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EXTREME VALUE THEORY FOR A SEQUENCE OF SUPREMA OF A CLASS OF GAUSSIAN PROCESSES WITH TREND

LANPENG JI AND XIAOFAN PENG

Abstract: We investigate extreme value theory of a class of random sequences defined by the all-time suprema of aggregated self-similar Gaussian processes with trend. This study is motivated by its potential applications in various areas and its theoretical interestingness. We consider both stationary sequences and non-stationary sequences obtained by considering whether the trend functions are identical or not. We show that a sequence of suitably normalised k th order statistics converges in distribution to a limiting random variable which can be a negative log transformed Erlang distributed random variable, a Normal random variable or a mixture of them, according to three conditions deduced through the model parameters. Remarkably, this phenomenon resembles that for the stationary Normal sequence. We also show that various moments of the normalised k th order statistics converge to the moments of the corresponding limiting random variable. The obtained results enable us to analyze various properties of these random sequences, which reveals the interesting particularities of this class of random sequences in extreme value theory.

Key Words: Extreme value; self-similarity; Gaussian processes; fractional Brownian motion; generalized Weibull-like distribution; moments; Pickands constant; Poisson convergence; order statistics; phantom distribution function; extremal index.

AMS Classification: Primary 60G15; secondary 60G70

1. INTRODUCTION

Let $\{X_i(t), t \geq 0\}, i = 1, 2, \dots$, be independent copies of a centered self-similar Gaussian process with almost surely (a.s.) continuous sample paths, self-similarity index $H \in (0, 1)$ and variance function t^{2H} , and let $\{X(t), t \geq 0\}$ be another independent centered self-similar Gaussian processes with a.s. continuous sample paths, self-similarity index $H_0 \in (0, 1)$ and variance function t^{2H_0} . We refer to the recent contribution [9] for a nice discussion on properties and examples of self-similar Gaussian processes. We define, for positive constants $\sigma, \sigma_0, c_i, i \geq 1$ and $\beta > \max(H, H_0)$,

$$(1) \quad Q_i := \sup_{t \geq 0} (\sigma X_i(t) + \sigma_0 X(t) - c_i t^\beta), \quad i = 1, 2, \dots$$

Studies on distributional properties of a single extremum, Q_i , or joint extrema (e.g., (Q_1, \dots, Q_d) for some $d \in \mathbb{N}$) have attracted growing interest in recent literature. On one side, it is a natural object of interest in the extreme value theory of stochastic processes. On the other side, strong motivation for this investigation stems, for example, from (multivariate) stochastic models applied to modern risk theory, advanced communication networks and financial mathematics, to name some of the applied-probability areas. Since explicit formulas for the (joint) distributions are out of reach except for some very special cases, current studies have been focused on deriving (joint) tail asymptotics for these (joint) extrema; see e.g., [8, 15, 22, 25] on the single extremum and [12, 13, 26] on the joint extrema of some Gaussian processes.

In this paper, we are interested in developing extreme value theory for the sequence of random variables $\{Q_i\}_{i \geq 1}$ defined in (1). Precisely, denoting the k th largest value (or k th order statistics) of Q_1, \dots, Q_n by

$$(2) \quad M_n^{(k)} := \max_{i \leq n}^{(k)} Q_i, \quad 1 \leq k < n \in \mathbb{N},$$

we aim to establish limit theorems for $e_n^{-1}(M_n^{(k)} - d_n)$, as $n \rightarrow \infty$, for some suitably chosen normalising deterministic functions $e_n, d_n, n \in \mathbb{N}$. To this end, it is easily seen that we can simply assume $\sigma = 1$ without loss of generality. For notational simplicity, we will make this assumption throughout the rest of the paper but we should bare in mind that general results for $\sigma \neq 1$ can be derived by an easy adjustment of the normalizing functions e_n, d_n . Note that by our notation $M_n^{(1)} = \max_{i \leq n}^{(1)} Q_i$ should be understood as $M_n = \max_{i \leq n} Q_i$.

One of motivations for this study stems from the increasing interest in developing extreme value theory for Gaussian processes in random environment. We refer to a series of papers by Piterbarg and his co-authors, e.g., [24], [35] and [36], for recent developments on this topic. The model in (1) can be seen as a multi-dimensional counterpart and may be interpreted as follows: The processes $\{\sigma X_i(t) - c_i t^\beta, t \geq 0\}$, $i = 1, 2, \dots, n$, are the subject of primary interest that are affected (or perturbed) by a common random environment modelled by $\{\sigma_0 X(t), t \geq 0\}$. For example, we consider an insurance company running n lines of business, where the claim surplus processes are modelled by $\{\sigma X_i(t) - c_i t^\beta, t \geq 0\}$, $i = 1, 2, \dots, n$, which are affected by the common random environment $\{\sigma_0 X(t), t \geq 0\}$. In this context, one could think of environmental factors affecting the claim surplus processes, such as the state of the economy, the political situation, weather conditions, and policy regulations. For this model, the obtained results below can give an approximation for the probability

$$\mathbb{P} \left\{ M_n^{(k)} > d_n + e_n x \right\}$$

for any $x \in \mathbb{R}$, and large n . This approximation can be used to evaluate the probability that at least k lines of business will ultimately get ruined, if ruin is defined to occur when the claim surplus process exceeds the capital $d_n + e_n x$ for a chosen x and a chosen large n .

Another motivation of our study comes from a recent contribution [29], where the authors investigate a particular model with all the Gaussian processes involved in (1)-(2) being Brownian motions, $c = c_i, i \geq 1$, and $\beta = 1$ (hereafter, referred to as *Brownian model with linear drift*). In their context, M_n models the maximum queue length in a fork-join network of n statistically identical queues (i.e., Q_1, \dots, Q_n) which are driven by a common Brownian motion perturbed arrival process and independent Brownian motion perturbed service processes, respectively. The obtained theoretical limit result on M_n therein is the key to developing structural insights into the dimensioning of assembly systems; interested readers are referred to [29] for more details on this application. We extend the model of [29] by considering general self-similar Gaussian processes with (non-identical) non-linear drifts, and study the limit properties for the k th order statistics of Q_1, \dots, Q_n . It is known that Gaussian processes play an important role in network modelling, see e.g., [28], we expect that the obtained results potentially have general applications in this area, particularly for fork-joint networks.

Besides the motivations in the application areas as above, the study of the random sequence $\{Q_i\}_{i \geq 1}$ is of interest from a theoretical point of view. Note that the random variables $Q_i, i \geq 1$ are mutually dependent with a dependence structure induced by the common stochastic process $\{\sigma_0 X(t), t \geq 0\}$. The random sequence $\{Q_i\}_{i \geq 1}$ is a *stationary sequence* if $c = c_i, i \geq 1$, and a *non-stationary sequence*, otherwise. Most of the work in extreme value theory has been done for independent and identically distributed (IID) sequences; see, e.g., [10, 18, 37]. We refer to Chapters 3 and 5-6 in [27] and Chapters 8-9 in [19] for some early discussions on extreme value theory for general dependent (stationary or non-stationary) sequence. Extreme value theory for general class of (non-)stationary sequences normally involves an asymptotic independence condition $D(u_n)$ of mixing type. Clearly, the stationary sequence $\{Q_i\}_{i \geq 1}$, with $c = c_i, i \geq 1$, is non-ergodic (in fact, exchangeable) and thus the mixing condition $D(u_n)$ can not be verified. This means that many results in the classical

theory may not be proved using the classical point process approach. For example, a Poisson point process approximation for the number of high-level exceedances by $\{Q_i\}_{i \geq 1}$ may be impossible (or meaningless as commented in Example 8.2.1 in [19]), since a proof of such a result normally relies on the mixing condition. Although the dependence structure of the sequence $\{Q_i\}_{i \geq 1}$ is generally hard to analyze (see [12] for some remarks on the Brownian model with linear drift), this sequence exhibits some interesting (sometimes uncommon) properties as follows:

