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Inverse problems of damped wave equations with Robin boundary conditions: an application to blood perfusion

Yan-Long Fang[1,2,*], **Daniel Lesnic**[2] **and Moataz Alosaimi**[3]

[1] Department of Mathematics, University College London, London, WC1H 0AY, United Kingdom
[2] Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom
[3] Department of Mathematics and Statistics, College of Science, Taif University, PO Box 11099, Taif 21944, Saudi Arabia

E-mail: yanlong.fang@ucl.ac.uk

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Abstract

Knowledge of the properties of biological tissues is essential in monitoring any abnormalities that may be forming and have a major impact on organs malfunctioning. Therefore, these disorders must be detected and treated early to save lives and improve the general health. Within the framework of thermal therapies, e.g. hyperthermia or cryoablation, the knowledge of the tissue temperature and of the blood perfusion rate are of utmost importance. Therefore, motivated by such a significant biomedical application, this paper investigates, for the first time, the uniqueness and stable reconstruction of the space-dependent (heterogeneous) perfusion coefficient in the thermal-wave hyperbolic model of bio-heat transfer from Cauchy boundary data using the powerful technique of Carleman estimates. Additional novelties consist in the consideration of Robin boundary conditions, as well as developing a mathematical analysis that leads to stronger stability estimates valid over a shorter time interval than usually reported in the literature of coefficient identification problems for hyperbolic partial differential equations. Numerically, the inverse coefficient problem is recast as a nonlinear least-squares minimization that is solved using the

* Author to whom any correspondence should be addressed.

conjugate gradient method (CGM). Both exact and noisy data are inverted. To achieve stability, the CGM is stopped according to the discrepancy principle. Numerical results for a physical example are presented and discussed, showing the convergence, accuracy and stability of the inversion procedure.

Keywords: inverse coefficient problem, bio-heat transfer, thermal-wave model, Carleman estimates, conjugate gradient method

1. Introduction

1.1. Physical background

Understanding the heat transfer in biological tissues is of crucial importance in biomedical applications such as hyperthermia, thermal ablation or microwave heating, which are typical examples of thermal therapies used to treat cancer, menorrhagia and benign prostate hyperplasia [1]. When mathematically modelling such applications, care should be taken to include the underlying processes that are taking place such as heat conduction, blood perfusion and heat generation due to metabolism. One formulation that takes into account these mechanisms is based on the much celebrated Pennes' parabolic reaction-diffusion equation (obtained by taking $\tau = 0$ in equation (1) below), which was proposed to model the temperature evolution during cancer hyperthermia treatment [51], the thermal radiation from cellular phones [55] and the ablation of afflicted tissues [28], among others. However, although still widely used, the Pennes parabolic model of heat transfer implies infinite speed of heat propagation. This characteristic becomes practically unrealistic when modelling heat propagation in biological tissues for which a non-negligible relaxation time-lag τ (typically between 15–30 seconds) occurs [49]. Therefore, a more appropriate model taking into account that in biological tissues thermal waves propagate with a finite speed is given by the hyperbolic equation [46],

$$\tau C_{\text{tissue}} \mathrm{u}_{\mathrm{tt}} + (C_{\text{tissue}} + \tau C_b w_b)\mathrm{u}_{\mathrm{t}} = \kappa \Delta \mathrm{u} + C_b w_b(\mathrm{u} - \mathrm{u}_b) + \mathrm{Q} + \tau \mathrm{Q}_{\mathrm{t}}, \quad (\mathrm{t}, \mathrm{x}) \in (0, \mathrm{T}) \times \Omega, \tag{1}$$

where Ω is the spatial domain occupied by the tissue, T is a final time (in s) of interest, u is the tissue temperature (in °C), u_b is the (arterial) blood temperature (in °C), κ and C_{tissue} are the thermal conductivity (in $\mathrm{W}\,(\mathrm{m}\,^\circ\mathrm{C})^{-1}$) and heat capacity (in $\mathrm{J}\,(\mathrm{kg}\,^\circ\mathrm{C})^{-1}$) of the tissue, respectively, C_b is the heat capacity (in $\mathrm{J}\,(\mathrm{kg}\,^\circ\mathrm{C})^{-1}$) of the blood, w_b is the bood perfusion rate (in s^{-1}) and Q contains the heat generations (in $\mathrm{W\,m^{-3}}$) due to metabolism and external heating. Equation (1) is supplied with the initial conditions

$$\mathrm{u} = \Phi_0, \quad \mathrm{u}_{\mathrm{t}} = 0 \quad \text{in } \Omega \text{ at } \mathrm{t} = 0$$

and the Robin boundary condition

$$\kappa \partial_\nu \mathrm{u} = \mathrm{h}(\mathrm{u}_{amb} - \mathrm{u}) \quad \text{on } (0, \mathrm{T}) \times \partial\Omega, \tag{3}$$

where Φ_0 is prescribed initial temperature (in °C), ν is the outward unit normal to the boundary $\partial\Omega$ and h is the heat transfer coefficient (in $\mathrm{W}\,(\mathrm{m}^3\,^\circ\mathrm{C})^{-1}$) between the tissue and the surrounding ambient having a temperature u_{amb} (in °C). In case $\mathrm{h} \to 0$, equation (3) becomes the homogeneous Neumann adiabatic boundary condition

$$\partial_\nu \mathrm{u} = 0 \quad \text{on } (0, \mathrm{T}) \times \partial\Omega, \tag{4}$$

whilst in case $|\mathrm{h}| \to \infty$, equation (3) becomes the Dirichlet boundary condition

$$\mathrm{u} = \mathrm{u}_{amb} \quad \text{on } (0,\mathrm{T}) \times \partial\Omega. \tag{5}$$

In practical applications concerned with hyperthermia treatment or assessing burn injuries, evaluating the blood perfusion rate w_b is of main interest. In this paper, we consider the case when the unknown coefficient $w_b = w_b(\mathrm{x})$ is space-dependent across the tissue and we aim to recontruct it from non-intrusive temperature measurements over the whole boundary $\partial\Omega$ or on a portion of it $\Gamma_0 \subset \partial\Omega$, namely,

$$\mathrm{u} = \Theta \quad \text{on } (0,\mathrm{T}) \times \Gamma_0. \tag{6}$$

Before performing any analysis, equations (1)–(6) have to be non-dimensionalised. Assuming that u_b is a non-zero constant, we introduce the following dimensionless variables:

$$\begin{aligned} &x = \mathrm{x}\sqrt{\frac{C_{\text{tissue}}}{\kappa\tau}}, \quad t = \frac{\mathrm{t}}{\tau}, \quad T = \frac{\mathrm{T}}{\tau}, \quad u = \frac{\mathrm{u}-\mathrm{u}_b}{\mathrm{u}_b}, \\ &\varphi_0 = \frac{\Phi_0 - \mathrm{u}_b}{\mathrm{u}_b}, \quad u_{amb} = \frac{\mathrm{u}_{amb} - \mathrm{u}_b}{\mathrm{u}_b}, \quad h = \mathrm{h}\sqrt{\frac{\tau}{\kappa C_{\text{tissue}}}}, \\ &Q = \frac{\tau \mathrm{Q}}{\mathrm{u}_b C_{\text{tissue}}}, \quad w = \frac{\tau C_b w_b}{C_{\text{tissue}}}, \quad \theta = \frac{\Theta - \mathrm{u}_b}{\mathrm{u}_b}. \end{aligned} \tag{7}$$

With this non-dimensionalisation, equations (1)–(3) and (6) recast as (denoting $F := Q + Q_t$)

$$u_{tt} + (1 + w(x))u_t = \Delta u - w(x)u + F(t,x), \quad (t,x) \in (0,T) \times \Omega =: \Omega_T, \tag{8}$$

$$u(0,\cdot) = \varphi_0, \quad u_t(0,\cdot) = 0 \quad \text{in } \Omega, \tag{9}$$

$$\partial_\nu u = h(u_{amb} - u) \quad \text{on } (0,T) \times \partial\Omega, \tag{10}$$

$$u = \theta \quad \text{on } (0,T) \times \Gamma_0, \tag{11}$$

where, without any confusion, the dimensionless ν, Γ_0, Ω and $\partial\Omega$ have been denoted by the same symbols as their dimensional counterparts. Equations (4) and (5) also take the non-dimensional form

$$\partial_\nu u = 0 \quad \text{on } (0,T) \times \partial\Omega, \tag{12}$$

$$u = u_{amb} \quad \text{on } (0,T) \times \partial\Omega. \tag{13}$$

Certain thermal experiments [7] may be designed so that the ambient temperature u_{amb} is equal to the arterial blood temperature u_b. In such a situation, from (7) it follows that $u_{amb} = 0$ and equations (10) and (13) simplify, respectively, as

$$\partial_\nu u + hu = 0 \quad \text{on } (0,T) \times \partial\Omega. \tag{14}$$

and

$$u = 0 \quad \text{on } (0,T) \times \partial\Omega. \tag{15}$$

1.2. Mathematical background

In the mathematical analysis we consider the exposition in a more general setup than the particular physical model in equations (8) and (9). To start with, assume $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$, is an open and bounded C^2-domain. We mention that a one-dimensional related, but different inverse problem has recently been analysed in [52].

Consider the (homogeneous) Robin problem related to (8), (9) and (14) given by

$$\begin{cases} u_{tt} - \Delta u + Q_0 u + Q_1 u_t = F & \text{in } \Omega_T, \\ \partial_\nu u + hu = 0 & \text{on } (0,T) \times \partial\Omega, \\ u = \varphi_0, \ u_t = \varphi_1 & \text{in } \{t=0\} \times \Omega, \end{cases} \tag{16}$$

where the functions Q_0 and Q_1 are in $L^\infty(\Omega)$. Moreover, assume $\varphi_0 \in H^1(\Omega)$, $\varphi_1 \in L^2(\Omega)$, $F \in L^2(\Omega_T)$ and $h = h(x) \in L^\infty(\partial\Omega)$. In most of the mathematical analysis presented below, Q_0 and Q_1 are independent of each other, but we shall also consider the physical case of equation (1) in which $Q_0(x) = w(x)$ and $Q_1(x) = 1 + w(x)$ in theorems 4.4 and 4.5. In the limit that $h \to 0$ we obtain the (homogeneous) Neumann problem in which the Robin boundary condition in (16) is being replaced by (12).

This paper is organized as follows. In section 2, the mathematical formulation of the direct problem and its well-posedness are presented. In section 3, the poweful technique of Carleman estimates is discussed and the main theorem 3.3 is established. Section 4 is fully devoted to theoretically investigating the inverse coefficient identification problems of interest and establishing conditional stability estimates. We mention herein that Lipschitz stability estimates, similar to those established in theorems 4.1, 4.3 and 4.4, but for the wave equation $c(x)u_{tt} = \Delta u$ with unknown coefficient $c(x)$ were obtained in [36] and [35, theorem 3.6]. In section 5, the inverse problem is recast as a variational problem. The resulting nonlinear least-squares minimizing functional is proved to be Fréchet differentiable and an explicit expression for its gradient is derived. Numerical results obtained using the conjugate gradient method (CGM) for a physical example are presented and discussed in section 6 to confirm the proposed inversion algorithm's convergence, accuracy and stability. Finally, conclusions are provided in section 7.

1.3. Comparisons with other works

Our approach to the well-posedness of direct problem is based on Evans' book [17] with adaptions to Robin boundary condition settings and some basic spectral theory of the Laplacian [5, 6]. The method of Carleman estimates is inspired by approach developed in the book of Bellassoued and Yamamoto [10], as well as the one of Lerner [44]. Theorem 3.3 extends the Carleman estimate in [10, chapter 4], which can also be viewed as an extension of [44] in the presence of boundary. There are several improvements obtained in this paper when comparing with other literature. One improvement compared with [10] is to include all other boundary contributions into the Carleman estimate (see remark 3.4). This can also be compared with the work of Lasiecka *et al* in [40], where authors used a different approach to obtain the Carleman estimate and their result looks different from ours. As an application of our Carleman estimate, the stability estimates presented in section 4 are sharper than the works of [10, 23, 40, 47] (see remark 4.2). This mathematical refinement is physically meaningful, as it results in a shorter time of physical measurements on the boundary that is required to determine the solution in the interior of the domain.

2. Direct problem

The well-posedness of the direct problem (16) is established in the literature, (see, for instance, [37, chapter 4], [17, chapter 7] and [45]). First, we define the time-dependent bilinear form

$$\mathrm{B}[u,v;t] := \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_\Omega (Q_1 u_t + Q_0 u) v \mathrm{d}x + \int_{\partial\Omega} huv \mathrm{d}y.$$

A function u in Λ_h, where

$$\Lambda_h := \left\{u \in L^2(0,T;H^1(\Omega,h)) \text{ with } u_t \in L^2(0,T;L^2(\Omega)) \text{ and } u_{tt} \in L^2(0,T;H^{-1}(\Omega,h))\right\}, \quad (17)$$

$H^1(\Omega,h) := \overline{\{v \in H^1(\Omega) \cap C_0^\infty(\overline{\Omega}) \text{ with } \int_{\partial\Omega} |h| v^2 \mathrm{d}y < \infty\}}$ and $H^{-1}(\Omega,h) \subset H^{-1}(\Omega)$ is the dual of $H^1(\Omega,h)$, is said to be a weak solution to (16) if

- $\langle u_{tt}, v\rangle + \mathrm{B}[u,v;t] = \langle F, v\rangle$ for every $v \in H^1(\Omega;h)$.
- $u(0,\cdot) = \varphi_0,\ \ u_t(0,\cdot) = \varphi_1$.

In the above, $\langle\cdot,\cdot\rangle$ denotes the real inner product in $L^2(\Omega)$. Note that the space $H^1(\Omega,h)$ becomes $H_0^1(\Omega)$ when $|h| = \infty$, and $H^1(\Omega)$ when $h = 0$. Moreover, the closure in the definition of $H^1(\Omega,h)$ should be understood in the sense of the closure of domains of the quadratic form given by $\mathcal{Q}(u,v) := \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_{\partial\Omega} huv\mathrm{d}y$, (see, for instance, [6]).

Theorem 2.1. *Let $\varphi_0 \in H^1(\Omega)$, $\varphi_1 \in L^2(\Omega)$, $F \in L^2(\Omega_T)$, $h \in L^\infty(\partial\Omega)$, $Q_0 \in L^\infty(\Omega)$ and $Q_1 \in L^\infty(\Omega)$ be given input data for the direct problem (16). Then there exists a unique weak solution $u \in \Lambda_h$ of the direct problem (16). Moreover, this solution satisfies the stability estimate*

$$\operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \left(\|u(t,\cdot)\|_{H^1(\Omega)} + \|u_t(t,\cdot)\|_{L^2(\Omega)}\right) \leqslant C\Bigg(\|F\|_{L^2(\Omega_T)} + \|\varphi_0\|_{H^1(\Omega)} + \|\varphi_1\|_{L^2(\Omega)}\Bigg), \quad (18)$$

where the positive constant C depends on the L^∞-norms of Q_0 and Q_1.

We defer the proof to appendix A since methods of proving theorem 2.1 can be found in various literature with some minor modifications. For instance, in [37, chapter IV, theorem 5.1], one can find a similar statement for the well-posedness of the Robin problem (16) but with $F \in L^1(0,T;L^2(\Omega))$ instead of $F \in L^2(\Omega_T)$, as in our theorem.

Remark 2.2. In the case of the homogeneous Neumann boundary condition (12) replacing the Robin boundary condition in (16), theorem 2.1 holds by simply taking $h = 0$. Also, the well-posedness of the solution $u \in H^1(\Omega_T)$ of the homogeneous Dirichlet direct problem obtained by replacing the Robin boundary condition in (16) by the zero Dirichlet condition (15) and assuming $\varphi_0 \in H_0^1(\Omega)$ can be found in [37, chapter IV, theorem 4.2] or [17, chapter 7]. For the inhomogeneous Dirichlet direct problem given by

$$\begin{cases} u_{tt} - \Delta u + Q_0 u + Q_1 u_t = F & \text{in } \Omega_T, \\ u = u_{amb} & \text{on } (0,T)\times\partial\Omega, \\ u = \varphi_0,\ \ u_t = \varphi_1 & \text{in } \{t=0\}\times\Omega, \end{cases} \quad (19)$$

a unique solvability result for

$$u \in \mathcal{V} := \{u \in C([0,T];H^1(\Omega)) \text{ with } u_t \in C([0,T];L^2(\Omega)) \text{ and } u_{tt} \in C([0,T];H^{-1}(\Omega))\}$$

can be found in [38, theorem 2.4 with $\theta = 0$], when the input data $F \in L^1(0,T;L^2(\Omega))$, $\varphi_0 \in H^1(\Omega)$, $\varphi_1 \in L^2(\Omega)$, $u_{amb} \in H^1((0,T)\times\partial\Omega)$ satisfies the compatibility condition

$$u_{amb} = \varphi_0 \quad \text{on } \{0\}\times\partial\Omega. \quad (20)$$

and $Q_0 = Q_1 = 0$. Furthermore, it was also obtained that $\partial_\nu u \in L^2((0,T)\times\partial\Omega)$. These results can be generalized when the governing hyperbolic equation (19) contains the lower-order terms Q_0 and $Q_1 \in L^\infty(\Omega)$. In the particular case that $F = 0$, then one can replace the condition $u_{amb} \in H^1((0,T)\times\partial\Omega)$ on the inhomogeneous Dirichlet data by $u_{amb} \in C([0,T];H^{1/2}(\partial\Omega)) \cap H^1(0,T;L^2(\partial\Omega))$, see [38, theorem 3.4].

