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Schanuel type conjectures and disjointness

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Abstract
Given a subfield F of $\mathbb{C}$, we study the linear disjointess of the field E generated by iterated exponentials of elements of $\overline{F}$, and the field L generated by iterated logarithms, in the presence of Schanuel's conjecture. We also obtain similar results replacing exp by the modular j-function, under an appropriate version of Schanuel's conjecture, where linear disjointness is replaced by a notion coming from the action of GL_2 on $\mathbb{C}$. We also show that for certain choices of F we obtain unconditional versions of these statements.

Keywords Complex exponential · j-Function · Linear disjointness · G-disjointness · Schanuel's conjecture · Ax–Schanuel

Mathematics Subject Classification 11F03 · 11J81 · 11J85 · 11J89

1 Introduction

Let $\exp(z)$ denote the usual complex exponential function ($\exp(z) = e^z$). Set $E_0 = L_0 = \overline{\mathbb{Q}}$ and for every positive integer n define

$$E_n := \overline{E_{n-1}(\{\exp(x) \mid x \in E_{n-1}\})} \quad \text{and} \quad L_n := \overline{L_{n-1}(\{x \mid \exp(x) \in L_{n-1}\})},$$

and let $E := \bigcup_{n=1}^{\infty} E_n$ and $L := \bigcup_{n=1}^{\infty} L_n$. It is shown in [6] that Schanuel's conjecture implies that E and L are linearly disjoint over $\overline{\mathbb{Q}}$. Similar results have been obtained

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more recently in [5, 15] where $\exp(z)$ is replaced by the exponential of an (semi-) abelian variety.

The aim of this paper is to do three things. The first objective is, still assuming Schanuel's conjecture, to find more general finitely generated subfields F of $\mathbb{C}$ such that if we set the initial step of the towers E_0 and L_0 to be the algebraic closure of F, then the resulting fields E and L are linearly disjoint over $\overline{F}$. This is achieved in Theorem 4.2.

The second objective is to show that under a version of Schanuel's conjecture for the modular j-function (that we call MSCD), one can produce an analogous result for the modular j-function. More specifically, set $J_0 = K_0 = \overline{F}$, where F is a finitely generated subfield of $\mathbb{C}$, and inductively define

$$J_n := \overline{J_{n-1}(\{j(z) \mid z \in J_{n-1} \cap \mathbb{H}^+\})} \quad \text{and} \quad K_n := \overline{K_{n-1}(\{z \in \mathbb{H}^+ \mid j(z) \in K_{n-1}\})}.$$

Set $J := \bigcup_{n=1}^{\infty} J_n$ and $K := \bigcup_{n=1}^{\infty} K_n$. Unlike exp, the j-function is not a group homomorphism, and so linear disjointness is not the right notion for proving a relation between J and K. Instead we will define a notion of disjointness that previously appeared in [7] that considers the action of the group $G := \mathrm{GL}_2$. This is achieved in Theorem 5.3

The final objective is to find initial fields F so that the linear disjointness of E and L is obtained unconditionally, that is, without having to rely on Schanuel's conjecture. We also obtain an analogous unconditional result for j. This is done in Theorems 4.4 and 5.4.

The methods used to prove all of our main results rely mostly on the work of [1] on *convenient generators*, which in turn rely heavily on the so-called Ax-Schanuel theorems: [2] in the case of exp and [16] in the case of j. We expect that similar constructions can be performed whenever an Ax-Schanuel theorem is available in differential form, and so the methods presented here can be expected to extend to other situations, such as the exponential maps of semi-abelian varieties (using [3] or [9]) or the uniformization maps of Shimura varieties (using [14]).

1.1 Structure of the paper

In Sect. 2, we introduce preliminary definitions and results regarding linear disjointness and G-disjointness. In Sect. 3, we review several Schanuel-type inequalities, in particular the Modular Schanuel Conjecture, and discuss the details of convenient generators. We prove the main results of this paper in Sects. 4 and 5.

2 Preliminaries

2.1 Basic notation

- If F is any subfield of $\mathbb{C}$, then $\overline{F}$ it is algebraic closure in $\mathbb{C}$.

- If $x_1, \ldots, x_m$ are elements of $\mathbb{C}$, then we use $\mathbf{x}$ to denote $(x_1, \ldots, x_m)$. Furthermore, if f denotes a function, then we write $f(\mathbf{x})$ to mean $(f(x_1), \ldots, f(x_m))$.
- As mentioned in the introduction, let G denote the linear group GL_2. For any subfield F of $\mathbb{C}$ there is a natural action of $G(F)$ on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ given by Möbius transformations as follows:
$$gx := \frac{ax+b}{cx+d}, \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
with $g \in G(F)$. Whenever we say that $G(F)$ *acts on* $\mathbb{C}$, it will be in this manner.

2.2 Closures, dimensions and disjointness

In this section we introduce the various notions of disjointess that will be used for our main results, and we review some of their properties.

Definition Let $F \subseteq \mathbb{C}$ be a subfield and let $A \subset \mathbb{C}$ be a finite subset.

(a) Thinking of $\mathbb{C}$ as an F-vector space, we denote by $\mathrm{l.dim}_F(A)$ the dimension of the F-vector space $\mathrm{Span}_F(A)$, the F-linear span of A.
Given another subset $B \subseteq \mathbb{C}$, let $\pi : \mathbb{C} \to \mathbb{C}/\mathrm{Span}_F(B)$ denote the quotient map. We write $\mathrm{l.dim}_F(A|B)$ to denote the dimension of the F-vector space $\pi(\mathrm{Span}_F(A))$.

(b) Considering the action of $G(F)$ on $\mathbb{C}$, we denote by $\dim_{G(F)}(A)$ the number of distinct $G(F)$-orbits generated by elements of A. We say that A is $G(F)$-*independent* if $|A| = \dim_{G(F)}(A)$.
Given another subset $B \subseteq \mathbb{C}$, we denote by $\dim_{G(F)}(A|B)$ the number of distinct $G(F)$-orbits generated by elements of A which do not contain elements from B.