- Some convergence results for suitably normalised order statistics $e_n^{-1}(M_n^{(k)} - d_n)$, as $n \rightarrow \infty$, are approachable directly without first deriving point process convergence as in the classical theory. More precisely, we show, in Theorem 3.1 and Theorem 3.5 that, for different scenarios (derived through H, H_0, β), there exist $e_n, d_n, n \in \mathbb{N}$ such that the weak limit of $e_n^{-1}(M_n^{(k)} - d_n)$ can be a negative log transformed Erlang distributed random variable, a Normal random variable or a mixture of them. This phenomenon remarkably resembles that of the stationary Normal sequences, for which it has been well known that a sequence of suitably normalised maxima of a stationary Normal sequence with correlation function r_n will converge to a Gumbel random variable, a Normal random variable or a mixture of them, according to different limiting values of $r_n \log n$; see [27]. By considering the k th order statistics we can obtain an equivalent (mixed) Poisson distribution convergence of the number of high-level exceedances by $\{Q_i\}_{i \geq 1}$; some result that might be useful in applications.
- The notion of phantom distribution function was introduced by O'Brien [31], and that the existence of such a distribution is a quite common phenomenon for stationary weakly dependent sequences. In [17] the authors derive equivalence statements for the existence of a continuous phantom distribution function for a stationary sequence, where it is claimed that the asymptotic independence of maxima (i.e., $D(u_n)$) is not really a necessary condition. They also constructed a non-ergodic stationary process which admits a continuous phantom distribution function; see Theorem 4 therein. The stationary sequence $\{Q_i\}_{i \geq 1}$ (with $c = c_i, i \geq 1$) here gives another example of non-ergodic stationary process which admits a continuous phantom distribution function under some scenario, while under other scenarios it does not admit a phantom distribution function. See Remarks 3.2 (d) below for some detailed discussions.
- It is well known that for IID sequences, the weak convergence of any one of the normalised order statistics is equivalent to the convergence of the corresponding maxima; see e.g., Theorem 2.2.2 in [27]. However, for general dependent random sequences this might not always be true, for example, Mori [30] provides an example of such a sequence for which convergence for lower order statistics does not guarantee the convergence for the higher order statistics. We refer to [21] for some related discussions. In particular, the author derives an interesting equivalence result on the (compound) Poisson point process approximation for exceedances and the convergence for all the order statistics (see Theorem 5.1 therein). Note again that a mixing condition is crucial in the proof of this result. The sequence $\{Q_i\}_{i \geq 1}$ here, regardless of stationarity, gives an example where the convergence of all the order statistics can be established. Since the results obtained in Theorem 3.1 and Theorem 3.5 are scenario specific, it is easily seen that the convergence for any one of the order statistics is equivalent to the convergence for the maxima.
- An overview of extreme value theory for general non-stationary sequences can be found in Chapter 9 in [19] which was developed on the basis of assuming some general mixing condition $D(u_{ni})$. As discussed earlier, the non-stationary sequence $\{Q_i\}_{i \geq 1}$ (with different c_i 's) may not satisfy this mixing condition. That being said, we can derive, under a mild restriction on c_i 's, some convergence results for the order statistics $M_n^{(k)}$ with some suitable normalisation, which can be seen as a thinning version of the results for stationary cases where all the c_i 's are the same.

It is well known that weak convergence of a sequence of random variables does not imply convergence of moments. In the classical extreme value theory for IID sequences, it has been shown that such convergence of moments of normalised maxima is valid provided that some moment conditions are satisfied; see, e.g., [33] or Section 2.1 of [37]. More recently, in [4] and [32] the authors discuss moment convergence of extremes under power normalization. A natural question here is whether various moments of $e_n^{-1}(M_n^{(k)} - d_n)$ converge to the moments of the corresponding limiting random variable. In Theorem 3.3 (see also Theorem 3.5), this question is answered affirmatively without imposing any further conditions. As a by-product of this study, we derive, in Proposition 2.3, some moment convergence result for the k th order statistics of IID generalized Weibull-like random variables. This problem is of independent interest and has not been extensively explored in the existing literature as far as we are aware.

It is worth mentioning that in [29], the stationary and independent increment property of Brownian motion is the key to their proof for the weak convergence. Whereas, our proof mainly relies on the asymptotic theory of Gaussian processes, particularly, the Borell-TIS inequality and the tail asymptotics of supremum. Some ideas in our proof for the weak convergence results are stimulated by the intuitive interpretations provided for the Brownian model with linear drift in Section 5.1 of [29].

Brief outline of the paper: In Section 2 we present some preliminary results concerning the tail asymptotics of the all-time supremum of a class of self-similar Gaussian processes with trend and some limit results for generalized Weibull-like random variables. The main results on the convergence of the suitably normalised order statistics are given in Section 3, with the proofs displayed in Section 4. Some technical proofs for Section 2.2 and some frequently used C_r inequalities are presented in an Appendix.

2. PRELIMINARIES

2.1. Extremes of self-similar Gaussian processes with trend. Let $\{X_H(t), t \geq 0\}$ be a centered self-similar Gaussian process with a.s. continuous sample paths, self-similarity index $H \in (0, 1)$ and variance function t^{2H} . Define, for $\beta > H$ and $c > 0$,

$$\tau_u = \inf\{t \geq 0 : X_H(t) - ct^\beta > u\}$$

to be the first hitting time of a level $u > 0$ by the stochastic process $\{X_H(t) - ct^\beta, t \geq 0\}$.

By self-similarity of X_H , the probability of ultimately crossing an upper level $u > 0$ by the process $\{X_H(t) - ct^\beta, t \geq 0\}$ is given as

$$\mathbb{P}\{\tau_u < \infty\} = \mathbb{P}\left\{\sup_{t \geq 0} (X_H(t) - ct^\beta) > u\right\} = \mathbb{P}\left\{\sup_{t \geq 0} Z(t) > u^{1-H/\beta}\right\},$$

with

$$Z(t) = \frac{X_H(t)}{1 + ct^\beta}, \quad t \geq 0.$$

It follows from Proposition 3 of [15] that

$$\lim_{t \rightarrow \infty} Z(t) = 0 \quad a.s.$$

which means that the sample paths of $\{Z(t), t \geq 0\}$ are bounded a.s. This ensures $\sup_{t \geq 0} (X_H(t) - ct^\beta) < \infty$ a.s. and is important when we apply the Borell-TIS inequality later in some of the proofs. Next, from [22] or [23], we have that the standard deviation function

$$\sigma_Z(t) = \sqrt{\text{Var}(Z(t))} = \frac{t^H}{1 + ct^\beta}, \quad t \geq 0,$$

attains its maximum on $[0, \infty)$ at the unique point $t_0 = \left(\frac{H}{c(\beta-H)}\right)^{\frac{1}{\beta}}$ and

$$\sigma_Z(t) = A - \frac{BA^2}{2}(t-t_0)^2 + o((t-t_0)^2), \quad t \rightarrow t_0,$$

where

$$(3) \quad A = \frac{t_0^H}{1+ct_0^\beta} = \frac{\beta-H}{\beta} \left(\frac{H}{c(\beta-H)}\right)^{\frac{H}{\beta}}, \quad B = \left(\frac{H}{c(\beta-H)}\right)^{-\frac{H+2}{\beta}} H\beta.$$