With higher regularity on the data F, φ_0 and φ_1, one could obtain higher regularity for the solution u of the direct problem (16), as given by the following theorem.

Theorem 2.3. *Let $h \in L^\infty(\partial\Omega)$, $Q_0 \in L^\infty(\Omega)$ and $Q_1 \in L^\infty(\Omega)$. Denote $\mathcal{L} := -\Delta + Q_0(x)I + Q_1(x)\partial_t$ and let $m \in \mathbb{N}$. If $\varphi_0 \in H^{m+1}(\Omega)$, $\varphi_1 \in H^m(\Omega)$ and $\partial_t^k F \in L^2(0,T;H^{m-k}(\Omega))$ for $k = 0,\ldots,m$, satisfy the following regularity conditions:*

$$\begin{cases} \varphi_{0,0} := \varphi_0 \in H^1(\Omega,h), \qquad \varphi_{1,0} := \varphi_1 \in H^1(\Omega,h), \\ \varphi_{0,2l} := \partial_t^{2l-2}F(0,\cdot) - \mathcal{L}\varphi_{0,2l-2} \in H^1(\Omega,h) \quad (\textit{if } m = 2l), \\ \varphi_{1,2l+1} := \partial_t^{2l-1}F(0,\cdot) - \mathcal{L}\varphi_{1,2l-1} \in H^1(\Omega,h) \quad (\textit{if } m = 2l+1), \end{cases} \tag{21}$$

then there exists a unique solution $u \in \Lambda_h$ to problem (16), which, in addition, satisfies the regularity $\partial_t^k u \in L^\infty(0,T;H^{m+1-k}(\Omega))$ for $k = 0,\ldots,m+1$ and

$$\operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \sum_{k=0}^{m+1} \|\partial_t^k u\|_{H^{m+1-k}(\Omega)} \leqslant C\left(\sum_{k=0}^{m} \|\partial_t^k F\|_{L^2(0,T;H^{m-k}(\Omega))} + \|\varphi_0\|_{H^{m+1}(\Omega)} + \|\varphi_1\|_{H^m(\Omega)}\right), \tag{22}$$

where the positive constant C depends on the L^∞-norms of Q_0 and Q_1.

Proof. The proof is a minor change of theorem 6 in [17, chapter 7, section 7.2.3]. □

Remark that in case $m = 0$, the condition $\varphi_1 \in H^1(\Omega,h)$ in theorem 21 can be relaxed to be $\varphi_1 \in L^2(\Omega)$, as in theorem 2.1. Applying theorem 2.3 to the problem (8), (9) and (14), we also obtain the following corollary, which will be useful in section 5 in the proof of theorem 5.1.

Corollary 2.4. *Let $h \in L^\infty(\partial\Omega)$ and $w \in L^\infty(\Omega)$. Denote $\mathcal{L} := -\Delta + w(x)I + (1+w(x))\partial_t$ and let $m \in \mathbb{N}$. If $\varphi_1 = 0$, $\varphi_0 \in H^{m+1}(\Omega)$ and $\partial_t^k F \in L^2(0,T;H^{m-k}(\Omega))$ for $k = 0,\ldots,m$, satisfy the following regularity conditions:*

$$\begin{cases} \varphi_{0,0} := \varphi_0 \in H^1(\Omega,h), \\ \varphi_{0,2l} := \partial_t^{2l-2}F(0,\cdot) - \mathcal{L}\varphi_{0,2l-2} \in H^1(\Omega,h) \quad (\textit{if } m = 2l), \\ \varphi_{1,2l+1} := \partial_t^{2l-1}F(0,\cdot) \in H^1(\Omega,h) \quad (\textit{if } m = 2l+1), \end{cases} \tag{23}$$

then there exists a unique solution $u \in \Lambda_h$ to problem (8), (9) and (14), which, in addition, satisfies the regularity $\partial_t^k u \in L^\infty(0,T;H^{m+1-k}(\Omega))$ for $k = 0,\ldots,m+1$ and

$$\operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \sum_{k=0}^{m+1} \|\partial_t^k u\|_{H^{m+1-k}(\Omega)} \leqslant C\left(\sum_{k=0}^{m} \|\partial_t^k F\|_{L^2(0,T;H^{m-k}(\Omega))} + \|\varphi_0\|_{H^{m+1}(\Omega)}\right), \tag{24}$$

where the positive constant C depends on the L^∞-norm of w.

The rest of the paper concerns the analysis of inverse coefficient problems. The main ingedient that is used for proving stability estimates of these inverse problems is based on Carleman estimates, which are introduced and discussed in the next section.

3. Carleman estimates

A brief review of literature. Applications of Carleman-type estimates for obtaining various observability and controllability results in inverse problems were initiated by the groundbreaking publication of Bukhgeim and Klibanov [12] and have subsequently been tremendously successful in the last few decades [30, 31, 33–35, 57]. For instance, a Carleman-type estimate can be obtained as in [42] for a parabolic setting or as in [43] for a hyperbolic setting. A systematic study of various Carleman estimates without boundary conditions, but under a general pseudo-convexity assumption, can be found in the book by Lerner [44]. As we will be focusing on hyperbolic inverse problems, we will mainly employ tools developed by Bellassoued and Yamamoto in [10], which can also be found in [24, 25]. Even though the book [10] focuses on hyperbolic Carleman estimates under Dirichlet boundary conditions, the tool can be extended to cope with Neumann conditions and, more importantly, Robin boundary conditions in our setting modelling the heat transfer with the surrounding ambient environment, as given by equation (10). Note that Carleman estimates with boundary conditions can be traced back to the work of Isakov in [27], Lavrent'ev *et al* in [41], as well as Tataru's works in [53, 54].

Notations. The approach presented in this section can be adapted to more general manifolds. For the sake of simplicity, we will assume that our n-dimensional time-space manifold, denoted by X, is Minkowski, where the metric is given by $\eta = \mathrm{d}t^2 - g$ and g is the Euclidean metric. Therefore, we can identify vectors with covectors. That is, for vector fields $V = (V_0, \tilde{V})$ and $W = (W_0, \tilde{W})$ on X, we have $\eta(V, W) = V_0 W_0 - g(\tilde{V}, \tilde{W})$. We will also denote $\eta(V, V)$ by $\eta(V)$ and $g(\tilde{V}, \tilde{V})$ by $|\tilde{V}|^2$. For a given function f defined in the time-space, we will use $\mathrm{d}f$ to denote its exterior derivative, which means it is an n-component-wise vector. We will use ∇f to denote the vector containing the spatial components of $\mathrm{d}f$. In other words, ∇f is a d-component-wise vector, where $d = n - 1$. The Hessian with respect to g of a function, f, is denoted by $\nabla^2 f$ and its action on vectors is given by $\nabla^2 f(\tilde{V}, \tilde{W}) := \sum_{k,l=1}^{d} \tilde{V}^k \tilde{W}^l \partial_k \partial_l f$.

3.1. Carleman estimate: a control from interior

In this subsection, we will review a derivation of a classical Carleman estimate via the method developed in [22, 44]. This will serve us as a guidance in the next subsection, when we extend the method to cope with boundary conditions.

Let $\mathrm{m} \in \mathbb{N}^*$, a typical Carleman estimate (inequality) for an m-th order differential operator P acting on $\mathbb{R}^n$ and a suitable function ϕ is given by

$$\begin{aligned} &\exists C > 0,\ \exists \lambda_0 \geqslant 1, \text{ such that } \forall \lambda \geqslant \lambda_0,\ \forall v \in C_0^\infty(\mathbb{R}^n), \\ &C\lambda \|v\|^2_{H_\lambda^{\mathrm{m}-1}(\mathbb{R}^n)} \leqslant \|p_{\mathrm{m}}(\zeta, D + \mathrm{i}\lambda \mathrm{d}\phi)v\|^2_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{25}$$

where $D = -\mathrm{i}\nabla_\zeta$, $p_{\mathrm{m}}(\zeta, \xi)$ is the principal symbol of P and $H_\lambda^{\mathrm{m}}(\mathbb{R}^n)$ is the λ-weighted Sobolev space with norm

$$\|v\|^2_{H_\lambda^{\mathrm{m}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(|\xi|^2 + \lambda^2\right)^{\mathrm{m}} |\hat{v}(\xi)|^2 \mathrm{d}\xi,$$

where $\hat{v}$ denotes the Fourier transform of the function v. Note that the main role in (25) is represented by the principal part of the operator P because contributions from lower-order terms can be absorbed into $\lambda \|v\|^2_{H_\lambda^{\mathrm{m}-1}(\mathbb{R}^n)}$. When working with λ-weighted Sobolev spaces and λ-dependent operators P_λ, we will treat λ the same as ξ, e.g. the symbol of P_λ will be a polynomial in (ξ, λ).

A strategy to prove the celebrated inequality (25) via pseudo-differential operators is explored in great details in [44] and it is summarised below.

(1) Rewrite the right-hand-side of (25) as

$$\|p_{\mathrm{m}}(\zeta, D + \mathrm{i}\lambda \mathrm{d}\phi)v\|^2_{L^2(\mathbb{R}^n)} = \|p^+_{\phi,\lambda} v\|^2_{L^2(\mathbb{R}^n)} + \|p^-_{\phi,\lambda} v\|^2_{L^2(\mathbb{R}^n)} + \mathrm{Re}\langle [p^+_{\phi,\lambda}, p^-_{\phi,\lambda}]v, v\rangle, \tag{26}$$

where $p^+_{\phi,\lambda}$ and $p^-_{\phi,\lambda}$ are the formally self-adjoint and skew-adjoint parts of the operator $p_{\mathrm{m}}(\zeta, D + \mathrm{i}\lambda \mathrm{d}\phi)$, respectively, and $[\cdot, \cdot]$ denotes the commutator.

(2) Let $\tilde{\mathrm{m}} \in \mathbb{R}$ and denote $\omega = (|\xi|^2+\lambda^2)^{\tilde{\mathrm{m}}}$. Let $p_{\phi,\lambda,\tilde{m}}(\zeta,\xi,\lambda)$ be the sum of the principal symbols of p_1 and p_2, where

$$\begin{cases} \mathrm{p}_1 = \mathrm{Op}(\omega)\left((p^+_{\phi,\lambda})^* p^+_{\phi,\lambda} + (p^-_{\phi,\lambda})^* p^-_{\phi,\lambda}\right), \\ \mathrm{p}_2 = \dfrac{1}{2}\left([p^+_{\phi,\lambda}, p^-_{\phi,\lambda}] + [p^+_{\phi,\lambda}, p^-_{\phi,\lambda}]^*\right), \end{cases} \tag{27}$$

where * denotes the adjoint and $\mathrm{Op}(\omega)$ is the classical quantization of the symbol ω in ξ variable, i.e. $\mathrm{Op}(\omega)v := (|D|^2+\lambda^2)^{\tilde{\mathrm{m}}}v$. The principal symbol of p_1 is equal to $\omega|p_{\mathrm{m}}(\zeta,\xi+\mathrm{i}\lambda\mathrm{d}\phi)|^2$ and it is homogeneous in (ξ,λ) with degree $2(\mathrm{m}+\tilde{\mathrm{m}})$. Meanwhile, the principal symbol of p_2 will be denoted by $c_{2\mathrm{m}-1,\phi}$ and it is a homogeneous polynomial of degree $2\mathrm{m}-1$ given by

$$c_{2\mathrm{m}-1,\phi} = \{\mathrm{Re}\, p_{\mathrm{m}}(\zeta,\xi+\mathrm{i}\lambda\mathrm{d}\phi), \mathrm{Im}\, p_{\mathrm{m}}(\zeta,\xi+\mathrm{i}\lambda\mathrm{d}\phi)\},$$

where $\{\cdot,\cdot\}$ is the Poisson bracket. Choose some suitable function ϕ such that

$$C\lambda(|\xi|^2+\lambda^2)^{\mathrm{m}-1} \leqslant p_{\phi,\lambda,\tilde{\mathrm{m}}}(\zeta,\xi,\lambda). \tag{28}$$

(3) Apply the Fefferman-Phong inequality [18] to (28) and obtain

$$C\lambda\|v\|^2_{H^{\mathrm{m}-1}_\lambda(\mathbb{R}^n)} \leqslant \|p_{\mathrm{m}}(\zeta,D+\mathrm{i}\lambda\mathrm{d}\phi)v\|^2_{L^2(\mathbb{R}^n)} + C'\|v\|^2_{H^{\mathrm{m}-1}_\lambda(\mathbb{R}^n)}, \tag{29}$$

which is essentially the estimate (25) for sufficiently large λ.

Remark 3.1. We will briefly explain the reasons behind each step.

(1) The reason of the splitting in equation (26) is to involve the third term, which is one order lower than p_1 in equation (27) for $\tilde{\mathrm{m}}=0$. Moreover, symbolic calculus of pseudo-differential operators can be applied easily to the commutator.
(2) When $\tilde{\mathrm{m}}=0$, inequality (28) is the same as the sufficient condition of Carleman estimate proved in theorem 28.2.3 of [22].
(3) The advantage of using the Fefferman-Phong inequality is that (29) holds for principally normal operators, which include operators with complex-valued principal symbols. For operators with real-valued principal symbols, one observes that (29) can be derived via

$$\begin{aligned} &\|p_{\mathrm{m}}(\zeta,D+\mathrm{i}\lambda\mathrm{d}\phi)v\|^2_{L^2(\mathbb{R}^n)} + C_0\|v\|^2_{H^{\mathrm{m}-1}_\lambda(\mathbb{R}^n)} \\ &\quad \geqslant \lambda^{-2\tilde{\mathrm{m}}}\|p_{\mathrm{m}}(\zeta,D+\mathrm{i}\lambda\mathrm{d}\phi)v\|^2_{H^{\tilde{\mathrm{m}}}_\lambda(\mathbb{R}^n)} + C_0\|v\|^2_{H^{\mathrm{m}-1}_\lambda(\mathbb{R}^n)} \geqslant C_1\lambda\|v\|^2_{H^{\mathrm{m}-1}_\lambda(\mathbb{R}^n)} \end{aligned}$$

for some $\tilde{\mathrm{m}} \leqslant 0$, which can then be achieved by applying Gårding's inequality to (28) with $\tilde{\mathrm{m}} = -1/2$.

This leads us to introduce the concept of strong pseudo-convexity.

3.1.1. Pseudo-convex functions. As the pseudo-convexity of a function is important in establishing Carleman estimates, we will now give its definition.

Definition 3.2. Let X be a n-dimensional manifold and P be an m-th order differential operator with $C^1(X)$ principal coefficients and $L^\infty_{\mathrm{loc}}(X)$ complex-valued lower-order terms. A function ϕ is called strongly pseudo-convex at $\zeta_0 \in X$ with respect to P if $\mathrm{d}\phi(\zeta_0) \neq 0$ and for all $(\xi, \lambda) \in (\mathbb{R}^n \times \mathbb{R}_+)\backslash\{0\}$, we have that

$$\begin{aligned} p_{\mathrm{m}}(\zeta_0, \xi_{\lambda,0}) = \nabla_\xi p_{\mathrm{m}}(\zeta_0, \xi_{\lambda,0}) \cdot \nabla\phi(\zeta_0) = 0 \text{ implies}& \\ \lim_{\epsilon\to 0+} \frac{1}{\lambda+\epsilon} \operatorname{Im}\left(\overline{\nabla_\xi p_{\mathrm{m}}(\zeta_0, \xi_{\lambda,\epsilon})} \cdot \nabla_\zeta p_{\mathrm{m}}(\zeta_0, \xi_{\lambda,\epsilon})\right)& \\ + \nabla^2\phi(\zeta_0)\left(\overline{\nabla_\xi p_{\mathrm{m}}(\zeta_0, \xi_{\lambda,0})}, \nabla_\xi p_{\mathrm{m}}(\zeta_0, \xi_{\lambda,0})\right) > 0,& \end{aligned} \tag{30}$$

where $\xi_{\lambda,\epsilon} = \xi + \mathrm{i}(\lambda+\epsilon)\mathrm{d}\phi(\zeta_0)$. Moreover, we say that ϕ is strongly pseudo-convex with respect to P on a set $\mathcal{U} \subset X$, if it is strongly pseudo-convex for each $\zeta_0 \in \mathcal{U}$.

The operator in (16) defined by

$$P := \Box + Q_1\partial_t + Q_0 I = \partial_t^2 - \Delta + Q_1\partial_t + Q_0 I \tag{31}$$

satisfies the criteria of the operator in definition 3.2 for $n = 1+d$. Moreover, P is a second-order ($\mathrm{m} = 2$) differential operator and its principal symbol is given by $p_2(t,x;t',x') = -t'^2 + |x'|^2$, where (t',x') are the dual variables of (t,x). As quadratic polynomials can only have complex roots in conjugate pairs, (30) is void for $\lambda \neq 0$. Therefore, it simplifies the pseudo-convexity criterion for second-order differential operators to

$$p_2(t_0,x_0;t',x') = (H_{p_2}\phi)(t_0,x_0;t',x') = 0 \text{ implies } (H_{p_2}^2\phi)(t_0,x_0;t',x') > 0,$$

for all $(t',x') \in \mathbb{R}^{1+d}\backslash\{0\}$ and H_{p_2} is the Hamiltonian vector field generated by p_2.

As for the choice of a pseudo-convex function ϕ, the Carleman weight function,

$$\phi(t,x) = e^{\psi(t,x)} \quad \text{with} \quad \psi(t,x) = \gamma(|x-x_0|^2 - \beta|t-t_0|^2), \tag{32}$$

is one of the mostly used (see, e.g. [36, formula (4.1)] or [35, formula (2.76)]) for hyperbolic inverse problems in Minkowski time-space. Indeed for $\gamma > 0$, $\beta \in (0,1)$ and $(t_0,x_0) \notin \Omega_T$, ϕ will be strongly pseudo-convex with respect to $\Box$ on Ω_T (see appendix B).

