Proceedings Paper:

Willis, T. and Punzo, G. orcid.org/0000-0003-4246-9045 (2023) Multi-layer Cournot-congestion model. In: Perdon, A.M., Zattoni, E. and Loiseau, J.J., (eds.) IFAC-PapersOnLine. 1st IFAC Workshop on Control of Complex Systems (COSY 2022), 24-25 Nov 2022, Bologna, Italy. Elsevier BV , pp. 61-66.

https://doi.org/10.1016/j.ifacol.2023.01.049

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IFAC PapersOnLine 55-40 (2022) 61–66

Multi-Layer Cournot-Congestion Model ⋆

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Abstract: Geopolitical instability, climate change and black swan events disrupt the trade and logistics of resources around the globe. Events such as the unforeseen closure of the Suez Canal or the cessation of trade between some players due to wars or embargos are some examples of this. The problem of predicting local price change under modification of an underlying transport network unites elements of game theory, network theory and transport theory. The micro-economic Cournot oligopoly game involves modelling economic actors as rational players attempting to maximise profit. Under fixed transport conditions, analytical results can be found on the equilibria. Similarly, the transport layer can be analytically solved using techniques for selfish routing. Where trade and transport layers are linked together there is inter-layer feedback wherein players attempt to maximise their utility. We looked at the nature of the approach towards a new equilibrium under this instantaneous network change. In this respect our findings indicate that players benefit significantly from taking advantage of the non-simultaneous responses to the market rather than from moving to a new equilibrium utility. We found that when uncoupled, both the upper and lower layers have concave utility curves meaning there exist unique and stable equilibria in both cases. This leads to the multilayer model having non-unique stable equilibria for which general solutions exist.

Keywords: Multilayer Network Theory, Oligopoly, Wardrop Equilibrium, Nash Equilibrium, Transport Theory, Social Optimum, Game Theory, Algorithmic Convergence

1. INTRODUCTION

Originally conceived to model duopolies, a Cournot competition is an economic model which applies to oligopolies competing on the same products and markets. In Cournot competitions, the reward a player gets by selling goods depends on the actions of the other players which change the price at which such goods are sold. The profits received by players are affected by the actions of individual players and strategies are chosen in response to those (see Cournot (1897)). Cournot competitions are also known as Cournot games given their strategic nature. With respect to market dynamics, single Cournot games have been widely studied and results on the nature of equilibria are available in the literature by (Varian (1992), Mas-Collel (1995)). By dividing the players from the market, Bimpikis et al. (2014) obtained equilibria for the Cournot competition in which players are associated to specific markets via a bipartite network. Results were also obtained by Kyparisis and Qiu (1990), Qiu (1991) and Abolhassani et al. (2014) each of whom defined an equilibrium allocation to be a set of strategies in which firms make zero marginal profit. Cournot competitions do not have elements linked to variable costs of transportation or associated to the physical distance between players and markets in which they compete.
The well known selfish routing problem considers the cost of moving across physical distances. In these, players have to navigate a network from an origin to a destination node where each edge has a cost, which is a function of the number of players using it, and which eventually diminishes the utility of the players as more of them use the same edge (Roughgarden (2006)). In selfish routing problems, a Wardrop equilibrium is defined as the set of routing strategies adopted by the player such that no player has any incentive in changing their routing unilaterally. Such an equilibrium concept is fundamental in the traffic assignment problem, to the point that measured flows are often taken as the naturally emerging equilibrium and a starting point to obtain the origin-destination matrix by reconstructing the costs associated to each edge through congestion functions (Zhang et al. (2019)). Congestion functions are convex, monotonically increasing functions relating the travel time along a road segment (a popular proxy for the cost) to the volume of traffic on that segment. Although different from Cournot games, selfish routing problems have been investigated through multilayer networks too, where each layer represents different transport means or performance (e.g. long range fast journeys as opposed to short range, slower transfers). Ibrahim and De Bacco (2021) examined such a multilayer transport setting providing algorithms to obtain Wardrop equilibria.

The multilayer setting is amenable to represent different dynamical systems, each abstracted as a network, interacting between them. In particular, the network Cournot competition as explored by Bimpikis et al. (2014) can be connected to a separate transport layer, where the selfish routing problem of shipping the goods from players to markets is explored. The Nash equilibrium of the Cournot game will have to be considered in conjunction with the Wardrop equilibrium of the selfish routing problem, as

⋆ This paper was developed under PhD funding by the ESPRC.