Furthermore, we assume a *local stationarity* of the standardized Gaussian process $\bar{X}_H(t) := X_H(t)/t^H, t > 0$ in a neighbourhood of the point t_0 , i.e.,

$$(4) \quad \lim_{s,t \rightarrow t_0} \frac{\mathbb{E}\{(\bar{X}_H(s) - \bar{X}_H(t))^2\}}{K^2(|s-t|)} = 1$$

holds for some positive function $K(\cdot)$ which is regularly varying at 0 with index $\alpha/2 \in (0, 1)$. Condition (4) is a common assumption in the literature; see, e.g., [15] and [22]. It is worth noting that the assumption (4) is slightly general than the S2 in the definition of self-similar Gaussian processes in [9], and in [22] a slightly larger class of Gaussian processes is also discussed. Throughout this paper, we denote by $\overleftarrow{K}(\cdot)$ the asymptotic inverse of $K(\cdot)$, and thus

$$\overleftarrow{K}(K(t)) = K(\overleftarrow{K}(t))(1+o(1)) = t(1+o(1)), \quad t \downarrow 0.$$

It follows that $\overleftarrow{K}(\cdot)$ is regularly varying at 0 with index $2/\alpha$; see, e.g., [18].

Below, by $\{B_{\alpha/2}(t), t \geq 0\}$ we denote a standard fractional Brownian motion (sfBm) with Hurst index $\alpha/2 \in (0, 1)$, and

$$\text{Cov}(B_{\alpha/2}(t), B_{\alpha/2}(s)) = \frac{1}{2}(t^\alpha + s^\alpha - |t-s|^\alpha), \quad t, s \geq 0.$$

The well known Pickands constant \mathcal{H}_α in the Gaussian theory is defined by

$$\mathcal{H}_\alpha = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left\{ \exp \left(\sup_{t \in [0, T]} (\sqrt{2}B_{\alpha/2}(t) - t^\alpha) \right) \right\} \in (0, \infty).$$

We refer to [6, 14, 16, 34] and references therein for basic properties of the Pickands and related constants.

The following proposition gathers some useful results from [22] and [23] (see also [11]).

Proposition 2.1. *Let $\{X_H(t), t \geq 0\}$ be a centered self-similar Gaussian process defined as above satisfying (4) and let $c > 0$. Assume $\beta > H$. Then, for any $\varepsilon_0 \in (0, t_0)$ and any $T > t_0$,*

$$(5) \quad \begin{aligned} \mathbb{P} \left\{ \sup_{t \geq 0} (X_H(t) - ct^\beta) > u \right\} &= \mathbb{P} \left\{ \sup_{t_0 - \varepsilon_0 \leq t \leq t_0 + \varepsilon_0} \frac{X_H(t)}{1+ct^\beta} > u^{1-H/\beta} \right\} (1+o(1)) \\ &= \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \frac{X_H(t)}{1+ct^\beta} > u^{1-H/\beta} \right\} (1+o(1)) \\ &= R(u) \exp \left(-\frac{u^{2(1-\frac{H}{\beta})}}{2A^2} \right) (1+o(1)), \quad u \rightarrow \infty, \end{aligned}$$

where (with A, B given in (3))

$$(6) \quad R(u) = \frac{A^{\frac{3}{2}-\frac{2}{\alpha}} \mathcal{H}_\alpha}{2^{\frac{1}{\alpha}} B^{\frac{1}{2}}} \frac{u^{\frac{2H}{\beta}-2}}{\overleftarrow{K}(u^{\frac{H}{\beta}-1})}, \quad u > 0.$$

2.2. Limit theorems for order statistics of Weibull-like random variables. As in [7], a probability distribution function F is called a *generalized Weibull-like distribution* if

$$(7) \quad F(x) = 1 - \rho(x) \exp(-Cx^\tau), \quad x \geq x_0$$

for some $x_0 > 0$, where C, τ are two positive constants and $\rho(x) > 0$ is a regularly varying function at infinity with index $\gamma \in \mathbb{R}$. Note that in [3, 18, 20], the special case $\rho(x) = \rho_0 x^\gamma$, for some $\rho_0 > 0$, is discussed.

Let $\{Y_i\}_{i \geq 1}$ be a sequence of IID random variables which are right tail equivalent to a generalized Weibull-like distribution function of the form (7). The following result gives a limit theorem for the k th order statistics $Y_n^{(k)} := \max_{i \leq n}^{(k)} Y_i$. Hereafter, \xrightarrow{d} denotes convergence in distribution and $\stackrel{d}{=}$ means equivalence in (finite-dimensional) distribution.

Proposition 2.2. *Let*

$$(8) \quad \begin{aligned} \mu_n &= (C^{-1} \log n)^{1/\tau} + \frac{1}{\tau} (C^{-1} \log n)^{1/\tau-1} \left(C^{-1} \log(\rho((C^{-1} \log n)^{1/\tau})) \right), \quad n \in \mathbb{N}, \\ \nu_n &= (C\tau)^{-1} (C^{-1} \log n)^{1/\tau-1}, \quad n \in \mathbb{N}. \end{aligned}$$

We have, for any fixed integer $k > 0$,

$$\nu_n^{-1} \left(Y_n^{(k)} - \mu_n \right) \xrightarrow{d} \Lambda^{(k)}, \quad n \rightarrow \infty,$$

where $\Lambda^{(k)} = -\ln E_k$, with E_k being an Erlang distributed random variable with shape parameter k and rate parameter equal to 1. In particular, $\Lambda^{(1)}$ is the standard Gumbel random variable, i.e., $\mathbb{P}\{\Lambda^{(1)} \leq x\} = \exp(-e^{-x})$, $x \in \mathbb{R}$.

The next result is about the (absolute) moment convergence of the normalized k th order-statistics $\nu_n^{-1}(Y_n^{(k)} - \mu_n)$ defined in Proposition 2.2. To this end, we need to control the left tail of the generalized Weibull-like random variables Y_i . This problem does not seem to have been explored in the existing literature. Some results exist only when $k = 1$, that is, for the maximum; see, e.g., [33] or [37].

Proposition 2.3. *Suppose*

$$(9) \quad \limsup_{x \rightarrow \infty} \mathbb{P}\{Y_1 < -x\} x^\eta < \infty$$

holds for some $\eta > 0$. We have, for any $\lambda > 0$, that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left\{ \left| \nu_n^{-1} \left(Y_n^{(k)} - \mu_n \right) \right|^\lambda \right\} = \mathbb{E} \left\{ \left| \Lambda^{(k)} \right|^\lambda \right\}.$$

As an application of the above results, we consider order statistics of independent random variables obtained by removing the process $\{\sigma_0 X(t), t \geq 0\}$ from (1)-(2) which are defined as

$$(10) \quad \widetilde{M}_n^{(k)} := \max_{i \leq n}^{(k)} \widetilde{Q}_i := \max_{i \leq n} \sup_{t \geq 0} (X_i(t) - c_i t^\beta), \quad n > k.$$

Recall that we have assumed $\sigma = 1$. Without loss of generality, for any fixed n we assume that the constants c_i 's are of ascending order with

$$(11) \quad c := c_1 = \cdots = c_{m_n} < c_{m_n+1} \leq \cdots \leq c_n,$$

where $m_n \leq n$ is some integer such that $\lim_{n \rightarrow \infty} m_n/n = p \in (0, 1]$, i.e., the number of minimal drifts is proportional to the total number n . In what follows, when we say $m_n = n$ we simply mean that all the c_i 's are equal to c and thus assuming $\{\widetilde{Q}_i\}_{i \geq 1}$ is an IID sequence.