3.2. Carleman estimate with boundary conditions

If we can write a second-order differential operator as $P = P_2 + P_1$, where P_1 is a first-order differential operator with $L^\infty(\Omega_T)$ coefficients, then $\|P_1 u\|_{L^2(\Omega_T)}$ can be absorbed by choosing sufficiently large constant C in (25). In other words, for the case of P being (31), it suffices to establish a Carleman estimate for $P = P_2 = \Box$. Therefore, we will first establish a Carleman estimate for the conjugate operator of $\Box$, i.e. $P_{\phi,\lambda} = e^{\lambda\phi}\Box e^{-\lambda\phi}$, where ϕ is a function in

$$\begin{aligned} \mathcal{P} := \Big\{ &\phi(t,x) = e^{\gamma\psi(t,x)}, \text{ where } \gamma > 0,\ \psi(t,x) = \psi_0(x) - \beta\psi_1(t) \text{ with } \beta \in (0,1), \\ &\text{such that } \mathrm{d}\psi \neq 0 \text{ in } \Omega_T \text{ and the Hessian of } \psi_0 \text{ with respect to } g \text{ is positive in } \Omega, \\ &\text{i.e. } \nabla^2\psi_0(x',x') > \rho\alpha|x'|^2,\ \forall x' \in \mathbb{R}^d, \text{ for some } \rho > \beta \Big\}, \end{aligned} \tag{33}$$

where $\alpha = \sup_{t\in(0,T)} |\psi_1''(t)|$. Here, the small positive parameter β and the large positive parameter γ will be chosen so that Hörmander's pseudo-convexity property (30) is satisfied. It turns out that, for sufficiently large γ and sufficiently small β, a function $\phi \in \mathcal{P}$ is strongly pseudo-convex with respect to P on Ω_T, see appendix B for the justification. A sub-class of $\mathcal{P}$ is given by

$$\mathcal{P}_0 = \{\phi \in \mathcal{P} \text{ with } |\nabla\psi_0| > 2\delta > 0 \text{ in } \Omega\},$$

for some positive constant δ. We will use this sub-class of functions to obtain the Carleman estimate (35) below.

For $\Omega_T \subset X$, one can decompose $\|P_{\phi,\lambda}u\|^2_{L^2(\Omega_T)}$ into

$$\|P_{\phi,\lambda}u\|^2_{L^2(\Omega_T)} = \|P^+_{\phi,\lambda}u\|^2_{L^2(\Omega_T)} + \|P^-_{\phi,\lambda}u\|^2_{L^2(\Omega_T)} + 2\left(P^+_{\phi,\lambda}u, P^-_{\phi,\lambda}u\right), \tag{34}$$

where $P^+_{\phi,\lambda}u = \Box u + \lambda^2\eta(\mathrm{d}\phi,\mathrm{d}\phi)u$ and $P^-_{\phi,\lambda}u = -2\lambda\eta(\mathrm{d}\phi,\mathrm{d}u) - \lambda(\Box\phi)u$. Note that equation (34) is an analogue of (26) up to a change of lower-order terms. In the following theorem, we extend the results in [10, chapter 4] to include the lower ($t=0$) and upper ($t=T$) surface contributions.

Theorem 3.3. *Let $u \in H^2(\Omega_T)$ and ϕ be a function in the set $\mathcal{P}_0$. Then there exists a positive constant $\gamma_0(\beta,\delta,\rho)$, such that, for any $\gamma > \gamma_0$, there exist positive constants $C(\alpha,\beta,\gamma,\delta,\rho,\Omega_T)$ and $\lambda_0(\gamma)$, for which the following estimate hold:*

$$C\lambda\|u\|^2_{H^1_\lambda(\Omega_T)} \leqslant \|P_{\phi,\lambda}u\|^2_{L^2(\Omega_T)} + \mathcal{B}_1 + \mathcal{B}_2(T) - \mathcal{B}_2(0), \quad \forall\lambda \geqslant \lambda_0, \tag{35}$$

where

$$\begin{aligned}
\mathcal{B}_1 = {}& \frac{1}{2}\gamma^2\lambda\int_{(0,T)\times\partial\Omega}\phi|u|^2(\Box\psi + \gamma\eta(\mathrm{d}\psi))\partial_\nu\psi_0\mathrm{d}y\mathrm{d}t \\
&- \frac{1}{2}\gamma\lambda\int_{(0,T)\times\partial\Omega}\phi|u|^2\partial_\nu\left(\Delta\psi_0 + \gamma|\nabla\psi_0|^2\right)\mathrm{d}y\mathrm{d}t \\
&- 2\gamma\lambda\int_{(0,T)\times\partial\Omega}\phi\partial_\nu u\eta(\mathrm{d}u,\mathrm{d}\psi)\mathrm{d}y\mathrm{d}t \\
&+ \gamma\lambda\int_{(0,T)\times\partial\Omega}\phi\eta(\mathrm{d}u)\partial_\nu\psi_0\mathrm{d}y\mathrm{d}t - \gamma\lambda\int_{(0,T)\times\partial\Omega}\phi u\partial_\nu u(\Box\psi + \gamma\eta(\mathrm{d}\psi))\mathrm{d}y\mathrm{d}t \\
&+ \alpha(\rho+\beta)\gamma\lambda\int_{(0,T)\times\partial\Omega}\phi\left(\gamma|u|^2\partial_\nu\psi_0 - u\partial_\nu u\right)\mathrm{d}y\mathrm{d}t \\
&- (\gamma\lambda)^3\int_{(0,T)\times\partial\Omega}\phi^3|u|^2\eta(\mathrm{d}\psi)\partial_\nu\psi_0\mathrm{d}y\mathrm{d}t,
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
\mathcal{B}_2(t) = {}& \alpha(\rho+\beta)\gamma\lambda\int_\Omega\phi uu_t\mathrm{d}x - \frac{1}{2}\alpha(\rho+\beta)\gamma^2\lambda\int_\Omega\phi|u|^2\mathrm{d}x \\
&- \gamma\lambda\int_\Omega\phi\left(\frac{1}{2}|u|^2\left(2\gamma\beta^2\psi_1'\psi_1'' - \beta\psi_1''' - \gamma\beta\psi_1'(\Box\psi + \gamma\eta(\mathrm{d}\psi))\right)\right. \\
&\left. + 2u_tg(\nabla u,\nabla\psi_0) - uu_t(\Box\psi + \gamma\eta(\mathrm{d}\psi)) + \beta\psi_1'|u_t|^2\right)\mathrm{d}x \\
&- \beta(\gamma\lambda)^3\int_\Omega\phi^3\psi_1'|u|^2\eta(\mathrm{d}\psi)\mathrm{d}x.
\end{aligned} \tag{37}$$

The same estimate holds for P being replaced by $P + P_1$, where P_1 is a first-order differential operator with $L^\infty(\Omega_T)$ coefficients. The only difference is that λ_0 will also be dependent on the $L^\infty(\Omega_T)$-norm of the coefficients.

Remark 3.4. In the special case $\psi_1(t) = (t-t_0)^2$ for some $0 < t_0 < T$, then the boundary term derived in [10, chapter 4] is different from $\mathcal{B}_1$ only by the first term in equation (36). The second term $\mathcal{B}_2$ in (37) is a new boundary contribution that is absent under the vanishing conditions of u at $t = 0$ and $t = T$, which was imposed in [10, chapter 4].

Remark 3.5. An analogue of theorem 3.3 can be obtained by using the method developed in [35, chapter 2], where a global Carleman estimate was obtained via a local version of it. The boundary terms can then be achieved by calculating the divergence term and the time derivative term in this local formula (see [35, theorem 2.5.1]). Moreover, the analysis can be simplified by using only the explicit function (32), as in [35]. However, we have stated (and proved) theorem 3.3 for any function $\phi \in \mathcal{P}_0$ for generality and in order to be consistent with Hörmander's approach to Carleman estimates (see [22]).

Proof of theorem 3.3. We will focus on $\left(P^+_{\phi,\lambda}u, P^-_{\phi,\lambda}u\right)$ in (34). As in [10, chapter 4], one has

$$\begin{aligned}\left(P^+_{\phi,\lambda}u, P^-_{\phi,\lambda}u\right) = &-2\lambda\int_{\Omega_T} u_{tt}\eta(\mathrm{d}u,\mathrm{d}\phi)\mathrm{d}x\mathrm{d}t - \lambda\int_{\Omega_T} uu_{tt}\Box\phi\,\mathrm{d}x\mathrm{d}t\\ &+2\lambda\int_{\Omega_T}(\Delta u)\eta(\mathrm{d}u,\mathrm{d}\phi)\mathrm{d}x\mathrm{d}t + \lambda\int_{\Omega_T}(u\Delta u)\Box\phi\mathrm{d}x\mathrm{d}t\\ &-2\lambda^3\int_{\Omega_T} u\eta(\mathrm{d}u,\mathrm{d}\phi)\eta(\mathrm{d}\phi)\mathrm{d}x\mathrm{d}t - \lambda^3\int_{\Omega_T}|u|^2\eta(\mathrm{d}\phi)\Box\phi\,\mathrm{d}x\mathrm{d}t\\ :=&\sum_{k=1}^{6} I_k.\end{aligned}$$

Applying integration by parts to eliminate all the second-order derivatives of u in I_k, we have

$$\begin{aligned}I_1 =& \lambda\int_{\Omega_T}|u_t|^2(\phi_{tt}+\Delta\phi)\mathrm{d}x\mathrm{d}t - 2\lambda\int_{\Omega_T}u_t g(\nabla u,\nabla\phi_t)\mathrm{d}x\mathrm{d}t\\ &-\lambda\int_{(0,T)\times\partial\Omega}|u_t|^2\partial_\nu\phi\,\mathrm{d}y\mathrm{d}t - \lambda\int_\Omega\left(|u_t|^2\phi_t\right)\Big|_0^T\mathrm{d}x + 2\lambda\int_\Omega\left(u_t g\left(\nabla u,\nabla\phi\right)\right)|_0^T\mathrm{d}x,\\ I_2 =& \lambda\int_{\Omega_T}|u_t|^2\Box\phi\,\mathrm{d}x\mathrm{d}t - \frac{\lambda}{2}\int_{\Omega_T}|u|^2\Box\phi_{tt}\mathrm{d}x\mathrm{d}t - \lambda\int_\Omega(uu_t\Box\phi)|_0^T\mathrm{d}x + \frac{\lambda}{2}\int_\Omega\left(|u|^2\Box\phi_t\right)\Big|_0^T\mathrm{d}x,\\ I_3 =& \lambda\int_{\Omega_T}\left(|\nabla u|^2\Box\phi - 2u_t g(\nabla u,\nabla\phi_t) + 2\nabla^2\phi(\nabla u,\nabla u)\right)\mathrm{d}x\mathrm{d}t\\ &+\lambda\int_{(0,T)\times\partial\Omega}\left(2\partial_\nu u\eta(\mathrm{d}u,\mathrm{d}\phi) + |\nabla u|^2\partial_\nu\phi\right)\mathrm{d}y\mathrm{d}t,\\ I_4 =& \lambda\int_{\Omega_T}\left(\frac{1}{2}|u|^2\Delta\Box\phi - |\nabla u|^2\Box\phi\right)\mathrm{d}x\mathrm{d}t + \lambda\int_{(0,T)\times\partial\Omega}\left(u\partial_\nu u\Box\phi - \frac{1}{2}|u|^2\partial_\nu\Box\phi\right)\mathrm{d}y\mathrm{d}t,\\ I_5 =& 2\lambda^3\int_{\Omega_T}|u|^2\left(\frac{1}{2}\eta(\mathrm{d}\phi)\Box\phi + |\phi_t|^2\phi_{tt} + \nabla^2\phi(\nabla\phi,\nabla\phi) - 2\phi_t g(\nabla\phi_t,\nabla\phi)\right)\mathrm{d}x\mathrm{d}t\\ &+\lambda^3\int_{(0,T)\times\partial\Omega}|u|^2\eta(\mathrm{d}\phi)\partial_\nu\phi\,\mathrm{d}y\mathrm{d}t - \lambda^3\int_\Omega\left(|u|^2\phi_t\eta(\mathrm{d}\phi)\right)|_0^T\mathrm{d}x.\end{aligned}$$

Summing the above expression for $I_1,\ldots,I_5$ and I_6, we have

$$\begin{aligned}\left(P^+_{\phi,\lambda}u,P^-_{\phi,\lambda}u\right) &= 2\lambda\int_{\Omega_T}\left(\phi_{tt}|u_t|^2-2u_tg(\nabla u,\nabla\phi_t)+\nabla^2\phi(\nabla u,\nabla u)\right)\mathrm{d}x\mathrm{d}t\\&\quad+2\lambda^3\int_{\Omega_T}|u|^2\left(\phi_{tt}|\phi_t|^2-2\phi_tg(\nabla\phi,\nabla\phi_t)+\nabla^2\phi(\nabla\phi,\nabla\phi)\right)\mathrm{d}x\mathrm{d}t\\&\quad-\frac{\lambda}{2}\int_{\Omega_T}|u|^2\Box^2\phi\,\mathrm{d}x\mathrm{d}t+B_0,\end{aligned}$$

where B_0 is the summation of boundary terms in I_k and it is given by

$$\begin{aligned}B_0 &= \lambda\int_{(0,T)\times\partial\Omega}(2\partial_\nu u\eta(\mathrm{d}u,\mathrm{d}\phi)-\eta(\mathrm{d}u)\partial_\nu\phi)\,\mathrm{d}y\mathrm{d}t\\&\quad+\lambda\int_{(0,T)\times\partial\Omega}\left(u\partial_\nu u\Box\phi-\frac{1}{2}|u|^2\partial_\nu\Box\phi\right)\mathrm{d}y\mathrm{d}t\\&\quad+\lambda^3\int_{(0,T)\times\partial\Omega}|u|^2\eta(\mathrm{d}\phi)\partial_\nu\phi\,\mathrm{d}y\mathrm{d}t\\&\quad+\lambda\int_\Omega\left(\frac{1}{2}|u|^2\Box\phi_t+2u_tg(\nabla u,\nabla\phi)-uu_t\Box\phi-|u_t|^2\phi_t\right)\Big|_0^T\mathrm{d}x\\&\quad-\lambda^3\int_\Omega\left(|u|^2\phi_t\eta(\mathrm{d}\phi)\right)\Big|_0^T\mathrm{d}x.\end{aligned}$$

Substituting $\phi(t,x)=e^{\gamma\psi(tx,x)}$, where $\psi(t,x)=\psi_0(x)-\beta\psi_1(t)$, one has

$$\begin{aligned}\left(P^+_{\phi,\lambda}u,P^-_{\phi,\lambda}u\right) &= 2\gamma\lambda\int_{\Omega_T}\phi\left(\nabla^2\psi_0(\nabla u,\nabla u)-\beta\psi_1''|u_t|^2+\gamma(\eta(\mathrm{d}u,\mathrm{d}\psi))^2\right)\mathrm{d}x\mathrm{d}t\\&\quad+2(\gamma\lambda)^3\int_{\Omega_T}\phi^3|u|^2\left(\nabla^2\psi_0(\nabla\psi_0,\nabla\psi_0)-\beta\psi_1''(\psi_1')^2+\gamma(\eta(\mathrm{d}\psi))^2\right)\mathrm{d}x\mathrm{d}t\\&\quad-\frac{\lambda}{2}\int_{\Omega_T}|u|^2\Box^2\phi\,\mathrm{d}x\mathrm{d}t+B_0.\end{aligned}\tag{38}$$

Now, B_0 becomes

$$\begin{aligned}B_0 &= \gamma\lambda\int_{(0,T)\times\partial\Omega}\phi\Bigg(2\partial_\nu u\eta(\mathrm{d}u,\mathrm{d}\psi)-\eta(\mathrm{d}u)\partial_\nu\psi_0+u\partial_\nu u(\Box\psi+\gamma\eta(\mathrm{d}\psi))\\&\quad-\frac{1}{2}\gamma|u|^2(\Box\psi+\gamma\eta(\mathrm{d}\psi))\partial_\nu\psi_0+\frac{1}{2}|u|^2\partial_\nu(\Delta\psi_0+\gamma|\nabla\psi_0|^2)\Bigg)\mathrm{d}y\mathrm{d}t\\&\quad+(\gamma\lambda)^3\int_{(0,T)\times\partial\Omega}\phi^3|u|^3\eta(\mathrm{d}\psi)\partial_\nu\psi_0\mathrm{d}y\mathrm{d}t\\&\quad+\gamma\lambda\int_\Omega\Bigg[\phi\Big(\frac{1}{2}|u|^2\left(2\gamma\beta^2\psi_1'\psi_1''-\beta\psi_1'''-\gamma\beta\psi_1'(\Box\psi+\gamma\eta(\mathrm{d}\psi))\right)\\&\quad+2u_tg(\nabla u,\nabla\psi_0)-uu_t(\Box\psi+\gamma\eta(\mathrm{d}\psi))+\beta\psi_1'|u_t|^2\Big)\Bigg]\Bigg|_0^T\mathrm{d}x\\&\quad+\beta(\gamma\lambda)^3\int_\Omega\left(\phi^3\psi_1'|u|^2\eta(\mathrm{d}\psi)\right)\Big|_0^T\mathrm{d}x.\end{aligned}$$