10.1016/j.ifacol.2023.01.049

one or both may be achieved. Where there is a Wardrop equilibrium in the selfish routing problem, under appropriate assumptions this is also a Nash-Cournot equilibrium (Haurie and Marcotte (1985)).

This work presents a bilayer model where a Cournot competition is set in the bipartite network of players and markets in the first layer and a selfish routing problem is presented in the second layer, as a congestion game played in the physically embedded transportation network.

An examination of a capacity constrained oligopoly problem was considered by Alsabah et al. (2021) finding that a reduction in transportation costs can negatively impact profit for all firms. Their approach differs from that of this paper as they do not consider transport on a network.

While the Cournot competition adopts the setting in Bimpikis et al. (2014) and the congestion game leverages on the results in Ibrahim and De Bacco (2021), the interaction between the two leads to original results in the emergence of equilibrium points where each player has no incentive to change the quantity of goods to sell in each market or the routing to ship such goods. We shall follow the notation in Boccaletti et al. (2014) for the multilayer structure.

The network Cournot case and transportation problems are both well studied subjects. A gap is present in the literature about the interaction between these problems. This work addresses such a gap and explores the dynamics of the bilayer model.

In this paper, we present the following original contributions:

- We introduce a new bi-layer model of market transport where a Cournot competition on a bipartite network is influenced by transport costs, which are the result of a congestion game in a second layer. The volume of traffic in the second layer are in turn influenced by the goods exchanged between sellers and markets in the Cournot competition.
- We prove existence and uniqueness of the equilibrium points in each layer individually in the hypothesis of stationary conditions in the other and show they are stable.
- We finally analysed the coupled co-evolving dynamics of both layer, find the equilibrium points and their features in terms of uniqueness and stability.

This work, while focussed on the analytical aspects of the Cournot-congestion game, looks also at more applied research questions around the relationship between market competition and transportation costs. This is a theme with wide reaching utility, impacting strategic choices for food security and supply chain resilience. One example of this is the global wheat market, where some countries are net producers and some are net consumers, and knowledge about the coupled dynamics of prices and transport costs is extremely valuable. The link from the model to the application is offered by considering the world market as a routing problem wherein firms are competing agents attempting to maximise their personal utility.

2. MODEL FORMULATION

2.1 Model Basis

The bi-layer model captures the effects of the goods' routing in the transportation layer on their price. Players are considered, each selling goods in order to maximise their own profit, taking into account the price of goods in each market and the cost of transporting goods to those markets. The aim of the players is to maximise income generated by the sale minus the transportation cost.

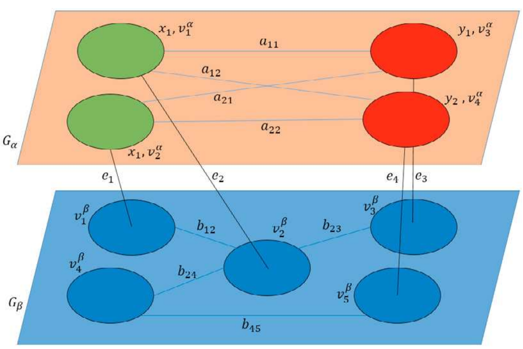

Fig. 1. An example model with 2 players, 2 markets and a 5 node transport layer

2.2 Network Formulation

We shall consider a multilayer network composed of two layers, hence referred to as a bilayer network. A bilayer network is a pair $\mathcal{M} = (\mathcal{G}, \mathcal{E}_{\alpha\beta})$ where α and β index the layers and $\mathcal{G} = (G_\alpha, G_\beta)$ is a family of undirected weighted graphs called 'layers' of $\mathcal{M}$. In layer α we have the graph $G_\alpha = (V_\alpha, E_\alpha)$ where V_α is a set of nodes $v_i^\alpha \in V_\alpha$ with $i \in \{1, 2...N_\alpha\}$ and $a_{i,j} = (v_i^\alpha, v_j^\alpha) \in E_\alpha$ is a set of edges. Likewise, we have $G_\beta = (V_\beta, E_\beta)$. with $v_i^\beta \in V_\beta$ with $i \in \{1, 2...N_\beta\}$ and $b_{i,j} = (v_i^\beta, v_j^\beta) \in E_\beta$.

