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Weight Distribution of Classical Codes Influences Robust Quantum Metrology

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Quantum metrology is expected to be a prominent use-case of quantum technologies. However, noise easily degrades these quantum probe states, and negates the quantum advantage they would have offered in a noiseless setting. Although quantum error correction (QEC) can help tackle noise, fault-tolerant methods are too resource intensive for near-term use. Hence, a strategy for (near-term) robust metrology that is easily adaptable to future QEC-based quantum metrology is desirable. Here, we propose such an architecture by studying the performance of quantum probe states that are constructed from $[n, k, d]$ binary block codes of minimum distance $d \geq t + 1$. Such states can be interpreted as the logical $|+ + \cdots +\rangle$ state of a CSS code whose logical X group is defined by the aforesaid binary code. When a constant, t, number of qubits of the quantum probe state are erased, using the quantum Fisher information (QFI) we show that the resultant noisy probe can give an estimate of the magnetic field with a precision that scales inversely with the variances of the weight distributions of the corresponding 2^t shortened codes. Moreover, we show that if C is any code concatenated with inner repetition codes of length linear in n, then a quantum advantage in quantum metrology is possible. This implies that, given any CSS code of constant length, concatenation with repetition codes of length linear in n is asymptotically optimal for quantum metrology with a constant number of erasure errors. Besides the fundamental QFI result, we also explicitly construct an observable that when measured on such noisy code-inspired probe states, yields a precision on the magnetic field strength that also exhibits a quantum advantage in the limit of vanishing magnetic field strength. We emphasize that, despite the use of coding-theoretic methods, our results do not involve syndrome measurements or error correction. We complement our results with examples of probe states constructed from Reed-Muller codes.

I. INTRODUCTION

Quantum metrology is important for applications ranging from enabling precision navigation to medical imaging (see also [1] and the references therein). Here we focus on quantum magnetometers, which estimate magnetic fields utilizing a quantum resource known as a *probe state*, which is essentially a specially chosen entangled state on a system of multiple spins. Quantum magnetometers are expected to be a prominent use-case of quantum technologies because they are examples of quantum sensors that can potentially estimate more precisely than what is classically possible [2–6]. However, to date, because of the fragility of quantum resources, quantum magnetometers have yet to realize their full potential.

In the complete absence of decoherence, quantum magnetometers can indeed make much more precise estimates of magnetic fields than classical magnetometers [2, 7]. However, once we consider any realistic scenario where decoherence must be present, the performance of quantum magnetometers degrades [8–13]. Indeed, the optimal probe state for quantum metrology in the noiseless setting, the GHZ state $(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$, is easily rendered useless in the presence of noise [2]. One approach to combat the effects of decoherence on a probe state is to use probe states that are chosen from quantum error correction codes, and perform active quantum error correction [14–21]. However, in lieu of active quantum error correction protocols, which remain challenging to implement, it is pertinent to understand the extent to which quantum metrology can prove advantageous using inherently noisy probe states.

Various noise models have been studied in the context of quantum metrology with noisy probe states, including non-Markovian noise models [22, 23], semigroup and non-semigroup quantum channels [24], erasure errors [25, 26], some physically motivated examples [27, 28], and errors that do not degrade the performance of probe states [29]. Of these noise models, we focus our attention on erasure errors. This is because erasure errors are one of the simplest types of errors considered in both classical and quantum coding theory [30, 31], and the connection between noisy quantum metrology and coding theory even in this simplest setting is not well-understood.

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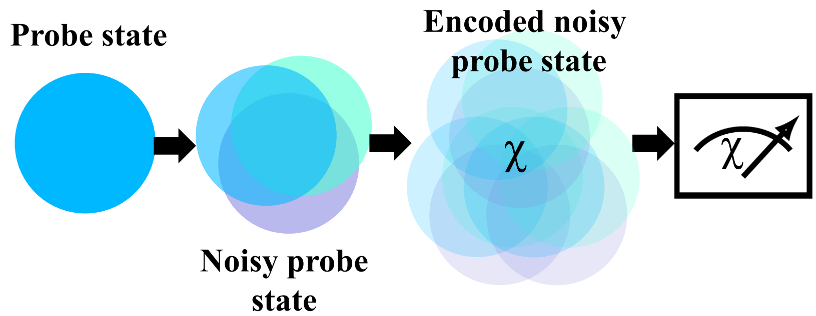

FIG. 1. A cartoon sketch of the robust metrology problem. First a probe state is prepared. Then noise occurs on the probe state. Subsequently, the probe state encodes information about a parameter χ that is to be estimated. Measurements on the encoded noisy probe state gives estimates on χ.

One obvious strategy to construct robust probe states with respect to erasure errors is to consider those that are randomly generated. This is because random qubit states almost surely lie within the codespace of quantum codes that have distance linear in the number of qubits. In fact, corresponding random quantum codes almost surely saturate the quantum Gilbert-Varshamov bound [32–35], and this scenario persists for certain random concatenated codes [36]. Since random qubit codes almost surely have large distance, and random qubit states almost surely have a large amount of entanglement [37], one might expect the distance of a quantum code to be related to the amount of entanglement contained in its codewords. Given that entanglement is a necessary condition for achieving a quantum advantage in quantum metrology [4], one might expect that random qubit states should also be good candidate probe states when a small number of erasures occur. Surprisingly however, random quantum states almost surely lose their quantum advantage for quantum metrology even when only one erasure occurs [25]. Hence, entanglement is necessary but not sufficient to ensure that probe states can remain robust with respect to erasure errors. In contrast to the noiseless setting, where the optimal probe state is known, i.e., the GHZ state as stated above, the sufficient conditions for the optimality of noisy probe states are not completely understood.

Given that random qubit codes are not good for robust quantum metrology, one might wonder what codes yield good states for quantum metrology. Given that random symmetric states are almost surely good for robust quantum metrology [25], one might wonder if quantum codes that lie within the symmetric subspace are good for robust quantum metrology . Such codes, known as permutation-invariant quantum codes, have been studied [38–40], and the potential of a code family in [38] has been investigated for its potential in robust quantum metrology [26]. Numerical approaches to searching for robust probe states for robust metrology have also been taken [41]. However, a deeper understanding on the connection between coding theory and the performance of robust quantum metrology remains to be better understood. In particular, the problem of using inherently noisy classical-code-inspired probe states directly for robust quantum metrology, without any active quantum error correction, remains open.

In this paper, we investigate the potential performance of quantum magnetometers using coding-theory inspired probe states where a constant number of qubits are erased. Given any length n binary classical code C, we let $|\psi_C\rangle$ denote a pure state of the form

$$|\psi_C\rangle \coloneqq \frac{1}{\sqrt{|C|}} \sum_{\mathbf{x}\in C} |\mathbf{x}\rangle. \tag{I.1}$$

When interpreted through the lens of quantum error correction (QEC), if C is taken to be a linear code, this corresponds to the logical $|+ + \cdots +\rangle$ state of a CSS code [42] whose logical X operators (including the X-type stabilizers) are defined by the code C. Similarly, for some $[[n, k, d]]$ CSS code, if C is taken to provide only the X-type stabilizers (resp. coset of the X-type stabilizers generated by the product of logical X operators from the set $\{\bar{X}_i : u_i = 1\}$ for a fixed $\mathbf{u} \in \{0, 1\}^k$), then the above is just the logical $|00 \cdots 0\rangle$ (resp. $|\mathbf{u}\rangle$) state [31, Section 10.4.2]. We propose to use $\rho_C = |\psi_C\rangle\langle\psi_C|$ as a probe state for quantum metrology. By interpreting every spin as a qubit, we use Z_j to denote the Pauli operator that applies $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ on qubit j and the identity operator on all other qubits. Treating the magnetic field as a classical field, and assuming that we know the alignment of the magnetic field, the interaction of the magnetic field with each spin is effectively equivalent to the Hamiltonian

$$\chi \bar{H} = \chi(Z_1 + \cdots + Z_n), \tag{I.2}$$

where n is the number of spins in the physical system, χ is the field strength to be estimated [4, Eq. (4)], and $\bar{H} \coloneqq Z_1 + \cdots + Z_n$ is the generator of the subsequent unitary evolution, $U_\chi = e^{-i\chi \bar{H}}$, in the noiseless case. The

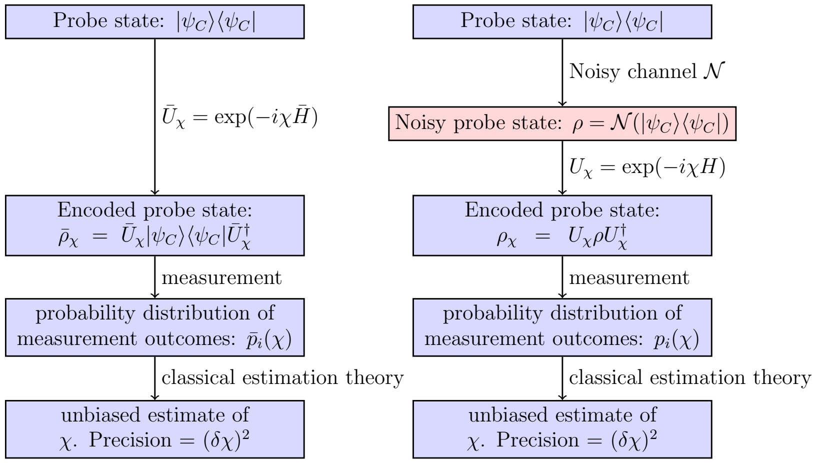

FIG. 2. Comparison of noiseless quantum metrology and robust quantum metrology.

quantum resource utilized here is ρ_C, which depends on our choice of a classical binary code C. The goal in quantum metrology is to have an unbiased estimate of χ that minimizes the variance $(\delta\chi)^2$.

We are interested in quantum metrology that uses noisy probe states. We consider the simplest scenario where noise only occurs on a probe state after it is prepared, and subsequent unitary evolution of the noisy probe state occurs perfectly. We compare the settings of noiseless quantum metrology and robust quantum metrology in Fig. 2.

First, we review the noiseless quantum metrology problem, a schematic of which is illustrated in the left side of Fig. 2. The unknown parameter to be estimated, χ, is embedded in a Hamiltonian $\chi\bar{H}$. Here, we set time to be equal to 1, because it can be absorbed into the parameter χ that we wish to estimate. To extract information about the unknown parameter χ, one prepares the probe state $|\psi_C\rangle\langle\psi_C|$ and allows information on χ to accumulate on the probe state. The unitary evolution $\bar{U}_\chi = \exp(-i\chi\bar{H})$ in accordance with the laws of quantum mechanics then takes the initial state $|\psi_C\rangle\langle\psi_C|$ to an encoded state $\bar{\rho}_\chi = \bar{U}_\chi|\psi_C\rangle\langle\psi_C|\bar{U}_\chi^\dagger$. One then measures the encoded probe state $\bar{\rho}_\chi$ with respect to an observable M that is an unbiased estimator of χ. This outputs a classical probability distribution $\bar{p}_i(\chi)$ that depends on χ. The classical Fisher information of this probability distribution quantifies the precision in which one can estimate χ. Namely, the variance of the parameter χ can be expressed in terms of the variance of M using the error propagation formula [6, (1)], which is

$$(\delta\chi)^2 = \frac{\mathrm{Tr}(\bar{\rho}_\chi M^2) - \mathrm{Tr}(\bar{\rho}_\chi M)^2}{\left|\frac{\partial}{\partial\chi}\mathrm{Tr}(\bar{\rho}_\chi M)\right|^2}. \tag{I.3}$$

The goal in quantum metrology is to minimize this variance with respect to all possible observables M. A celebrated result in quantum metrology is that such a minimum variance is given by the fundamental quantum Cramér-Rao bound which was proven by Helstrom and Holevo [43–45]. This minimum variance is simply the inverse of the quantum Fisher information (QFI) [4, 6, 43, 46], namely,

$$\min\{(\delta\chi)^2 : M = M^\dagger, \mathbb{E}(M) = \chi\} = \mathrm{QFI}^{-1}. \tag{I.4}$$

The QFI is a metric [47, 48], which in this case depends on the probe state and the generator $\bar{H}$. However, evaluating the QFI explicitly requires the full spectral decomposition of the probe state, and may thus be challenging to find in general.