Since $\nabla^2\psi_0(x',x') > \alpha\rho|x'|^2$ for all $x' \in \mathbb{R}^d$, together with equation (38), this shows that $\|\nabla u\|_{L^2(\Omega_T)}$ is controlled by $\left(P^+_{\phi,\lambda}u, P^-_{\phi,\lambda}u\right)$, and we are left to control the time-derivative term of u, i.e. $-2\beta\gamma\lambda\int_{\Omega_T}\phi\psi_1''|u_t|^2\mathrm{d}x\mathrm{d}t$. This means that another positive term alike $\int_{\Omega_T}\phi|u_t|^2\mathrm{d}x\mathrm{d}t$ is needed to overcome this latter term. This can be achieved by considering the following term:

$$\int_{\Omega_T}(P^+_{\phi,\lambda}u)u\phi\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}(\Box u + \lambda^2\eta(\mathrm{d}\phi)u)u\phi\,\mathrm{d}x\mathrm{d}t.$$

Again, using integration by parts to remove all the second-order derivatives of u, one deduces

$$\begin{aligned}\int_{\Omega_T}(P^+_{\phi,\lambda}u)u\phi\,\mathrm{d}x\mathrm{d}t = &-\int_{\Omega_T}\phi|u_t|^2\mathrm{d}x\mathrm{d}t + \frac{\gamma}{2}\int_{\Omega_T}\phi|u|^2(\Box\psi + \gamma\eta(\mathrm{d}\psi))\mathrm{d}x\mathrm{d}t\\ &+\int_{\Omega_T}\phi|\nabla u|^2\mathrm{d}x\mathrm{d}t + (\gamma\lambda)^2\int_{\Omega_T}\phi^3|u|^2\eta(\mathrm{d}\psi)\mathrm{d}x\mathrm{d}t + B_1,\end{aligned}$$

where B_1 is a boundary term given by

$$B_1 = \int_{(0,T)\times\partial\Omega}\phi\left(\gamma|u|^2\partial_\nu\psi - u\partial_\nu u\right)\mathrm{d}y\mathrm{d}t + \int_\Omega(\phi u u_t)|_0^T\,\mathrm{d}x - \frac{\gamma}{2}\int_\Omega\left(\phi|u|^2\psi_t\right)\Big|_0^T\mathrm{d}x.$$

Now, by Cauchy-Schwarz and Young's inequalities, we have

$$\begin{aligned}\left|B_1 - \int_{\Omega_T}\phi|u_t|^2\mathrm{d}x\mathrm{d}t\right| \leqslant{}& \epsilon\|P^+_{\phi,\lambda}u\|^2_{L^2(\Omega_T)} + \frac{1}{2\epsilon}\int_{\Omega_T}\phi^2|u|^2\mathrm{d}x\mathrm{d}t\\ &+\int_{\Omega_T}\phi|\nabla u|^2\mathrm{d}x\mathrm{d}t + (\gamma\lambda)^2\int_{\Omega_T}\phi^3|u|^2|\eta(\mathrm{d}\psi)|\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{39}$$

for any $0<\epsilon<\frac{e^{\mathsf{c}\gamma}}{2\gamma(\mathsf{a}+\mathsf{b}\gamma)}$, where $\mathsf{a} := \sup_{\Omega_T}|\Box\psi|$, $\mathsf{b} := \sup_{\Omega_T}|\eta(\mathrm{d}\psi)|$ and $\mathsf{c} := \inf_{\Omega_T}\psi$. Note that there exits $\mathsf{d} = \mathsf{d}(\beta,\gamma,\Omega_T)$ such that $|\Box^2\phi(t,x)| \leqslant \mathsf{d}\phi(t,x)$ for all $(t,x)\in\Omega_T$. Therefore, applying equation (39) and

$$\begin{cases}|\Box^2\phi| \leqslant \mathsf{d}\phi,\\ \nabla^2\psi_0(x',x') \geqslant \rho\alpha|x'|^2,\ \ \forall x'\in\mathbb{R}^d,\end{cases}$$

to equation (38), we obtain

$$\begin{aligned}&\left(P^+_{\phi,\lambda}u, P^-_{\phi,\lambda}u\right) + \alpha(\rho+\beta)\gamma\lambda\epsilon\|P^+_{\phi,\lambda}u\|^2_{L^2(\Omega_T)} + \alpha(\rho+\beta)\gamma\lambda B_1 - B_0\\ &\quad\geqslant \alpha(\rho-\beta)\gamma\lambda\int_{\Omega_T}\phi\left(|u_t|^2 + |\nabla u|^2\right)\mathrm{d}x\mathrm{d}t\\ &\quad+2(\gamma\lambda)^3\int_{\Omega_T}\phi^3|u|^2\mathrm{H}(\psi)\mathrm{d}x\mathrm{d}t - \frac{\mathsf{d}\lambda}{2}\int_{\Omega_T}\phi|u|^2\mathrm{d}x\mathrm{d}t\\ &\quad-\frac{\alpha(\rho+\beta)\gamma\lambda}{2\epsilon}\int_{\Omega_T}(u\phi)^2\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $\mathrm{H}(\psi) := \rho\alpha|\nabla\psi|^2 - \beta\alpha|\psi_t|^2 - \frac{\alpha(\rho+\beta)}{2}|\eta(\mathrm{d}\psi)| + \gamma\,(\eta(\mathrm{d}\psi))^2$. Note that

$$\begin{aligned}\mathrm{H}(\psi) &= \alpha(\rho-\beta)|\nabla\psi_0|^2 - \alpha\beta\eta(\mathrm{d}\psi) - \frac{\alpha(\rho+\beta)}{2}|\eta(\mathrm{d}\psi)| + \gamma\,(\eta(\mathrm{d}\psi))^2 \\ &\geqslant \alpha(\rho-\beta)|\nabla\psi_0|^2 - \frac{\alpha(\rho+3\beta)}{2}|\eta(\mathrm{d}\psi)| + \gamma\,(\eta(\mathrm{d}\psi))^2 \\ &\geqslant \alpha(\rho-\beta)|\nabla\psi_0|^2 - \frac{\alpha^2(\rho+3\beta)^2}{16\gamma}.\end{aligned}$$

Since $|\nabla\psi_0| > 2\delta$, we have that $\mathrm{H}(\psi) > 2\alpha(\rho-\beta)\delta^2$ for $\gamma > \frac{\alpha(\rho+3\beta)^2}{16(\rho-\beta)\delta^2}$. Therefore,

$$\begin{aligned}&\|P_{\phi,\lambda}u\|^2_{L^2(\Omega_T)} + \alpha(\rho+\beta)\gamma\lambda B_1 - B_0 \\ &\quad\geqslant \left(P^+_{\phi,\lambda}u, P^-_{\phi,\lambda}u\right) + \frac{1}{2}\|P^+_{\phi,\lambda}u\|^2_{L^2(\Omega_T)} + \alpha(\rho+\beta)\gamma\lambda B_1 - B_0 \\ &\quad\geqslant \alpha(\rho-\beta)\gamma\lambda\int_{\Omega_T}\phi\,(|u_t|^2 + |\nabla u|^2)\mathrm{d}x\mathrm{d}t + (\gamma\lambda)^3\int_{\Omega_T}\phi^3|u|^2\mathrm{H}(\psi)\mathrm{d}x\mathrm{d}t,\end{aligned}$$

for $\gamma > \gamma_0 := \max\left\{1, \frac{\alpha(\rho+3\beta)^2}{16(\rho-\beta)\delta^2}\right\}$ and $\lambda > \lambda_0 := e^{-\mathsf{c}\gamma}\max\left\{\frac{(\mathsf{a}+\mathsf{b}\gamma)}{\alpha(\rho+\beta)}, \left(\frac{\mathsf{d}}{2\alpha(\rho-\beta)\delta^2}\right)^{\frac{1}{2}}, \frac{\alpha(\rho+\beta)^2}{(\rho-\beta)\gamma\delta^2}\right\}$. This shows that

$$\mathcal{B}_1 + \mathcal{B}_2(T) - \mathcal{B}_2(0) = \alpha(\rho+\beta)\gamma\lambda B_1 - B_0,$$

where $\mathcal{B}_1$, defined by (36), is the contribution from time-like surfaces, and $\mathcal{B}_2(0)$ and $\mathcal{B}_2(T)$ are the contributions from the initial and final surfaces, respectively. Finally, using the fact that ϕ is positive and bounded in Ω_T, we have

$$\begin{aligned}&\|P_{\phi,\lambda}u\|^2_{L^2(\Omega_T)} + \alpha(\rho+\beta)\gamma\lambda B_1 - B_0 \\ &\quad\geqslant C(\alpha,\beta,\gamma,\delta,\rho,\Omega_T)\left(\lambda\int_{\Omega_T}(|u_t|^2 + |\nabla u|^2)\mathrm{d}x\mathrm{d}t + \lambda^3\int_{\Omega_T}|u|^2\mathrm{d}x\mathrm{d}t\right).\end{aligned}$$

This completes the proof of theorem 3.3.

□

4. Inverse problems

Many inverse problems can be reduced to recovering coefficients in (16). For instance, see [26] and [35, chapter 3] for recovering the wave speed and [56] for the damping coefficient. For our applications, we focus on the recovery of the damping $Q_1(x)$ or potential $Q_0(x)$ coefficient appearing in (16), as well as of the blood perfusion coefficient $w(x)$ appearing in (8). Thereby, it suffices to consider the function of the type

$$\phi(t,x) = e^{\gamma(\psi_0(x) - \beta(t-t_0)^2)}.$$

This gives $\alpha = 2$ in (33) and we are left with the freedom of choosing ψ_0 satisfying

$$\nabla^2\psi_0(x',x') > 2\rho|x'|^2,\ \forall x' \in \mathbb{R}^d \quad \text{and} \quad |\nabla\psi_0| > 2\delta > 0 \text{ on } \Omega. \tag{40}$$

As we would like to study observability or controllability through partial boundary data, we thereby define $\Gamma_+ := \{y \in \partial\Omega : \partial_\nu\psi_0(y) \geqslant 0\}$, $\Gamma_- := \{y \in \partial\Omega : \partial_\nu\psi_0(y) \leqslant 0\}$ and $\Gamma := \{y \in \partial\Omega : \partial_\nu\psi_0(y) \neq 0\}$. Moreover, $r_- = \inf_{x\in\Omega}\psi_0$ and $r_+ = \sup_{x\in\Omega}\psi_0$ will be used in our next theorem, which is an application of theorem 3.3. This is also an improved version of the main result in [23].

Theorem 4.1 (Dirichlet boundary condition (13), $Q_0(x)$ known, $Q_1(x)$ unknown). *Let $F \in L^1(0,T;L^2(\Omega))$, $Q_0 \in L^\infty(\Omega)$, $u_{amb} \in H^1((0,T)\times\partial\Omega)$, $\varphi_0 \in H^1(\Omega)$ and $\varphi_1 \in L^2(\Omega)$ satisfying the compatibility condition (20). For $i=1,2$, let $u^{(i)} \in \mathcal{V}$ be the solution to (19) with the space-dependent coefficient $Q_1 = Q_1^{(i)} \in L^\infty(\Omega)$. Choose a ψ_0 that satisfies condition (40). Assume also that the following conditions hold:*

(i) $|\varphi_1(x)| \geqslant c_1 > 0$ a.e. $x \in \Omega$.
(ii) $u^{(i)} \in H^3(0,T;H^{k+2}(\Omega)) \cap H^2(0,T;H^k(\Omega))$ for some $k > \frac{d}{2}$.

Denote by $q^{(i)} := \partial_\nu u^{(i)}|_{(0,T)\times\partial\Omega}$. Then, for $T > T_0 := \left(\frac{r_+ - r_-}{\beta}\right)^{\frac{1}{2}}$, we have the following stability estimate:

$$\|Q_1^{(1)} - Q_1^{(2)}\|_{L^2(\Omega)} \leqslant C\|q_t^{(1)} - q_t^{(2)}\|_{L^2((0,T)\times\Gamma_+)}, \tag{41}$$

where C depends on β, Ω_T, ψ_0, φ_0, φ_1 and the L^∞-norms of Q_0, $Q_1^{(i)}$ and $u^{(i)}$ for $i=1,2$.

Remark 4.2. The lower bound T_0 is $(r_+ - r_-)^{\frac{1}{2}}$ when taking the limit of $\beta \nearrow 1$. This is better than the ones in [10, 23, 40], which are greater or equal to $r_+^{\frac{1}{2}}$. The set of solutions at (ii) in theorem 4.1 is included in the admissible set Λ_h defined in equation (17).

Proof of theorem 4.1. Following the standard technique, we consider the difference of $u^{(1)}$ and $u^{(2)}$. Define $U := u^{(1)} - u^{(2)}, f(x) = Q_1^{(2)}(x) - Q_1^{(1)}(x)$ and $R = u_t^{(2)}$. By extending U, $Q_1^{(1)}$ and R from $(0,T)$ to $(-T,T)$, as in appendix C, one has

$$\begin{cases} \left(U_{tt} - \Delta U + Q_1^{(1)} U_t + Q_0 U\right)(t,x) = f(x)R(t,x), & (t,x) \in (-T,T)\times\Omega, \\ U(t,x) = 0,\ \partial_\nu U(t,x) = q^{(1)}(t,x) - q^{(2)}(t,x), & (t,x) \in (-T,T)\times\partial\Omega, \\ U(0,x) = 0,\ U_t(0,x) = 0 & x \in \Omega, \end{cases}$$

Now, take a function $\chi \in C^\infty[-T,T]$ that satisfy

$$\chi(t) = \begin{cases} 0, & |t-T| < \varepsilon \quad \text{or} \quad |t+T| < \varepsilon, \\ 1, & |t| < T - 2\varepsilon, \end{cases} \tag{42}$$

where ε is a small positive number. Then $\mathrm{w} := \chi U_t$ satisfies

$$\begin{cases} \mathrm{w}_{tt} - \Delta \mathrm{w} + Q_1^{(1)}\mathrm{w}_t + Q_0 \mathrm{w} = \chi f R_t + \chi'(2U_{tt} + Q_1^{(1)}U_t) + \chi'' U_t & \text{in } (-T,T)\times\Omega, \\ \mathrm{w} = 0,\ \partial_\nu \mathrm{w} = q_t^{(1)} - q_t^{(2)}, & \text{in } (-T,T)\times\partial\Omega, \\ \mathrm{w}(0,x) = 0,\ \mathrm{w}_t(0,x) = f(x)R(0,x), & \text{in } \{t=0\}\times\Omega. \end{cases} \tag{43}$$

Moreover, w vanishes in $\{|t \pm T| < \varepsilon\}\times\Omega$. In this proof, the letter C will be used to denote different constants. By setting $P := \partial_t^2 - \Delta + Q_1^{(1)}\partial_t + Q_0 I$, $u := e^{\lambda\phi}\mathrm{w}$ and $\phi(t,x) = e^{\gamma(\psi_0(x) - \beta t^2)}$ in the framework of theorem 3.3, one concludes that

$$\begin{aligned} C\lambda\|u\|^2_{H^1_\lambda((-T,T)\times\Omega)} \leqslant{}& \left\|e^{\lambda\phi}\left(\chi f R_t + B(\chi,U)\right)\right\|^2_{L^2((-T,T)\times\Omega)} \\ &+ \gamma\lambda\int_{(-T,T)\times\Gamma_+} \phi e^{2\lambda\phi}|\partial_\nu \mathrm{w}|^2 \partial_\nu\psi_0 \mathrm{d}y\mathrm{d}t, \end{aligned} \tag{44}$$

where $B(\chi,U) := \chi'\left(2U_{tt} + Q_1^{(1)}U_t\right) + \chi'' U_t$ and C depends on β, ψ_0 and Ω_T, whilst λ depends on ψ_0 and the $L^\infty(\Omega)$-norms of Q_0 and $Q_1^{(1)}$. Note that assumption (ii) in theorem

4.1 and proposition D.1 of appendix D imply that R and $R_t \in L^\infty((-T,T)\times\Omega)$. As the support of χ' and χ'' is contained in $|t\pm T|<2\varepsilon$, we have, by the estimate (18),

$$\begin{aligned}
&\left\|e^{\lambda\phi}B(\chi,U)\right\|^2_{L^2((-T,T)\times\Omega)}\\
&\quad\leqslant C\operatorname*{ess\,sup}_{|t-T|<2\varepsilon,x\in\Omega} e^{2\lambda\phi}\left(\|fR_t\|^2_{L^2((-T,T)\times\Omega)}+\|fR\|^2_{L^2((-T,T)\times\Omega)}+\|fR(0,\cdot)\|^2_{L^2(\Omega)}\right)\\
&\quad\leqslant Ce^{2\lambda e^{\gamma r_-(\varepsilon)}}\left(\operatorname*{ess\,sup}_{(t,x)\in(-T,T)\times\Omega}|R|^2+\operatorname*{ess\,sup}_{(t,x)\in(-T,T)\times\Omega}|R_t|^2\right)\|f\|^2_{L^2(\Omega)}\leqslant Ce^{2\lambda e^{\gamma r_-(\varepsilon)}}\|f\|^2_{L^2(\Omega)}
\end{aligned}\tag{45}$$

for $T>\left(\frac{r_+-r_-(\varepsilon)}{\beta}\right)^{\frac12}+2\varepsilon$, where $r_-(\varepsilon):=r_--\varepsilon$ for arbitrary small ε. Here, the constant C in (45) depends on T, χ, $Q_1^{(1)}$, R and R_t. Using the stationary phase method [21, section 7.7], we have

$$\begin{aligned}
\left\|e^{\lambda\phi}fR_t\right\|^2_{L^2((-T,T)\times\Omega)}&\leqslant\operatorname*{ess\,sup}_{(t,x)\in(-T,T)\times\Omega}|R_t|^2\left\|e^{\lambda\phi}f\right\|^2_{L^2((-T,T)\times\Omega)}\\
&=\operatorname*{ess\,sup}_{(t,x)\in(-T,T)\times\Omega}|R_t|^2\int_\Omega\left(e^{2\lambda\phi(0,x)}\int_{-T}^{T}e^{2\lambda(\phi(t,x)-\phi(0,x))}|f|^2\mathrm{d}t\right)\mathrm{d}x\\
&\leqslant\operatorname*{ess\,sup}_{(t,x)\in(-T,T)\times\Omega}|R_t|^2\int_\Omega\left(e^{2\lambda\phi(0,x)}\int_{-T}^{T}|f|^2O(\lambda^{-\frac12})\mathrm{d}t\right)\mathrm{d}x.
\end{aligned}\tag{46}$$