Each node on the upper layer (and accordingly every member of these sets) is associated to a node in G_β. The discrete 'location' map $L(\cdot) = v_i^\beta : \mathbf{V}_\alpha \to \mathbf{V}_\beta$ gives the geographical embedding of any player or market onto G_β and for each $v_k \in G_\alpha$, there exists $e_k \in \mathcal{E}_{\alpha\beta}$ with $e_k = (v_k^\alpha, L(v_k^\alpha))$.

E_α and E_β are the sets of intra-layer edges. Hence $\mathcal{E}_{\alpha\beta}$ is defined to be the set of inter-layer edges with $e_k = (v_k^\alpha, L(v_k^\alpha))$, with $k \in \{1, 2...N_\alpha\}$. Throughout, referring to the schematic representation in Figure 1, G_α is also referred to in text as the 'Upper Layer' and G_β as the 'Lower Layer'.

The nodes in the upper layer can be divided in a set of N players (sellers) and a set of M markets. The bipartite assumption is appropriate as no traffic passes between pairs of sellers or pairs of markets. $X = (x_1, x_2, \ldots, x_N)$ is the set of players and $Y = (y_1, y_2, \ldots, y_M)$ is the set of markets, with $X \cap Y = \emptyset$ and $V_\alpha = X \cup Y$.

x_i and y_j do not have layer-denoting superscripts as the sets X and Y are only in the upper layer.

There exists an ordering on the elements of V_α with $v_1 = x_1, \ldots, v_N = x_N, v_{N+1} = y_1 \ldots, v_{N_\alpha} = y_M$.

The weights of the edges E_α represent the amount of goods being sold by a player to a market and are given by $w(\cdot) : E_\alpha \to \mathbb{R}_+$.

As there are no edges between any two elements of X and no edges between any two elements of Y, G_α is a bipartite graph such that for $a_{(\cdot,\cdot)} \in E_\alpha, a_{(\cdot,\cdot)} = (x_i, y_j), x_i \in X, y_j \in Y$ and $(x_i, y_j) \in E_\alpha$ for all $x_i \in X, y_j \in Y$.

The lower layer G_β is the transportation layer and its nodes and edges represent a real world transportation system. The edges in the lower layer have lengths and capacities, $l(b_i) \in \mathbb{R}_+$, $c(b_i) \in \mathbb{R}_+$.

Finally, the set $E_{\alpha\beta}$ of inter-layer edges represents the geographical embedding of the elements of the upper layer into the lower layer.

2.3 Transport Paths

Transport on the lower layer must fulfil demand in the upper layer, that is, the amount of goods transferred between a player and a market in the upper layer must correspond to the travel demand between the location of the player and the location of the market in the transport layer. While there exists only a single path between x_i and y_j in the upper layer, there can exist multiple paths between $L(x_i)$ and $L(y_j)$ in the lower layer. This paper refers to each possible route as a path $p_{i,j}^k$ where i and j represent the index of the element of X and Y that path represents travel between and k is the index of the path. The set of all paths between $L(x_i)$ and $L(y_j)$ is $P_{i,j}$ and the collection of all sets of paths between $L(X)$ and $L(Y)$ in the lower layer is P such that $P = (P_{1,1}, \dots, P_{1,M}, P_{2,1}, \dots, P_{N,1}, \dots, P_{N,M})$ and $P_{i,j} = \left(p_{i,j}^1, p_{i,j}^2, \dots, p_{i,j}^{N_{P_{i,j}}}\right)$, where $N_{P_{i,j}}$ is the number of paths, in the lower layer, between node $L(x_i^\alpha)$ and $L(y_j^\alpha)$. The transport through the lower layer must fulfil the transport requirements as described by edge weight in the upper layer. As such $w(a_{i,j}) = \sum_{k=1}^{N_{P_{i,j}}} p_{i,j}^k$. Note that, with a slight abuse of notation, we defined $p_{i,j}^k$ as both the path (an ordered set of edges between $L(x_i)$ and $L(y_j)$) and as the flow of goods routed through them.

3. COURNOT GAME

Cournot competition describes the dynamics of the players in the upper layer with respect to each market as the following conditions are satisfied:

- The game has multiple players;
- there is no collusion between players;
- each player has market power, that is, each player strategy changes the price in the market they participate in;
- The number of players and markets are fixed and the goods sold by all players are homogeneous;
- players choose the quantity of goods to sell rather than their price, which is a consequence of the total amount of good sold;
- players engage in rational behaviour.

