{P_{(\beta)}} T_{P_{(\alpha)}} A_\mu. \tag{47}$$

It is obvious from this expression that $\Delta_{(\beta,\alpha)} A_\mu$ is given by the product of a "hidden" transformation (41) with respect to the translation Killing vector $P_{(\alpha)} = \partial_\alpha$ and a Lie derivative (40) with respect to the translation Killing vector $P_{(\beta)} = \partial_\beta$. This makes clear that the zilch symmetry transformation $\Delta_{(\beta,\alpha)} A_\mu$ belongs to the enveloping algebra of our "hidden" invariance algebra [and so do all transformations of the form (25)].

B. "Hidden" symmetries of Maxwell's equations in the presence of a material four-current

In the presence of a material four-current Maxwell's equations are

$$\Box A_\mu - \partial_\mu \partial^\nu A_\nu = -J_\mu, \tag{48}$$

where $J^\mu = (\rho, \boldsymbol{j})$. Maxwell's equations remain invariant under simultaneous infinitesimal conformal transformations of A_μ and J_μ, i.e. Eq. (48) will still be satisfied if we make the following replacements:

$$\begin{aligned} &A_\mu \to L_\xi A_\mu, \\ &J_\mu \to L_\xi J_\mu + \frac{\partial_\alpha \xi^\alpha}{2} J_\mu, \quad \xi \in so(4,2), \end{aligned} \tag{49}$$

where L_ξ is the Lie derivative (40). It is interesting to investigate whether the "hidden" symmetries (41) of free Maxwell's equations also persist in the presence of matter. Indeed, we find that if A_μ and J_μ satisfy Eq. (48), then Eq. (48) will still be satisfied if we make the following replacements:

$$\begin{aligned} &A_\mu \to T_\xi A_\mu, \\ &J_\mu \to \delta^{hid}_\xi J_\mu = \epsilon_{\rho\mu\sigma\lambda} \partial^\sigma \left(\xi^\rho J^\lambda \right), \quad \xi \in so(4,2), \end{aligned} \tag{50}$$

where $T_\xi A_\mu$ is given by Eq. (41), while we call $\delta^{hid}_\xi J_\mu$ in the second line the "hidden" transformation of the four-current. Equation (50) describes the "hidden" symmetries of Maxwell's equations in the presence of a material four-current.

Unlike the free case, in the presence of matter, the algebra does not close on $so(6,\mathbb{C})_\mathbb{R}$ up to gauge transformations of the four-potential [45]. This can be readily understood from the following example. By calculating the commutators between "hidden" symmetries generated by translation Killing vectors (36), we find:

$$\begin{aligned} &[T_{P_{(\alpha)}}, T_{P_{(\beta)}}] A_\mu \\ &= \partial_\mu \left(-P^\sigma_{(\alpha)} P^\rho_{(\beta)} F_{\sigma\rho} \right) + \left(P_{(\alpha)\mu} P^\rho_{(\beta)} - P^\rho_{(\alpha)} P_{(\beta)\mu} \right) \partial^\sigma F_{\sigma\rho} \end{aligned} \tag{51}$$

(compare this equation with Eq. (46)) and

$$\begin{aligned} [\delta^{hid}_{P_{(\alpha)}}, \delta^{hid}_{P_{(\beta)}}] J_\mu &= \partial_\mu \left(-P^\sigma_{(\alpha)} P^\rho_{(\beta)} (\partial_\sigma J_\rho - \partial_\rho J_\sigma) \right) \\ &+ \left(P_{(\alpha)\mu} P^\rho_{(\beta)} - P^\rho_{(\alpha)} P_{(\beta)\mu} \right) \Box J_\rho. \end{aligned} \tag{52}$$

Also, from these commutators, it follows that Maxwell's equations (48) will still be satisfied if we make the replacements (this is easy to check):

$$A_\mu \to \left(P_{(\alpha)\mu} P^\rho_{(\beta)} - P^\rho_{(\alpha)} P_{(\beta)\mu} \right) \partial^\sigma F_{\sigma\rho} \tag{53}$$

and

$$\begin{aligned} J_\mu \to\ &\partial_\mu \left(-P^\sigma_{(\alpha)} P^\rho_{(\beta)} (\partial_\sigma J_\rho - \partial_\rho J_\sigma) \right) \\ &+ \left(P_{(\alpha)\mu} P^\rho_{(\beta)} - P^\rho_{(\alpha)} P_{(\beta)\mu} \right) \Box J_\rho. \end{aligned} \tag{54}$$

The study of the full structure of the algebra in the presence of matter is something that we leave for future work.