Using the inequalities (44)–(46), we obtain

$$\begin{aligned}
\lambda\|e^{\lambda\phi}\mathrm{w}\|^2_{H^1_\lambda((-T,T)\times\Omega)}\leqslant{}&C\left(O(\lambda^{-\frac12})\|e^{2\lambda\phi(0,\cdot)}f\|_{L^2(\Omega)}+e^{2\lambda e^{\gamma r_-(\epsilon)}}\|f\|^2_{L^2((-T,T)\times\Omega)}\right)\\
&+\gamma\lambda\int_{(-T,T)\times\Gamma_+}\phi e^{2\lambda\phi}|\partial_\nu\mathrm{w}|^2\partial_\nu\psi_0\mathrm{d}y\mathrm{d}t.
\end{aligned}\tag{47}$$

Now consider

$$\begin{aligned}
\int_\Omega u_tPu\mathrm{d}x={}&\int_\Omega u_t\chi fR_te^{\lambda\phi}\mathrm{d}x+\lambda\int_\Omega u_tA(u)\mathrm{d}x\\
&+\int_\Omega u_t(\lambda b_0+\lambda^2b_1)u\mathrm{d}x+\int_\Omega u_te^{\lambda\phi}B(\chi,U)\mathrm{d}x,
\end{aligned}\tag{48}$$

where A is a first-order operator acting on u with $L^\infty(X)$ coefficients $b_0\in L^\infty(X)$ and $b_1\in L^\infty(X)$. Integrating the left-hand-side of equation (48) from 0 to T and using the boundary conditions in (43), we have

$$\begin{aligned}
\int_0^T\int_\Omega u_tPu\mathrm{d}x\mathrm{d}t&=\frac12\int_0^T\int_\Omega\partial_t\left((\partial_tu)^2+\nabla u\cdot\nabla u+Q_0u^2\right)\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega Q_1^{(1)}(\partial_tu)^2\mathrm{d}x\mathrm{d}t\\
&=-\frac12\int_\Omega\left(e^{\lambda\phi(0,x)}f(x)R(0,x)\right)^2\mathrm{d}x+\int_0^T\int_\Omega Q_1^{(1)}(\partial_tu)^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

This means

$$\begin{aligned}
&\int_\Omega|f(x)|^2|R(0,x)|^2e^{2\lambda\phi(0,x)}\mathrm{d}x\\
&\quad\leqslant C\left(\|fe^{\lambda\phi}\|_{L^2((-T,T)\times\Omega)}+\left\|e^{\lambda\phi}B(\chi,U)\right\|^2_{L^2((-T,T)\times\Omega)}+\lambda\|u\|^2_{H^1_\lambda((-T,T)\times\Omega)}\right).
\end{aligned}$$

Applying estimates (45) and (47) to the above inequality, we conclude

$$\int_{\Omega}|f(x)|^2|R(0,x)|^2e^{2\lambda\phi(0,x)}\mathrm{d}x \leqslant C\Big(\|fe^{\lambda\phi}\|_{L^2((-T,T)\times\Omega)}+e^{2\lambda e^{\gamma r_-(\epsilon)}}\|f\|^2_{L^2(\Omega)} + \gamma\lambda\int_{(-T,T)\times\Gamma_+}\phi e^{2\lambda\phi}|\partial_\nu \mathrm{w}|^2\partial_\nu\psi_0\mathrm{d}y\mathrm{d}t\Big).$$

Note that $\|fe^{\lambda\phi}\|_{L^2((-T,T)\times\Omega)}$ can be estimated by the same factor as in (46). This reduces the estimate to

$$\int_{\Omega}|f(x)|^2|R(0,x)|^2e^{2\lambda\phi(0,x)}\mathrm{d}x \leqslant C\Big(e^{2\lambda e^{\gamma r_-(\epsilon)}}\|f\|^2_{L^2(\Omega)}+\gamma\lambda\int_{(-T,T)\times\Gamma_+}\phi e^{2\lambda\phi}|\partial_\nu \mathrm{w}|^2\partial_\nu\psi_0\mathrm{d}y\mathrm{d}t\Big). \tag{49}$$

As assumption (i) in theorem 4.1 says that $|R(0,x)|=|\varphi_1(x)|\geqslant c_1>0$, we have

$$\int_{\Omega}|f(x)|^2e^{2\lambda\phi(0,x)}\mathrm{d}x\leqslant C\int_{\Omega}|f(x)|^2|R(0,x)|^2e^{2\lambda\phi(0,x)}\mathrm{d}x.$$

Together with estimate (49) this yields

$$\begin{aligned}e^{2\lambda e^{\gamma r_-}}\|f\|^2_{L^2(\Omega)}&\leqslant\int_{\Omega}|f(x)|^2e^{2\lambda\phi(0,x)}\mathrm{d}x\leqslant C\int_{\Omega}|f(x)|^2|R(0,x)|e^{2\lambda\phi(0,x)}\mathrm{d}x\\&\leqslant C\Big(e^{2\lambda e^{\gamma r_-(\epsilon)}}\|f\|^2_{L^2(\Omega)}+\gamma\lambda\int_{(-T,T)\times\Gamma_+}\phi e^{2\lambda\phi}|\partial_\nu \mathrm{w}|^2\partial_\nu\psi_0\mathrm{d}y\mathrm{d}t\Big).\end{aligned}\tag{50}$$

Since $r_->r_-(\varepsilon)$, we can choose λ sufficiently large to absorb the first term in the right-hand side of (50). Finally, we let $\varepsilon\searrow 0$, which completes the proof. □

Theorem 4.3 (Dirichlet boundary condition (13), $Q_1(x)$ known, $Q_0(x)$ unknown). *Let $F\in L^1(0,T;L^2(\Omega))$, $Q_1\in L^\infty(\Omega)$, $\varphi_0\in H^1(\Omega)$, $\varphi_1\in L^2(\Omega)$ and $u_{amb}\in H^1((0,T)\times\partial\Omega)$ satisfying the compatibility condition (20). For $i=1,2$, let $u^{(i)}\in\mathcal{V}$ be the solution to (19) with space-dependent coefficient $Q_0=Q_0^{(i)}\in L^\infty(\Omega)$. Choose a ψ_0 that satisfies condition (40). Assume also that the following conditions hold:*

(i) $|\varphi_0(x)|\geqslant c_0>0$ a.e. $x\in\Omega$.
(ii) $u^{(i)}\in H^2(0,T;H^{k+2}(\Omega))\cap H^1(0,T;H^k(\Omega))$ for some $k>\frac{d}{2}$.

Denote by $q^{(i)}:=\partial_\nu u^{(i)}|_{(0,T)\times\partial\Omega}$. Then, for $T>T_0:=\left(\frac{r_+-r_-}{\beta}\right)^{\frac12}$, we have the following stability estimate:

$$\|Q_0^{(1)}-Q_0^{(2)}\|_{L^2(\Omega)}\leqslant C\|q_t^{(1)}-q_t^{(2)}\|_{L^2((0,T)\times\Gamma_+)},\tag{51}$$

where C depends on β, Ω_T, ψ_0, φ_0, φ_1 and the L^∞-norms of Q_0, $Q_1^{(i)}$ and $u^{(i)}$ for $i=1,2$.

Proof. We will use the same strategy as that employed in proving theorem 4.1. Let $U:=u^{(1)}-u^{(2)}$, $f(x)=Q_0^{(2)}(x)-Q_0^{(1)}(x)$ and $R=u^{(2)}$. As before, one has

$$\begin{cases}\left(U_{tt}-\Delta U+Q_1U_t+Q_0^{(1)}U\right)(t,x)=f(x)R(t,x), & (t,x)\in(-T,T)\times\Omega,\\ U(t,x)=0,\ \ \partial_\nu U(t,x)=q^{(1)}(t,x)-q^{(2)}(t,x) & (t,x)\in(-T,T)\times\partial\Omega,\\ U(0,x)=0,\ \ U_t(0,x)=0. & x\in\Omega,\end{cases}\tag{52}$$

Again, let χ be the same as in equation (42) and then, $\mathrm{w} = \chi U_t$ satisfies

$$\begin{cases} \mathrm{w}_{tt} - \Delta \mathrm{w} + Q_1 \mathrm{w}_t + Q_0^{(1)} \mathrm{w} = \chi f R_t + \chi'(2U_{tt} + Q_1 U_t) + \chi'' U_t & \text{in } (-T,T)\times\Omega, \\ \mathrm{w} = 0,\ \ \partial_\nu \mathrm{w} = q_t^{(1)} - q_t^{(2)} & \text{on } (-T,T)\times\partial\Omega, \\ \mathrm{w}(0,x) = 0,\ \ \mathrm{w}_t(0,x) = f(x)R(0,x), & \text{in } \{t=0\}\times\Omega. \end{cases}$$

Now, set $P := \partial_t^2 - \Delta + Q_1\partial_t + Q_0^{(1)} I$, $u = e^{\lambda\phi}\mathrm{w}$ and $\phi(t,x) = e^{\gamma(\psi_0(x)-\beta t^2)}$. Then, theorem 3.3 gives

$$C\lambda \|e^{\lambda\phi}\mathrm{w}\|^2_{H^1_\lambda((-T,T)\times\Omega)} \leqslant \left\|e^{\lambda\phi}\left(fR_t + \chi'\left(2U_{tt} + Q_1U_t\right) + \chi'' U_t\right)\right\|^2_{L^2((-T,T)\times\Omega)} + \gamma\lambda \int_{(-T,T)\times\Gamma_+} \phi e^{2\lambda\phi} |\partial_\nu \mathrm{w}|^2 \partial_\nu\psi_0 \mathrm{d}y\mathrm{d}t,$$

where C depends on β, ψ_0 and Ω_T, whilst λ depends on ψ_0 and L^∞-norms of Q_1 and $Q_0^{(1)}$. Now, assumption (ii) in theorem 4.3 and proposition D.1 of appendix D give $R, R_t \in L^\infty((-T,T)\times\Omega))$, which implies the estimate (47). As assumption (i) in theorem 4.3 means that $|R(0,x)| = |\varphi_0(x)| \geqslant c_0 > 0$, we thereby have the estimate (49). This completes the proof via the same method shown in the proof of theorem 4.1. □

The same method gives the following theorem for the physical problem given by equations (8), (9) and (13).

Theorem 4.4 (Dirichlet boundary condition (13), $Q_0(x)$ and $Q_1(x) = aQ_0(x) + b$ unknown). *Let $F \in L^1(0,T;L^2(\Omega))$, $u_{amb} \in H^1((0,T)\times\partial\Omega)$, $\varphi_0 \in H^1(\Omega)$ and $\varphi_1 \in L^2(\Omega)$ satisfying the compatibility condition (20). Let a and $b \in \mathbb{R}$ be given numbers. For $i = 1,2$, let $u^{(i)} \in \mathcal{V}$ be the solution to (19) with space-dependent coefficients $Q_0 = Q_0^{(i)} \in L^\infty(\Omega)$ and $Q_1^{(i)} = aQ_0^{(i)} + b$. Choose a ψ_0 that satisfies condition (40). Assume also that the following conditions hold:*

(i) $|a\varphi_1(x) + \varphi_0(x)| \geqslant c_{01} > 0$ a.e. $x \in \Omega$.

(ii) $u^{(i)} \in H^3(0,T;H^{k+2}(\Omega)) \cap H^2(0,T;H^k(\Omega))$ for some $k > \frac{d}{2}$.

Denote by $q^{(i)} := \partial_\nu u^{(i)}|_{(0,T)\times\partial\Omega}$. Then, for $T > T_0 := \left(\frac{r_+ - r_-}{\beta}\right)^{\frac{1}{2}}$, the stability estimate (41) holds.

Proof. The main difference to the proof of theorem 4.3 is that now we have $R = au_t^{(2)} + u^{(2)}$. Therefore, we require $|R(0,x)| = |a\varphi_1(x) + \varphi_0(x)| \geqslant c_{01} > 0$ a.e. $x \in \Omega$, as assumed in (i). □

For the Robin boundary condition (14), a control estimate based on the Carleman inequality (35) is given by the following theorem.

Theorem 4.5 (Robin boundary condition (14), $Q_0(x)$ and $Q_1(x) = aQ_0(x) + b$ unknown). *Let $F \in L^2(\Omega_T)$, $\varphi_0 \in H^1(\Omega)$, $\varphi_1 \in L^2(\Omega)$ and $h \in L^\infty(\partial\Omega)$. Let a and $b \in \mathbb{R}$ be given numbers. For $i = 1,2$, let $u^{(i)} \in \Lambda_h$ be the solution to (16) with space-dependent coefficients $Q_0 = Q_0^{(i)} \in L^\infty(\Omega)$ and $Q_1^{(i)} = aQ_0^{(i)} + b$. Choose a ψ_0 that satisfies condition (40). Assume also that the following conditions hold:*

(i) $|a\varphi_1(x) + \varphi_0(x)| \geqslant c_{01} > 0$ a.e. $x \in \Omega$.

(ii) $u^{(i)} \in H^3(0,T;H^{k+2}(\Omega)) \cap H^2(0,T;H^k(\Omega))$ for some $k > \frac{d}{2}$.

Denote by $p^{(i)} := u^{(i)}|_{(0,T)\times\partial\Omega}$. Then, for $T > T_0 := \left(\frac{r_+ - r_-}{\beta}\right)^{\frac{1}{2}}$, we have the following stability estimate:

$$\|Q_0^{(1)} - Q_0^{(2)}\|_{L^2(\Omega)} \leqslant C\left(\|p_t^{(1)} - p_t^{(2)}\|_{L^2(0,T;H^1(\Gamma))} + \|p_{tt}^{(1)} - p_{tt}^{(2)}\|_{L^2((0,T)\times\Gamma)}\right), \quad (53)$$

where C depends on β, Ω_T, ψ_0, φ_0, φ_1 and the L^∞-norms of h, $Q_0^{(i)}$ and $u^{(i)}$ for $i = 1,2$.

Remark 4.6. For the zero-Neumann adiabatic boundary condition (12) instead of (14), the same stability estimate (53) holds.

For higher regularity data, we have the following proposition related to theorems 2.3 and 4.5.

Proposition 4.7. *Let $\mathbb{N} \ni m > \frac{d}{2} + 1$, and a and $b \in \mathbb{R}$ be given numbers. Assume that $h \in L^\infty(\partial\Omega)$, $\varphi_0 \in H^{m+1}(\Omega)$ and $\varphi_1 \in H^m(\Omega)$ satisfy condition (i) of theorem 4.5, and $\partial_t^k F \in L^2(0,T;H^{m-k}(\Omega))$ for $k = 0,\ldots,m$. Moreover, assume that the regularity conditions (21) are satisfied. Then, for $i = 1,2$, the solutions, denoted by $u^{(i)} \in \Lambda_h$, which, in addition, satisfies the regularity $\partial_t^k u^{(i)} \in L^\infty(0,T;H^{m+1-k}(\Omega))$ for $k = 0,\ldots,m+1$, to (16) with the space-dependent coefficients $Q_0 = Q_0^{(i)} \in L^\infty(\Omega)$ and $Q_1^{(i)} = aQ_0^{(i)} + b$, satisfy the estimates (22) and (53).*