However, as there are a number of markets, they end up engaging in parallel Cournot competition.

3.1 Game Dynamics

Each player $x \in X$ attempts to sell their goods in order to maximise the difference between their income and the costs they incur.

- The costs they incur are for the utilisation of transport links. The transport links become congested as they are used more and this influences the total costs.
- The income generated comes from selling goods to markets. They receive profit depending on the supply and demand to each market they sell to.

The utility is hence given by the function: $u(s_i, s_{-i}) : S_i, S_{-i} \to\in \mathbb{R}_+$. s_i is the current strategy of player x_i, S_i is the set of all possible strategies for player x_i, s_{-i} is the current strategy of all other players and S_{-i} is the set of all possible strategies of all players. The game is played asynchronously with players sequentially updating their strategy to the best response for the strategies played by all other players. The players are concerned with the maximisation of their utility over multiple rounds rather than just in the immediate future. Accordingly, they consider the future strategies of other players when deciding their own best response. It is assumed that player 1 (x_1) updates their strategy first, followed by player 2, progressing in order and returning to player 1 after player N has updated their strategy. Each player's strategy is the union of an upper layer and a lower layer set of actions, which consist of:

- Choosing an allocation of sellers to sell goods to in the upper layer. This is a mixed strategy of the form $(w(a_{i,1}), w(a_{i,2}), \dots, w(a_{i,M}))$ for $a_{i,j} \in E_\alpha$.
- Selecting a mixed-strategy of paths they utilise to transport these goods. The paths they select must appropriately fulfil their origin destination pairing.

This gives a strategy

$$s_i = \begin{pmatrix} [a_{i,1}, a_{i,2}, \dots, a_{i,M}] \\ [p_{i,1}^1, \dots, p_{i,1}^{(N_{P_{i,1}})}, \dots, p_{i,m}^1, \dots, p_{i,N_c}^{(N_{P_{i,m}})}] \end{pmatrix} \tag{1}$$

The utility for a player x_i is given by

$$u(s_i, s_{-i}) = A(s_i, s_{-i}) - B(s_i, s_{-i}) \tag{2}$$

$$A(s_i, s_{-i}) = \sum_{j=1}^{M} A_{m_j}(s_i, s_{-i}) \tag{3}$$

where $A(s_i, s_{-i})$ is the total profit made in the upper layer and $A_{m_j}(s_i, s_{-i})$ is the profit made by i in market j. $B(s_i, s_{-i})$ is the congestion function. We do not impose specific restrictions on the choice of the congestion functions other than continuity and being strictly monotonically increasing, from which convexity follows.

4. UNCOUPLED DYNAMICS RESULTS

4.1 Approach to Cournot Equilibrium

First considering that the distribution of flows across all the paths between the same origin and destination is uniform and fixed (such that $p_{ij}^k = p_{ij}^l$ for all $k, l \in \{1, 2, ...N_{p_{ij}}\}$ and $i \in \{1, 2, ...N\}$, $j \in \{1, 2, ...M\}$) and does not depend on the amount of goods the player sells in that particular market. This makes the transportation costs influence the player's utility but makes the lower layer's dynamics insensitive to the dynamics in the upper layer. In other words, a player can change the amount of goods they sell in each market but they cannot change the share of flow between paths with the same origin and destination. The game is therefore played in the upper layer only, albeit influenced by the costs generated in the lower layer. Assuming the game starts from a non-equilibrium state, with the players progressively playing the optimal response to the other players' strategies, the utilities across multiple rounds can be examined.

Bimpikis et al. (2014) proved that, where the profit function is twice differentiable, concave, and strictly decreasing and the costs of production are twice differentiable, convex and increasing, the game has a unique equilibrium. In the present case, there is no cost of production in the upper layer. The 0-function can however be substituted and fulfils the conditions of being twice differentiable and weakly monotonically increasing. Consider the two player case with $A_{m_j}(s_i, s_{-i})$ generally defined as $A_{m_j}(s_i, s_{-i}) = w(a_{i,j}) \cdot \left(Q - \sum_{k=1}^{N} w(a_{k,j})\right)$, where Q is the profit where there is no supply. A profit function does exist which fulfils the conditions outlined by Bimpikis et al. (2014). As such, the game has a unique equilibrium as proved therein.