C. "Hidden" symmetries for the complex Abelian gauge field

The free action for the complex Abelian gauge field, $\mathscr{A}_\mu$, is

$$-\frac{1}{8}\int d^4x \mathscr{F}^\dagger_{\mu\nu}\mathscr{F}^{\mu\nu}, \tag{55}$$

where $\dagger$ denotes complex conjugation, while $\mathscr{F}_{\mu\nu} = \partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu$. Expressing $\mathscr{A}_\mu$ as $\mathscr{A}_\mu = A_\mu + iC_\mu$, the action (55) takes the form of the duality-symmetric action (12).

The field equation for the complex potential is

$$\Box\mathscr{A}_\mu - \partial_\mu\partial^\nu\mathscr{A}_\nu = 0. \tag{56}$$

This equation is invariant under the infinitesimal conformal transformations $L_\xi\mathscr{A}_\mu$ in Eq. (40) (with A_μ replaced by $\mathscr{A}_\mu$), as well as under the "hidden" transformations $T_\xi\mathscr{A}_\mu$ in Eq. (41) (with A_μ replaced by $\mathscr{A}_\mu$). As in the case of the real potential, the structure of the algebra generated by the conformal and the "hidden" transformations is determined by the commutators in Eqs. (44)-(46) with A_μ replaced by $\mathscr{A}_\mu$ and $F^{\sigma\rho}$ replaced by $\mathscr{F}^{\sigma\rho}$.

Now, we will show that if we redefine the "hidden" transformations $T_\xi\mathscr{A}_\mu$, the "hidden" algebra of Eq. (56) will be isomorphic to $so(4,2)\bigoplus so(4,2)$. Let us redefine $T_\xi\mathscr{A}_\mu$ by multiplying with i as

$$T'_\xi\mathscr{A}_\mu \equiv i\,T_\xi\mathscr{A}_\mu = \,i\,\xi^\rho\epsilon_{\rho\mu\sigma\lambda}\partial^\sigma\mathscr{A}^\lambda, \qquad \xi\in so(4,2). \tag{57}$$

These transformations leave both the action (55) and Eq. (56) invariant (on the other hand, T_ξ is a symmetry of the field equation only). Now, the "hidden" algebra of the field equation is generated by the conformal transformations $L_\xi\mathscr{A}_\mu$ and the redefined "hidden" transformations $T'_\xi\mathscr{A}_\mu$. If we now define the new set of generators:

$$\mathscr{T}^\pm_\xi\mathscr{A}_\mu \equiv \frac{1}{\sqrt{2}}\left(L_\xi \pm T'_\xi\right)\mathscr{A}_\mu, \qquad \xi\in so(4,2), \tag{58}$$

it is easy to see that the $\mathscr{T}^+_\xi$'s generate a $so(4,2)$ algebra on their own, and so do the transformations $\mathscr{T}^-_\xi$, while $[\mathscr{T}^+_\xi,\mathscr{T}^-_{\xi'}] = 0$ for any $\xi,\xi'\in so(4,2)$ [these follow directly from Eqs. (44)-(46)]. Thus, the "hidden" algebra is now isomorphic to $so(4,2)\bigoplus so(4,2)$ and closes up to gauge transformations of the complex potential $\mathscr{A}_\mu$.

V. ZILCH CONTINUITY EQUATIONS IN THE PRESENCE OF ELECTRIC CHARGES AND CURRENTS FROM SYMMETRIES OF THE STANDARD INTERACTING ACTION

In the presence of a non-dynamical material four-current, $J^\mu = (\rho,\boldsymbol{j})$, the standard interacting EM action is

$$S' = S + \int d^4x\, J^\nu A_\nu = \int d^4x\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\nu A_\nu\right). \tag{59}$$