Comparing (5) and (7), we see that each \widetilde{Q}_i is right tail equivalent to a generalized Weibull-like distribution. Particularly, for \widetilde{Q}_1 we have

$$(12) \quad \rho(u) = R(u), \quad \tau = 2(1 - H/\beta), \quad C = \frac{1}{2A^2}.$$

With A given in (3), $R(u)$ given in (6) and τ given in (12), we define

$$(13) \quad \begin{aligned} b_n &:= (2A^2 \log n)^{1/\tau} + \frac{1}{\tau} (2A^2 \log n)^{1/\tau-1} \left(2A^2 \log(R((2A^2 \log n)^{1/\tau})) \right), \quad n \in \mathbb{N}, \\ a_n &:= 2A^2 \tau^{-1} (2A^2 \log n)^{1/\tau-1}, \quad n \in \mathbb{N}. \end{aligned}$$

Proposition 2.4. *Assume that (11) holds with some $m_n \leq n$ such that $\lim_{n \rightarrow \infty} m_n/n = p \in (0, 1]$. We have,*

$$(14) \quad a_{m_n}^{-1} (\widetilde{M}_n^{(k)} - b_{m_n}) \xrightarrow{d} \Lambda^{(k)}, \quad n \rightarrow \infty,$$

and, for any $\lambda > 0$,

$$(15) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left\{ \left| a_{m_n}^{-1} (\widetilde{M}_n^{(k)} - b_{m_n}) \right|^\lambda \right\} = \mathbb{E} \left\{ \left| \Lambda^{(k)} \right|^\lambda \right\}.$$

3. MAIN RESULTS

In this section, we shall first consider the stationary sequence $\{Q_i\}_{i \geq 1}$ where $c = c_i, i \geq 1$, and then present results for the general non-stationary case where the constants c_i 's may not be the same. Finally, as an application a fractional Brownian model with linear drift is discussed.

Below is one of the principal results on the weak convergence of suitably normalised $M_n^{(k)}$ defined in (2) for the stationary sequence $\{Q_i\}_{i \geq 1}$. This result extends one of the main results in [29] where only the Brownian model with linear drift is discussed. We also present an equivalent (mixed) Poisson distribution convergence result on the number of exceedances of a level $u_n(x) = b_n + a_n x$ by Q_1, \dots, Q_n , denoted by $N_n(x)$, for any $x \in \mathbb{R}$. In what follows, we denote by \mathcal{N} a standard Normal random variable, independent of $\Lambda^{(k)}$.

Theorem 3.1. *Let $M_n^{(k)}, n \in \mathbb{N}$ be defined in (1)-(2) with $\sigma = 1$ and $c = c_i, i \geq 1$, and let $b_n, a_n, n \in \mathbb{N}$ be given as in (13). Assume $\beta > \max(H, H_0)$. We have, for any $k \in \mathbb{N}$,*

(i). *If $\beta > 2H - H_0$, then*

$$\sigma_0^{-1} t_0^{-H_0} b_n^{-H_0/\beta} (M_n^{(k)} - b_n) \xrightarrow{d} \mathcal{N}, \quad n \rightarrow \infty.$$

(ii). *If $\beta < 2H - H_0$, then*

$$a_n^{-1} (M_n^{(k)} - b_n) \xrightarrow{d} \Lambda^{(k)}, \quad n \rightarrow \infty,$$

or equivalently, for any $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \{N_n(x) < k\} = \exp(-e^{-x}) \sum_{l=0}^{k-1} \frac{e^{-lx}}{l!}.$$

That is, the number of exceedances $N_n(x)$ is approximately Poisson distributed with intensity $\lambda(x) = e^{-x}$.

(iii). *If $\beta = 2H - H_0$, then*

$$a_n^{-1} (M_n^{(k)} - b_n) \xrightarrow{d} \Lambda^{(k)} + \frac{\sigma_0 c \beta}{H} \mathcal{N}, \quad n \rightarrow \infty,$$

or equivalently, for any $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \{N_n(x) < k\} = \int_{-\infty}^{\infty} \exp\left(-e^{-x+y\sigma_0 c \beta/H}\right) \sum_{l=0}^{k-1} \frac{e^{l(-x+y\sigma_0 c \beta/H)}}{l!} \varphi(y) dy,$$

where $\varphi(y) = (2\pi)^{-1/2} e^{-y^2/2}, y \in \mathbb{R}$, is the density function of the standard Normal distribution. That is, the number of exceedances $N_n(x)$ is approximately mixed Poisson distributed with random intensity $\lambda(x) = e^{-x + \mathcal{N} \sigma_0 c \beta/H}$.

Remarks 3.2. (a). In the case $\beta > 2H - H_0$, it can be understood as that the dependence among $\{Q_i\}_{i \geq 1}$ is so strong that in the limit the sequence will have either infinitely many or no exceedances of a high-level.

(b). It is interesting to notice that the above three types of limiting result for $k = 1$ (i.e., Normal, Gumbel and a mixture of them) resemble the classical results for the stationary Normal sequences.

(c). It is worth noting that in the case of stationary Normal sequence (and many other general stationary sequences) it is the the Poisson (or Cox) point process convergence that is first obtained which implies the convergence for order statistics. As discussed in the Introduction this approach might not work here due to non-existence of a mixing condition for the stationary sequence $\{Q_i\}_{i \geq 1}$. Here we directly prove a weak convergence result for the order statistics which is equivalent to a (mixed) Poisson distribution convergence under the last two scenarios.

(d). A stationary sequence $\{\xi_i\}_{i \geq 1}$ is said to admit a phantom distribution function G if

$$\mathbb{P} \left\{ \max_{i \leq n} \xi_i \leq u_n \right\} - G^n(u_n) \rightarrow 0, \quad n \rightarrow \infty,$$

for every sequence $\{u_n\}_{n \geq 1} \subset \mathbb{R}$, see e.g., [17] and references therein. It can be shown that under scenarios (i) and (iii) the stationary sequence $\{Q_i\}_{i \geq 1}$ does not admit a phantom distribution function, whereas under scenario (ii) it admits a continuous phantom distribution function. A proof is given in Section 4 following the proof of Theorem 3.1.

(e). It is of interest to study the existence and value (if exists) of extremal index θ of the stationary sequence $\{Q_i\}_{i \geq 1}$; see Section 3.7 of [27] for a definition of extremal index. To this end, the asymptotics of $\mathbb{P}\{Q_1 > u\}$, as $u \rightarrow \infty$, seems to be a key tool; some results regarding this asymptotics have been obtained in [25] under some additional conditions (see A1 and A2 therein for slightly general Gaussian processes) which are assumed to hold here for simplicity. In order to save some space we only give some comments, omitting technical assumptions and derivations, for this remark. We can show that under scenario (ii), the extremal index $\theta = 1$. This can be checked by choosing $u_n(x) = a_n x + b_n$ and using the asymptotics of Theorem 2.1 combined with formulas (5) and (7) in [25]. In fact, it is quite intuitive that when H is large enough (in the sense of scenario (ii)) the stationary sequence $\{Q_i\}_{i \geq 1}$ shows a strong independence which allows it to have an associated independent sequence in the sense of [27] and thus $\theta = 1$. Similarly, we can check that the extremal index does not seem to make sense under scenario (iii), this is understandable due to the mixture type of the limiting distribution in (iii) of Theorem 3.1. Finally, under scenario (i) we conjecture that $\theta = 0$, this is understandable intuitively due to some strong clustering property discussed in remark (a) above. It seems hard to confirm such a result in general because of the complicated higher than first order asymptotics for the function $f_u^2(s)$, as $u \rightarrow \infty$, in (5) of [25] under this scenario. However, we can easily verify this conjecture for the Brownian model with linear drift, using explicit formulas.