Definition Let E, F, L be subfields of $\mathbb{C}$ such that $E \subseteq F \cap L$.

(a) F is *linearly disjoint from* L *over* E, denoted $F \perp^l_E L$, if every finite tuple of elements in L that is E-linearly independent is also F-linearly independent. Equivalently, $F \perp^l_E L$ if and only if for any tuple $\boldsymbol{\ell}$ from L, $\mathrm{l.dim}_F(\boldsymbol{\ell}) = \mathrm{l.dim}_E(\boldsymbol{\ell})$.

(b) F and L are E-*free*, denoted $F \perp^f_E L$, if every finite set of elements of L which is algebraically independent over E is also algebraically independent over F. Equivalently, $F \perp^f_E L$ if and only if for any tuple $\boldsymbol{\ell}$ from L, $\mathrm{tr.deg.}_F F(\boldsymbol{\ell}) = \mathrm{tr.deg.}_E E(\boldsymbol{\ell})$.

(c) We say that F is $G(E)$-*disjoint from* L, denoted $F \perp^G_E L$, if for every finite subset of elements of L that is $G(E)$-independent is also $G(F)$-independent. Equivalently, $F \perp^G_E L$ if and only if for any tuple $\boldsymbol{\ell}$ from L, $\dim_{G(F)}(\boldsymbol{\ell}) = \dim_{G(E)}(\boldsymbol{\ell})$.
Equivalently, $F \perp^G_E L$ if and only if for any pair of elements ℓ_1, ℓ_2 from L for which there exists $g \in G(F)$ such that $g\ell_1 = \ell_2$, there is $h \in G(E)$ such that $h\ell_1 = \ell_2$.

Remark 2.1 Although the definitions are not stated in a symmetric way, both $\perp^l$ and $\perp^f$ are symmetric relations (see [12, p. 360] and [12, p. 362]). This leads naturally to the following question (which seems to be open).

Question Is G-disjointness a symmetric relation, that is, is it true that for any three subfields $E, F, L \subseteq \mathbb{C}$ satisfying $E \subseteq F \cap L$ we have $F \perp_E^G L \iff L \perp_E^G F$?

While the general case remains open, the following lemma provides a special case in which symmetry holds:

Lemma 2.2 *Suppose that $E = L \cap F$. If F is linearly disjoint from L over E, then $F \perp_E^G L$ and $L \perp_E^G F$.*

Proof We will show that $F \perp_E^l L$ implies $F \perp_E^G L$, and since linear disjointness is a symmetric condition, this is enough. Suppose that $\ell_1, \ell_2 \in L$ and $g \in G(F)$, such that $g\ell_1 = \ell_2$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a, b, c, d \in F$ satisfying $ad \neq bc$. Then we have $0 = c\ell_1\ell_2 - a\ell_1 + d\ell_2 - b$, which is a non-trivial F-linear combination of the elements $1, \ell_1, \ell_2, \ell_1\ell_2$ (all of which are in L). As F is linearly disjoint from L over E, there exist $\alpha, \beta, \gamma, \delta \in E$ (not all zero) preserving the linear dependence:

$$0 = \gamma\ell_1\ell_2 - \alpha\ell_1 + \delta\ell_2 - \beta.$$

We now consider the possible cases.

(a) Suppose that $\alpha\ell_1 + \beta = 0$. It follows that $\ell_1 \in E$, and as $g \in G(F)$, then $\ell_2 = g\ell_1 \in F$. This shows that $\ell_2 \in L \cap F$. Since $\ell_1, \ell_2 \in E$, then there exists $h \in G(E)$ such that $h\ell_1 = \ell_2$.
(b) Suppose that $\gamma\ell_1 + \delta = 0$. Again it follows that $\ell_1 \in E$, so we conclude as in (a).
(c) Suppose that $\alpha\ell_1 + \beta \neq 0$, $\gamma\ell_1 + \delta \neq 0$, and $\alpha\delta - \beta\gamma = 0$. Then there exists $e \in E$ such that $\alpha = e\gamma$ and $\beta = e\delta$. Then $\ell_2 = \frac{\alpha\ell_1+\beta}{\gamma\ell_1+\delta} = e$. Thus $\ell_2 \in E$, and we can conclude with a similar argument to (a).
(d) If none of the cases (a), (b), (c) are satisfied, then the matrix $h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is in $G(E)$ and satisfies $h\ell_1 = \ell_2$.

□

Finally, we will make use of the following lemma throughout Sect. 5:

Lemma 2.3 *If $F \perp_E^G L$, then $F \cap L = E$.*

Proof Suppose $t \in L \setminus E$, and by way of contradiction suppose $t \in F$. It follows that $g = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \in G(F)$. Since $g1 = t$ and we are assuming $F \perp_E^G L$, there exists $h \in G(E)$ such that $h1 = t$. But t is not in E, which is a contradiction. □

2.3 The *j*-function

First, we define a complex lattice, $\Lambda \subseteq \mathbb{C}$ is the additive subgroup of $\mathbb{C}$ generated by $\omega_1, \omega_2 \in \mathbb{C}$. In otherwords, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. An elliptic curve over $\mathbb{C}$ is $E_\Lambda := \mathbb{C}/\Lambda$. Given two elliptic curves E_Λ and $E_{\Lambda'}$, an isogeny is a nonzero analytic homomorphism mapping of the elliptic curves. There exists an isogeny between any two elliptic curves E_Λ and $E_{\Lambda'}$ if and only if there exists a nonzero $z \in \mathbb{C}$ such that $z\Lambda = \Lambda'$. Hence studying the $G(\mathbb{Q})$ disjointness of the lattices provides insight into the isogenies between the corresponding elliptic curves. This motivates the following definition:

Definition Let $j : \mathbb{H}^+ \to \mathbb{C}$ be the holomorphic map given by $j(z) = j(E_{\Lambda_z})$, where Λ_z is the integer lattice in $\mathbb{C}$, formed using z and $j(E_{\Lambda_z})$ is the j-invariant of E_{Λ_z}.