Proof of theorem 4.5. As the proof is similar to the one of theorems 4.1 and 4.3, we will only highlight the main differences. Let $U := u^{(1)} - u^{(2)}$, $f(x) = Q_0^{(2)}(x) - Q_0^{(1)}(x)$ and $R = au_t^{(2)} + u^{(2)}$, so that the form of (52) changes to

$$\begin{cases} \left(U_{tt} - \Delta U + Q_1^{(1)}U_t + Q_0 U\right)(t,x) = f(x)R(t,x), & (t,x) \in (-T,T)\times\Omega, \\ \partial_\nu U(t,x) + h(x)U(t,x) = 0,\ \ U(t,x) = p^{(1)}(t,x) - p^{(2)}(t,x), & (t,x) \in (-T,T)\times\partial\Omega, \\ U(0,x) = 0,\ \ U_t(0,x) = 0 & x \in \Omega, \end{cases}$$

and the Carleman estimate (35) for $e^{\lambda\phi}\mathrm{w}$ becomes

$$\begin{aligned} &C\lambda\|e^{\lambda\phi}\mathrm{w}\|^2_{H^1_\lambda((-T,T)\times\Omega)} \\ &\quad\leqslant \left\|e^{\lambda\phi}\left(fR_t + \chi'\left(2U_{tt} + Q_1^{(1)}U_t\right) + \chi''U_t\right)\right\|^2_{L^2((-T,T)\times\Omega)} \\ &\qquad + \gamma\lambda\int_{(-T,T)\times\partial\Omega}\phi e^{2\lambda\phi}|\mathrm{w}|^2 G(h,\phi)\mathrm{d}y\mathrm{d}t + \gamma\lambda\int_{(-T,T)\times\partial\Omega}\phi e^{2\lambda\phi}|\mathrm{w}_t|^2\partial_\nu\psi_0\mathrm{d}y\mathrm{d}t \\ &\qquad - \gamma\lambda\int_{(-T,T)\times\partial\Omega}\phi e^{2\lambda\phi}|\partial_{\mathbf{T}}\mathrm{w}|^2\partial_\nu\psi_0\mathrm{d}y\mathrm{d}t, \end{aligned} \quad (54)$$

where $|\partial_{\mathbf{T}}\mathrm{w}|^2$ is the tangential part of $|\nabla\mathrm{w}|^2$ and $G(h,\phi)$ is some smooth function of h and ϕ. Note that, the boundary control in (54) is over the whole boundary $\partial\Omega$, whilst for the Dirichlet boundary condition (13) the control in (44) is over the portion Γ_+ of the boundary $\partial\Omega$. Since

$$\begin{aligned} &\gamma\lambda\int_{(-T,T)\times\partial\Omega}\phi e^{2\lambda\phi}|\mathrm{w}|^2 G(h,\phi)\mathrm{d}y\mathrm{d}t + \gamma\lambda\int_{(-T,T)\times\partial\Omega}\phi e^{2\lambda\phi}|\mathrm{w}_t|^2\partial_\nu\psi_0\mathrm{d}y\mathrm{d}t \\ &\quad - \gamma\lambda\int_{(-T,T)\times\partial\Omega}\phi e^{2\lambda\phi}|\partial_{\mathbf{T}}\mathrm{w}|^2\partial_\nu\psi_0\mathrm{d}y\mathrm{d}t \leqslant C\left(\|\mathrm{w}\|^2_{L^2(-T,T;H^1(\Gamma))} + \|\mathrm{w}_t\|^2_{L^2(-T,T;L^2(\Gamma))}\right), \end{aligned}$$

the right-hand-side of the above inequality shall be the replacement for the boundary term in (47). The rest is the same as in the proof of theorem 4.1. □

It is not hard to see that the initial data at $t = 0$ in (43) is key to obtain the stability control of Q_0 or Q_1. This inspires us to study, for example, $v = \chi U_{tt}$ or higher-order time derivatives of U, which, in principle, will give us some estimates of Q_0 and Q_1. However, in contrast to (43),

Pv fails to be in $L^2((-T,T)\times\Omega)$ (see appendix C). Therefore, the Carleman estimate (35) will no longer be useful. As shown in appendix C, the obstruction is exactly caused by extensions of U, Q_0, Q_1 and R from $(0,T)$ to $(-T,T)$. However, if we would like to obtain a control of both Q_0 and Q_1 at an intermediate time, say $t=T/2$ as in [47], then we do not need to worry about singularities caused by extensions. Alternatively, one could use the method mentioned in remark 3.5 to avoid the symmetric extension.

Remark 4.8. Although not detailed herein, it is worth remarking that by employing the microlocal analysis of Lasiecka and Triggiani [39] to study the property of the operator P (defined by (31)) around the boundary in the Robin boundary condition setting, one can improve the stability estimate (53) to read as

$$\|Q_0^{(1)}-Q_0^{(2)}\|_{L^2(\Omega)}\leqslant C\left(\|p_t^{(1)}-p_t^{(2)}\|_{L^2((0,T)\times\Gamma)}+\|p_{tt}^{(1)}-p_{tt}^{(2)}\|_{L^2((0,T)\times\Gamma)}\right), \tag{55}$$

where the first term $\|p_t^{(1)}-p_t^{(2)}\|_{L^2((0,T);H^1(\Gamma))}$ in the right-hand side of (53) has been replaced by the sharper term $\|p_t^{(1)}-p_t^{(2)}\|_{L^2((0,T)\times\Gamma)}$ in (55).

Section 4 was devoted to establish the uniqueness and several conditional stability estimates for the inverse coefficient problems. However, the stability results expressed by the inequalities (41), (51), (53) and (55) involve the time derivatives of the measured boundary data, which, when polluted by noise, give rise in itself to an ill-posed problem of numerical differentiation that needs to be regularized. As such, the next section presents a variational formulation of the inverse problem (8), (9), (11) and (14), which enables an iterative regularizing numerical development based on the CGM.

5. Variational formulation

In order to solve the inverse problem (8), (9), (11) and (14) for the reconstruction of the unknown dimensionless blood perfusion coefficient $w(x)$ along with the dimensionless temperature $u(x,t)$, we minimize the nonlinear least-squares objective functional $J:L^2(\Omega)\to\mathbb{R}_+$ defined by

$$J(w):=\frac{1}{2}\|u(\cdot,\cdot;w)-\theta\|^2_{L^2((0,T)\times\Gamma_0)}, \tag{56}$$

where $u(\cdot,\cdot;w)$ denotes the solution of the direct problem (8), (9) and (14) for a given $w\in L^2(\Omega)$. In the next theorem, we prove that the objective functional (56) is Fréchet differentiable and derive an expression for its gradient.

Theorem 5.1. *The objective functional J defined in (56) is Fréchet differentiable and its gradient is given by*

$$J'(w)=-\int_0^T(u(t,x)+u_t(t,x))\,v(t,x)dt, \tag{57}$$

where $v(x,t)$ is the solution of the following adjoint problem:

$$\begin{cases} v_{tt}(t,x)-(1+w(x))v_t(t,x)=\Delta v(t,x)-w(x)v(t,x) \\ +\int_{\Gamma_0}(u(t,y)-\theta(t,y))\delta(x-y)dy, \quad (t,x)\in(0,T)\times\overline{\Omega}, \\ v(T,x)=v_t(T,x)=0, \quad x\in\Omega, \\ \partial_\nu v(t,x)=h(x)v(t,x), \quad (t,x)\in(0,T)\times\partial\Omega, \end{cases} \tag{58}$$

where δ is the Dirac delta function, which in numerical computation is approximated by

$$\delta(x-y) \approx \frac{1}{\sigma\sqrt{\pi}} \exp\left(-\frac{|x-y|^2}{\sigma^2}\right),$$

where σ is a small positive constant, typically 10^{-3}.

Proof. Taking a small variation $\delta w \in L^2(\Omega)$ of w, we have

$$J(w+\delta w) - J(w) = \int_0^T \int_{\Gamma_0} (u(t,y;w) - \theta(t,y))\delta u(t,y;w)dydt + \frac{1}{2}\|\delta u\|^2_{L^2((0,T)\times\Gamma_0)}, \tag{59}$$

where δu is the solution of the sensitivity problem:

$$\begin{cases} (\delta u)_{tt} + (1+w(x))(\delta u)_t = \Delta(\delta u) - w(x)\delta u - (u+u_t)\delta w(x), & (t,x) \in (0,T)\times\overline{\Omega}, \\ \delta u(0,x) = (\delta u)_t(0,x) = 0, \quad x \in \Omega, \\ \partial_\nu(\delta u)(t,x) = h(x)\delta u(t,x), \quad (t,x) \in (0,T)\times\partial\Omega. \end{cases} \tag{60}$$

For the sensitivity problem (60), we wish to establish that

$$\|\delta u\|^2_{L^2((0,T)\times\Gamma_0)} = o\left(\|\delta w\|_{L^2(\Omega)}\right), \quad \text{as } \|\delta w\|_{L^2(\Omega)} \to 0. \tag{61}$$

For this, we start with

$$\|\delta u\|_{L^2((0,T)\times\Gamma_0)} \leqslant C\|\delta u\|_{L^\infty(0,T;L^2(\partial\Omega))} \leqslant C\|\delta u\|_{L^\infty(0,T;H^{1/2}(\partial\Omega))} \leqslant C\|\delta u\|_{L^\infty(0,T;H^1(\Omega))}. \tag{62}$$

By applying theorem 2.1 to δu satisfying the problem (60), we obtain

$$\|\delta u\|_{L^\infty(0,T;H^1(\Omega))} \leqslant C\|(u+u_t)\delta w\|_{L^2(\Omega_T)}, \tag{63}$$

Now, from corollary 2.4 and Sobolev's inequality, we have that

$$\begin{aligned} \|u\|_{L^\infty(\Omega_T)} + \|u_t\|_{L^\infty(\Omega_T)} &\leqslant C\left(\|u\|_{L^\infty(0,T;H^{m+1}(\Omega))} + \|u_t\|_{L^\infty(0,T;H^m(\Omega))}\right) \\ &\leqslant C\left(\sum_{k=0}^m \|\partial_t^k F\|_{L^2(0,T;H^{m-k}(\Omega))} + \|\varphi_0\|_{H^{m+1}(\Omega)}\right). \end{aligned} \tag{64}$$

for any $m \geqslant \max\{\frac{d}{2},1\}$. Finally, combining equations (62)–(64), we conclude that

$$\|\delta u\|_{L^2((0,T)\times\Gamma_0)} \leqslant C\|\delta w\|_{L^2(\Omega)}, \tag{65}$$

where the positive constant C depends on the norms of φ_0 and F, as in equation (24), and of the $L^\infty(\Omega)$-norm of w. Then, the inequality (65) implies the required statement (61).

Now, multiplying the PDE in (58) by $\delta u(x,t)$ and integrating over the solution domain $(0,T)\times\Omega$, using the terminal and boundary conditions in (58) and also using (60), yield

$$\int_\Gamma \int_0^T (u(t,y;w) - \theta(t,y))\delta u(t,y;w)dtdy = -\int_\Omega \delta w(x) \int_0^T (u(t,x) + u_t(t,x))\, v(t,x)dtdx, \tag{66}$$

From (59), (61) and (66) we obtain that J is Fréchet differentiable, and its gradient at w is given by (57).

□

In the following subsection, the CGM is described for the minimization of the objective functional (56).

5.1. Iterative procedure

The CGM employed to minimize the objective functional J for the reconstruction of the space-dependent perfusion coefficient $w(x)$ is based on the recursive relation:

$$w^{n+1}(x) = w^n(x) - \zeta_n R_n(x), \quad n = 0, 1, 2, \ldots, \tag{67}$$

where the direction of descent R_n is given by

$$R_n(x) = \begin{cases} -J'(w^n), & \text{for } n = 0, \\ -J'(w^n) + \beta_n R_{n-1}, & \text{for } n = 1, 2, \ldots, \end{cases} \tag{68}$$

the Fletcher–Reeves conjugate coefficient β_n is given by

$$\beta_0 = 0, \quad \beta_n = \frac{\|J'(w^n)\|^2_{L^2(\Omega)}}{\|J'(w^{n-1})\|^2_{L^2(\Omega)}}, \quad n = 1, 2, \ldots, \tag{69}$$

and the search step size ζ_n is computed as the minimizer

$$\zeta_n = \arg\min_{\zeta \geqslant 0} J(w^n - \zeta R_n), \quad n = 0, 1, \ldots. \tag{70}$$

To evaluate ζ_n from (70), we have

$$J(w^n - \zeta R_n) = \frac{1}{2}\|u(\cdot,\cdot;w^n - \zeta R_n) - \theta\|^2_{L^2((0,T)\times\Gamma_0)},$$

We set $\delta w^n = R_n$ and linearize $u(x,t;w^n - \zeta R_n)$ by a first-order Taylor series expression to obtain

$$u(\cdot,\cdot;w^n - \zeta R_n) \approx u(\cdot,\cdot;w^n) - \zeta R_n \frac{\partial u}{\partial w^n}(\cdot,\cdot;w^n) \approx u(\cdot,\cdot;w^n) - \zeta \delta u(\cdot,\cdot;w^n),$$

where $\delta u(x,t;w^n)$ is obtained by solving the sensitivity problem (60) with $\delta w^n = R_n$. Then, differentiating $J(w^n - \zeta R_n)$ with respect to ζ and making it vanish yield

$$\zeta_n = \frac{\left|\int_{\Gamma_0}\int_0^T (u(t,y;w^n) - \theta(t,y))\delta u(t,y;w^n)dtdy\right|}{\|\delta u(\cdot,\cdot;w^n)\|^2_{L^2((0,T)\times\Gamma_0)}}. \tag{71}$$

5.2. Stopping criterion

For ensuring stability, we need to stop the iterations according to the discrepancy principle at the first iteration n_* for which

$$J(w^{n_*}) \leqslant \bar{\epsilon}, \tag{72}$$

where $\bar{\epsilon}$ is a small positive value, e.g. $\bar{\epsilon} = 10^{-5}$, for exact data or

$$\bar{\epsilon} = \frac{1}{2}\|\theta^\epsilon - \theta\|^2_{L^2((0,T)\times\Gamma_0)}$$

for noisy data θ^ϵ.

5.3. Algorithm

The CGM proceeds as described in the following steps:

(1) Set $n = 0$ and select an initial guess $w^0 \in L^2(\Omega)$.
(2) Solve the direct problem given by (8), (9) and (14) to obtain $u(\cdot,\cdot;w^n)$ and compute $J(w^n)$ from equation (56).

(3) Stop if the criterion (72) is satisfied. Else go to step 4.
(4) Solve the adjoint problem given by (58) to find $v(\cdot,\cdot;w^n)$. Compute the gradient $J'(w^n)$ from equation (57), the conjugate coefficient β^n from equation (69), and the direction of descent R_n from equation (68).
(5) Solve the sensitivity problem given by (60) to obtain $\delta u(\cdot,\cdot;w^n)$ by taking $\delta w^n = R_n$ and compute the search step size ζ_n from equation (71).
(6) Update w^n from equation (67), set $n = n+1$ and go to step 2.

The dimensional temperature $\mathrm{u(t,x)}$ and space-dependent perfusion coefficient $w_b(\mathrm{x})$ can be obtained, via (7), after $u(t,x)$ and $w(x)$ have been reconstructed.

In the next section, the proposed inversion algorithm's convergence, accuracy and stability are illustrated and discussed on a physical example.

6. Numerical results and discussion

We consider a physical example concerning the recovery of the blood perfusion rate $w_b(\mathrm{x})$ of a one-dimensional, one-layered biological skin tissue $\Omega = (0,L)$, which undergoes a laser irradiation of the form, [13],

$$\mathrm{Q(x)} = \mu I_0 e^{-\mu \mathrm{x}}, \quad \mathrm{x} \in [0,\mathrm{L}],$$

where $\mu = 700\ \mathrm{m}^{-1}$ is the extinction coefficient of the tissue, $I_0 = 500\ \mathrm{W\ m}^{-2}$ denotes the intensity of the laser and, for simplicity, the heat generation due to metabolism has been neglected.

The initial and adiabatic boundary conditions, i.e. $\mathrm{h} = 0$, taken from [2, 20], respectively, are

$$\mathrm{u}(0,x) = \Phi_0(x) = 37^\circ\mathrm{C}, \quad \mathrm{u_t(0,x)} = 0, \quad \mathrm{x} \in [0,\mathrm{L}],$$

and

$$-\mathrm{u_x(t,0)} = \mathrm{u_x(t,L)} = 0, \quad \mathrm{t} \in [0,\mathrm{T}].$$

The properties of the tissue are taken as: $\kappa = 0.4\ \mathrm{W\,(m\ ^\circ C)^{-1}}$, $C_{\mathrm{tissue}} = 325 \times 10^4\ \mathrm{J\,(m^3\ ^\circ C)^{-1}}$ and $\mathrm{L} = 0.003$ m, [19]. The properties of the blood are taken as: $C_b = 399 \times 10^4\ \mathrm{J\,(m^3\ ^\circ C)^{-1}}$, $w_b(\mathrm{x}) = 0.04\ \mathrm{s}^{-1}$ and $\mathrm{u}_b = 37^\circ\mathrm{C}$, [4]. We also take $\tau = 20$ s from [50] and $\mathrm{T} = 100$ s.

The above dimensional quantities transform, via (7), into the following dimensionless input data:

$$L = \mathrm{L}\sqrt{\frac{C_{\mathrm{tissue}}}{\kappa\tau}} = 1.9121,\ T = 5,\ w(x) = 0.9822,$$
$$u(0,x) = \varphi_0(x) = 0,\ u_t(0,x) = 0,\ h = 0,\ F(t,x) = 0.0582e^{-1.0983x}. \tag{73}$$

We wish to recover the dimensionless solution $w(x)$ and $u(t,x)$, allowing us to obtain, via (7), the dimensional blood perfusion rate $w_b(\mathrm{x})$ and the tissue temperature $\mathrm{u(t,x)}$.

The direct, adjoint and sensitivity problems present in the CGM described in section 5.1 are solved using the Crank-Nicolson finite-difference method (FDM) [14] in one-dimension ($d = 1$) with a uniform mesh size L/M and time step T/N. The two-point first-order backward finite difference formula is used to approximate the time-derivative $u_t(t,x)$ in (57). The trapezoidal rule is used for discretizing all the integrals present. The accuracy error, as a function of the number of iterations n, is defined as

$$E(w^n) = \|w^n - w\|_{L^2(\Omega)}, \tag{74}$$

where w^n stands for the numerical result obtained by the CGM at the iteration number n and w denotes the true dimensionless blood perfusion coefficient (if available).

In the absence of an analytical solution for $u(t,x)$ being available, we generate the input measured data θ in (11) numerically by solving first the direct problem given by

$$\begin{cases} u_{tt} + (1+w(x))u_t = u_{xx} - w(x)u + F(t,x), & (t,x) \in (0,T) \times (0,L), \\ u(0,x) = u_t(0,x) = 0, & x \in (0,L), \\ -u_x(t,0) = u_x(t,L) = 0, & t \in (0,T), \end{cases}$$

using the FDM, with the input data (73) and the true coefficient $w(x) = 0.9822$ assumed known. We then consider only half of the boundary data for each $u(t,0) = \theta(t,0)$ and $u(t,L) = \theta(t,L)$ obtained from solving the direct problem with $M = N = 200$ as our input data θ in (11), and solve the inverse problem with a coarser mesh of $M = N = 100$ in order to avoid committing an inverse crime. The data θ is further perturbed by additive random noise, which is numerically simulated as $\theta^\epsilon = \theta + \epsilon p \|\theta\|_{L^\infty((0,T)\times\Gamma_0)}$, where p represents the percentage of noise and ε are random variables generated from a Gaussian normal distribution with mean 0 and variance 1.