This equilibrium can be found by considering the sum of $A_{m_1}(h, s_{-i})$ and $A_{m_2}(1-h, s_{-i})$ for variable $h \in [0, 1]$. $h \in [0, 1]$ implies the production by each player is 1 in total, allocated among all markets. It can be seen that this function is concave (where h is $w(a_{1,1})$). The equilibrium occurs at the maximum of this graph.

The approach towards an equilibrium happens in a consistent manner. First define an equilibrium-mimicking strategy to be one in which players move so that all markets receive the same supply as they would if the game was in equilibrium. Consider the myopic best response as one concerned only with the utility received after one round of the game. The myopic best response by a player lies at the midpoint between the final equilibrium strategy and the equilibrium mimicking strategy for that player. In the two player case this is easy to identify as there is only a single player's strategy to respond to. This leads at our first Theorem.

Theorem 1. Let $A(\cdot, \cdot)$ be as defined in Equation (3), and let $g \geq 0$ be the difference in utility gained by the first player moving as they play their myopic best response, compared to the utility they would receive at equilibrium. Then the utility of the first player through successive myopic best responses, compared to the final unique equilibrium strategy is $\frac{16g}{30} > 0$.

This means that rational players will have no incentive to play an equilibrium strategy if all the other players have not already done so.

Proof. The two possible strategies are playing the myopic best response to the opponents in order to take advantage of their non-optimal strategy or moving to the equilibrium position immediately. Normalising with respect to the equilibrium strategy, with g the additional utility gained by the myopic responder relative to the equilibrium, the utility gained by the player at each subsequent change in strategy by either player is given by the sequence $g, -\frac{g}{2}, \frac{g}{16}, -\frac{g}{32}, \ldots$

This is an infinite sequence which, using the formula $\sum_{n=1}^{\infty} s_n = \frac{s_1}{1-(\frac{s_2}{s_1})}$, sums to $\frac{16g}{30} > 0$.

□

Accordingly, each player will play the myopic best response given their opponents current strategy rather than playing the equilibrium strategy immediately. As such, the myopic best response is also the best strategy over any time frame. The game dynamics therefore approaches the equilibrium as shown in Figure 2.

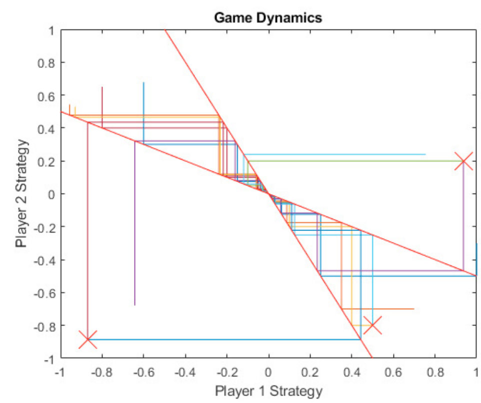

Fig. 2. The Progressive Strategy positions of both players as they approach the equilibrium from a variety of start positions. Bounds on convergence and start points from which both convergence paths are shown.

4.2 Transport-only solutions

We now consider the equilibrium in the lower layer when the volume of goods to each market is fixed in the upper layer. Considering a congestion function $B(f)$ which increases monotonically with flow f, we can now state the following

Theorem 2. Consider two values of the flows f and f' between any two nodes in the upper layer, with $f > f'$. When $B(f) \geq B(f')$ *and* $w(a_{i,j}) \in \mathbb{R}_+$, then *Wardrop Equilibria exist in the lower layer.*

In fact, where the transportation costs are monotonically increasing as flow increases and assuming fixed edge weights in the upper layer, Wardrop equilibria can be found Wardrop (1952).

Proof. Consider the rate of change of utility with respect to changes made in a player's transport strategy, given by $\frac{\partial u(s_i, s_{-i})}{\partial p_{i,j}^k}$ When considering a change in the paths selected a player can examine the change in congestion cost from doing so. Increasing the amount the player uses a path will always result in an increase in costs. Due to the fixed upper layer however, $w(a_{i,j})$ is constant and

$\sum_{k=1}^{N_{P_{i,j}}} p_{i,j}^k = w(a_{i,j})$ is fixed. As such any additional traffic along one path must result in a reduction in traffic along a parallel path. The path allocation is therefore in equilibrium when $\frac{\partial u(s_i, s_{-i})}{\partial p_{ij}^k} = q, q \in \mathbb{R}_+, p_{ij}^k \neq 0$ for all i, j, k.