Second, we consider a robust metrology problem, which is the focus of this paper. We illustrate this schematically in the right side of Fig. 2. We model the introduction of noise with a noisy quantum channel $\mathcal{N}$ that acts on the initial probe state $|\psi_C\rangle\langle\psi_C|$, and denote the resultant noisy probe state as $\rho = \mathcal{N}(|\psi_C\rangle\langle\psi_C|)$. In our case, since $\mathcal{N}$ is an erasure channel, it is mathematically equivalent to a partial trace on ρ. We denote the Hamiltonian that acts on ρ as H_χ. Note that H_χ is not necessarily equal to $\bar{H}_\chi$, because the number of qubits in the state ρ can differ from that of $|\psi_C\rangle\langle\psi_C|$. This is indeed the case when $\mathcal{N}$ is an erasure channel. In particular, if $\mathcal{N}$ erases t qubits, then $H_\chi = \chi(Z_1 + \cdots + Z_{n-t})$, and we denote the corresponding generator as

$$H = H_\chi/\chi = Z_1 + \cdots + Z_{n-t}. \tag{I.5}$$

Denoting $U_\chi = e^{-i\chi H}$, one then measures the encoded probe state $\rho_\chi = U_\chi\rho U_\chi^\dagger$ with respect to an observable M that is an unbiased estimator of χ. This outputs a classical probability distribution $p_i'(\chi)$. The classical Fisher information of this probability distribution quantifies the precision in which one can estimate χ, and the optimal $(\delta\chi)^2$ is given by the inverse of the QFI of ρ_χ.

Crucially, a celebrated result of quantum metrology is that quantum resources can, in a noiseless setting, offer a quantum advantage for metrology. Using only classical resources, the optimal $1/(\delta\chi)^2$ scales linearly with the number of spins n. Once we are allowed to use a quantum probe state, there can be a quadratic scaling in the optimal $(\delta\chi)^2$. Namely, in a noiseless scenario, the optimal $1/(\delta\chi)^2$ is $\Omega(n^2)$. Moreover, the optimal probe state is simply an n-qubit GHZ state. Unfortunately, once a GHZ state decoheres, $1/(\delta\chi)^2$ once again scales linearly with n. A very useful feature to have is for the QFI of ρ_χ to scale quadratically with n, when the noise introduced by $\mathcal{N}$ is not too severe.

In this paper, given any classical code C with a minimum distance that is at least $t+1$, we obtain upper and lower bounds on $(\delta\chi)^2$ after any t qubits are erased from the quantum probe state (II.1). (The minimum distance is used indirectly via the "t-disjointness" property defined in Section II C.) In this scenario, we find that using our code-inspired probe states, even after t erasure errors have occurred, we can estimate χ so that $1/(\delta\chi)^2$ scales with the variances of the weight distributions of the corresponding 2^t shortened codes. We emphasize that our result applies in a very general setting, as aside from a distance criterion we impose on the classical code C, no other assumptions are made. Hence, C can in general be a non-linear binary code. Moreover, we show that if C is any constant-length code concatenated with repetition codes of length linear in n, then $1/(\delta\chi)^2$ is at least quadratic in n.

An implication of our result is that, given any CSS code of constant length, we can concatenate it with repetition codes of length linear in n. If we do so and pick an appropriate state in the CSS codespace, the QFI under erasure errors is boosted by the concatenation with the inner repetition codes. The operational significance is that (a) CSS codes are well-understood in quantum coding theory; (b) we can achieve robust quantum metrology with CSS states, without any error correction performed; and (c) our framework for robust quantum metrology is compatible with subsequent protocols where quantum error correction on CSS codes is required [14–21].

In practice, it is not only important to obtain bounds on the minimum $(\delta\chi)^2$ possible for quantum metrology with noisy probe states, but also important to know what measurements to make to get close to these bounds. We address this by giving an explicit observable that one can measure. When one measures this observable on our noisy code-inspired probe state that has encoded information about χ, we show that $(\delta\chi)^2$ depends on the variances of the weight distributions of the corresponding shortened codes in the limit of small magnetic field strength χ. We illustrate the performance of measuring this explicit observable with our general lower bound on the optimal $(\delta\chi)^2$ in Fig. 3.

Now let us outline the structure of the paper. In Section II, we give the main results of our paper, which are bounds on the precision of estimating χ after t erasures have occurred on a probe state $|\psi_C\rangle$ constructed from a classical code C. In Section II A, we revisit the probe state $|\psi_C\rangle$ and give the (generator based) upper and lower bounds in the literature that we use to bound the QFI. In Section II B, we consider the example where C is a binary Reed-Muller code with parameters RM(1,3), and evaluate upper and lower bounds on its QFI in the noiseless case as well as when there is a single erasure. In Section II C, we introduce notation related to having multiple erasures, and define a disjointness property for partitions of C that we need to establish our main results. In Section II D, we give a lower bound on the QFI of a probe state $|\psi_C\rangle$ after t erasure errors have occurred. This lower bound is related to the variances of the weight distributions of 2^t shortened codes of C. We also show how concatenating C with inner repetition codes can allow the QFI to scale quadratically with n as long as the length of the outer code is held constant and the number of erasures t remains bounded by the disjointness criterion. Our results imply that our probe states can also tolerate a linear number of burst erasures. In Section II E, we give corresponding upper bounds on the QFI. In Section III, we give an explicit observable that when measured on the Hamiltonian-evolved noisy probe state gives results that are consistent with those in Section II D. In Section IV, we explain how our results can work with explicit codes. Finally in Section V, we summarize our results and discuss what we think are interesting problems to consider in the future.

II. BOUNDS ON THE QFI AFTER ERASURES

In this section, we investigate bounds on the QFI of our candidate probe state after t erasure errors have occurred. An erasure error occurs if we know which qubit has been completely destroyed. Similarly, t erasure errors occur if we know which t qubits have been completely destroyed. When t erasure errors occur, we can always write down the labels of the erased qubits to be $j_1, \dots, j_t$ and let $E = \{j_1, \dots, j_t\}$ denote the corresponding set of erasures. Without loss of generality, we can assume that $j_1 < \dots < j_t$.

A. Probe states from classical codes

Recall that given any length n binary linear code C, $\rho_C = |\psi_C\rangle\langle\psi_C|$ is the probe state that we plan to use to estimate the unknown parameter χ corresponding to the generator $\bar{H} = Z_1 + \cdots + Z_n$, where

$$|\psi_C\rangle := \frac{1}{\sqrt{|C|}} \sum_{\mathbf{x}\in C} |\mathbf{x}\rangle. \tag{II.1}$$

It will be clear from the following example in Section II B that, in general, when C is a linear code, $|\psi_C\rangle$ is the logical $|+ + \cdots +\rangle$ state of a CSS code whose logical X operators (including all X-type stabilizers) are given by the classical code C. When erasure errors occur on a subset $E = \{j_1, \dots, j_t\}$ of qubits in the probe state ρ_C, we obtain a state that is just the partial trace of ρ_C with the qubits labeled by E being traced out. We denote this state as $\rho_C[E]$. In the robust metrology problem, we estimate the unknown parameter χ corresponding to the generator $H = Z_1 + \cdots + Z_{n-t}$ on the corrupted probe state $\rho_C[E]$. For notational simplicity, we will denote $\rho = \rho_C[E]$ in the rest of the paper whenever the set E and the code C is clear from the context.

Since we focus on robust quantum metrology, the QFI that we wish to bound is not the one for the noiseless probe state, but that of the noisy probe state when t erasure errors have occurred. The QFI is a quantity that depends on the probe state and the generator, and evaluating it explicitly requires the spectral decomposition of the probe state. Since it is not always possible to analytically determine the spectral decomposition of a probe state, we rely on generator bounds, which give upper and lower bounds on the QFI. In particular, we have

$$\mathrm{QFI} \geq \|[\rho, H]\|_2^2 = 2\,\mathrm{Tr}(\rho^2 H^2) - 2\,\mathrm{Tr}(\rho H \rho H), \tag{II.2}$$

and this bound is tight when ρ is a pure state, which corresponds to the case where zero erasures occur. There is a also an upper bound for the QFI that is tight when ρ is a pure state [4], namely,

$$\mathrm{QFI} \leq 4\,\mathrm{Tr}(\rho H^2) - 4\,\mathrm{Tr}(\rho H)^2. \tag{II.3}$$

B. Example: A probe state from a [[8,3,2]] Reed Muller code

Consider the $[[8,3,2]]$ quantum Reed-Muller code [49, 50] described by two classical binary codes C_2 and C_1 defined as $C_2 := \mathrm{RM}(0,3)$ and $C_1 := \mathrm{RM}(1,3)$, respectively. Due to the properties of RM codes, we have $C_2 \subset C_1$ and the dimensions are $\dim(C_2) = 1$ and $\dim(C_1) = 4$. The standard generator matrix for C_1 is given by

$$G(C_1) = \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] = \left[\begin{array}{c} G(C_2) \\ \hline G(C_1/C_2) \end{array}\right]. \tag{II.4}$$

The X-stabilizers are given by C_2, the length 8 repetition code, and the Z-stabilizers are given by $C_1^{\perp} = C_1$ since C_1 is the $[8,4,4]$ extended Hamming code that is self-dual. The canonical generators for the logical X operators correspond to degree-1 monomials that generate the space C_1/C_2, namely $\bar{X}_1 = X_2X_4X_6X_8$, $\bar{X}_2 = X_3X_4X_7X_8$, $\bar{X}_3 = X_5X_6X_7X_8$. Therefore, for $x_1, x_2, x_3 \in \{0,1\}$, the logical computational basis states can be written as

$$|x_1x_2x_3\rangle_L \equiv \frac{1}{\sqrt{2}} \bar{X}_1^{x_1}\bar{X}_2^{x_2}\bar{X}_3^{x_3} \left(|00000000\rangle + |11111111\rangle\right). \tag{II.5}$$

We choose the probe state for metrology as

$$|\psi\rangle = |+++\rangle_L \equiv \frac{1}{4}\sum_{c\in C_1} |c\rangle\,. \tag{II.6}$$