It is well-known (see e.g. [13, p. 20]) that j satisfies the following differential equation (and none of lower order):

$$0 = \frac{j'''}{j'} - \frac{3}{2}\left(\frac{j''}{j'}\right)^2 + \frac{j^2 - 1968j + 2654208}{j^2(j-1728)^2}\left(j'\right)^2. \tag{2.1}$$

We recall that there is a family of polynomials $\{\Phi_N(X,Y)\}_{N=1}^{\infty} \subseteq \mathbb{Z}[X,Y]$ called the *modular polynomials* associated with j (see [11, Chap. 5, Sect. 2] for definitions and properties). Each $\Phi_N(X,Y)$ is irreducible in $\mathbb{C}[X,Y]$, $\Phi_1(X,Y) = X - Y$, and for $N \geq 2$, $\Phi_N(X,Y)$ is symmetric of total degree $\geq 2N$.

We will often make use of the following fact. For every g in $G(\mathbb{Q})$ we can define $\widetilde{g}$ as the unique matrix of the form rg with $r \in \mathbb{Q}$ and $r > 0$, so that the entries of $\widetilde{g}$ are all integers and relatively prime. Then, for every x and y in $\mathbb{H}$ the following statements are equivalent:

- $\Phi_N(j(x), j(y)) = 0$;
- There exists g in G with $gx = y$ and $\det(\widetilde{g}) = N$.

Definition A point $z \in \mathbb{H}$ is said to be *special* if there exists a non-scalar matrix $g \in G(\mathbb{Q})$, such that $gx = x$. We denote the set of all special points as Σ.

Remark A classical theorem of Schneider [17] states that $\text{tr.deg.}_{\mathbb{Q}}\mathbb{Q}(x, j(x)) = 0$ if and only if $x \in \Sigma$.

3 Schanuel-type conjectures

We start by recalling the now classical conjecture of Schanuel on complex exponentiation.

Conjecture 3.1 (Schanuel: SC) *For every* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n$ *we have*

$$\text{tr.deg.}_{\mathbb{Q}}\mathbb{Q}(\mathbf{x}, \exp(\mathbf{x})) \geq \text{l.dim}_{\mathbb{Q}}(\mathbf{x}). \tag{3.1}$$

We remark that SC gives an inequality statement for the transcendence degree measured over $\mathbb{Q}$. Since one of our objectives is to obtain results about linear disjointness over arbitrary fields, we need to first find a version of SC which works over a given finitely generated field. This will require the use of "convenient generators" for the fields, and the details will be explained in the next section. First we recall variants of SC for the j-function. For a detail of the origins of these variants, see [1, Sect. 6.3] and references therein.

Conjecture 3.2 (Modular Schanuel with derivative: MSCD) *For every* $z_1, \ldots, z_n \in \mathbb{H}^+$ *we have:*

$$\operatorname{tr.deg.}_{\mathbb{Q}}\mathbb{Q}\left(\mathbf{z}, j(\mathbf{z}), j'(\mathbf{z}), j''(\mathbf{z})\right) \geq 3 \dim_{G(\mathbb{Q})}(\mathbf{z}|\Sigma).$$

It is easy to see that MSCD implies the following statement without derivatives.

Conjecture 3.3 (Modular Schanuel) *For every* $z_1, \ldots, z_n \in \mathbb{H}^+$ *we have:*

$$\operatorname{tr.deg.}_{\mathbb{Q}}\mathbb{Q}\left(\mathbf{z}, j(\mathbf{z})\right) \geq \dim_{G(\mathbb{Q})}(\mathbf{z}|\Sigma).$$

3.1 Field derivations

Definition A map $\partial : \mathbb{C} \to \mathbb{C}$ is a called a *derivation* if it satisfies the following two conditions:

(1) $\partial(a+b) = \partial(a) + \partial(b)$ for every $a, b \in \mathbb{C}$.
(2) $\partial(ab) = a\partial(b) + b\partial(a)$ for every $a, b \in \mathbb{C}$.

A derivation $\partial : \mathbb{C} \to \mathbb{C}$ is called an *exponential derivation* if it satisfies:

$$\partial(\exp(z)) = \exp(z)\partial(z)$$

for all $z \in \mathbb{C}$. Let EDer denote the set of all exponential derivations.

A derivation $\partial : \mathbb{C} \to \mathbb{C}$ is called a j*-derivation* if it satisfies[1]:

$$\partial(j(z)) = j'(z)\partial(z) \wedge \partial(j'(z)) = j''(z)\partial(z) \wedge \partial(j''(z)) = j'''(z)\partial(z),$$

for all $z \in \mathbb{H}^+$. Let jDer denote the set of all j-derivations.

Define

$$C_{\exp} := \bigcap_{\partial \in \mathrm{EDer}} \ker \partial \quad \text{and} \quad C_j := \bigcap_{\partial \in j\mathrm{Der}} \ker \partial.$$

Using some techniques from o-minimality, one can show that there are $|\mathbb{C}|$-many $\mathbb{C}$-linearly independent exponential derivations, and the same is true about j-derivations (see [4] for the details in the case of exp and [7, Sect. 5] for the case of j). One can find more explicit descriptions of the sets $C_{\exp}$ and C_j by using *Khovanskii systems* of equations (see [1, Sect. 6] for j and [10, Sect. 3] for exp).

These types of derivations can also be used to define certain closure operators called *pregeometries* which have associated well-defined notions of dimension (see [18, Appendix C] for the basic definitions and properties concerning pregeometries).