This fulfils the conditions of a Wardrop equilibrium wherein all paths have equal cost to all players.

□

5. GRAPH REDUCTION

A path $p^k_{i,j}$ on a graph (introduced in section 2.3) gives a set of edges which can be used to move from i to j. Let $\mu_{q,r}$ be the set of paths which use edge $b_{q,r} \in E_\beta$. $\mu_{q,r} = [a^{q,r}_{1,1,1}, a^{q,r}_{1,1,2}, a^{q,r}_{1,2,1}, \dots]$ where $a^{q,r}_{i,j,k} \in \{0,1\}$ indicates whether path k between player i and market j uses edge q,r.

As such, given an edge $a_{i,j}$ in the upper layer, $P_{i,j}$ is a mapping from $a_{i,j}$ to E_β and is a collection of the sets of edges in the lower layer which are used as part of the transport solution running parallel to the edge, $a_{i,j}$.
Hence, $\mu_{q,r}$ is a mapping from $b_{q,r}$ to E_α giving the set of edges in the upper layer which use $b_{q,r}$ in their transport solution. More formally, we have $P_{(\cdot,\cdot)} : E_\alpha \to \{E_\beta, E_\beta, \dots\}$ and $\mu_{(\cdot,\cdot)} : E_\beta \to \{E_\alpha, E_\alpha, \dots\}$. Defining a fulfilment as a set of all parallel paths between $L(x_i)$ and $L(y_j)$.

When all the edges in G_β have the same capacity, the graph can be reduced from a set of nodes and edges to a Λ representation on which simultaneous equations can be used to find equilibria. Uniform capacity is required, as the aim of this representation is to classify edges by their Λ's to reduce the complexity of the graph and edges with different capacities cannot be grouped together.

Where every edge can be classified as being in $\lambda_{(\cdot,\dots,\cdot)}$, all edges which are used by the same set of fulfilments can be considered to be a single edge whose length is the sum of the lengths of the edges within it, and whose capacity is equal to the capacity of each edge in it. This gives a graph representation $G_\beta \to \Lambda$.

Theorem 3. Let the network in the lower layer G_β be acyclic and let G^1_β and G^2_β be two subnetworks included in G_β such that $G^1_\beta \cap G^2_\beta = \emptyset$, and $G^1_\beta \cup G^2_\beta = G_\beta - \{b\}$, where $b \in E_\beta$. Let N_1 and M_1 be the number of nodes in G^1_β directly connected respectively to a player and to a market in G_α. Also define $N_2 = N - N_1$ and $M_2 = M - M_1$. For any vertex $v_i \in G^1_\beta$ and $v_j \in G^2_\beta$ any path connecting them will use b. Then the multiplicity of fulfilments which use the edge e will be such that $N_1M_2 + N_2M_1$.

Proof. Consider an edge b. There exists subgraphs G^1_β and G^2_β such that for $v_i \in G^1_\beta$ and $v_j \in G^2_\beta$ any path connecting them will use b. As each element of the upper layer is embedded into only a single node of the lower layer, $L(v^\alpha_g)$ is either in G^1_β or G^2_β but not both. Of the N players, if i are in G^1_β then $N-i$ are in G^2_β. Similarly of the M markets, if j are in G^2_β, $M-j$ are in G^1_β. b is used $i \cdot j$ times from G^1_β to G^2_β as every $L(x_a)$ has one edge to $L(y_b)$ and $(N-i)\cdot(M-j)$ times from G^2_β to G^1_β.

□

Theorem 3 can then be used to reduce the graph to a smaller set of $\lambda_{\dots}$ as $\lambda_{\dots}$ whose index multiplicity does not exist in the set of possible $N_1M_2 + N_2M_1$ need not be considered. For example in the case with two players and two markets $\lambda_a = \emptyset$ as there exist no edges which are only used by 1 fulfilment.

As means of an example, this becomes $G_\beta \to [\lambda_{(a_{11},a_{12})}, \lambda_{(a_{11},a_{21})}, \dots, \lambda_{(a_{11},a_{12},a_{21},a_{22})}]$ in the two player, two market sub-case. Note that the number of elements in the vector is independent on the number of actual edges in G_β.