First, let us assume that the channel erases the first qubit, so that the resulting reduced density matrix is $\rho = \mathrm{Tr}_1\left[|\psi\rangle\langle\psi|\right]$. The generator matrix, $G_1^p = G(C_1^p) \in \{0,1\}^{4\times 7}$, for the code C_1 punctured in the first position, which is the standard $[7,4,3]$ Hamming code, is the matrix $G(C_1)$ with the first column removed. The last 3 rows of G_1^p, denoted as the matrix $G_1^s = G(C_1^s) \in \{0,1\}^{3\times 7}$, is a generator matrix for the dual of the Hamming code, which is the shortened $\mathrm{RM}(1,3)$ code, also called the $[7,3,4]$ simplex code. Therefore, the aforesaid reduced density matrix is

$$\rho = \frac{1}{16}\sum_{c_1,c_2\in C_1^s} |c_1\rangle\langle c_2| + \frac{1}{16}\sum_{c_1,c_2\in C_1^s} |\underline{1}\oplus c_1\rangle\langle \underline{1}\oplus c_2|\,, \tag{II.7}$$

where $\underline{1}$ denotes the length 7 vector with all entries equal to 1. In this example, $H = \sum_{i=1}^{7} Z_i$. To obtain a lower bound on the QFI of the probe state after the first qubit is erased, it suffices to calculate $2\operatorname{Tr}\left(\rho^2 H^2\right) - 2\operatorname{Tr}(\rho H \rho H) = \|[\rho, H]\|_2^2$. We first observe that

$$H^2 = 7I_{128} + 2\sum_{\substack{i,j=1\\ i<j}}^{7} Z_i Z_j, \tag{II.8}$$

$$\rho^2 = \frac{1}{256}\sum_{c_1,c_2,c_1',c_2' \in C_1^s}\Big[|c_1\rangle \langle c_2|c_1'\rangle \langle c_2'| + |\underline{1}\oplus c_1\rangle \langle \underline{1}\oplus c_2|\underline{1}\oplus c_1'\rangle \langle \underline{1}\oplus c_2'| \Big] \tag{II.9}$$

$$= \frac{8}{256}\sum_{c_1,c_2' \in C_1^s} [|c_1\rangle \langle c_2'| + |\underline{1}\oplus c_1\rangle \langle \underline{1}\oplus c_2'|] \tag{II.10}$$

$$= \frac{1}{32}\sum_{c_1,c_2 \in C_1^s}\Big[|c_1\rangle \langle c_2| + |\underline{1}\oplus c_1\rangle \langle \underline{1}\oplus c_2| \Big]. \tag{II.11}$$

Then we can calculate

$$\rho^2 H^2 = \underbrace{\frac{7}{32}\sum_{c_1,c_2\in C_1^s}\Big[|c_1\rangle \langle c_2| + |\underline{1}\oplus c_1\rangle \langle \underline{1}\oplus c_2| \Big]}_{A} + \underbrace{\frac{2}{32}\sum_{c_1,c_2\in C_1^s}\sum_{\substack{i,j=1\\ i<j}}^{7}\Big[|c_1\rangle \langle c_2| Z_iZ_j + |\underline{1}\oplus c_1\rangle \langle \underline{1}\oplus c_2| Z_iZ_j \Big]}_{B}, \tag{II.12}$$

$$\operatorname{Tr}(A) = \frac{7}{32}\times(2\times 8) = \frac{7}{2}, \tag{II.13}$$

$$\operatorname{Tr}(B) \overset{\text{(a)}}{=} \frac{1}{16}\sum_{c_1\in C_1^s}\sum_{\substack{i,j=1\\ i<j}}^{7} 2(-1)^{c_1(e_i+e_j)^T} \tag{II.14}$$

$$\overset{\text{(b)}}{=} \frac{1}{8}\left[\binom{7}{2} + \left\{\binom{4}{2} + \binom{3}{2} - \left(21 - \binom{4}{2} - \binom{3}{2}\right)\right\}\times 7\right] \tag{II.15}$$

$$= 0 \tag{II.16}$$

$$\Rightarrow 2\operatorname{Tr}(\rho^2 H^2) = 7. \tag{II.17}$$

In step (a) the vectors e_i and e_j denote the standard basis vectors for $\{0,1\}^7$ with a single entry 1 in the i-th and j-th entry, respectively, and zeros elsewhere. We have used the fact that $Z_iZ_j|c_2\rangle = (-1)^{c_2(e_i+e_j)^T}|c_2\rangle$ and $c_1 = c_2$ for the trace to be non-zero. Furthermore, in step (b) we have used the fact that all non-zero codewords of the simplex code C_1^s have weight exactly 4. Next we calculate

$$\rho H = \frac{1}{16}\sum_{c_1,c_2\in C_1^s}\sum_{i=1}^{7}\Big[|c_1\rangle \langle c_2| Z_i + |\underline{1}\oplus c_1\rangle \langle \underline{1}\oplus c_2| Z_i \Big] \tag{II.18}$$

$$= \frac{1}{16}\sum_{c_1,c_2\in C_1^s}\left(\sum_{i=1}^{7}(-1)^{c_{2,i}}\right)\Big[|c_1\rangle \langle c_2| - |\underline{1}\oplus c_1\rangle \langle \underline{1}\oplus c_2| \Big]. \tag{II.19}$$

This implies that

$$\rho H \rho H = \frac{1}{256}\left[\sum_{c_1,c_2\in C_1^s}\sum_{i=1}^{7}(-1)^{c_{2,i}}\left(|c_1\rangle \langle c_2| - |\underline{1}\oplus c_1\rangle \langle \underline{1}\oplus c_2|\right)\right]\left[\sum_{c_1',c_2'\in C_1^s}\sum_{j=1}^{7}(-1)^{c_{2,j}'}\left(|c_1'\rangle \langle c_2'| - |\underline{1}\oplus c_1'\rangle \langle \underline{1}\oplus c_2'|\right)\right] \tag{II.20}$$

$$= \frac{1}{256} \sum_{c_1, c_2, c_2' \in C_1^s} \left(\sum_{i,j=1}^{7} (-1)^{c_{2,i} + c'_{2,j}} \right) \left(|c_1\rangle \langle c_2'| + |\underline{1} \oplus c_1\rangle \langle \underline{1} \oplus c_2'| \right) \tag{II.21}$$

$$= \frac{1}{256} \left[\sum_{c_2 \in C_1^s} \sum_{i=1}^{7} (-1)^{c_{2,i}} \right] \left[\sum_{c_1, c_2' \in C_1^s} \left(\sum_{j=1}^{7} (-1)^{c'_{2,j}} \right) \left(|c_1\rangle \langle c_2'| + |\underline{1} \oplus c_1\rangle \langle \underline{1} \oplus c_2'| \right) \right] \tag{II.22}$$

$$= 0. \tag{II.23}$$

Hence, it follows that $2\,\mathrm{Tr}(\rho H \rho H) = 0$, from which it follows that

$$\|[\rho, H]\|_2^2 = 2\,\mathrm{Tr}(\rho^2 H^2) - 2\,\mathrm{Tr}(\rho H \rho H) = 7. \tag{II.24}$$

If any qubit other than the first is erased, it can be easily verified that the resulting shortened code C_1^s, where the punctured bit takes the value 0 in all codewords, has an identical weight distribution as for the above case of the first bit being erased. A similar statement is true for the coset of this shortened code generated by adding the all 1s vector to all codewords, corresponding to the second summation in ρ above. Hence, if exactly one qubit out of the 8 are erased, then

$$\|[\rho, H]\|_2^2 = 7. \tag{II.25}$$

Let us now calculate the QFI lower bound for the state $|\psi\rangle$ when there are no erasures. First, we have

$$\rho_C = |\psi\rangle \langle\psi| = \frac{1}{16} \sum_{c_1, c_2 \in C_1} |c_1\rangle \langle c_2| = \rho^2. \tag{II.26}$$

In this case, we can take $H = \sum_{i=1}^{8} Z_i$. Then $H^2 = 8 I_{256} + 2 \sum_{i<j}^{8} Z_i Z_j$. Hence,

$$\rho^2 H^2 = \frac{8}{16} \sum_{c_1, c_2 \in C_1} |c_1\rangle \langle c_2| + \frac{2}{16} \sum_{c_1, c_2 \in C_1} \sum_{\substack{i,j=1 \\ i<j}}^{8} |c_1\rangle \langle c_2| Z_i Z_j \tag{II.27}$$

$$= \frac{1}{2} \sum_{c_1, c_2 \in C_1} |c_1\rangle \langle c_2| + \frac{1}{8} \sum_{c_1, c_2 \in C_1} \sum_{\substack{i,j=1 \\ i<j}}^{8} (-1)^{c_2 (e_i + e_j)^T} |c_1\rangle \langle c_2|. \tag{II.28}$$

It is clear that the trace of the first term is $16/2 = 8$. For the trace of the second term, we calculate

$$\frac{1}{8} \sum_{c_1 \in C_1} \sum_{\substack{i,j=1 \\ i<j}}^{8} (-1)^{c_1 (e_i + e_j)^T} \overset{(a)}{=} \frac{1}{8} \left[2 \times \binom{8}{2} + 14 \times \left\{ \binom{4}{2} + \binom{4}{2} - \left(\binom{8}{2} - 2 \times \binom{4}{2} \right) \right\} \right] \tag{II.29}$$

$$= 7 - 7 \tag{II.30}$$

$$= 0. \tag{II.31}$$

Here, in step (a) we have used the fact that except the all-zeros codeword and the all-ones codeword, all codewords in C_1 have weight exactly 4. Therefore, we have $2\,\mathrm{Tr}(\rho^2 H^2) = 16$. Next, we observe that

$$\rho H = \frac{1}{16} \sum_{c_1, c_2 \in C_1} \sum_{i=1}^{8} (-1)^{c_{2,i}} |c_1\rangle \langle c_2|. \tag{II.32}$$

So, we can calculate

$$\rho H \rho H = \frac{1}{256} \sum_{c_1, c_2, c_1', c_2' \in C_1} \sum_{i,j=1}^{8} (-1)^{c_{2,i} + c'_{2,j}} |c_1\rangle \langle c_2 | c_1'\rangle \langle c_2'| \tag{II.33}$$

$$= \frac{1}{256}\left[\sum_{c_2\in C_1}\sum_{i=1}^{8}(-1)^{c_{2,i}}\right]\sum_{c_1,c_2'\in C_1}\sum_{j=1}^{8}(-1)^{c'_{2,j}}\left|c_1\right\rangle\left\langle c_2'\right| \tag{II.34}$$

$$= 0, \tag{II.35}$$

again because the only codeword weights in C_1 are $0, 4, 8$. This implies $2\operatorname{Tr}(\rho H\rho H) = 0$ and thus

$$\| [\left|\psi\right\rangle\left\langle\psi\right|, H] \|_2^2 = 2\operatorname{Tr}(\rho^2 H^2) - 2\operatorname{Tr}(\rho H\rho H) = 16. \tag{II.36}$$

Since this generator bound is tight for pure states, we observe that the introduction of a single erasure has more than halved the QFI lower bound for this Reed-Muller probe state. We will show later that this situation can be improved by concatenating the chosen code $C = C_1$ with an inner repetition code. Namely, the QFI lower bound becomes $7r^2$ when we use inner repetition codes of length r. For example, concatenating with repetition codes of lengths $3, 5$, and 8 produces probe states consisting of $24, 40$, and 64 qubits, with QFI lower bounds of $63, 175$, and 448 respectively, assuming a single qubit is erased.

C. Multiple erasures on a general probe state

We will now show that the corrupted probe state $\rho_C[E]$ has a particularly simple form. But first, we have to introduce some notation that corresponds to the partition of the code C into 2^t sets, each labeled by $C_{\mathbf{z},E}$, where $\mathbf{z} = (z_1, \ldots, z_t)$ denotes a length t binary string. The set $C_{\mathbf{z},E}$ consists of all codewords $\mathbf{c}$ in C that satisfy $c_{j_i} = z_i$ for all $i \in \{1, 2, \ldots, t\}$, where $E = \{j_1, j_2, \ldots, j_t\} \subsetneq \{1, 2, \ldots, n\}$ as defined earlier. Now, given a vector $\mathbf{x}$, let $\mathbf{x}[E]$ denote the vector obtained from $\mathbf{x}$ after deleting (puncturing) all components labeled by the set E. Then for all $\mathbf{z} \in \{0, 1\}^t$, we define the length $n - t$ shortened codes of C,

$$C_{\mathbf{z}}[E] \coloneqq \{\mathbf{x}[E] : \mathbf{x} \in C_{\mathbf{z},E}\}. \tag{II.37}$$

Note that $C_{\mathbf{z}}[E]$ is a non-linear code except when $\mathbf{z}$ is the all-zeros vector. Also, we define

$$p_{\mathbf{z}} \coloneqq \frac{|C_{\mathbf{z}}[E]|}{|C|} = \frac{|C_{\mathbf{z},E}|}{|C|}, \tag{II.38}$$

$$\left|\psi_{\mathbf{z}}\right\rangle \coloneqq \frac{1}{\sqrt{|C_{\mathbf{z}}[E]|}}\sum_{\mathbf{x}\in C_{\mathbf{z}}[E]}\left|\mathbf{x}\right\rangle. \tag{II.39}$$

Using this notation, we will now obtain a simple expression for $\rho_C[E]$, the probe state after qubits in E are erased.