1 As shown in [7, Sect. 5] these conditions already imply that ∂ will respect all the derivatives of j.

Definition Let $A \subseteq \mathbb{C}$ be any set. We define the set $e\mathrm{cl}(A)$ by the property: $x \in e\mathrm{cl}(A)$ if and only if $\partial(x) = 0$ for every exponential derivation ∂ with $A \subseteq \ker \partial$. If $A = e\mathrm{cl}(A)$, then we say that A is $e$cl-*closed*.

Similarly, we define the set $j\mathrm{cl}(A)$ by the property: $x \in j\mathrm{cl}(A)$ if and only if $\partial(x) = 0$ for every j-derivation ∂ with $A \subseteq \ker \partial$. If $A = j\mathrm{cl}(A)$, then we say that A is $j$cl-*closed*.

Every $e$cl-closed and every $j$cl-closed subset of $\mathbb{C}$ is an algebraically closed subfield. We denote by $\dim^e$ the dimension associated with $e$cl, and by $\dim^j$ the dimension associated with $j$cl. For reference, $\dim^e$ can be defined in the following way. For any subsets $A, B \subseteq \mathbb{C}$ and for every non-negative integer n, $\dim^e(A|B) \geq n$ if and only if there exist $a_1, \ldots, a_n \in e\mathrm{cl}(A)$ and $\partial_1, \ldots, \partial_n \in \mathrm{EDer}$ such that $B \subseteq \ker \partial_i$ for $i = 1, \ldots, n$ and

$$\partial_i\,(a_k) = \begin{cases} 1\ , & \text{if } i = k \\ 0\ , & \text{else} \end{cases}$$

for every $i, k = 1, \ldots, n$. The dimension $\dim^j$ can be defined in an analogous way.

Lemma 3.4 (see [10] and [7]) *$C_{\exp}$ and C_j are countable algebraically closed subfields of $\mathbb{C}$. Furthermore,*

(a) *For every $z \in \mathbb{C}$, z is in $C_{\exp}$ if and only if* $\exp(z)$ *is in $C_{\exp}$.*
(b) *For every $z \in \mathbb{H}^+$, z is in C_j if and only if $j(z)$, $j'(z)$ or $j''(z)$ is in C_j.*

3.2 Convenient generators

Definition We will say that a tuple $\mathbf{t} = (t_1, \ldots, t_m)$ of elements of $\mathbb{C}$ is *convenient for* exp if

$$\mathrm{tr.deg.}_{\mathbb{Q}}\mathbb{Q}(\mathbf{t}, \exp(\mathbf{t})) = \mathrm{l.dim}_{\mathbb{Q}}(\mathbf{t}) + \dim^e(\mathbf{t}).$$

We will say that a tuple $\mathbf{t} = (t_1, \ldots, t_m)$ of elements of $\mathbb{H}^+$ is *convenient for* j if

$$\mathrm{tr.deg.}_{\mathbb{Q}}\mathbb{Q}(\mathbf{t}, J(\mathbf{t})) = 3\dim_G(\mathbf{t}|\Sigma) + \dim^j(\mathbf{t}).$$

Convenient tuples allow us to get Schanuel-type inequalities.

Lemma 3.5 *Suppose $\mathbf{t} \in \mathbb{C}^m$ is convenient for* exp. *Set $F = \mathbb{Q}(\mathbf{t}, \exp(\mathbf{t}))$. Then SC implies that for any $\mathbf{x} = (x_1, \ldots x_n) \in \mathbb{C}^n$ we have:*

$$\mathrm{tr.deg.}_F F(\mathbf{x}, \exp(\mathbf{x})) \geq \mathrm{l.dim}_{\mathbb{Q}}(\mathbf{x}|\mathbf{t}).$$

Proof By SC we have that

$$\mathrm{tr.deg.}_{\mathbb{Q}}\mathbb{Q}(\mathbf{x}, \mathbf{t}, \exp(\mathbf{x}), \exp(\mathbf{t})) \geq \mathrm{l.dim}_{\mathbb{Q}}(\mathbf{x}, \mathbf{t}).$$

Using the addition formula and the fact that $\mathbf{t}$ is convenient for exp as in the proof of [8, Lemma 4.9], we get the result. $\square$

An analogous statement for j can be found in [8, Lemma 4.9]. Of course, we need to address the question of whether convenient tuples exist. For the case of j this was shown in [8, Lemma 4.13], under the assumption of MSCD. A similar proof gives us the result for exp.

Proposition 3.6 *Let $F \subset \mathbb{C}$ be a subfield such that* $\operatorname{tr.deg.}_{\mathbb{Q}} F$ *is finite. Then SC implies that there exist* $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{C}^m$ *such that*

(c1) $F \subseteq \overline{\mathbb{Q}(\mathbf{t}, \exp(\mathbf{t}))}$*, and*
(c2) $\mathbf{t}$ *is convenient for* exp.

Furthermore, without loss of generality we may assume that $\operatorname{l.dim}_{\mathbb{Q}}(\mathbf{t}) = m$.

Proof By [1, Theorem 5.6] there exist $\mathbf{t}_1 := t_1, \ldots, t_k \in \mathbb{C} \backslash C_{\exp}$ such that

(a) $K \subseteq \overline{C_{\exp}(\mathbf{t}_1, \exp(\mathbf{t}_1))}$,
(b) $\operatorname{tr.deg.}_{C_{\exp}} C_{\exp}(\mathbf{t}_1, \exp(\mathbf{t}_1)) = \operatorname{l.dim}_{\mathbb{Q}}\left(\mathbf{t}_1 | C_{\exp}\right) + \dim^e(\mathbf{t}_1)$.