(f). We remark that extensions of Theorem 3.1 to multivariate order statistics of the form

$$\mathbf{M}_n^{(k)} := \left(\max_{i \leq n}^{(k)} Q_{1,i}, \max_{i \leq n}^{(k)} Q_{2,i}, \dots, \max_{i \leq n}^{(k)} Q_{d,i} \right),$$

can be done similarly, where $Q_{l,i} = \sup_{t \geq 0} (X_i^{(l)}(t) + \sigma_0 X^{(l)}(t) - ct^\beta)$ with $\{X_i^{(l)}(t), t \geq 0\}$, $l = 1, \dots, d$, $i = 1, \dots, n$ being independent copies of a self-similar Gaussian process and $\{(X^{(1)}(t), \dots, X^{(d)}(t)), t \geq 0\}$ being a d -dimensional self-similar Gaussian process. We refer to [2] for examples of multivariate self-similar Gaussian processes which include some multivariate fBm as special case.

The next result shows that for the stationary sequence $\{Q_i\}_{i \geq 1}$, the (absolute) moments of the normalised order statistics converge to the (absolute) moments of the corresponding limiting random variable.

Theorem 3.3. Under the assumptions of Theorem 3.1, we have, for any $\lambda > 0$,

(i). If $\beta > 2H - H_0$, then

$$\lim_{n \rightarrow \infty} \mathbb{E} \left\{ \left| \sigma_0^{-1} t_0^{-H_0} b_n^{-H_0/\beta} \left(M_n^{(k)} - b_n \right) \right|^\lambda \right\} = \mathbb{E} \left\{ |\mathcal{N}|^\lambda \right\}.$$

(ii). If $\beta < 2H - H_0$, then

$$\lim_{n \rightarrow \infty} \mathbb{E} \left\{ \left| a_n^{-1} \left(M_n^{(k)} - b_n \right) \right|^\lambda \right\} = \mathbb{E} \left\{ \left| \Lambda^{(k)} \right|^\lambda \right\}.$$

(iii). If $\beta = 2H - H_0$, then

$$\lim_{n \rightarrow \infty} \mathbb{E} \left\{ \left| a_n^{-1} \left(M_n^{(k)} - b_n \right) \right|^\lambda \right\} = \mathbb{E} \left\{ \left| \Lambda^{(k)} + \frac{\sigma_0 c \beta}{H} \mathcal{N} \right|^\lambda \right\}.$$

Remark 3.4. We can see from the proof of Theorem 3.3 that, when λ is an integer, the above convergence results still hold for moments without the modulus. Absolute moments of the limiting distributions can sometimes be given more explicitly, for example, it follows from [38] that $\mathbb{E} \left\{ |\mathcal{N}|^\lambda \right\} = \frac{2^{\lambda/2}}{\sqrt{\pi}} \Gamma \left(\frac{\lambda+1}{2} \right)$, with $\Gamma(\cdot)$ the Gamma function. Furthermore, by a change a variable formula, we can obtain $\mathbb{E} \left\{ |\Lambda^{(1)}|^\lambda \right\} = \int_0^\infty |\log y|^\lambda e^{-y} dy$. The formula for other distributions seems to be complicate and thus omitted here. Moreover, these moments can be easily approximated by using Monte Carlo simulations.

The following theorem presents analogues of Theorem 3.1 and Theorem 3.3 for a non-stationary sequence $\{Q_i\}_{i \geq 1}$ with general c_i 's.

Theorem 3.5. Let $M_n^{(k)}, n \in \mathbb{N}$ be defined in (1)-(2) with $\sigma = 1$, and (11) holds with some $m_n < n$ such that $\lim_{n \rightarrow \infty} m_n/n = p \in (0, 1]$, and let $b_n, a_n, n \in \mathbb{N}$ be given as in (13). Assume $\beta > \max(H, H_0)$. Then, the claims of (i)-(iii) in Theorem 3.1 and Theorem 3.3 hold true when replacing a_n, b_n with a_{m_n}, b_{m_n} , respectively.

Remark 3.6. The above result is understandable intuitively as follows: The probability of exceeding a high-level threshold by Q_i , for any $i > m_n$ is much less than that of $Q_i, i \leq m_n$, so a lower threshold $u_{m_n}(x)$ (defined through m_n instead of n) is needed in order to derive the same limiting distribution as for the stationary case. In this sense, the above results for the non-stationary sequence $\{Q_i\}_{i \geq 1}$ can be seen as a thinning version of the results in Theorem 3.1 and Theorem 3.3.

We conclude this section with an example, where we derive corresponding results for the fBm model with a linear drift (i.e., $\beta = 1$). For a sfBm $\{X_1(t), t \geq 0\}$ with Hurst index $H \in (0, 1)$,

$$\text{Cov}(X_1(t), X_1(s)) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0.$$

One can check that sfBm X_1 fulfills (4) with $K(t) = t_0^{-H} t^H = (H/(c(1-H)))^{-H/\beta} t^H, t \geq 0$. Thus, by Proposition 2.1, we have

$$\mathbb{P} \left\{ \sup_{t \geq 0} (X_1(t) - ct) > u \right\} = R(u) \exp \left(-\frac{u^\tau}{2A^2} \right) (1 + o(1))$$

as $u \rightarrow \infty$, where

$$(16) \quad A = \frac{H^H (1-H)^{1-H}}{c^H}, \quad \tau = 2(1-H), \quad R(u) = \frac{2^{-\frac{1}{2H}} \mathcal{H}_{2H}}{\sqrt{H(1-H)}} \left(\frac{c^H u^{1-H}}{H^H (1-H)^{1-H}} \right)^{\frac{1}{H}-2}.$$

Corollary 3.7. Let $\{X_i(t), t \geq 0\}, i = 1, 2, \dots$, be independent sfBm's with common Hurst index $H \in (0, 1)$ and $\{X(t), t \geq 0\}$ be another independent sfBm with Hurst index $H_0 \in (0, 1)$. Assume $\sigma = 1$ and $\beta = 1$. Then, the claims in Theorems 3.1, 3.3 and 3.5 are valid, with b_n, a_n in (13) defined through (16).

Remark 3.8. Particularly, if $H = H_0 = 1/2, c = c_i, i \geq 1$ and $k = 1$, we recover Theorem 5.2 of [29].

4. FURTHER RESULTS AND PROOFS

Before starting the proof, we first give some auxiliary results which will be used later. Recall the key point $t_0 = \left(\frac{H}{c(\beta-H)}\right)^{\frac{1}{\beta}}$ as given above the formula (3). The lemma below is about limiting properties of a_n and b_n which can be obtained immediately from their definition.