Proposition 1. *Let C be a binary code of length n, and let $E = \{j_1, j_2, \ldots, j_t\} \subsetneq \{1, 2, \ldots, n\}$. Let $\rho_C = |\psi_C\rangle\langle\psi_C|$, where $|\psi_C\rangle$ is as given in* (II.1)*. When the qubits belonging to E are erased from ρ_C, the candidate probe state becomes*

$$\rho_C[E] = \frac{1}{|C|}\sum_{\mathbf{z}\in\{0,1\}^t}\sum_{\mathbf{x},\mathbf{y}\in C_{\mathbf{z}}[E]}|\mathbf{x}\rangle\langle\mathbf{y}| = \sum_{\mathbf{z}\in\{0,1\}^t} p_{\mathbf{z}}|\psi_{\mathbf{z}}\rangle\langle\psi_{\mathbf{z}}|. \tag{II.40}$$

In general, the codes $C_{\mathbf{z}}[E]$ for distinct values of $\mathbf{z} \in \{0, 1\}^t$ need not be disjoint, and the states $|\psi_{\mathbf{y}}\rangle$ and $|\psi_{\mathbf{z}}\rangle$ need not be pairwise orthogonal. Proposition 1 only gives a spectral decomposition of $\rho_C[E]$ when the codes $C_{\mathbf{z}}[E]$ are disjoint codes for distinct values of $\mathbf{z} \in \{0, 1\}^t$. This disjointness condition is guaranteed whenever C has distance strictly larger than t, though this distance criteria is not a necessary condition. We make this condition explicit in the following definition.

Definition 2. Let C be a code and E be a t-set such that any pair of codes $C_{\mathbf{y}}[E]$ and $C_{\mathbf{z}}[E]$ are disjoint for distinct $\mathbf{y}, \mathbf{z} \in \{0, 1\}^t$. Then we say that C is t-disjoint with respect to E.

Subsequent subsections obtain upper and lower bounds on the QFI of $U_\chi\, \rho_C[E]\, U_\chi^\dagger$ in terms of the properties of the classical code C. Crucial to the development of these bounds are the *weight enumerators* of $C_{\mathbf{z},E}$ and $C_{\mathbf{z}}[E]$, given respectively by

$$A_{C,k,\mathbf{z},E} = |\{\mathbf{x} \in C_{\mathbf{z},E} \colon \operatorname{wt}(\mathbf{x}) = k\}|, \tag{II.41}$$

$$A_{C,k,\mathbf{z}}[E] = |\{\mathbf{x} \in C_{\mathbf{z}}[E] \colon \operatorname{wt}(\mathbf{x}) = k\}|. \tag{II.42}$$

We correspondingly define $X_{C,\mathbf{z},E}$ and $X_{C,\mathbf{z}}[E]$ to be random variables such that

$$\Pr[X_{C,\mathbf{z},E} = k] = \frac{A_{C,k,\mathbf{z},E}}{|C_{\mathbf{z},E}|}, \tag{II.43}$$

$$\Pr[X_{C,\mathbf{z}}[E] = k] = \frac{A_{C,k,\mathbf{z}}[E]}{|C_{\mathbf{z}}[E]|}. \tag{II.44}$$

In Proposition 3, we prove that the variances of the random variables $X_{C,\mathbf{z},E}$ and $X_{C,\mathbf{z}}[E]$ are equal.

Proposition 3. *Let C be any binary code of length n. Then, for any $E \subsetneq \{1,\dots,n\}$ and $\mathbf{z} \in \{0,1\}^{|E|}$, we have* $\mathrm{Var}(X_{C,\mathbf{z},E}) = \mathrm{Var}(X_{C,\mathbf{z}}[E])$.

Proof. Given any codeword $\mathbf{x} \in C_{\mathbf{z},E}$, there is a corresponding codeword $\mathbf{x}[E] \in C_{\mathbf{z}}[E]$ such that $\mathrm{wt}(\mathbf{x}) = \mathrm{wt}(\mathbf{x}[E]) + \mathrm{wt}(\mathbf{z})$. So, all weights of $C_{\mathbf{z},E}$ are a constant $\mathrm{wt}(\mathbf{z})$ away from the weights of $C_{\mathbf{z}}[E]$, i.e., $X_{C,\mathbf{z},E} = X_{C,\mathbf{z}}[E] + \mathrm{wt}(\mathbf{z})$. Hence, this constant does not affect the variance of these associated random variables. □

D. Lower bounds using the variances of weights

Let us define $Q_E(C)$ as the QFI of the probe state $\rho_C[E]$ with respect to the generator $H = Z_1 + \cdots + Z_{n-t}$. Our first key result in this section is a lower bound for $Q_E(C)$ in terms of the variances of the weight distributions of the codes $C_{\mathbf{z},E}$.

Theorem 4. *Let C be a binary code of length n, and let $E = \{j_1,\dots,j_t\} \subsetneq \{1,\dots,n\}$ label a set of qubits that will be erased. Suppose that C is t-disjoint with respect to E. Then we have*

$$Q_E(C) \geq 8 \sum_{\mathbf{z}\in\{0,1\}^t} p_{\mathbf{z}}^2 \,\mathrm{Var}(X_{C,\mathbf{z},E}). \tag{II.45}$$

Proof. Recall that $\rho = \rho_C[E]$ and $H = Z_1 + \cdots + Z_{n-t}$. Using the disjointness of the codes $C_{\mathbf{z}}$ along with the form of ρ from Proposition 1, we see that

$$\rho^2 = \frac{1}{|C|^2} \sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]| \sum_{\mathbf{x},\mathbf{y}\in C_{\mathbf{z}}[E]} |\mathbf{x}\rangle\langle\mathbf{y}|. \tag{II.46}$$

Now, for every $(n-t)$-bit string $\mathbf{x}$, we have

$$\begin{aligned} H|\mathbf{x}\rangle &= (-\,\mathrm{wt}(\mathbf{x}) + (n-t-\mathrm{wt}(\mathbf{x}))|\mathbf{x}\rangle \\ &= (n-t-2\,\mathrm{wt}(\mathbf{x}))|\mathbf{x}\rangle. \end{aligned} \tag{II.47}$$

The cyclic property of the trace implies that $\mathrm{Tr}(\rho^2 H^2) = \mathrm{Tr}(H\rho^2 H)$. Using (II.46), and taking the trace, we get

$$\mathrm{Tr}(\rho^2 H^2) = \frac{1}{|C|^2} \sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]| \sum_{\mathbf{x},\mathbf{y},\mathbf{u}\in C_{\mathbf{z}}[E]} \langle\mathbf{u}|H|\mathbf{x}\rangle\langle\mathbf{y}|H|\mathbf{u}\rangle. \tag{II.48}$$

Since H is a diagonal operator in the computational basis it follows that

$$\mathrm{Tr}(\rho^2 H^2) = \frac{1}{|C|^2} \sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]| \sum_{\mathbf{x}\in C_{\mathbf{z}}[E]} \langle\mathbf{x}|H|\mathbf{x}\rangle\langle\mathbf{x}|H|\mathbf{x}\rangle. \tag{II.49}$$

Since $\langle\mathbf{x}|H|\mathbf{x}\rangle = (n-t-2\,\mathrm{wt}(\mathbf{x}))$, we establish a connection between $\mathrm{Tr}(\rho^2 H^2)$ and the weight enumerator through the equation

$$\mathrm{Tr}(\rho^2 H^2) = \frac{1}{|C|^2} \sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]|\, q_{\mathbf{z}}, \tag{II.50}$$

where

$$q_{\mathbf{z}} := \sum_{k=0}^{n-t} A_{C,k,\mathbf{z}}[E] \left((n-t)^2 - 4(n-t)k + 4k^2\right). \tag{II.51}$$

Similarly, we can see that

$$
\begin{aligned}
\operatorname{Tr}(\rho H\rho H) =& \frac{1}{|C|^2} \sum_{\mathbf{z}\in\{0,1\}^t} \sum_{\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}\in C_{\mathbf{z}}[E]} \langle \mathbf{v}|H|\mathbf{x}\rangle\langle \mathbf{y}|H|\mathbf{u}\rangle \\
=& \frac{1}{|C|^2} \sum_{\mathbf{z}\in\{0,1\}^t} \sum_{\mathbf{x},\mathbf{y}\in C_{\mathbf{z}}[E]} \langle \mathbf{x}|H|\mathbf{x}\rangle\langle \mathbf{y}|H|\mathbf{y}\rangle \\
=& \frac{1}{|C|^2} \sum_{\mathbf{z}\in\{0,1\}^t} \sum_{\mathbf{x},\mathbf{y}\in C_{\mathbf{z}}[E]} (n-t-2\operatorname{wt}(\mathbf{x}))(n-t-2\operatorname{wt}(\mathbf{y})) \\
=& \frac{1}{|C|^2} \sum_{\mathbf{z}\in\{0,1\}^t} \left(\sum_{\mathbf{x}\in C_{\mathbf{z}}[E]} (n-t-2\operatorname{wt}(\mathbf{x})) \right)^2 . \qquad \text{(II.52)}
\end{aligned}
$$

Then, by direct usage of the definitions of the classical weight enumerators, we get

$$
\operatorname{Tr}(\rho H\rho H) = \sum_{\mathbf{z}\in\{0,1\}^t} \frac{r_{\mathbf{z}}^2}{|C|^2}, \qquad \text{(II.53)}
$$

where

$$
r_{\mathbf{z}} := \sum_{k=0}^{n-t} A_{C,k,\mathbf{z}}[E](n-t-2k). \qquad \text{(II.54)}
$$

Since we have the generator bound $Q_E(C) \geq 2\operatorname{Tr}(\rho^2 H^2) - 2\operatorname{Tr}(\rho H\rho H)$, we can use (II.50) and (II.53) to get

$$
Q_E(C) \geq \frac{2}{|C|^2} \sum_{\mathbf{z}\in\{0,1\}^t} \left(|C_{\mathbf{z}}[E]| q_{\mathbf{z}} - r_{\mathbf{z}}^2 \right). \qquad \text{(II.55)}
$$

Using (II.44) with (II.51) and (II.54) respectively, we get

$$
\begin{aligned}
q_{\mathbf{z}} &= |C_{\mathbf{z}}[E]| \left((n-t)^2 - 4(n-t)\mathbb{E}(X_{C,\mathbf{z}}[E]) + 4\mathbb{E}(X_{C,\mathbf{z}}[E]^2) \right), \qquad \text{(II.56)} \\
r_{\mathbf{z}} &= |C_{\mathbf{z}}[E]| \left((n-t) - 2\mathbb{E}(X_{C,\mathbf{z}}[E]) \right). \qquad \text{(II.57)}
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\frac{r_{\mathbf{z}}^2}{|C_{\mathbf{z}}[E]|^2} &= ((n-t) - 2\mathbb{E}(X_{C,\mathbf{z}}[E]))^2 \\
&= \left((n-t)^2 - 4(n-t)\mathbb{E}(X_{C,\mathbf{z}}[E]) + 4\mathbb{E}(X_{C,\mathbf{z}}[E])^2 \right),
\end{aligned}
$$

it follows that

$$
|C_{\mathbf{z}}[E]| q_{\mathbf{z}} - r_{\mathbf{z}}^2 = 4|C_{\mathbf{z}}[E]|^2 \operatorname{Var}(X_{C,\mathbf{z}}[E]). \qquad \text{(II.58)}
$$