Let us consider the variation of S' under the following simultaneous transformations of A_ν and J^ν:

$$\Delta A_\nu = n^\rho\tilde{n}^\mu\,\epsilon_{\mu\nu\sigma\lambda}\,\partial^\sigma\partial_\rho A^\lambda, \tag{60}$$

$$\Delta J^\nu = n^\rho\tilde{n}^\mu\,\epsilon_\mu{}^\nu{}_{\sigma\lambda}\,\partial^\sigma\partial_\rho J^\lambda, \tag{61}$$

where n^ρ and $\tilde{n}^\mu$ are two arbitrary constant four-vectors, while Eq. (60) coincides with the zilch symmetry transformation (25). The variation of the free action, S, is already known to be a total divergence [see Eq. (29)]. Also, after a straightforward calculation, we find that the variation of the interaction term is a total divergence, as

$$\Delta\Big(\int d^4x\, J^\nu A_\nu\Big) = \int d^4x\,\partial_\nu D^\nu_{int}, \tag{62}$$

where

$$D^\nu_{int} = \tilde{n}^\mu n^\rho(\delta^\nu_\rho\, J^\lambda\,{}^\star F_{\mu\lambda} - \partial_\rho J^\lambda A^\alpha\epsilon_{\mu\lambda}{}^\nu{}_\alpha). \tag{63}$$

Thus, the variation of the interacting action is

$$\Delta S' = \int d^4x\,\partial_\nu\left(D^\nu + D^\nu_{int}\right), \tag{64}$$

where D^ν is given by Eq. (30).

Now, by applying the standard Noether algorithm [35], we find the following continuity equations for the zilch tensor:

$$\partial_\lambda Z^{\mu\lambda\nu} = J_\lambda\,\partial^\nu\,{}^\star F^{\mu\lambda} - {}^\star F^{\mu\lambda}\,\partial^\nu J_\lambda. \tag{65}$$

These continuity equations determine the rate of gain or loss of the quantity $\int d^3x Z^{\mu 0\nu}$, with spatial density given by $Z^{\mu0\nu}$ and flux components given by $Z^{\mu j\nu}$.

The continuity equations (65) can be re-expressed in the form of continuity equations for the zilches (21), with spatial density given by $Z^{\mu\nu0}$ and flux components given by $Z^{\mu\nu j}$, as follows. First, we observe that, although in the presence of electric charges and currents the quantity $\int d^3x Z^{\mu0\nu}$ and the $\mu\nu$-zilch, $\int d^3x Z^{\mu\nu0}$ [Eq. (21)] are not equal to each other unless $\mu=\nu=0$ (because the symmetry property (22) no longer holds), they are related to each other by [46]

$$\begin{aligned}Z^{\mu\nu\rho} - Z^{\mu\rho\nu} = \frac{1}{2}\Big(&\epsilon^{\kappa\mu\nu\rho}\,\partial_\sigma T^\sigma{}_\kappa - \epsilon^{\kappa\lambda\nu\rho}\partial_\lambda T^\mu{}_\kappa \\ &- \epsilon^{\kappa\rho\lambda\mu}\partial_\lambda T^\nu{}_\kappa + \epsilon^{\kappa\nu\lambda\mu}\partial_\lambda T^\rho{}_\kappa \\ &- 2F^\mu{}_\lambda\,\epsilon^{\rho\lambda\nu\sigma}J_\sigma + 2J^\mu\,{}^\star F^{\nu\rho}\Big),\end{aligned} \tag{66}$$

where

$$T^\alpha{}_\beta = -F^{\alpha\lambda}F_{\lambda\beta} - \frac{1}{4}\delta^\alpha_\beta F^{\kappa\lambda}F_{\kappa\lambda} \tag{67}$$

is the Maxwell stress-energy tensor (with $\partial_\alpha T^\alpha{}_\beta = J^\alpha F_{\alpha\beta}$). Then, by taking the divergence of Eq. (66) with respect to the index ρ and using the continuity equation (65) we find

$$\begin{aligned}\partial_\rho Z^{\mu\nu\rho} = \eta^{\mu\nu}\,{}^\star F_{\lambda\sigma}\,\partial^\lambda J^\sigma - {}^\star F^{\mu\sigma}\,(\partial^\nu J_\sigma - \partial_\sigma J^\nu) \\ - {}^\star F^{\nu\sigma}\,(\partial^\mu J_\sigma - \partial_\sigma J^\mu).\end{aligned} \tag{68}$$

These are the ten continuity equations determining the rate of gain (or loss) for the ten zilches (21) in the presence of electric charges and currents. For $\mu = \nu = 0$, both continuity equations (65) and (68) coincide with the known equation (17) for optical chirality. To the best of our knowledge, the other nine continuity equations in Eq. (68) are presented here for the first time.