Lemma 4.1. *For a_n and b_n in (13), we have*

$$(17) \quad \lim_{n \rightarrow \infty} \frac{b_n^{H_0/\beta}}{a_n} = \tau(2A^2)^{\frac{H_0-\beta}{2(\beta-H)}} \lim_{n \rightarrow \infty} (\log n)^{\frac{\beta-2H+H_0}{2(\beta-H)}} = \begin{cases} \infty, & \text{if } \beta > 2H - H_0, \\ 0, & \text{if } \beta < 2H - H_0, \\ \tau/(2A^2), & \text{if } \beta = 2H - H_0. \end{cases}$$

Furthermore, as $n \rightarrow \infty$,

$$(18) \quad b_n^{1-H/\beta} = A\sqrt{2\log n} \left(1 + \frac{1}{2}(\log n)^{-1} \log(R((2A^2 \log n)^{\frac{\beta}{2(\beta-H)}}))\right)(1 + o(1)).$$

Lemma 4.2. *For any $\varepsilon_0 \in (0, t_0)$, there exists some small $\hat{c} \in (0, \min(1, c))$ such that*

$$(19) \quad \sigma_* = \sigma_*(\varepsilon_0, \hat{c}) := \max \left\{ \max_{t \in [0, t_0 - \varepsilon_0]} \frac{t^H}{1 - \hat{c} + ct^\beta}, \max_{t \geq t_0 + \varepsilon_0} \frac{t^H}{1 - \hat{c} + (c - \hat{c})t^\beta} \right\} < \frac{t_0^H}{1 + ct_0^\beta} = A.$$

Proof of Lemma 4.2: We only show the proof for the maximum taken over $[0, t_0 - \varepsilon_0]$, since similar arguments also apply to the second maximum taken over $[t_0 + \varepsilon_0, \infty)$. Note that, for any $\varepsilon_1 > -1$ and $\varepsilon_2 \in [0, c)$,

$$(20) \quad \arg \max_{t \geq 0} \frac{t^H}{1 + \varepsilon_1 + (c - \varepsilon_2)t^\beta} = t_0 \left(\frac{c}{c - \varepsilon_2} (1 + \varepsilon_1) \right)^{1/\beta}.$$

[This formula is given in a general form which is also helpful for later.] Thus, for any $\varepsilon_0 \in (0, t_0)$, we can find some small enough \hat{c} such that

$$\max_{t \in [0, t_0 - \varepsilon_0]} \frac{t^H}{1 - \hat{c} + ct^\beta} = \frac{(t_0 - \varepsilon_0)^H}{1 - \hat{c} + c(t_0 - \varepsilon_0)^\beta} < \frac{t_0^H}{1 + ct_0^\beta},$$

where the last inequality follows by (20) with $\varepsilon_1 = \varepsilon_2 = 0$. This completes the proof. \square

4.1. Proof of Theorem 3.1. In the following subsections, we first present the proof for scenario (i) and then a generic proof for scenarios (ii)-(iii).

4.1.1. Proof for (i). We need to show that, for any $x \in \mathbb{R}$,

$$\mathbb{P} \left\{ b_n^{-H_0/\beta} \left(M_n^{(k)} - b_n \right) > x \right\} \rightarrow \mathbb{P} \left\{ \sigma_0 t_0^{H_0} \mathcal{N} > x \right\}, \quad n \rightarrow \infty.$$

We will consider asymptotic lower and upper bounds, respectively. First, we have, from Lemma 4.3 below, that

$$(21) \quad \begin{aligned} \mathbb{P} \left\{ b_n^{-H_0/\beta} \left(M_n^{(k)} - b_n \right) > x \right\} &\geq \mathbb{P} \left\{ b_n^{-H_0/\beta} \left(\max_{i \leq n}^{(k)} X_i(t_0 b_n^{1/\beta}) + \sigma_0 X(t_0 b_n^{1/\beta}) - c(t_0 b_n^{1/\beta})^\beta - b_n \right) > x \right\} \\ &\rightarrow \mathbb{P} \left\{ \sigma_0 t_0^{H_0} \mathcal{N} > x \right\}, \quad n \rightarrow \infty, \end{aligned}$$

which yields the required lower bound. Next, for any $\varepsilon_0 \in (0, t_0)$, we introduce the following notation,

$$\begin{aligned} A_{1,i} &= \left\{ b_n^{-H_0/\beta} \left(\sup_{(t_0-\varepsilon_0)b_n^{1/\beta} \leq t \leq (t_0+\varepsilon_0)b_n^{1/\beta}} (X_i(t) + \sigma_0 X(t) - ct^\beta) - b_n \right) > x \right\}, \quad 1 \leq i \leq n, \\ A_{2,i} &= \left\{ b_n^{-H_0/\beta} \left(\sup_{0 \leq t \leq (t_0-\varepsilon_0)b_n^{1/\beta}} (X_i(t) + \sigma_0 X(t) - ct^\beta) - b_n \right) > x \right\}, \quad 1 \leq i \leq n, \\ A_{3,i} &= \left\{ b_n^{-H_0/\beta} \left(\sup_{t \geq (t_0+\varepsilon_0)b_n^{1/\beta}} (X_i(t) + \sigma_0 X(t) - ct^\beta) - b_n \right) > x \right\}, \quad 1 \leq i \leq n, \\ B_2 &= \cup_{i \leq n} A_{2,i}, \quad B_3 = \cup_{i \leq n} A_{3,i}. \end{aligned}$$

We derive that

$$\begin{aligned} \mathbb{P} \left\{ b_n^{-H_0/\beta} (M_n^{(k)} - b_n) > x \right\} &= \mathbb{P} \left\{ \bigcup_{\substack{\{j_1, \dots, j_k\} \\ \subseteq \{1, \dots, n\}}} \left((\cup_{i=1}^3 A_{i,j_1}) \cap \dots \cap (\cup_{i=1}^3 A_{i,j_k}) \right) \right\} \\ &= \mathbb{P} \left\{ \bigcup_{\substack{\{j_1, \dots, j_k\} \\ \subseteq \{1, \dots, n\}}} \left((\cup_{i=1}^3 A_{i,j_1}) \cap \dots \cap (\cup_{i=1}^3 A_{i,j_k}) \right), (B_2 \cup B_3)^c \right\} \\ (22) \quad &+ \mathbb{P} \left\{ \bigcup_{\substack{\{j_1, \dots, j_k\} \\ \subseteq \{1, \dots, n\}}} \left((\cup_{i=1}^3 A_{i,j_1}) \cap \dots \cap (\cup_{i=1}^3 A_{i,j_k}) \right), (B_2 \cup B_3) \right\} \\ &\leq \mathbb{P} \left\{ \bigcup_{\substack{\{j_1, \dots, j_k\} \\ \subseteq \{1, \dots, n\}}} (A_{1,j_1} \cap \dots \cap A_{1,j_k}) \right\} + \mathbb{P} \{B_2\} + \mathbb{P} \{B_3\}, \end{aligned}$$

where $\bigcup_{\substack{\{j_1, \dots, j_k\} \\ \subseteq \{1, \dots, n\}}}$ denotes the union of all the possible combinations of j_1, \dots, j_k drawn without replacement from $\{1, \dots, n\}$. Thus, the above inequality can be re-written as

$$\begin{aligned} \mathbb{P} \left\{ b_n^{-H_0/\beta} (M_n^{(k)} - b_n) > x \right\} &\leq \mathbb{P} \left\{ \max_{i \leq n} \sup_{(t_0-\varepsilon_0)b_n^{1/\beta} \leq t \leq (t_0+\varepsilon_0)b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} > x \right\} \\ &+ \mathbb{P} \left\{ \max_{i \leq n} \sup_{0 \leq t \leq (t_0-\varepsilon_0)b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} > x \right\} \\ &+ \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq (t_0+\varepsilon_0)b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} > x \right\}. \end{aligned}$$