From Proposition 3, it follows from (II.58)

$$
|C_{\mathbf{z}}[E]| q_{\mathbf{z}} - r_{\mathbf{z}}^2 = 4|C_{\mathbf{z}}[E]|^2 \operatorname{Var}(X_{C,\mathbf{z},E}). \qquad \text{(II.59)}
$$

Using (II.59) on the inequality (II.55) then gives the result. □

Remark: Suppose that C is a linear code with distance at least 2, and has no zero columns in its generator matrix. Then for all $z = 0, 1$, and $j = 1, \ldots, n$, the cardinality of $C_{(z),\{j\}}$ is equal to $|C|/2$. Therefore,

$$
Q_{\{j\}}(C) \geq 2(\operatorname{Var}(X_{C,(0),\{j\}}) + \operatorname{Var}(X_{C,(1),\{j\}})). \qquad \text{(II.60)}
$$

Theorem 4 shows an explicit relation between a lower bound for QFI and the classical weight enumerators of the relevant punctured codes under the t-qubit erasure model. We recollect that for quantum metrology we desire the QFI to grow quadratically, or at least superlinearly, with the number of (physical) qubits n. Next we show that concatenating a fixed length code C with a repetition code can boost the QFI and hence potentially lead us towards our goal.

Lemma 5 (Boosting Lemma). *Let C^{outer} be a binary code of of length m. Denote by C the concatenated code of length $n = mr$, where C is the concatenation of C^{outer} with a repetition code of length r as the inner code, i.e., replace each bit of codewords in C^{outer} with r copies of that bit. Let $E = \{i_1, \dots, i_{t'}\} \subsetneq \{1, \dots, n\}$ be a set that labels which t' qubits are to be erased, so that the set of outer blocks with at least one erasure is given by $E_g = \{\lfloor (i-1)/r \rfloor + 1 : i \in E\} = \{j_1, \dots, j_t\} \subsetneq \{1, \dots, m\}$, which has t distinct elements. Suppose that C^{outer} is t-disjoint with respect to E_g. Then we have*

$$Q_E(C) \geq 8r^2 \sum_{\mathbf{z}_g \in \{0,1\}^t} p^2_{\mathbf{z}_g} \operatorname{Var}(X_{C^{\text{outer}}, \mathbf{z}_g, E}). \tag{II.61}$$

Proof. From Theorem 4 we have

$$Q_E(C) \geq \frac{8}{|C|^2} \sum_{\mathbf{z} \in \{0,1\}^{t'}} |C_{\mathbf{z}}[E]|^2 \operatorname{Var}(X_{C, \mathbf{z}, E}). \tag{II.62}$$

Due to the concatenation structure, for many values of $\mathbf{z}$, $|C_{\mathbf{z}}[E]|$ is equal to zero. Hence, in the above summation, we only need to count terms where $\mathbf{z}$ respects the concatenation structure of C. In particular, whenever i_k and $i_{k'}$ are elements of E such that they belong to the same outer block, i.e.,

$$\lfloor (i_k - 1)/r \rfloor = \lfloor (i_{k'} - 1)/r \rfloor, \tag{II.63}$$

we must have $z_{i_k} = z_{i_{k'}}$. Now let us label the qubits in E that occur on the jth outer block as

$$S_j = \{i \in E : \lfloor (i-1)/r \rfloor + 1 = j\}. \tag{II.64}$$

Then, for a given $\mathbf{z} \in \{0,1\}^{t'}$, we know that if "$z_{i_k} = z_{i_{k'}}$ for all $i_k, i_{k'} \in S_j$" holds for all $j \in E_g$, then $|C_{\mathbf{z}}[E]| = |C^{\text{outer}}_{\mathbf{z}_g}[E_g]|$; otherwise, $|C_{\mathbf{z}}[E]| = 0$. Hence, the number of terms in the summation that are non-zero is at most 2^t instead of $2^{t'}$. Next, let $\mathbf{x} = (x_1, \dots, x_m)$ be a codeword of C^{outer}. Then the corresponding concatenated codeword in C is

$$\mathbf{x}' = (\underbrace{x_1, \dots, x_1}_{r}, \overbrace{\dots, \dots, \dots}^{(m-2)r}, \underbrace{x_m, \dots, x_m}_{r}), \tag{II.65}$$

and it comprises of m blocks of repeated indices. The weight of $\mathbf{x}'$ is r times of $\operatorname{wt}(\mathbf{x})$ for every $\mathbf{x} \in C^{\text{outer}}$. Hence, it follows that

$$\operatorname{Var}(X_{C, \mathbf{z}, E}) = r^2 \operatorname{Var}(X_{C^{\text{outer}}, \mathbf{z}_g, S}). \tag{II.66}$$

Since we also have $|C| = |C^{\text{outer}}|$, the lemma follows. $\square$

Now note that the QFI of the concatenated code scales quadratically with the lengths of the inner repetition codes r. Hence, when the outer code is fixed and the length of the inner repetition codes are allowed to grow, the QFI scales like $\Omega(r^2) = \Omega(n^2/m^2)$, and this is quadratic in n since m is a constant.

Remark 1. We also emphasize that another strength of the Boosting Lemma is that it demonstrates that we can sustain a linear number of burst erasures. We say that t' burst erasures occur if t' consecutive qubits are erased. Since t' in the boosting lemma is at most tr, and C being t-disjoint fixes $t < m$, it follows that $t' = \Omega(n)$, since $r = n/m$ scales linearly in n by definition once m is fixed. Hence, following the arguments in the previous paragraph, we can remain robust to a linear number of burst erasures while also having the QFI scale quadratically with n.

So far, we have been discussing the asymptotic performance of the QFI under arbitrary erasure errors and burst erasure errors . Later in Section IV A, we give more explicit lower bounds on the QFI using noisy probe states which arise from Reed Muller codes concatenated with inner repetition codes.

E. Upper bounds using the variances of weights

In the previous section, we have analyzed the minimum scaling of QFI for probe states based on classical codes and also shown that concatenation with repetition codes helps achieve the Heisenberg scaling. However, it is still an interesting problem to explore the upper limit on QFI under mild assumptions, without any concatenation. The following result establishes such an upper bound for code-inspired probe states.

Theorem 6. *Let C be a length n binary code, and let $E = \{j_1, \ldots, j_t\} \subsetneq \{1, 2, \ldots, n\}$. Suppose further that C is t-disjoint with respect to E. Then we have*

$$Q_E(C) \leq 16 \sum_{\mathbf{z}\in\{0,1\}^t} p_{\mathbf{z}}\, \mathbb{E}(X_{C,\mathbf{z}}[E]^2) - 16 \left(\sum_{\mathbf{z}\in\{0,1\}^t} p_{\mathbf{z}}\, \mathbb{E}(X_{C,\mathbf{z}}[E]) \right)^2 . \tag{II.67}$$

Proof. To simplify notation, let $\rho = \rho_C[E]$. For every $(n-1)$-bit string $\mathbf{x}$, we have

$$H|\mathbf{x}\rangle = (-\operatorname{wt}(\mathbf{x}) + (n-1-\operatorname{wt}(\mathbf{x}))|\mathbf{x}\rangle = (n-1-2\operatorname{wt}(\mathbf{x}))|\mathbf{x}\rangle. \tag{II.68}$$

The cyclic property of the trace implies that $\operatorname{Tr}(\rho H^2) = \operatorname{Tr}(H\rho H)$. Using the form of ρ from Proposition 1 we get

$$\operatorname{Tr}(\rho H^2) = \frac{1}{|C|} \sum_{\mathbf{z}\in\{0,1\}^t} \sum_{\mathbf{x},\mathbf{y},\mathbf{u}\in C_{\mathbf{z}}[E]} \langle \mathbf{u}|H|\mathbf{x}\rangle\langle \mathbf{y}|H|\mathbf{u}\rangle. \tag{II.69}$$

Since H is a diagonal operator in the computational basis, it follows that

$$\operatorname{Tr}(\rho H^2) = \frac{1}{|C|} \sum_{\mathbf{z}\in\{0,1\}^t} \sum_{\mathbf{x}\in C_{\mathbf{z}}[E]} \langle \mathbf{x}|H|\mathbf{x}\rangle\langle \mathbf{x}|H|\mathbf{x}\rangle. \tag{II.70}$$

Since $\langle \mathbf{x}|H|\mathbf{x}\rangle = (n-t-2\operatorname{wt}(\mathbf{x}))$, we establish a connection between $\operatorname{Tr}(\rho H^2)$ and weight enumerators through the equation

$$\operatorname{Tr}(\rho H^2) = \frac{1}{|C|} \sum_{\mathbf{z}\in\{0,1\}^t} q_{\mathbf{z}}, \tag{II.71}$$

where $q_{\mathbf{z}}$ is as given in (II.51). Similarly, we can see that

$$\operatorname{Tr}(\rho H) = \frac{1}{|C|} \sum_{\mathbf{z}\in\{0,1\}^t} \sum_{\mathbf{x}\in C_{\mathbf{z}}[E]} \langle \mathbf{x}|H|\mathbf{x}\rangle. \tag{II.72}$$

It follows that

$$\begin{aligned} \operatorname{Tr}(\rho H) &= \frac{1}{|C|} \sum_{\mathbf{z}\in\{0,1\}^t} \sum_{\mathbf{x}\in C_{\mathbf{z}}[E]} (n-t-2\operatorname{wt}(\mathbf{x})) \\ &= \frac{1}{|C|} \sum_{\mathbf{z}\in\{0,1\}^t} r_{\mathbf{z}}, \end{aligned} \tag{II.73}$$

where $r_{\mathbf{z}}$ is as given in (II.54). Since we have the generator bound $Q_E(C) \leq 4\operatorname{Tr}(\rho H^2) - 4\operatorname{Tr}(\rho H)^2$, we can use (II.71) and (II.73) to get

$$Q_E(C) \leq \frac{4}{|C|} \left(\sum_{\mathbf{z}\in\{0,1\}^t} q_{\mathbf{z}} - \frac{1}{|C|} \sum_{\mathbf{y},\mathbf{z}\in\{0,1\}^t} r_{\mathbf{y}} r_{\mathbf{z}} \right). \tag{II.74}$$

Now note that

$$\sum_{\mathbf{z}\in\{0,1\}^t} q_{\mathbf{z}} = (n-t)^2 \sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]| - 4(n-t) \sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]|\, \mathbb{E}(X_{C,\mathbf{z}}[E]) + 4 \sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]|\, \mathbb{E}(X_{C,\mathbf{z}}[E]^2) \tag{II.75}$$

$$= (n-t)^2|C| - 4(n-t) \sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]|\, \mathbb{E}(X_{C,\mathbf{z}}[E]) + 4 \sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]|\, \mathbb{E}(X_{C,\mathbf{z}}[E]^2), \tag{II.76}$$

and

$$\frac{1}{|C|} \left(\sum_{\mathbf{z}\in\{0,1\}^t} r_{\mathbf{z}} \right)^2 = \frac{1}{|C|} \left((n-t) \sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]| - 2 \sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]|\, \mathbb{E}(X_{C,\mathbf{z}}[E]) \right)^2 \tag{II.77}$$

$$= \frac{1}{|C|}\left((n-t)|C| - 2\sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]|\,\mathbb{E}(X_{C,\mathbf{z}}[E])\right)^2 \tag{II.78}$$

$$= (n-t)^2|C| - 4(n-t)\sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]|\,\mathbb{E}(X_{C,\mathbf{z}}[E]) + \frac{4}{|C|}\left(\sum_{\mathbf{z}\in\{0,1\}^t} |C_{\mathbf{z}}[E]|\,\mathbb{E}(X_{C,\mathbf{z}}[E])\right)^2. \tag{II.79}$$