The change in utility a player i gets from selling more to a specific market, hence increasing the weight of any edge a in the upper layer is $\left(\frac{\partial u(s_i,s_{-i})}{\partial a}\right)$. An equilibrium, in this case, corresponds to no incentive to change the amount sold, therefore, in the 2 player, two market case, this means solving $\frac{\partial u(s_1,s_{-1})}{\partial a_{11}} = \frac{\partial u(s_1,s_{-1})}{\partial a_{12}}$, $\frac{\partial u(s_2,s_{-2})}{\partial a_{21}} = \frac{\partial u(s_2,s_{-2})}{\partial a_{22}}$, $\frac{\partial u(s_1,s_{-1})}{\partial a_{11}} = \eta_1$, $\frac{\partial u(s_2,s_{-2})}{\partial a_{21}} = \eta_2$ where η_1 and η_2 are constants representing the cost of production of goods to the players such that if players sell with costs greater than η_1 and η_2, they will have negative profits.

5.1 Null Transportation Case

Lemma 4. In the Cournot-congestion game, where transportation costs are null, the sale price is the same in every market.

Proof. The utility for a player is given in Equation (2). Transportation costs are 0 where $B(\cdot,\cdot) = 0$ and accordingly the utility of each player is $A(\cdot,\cdot)$. By examining the partial differential equations it has been found that

$$\frac{\partial u(s_1,s_{-1})}{\partial a_{11}} = \frac{\partial u(s_1,s_{-1})}{\partial a_{12}}, \frac{\partial u(s_1,s_{-1})}{\partial a_{11}} = \frac{\partial A_{m_1}(s_1,s_{-1})}{\partial a_{11}} \tag{4}$$

Accordingly from Equation (4) $\frac{\partial A_{m_1}(s_1,s_{-1})}{\partial a_{11}} = \frac{\partial A_{m_2}(s_2,s_{-2})}{\partial a_{21}}$ can be derived. As A_{m_1} uses the same utility equation as A_{m_2}, where their partial derivatives are the same, they hold the same value. As such, without loss of generality this can be applied to any 2 markets and accordingly the prices in all markets will be at equilibrium at a single constant value.

□

6. GENERAL MULTILAYER SOLUTIONS

When examining the coupled layers, the proof of the existence of unique equilibria given in Bimpikis et al. (2014) and used in section 4.1 does not hold. This is because while the profit functions remain twice differentiable, convex and strictly decreasing, the costs functions are not twice differentiable, concave and increasing.

Theorem 5. The Cournot-congestion game is in equilibrium if and only if, for each player, $\frac{\partial u(s_i,s_{-i})}{\partial a_{ij}} - \frac{\partial u(s_i,s_{-i})}{\partial p^k_{ij}} = r(i)$, for $i \in \{1,2,\dots,N\}$, $j \in \{1,2,\dots,M\}$, $k \in \{1,2,\dots,N_{P_{i,j}}\}$, $r(i) \in \mathbb{R}_+$ with $r(i)$ a fixed constant for all j,k.

This theorem means the system is at equilibrium if and only if each player has the same marginal utility for all (non-zero) strategies available to them.

Proof. The change in utility of a market and associated route is given by

$$\frac{\partial u(s_i, s_{-i})}{\partial a_{ij}} - \frac{\partial u(s_i, s_{-i})}{\partial p_{ij}^k}. \quad (5)$$

Where there exist no alternate markets or paths such that changing to that market and using an associated transport route in order to transport goods there has a utility increase greater in magnitude than the decrease in utility from no longer using the current strategy, a player will change strategies. This implies that for any $s \in S$ for which Equation (5) is not true, the system is not at equilibrium.

If all players have the same marginal utility for each strategy than no player has a benefit in marginal changes to their strategy. Accordingly the system must be in equilibrium.

7. DISCUSSION

This paper explored the dynamics of two coupled games that together capture dynamic behaviour observable in global trade. The strategic interaction of players in oligopolies is well captured by the Cournot competition, and studied under this light since the beginning of the 20^{th} century. Likewise, congestion games, are a classical operation research problem and characteristics of the Wardrop equilibrium are well known. However, when the two games are set on graphs and made to interact as a multilayer network, the dynamics becomes richer and the equilibria in each layer depend on the dynamics of the other.