We run the CGM, based on minimizing the least-squares functional (56) with $\Omega = (0,L)$ and $\Gamma_0 = \partial\Omega = \{0,L\}$ (full boundary temperature measurements) or $\{0\} = \Gamma_0 \subset \partial\Omega$ (partial boundary temperature measurements) from the initial guess $w^0(x) = 0.5$, which is reasonably far from the true solution $w(x) = 0.9822$. Of course, since the inverse problem under consideration is non-linear, the least-squares objective functional (56) is non-convex and a good initial guess is required; otherwise the iterative minimization algorithm can get stuck in a local minimum. Alternatively, one may employ the globally convexification method for coefficient inverse problems originated in [32], further stated in [35, chapters 5–11] and [36, formula (4.1)], and its second generation developed in [8, 9]. This method does not sufer from the phenomenon of local minima and works for both the case of vanishing initial conditions [35, chapters 7, 8, 10, 11], which are not in the framework of the Bukhgeim-Klibanov method, as well as for the cases that are in that framework, i.e. non-vanishing initial conditions, such as the one of this paper, [35, chapter 9], [8, 9, 36].

Figures 1(a) and 2(a) depict the monotonic decreasing convergence of the objective functional that is minimized for full Dirichlet data measured over the whole boundary $\Gamma_0 = \partial\Omega = \{0,L\}$ and for partial data measured over the portion $\Gamma_0 = \{0\}$, respectively, as a function of the number of iterations n, for $p \in \{0,1,3\}\%$ noise. For exact data, i.e. $p = 0$, the objective functionals in figures 1(a) and 2(a) rapidly attain very low values of 1.3×10^{-11} and 6.9×10^{-11}, respectively, in 20 iterations. For noisy data $p \in \{1,3\}\%$, the stopping iteration numbers $n^* \in \{8,7\}$ and $n^* \in \{7,4\}$ are generated according to the discrepancy principle (72) for $\Gamma_0 = \{0,L\}$ and $\Gamma_0 = \{0\}$, respectively. Figures 1(b) and 2(b) depict the accuracy error (74), as a function of the number of iterations n, and the optimal iteration numbers $n_{\text{opt}} \in \{8,6\}$ and $n_{\text{opt}} \in \{7,5\}$ for $\Gamma_0 = \{0,L\}$ and $\Gamma_0 = \{0\}$, respectively, can be inferred. It can be also observed that, the stopping iteration numbers n^* generated according to the discrepancy principle (72) are the same as the optimal ones n_{opt} for $p = 1\%$, while there is only one iteration difference between n^* and n_{opt} for $p = 3\%$. Of course, in practice only the values of n^* can be obtained by the discrepancy principle (72).

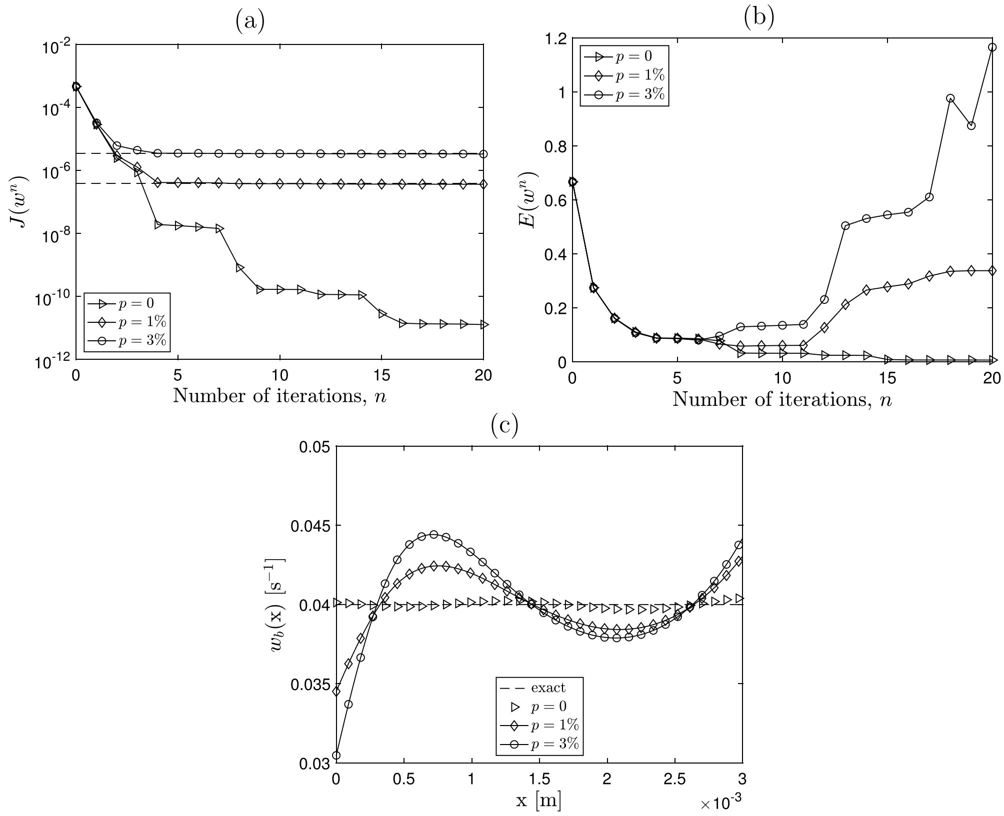

Figure 1. (a) The objective functional (56), (b) the accuracy error (74), and (c) the exact and numerical perfusion coefficient $w_b(x)$ for $p \in \{0, 1\%, 3\%\}$ noise, in case of inverting the temperature data (11) over the whole boundary, i.e. $\Gamma_0 = \partial\Omega = \{0, L\}$.

The corresponding numerical solutions for the dimensional space-dependent perfusion function $w_b(\mathrm{x})$ obtained via (7) are presented in figures 1(c) and 2(c) for $\Gamma_0 = \{0, L\}$ and $\Gamma_0 = \{0\}$, respectively. First, it can be seen that in the case of noiseless data, the numerical solutions for the perfusion coefficient $w_b(\mathrm{x})$ are very and reasonably accurate for $\Gamma_0 = \{0, L\}$ and $\Gamma_0 = \{0\}$, respectively. Second, in the case of noisy data, the numerical solutions are reasonably stable and become more accurate, as the percentage of noise p decreases from 3% to 1% and then to zero. The accuracy errors $E(w^{n_*}) \in \{0.006, 0.058, 0.095\}$ for $p \in \{0, 1\%, 3\%\}$ noise, obtained from figure 1(b), are smaller than $E(w^{n_*}) \in \{0.033, 0.10, 0.11\}$ for $p \in \{0, 1\%, 3\%\}$ noise, obtained from figure 2(b), indicating that, as expected, the numerical solution for the case of full boundary data $\Gamma_0 = \{0, L\}$ being inverted is more accurate than the numerical solution obtained by inverting limited partial data over $\Gamma_0 = \{0\}$ only.
In closing, we mention that although this section has illustrated the performance of the CGM only for a physical one-dimensional biological skin tissue, the CGM can also be numerically implemented in higher dimensions, as described recently in [3] for related inverse source problem in the thermal-wave model of bio-heat transfer.

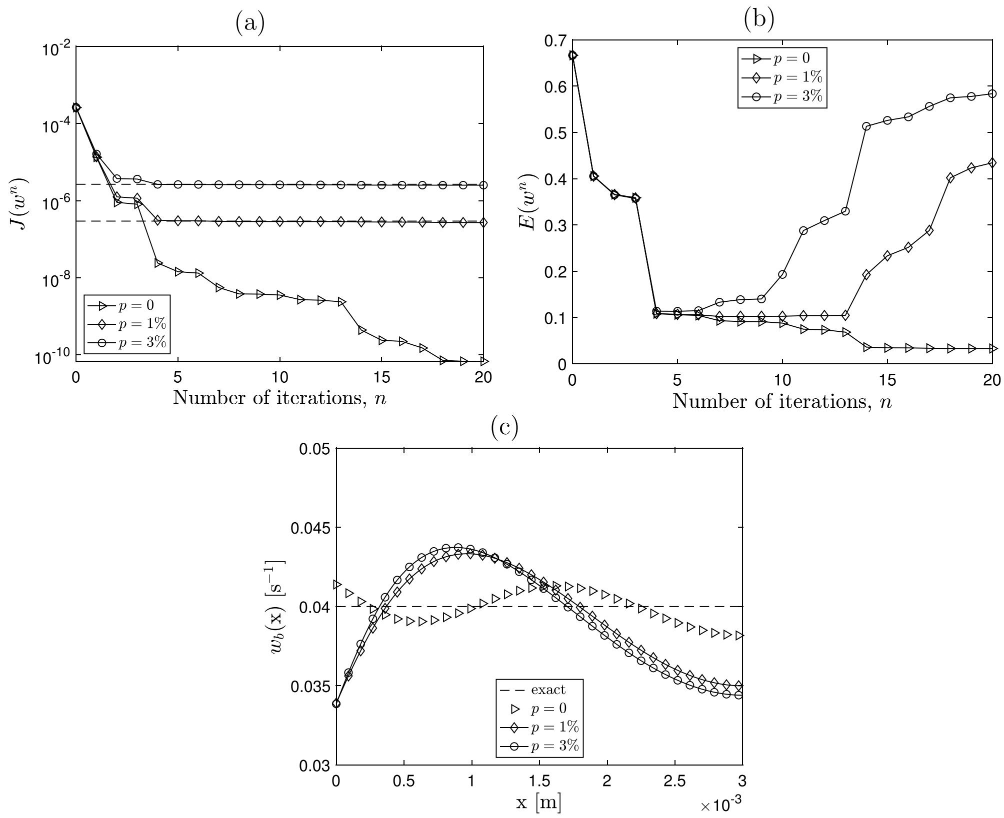

Figure 2. (a) The objective functional (56), (b) the accuracy error (74), and (c) the exact and numerical perfusion coefficient $w_b(\mathrm{x})$ for $p \in \{0, 1\%, 3\%\}$ noise, in case of inverting the partial temperature data (11) over a portion of the boundary $\Gamma_0 = \{0\}$.

7. Conclusions

The inverse coefficient problem of recovering the unknown space-dependent blood perfusion coefficient in the hyperbolic thermal-wave model of bio-heat transfer from boundary temperature measurements has been investigated. Uniqueness and conditional Lipschitz stability have been established using the technique of Carleman estimates. These have been found valid over a time interval $(0, T)$ with $T > (\frac{r_+ - r_-}{\beta})^{1/2}$ that, for $\beta \in (1 - \frac{r_-}{r_+}, 1)$, is shorter than the usual $T > r_+^{1/2}$ previously reported in the literature [23, 40]. Further, the problem has been reformulated as a nonlinear least-squares minimization problem and has been numerically solved using the CGM combined with the discrepancy principle for achieving stability. Numerical results associated with a physical example have been presented and discussed. Accurate and stable solutions have been obtained for both exact and random noisy data using the proposed iterative CGM.

Further work associated with the thermal-wave model of bio-heat transfer is possible, e.g. determining the shape, size and location of tumours within biological tissues using non-intrusive boundary observations.

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

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Appendix A. Proof of theorem 2.1

In this appendix, we modify the method given in [45] and [17, chapter 7] for proving theorem 2.1.

Proof of theorem 2.1. The proof of existence is similar to the construction of weak solutions via Galerkin's method in [17, chapter 7] or, in a more abstract setting, in [45]. Below we give a sketch of the proof. As we will focus on the case when $d = 2$ or 3, by Sobolev embeddings, we have

$$(-\Delta + iI)^{-1} : L^2(\Omega) \to H^2(\Omega) \cap H^1(\Omega; h) \hookrightarrow C^{0,\alpha}(\Omega) \hookrightarrow L^\infty(\Omega),$$

which is a compact operator. As $Q_0 \in L^\infty(\Omega)$ can be thought as a bounded operator from $L^\infty(\Omega)$ to $L^2(\Omega)$, it follows that $Q_0(-\Delta + iI)^{-1}$ is a compact operator and hence $-\Delta + Q_0$ is essentially self-adjoint and has a discrete spectrum. This way of establishing spectral properties of self-adjoint operators uses the idea of relative compactness, which can be founded in [15, chapter 4] or [29, section 1, chapter 4]. The idea of using compact embeddings to establish relative compactness of Q_0 can be found in [11]. Therefore, we can choose an orthogonal basis, denoted by $\{w_k\}_{k=1}^\infty$, of $H^1(\Omega; h)$ and $L^2(\Omega)$. For instance, we can choose $\{w_k\}_{k=1}^\infty$ to be the eigenfunctions of $-\Delta + Q_0$ subject to the Robin boundary condition in (16). Standard Galerkin's method gives that for a given $K \in \mathbb{N}^*$, the function $u_K(t,x) := \sum_{k=1}^K d_K^k(t) w_k(x)$ is an approximation solution to $u(t,x)$ that yields a weak solution to (16), where d_K^k is determined by satisfying $\langle (u_K)_{tt}, w_k\rangle + \mathrm{B}[u_K, w_k; t] = \langle F, w_k\rangle$ for $k = 1, \ldots, K$. The stability estimate (18) of u_K can be established via energy estimates [17, chapter 7]. The energy function of u_m associated with (16) is given by

$$E_K(t) = \int_\Omega \left((u_K)_t^2 + u_K^2 + |\nabla u_K|^2\right) \mathrm{d}x. \tag{75}$$

It is clear that $E_K(t)$ is differentiable in t, as the elliptic regularity theorem warrants the smoothness of w_k for $k = 1, \ldots, K$. Differentiating (75) yields

$$\begin{aligned}\frac{1}{2}E_K'(t) &= \langle (u_K)_{tt}, (u_K)_t\rangle_\Omega + \langle (u_K)_t, u_K\rangle_\Omega + \langle \nabla u_K, \nabla (u_K)_t\rangle_\Omega \\ &= \langle (u_K)_t, u_K\rangle_\Omega + \langle F, (u_K)_t\rangle_\Omega - \langle Q_0 u_K, (u_K)_t\rangle_\Omega - \langle Q_1 (u_K)_t, (u_K)_t\rangle_\Omega - \langle h u_K, (u_K)_t\rangle_{\partial\Omega}.\end{aligned}$$

Then, by the Cauchy-Schwarz and Sobolev trace inequalities, we have

$$E_K(t) = E_K(0) + \int_0^t E_K'(s)\mathrm{d}s \leqslant C_1\left(E_K(0) + \int_0^t \|F(s,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}s\right) + C_2\int_0^t E_K(s)\mathrm{d}s.$$

By Gronwall's inequality, we obtain

$$E_K(t) \leqslant C(t)\left(E_K(0) + \int_0^t \|F(s,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}s\right), \tag{76}$$

where $C(t) = C_1(1 + C_2 t e^{C_2 t})$. Inequality (76) gives the stability estimate (18) for u_K. Now, Banach-Alaoglu theorem gives the existence of a weak solution to (16) defined by $u := \lim_{K_l\to\infty} u_{K_l}$. This proves the existence of a weak solution. The uniqueness of u follows similarly, whilst the estimate (18) results immediately by taking the limit $K_l \to \infty$ in (76). This also implies the continuous dependence on the data $F \in L^2(\Omega_T)$, $\varphi_0 \in H^1(\Omega)$ and $\varphi_1 \in L^2(\Omega)$. $\square$

Appendix B. A class of pseudo-convex phase function

In this appendix, we will show that the class of functions $\phi(t,x) = e^{\gamma(\psi_0(x)-\beta\psi_1(t))}$ is strongly pseudo-convex with respect to the wave operator $P = \square_c = \frac{1}{c^2}\partial_t^2 - \Delta$ when γ and β satisfy condition (78). The principal symbol of $\square_c$, is given by $p_2(t,x;t',x') = -\frac{1}{c^2}t'^2 + |x'|^2$. Then the action of Hamiltonian vector field H_{p_2} on ϕ is given by

$$H_{p_2}\phi = \{p_2,\phi\} = 2\phi\left(\frac{\gamma\beta}{c^2}t'\psi_1' + \gamma g(x',\nabla\psi_0)\right),$$

$$H^2_{p_2}\phi = \{p_2,\{p_2,\phi\}\} = \phi\left[-4\frac{\gamma\beta t'^2}{c^4}\psi_1'' + 4\gamma\nabla^2\psi_0(x',x') + 4\left(\frac{\gamma\beta}{c^2}t'\psi_1' + \gamma g(x',\nabla\psi_0)\right)^2\right].$$

The strongly pseudo-convex condition in definition 3.2 on $\mathcal{U} \subset X$ is equivalent to ϕ satisfying the following two conditions on $\mathcal{U}$:

$$\begin{cases} \mathrm{d}\phi \neq 0, \\ p_2 = H_{p_2}\phi = 0 \quad \Longrightarrow \quad c^2\gamma\nabla^2\psi_0(x',x') > \gamma\beta\alpha|x'|^2, \ \forall x' \in \mathbb{R}^d. \end{cases} \tag{77}$$

Note that in the special case when $\psi_1(t) = |t - t_0|^2$, our class of functions reduces to the one in [10]. In this special case, i.e. $\phi(t,x) = e^{\gamma(\psi_0(x)-\beta|t-t_0|^2)}$, the strongly pseudo-convex condition (77) on $\mathcal{U}$ reduces to

$$\begin{cases} t \neq t_0 \quad \text{or} \quad \nabla\psi_0 \neq 0, \\ p_2 = H_{p_2}\phi = 0 \quad \Longrightarrow \quad c^2\gamma\nabla^2\psi_0(x',x') > 2\gamma\beta|x'|^2, \ \forall x' \in \mathbb{R}^d. \end{cases} \tag{78}$$

If $\nabla^2\psi_0(x',x') > 2\rho|x'|^2$ for all $x' \in \mathbb{R}^d$ on $\mathcal{U}$, then the conditions

$$\nabla\psi_0 \neq 0 \quad \text{and} \quad c^2\gamma\rho > \gamma\beta \quad \text{on} \quad U$$

will be sufficient for ϕ to be strongly pseudo-convex on U, which is exactly the assumption (A.1) in [10, chapter 4] when $c = \gamma = 1$. For the time-space Minkowski manifold, one can further choose $\psi_0(x) = |x - x_0|^2$, hence $\phi(t,x) = e^{\gamma(|x-x_0|^2-\beta(t-t_0)^2)}$ as in (32), and then

$$x_0 \notin U \quad \text{and} \quad c^2\gamma > \gamma\beta \quad \text{on} \quad \mathcal{U}$$

will be sufficient to achieve the simplified strongly pseudo-convex second condition in (78).