The result then follows. □

Remark 2. We can obtain a simpler, albeit looser, upper bound on the QFI that is expressed explicitly in terms of the variances of the weight distributions of the shortened codes. To arrive at a simpler bound, aside from the assumptions made in Theorem 6, we suppose that we additionally have

$$\sum_{\mathbf{z}\in\{0,1\}^t} p_{\mathbf{z}}\,\mathbb{E}(X_{C,\mathbf{z}}[E]) \ge s, \tag{II.80}$$

for some $s \ge 1$. Then

$$Q_E(C) \le 16\left(\frac{s}{n}\right)\sum_{\mathbf{z}\in\{0,1\}^t} p_{\mathbf{z}}\ \mathrm{Var}(X_{C,\mathbf{z}}[E]) + 16\left(1-\frac{s}{n}\right)\sum_{\mathbf{z}\in\{0,1\}^t} p_{\mathbf{z}}\,\mathbb{E}(X_{C,\mathbf{z}}[E]^2). \tag{II.81}$$

Proof of (II.81) *in Remark 2.* Now let us consider an inequality relating $\sum_{\mathbf{z}} p_{\mathbf{z}} x_{\mathbf{z}}^2$ and $n(\sum_{\mathbf{z}} p_{\mathbf{z}} x_{\mathbf{z}})^2$, where $p_{\mathbf{z}}$ are probabilities, and $x_{\mathbf{z}}$ are non-negative numbers in the interval $[0,n]$ for some positive number n. Suppose further that $\sum_{\mathbf{z}} p_{\mathbf{z}} x_{\mathbf{z}} \ge s \ge 1$. Then it follows that

$$\sum_{\mathbf{z}} p_{\mathbf{z}} x_{\mathbf{z}}^2 \le n\sum_{\mathbf{z}} p_{\mathbf{z}} x_{\mathbf{z}} \le \left(\frac{n}{s}\right)\left(\sum_{\mathbf{z}} p_{\mathbf{z}} x_{\mathbf{z}}\right)^2. \tag{II.82}$$

Using (II.82) and identifying $x_{\mathbf{z}}$ with $\mathbb{E}(X_{C,\mathbf{z}}[E])$, we find that $-\left(\sum_{\mathbf{z}} p_{\mathbf{z}} x_{\mathbf{z}}\right)^2 \le \left(\frac{s}{n}\right)\sum_{\mathbf{z}} p_{\mathbf{z}} x_{\mathbf{z}}^2$. Substituting this inequality into Theorem 6 gives the upper bound (II.81). □

Theorem 6 shows that the QFI upper bound depends on a variance-like quantity on the weight distributions of the 2^t shortened codes. If $t \le k$ and the submatrix comprising the columns corresponding to E of any generator matrix for C has full rank, then we will indeed have $p_{\mathbf{z}} = \frac{1}{2^t}$ by symmetry of binary subspaces. Furthermore, both the lower and upper bounds on QFI indicate that we need codes with a large variation in codeword weights. So, for such codes it is reasonable to expect that $\mathbb{E}(X_{C,\mathbf{z}}[E]) \approx \frac{n}{2}$ whenever $t \ll n$, in which case the QFI lower and upper bounds simplify to

$$\frac{2}{2^t}\left[\frac{4}{2^t}\sum_{\mathbf{z}\in\{0,1\}^t} \mathbb{E}(X_{C,\mathbf{z}}[E]^2) - n^2\right] \le Q_E(C) \le 4\left[\frac{4}{2^t}\sum_{\mathbf{z}\in\{0,1\}^t} \mathbb{E}(X_{C,\mathbf{z}}[E]^2) - n^2\right]. \tag{II.83}$$

Our comparison shows quite clearly that the generator-based QFI lower and upper bounds from the literature are away by a factor 2^{-t-1} in our setting of code-inspired probe states. But, it also explicitly shows that codes with quadratically scaling second moments on the weight distributions of their shortened versions are highly desirable for robust metrology.

III. AN EXPLICIT OBSERVABLE

We first review some facts about the symmetric logarithm derivative in quantum metrology [4]. When an encoded probe state $\rho_\chi = U_\chi \rho U_\chi^\dagger$ is a pure state, the QFI of ρ_χ for unitary dynamics U_χ generated by H is given by

$$\mathrm{QFI} = \mathrm{Tr}(\rho_\chi L_\chi^2), \tag{III.1}$$

where

$$L_\chi := 2i(\rho_\chi H - H\rho_\chi). \tag{III.2}$$

Measuring in the eigenbasis of the operator L_χ then gives the optimal QFI. The operator L_χ is known as the symmetric logarithm derivative (SLD) of ρ_χ with respect to the generator H, because

$$\frac{d\rho_\chi}{d\chi} = \frac{1}{2}\left(L_\chi \rho_\chi - \rho_\chi L_\chi\right). \tag{III.3}$$

Now note that because the trace is basis-invariant, we can also write QFI $= \operatorname{Tr}\rho(U_\chi^\dagger L_\chi U_\chi)^2$. Since the generator H is invariant under conjugation by U_χ or $U_\chi^\dagger$, we can see that $U_\chi^\dagger L U_\chi = 2i(\rho H - H\rho) =: L$. Hence, we can rewrite the QFI as $\operatorname{Tr}(\rho L^2)$ where L is by definition independent of χ. When χ is close to zero, ρ_χ will be close to ρ and L will become close to the optimal observable.

For us, ρ is not a pure state, but a mixed state with the form

$$\rho = \sum_{\mathbf{z}} p_{\mathbf{z}} |\psi_{\mathbf{z}}\rangle\langle\psi_{\mathbf{z}}|. \tag{III.4}$$

Unfortunately, the symmetric logarithmic derivative (of $\rho_\chi = U_\chi \rho U_\chi^\dagger$) becomes more complicated in this case. We can nonetheless consider the operator

$$L = 2i\sum_{\mathbf{z}} p_{\mathbf{z}}(|\psi_{\mathbf{z}}\rangle\langle\psi_{\mathbf{z}}|H - H|\psi_{\mathbf{z}}\rangle\langle\psi_{\mathbf{z}}|) \tag{III.5}$$

as the observable to measure on the encoded probe state

$$\rho_\chi = U_\chi \rho U_\chi^\dagger = \sum_{\mathbf{z}} p_{\mathbf{z}} |\psi_{\mathbf{z}}^\chi\rangle\langle\psi_{\mathbf{z}}^\chi|, \tag{III.6}$$

where $H = Z_1 + \cdots + Z_{n-t}$ and $|\psi_{\mathbf{z}}^\chi\rangle := U_\chi|\psi_{\mathbf{z}}\rangle$. Because L in (III.5) has a form that is reminiscent of (III.2), we can intuitively expect L to be an observable that gives estimates of χ with $(\delta\chi)^2$ that scales similarly with that given by the optimal observable. To evaluate the performance of our observable L, we must evaluate the quantities

$$\operatorname{Tr}(\rho_\chi L), \quad \frac{\partial}{\partial\chi}\operatorname{Tr}(\rho_\chi L), \quad \operatorname{Tr}(\rho_\chi L^2). \tag{III.7}$$

These quantities can be calculated using the following lemma.

Lemma 7. *For every $\mathbf{y} \in \{0,1\}^t$, let $|\psi_{\mathbf{y}}\rangle = \frac{1}{\sqrt{|C_{\mathbf{y}}|}}\sum_{\mathbf{x}\in C_{\mathbf{y}}}|\mathbf{x}\rangle$, where $C_{\mathbf{y}}$ is a binary code of length $n-t$. Let $|\psi_{\mathbf{y}}^\chi\rangle = e^{-i\chi H}|\psi_{\mathbf{y}}\rangle$, where $H = Z_1 + \cdots + Z_{n-t}$. Then*

$$\langle\psi_{\mathbf{y}}^\chi|\psi_{\mathbf{y}}\rangle = \frac{1}{|C_{\mathbf{y}}|}e^{i\chi(n-t)}\sum_{\mathbf{x}\in C_{\mathbf{y}}} e^{-2i\chi\,\mathrm{wt}(\mathbf{x})}, \tag{III.8}$$

$$\langle\psi_{\mathbf{y}}^\chi|H|\psi_{\mathbf{y}}\rangle = \frac{1}{|C_{\mathbf{y}}|}e^{i\chi(n-t)}\sum_{\mathbf{x}\in C_{\mathbf{y}}} ((n-t) - 2\,\mathrm{wt}(x))e^{-2i\chi\,\mathrm{wt}(\mathbf{x})}, \tag{III.9}$$

$$\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}\rangle = (n-t) - \frac{2}{|C_{\mathbf{y}}|}\sum_{\mathbf{x}\in C_{\mathbf{y}}}\mathrm{wt}(\mathbf{x}), \tag{III.10}$$

$$\langle\psi_{\mathbf{y}}|H^2|\psi_{\mathbf{y}}\rangle = (n-t)^2 - \frac{4(n-t)}{|C_{\mathbf{y}}|}\sum_{\mathbf{x}\in C_{\mathbf{y}}}\mathrm{wt}(\mathbf{x}) + \frac{4}{|C_{\mathbf{y}}|}\sum_{\mathbf{x}\in C_{\mathbf{y}}}\mathrm{wt}(\mathbf{x})^2. \tag{III.11}$$

Proof. It is easy to see that

$$H|\mathbf{x}\rangle = (n-t) - 2\,\mathrm{wt}(\mathbf{x})|\mathbf{x}\rangle, \tag{III.12}$$

$$H^2|\mathbf{x}\rangle = ((n-t) - 2\,\mathrm{wt}(\mathbf{x}))^2|\mathbf{x}\rangle, \tag{III.13}$$

$$e^{-i\chi H}|\mathbf{x}\rangle = \exp[-i\chi((n-t) - 2\,\mathrm{wt}(\mathbf{x}))]|\mathbf{x}\rangle. \tag{III.14}$$

The results then directly follow from these observations. □

Now we present the main result of this section that makes use of the above lemma.

Theorem 8. *Let $\rho = \sum_{\mathbf{y}\in\{0,1\}^t} p_{\mathbf{y}} |\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|$, where $|\psi_{\mathbf{y}}\rangle$ satisfy the assumptions of Lemma 7. Assume that the length $(n-t)$ (potentially non-linear) codes $C_{\mathbf{y}}$ satisfy $C_{\mathbf{y}} \cap C_{\mathbf{z}} = \emptyset$ for distinct $\mathbf{y}$ and $\mathbf{z}$, so that $|\psi_{\mathbf{y}}\rangle$ and $|\psi_{\mathbf{z}}\rangle$ are pairwise orthogonal, term by term. Let ρ_χ and L be as defined in* (III.6) *and* (III.5) *respectively. Then we have*

$$\lim_{\chi\to 0} (\delta\chi)^2 \leq \frac{1}{16\sum_{\mathbf{y}\in\{0,1\}^t} p_{\mathbf{y}}^2 V_{\mathbf{y}}}, \tag{III.15}$$

where $V_{\mathbf{y}}$ denotes the variance of the weight distribution of $C_{\mathbf{y}}$.