A recent attempt of introducing the transport dynamics as a cost in the Cournot competition was made by Bimpikis et al. (2014) where, to each player-market interaction, a congestion cost is associated . By separating the transport layer from the Cournot competition, our work opens up to the possibility of multiple paths, meaning that a strategy for each player includes the choice of routing to the market. The travel costs were already included in the Cournot competition by Alsabah et al. (2021), however this approach to routing considers only functional transport costs. The multilayer path based transport routing that appears in this paper was not found previous literature by the authors. By analysing the two layers separately we found our results aligned to the existing literature on single layer network oligopolies as well as selfish routing problems, which we leverage on. However a difference exists between the fixed travel dynamics discussed in Section 4.1 and the classical Cournot competition. In our case, transportation cost still exist but players are not able to influence them by choosing alternative routing.

Under mild assumptions about convexity and concavity of costs and profits, the model offers insights about the gain in playing a selfish strategy while approaching the equilibrium. This was only proved for the case of 2 players and any number of markets, but an extension to multiple players is possible and subject of the current research.

Finally, results are achieved using a graph representation which makes use of a notation based on the links used in the paths. This representation stems from transportation research and is alike to the link-route incidence matrix representation. In our framework it offers the tool to reduce the multiplicity of the solution space making the identification of equilibria of the complete Cournot-congestion game feasible.

8. CONCLUSIONS

In this work, a multilayer network was used to couple the network Cournot competition (on the upper layer) to a selfish routing problem (on the lower layer). The nature of the behaviour of players on the upper layer when uncoupled from the lower layer has been found to manifest as a unique stable equilibrium. Similarly it was found that there exists a unique stable equilibrium on the lower layer. When the layers are coupled there no longer exists unique stable equilibria as the concave behaviour in the upper layer and the convex behaviour in the lower layer when added give a non-concave utility function. Existence of equilibria was however proved and the dynamics examined via a link-route representation of the lower layer.

REFERENCES

Abolhassani, M., Bateni, M., Hajiaghayi, M., Mahini, H., and Sawant, A. (2014). Network cournot competition. *CoRR*, abs/1405.1794.

Alsabah, H., Bernard, B., Capponi, A., Iyengar, G., and Sethuraman, J. (2021). Multiregional oligopoly with capacity constraints. *Management Science*, 67(8), 4789–4808. doi:10.1287/mnsc.2020.3728.

Bimpikis, K., Ehsani, S., and Ilkiliç, R. (2014). Cournot competition in networked markets. In *EC 2014 - Proceedings of the 15th ACM Conference on Economics and Computation*, 733. Macmillan and co ltd.

Boccaletti, S., Bianconi, G., Criado, R., del Genio, C.I., Gómez-Gardeñes, J., Romance, M., Sendiña-Nadal, I., Wang, Z., and Zanin, M. (2014). The structure and dynamics of multilayer networks. *Physics Reports*, 544(1), 1–122. doi:10.1016/J.PHYSREP.2014.07.001.

Cournot, A.A. (1897). *Researches into the mathematical principles of the theory of wealth.*

Haurie, A. and Marcotte, P. (1985). *On the Relationship Between Nash-Cournot and Wardrop Equilibria.* J. Wiley.

Ibrahim, A., L.A. and De Bacco, C. (2021). Optimal transport in multilayer networks for traffic flow optimization. *Algorithms*, 14(7). doi:10.3390/a14070189.

Kyparisis, J. and Qiu, Y. (1990). Solution differentiability for oligopolistic network equilibria. *Operations Research Letters*, 9(6), 395–402.

Mas-Collel, A. (1995). *Microeconomic Thoery.* Oxford University Press Inc.

Qiu, Y. (1991). Solution properties of oligopolistic network equilibria. *Networks*, 21(5), 565–580. doi: https://doi.org/10.1002/net.3230210506.

Roughgarden, T. (2006). Potential functions and the inefficiency of equilibria. *International Congress of Mathematicians, ICM 2006*, 3, 1071–1094. doi:10.4171/022-3/52.

Varian, H. (1992). Microeconomic analysis third edition.

Wardrop, J.G. (1952). Some theoretical aspects of road traffic resarch.

Zhang, J., Liu, M., and Zhou, B. (2019). Analytical Model for Travel Time-Based BPR Function with Demand Fluctuation and Capacity Degradation. *Mathematical Problems in Engineering*, 2019. doi: 10.1155/2019/5916479.