In view of (28) and (29) in Lemma 4.4 below, we know that the last two terms on the right-hand side converge to 0, as $n \rightarrow \infty$. For the remaining first term, it follows, by self-similarity, that

$$\begin{aligned} &\mathbb{P} \left\{ \max_{i \leq n} \sup_{(t_0-\varepsilon_0)b_n^{1/\beta} \leq t \leq (t_0+\varepsilon_0)b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} > x \right\} \\ &\leq \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq 0} \frac{X_i(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} + \sup_{(t_0-\varepsilon_0)b_n^{1/\beta} \leq t \leq (t_0+\varepsilon_0)b_n^{1/\beta}} \frac{\sigma_0 X(t)}{b_n^{H_0/\beta}} > x \right\} \\ &= \mathbb{P} \left\{ b_n^{-H_0/\beta} (\widetilde{M}_n^{(k)} - b_n) + \sup_{1 \leq t \leq (t_0+\varepsilon_0)/(t_0-\varepsilon_0)} \sigma_0 X(t) (t_0 - \varepsilon)^{H_0} > x \right\} \end{aligned}$$

with $\widetilde{M}_n^{(k)}$ defined in (10) with $c = c_i, i \geq 1$ in the stationary case. Therefore, we derive from (14) and (17) that, for $\beta > 2H - H_0$,

$$b_n^{-H_0/\beta}(\widetilde{M}_n^{(k)} - b_n) = \frac{a_n}{b_n^{H_0/\beta}} \frac{\widetilde{M}_n^{(k)} - b_n}{a_n} \xrightarrow{d} 0, \quad n \rightarrow \infty,$$

and thus

$$\begin{aligned} & \lim_{\varepsilon_0 \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0)b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0)b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} > x \right\} \\ & \leq \lim_{\varepsilon_0 \rightarrow 0} \mathbb{P} \left\{ \sup_{1 \leq t \leq (t_0 + \varepsilon_0)/(t_0 - \varepsilon_0)} \sigma_0 X(t)(t_0 - \varepsilon_0)^{H_0} > x \right\} = \mathbb{P} \left\{ \sigma_0 t_0^{H_0} \mathcal{N} > x \right\}, \end{aligned}$$

which gives the required upper bound. The proof is complete. \square

Below we present the two lemmas used in this proof.

Lemma 4.3. *Under the assumption of Theorem 3.1 and the condition in (i) (i.e., $\beta > 2H - H_0$), we have, as $n \rightarrow \infty$,*

$$b_n^{-H_0/\beta} \left(\max_{i \leq n}^{(k)} X_i(t_0 b_n^{1/\beta}) + \sigma_0 X(t_0 b_n^{1/\beta}) - c(t_0 b_n^{1/\beta})^\beta - b_n \right) \xrightarrow{d} \sigma_0 t_0^{H_0} X(1).$$

Proof of Lemma 4.3: First, by self-similarity,

$$\begin{aligned} & b_n^{-H_0/\beta} \left(\max_{i \leq n}^{(k)} X_i(t_0 b_n^{1/\beta}) + \sigma_0 X(t_0 b_n^{1/\beta}) - c(t_0 b_n^{1/\beta})^\beta - b_n \right) \\ & \stackrel{d}{=} t_0^H \left(\max_{i \leq n}^{(k)} X_i(1) - \frac{1 + ct_0^\beta}{t_0^H} b_n^{1-H/\beta} \right) b_n^{(H-H_0)/\beta} + \sigma_0 t_0^{H_0} X(1). \end{aligned}$$

It is sufficient to show that

$$(23) \quad \left(\max_{i \leq n}^{(k)} X_i(1) - \frac{1 + ct_0^\beta}{t_0^H} b_n^{1-H/\beta} \right) b_n^{(H-H_0)/\beta} \xrightarrow{d} 0, \quad n \rightarrow \infty.$$

For the IID standard Normal sequence $X_i(1), i = 1, 2, \dots$, we have from Proposition 2.2 (see also Theorem 1.5.3 in [27]) that

$$(24) \quad Z_n^{(k)} := \sqrt{2 \log n} \left(\max_{i \leq n}^{(k)} X_i(1) - \left(\sqrt{2 \log n} - \frac{\log(4\pi \log n)}{2\sqrt{2 \log n}} \right) \right) \xrightarrow{d} \Lambda^{(k)}, \quad n \rightarrow \infty.$$

Then, we can rewrite the left-hand side of (23) as

$$\begin{aligned} & \left(\max_{i \leq n}^{(k)} X_i(1) - \frac{1 + ct_0^\beta}{t_0^H} b_n^{1-H/\beta} \right) b_n^{(H-H_0)/\beta} \\ & = \frac{\sqrt{2 \log n} \left(\max_{i \leq n}^{(k)} X_i(1) - \left(\sqrt{2 \log n} - \frac{\log(4\pi \log n)}{2\sqrt{2 \log n}} \right) \right)}{\sqrt{2 \log n} b_n^{(H_0-H)/\beta}} - \frac{\frac{1 + ct_0^\beta}{t_0^H} b_n^{1-H/\beta} - \left(\sqrt{2 \log n} - \frac{\log(4\pi \log n)}{2\sqrt{2 \log n}} \right)}{b_n^{(H_0-H)/\beta}} \\ (25) \quad & =: \frac{Z_n^{(k)}}{\sqrt{2 \log n} b_n^{(H_0-H)/\beta}} - r_n. \end{aligned}$$

By the definition of b_n in (13) and the assumption $\beta > 2H - H_0$, we have

$$(26) \quad \lim_{n \rightarrow \infty} \sqrt{2 \log n} b_n^{(H_0-H)/\beta} = \lim_{n \rightarrow \infty} A^{\frac{H_0-H}{\beta-H}} (2 \log n)^{\frac{\beta+H_0-2H}{2(\beta-H)}} (1 + o(1))^{(H_0-H)/\beta} = \infty,$$

which together with Taylor expression (18) implies that

$$(27) \quad \lim_{n \rightarrow \infty} r_n = \frac{1}{\sqrt{2 \log n} b_n^{(H_0-H)/\beta}} \left(\log(R((2A^2 \log n)^{1/\tau})) (1 + o(1)) + 2^{-1} \log(4\pi \log n) \right) = 0.$$

Consequently, substituting (24) and (26)-(27) into (25), we get (23). This completes the proof. \square

Lemma 4.4. *Under the assumption of Theorem 3.1, we have, for any $\varepsilon_0 \in (0, t_0)$ and any $x \in \mathbb{R}$,*

$$(28) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \max_{i \leq n} \sup_{0 \leq t \leq (t_0 - \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} > x \right\} = 0,$$

$$(29) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} > x \right\} = 0.$$

Proof of Lemma 4.4: We first prove (28). Note, by self-similarity,

$$(30) \quad \begin{aligned} & \mathbb{P} \left\{ \max_{i \leq n} \sup_{0 \leq t \leq (t_0 - \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} > x \right\} \\ & \leq \mathbb{P} \left\{ \max_{i \leq n} \sup_{0 \leq t \leq (t_0 - \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} + \sup_{0 \leq t \leq (t_0 - \varepsilon_0) b_n^{1/\beta}} \frac{\sigma_0 X(t)}{b_n^{H_0/\beta}} > x \right\} \\ & = \mathbb{P} \left\{ \max_{i \leq n} \sup_{0 \leq t \leq (t_0 - \varepsilon_0)} \left(b_n^{(H-H_0)/\beta} X_i(t) - (1 + ct^\beta) b_n^{1-H_0/\beta} \right) + \sup_{0 \leq t \leq (t_0 - \varepsilon_0)} \sigma_0 X(t) > x \right\}. \end{aligned}$$