Proof. Since H is a diagonal matrix and we expand $|\psi_{\mathbf{y}}\rangle$ in the computational basis, we have

$$\begin{aligned} \mathrm{Tr}(\rho_\chi L) &= 2i \sum_{\mathbf{u},\mathbf{y}} p_{\mathbf{u}} p_{\mathbf{y}} \,\mathrm{Tr}\left[|\psi_{\mathbf{u}}^\chi\rangle\langle\psi_{\mathbf{u}}^\chi| \left(|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H - H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}| \right)\right] \\ &= 2i \sum_{\mathbf{y}} p_{\mathbf{y}}^2 \left[\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}^\chi\rangle\langle\psi_{\mathbf{y}}^\chi|\psi_{\mathbf{y}}\rangle - \langle\psi_{\mathbf{y}}|\psi_{\mathbf{y}}^\chi\rangle\langle\psi_{\mathbf{y}}^\chi|H|\psi_{\mathbf{y}}\rangle \right]. \end{aligned} \tag{III.16}$$

Now by interpreting $\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}^\chi\rangle\langle\psi_{\mathbf{y}}^\chi|\psi_{\mathbf{y}}\rangle$ as a complex number z, and noting that $z - z^* = 2i\mathrm{Im}(z)$, we can use the results of Lemma 7 to get

$$\begin{aligned} \mathrm{Tr}(\rho_\chi L) &= -4 \sum_{\mathbf{y}} \frac{p_{\mathbf{y}}^2}{|C_{\mathbf{y}}|^2} \mathrm{Im}\left[\sum_{\mathbf{x}\in C_{\mathbf{y}}} e^{-2i\chi\,\mathrm{wt}(\mathbf{x})} \sum_{\mathbf{v}\in C_{\mathbf{y}}} ((n-t) - 2\,\mathrm{wt}(\mathbf{v})) e^{2i\chi\,\mathrm{wt}(\mathbf{v})} \right] \\ &= -4 \sum_{\mathbf{y}} \frac{p_{\mathbf{y}}^2}{|C_{\mathbf{y}}|^2} \sum_{\mathbf{x},\mathbf{v}\in C_{\mathbf{y}}} (n-t-2\,\mathrm{wt}(\mathbf{v})) \Big[\cos(2\chi\,\mathrm{wt}(\mathbf{x}))\sin(2\chi\,\mathrm{wt}(\mathbf{v})) - \sin(2\chi\,\mathrm{wt}(\mathbf{x}))\cos(2\chi\,\mathrm{wt}(\mathbf{v})) \Big] \\ &= -4 \sum_{\mathbf{y}} \frac{p_{\mathbf{y}}^2}{|C_{\mathbf{y}}|^2} \sum_{\mathbf{x},\mathbf{v}\in C_{\mathbf{y}}} (n-t-2\,\mathrm{wt}(\mathbf{v})) \sin(2\chi(\mathrm{wt}(\mathbf{v}) - \mathrm{wt}(\mathbf{x}))). \end{aligned} \tag{III.17}$$

It follows that

$$\lim_{\chi\to 0} \mathrm{Tr}(\rho_\chi L) = 0. \tag{III.18}$$

Differentiating with respect to χ, we get

$$\frac{\partial}{\partial\chi} \mathrm{Tr}(\rho_\chi L) = -8 \sum_{\mathbf{y}} \frac{p_{\mathbf{y}}^2}{|C_{\mathbf{y}}|^2} \sum_{\mathbf{x},\mathbf{v}\in C_{\mathbf{y}}} (n-t-2\,\mathrm{wt}(\mathbf{v}))(\mathrm{wt}(\mathbf{v}) - \mathrm{wt}(\mathbf{x})) \cos(2\chi(\mathrm{wt}(\mathbf{v}) - \mathrm{wt}(\mathbf{x})). \tag{III.19}$$

Hence,

$$\begin{aligned} \lim_{\chi\to 0} \frac{\partial}{\partial\chi} \mathrm{Tr}(\rho_\chi L) &= 16 \sum_{\mathbf{y}} \frac{p_{\mathbf{y}}^2}{|C_{\mathbf{y}}|^2} \sum_{\mathbf{x},\mathbf{v}\in C_{\mathbf{y}}} \mathrm{wt}(\mathbf{v})(\mathrm{wt}(\mathbf{v}) - \mathrm{wt}(\mathbf{x})) \\ &= 16 \sum_{\mathbf{y}} p_{\mathbf{y}}^2 V_{\mathbf{y}}, \end{aligned} \tag{III.20}$$

where

$$\mu_{\mathbf{y}} := \frac{1}{|C_{\mathbf{y}}|} \sum_{\mathbf{x}\in C_{\mathbf{y}}} \mathrm{wt}(\mathbf{x}), \tag{III.21}$$

$$V_{\mathbf{y}} := \frac{1}{|C_{\mathbf{y}}|} \sum_{\mathbf{v}\in C_{\mathbf{y}}} \left(\mathrm{wt}(\mathbf{v})^2\right) - \mu_{\mathbf{y}}^2. \tag{III.22}$$

Next, using the assumption that $|\psi_{\mathbf{y}}\rangle$ and $|\psi_{\mathbf{z}}\rangle$ are term-wise orthogonal since $C_{\mathbf{y}} \cap C_{\mathbf{z}} = \emptyset$, we find that

$$\mathrm{Tr}(\rho_\chi L^2) = -4 \sum_{\mathbf{u},\mathbf{y},\mathbf{z}} p_{\mathbf{u}} p_{\mathbf{y}} p_{\mathbf{z}} \,\mathrm{Tr}\left[|\psi_{\mathbf{u}}^\chi\rangle\langle\psi_{\mathbf{u}}^\chi| \left(|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H - H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}| \right) \left(|\psi_{\mathbf{z}}\rangle\langle\psi_{\mathbf{z}}|H - H|\psi_{\mathbf{z}}\rangle\langle\psi_{\mathbf{z}}| \right)\right]$$

$$
\begin{aligned}
&= -4\sum_{\mathbf{y}} p_{\mathbf{y}}^3 \operatorname{Tr}\left[|\psi_{\mathbf{y}}^{\chi}\rangle\langle\psi_{\mathbf{y}}^{\chi}| \left(|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H - H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|\right) \left(|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H - H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|\right)\right] \\
&= -4\sum_{\mathbf{y}} p_{\mathbf{y}}^3 \langle\psi_{\mathbf{y}}^{\chi}| \left[\left(|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H\right) \left(|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H\right) - \left(|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H\right) \left(H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|\right)\right. \\
&\qquad \left. - \left(H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|\right) \left(|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H\right) + \left(H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|\right) \left(H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|\right)\right]|\psi_{\mathbf{y}}^{\chi}\rangle \\
&= -4\sum_{\mathbf{y}} p_{\mathbf{y}}^3 \Big[\langle\psi_{\mathbf{y}}^{\chi}|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}^{\chi}\rangle - \langle\psi_{\mathbf{y}}^{\chi}|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H^2|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|\psi_{\mathbf{y}}^{\chi}\rangle \\
&\qquad - \langle\psi_{\mathbf{y}}^{\chi}|H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}^{\chi}\rangle + \langle\psi_{\mathbf{y}}^{\chi}|H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|\psi_{\mathbf{y}}^{\chi}\rangle\Big] \\
&= -4\sum_{\mathbf{y}} p_{\mathbf{y}}^3 \Big[2\mathrm{Re}\left(\langle\psi_{\mathbf{y}}^{\chi}|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}^{\chi}\rangle\right) - |\langle\psi_{\mathbf{y}}^{\chi}|\psi_{\mathbf{y}}\rangle|^2\langle\psi_{\mathbf{y}}|H^2|\psi_{\mathbf{y}}\rangle - |\langle\psi_{\mathbf{y}}^{\chi}|H|\psi_{\mathbf{y}}\rangle|^2\Big]. \qquad \text{(III.23)}
\end{aligned}
$$

Then it follows from Lemma 7 that

$$
\begin{aligned}
&\mathrm{Re}\left(\langle\psi_{\mathbf{y}}^{\chi}|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}^{\chi}\rangle\right) \\
&= \frac{1}{|C_{\mathbf{y}}|^2}\mathrm{Re}\left(\sum_{\mathbf{u}\in C_{\mathbf{y}}} e^{-2i\chi \operatorname{wt}(\mathbf{u})}(n-t-2\mu_{\mathbf{y}}) \sum_{\mathbf{x}\in C_{\mathbf{y}}} (n-t-2\operatorname{wt}(\mathbf{x}))e^{2i\chi \operatorname{wt}(\mathbf{x})}\right) \\
&= \frac{(n-t-2\mu_{\mathbf{y}})}{|C_{\mathbf{y}}|^2} \sum_{\mathbf{u},\mathbf{x}\in C_{\mathbf{y}}} (n-t-2\operatorname{wt}(\mathbf{x}))\Big[\cos(2\chi \operatorname{wt}(\mathbf{u}))\cos(2\chi \operatorname{wt}(\mathbf{x})) + \sin(2\chi \operatorname{wt}(\mathbf{u}))\sin(2\chi \operatorname{wt}(\mathbf{x}))\Big] \\
&= \frac{(n-t-2\mu_{\mathbf{y}})}{|C_{\mathbf{y}}|^2} \sum_{\mathbf{u},\mathbf{x}\in C_{\mathbf{y}}} (n-t-2\operatorname{wt}(\mathbf{x}))\cos(2\chi(\operatorname{wt}(\mathbf{u}) - \operatorname{wt}(\mathbf{x})). \qquad \text{(III.24)}
\end{aligned}
$$

Thus,

$$
\lim_{\chi\to 0} \mathrm{Re}\left(\langle\psi_{\mathbf{y}}^{\chi}|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}|H|\psi_{\mathbf{y}}^{\chi}\rangle\right) = (n-t-2\mu_{\mathbf{y}})^2. \qquad \text{(III.25)}
$$

Next we find that

$$
|\langle\psi_{\mathbf{y}}^{\chi}|\psi_{\mathbf{y}}\rangle|^2\langle\psi_{\mathbf{y}}|H^2|\psi_{\mathbf{y}}\rangle = \frac{1}{|C_{\mathbf{y}}|^2} \sum_{\mathbf{v},\mathbf{u}\in C_{\mathbf{y}}} e^{2i\chi \operatorname{wt}(\mathbf{v})} e^{-2i\chi \operatorname{wt}(\mathbf{u})} \left((n-t)^2 - 4(n-t)\mu_{\mathbf{y}} + \frac{4}{|C_{\mathbf{y}}|}\sum_{\mathbf{x}\in C_{\mathbf{y}}} \operatorname{wt}(\mathbf{x})^2\right). \qquad \text{(III.26)}
$$

Hence, it follows that

$$
\lim_{\chi\to 0} |\langle\psi_{\mathbf{y}}^{\chi}|\psi_{\mathbf{y}}\rangle|^2\langle\psi_{\mathbf{y}}|H^2|\psi_{\mathbf{y}}\rangle = (n-t)^2 - 4(n-t)\mu_{\mathbf{y}} + \frac{4}{|C_{\mathbf{y}}|}\sum_{\mathbf{x}\in C_{\mathbf{y}}} \operatorname{wt}(\mathbf{x})^2. \qquad \text{(III.27)}
$$

Also,

$$
|\langle\psi_{\mathbf{y}}^{\chi}|H|\psi_{\mathbf{y}}\rangle|^2 = \left|\frac{1}{|C_{\mathbf{y}}|}\sum_{\mathbf{x}\in C_{\mathbf{y}}} ((n-t) - 2\operatorname{wt}(\mathbf{x}))\ e^{2i\chi \operatorname{wt}(\mathbf{x})}\right|^2, \qquad \text{(III.28)}
$$

and hence

$$
\lim_{\chi\to 0} |\langle\psi_{\mathbf{y}}^{\chi}|H|\psi_{\mathbf{y}}\rangle|^2 = (n-t-2\mu_{\mathbf{y}})^2. \qquad \text{(III.29)}
$$