VI. AN INTERESTING OPEN QUESTION

Let us suppose that the EM field interacts with a dynamical matter field with corresponding four-current $\tilde{J}^{\mu}$. Now, the action of the full interacting theory is

$$\int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tilde{J}^{\nu} A_{\nu} \right) + S_{\text{matter}}, \qquad (69)$$

where S_{matter} is the action corresponding to the free matter field. According to our earlier discussion, the simultaneous transformations (60) and (61) (with J^{ν} replaced by $\tilde{J}^{\nu}$) are symmetries of the first two terms in Eq. (69). Motivated by this observation, we may pose the question of whether one could identify symmetries of the full interacting theory (i.e. symmetries of all three terms in Eq. (69)). In other words, is it possible to identify a transformation of the matter field such that: this transformation is a symmetry of S_{matter}, while the four-current $\tilde{J}^{\mu}$ transforms as in Eq. (61)?

VII. DISCUSSION

The results of the present Letter establish a clear connection between all zilch continuity equations and symmetries of the standard EM action via Noether's theorem. Having identified all zilches with Noether charges, we can interpret them as the generators of the corresponding symmetry transformations (25) of the four-potential in the standard (classical or quantum) EM theory [35, 36, 47]. In the case of optical chirality, the explicit knowledge of the underlying symmetry generator is known to offer physical insight, since it allows the identification of the optical chirality eigenstates with plane waves of circular polarization [36]. Similarly, the symmetry transformations (25) can be used to identify the eigenstates of all zilches, which is something that we leave for future work.

A particularly interesting uninvestigated question is the one concerning the role of all zilches in light-matter interactions - the case of optical chirality is the only exception since its role has been studied [3]. The importance of this question becomes manifest by considering the fact that a physical interpretation for all zilches has been recently provided [21]. In particular, in Ref. [21] it was found that the zilches of a certain class of topologically non-trivial EM fields in vacuum can be expressed in terms of energy, momentum, angular momentum and helicity of the fields. Also, it was demonstrated that the zilches of these fields encode information about the topology of the field lines. We hope that the results presented in this Letter will be useful in future attempts to study the role of all zilches in light-matter interactions. More specifically, motivated by the interpretation and applications of the known continuity equation (17) for optical chirality [3, 39, 47–49], it is natural to interpret each of our new zilch continuity equations [Eq. (68)] as determining the rate of loss or gain of the corresponding "zilch quantity" of the EM field. Electrically charged matter acts as a source or sink for the zilch quantities.

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Data availability - Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Appendix A: Invariance of the standard free EM action under the zilch symmetries

Here we present some details for the calculation concerning the invariance of the standard free EM action (6) under the zilch symmetry transformation (25). For convenience, we focus only on the invariance of the action and not on the derivation of the associated Noether current (32). Also, we drop all terms that are total divergences in order to simplify the presentation. However, note that one needs to keep all such terms if they wish to re-derive Eq. (29).

Varying the action (6) with respect to the zilch symmetry transformation (25) we find

$$\begin{aligned} -2\,\Delta S &= \int d^4x\, F^{\mu\nu}\,\Delta F_{\mu\nu} \\ &= \int d^4x\, \left(F^{\mu\nu}\tilde{n}^\alpha n^\rho\,\partial_\rho\partial_\alpha{}^\star F_{\mu\nu} - F^{\mu\nu}\tilde{n}^\alpha n^\rho \epsilon_{\alpha\mu\nu\sigma}\partial_\rho\partial_\lambda F^{\lambda\sigma}\right), \end{aligned} \tag{A1}$$

where in the second line we used Eq. (26). The first term is readily shown to be equal to a total divergence as follows:

$$\begin{aligned} \int d^4x\, F^{\mu\nu}\tilde{n}^\alpha n^\rho\,\partial_\rho\partial_\alpha{}^\star F_{\mu\nu} &= -\int d^4x\;\partial_\rho F^{\mu\nu}\tilde{n}^\alpha n^\rho\,\partial_\alpha{}^\star F_{\mu\nu} \\ &= 2\int d^4x\;\partial^\nu F_\rho{}^\mu\;\tilde{n}^\alpha n^\rho\,\partial_\alpha{}^\star F_{\mu\nu} \\ &= 2\int d^4x\;\partial^\nu\left(F_\rho{}^\mu\;\tilde{n}^\alpha n^\rho\,\partial_\alpha{}^\star F_{\mu\nu}\right), \end{aligned}$$