As $\operatorname{tr.deg.}_{C_{\exp}} C_{\exp}(\mathbf{t}_1, \exp(\mathbf{t}_1))$ is finite, then there is a finitely generated field $L \subseteq C$ such that $\operatorname{tr.deg.}_{C_{\exp}} C_{\exp}(\mathbf{t}_1, \exp(\mathbf{t}_1)) = \operatorname{tr.deg.}_L L(\mathbf{t}_1, \exp(\mathbf{t}_1))$. As F is finitely generated, if M denotes the compositum of L and $F \cap C_{\exp}$, then M has finite transcendence degree over $\mathbb{Q}$.

Claim 3.7 *SC implies that there exist* $\mathbf{t}_2 = t_{k+1}, \ldots, t_m \in \mathbb{C} \cap C_{\exp}$ *such that*

(i) $M \subseteq \overline{\mathbb{Q}(\mathbf{t}_2, \exp(\mathbf{t}_2))}$*, and*
(ii) $\mathbf{t}_2$ *is convenient for* exp.

Proof If $M \subseteq \overline{\mathbb{Q}}$, then we are done. So suppose that M has positive transcendence degree over $\mathbb{Q}$, and let T be a transcendence basis for M over $\mathbb{Q}$. As $M \subseteq C_{\exp}$, then by [10, Theorem 1.1], for every $y \in T$ there are $y = y_1, \ldots, y_n \in C_{\exp}$ such that they are a solution to a Khovanskii system of exponential polynomials. Although the definition of Khovanskii systems used in [10, Sect. 3] allows the exponential polynomials to have iterated exponentials, we appeal to [1, Remark 6.1] to ensure that no iterations of exp occur (by increasing the number of variables if necessary). Thus the Khovanskii system we obtain is made up of polynomials $p_1, \ldots, p_n \in \mathbb{Q}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ so that if we set $f_i(Z_1, \ldots, Z_n) := p_i(Z_1, \ldots, Z_n, \exp(Z_1), \ldots, \exp(Z_n))$, then

$$f_i(y_1, \ldots, y_n) = 0 \quad \text{for all } i \in \{1, \ldots, n\},$$

and

$$\begin{vmatrix} \frac{\partial f_1}{\partial Z_1} & \cdots & \frac{\partial f_1}{\partial Z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial Z_1} & \cdots & \frac{\partial f_n}{\partial Z_n} \end{vmatrix} (y_1, \ldots, y_n) \neq 0.$$

If we choose n minimal with this property, we can guarantee that $\mathrm{l.dim}_{\mathbb{Q}}(y_1, \ldots, y_n) = n$. Having this Khovanskii system guarantees that

$$\mathrm{tr.deg.}_{\mathbb{Q}}\mathbb{Q}(y_1, \ldots, y_n, \exp(y_1), \ldots, \exp(y_n)) \leq n,$$

which combined with SC guarantees that $y_1, \ldots, y_n$ is convenient for exp. Since T is a transcendence basis, it's elements are $\mathbb{Q}$-linearly disjoint, so by repeating the above argument for every element of T and combining the solutions of the various Khovanskii systems, we get the desired tuple $\mathbf{t}_2$. □

Let $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$. By construction, the elements of $\mathbf{t}_1$ are linearly disjoint with element of $\mathbf{t}_2$. Condition (c1) is satisfied by (a) and (i). As L is contained in M, then condition (c2) is satisfied by (ii) and (b). □

4 Main results for exp

Throughout this section F will denote some specific choice of subfield of $\mathbb{C}$. Set $E_0 = L_0 = \overline{F}$, and then we define, as stated in the introduction, the towers of extensions

$$E_n := \overline{E_{n-1}(\{\exp(x) \mid x \in E_{n-1}\})} \quad \text{and} \quad L_n := \overline{L_{n-1}(\{x \mid \exp(x) \in L_{n-1}\})}.$$

Finally we define $E := \bigcup_{n=1}^{\infty} E_n$ and $L := \bigcup_{n=1}^{\infty} L_n$.

Lemma 4.1 *Let F be any subfield of $\mathbb{C}$. For all $x \in E_{n-1}$, there exists $A \subseteq E_{n-1}$ such that $A \cup \{x\}$ is algebraic over $F(\exp(A))$. Likewise, for any $x \in L_{n-1}$ there exists $C \subseteq \mathbb{C}$ such that $\exp(C) \subseteq L_{n-1}$ then $\exp(C) \cup \{x\}$ is algebraic over $F(C)$.*

Proof Repeat the proof of [6, Lemma], or see the proof of Lemma 5.2. □

Theorem 4.2 *Let $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{C}^s$ be a convenient tuple for* exp *and set $F := \mathbb{Q}(\mathbf{t}, \exp(\mathbf{t}))$. Assume SC is true. Then E is linearly disjoint from L over $\overline{F}$.*

Proof With all we have done, the proof is now a small modification of the one given in [6, Theorem] with the role of SC appearing in the form of Lemma 3.5. We proceed by induction and assume that E_{m-1} and L_n are linearly disjoint over $\overline{F}$. Suppose by way of contradiction that E_m and L_n are not linearly disjoint over $\overline{F}$, so take a finite subset $\{l_1, \ldots, l_k\} \subseteq L_n$, which is linearly independent over $\overline{F}$, and assume that there are $\{e_1, \ldots, e_k\} \subseteq E_m$ such that $\sum_{i=1}^{k} l_i e_i = 0$, where at least one $e_i \neq 0$. By Lemma 4.1 there exists a finite set $A \subseteq E_{m-1}$ such that $A \cup \{e_i\}_{i=1}^{k}$ is algebraic over $F(\exp(A))$, and a finite set $C \subseteq L_n$ such that $\exp(C) \cup \{l_i\}_{i=1}^{k}$ is algebraic over $F(C)$.