Therefore, we observe that

$$
\lim_{\chi\to 0} \mathrm{Tr}(\rho_\chi L^2) = 16\sum_{\mathbf{y}} p_{\mathbf{y}}^3 V_{\mathbf{y}}, \qquad \text{(III.30)}
$$

from which it follows that

$$\lim_{\chi \to 0} (\delta\chi)^2 = \lim_{\chi \to 0} \frac{\mathrm{Tr}(\rho_\chi L^2) - \mathrm{Tr}(\rho_\chi L)^2}{\left| \frac{\partial}{\partial \chi} \mathrm{Tr}(\rho_\chi L) \right|^2} \tag{III.31}$$
$$= \frac{16 \sum_{\mathbf{y}} p_{\mathbf{y}}^3 V_{\mathbf{y}}}{16^2 \left(\sum_{\mathbf{y}} p_{\mathbf{y}}^2 V_{\mathbf{y}} \right)^2}$$
$$\leq \frac{1}{16 \sum_{\mathbf{y}} p_{\mathbf{y}}^2 V_{\mathbf{y}}}. \qquad \square$$

Let us now identify $C_{\mathbf{y}}$ with $C_{\mathbf{y}}[E]$ for each $\mathbf{y} \in \{0,1\}^t$ and thus ρ with $\rho_C[E]$, the classical code-inspired probe state after the subset E of qubits is erased. Then, we can compare the result of Theorem 8 on the variance of χ with respect to the observable L with the lower bound on QFI obtained in Theorem 4. Recollecting that the minimum variance is given by the inverse of QFI, we see that measuring the observable L is optimal asymptotically, i.e., in the limit of $\chi \to 0$. Furthermore, when combined with the boosting lemma, this implies that measuring L on a probe state constructed from a fixed length (m) code concatenated with a length $\Omega(n)$ (inner) repetition code will have the desired Heisenberg scaling in the variance of χ as $\chi \to 0$. Finally, we note that the above result holds also when χ approaches integer multiples of π.

IV. EXAMPLES

A. Boosted Reed-Muller codes

Let us revisit the $[[8,3,2]]$ quantum Reed-Muller code based probe state discussed in Section II B. In this case the classical code corresponding to the probe state is the $[8,4,4]$ self-dual Reed-Muller code $C = \mathrm{RM}(1,3)$. We observed earlier that under no erasure the state has a QFI lower bound of 16 but under a single erasure this drops to 7. Assume we concatenate $C^{\text{outer}} = C$ with an inner repetition code of length $r = 3$ to get a code of length $n = 24$. According to the boosting lemma (Lemma 5), the QFI lower bound for the probe state constructed from the concatenated code only depends on the projection of the erasures to the outer code C. Since we earlier considered a single erasure on the non-concatenated code, in order to make a fair comparison let us fix the erasure rate as $1/8$. Thus, approximately three qubits get erased on the 24-qubit probe state. If these qubit indices belong to the same "block" of repeated bits in the concatenated code, then the projection to $C^{\text{outer}} = C$ produces a single qubit erasure. While this produced a QFI lower bound of 7 for the non-concatenated code, for the 24-qubit probe state this is enhanced by $r^2 = 9$ to 63, according to the boosting lemma. Thus, the normalized QFI per qubit has enhanced from $7/8$ to $63/24$. Similarly, if the projection produces two (resp. three) erasures on the outer code, then the QFI for the 24-qubit probe state is 27 (resp. 9). In general, for the outer code C, if one, two or three qubits are erased, then the normalized QFI lower bound is $7/8, 3/8$, and $1/8$, respectively. If four or more qubits are erased then the QFI lower bound is trivial (i.e., 0). Therefore, when concatenated with a repetition code of length r, the normalized bound increases to $7r/8, 3r/8$, and $r/8$, respectively, depending on the size of the projection of the erasures on the concatenated code to just the outer code.

In Figure 3 we compare the performance of our probe state from the concatenated RM(1,3) code with that of previously studied GNU probe states [26]. GNU probe states arise from the codespace of a specific family of permutation-invariant quantum error correction codes called GNU codes [38]. These codes on GNU qubits have three parameters, given by G, N and U. Here, G relates to the correctible number of bit-flips, N corresponds to the number of correctible phase-flips, and U is an unimportant scaling factor that is at least 1. These permutation-invariant codes however cannot be studied using the framework in our paper, as the distance of the corresponding classical code is equal to 1.

B. Boosted CSS codes

As mentioned earlier, the general code-inspired probe state we have considered is always the logical $|+ + \cdots +\rangle$ state of a CSS code whose logical X group (including the X-type stabilizers) is given by the chosen classical binary code C, as long as C is a linear code. So, if we used $C = \mathrm{RM}(1,3)$ above, then in the future when QEC-based metrology becomes feasible, we will only be able to detect a single error since the corresponding CSS code has parameters $[[8,3,2]]$. However, if we chose the logical $|+\rangle$ state of the $[[15,1,3]]$ quantum Reed-Muller code, then we can make use of Reed-Muller properties while also being able to correct a single error. Some properties that could be leveraged are

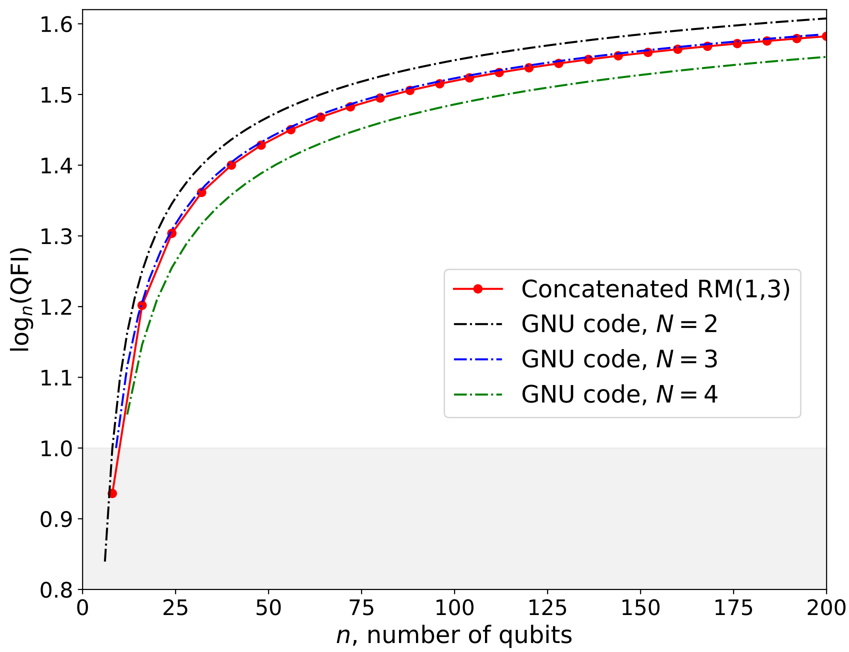

FIG. 3. We plot lower bounds on $\log_n$(QFI) for various code-inspired probe states for the robust quantum metrology problem. We compare the lower bounds that we have for the concatenated RM(1,3) code with that of previously studied GNU probe states [26] after one erasure error has occurred. Whenever the lower bound is above 1, there is a quantum advantage in using these code-inspired probe states.

the large symmetry group of (classical) Reed-Muller codes [51] and the fact that this quantum code has a transversal T property [50, 52]. Since transversal T realizes logical T^{-1} on this $[[15, 1, 3]]$ code, it does not take the logical $|+\rangle$ state to a code state that is orthogonal to it, so it remains unclear how this symmetry can be leveraged. However, since the unitary induced by the generator $H = \sum_i Z_i$ produces a transversal Z-rotation, it will be interesting to explore the utility of the transversal Z-rotation property. If this is found to be useful for quantum metrology, then one can easily incorporate well-known families of CSS codes, such as triorthogonal codes [52], that possess such a property into our code-inspired probe state framework [50, 53].

Surface codes form a popular family of CSS codes that are thought to be attractive candidates for quantum error correction in the near-term [54]. Although these codes encode a fixed number of qubits, with typical parameters being $[[2d^2, 2, d]]$ on a $d \times d$ square lattice, for metrology purposes our results show that only the variances of the weight distributions of the corresponding shortened classical codes matter. It is known that surface codes can be constructed as a hypergraph product of two classical length d repetition codes [55, 56]. Since repetition codes only have codeword lengths 0 and n, they have a quadratically scaling variance even under erasures. However, the logical X group for surface codes is not given by a repetition code, so one needs to analyze the weight distribution of this group to assess the utility of the resultant probe state for robust metrology. As surface codes are highly likely to be practically realized, this approach would naturally be adaptable to fault-tolerant quantum error correction based metrology when that becomes feasible.

Our scheme has some interesting connections with [15], where a scheme for quantum metrology with active quantum error correction was proposed. There, the probe states were of the form $(|0_L\rangle^{\otimes m} + |1_L\rangle^{\otimes m})$, where $|0_L\rangle$ and $|1_L\rangle$ are logical codewords from any quantum error correction code. So the concatenation has the repetition code as the outer code, and other quantum error correction codes as inner codes, opposite to the case we considered.

Another related work is in [29], where the authors derive some conditions for the noise model under which the QFI has absolutely no degradation. In contrast, we consider a weaker condition, where the QFI can degrade under the effects of erasure errors. In [29], the authors revisited the metrology problem using the probe states $(|0_L\rangle^{\otimes m} + |1_L\rangle^{\otimes m})$, and obtained heuristically the same conclusion as we do. Namely, they also find that concatenation of quantum error correction codes with repetition codes is advantageous. In our work however, we have several additional key findings. First, we have explicit bounds for the QFI in this scenario that are absent in [29], which applies to any quantum error correction code concatenated with repetition codes. Second, to the best of our knowledge, our work is the first to establish the connections of the problem of robust quantum metrology with that of coding theory.

V. DISCUSSIONS

In summary, we have studied the performance of code-inspired probe states for the problem of noisy quantum metrology. We find that there is a strong connection between noisy quantum metrology and classical coding theory. Namely, the QFI is related to the variances of the weight distributions of shortened codes. The larger the variance, the larger the corresponding QFI. Moreover, we have a boosting lemma that implies that any CSS code, when concatenated with repetition codes of linear length can be useful for robust metrology with a constant number of erasure errors [57]. We thereby side-step the no-go result of random codes for robust metrology by having these CSS codes to have asymptotically vanishing relative distance. We also expect that when the CSS codes are concatenated with repetition codes, we will also do very well for burst erasure errors, but we leave this for future work.

There are many open problems that remain to be solved. First, continuous quantum error correction protocols have previously been studied [58–60]. It will be interesting to extend our work further in this direction, to see how continuous time quantum error correction can be integrated with robust quantum metrology. Second, the potential of using QRM states for fault tolerant quantum metrology has recently been investigated [61]. Since QRM codes are CSS codes, it is interesting to see how QRM codes concatenated with repetition codes would perform correspondingly in a fault-tolerant setting for quantum metrology. Third, it will also be interesting to see how concatenation of random codes with specific families of codes with structure will perform for robust quantum metrology, as this will correspondingly extend the work of [25] which studied noisy quantum metrology for fully random quantum states. Fourth, it will be interesting to extend our results to a multiparameter setting, using recent developments on obtaining tight bounds for the robust estimation of incompatible observables [62].

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