where in the second line we used Eq. (8), and in the third line we used that the divergence of ${}^\star F_{\mu\nu}$ vanishes identically because of Eq. (8). We now drop the first term in Eq. (A1) (since we showed that it is a total divergence) and we express Eq. (A1) as

$$-2\,\Delta S = -2\int d^4x\;{}^\star F_{\alpha\sigma}\;\tilde{n}^\alpha n^\rho\quad\partial_\rho\partial_\lambda F^{\lambda\sigma}. \tag{A2}$$

On the other hand, keeping both terms in Eq. (A1) and using the off-shell identity (27), Eq. (A1) is re-written as

$$-2\,\Delta S = 2\int d^4x\;F^{\nu\sigma}\tilde{n}^\alpha n^\rho\;\partial_\rho\partial_\nu{}^\star F_{\alpha\sigma}. \tag{A3}$$

Integrating by parts twice, we find

$$-2\,\Delta S = 2\int d^4x\;{}^\star F_{\alpha\sigma}\;\tilde{n}^\alpha n^\rho\quad\partial_\rho\partial_\lambda F^{\lambda\sigma}. \tag{A4}$$

Comparing this equation with Eq. (A2), we find $\Delta S = 0$ (i.e. ΔS is equal to the integral of a total divergence), as required.

[1] E. Noether, Invariante Variationsprobleme, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse , 235 (1918).

[2] D. M. Lipkin, Existence of a new conservation law in electromagnetic theory, Journal of Mathematical Physics **5**, 696 (1964), https://doi.org/10.1063/1.1704165.

[3] Y. Tang and A. E. Cohen, Optical chirality and its interaction with matter, Phys. Rev. Lett. **104**, 163901 (2010).

[4] An alternative expression for optical chirality density that appears in the literature is $C = \frac{1}{2}\left(\boldsymbol{E}\cdot\boldsymbol{\nabla}\times\boldsymbol{E}+\boldsymbol{B}\cdot\boldsymbol{\nabla}\times\boldsymbol{B}\right)$ [3]. This expression is equal to Eq. (1) only in the absence of electric charges and currents. In this Letter, we use the expression (1) when electric charges and currents are present. The justification for our choice is that the expression (1) is equal to the 000-component of the zilch tensor (10) (up to a factor of 1/2), while, as we show in this Letter, the zilch tensor is the Noether current corresponding to the zilches in free electromagnetism. Also, arguments in favor of defining the optical chirality density in the presence of electric charges and currents using Eq. (1) can be found in Ref. [39].

[5] N. Yang and A. E. Cohen, Local Geometry of Electromagnetic Fields and Its Role in Molecular Multipole Transitions, J. Phys. Chem. B **115**, 5304–5311 (2011).

[6] E. Hendry, T. Carpy, J. Johnston, M. Popland, R. V. Mikhaylovskiy, A. J. Lapthorn, S. M. Kelly, L. D. Barron, N. Gadegaard, and M. Kadodwala, Ultrasensitive detection and characterization of biomolecules using superchiral fields, Nature Nanotech **5**, 783 (2010).

[7] Y. Tang and A. E. Cohen, Enhanced enantioselectivity in excitation of chiral molecules by superchiral light, Science (New York, N.Y.) **332**, 333 (2011).

[8] E. Hendry, R. V. Mikhaylovskiy, L. D. Barron, M. Kadodwala, and T. J. Davis, Chiral Electromagnetic Fields Generated by Arrays of Nanoslits, Nano Letters **12**, 3640 (2012).