Now take $B \subseteq A$ and $D \subseteq C$ such that $\exp(B)$ is a transcendence basis for $F(\exp(A))$ over F and D is a transcendence basis for $F(C)$ over F. We first show that $\mathrm{l.dim}_{\mathbb{Q}}(B \cup D|\mathbf{t}) = |B| + |D|$. To this end, consider an expression of the form

$$\sum_{b \in B} p_b b + \sum_{d \in D} q_d d + \sum_{i=1}^{s} r_i t_i = 0 \tag{4.1}$$

with $p_b, q_d, r_i \in \mathbb{Z}$. Observe that $\sum_{b\in B} p_b b \in E_{m-1}$, $\sum_{d\in D} q_d d \in L_n$ and $\sum_{i=1}^{s} r_i t_i \in \overline{F}$, so (4.1) shows that $\sum_{b\in B} p_b b \in L_n$ and $\sum_{d\in D} q_d d \in E_{m-1}$. By the induction hypothesis, E_{m-1} and L_n are linearly disjoint over $\overline{F}$, so $E_{m-1} \cap L_n = \overline{F}$. Since D is a transcendence basis over F, the condition $\sum_{d\in D} q_d d \in \overline{F}$ implies that the coefficients $q_d = 0$ for each $d \in D$. Hence $\displaystyle\sum_{b\in B} p_b b + \sum_{i=1}^{s} r_i t_i = 0$ and thus $\displaystyle\prod_{b\in B} (\exp b)^{p_b} \in \overline{F}$. But $\exp B$ is a transcendence basis, so $p_b = 0$ for every $b \in B$. This proves that $\mathrm{l.dim}_{\mathbb{Q}}(B \cup D|\mathbf{t}) = |B| + |D|$.

By Lemma 3.5 we have $\mathrm{tr.deg.}_F F(B, D, \exp(B), \exp(D)) \geq |B| + |D|$. We also have that:

$$\begin{aligned}\mathrm{tr.deg.}_F F(B, D, \exp(B), \exp(D)) &= \mathrm{tr.deg.}_F F(D, \exp(B))\\ &\leq |B| + |D|.\end{aligned}$$

Therefore $\mathrm{tr.deg.}_F F(D, \exp(B)) = |B| + |D|$, so $\overline{F(\exp(B))}$ and $\overline{F(D)}$ are $\overline{F}$-free, and hence they are linearly disjoint over $\overline{F}$. Since $\{e_1, \ldots, e_k\} \subset \overline{F(\exp(B))}$ and $\{l_1, \ldots, l_k\} \subset \overline{F(D)}$, we reach a contradiction. □

4.1 Unconditional result

Let $t_1, \ldots, t_s \in \mathbb{C} \setminus C_{\exp}$ satisfy

$$\mathrm{tr.deg.}_{C_{\exp}} C_{\exp}(\mathbf{t}, \exp(\mathbf{t})) = \mathrm{l.dim}_{\mathbb{Q}}(\mathbf{t}|C_{\exp}) + \dim^e(\mathbf{t}).$$

As we explained in the proof of Proposition 3.6, the existence of tuples $\mathbf{t}$ satisfying the above equation is given by [1, Theorem 5.6]. Set $F:=C_{\exp}(\mathbf{t}, \exp(\mathbf{t}))$ and define E and L accordingly.

Lemma 4.3 *Then for any* $\mathbf{x} = (x_1, \ldots x_n) \in \mathbb{C}^n$ *we have:*

$$\mathrm{tr.deg.}_F F(\mathbf{x}, \exp(\mathbf{x})) \geq \mathrm{l.dim}_{\mathbb{Q}}(\mathbf{x}|\mathbf{t} \cup C_{\exp}) + \dim^e(\mathbf{x}|\mathbf{t}).$$

Proof By [10, Corollary 5.2] (a consequence of a theorem of Ax [2, Theorem 3]) we have that

$$\mathrm{tr.deg.}_{C_{\exp}} C_{\exp}(\mathbf{x}, \mathbf{t}, \exp(\mathbf{x}), \exp(\mathbf{t})) \geq \mathrm{l.dim}_{\mathbb{Q}}(\mathbf{x}, \mathbf{t}|C_{\exp}) + \dim^e(\mathbf{x}, \mathbf{t}|C_{\exp}).$$

Using the addition formula and the fact that $\mathbf{t}$ is convenient for exp as in the proof of [8, Lemma 5.2], we obtain the desired result. □

We can now prove the following unconditional version of Theorem 4.2.

Theorem 4.4 *With F as above, $E \perp^{l}_{\overline{F}} L$.*

Proof The proof is nearly the same as the proof of Theorem 4.2. In what follows we will only focus on the step that require extra attention. Instead of assuming SC, we will use Lemma 4.3.

As before, we assume that $E_{m-1} \perp^{l}_{\overline{F}} L_n$ and that E_m and L_n are not linearly disjoint over $\overline{F}$. Suppose the set $\{l_1, \dots, l_k\} \subseteq L_n$ is linearly independent over $\overline{F}$, and that there are $\{e_1, \dots, e_k\} \subseteq E_m$ such that $\sum_{i=1}^{k} l_i e_i = 0$, where some $e_i \neq 0$. Choose $A \subseteq E_{m-1}$ and $C \subseteq L_n$ using Lemma 4.1 just as before.

Take $B \subseteq A$ and $D \subseteq C$ as above. We show that $\mathrm{l.dim}_{\mathbb{Q}}(B \cup D | \mathbf{t} \cup C_{\exp}) = |B| + |D|$. This time we need to consider an expression of the form

$$\sum_{b \in B} p_b b + \sum_{d \in D} q_d d + \sum_{i=1}^{s} r_i t_i = \gamma, \tag{4.2}$$

for some $\gamma \in C_{\exp}$, with $p_b, q_d, r_i \in \mathbb{Z}$. We have that $\sum_{b \in B} p_b b \in E_{m-1}$, $\sum_{d \in D} q_d d \in L_n$ and $\gamma - \sum_{i=1}^{s} r_i t_i \in \overline{F}$, so (4.2) shows that $\sum_{b \in B} p_b b \in L_n$ and $\sum_{d \in D} q_d d \in E_{m-1}$. Use Lemma 3.4, apply the induction hypothesis and finish as before. □

5 Main results for *j*

Throughout this section F will a subfield of $\mathbb{C}$. As in the previous section, we start with $E_0 = L_0 = \overline{F}$ and then proceed inductively as follows:

$$J_n := \overline{J_{n-1}(\{j(z) \mid z \in J_{n-1} \cap \mathbb{H}^+\})} \quad \text{and} \quad K_n := \overline{K_{n-1}(\{z \in \mathbb{H}^+ \mid j(z) \in K_{n-1}\})},$$

and set $J := \bigcup_{n=1}^{\infty} J_n$ and $K := \bigcup_{n=1}^{\infty} K_n$. We will keep this notation for the rest of the section.