Appendix C. Symmetric extensions

In this appendix, we study some properties of extensions from $(0,T)\times\Omega$ to $(-T,T)\times\Omega$ of solutions to

$$\begin{cases} PU := (\partial_t^2 - \Delta + Q_1\partial_t + Q_0 I)U = F & \text{in } (0,T)\times\Omega, \\ \partial_\nu U + h(x)U = 0 & \text{in } (0,T)\times\partial\Omega, \\ U = 0,\ \ U_t = 0 & \text{in } \{t=0\}\times\Omega. \end{cases}$$

Denote the extensions of U and F by $\tilde U$ and $\tilde F$, respectively. We want $\tilde U$ to be as regular as possible when $\tilde P$ acts on it, therefore it is natural to consider an even-time extension of U. That is,

$$\tilde U(t,\cdot) = U_{\text{even}}(t,\cdot) := \begin{cases} U(t,\cdot) & \text{for } t>0, \\ U(-t,\cdot) & \text{for } t<0. \end{cases}$$

Similarly, we have odd extensions of functions in the time variable. This implies $\partial_t^k\tilde U$ is even (odd) in time if k is even (odd). For instance, $\tilde U_t = (U_t)_{\text{odd}}$. To match the even parity of $\tilde F$, we need to have $\tilde Q_0 = (Q_0)_{\text{even}}$, $\tilde Q_1 = (Q_1)_{\text{odd}}$ and hence $\tilde P = \partial_{tt} - \Delta + \tilde Q_1\partial_t + \tilde Q_0 I$. That is,

$$\begin{cases} \tilde P\tilde U := (\partial_t^2 - \Delta + \tilde Q_1\partial_t + \tilde Q_0 I)\tilde U = \tilde F & \text{in } (-T,T)\times\Omega, \\ \partial_\nu \tilde U + h(x)\tilde U = 0 & \text{on } (-T,T)\times\partial\Omega, \\ \tilde U = 0,\ \ \tilde U_t = 0 & \text{in } \{t=0\}\times\Omega. \end{cases}$$

Since the principal symbol of $\tilde P$ is unchanged, $\tilde P$ meets the criteria of pseudo-convexity in definition 3.2. In other words, we are safe to apply the Carleman estimate to both $\tilde P$ and $\tilde U$. Thanks to the vanishing property of U at $t=0$, the action of $\tilde P$ on $\tilde w := \tilde U_t$ is still in $L^2((-T,T)\times\Omega)$. However, if we denote $\tilde v := \tilde U_{tt}$, then $\tilde P\tilde v$ has a delta singularity at $t=0$, regardless of the odd or even extensions of U.

Appendix D.

In this appendix, we prove the following proposition that is needed in the proof of theorem 4.1. From [17, chapter 5], one has

Proposition D.1. *Let $m,k,k_1,k_2\in\mathbb{N}$, $1\leqslant p_1,p_2\leqslant\infty$ and $-\frac12\leqslant s\leqslant\frac12$. Define*

$$W^{k_1,p_1}(0,T;W^{k_2,p_2}(\Omega)) := \left\{ u\in\mathcal{D}'(\Omega_T) \text{ with } \left\| \|\partial_t^k u\|_{W^{k_2,p_2}(\Omega)} \right\|_{L^{p_1}(0,T)} < \infty \text{ for } k\leqslant k_1 \right\}$$

and

$$\Lambda_{k,m,s}(\Omega_T) := \Big\{ u\in\mathcal{D}'(\Omega_T) \text{ with } \partial_t^{k'}u\in L^2(0,T;H^{m+1+s}(\Omega)) \\ \text{and } \partial_t^{k'+1}u\in L^2(0,T;H^{m-1-s}(\Omega)) \text{ for } k'\leqslant k \Big\}.$$

Then we have

$$\Lambda_{k,m,s}(\Omega_T)\subset W^{k,\infty}(0,T;W^{m+1,2}(\Omega)).$$

Proof. We first show the case $k=0$ and $m=0$, i.e. that $\Lambda_{0,0,s}(\Omega_T)\subset W^{0,\infty}(0,T;W^{1,2}(\Omega))$. For $u\in\mathcal{D}'(\Omega_T)$, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t,\cdot)\|^2_{L^2(\Omega)}\right| = 2\left|\int_\Omega \nabla u\cdot\nabla u_t \mathrm{d}x\right| \leqslant C\left|\int_\Omega \Delta u\cdot u_t \mathrm{d}x\right| \leqslant C\|\Delta u\|_{H^s(\Omega)}\|u\|_{H^{-s}(\Omega)}$$
$$\leqslant \frac{C}{2}\left(\|\Delta u\|^2_{H^s(\Omega)} + \|u_t\|^2_{H^{-s}(\Omega)}\right) \leqslant C\left(\|u\|^2_{H^{s+2}(\Omega)} + \|u_t\|^2_{H^{-s}(\Omega)}\right),$$

where we have used the fact that the Laplace operator satisfies the strengthened transmission condition with respect to Ω and it is continuous from $H^{s+2}(\Omega)$ to $H^s(\Omega)$ for $s\geqslant -\frac{1}{2}$ (see [16, theorem 9]), $|\langle v_1, v_2\rangle| \leqslant \|v_1\|_{H^s(\Omega)}\|v_2\|_{H^{-s}(\Omega)}$ for any $v_1\in H^s(\Omega)$, $v_2\in H^{-s}(\Omega)$, and $H^s_0(\Omega) = H^s(\Omega)$ for $-\frac{1}{2}\leqslant s\leqslant\frac{1}{2}$ (see [48, theorem 3.40]). Therefore, we have

$$\|u\|_{W^{0,\infty}(0,T;W^{1,2}(\Omega))} = \operatorname*{ess\,sup}_{0\leqslant t\leqslant T}\|u(t,\cdot)\|_{H^1(\Omega)}$$
$$\leqslant C\left(\|u\|_{L^2(0,T;H^{s+2}(\Omega))} + \|u_t\|_{L^2(0,T;H^{-s}(\Omega))}\right) = C\|u\|_{\Lambda_{0,0,s}(\Omega_T)},$$

where we have identified $W^{k,2}(\Omega)$ with $H^k(\Omega)$ for $k\geqslant 0$ and Lipschitz domains Ω. This proves the statement of the proposition for $k=0$ and $m=0$. For other values of k and m, we only need to substitute $v := \partial_t^k D_x^\alpha u$ with $|\alpha|\leqslant m$ and apply the same estimate to v. □

ORCID iDs

Yan-Long Fang https://orcid.org/0000-0002-6081-8088
Daniel Lesnic https://orcid.org/0000-0003-3025-2770
Moataz Alosaimi https://orcid.org/0000-0001-5778-6794

References

[1] Abraham J P and Sparrow E M 2007 A thermal-ablation bioheat model including liquid-to-vapor phase change, pressure and necrosis-dependent perfusion and moisture-dependent properties *Int. J. Heat Mass Transfer* **50** 2537–44
[2] Ahmadikia H, Fazlali R and Moradi A 2012 Analytical solution of the parabolic and hyperbolic heat transfer equations with constant and transient heat flux conditions on skin tissue *Int. Commun. Heat Mass Transfer* **39** 121–30
[3] Alosaimi M and Lesnic D 2023 Determination of the space-dependent source in the thermal-wave model of bio-heat transfer *Comput. Math. Appl.* **129** 34–49
[4] Alkhwaji A, Vick B and Diller T 2012 New mathematical model to estimate tissue blood perfusion, thermal contact resistance and core temperature *Trans. ASME, J. Biomech. Eng.* **134** 081004
[5] Arendt W and Mazzeo R 2012 Friedlander's eigenvalue inequalities and the Dirichlet-to-Neumann semigroup *Commun. Pure Appl. Anal.* **11** 2201–12
[6] Arendt W 2003 The Laplacian with Robin boundary conditions on arbitrary domains *Potential Anal.* **19** 341–63
[7] Autrique L and Lormel C 2008 Numerical design of experiment for sensitivity analysis - application to skin burn injury prediction *IEEE Trans. Biomed. Eng.* **55** 1279–90
[8] Baudouin L, de Buhan M and Ervedoza S 2017 Convergent algorithm based on Carleman estimates for the recovery of a potential in the wave equation *SIAM J. Numer. Anal.* **55** 1578–613
[9] Baudouin L, de Buhan M, Ervedoza S and Osses A 2021 Carleman-based reconstruction algorithm for the wave *SIAM J. Numer. Anal.* **59** 998–1039
[10] Bellassoued M and Yamamoto M 2017 *Carleman Estimates and Applications to Inverse Problems for Hyperbolic Systems (Springer Monographs in Mathematics)* (Tokyo: Springer) pp xii+260
[11] Bourgain J, Burq N and Zworski M 2013 Control for Schrödinger operators on 2-tori: rough potentials *J. Eur. Math. Soc.* **15** 1597–628
[12] Bukhgeim A I and Klibanov M V 1981 Uniqueness in the large of a class of multidimensional inverse problems *Sov. Math. Dokl.* **17** 244–7
[13] Cao L, Qin Q-H and Zhao N 2010 An RBF-MFS model for analysing thermal behaviour of skin tissues *Int. J. Heat Mass Transfer* **53** 1298–307

[14] Dai W and Nassar R 1999 A finite difference scheme for solving the heat transport equation at the microscale *Numer. Methods Partial Diff. Equ.* **15** 697–708
[15] Dimassi M and Sjöstrand J 1999 *Spectral Asymptotics in the Semiclassical Limit (London Mathematical Society Lecture Note Series)* (Cambridge: Cambridge University Press) pp xii+227
[16] Epstein L C 2007 Pseudo-differential methods for boundary value problems *Fields Inst. Commun.* **52** 171–200
[17] Evans L C 2010 *Partial Differential Equations (Graduate Studies in Mathematics)* 2nd edn (Providence, RI: American Mathematical Society) pp xxii+749
[18] Fefferman C 1978 On positivity of pseudo-differential operators *Proc. Natl Acad. Sci.* **75** 4673–4
[19] Grysa K and Maciag A 2020 Trefftz method in solving the Pennes' and single-phase-lag heat conduction problems with perfusion in the skin *Int. J. Numer. Methods Heat Fluid Flow* **30** 3231–46
[20] Ho J-R, Kuo C-P and Jiaung W-S 2003 Study of heat transfer in multilayered structure within the framework of dual-phase-lag heat conduction model using lattice Boltzmann method *Int. J. Heat Mass Transfer* **46** 55–69
[21] Hörmander L 2003 *The Analysis of Linear Partial Differential Operators I (Classics in Mathematics)* 2nd edn (Berlin: Springer) pp xi+440
[22] Hörmander L 2009 *The Analysis of Linear Partial Differential Operators IV (Grundlehren der Mathematischen Wissenschaften)* 2nd edn (Berlin: Springer) pp vii+352
[23] Hussein S O, Lesnic D and Yamamoto M 2017 Reconstruction of space-dependent potential and/or damping coefficients in the wave equation *Comput. Math. Appl.* **74** 1435–54
[24] Imanuvilov O Y and Yamamoto M 2001 Global Lipschitz stability in an inverse hyperbolic problem by interior observations *Inverse Problems* **17** 717–28
[25] Imanuvilov O Y and Yamamoto M 2001 Global uniqueness and stability in determining coefficients of wave equations *Commun. PDE* **26** 1409–25
[26] Ilmavirta J and Waters A 2018 Recovery of the sound speed for the Acoustic wave equation from phaseless measurements *Commun. Math. Sci.* **16** 1017–41
[27] Isakov V 2005 *Inverse Problems for Partial Differential Equations* 2nd edn (New York: Springer) pp xi+286
[28] Jalali A, Ayani M-B and Baghban M 2014 Simultaneous estimation of controllable parameters in a living tissue during thermal therapy *J. Therm. Biol.* **45** 37–42
[29] Kato T 1995 *Perturbation Theory for Linear Operators* 2nd edn (Berlin: Springer)
[30] Klibanov M V 1984 Inverse problems in the 'large' and Carleman bounds *Differ. Equ.* **20** 755–60
[31] Klibanov M V 1992 Inverse problems and Carleman estimates *Inverse Problems* **8** 575–96
[32] Klibanov M V 1997 Global convexity in a three-dimensional inverse acoustic problem *SIAM J. Math. Anal.* **28** 1371–88
[33] Klibanov M V and Timonov A A 2004 *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications (Inverse and Ill-Posed Problems Series)* (Leiden: VSP)
[34] Klibanov M V 2013 Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems *J. Inverse Ill-Posed Problems* **21** 477–510
[35] Klibanov M V and Li J 2021 *Inverse Problems and Carleman Estimates: Global Uniqueness, Global Convergence and Experimental Data* (Berlin: de Gruyter)
[36] Klibanov M V, Li J and Zhang W 2022 A globally convergent numerical method for a 3D coefficient inverse problem for a wave-like equation *SIAM J. Sci. Comput.* **44** A3341–65
[37] Ladyzhenskaya O A 1985 *Boundary Value Problems of Mathematical Physics* (Berlin: Springer)
[38] Lasiecka I, Lions J-L and Triggiani R 1986 Non homogeneous boundary value problems for second order hyperbolic operators *J. Math. Pure Appl.* **65** 149–92
[39] Lasiecka I and Triggiani R 1992 Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions *Appl. Math. Optim.* **25** 189–224
[40] Lasiecka I, Triggiani R and Zhang X 2000 Nonconservative wave equations with unobserved Neumann BC: global uniqueness and observability in one shot *Contemp. Math.* **268** 227–325
[41] Lavrent'ev M M, Romanov V G and Shishatskii S P 1986 *Ill-Posed Problems of Mathematical Physics and Analysis* (Providence, RI: American Mathematical Society) pp 290
[42] Lebeau G and Robbiano L 1995 Contrôle exact de l'équatio de la chaleur *Commun. PDE* **20** 335–56
[43] Lebeau G and Robbiano L 1997 Stabilisation de l'équation des ondes par le bord *Duke Math. J.* **86** 465–91
[44] Lerner N 2019 *Carleman Inequalities (Grundlehren der Mathematischen Wissenschaften)* (Cham: Springer) pp xxvii+557

[45] Lions J-L and Strauss W A 1965 Some non-linear evolution equations *Bull. de la Soc. Math. de France* **93** 43–96
[46] Liu J, Ren Z and Wang C 1995 Interpretation of living tissue's temperature oscillations by thermal wave theory *Chin. Sci. Bull.* **40** 1493–5
[47] Liu S and Triggiani R 2011 Global uniqueness and stability in determining the damping and potential coefficients of an inverse hyperbolic problem *Nonlinear Anal.: Real-World Appl.* **12** 1562–90
[48] McLean W 2000 *Strongly Elliptic Systems and Boundary Integral Equations* (Cambridge: Cambridge University Press) pp x+372
[49] Mitra K, Kumar S, Vedevarz A and Moallemi M 1995 Experimental evidence of hyperbolic heat conduction in processed meat *J. Heat Transfer* **117** 568–73
[50] Ozen S, Helhel S and Cerezci O 2008 Heat analysis of biological tissue exposed to microwave by using thermal wave model of bio-heat transfer (TWMBT) *Burns* **34** 45–49
[51] Ren Z P, Liu J, Wang C C and Jiang P X 1995 Boundary element method (BEM) for solving normal or inverse bio-heat transfer problem of biological bodies with complex shape *J. Therm. Sci.* **4** 117–24
[52] Romanov V and Hasanov A 2020 Recovering a potential in damped wave equation from Neumann-to-Dirichlet operator *Inverse Problems* **36** 115011
[53] Tataru D 1994 A priori estimates of Carleman's type in domains with boundary *J. Math. Pures Appl.* **73** 355–87
[54] Tataru D 1996 Carleman estimates and unique continuation for solutions to boundary value problems *J. Math. Pures Appl.* **75** 367–408
[55] Wainwright P 2000 Thermal effects of radiation from cellular telephones *Phys. Med. Biol.* **45** 2363
[56] Waters A 2018 Recovery of attenuation coefficients from phaseless measurements for the Helmholtz equation *Commun. Math. Sci.* **16** 579–87
[57] Yamamoto M 2018 Carleman estimates for parabolic equations and applications *Inverse Problems* **25** 123013