[9] M. Schäferling, D. Dregely, M. Hentschel, and H. Giessen, Tailoring Enhanced Optical Chirality: Design Principles for Chiral Plasmonic Nanostructures, Phys. Rev. X **2**,

031010 (2012).
[10] C. Rosales-Guzmán, K. Volke-Sepulveda, and J. P. Torres, Light with enhanced optical chirality, Opt. Lett. **37**, 3486 (2012).
[11] G. Schnoering, L. V. Poulikakos, Y. Rosales-Cabara, A. Canaguier-Durand, D. J. Norris, and C. Genet, Three-Dimensional Enantiomeric Recognition of Optically Trapped Single Chiral Nanoparticles, Phys. Rev. Lett. **121**, 023902 (2018).
[12] M. L. Solomon, J. M. Abendroth, L. V. Poulikakos, J. Hu, and J. A. Dionne, Fluorescence-Detected Circular Dichroism of a Chiral Molecular Monolayer with Dielectric Metasurfaces, Journal of the American Chemical Society **142**, 18304 (2020), pMID: 33048539, https://doi.org/10.1021/jacs.0c07140.
[13] L. V. Poulikakos, P. Gutsche, K. M. McPeak, S. Burger, J. Niegemann, C. Hafner, and D. J. Norris, Optical Chirality Flux as a Useful Far-Field Probe of Chiral Near Fields, ACS Photonics **3**, 1619 (2016), https://doi.org/10.1021/acsphotonics.6b00201.
[14] L. V. Poulikakos, P. Thureja, A. Stollmann, E. De Leo, and D. J. Norris, Chiral Light Design and Detection Inspired by Optical Antenna Theory, Nano Letters **18**, 4633 (2018), pMID: 29533637, https://doi.org/10.1021/acs.nanolett.8b00083.
[15] K. M. McPeak, C. D. van Engers, S. Bianchi, A. Rossinelli, L. V. Poulikakos, L. Bernard, S. Herrmann, D. K. Kim, S. Burger, M. Blome, S. V. Jayanti, and D. J. Norris, Ultraviolet Plasmonic Chirality from Colloidal Aluminum Nanoparticles Exhibiting Charge-Selective Protein Detection, Advanced Materials **27**, 6244 (2015).
[16] L. V. Poulikakos, Chiral Light–Matter Interactions in the Near and Far Field, Ph.D. thesis, ETH Zürich, Zürich, Switzerland https://doi.org/10.3929/ethz-b-000296342 (2018).
[17] S. K. Teo, W. A. Colburn, W. G. Tracewell, K. A. Kook, D. I. Stirling, M. S. Jaworsky, M. A. Scheffler, S. D. Thomas, and O. L. Lasklin, Clinical Pharmacokinetics of Thalidomide, Clin Pharmacokinet **43**, 311 (2004).
[18] J. S. Seo, D. Whang, H. Lee, S. Im Jun, J. Oh, Y. J. Jeon, and K. Kim, A homochiral metal–organic porous material for enantioselective separation and catalysis, Nature **404**, 982 (2000).
[19] L. Ma, C. Abney, and W. Lin, Enantioselective catalysis with homochiral metal-organic frameworks, Chem. Soc. Rev. **38**, 1248 (2009).
[20] H. U. Blaser and H. J. Federsel, *Asymmetric Catalysis on Industrial Scale: Challenges, Approaches and Solutions* (Wiley-VCH Verlag GmbH & Co. KGaA, Weinheim, Germany, 2004).
[21] G. Smith and P. Strange, Lipkin's conservation law in vacuum electromagnetic fields, Journal of Physics A: Mathematical and Theoretical **51**, 435204 (2018).
[22] N. H. Ibragimov, Symmetries, Lagrangian and Conservation Laws for the Maxwell Equations, Acta Appl. Math. **105**, 157 (2009).
[23] R. P. Cameron and S. M. Barnett, Electric-magnetic symmetry and Noether's theorem, New Journal of Physics **14**, 123019 (2012).
[24] S. Aghapour, L. Andersson, and K. Rosquist, The zilch electromagnetic conservation law revisited, Journal of Mathematical Physics **61**, 122902 (2020), https://doi.org/10.1063/1.5126487.
[25] I. Yu. Krivskii and V. M. Simulik, Noether analysis of zilch conservation laws and their generalization for the electromagnetic field. I. Use of different formulations of the principle of least action, Teor. Mat. Fiz. **80**, 864 (1989).
[26] I. Yu. Krivskii and V. M. Simulik, Noether analysis of zilch conservation laws and their generalization for the electromagnetic field. II. Use of Poincaré-invariant formulation of the principle of least action, Teor. Mat. Fiz. **80**, 912 (1989).
[27] M. Przanowski, B. Rajca, and J. Tosiek, On some conservation laws in the Maxwell electrodynamics in vacuum, Acta Phys. Pol. B **25**, 1065 (1994).
[28] A. Sudbery, A vector Lagrangian for the electromagnetic field, Journal of Physics A: Mathematical and General **19**, L33 (1986).
[29] M. G. Calkin, An invariance property of the free electromagnetic field, American Journal of Physics **33**, 958 (1965), https://doi.org/10.1119/1.1971089.
[30] S. Anco and J. Pohjanpelto, Classification of Local Conservation Laws of Maxwell's equations, Acta Applicandae Mathematicae **69**, 285 (2001).
[31] I. Yu. Krivsky, V. M. Simulik, and Z. Z. Torich, A covariant form for the complete set of first-order electromagnetic conservation laws, Physics Letters B **320**, 96 (1994).
[32] T. W. B. Kibble, Conservation laws for free fields, Journal of Mathematical Physics **6**, 1022 (1965), https://doi.org/10.1063/1.1704363.
[33] P. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, 1993).
[34] S. C. Anco and G. Bluman, Direct construction of conservation laws from field equations, Phys. Rev. Lett. **78**, 2869 (1997).
[35] S. Weinberg, *The Quantum Theory of Fields*, Vol. 1 (Cambridge University Press, Cambridge, UK, 1995).
[36] T. G. Philbin, Lipkin's conservation law, Noether's theorem, and the relation to optical helicity, Phys. Rev. A **87**, 043843 (2013).
[37] J. Pohjanpelto, First order generalized symmetries of maxwell's equations, Physics Letters A **129**, 148 (1988).
[38] The fact that the product of a duality transformation with an infinitesimal conformal transformation is a symmetry of Maxwell's equations expressed in terms of the EM tensor was first observed by Krivskii and Simulik [25, 26].
[39] M. Fernández Guasti, Chirality, helicity and the rotational content of electromagnetic fields, Physics Letters A **383**, 3180 (2019).
[40] The interested reader can find the expression for Λ from equation (14) of Ref. [32] or they can set $J^{\mu} = 0$ and let $\rho = 0$ in Eq. (66) of the present article.
[41] Equation (27) can be readily proved by contracting with $\epsilon_{\gamma\delta}{}^{\mu\nu}$ and then using well-known properties of the totally antisymmetric tensor.
[42] P. Ramond, *Field Theory: a Modern Primer*, Frontiers in Physics (Second edition, Addison Wesley, 1981).
[43] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics (Springer-Verlag, New York, 1997).
[44] J. Pohjanpelto, *Symmetries, conservation laws, and Maxwell's equations*, in 'Advanced Electromagnetism: Foundations, Theory and Applications', Barrett, T. W. and Grimes, D.