Remark 5.1 Consider $\mathbb{C}$ as a degree two extension of the field of real numbers $\mathbb{R}$ (and not as an abstract field). Let $L \subseteq \mathbb{C}$ be an algebraically closed subfield. Then for every $z \in \mathbb{C}$ we have that $z \in L$ if and only if the real and imaginary parts of z are in L. Indeed, choose $z \in L$ and write $z = a + ib$. Let $\partial : \mathbb{C} \to \mathbb{C}$ be any derivation satisfying that $L = \ker \partial$ (which exists since L is algebraically closed). Using [19, Sect. 4] we know that there are derivations $\lambda, \mu : \mathbb{R} \to \mathbb{R}$ such that:

$$0 = \partial(z) = \lambda(a) - \mu(b) + i(\lambda(a) + \mu(b)),$$

which gives that $\lambda(a) = \mu(b) = 0$. So $\partial(a) = \lambda(a) + i\lambda(a) = 0$ and similarly $\partial(b) = 0$. Therefore $a, b \in L$.

Lemma 5.2 *For any $x \in J_n$, there exists a finite set $T \subseteq J_{n-1} \cap \mathbb{H}^+$ such that $T \cup \{x\}$ is algebraic over $F(j(T))$.*

Likewise, for all $x \in K_n$, there exists a finite set $R \subseteq \mathbb{H}^+$ such that $j(R) \cup \{x\}$ is algebraic over $F(R)$ and for every $z \in R$, $j(z) \in K_{n-1}$.

Proof By Remark 5.1, given $x \in J_n$ there is a finite set $S_{n-1} \subset J_{n-1} \cap \mathbb{H}^+$ (possibly empty) such that the set $T_{n-1} = \{j(z) \mid z \in S_{n-1}\}$ is contained in $\mathbb{H}^+$ and x is algebraic over $J_{n-1}(T_{n-1})$. Similarly, given $0 \leq i \leq n$ and a finite set $T_{n-i} \subset J_{n-i}$, there is a finite set $S_{n-i-1} \subset J_{n-i} \cap \mathbb{H}^+$ such that the set $T_{n-i-1} = \{j(z) \mid z \in S_{n-i-1}\}$ is contained in $\mathbb{H}^+$ and T_{n-1} is algebraic over $J_{n-i-1}(T_{n-i-1})$. So we can proceed inductively to obtain finite sets $T_0, \ldots, T_{n-1}$ such that if we set $T = \bigcup_{m<n} T_m$, then $T \cup \{x\}$ is algebraic over $F(j(T))$.

We now seek to prove the existence of the finite set R with the desired properties. Using Remark 5.1, given $x \in K_n$ there is a finite set $R_{n-1} \subset \mathbb{H}^+$ such that for every $z \in R_{n-1}$, $j(z) \in K_{n-1}$ and x is algebraic over $K_{n-1}(R_{n-1})$. Proceeding in analogous way to the previous paragraph, we are done. $\square$

Theorem 5.3 *Suppose that* $\mathbf{t} = (t_1, \ldots, t_s) \in \left(\mathbb{H}^+\right)^s$ *is a convenient tuple for* j. *Set*

$$F := \mathbb{Q}(\mathbf{t}, j(\mathbf{t}), j'(\mathbf{t}), j''(\mathbf{t})).$$

Assume MSCD is true. Then $J \perp^G_F K$ *and* $K \perp^G_F J$.

Proof We follow the same overall strategy as in the proof of Theorem 4.2. We show that $J \perp^G_F K$ (the proof of $K \perp^G_F J$ is the same with the roles of J and K reversed).

We proceed by induction, assuming J_{m-1} and K_n are $G\left(\overline{F}\right)$-disjoint. We wish to prove that J_m and K_n are $G\left(\overline{F}\right)$-disjoint. Proceed by contradiction. Suppose $k_1, k_2 \in K_n$ are in different $G\left(\overline{F}\right)$-orbits, but that there is $g \in G(J_m)$ such that $gk_1 = k_2$. Let $a, b, c, d \in J_m$ be the coefficients of g (in the usual way). Then by Lemma 5.2, we have that there exists a finite set $T \subseteq J_{n-1} \cap \mathbb{H}^+$ such that all the elements of $T \cup \{a, b, c, d\}$ are algebraic over $F(j(T))$. Similarly, there exists a finite set $R \subseteq \mathbb{H}^+$ such that $j(R) \cup \{k_1, k_2\}$ is algebraic over $F(R)$ and for every $z \in R$, $j(z) \in K_{n-1}$.

Let $P \subseteq T$ be a subset such that the set $j(P)$ is a transcendence basis for $F(j(T))$ over F, and let $Q \subseteq R$ be a transcendence basis for $F(R)$ over F. The definitions of P and Q immediately imply that $(P \cup Q) \cap \Sigma = \emptyset$. We will show that $\dim_{G(\mathbb{Q})}(P \cup Q|\mathbf{t}) = |P| + |Q|$. Choose two elements $x, y \in P \cup Q$. We consider the following cases.