M. eds. (World Scientific, Singapore, 1995) https://www.worldscientific.com/doi/pdf/10.1142/2599.
[45] If one considers only the familiar conformal transformations (49), then the algebra closes on $so(4,2)$.
[46] Equation (66) is obtained following analogous steps as the ones described by Kibble in order to derive equation (14) (for the free EM field) in Ref. [32]. Our Eq. (66) coincides with equation (14) of Ref. [32] if no electric charges and currents are present.
[47] G. Nienhuis, Conservation laws and symmetry transformations of the electromagnetic field with sources, Phys. Rev. A **93**, 023840 (2016).
[48] L. V. Poulikakos, J. A. Dionne, and A. García-Etxarri, Optical Helicity and Optical Chirality in Free Space and in the Presence of matter, Symmetry **11**, 10.3390/sym11091113 (2019).
[49] F. Crimin, N. Mackinnon, J. B. Götte, and S. M. Barnett, Optical Helicity and Chirality: Conservation and Sources, Applied Sciences **9**, 10.3390/app9050828 (2019).
[50] R. M. Wald, *Advanced Classical Electromagnetism* (Princeton University Press, 2022).
[51] I. Yu. Krivskii and V. M. Simulik, The lagrangian of the electromagnetic field in terms of intensities and conservation laws (in Russian), Ukr. Fiz. Zh. **30**, 1457 (1985).
[52] J. Petersen, J. Volz, and A. Rauschenbeutel, Chiral nanophotonic waveguide interface based on spin-orbit interaction of light, Science **346**, 67 (2014).