(a) Suppose that $x, y \in P$. If there is $g \in G(\mathbb{Q})$ such that $gx = y$, then there exists a modular polynomial $\Phi_N(X, Y)$, such that $\Phi_N(j(x), j(y)) = 0$, which shows that $j(x)$ and $j(y)$ are algebraically dependent over $\mathbb{Q}$. But this contradicts that $j(P)$ is a transcendence basis.
(b) Suppose that $x, y \in Q$. If these elements were in the same $G(\mathbb{Q})$-orbit, this would contradict that Q is a transcendence basis.
(c) Suppose that $x \in P$ and $y \in Q$. If there is $g \in G(\mathbb{Q})$ such that $gx = y$, this implies that x and y are in $J_{m-1} \cap K_n$. By Lemma 2.3 this implies that $x, y \in \overline{F}$, but that contradicts that Q is a transcendence basis over F.

This shows that $\dim_{G(\mathbb{Q})}(P \cup Q) = |P| + |Q|$. Now suppose that there is $x \in P \cup Q$ and $g \in G(\mathbb{Q})$ such that $gx = t_i$ for some $i \in \{1, \ldots, s\}$. This implies that x is not transcendental over F, so $x \notin Q$. But then it must be that $x \in P$, and as $gx = t_i$, the set $\{x, j(x)\}$ is algebraic over $\mathbb{Q}(t_i, j(t_i))$. This contradicts that $j(P)$ is a transcendence basis over F. Therefore $\dim_{G(\mathbb{Q})}(P \cup Q|\mathbf{t}) = |P| + |Q|$. Then by [8, Lemma 4.9] (which assumes MSCD):

$$\text{tr.deg.}_F F(P, j(P), Q, j(Q)) \geq |P| + |Q|.$$

We also have that

$$\begin{aligned}\text{tr.deg.}_F F(P, j(P), Q, j(Q)) &= \text{tr.deg.}_F F(P, R, j(T), j(Q)) \\ &= \text{tr.deg.}_F F(R, j(T)) \\ &= \text{tr.deg.}_F F(Q, j(P)),\end{aligned}$$

thus

$$\text{tr.deg.}_F F(P, j(P), Q, j(Q)) \leq |P| + |Q|.$$

Hence $F(j(P))$ and $F(Q)$ are $\overline{F}$-free, so $\overline{F(j(P))}$ and $\overline{F(Q)}$ are $\overline{F}$-free. Freedom over $\overline{F}$ implies that the fields are linearly disjoint over $\overline{F}$. Applying Lemma 2.2, it follows that $\overline{F(j(P))}$ and $\overline{F(Q)}$ are $G\left(\overline{F}\right)$-disjoint, which is a contradiction. □

In [6] it is shown that as a consequence of the main theorem, one can show (among other things) that the numbers π, $\log(\pi)$, $\log(\log(\pi))$, ... are E-linearly independent, where E is constructed as in Sect. 4 with $E_0 = \overline{\mathbb{Q}}$. However, obtaining a similar result about the j function and π is not expected. As shown in [1, Remark 6.21], the generalized period conjecture of Grothendieck–André which implies MSCD, also implies that $\pi \notin C_j$. Choosing $F = \overline{\mathbb{Q}}$ will give that $J, K \subseteq C_j$ (see [1, Proposition 6.6]), so this prevents π from being in either J or K. Of course, one can find elements in C_j which would make the statement true in place of π (as well as the other corollaries of [6]), but since these numbers are not so natural we have decided not to pursue this.

5.1 Unconditional result

Suppose that $t_1, \ldots, t_s \in \mathbb{H}^+ \setminus C_j$ satisfy:

$$\text{tr.deg.}_{C_j} C_j(\mathbf{t}, j(\mathbf{t})) = \dim_{G(\mathbb{Q})}(\mathbf{t}|C_j) + \dim^j(\mathbf{t}).$$

Set $F{:=}C_j(\mathbf{t}, j(\mathbf{t}), j'(\mathbf{t}), j''(\mathbf{t}))$, and define J and K as before with $J_0 = K_0 = F$.

Theorem 5.4 *With F as above, $J \perp^G_F K$ and $K \perp^G_F J$.*

Proof We only focus on the differences with the proof of Theorem 5.3. We assume J_{m-1} and K_n are $G\left(\overline{F}\right)$-disjoint and suppose $k_1, k_2 \in K_n$ are in different $G\left(\overline{F}\right)$-orbits, but that there is $g \in G(J_m)$ such that $gk_1 = k_2$. Let $a, b, c, d \in J_m$ be the coefficients of g. Choose $T \subseteq J_{n-1} \cap \mathbb{H}^+$ and $R \subseteq \mathbb{H}^+$ using Lemma 5.2 as before.

Let $P \subseteq T$ be a such that $j(P)$ is a transcendence basis for $F(j(T))$ over F, and let $Q \subseteq R$ be a transcendence basis for $F(R)$ over F. We will show that $\dim_{G(\mathbb{Q})}(P \cup Q|\mathbf{t} \cup C_j) = |P| + |Q|$. The same proof from Theorem 5.3 shows that $\dim_{G(\mathbb{Q})}(P \cup Q|\mathbf{t}) =$

$|P| + |Q|$. Now suppose that there is $x \in P \cup Q$ and $g \in G(\mathbb{Q})$ such that $gx \in C_j$. If $x \in Q$, that contradicts that Q is a transcendence basis over F. So then we would have that $x \in P$, but since $x \in C_j$ impies $j(x) \in C_j$ (by Lemma 3.4), this contradicts that $j(P)$ is a transcendence basis over F. So $\dim_{G(\mathbb{Q})}(P \cup Q|\mathbf{t} \cup C_j) = |P| + |Q|$.

Then by [1, Lemma 5.2]:

$$\mathrm{tr.deg.}_F F(P, j(P), Q, j(Q)) \geq |P| + |Q|.$$

The rest of the proof stays the same. □

Declarations

Conflict of interest The authors state that there is no conflict of interest.

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