Since $\sup_{0 \leq t \leq (t_0 - \varepsilon_0)} X(t) < \infty$ a.s., it is sufficient to show that, for any $x \in \mathbb{R}$,

$$(31) \quad J_1(n, x) := \mathbb{P} \left\{ \max_{i \leq n} \sup_{0 \leq t \leq (t_0 - \varepsilon_0)} \left(b_n^{(H-H_0)/\beta} X_i(t) - (1 + ct^\beta) b_n^{1-H_0/\beta} \right) > x \right\} \rightarrow 0,$$

as $n \rightarrow \infty$. For the fixed ε_0 , choosing a small $\hat{c} \in (0, 1)$ satisfying (19), then using Borell-TIS inequality (see, e.g., Theorem 2.1.1 in [1]), we have, for large enough n such that $x b_n^{H_0/\beta - 1} > -\hat{c}$,

$$\begin{aligned} J_1(n, x) & \leq n \mathbb{P} \left\{ \sup_{0 \leq t \leq (t_0 - \varepsilon_0)} \frac{X_i(t)}{1 + ct^\beta + x b_n^{H_0/\beta - 1}} > b_n^{1-H/\beta} \right\} \\ & \leq n \mathbb{P} \left\{ \sup_{0 \leq t \leq (t_0 - \varepsilon_0)} \frac{X_i(t)}{1 - \hat{c} + ct^\beta} > b_n^{1-H/\beta} \right\} \\ & \leq n \exp \left(- (b_n^{1-H/\beta} - K_1)^2 \left(\sup_{t \in [0, t_0 - \varepsilon_0]} \frac{2t^{2H}}{(1 - \hat{c} + ct^\beta)^2} \right)^{-1} \right) \\ & \leq \exp \left(- \frac{(b_n^{1-H/\beta} - K_1)^2}{2\sigma_*^2} + \log n \right), \end{aligned}$$

where $K_1 := \mathbb{E} \left\{ \sup_{0 \leq t \leq (t_0 - \varepsilon_0)} X_1(t) / (1 - \hat{c} + ct^\beta) \right\} < b_n^{1-H/\beta}$ for all large enough n , and the last inequality follows from (19). Furthermore, by (18) and (19) we have

$$\lim_{n \rightarrow \infty} \frac{b_n^{2-2H/\beta}}{2\sigma_*^2 \log n} > 1,$$

implying (31). Thus, (28) is established. Next, by a similar argument we derive, for some $\hat{c} \in (0, \min(1, c))$ satisfying (19), that

$$\begin{aligned} & \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} > x \right\} \\ & \leq \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) - (c - \hat{c})t^\beta - b_n}{b_n^{H_0/\beta}} + \sup_{t \geq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{\sigma_0 X(t) - \hat{c}t^\beta}{b_n^{H_0/\beta}} > x \right\} \\ & \leq \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq (t_0 + \varepsilon_0)} \left(b_n^{(H-H_0)/\beta} X_i(t) - (1 + (c - \hat{c})t^\beta) b_n^{1-H_0/\beta} \right) + b_n^{-H_0/\beta} \sup_{t \geq 0} (\sigma_0 X(t) - \hat{c}t^\beta) > x \right\}. \end{aligned}$$

Note that $\lim_{n \rightarrow \infty} b_n^{-H_0/\beta} \sup_{t \geq 0} (\sigma_0 X(t) - \hat{c}t^\beta) = 0$ a.s.. Thus, in order to prove (29), it is sufficient to show that, for any fixed $x \in \mathbb{R}$,

$$J_2(n, x) := \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq (t_0 + \varepsilon_0)} \left(b_n^{(H-H_0)/\beta} X_i(t) - (1 + (c - \hat{c})t^\beta) b_n^{1-H_0/\beta} \right) > x \right\} \rightarrow 0,$$

as $n \rightarrow \infty$. Again, by Borell-TIS inequality, we have,

$$\begin{aligned} J_2(n, x) &\leq n \mathbb{P} \left\{ \sup_{t \geq (t_0 + \varepsilon_0)} \frac{X_i(t)}{1 + (c - \hat{c})t^\beta + x b_n^{H_0/\beta - 1}} > b_n^{1-H/\beta} \right\} \\ &\leq n \mathbb{P} \left\{ \sup_{t \geq (t_0 + \varepsilon_0)} \frac{X_i(t)}{1 - \hat{c} + (c - \hat{c})t^\beta} > b_n^{1-H/\beta} \right\} \\ &\leq n \exp \left(- (b_n^{1-H/\beta} - K_2)^2 \left(\sup_{t \geq (t_0 + \varepsilon_0)} \frac{2t^{2H}}{(1 - \hat{c} + (c - \hat{c})t^\beta)^2} \right)^{-1} \right) \\ &\leq \exp \left(- \frac{(b_n^{1-H/\beta} - K_2)^2}{2\sigma_*^2} + \log n \right) \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

where $K_2 := \mathbb{E} \left\{ \sup_{t \geq (t_0 + \varepsilon_0)} \frac{X_1(t)}{1 + (c - \hat{c})t^\beta} \right\} < \infty$. Thus, the proof is complete. \square

Before proving scenarios (ii) and (iii), we shall derive two important lemmas below.

Lemma 4.5. *Under the assumptions of Theorem 3.1 and the conditions in (ii) and (iii) (i.e., $\beta \leq 2H - H_0$), we have, for any $\varepsilon_0 \in (0, t_0)$ and any $x \in \mathbb{R}$,*

$$(32) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \max_{i \leq n} \sup_{0 \leq t \leq (t_0 - \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{a_n} > x \right\} = 0,$$

$$(33) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{a_n} > x \right\} = 0.$$

Proof of Lemma 4.5: The claims follow by similar arguments as those used in the proof of Lemma 4.4, with $b_n^{H_0/\beta}$ replaced by a_n . The assumption $\beta \leq 2H - H_0$ is used to show that

$$\lim_{n \rightarrow \infty} b_n^{H_0/\beta} a_n^{-1} \sup_{0 \leq t \leq (t_0 - \varepsilon_0)} X(t) < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} a_n^{-1} \sup_{t \geq 0} (X(t) - \hat{c}t^\beta) = 0, \quad \text{a.s..}$$

The details are thus omitted. \square

Remark 4.6. *It is easy to check that the claims in (32) and (33) are still valid if we remove $\sigma_0 X(t)$ from the numerators and without assuming $\beta \leq 2H - H_0$. This observation is useful for the following result.*

Lemma 4.7. *Under the assumptions of Theorem 3.1, we have, for any $\varepsilon_0 \in (0, t_0)$,*

$$\max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0) b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{a_n} \xrightarrow{d} \Lambda^{(k)},$$

as $n \rightarrow \infty$.

Proof of Lemma 4.7: We need to show that, for any $x \in \mathbb{R}$,

$$(34) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0) b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{a_n} > x \right\} = \mathbb{P} \left\{ \Lambda^{(k)} > x \right\}.$$

First, from (14) we have for any $x \in \mathbb{R}$,

$$(35) \quad \begin{aligned} \mathbb{P} \left\{ \Lambda^{(k)} > x \right\} &= \lim_{n \rightarrow \infty} \mathbb{P} \left\{ a_n^{-1} \left(\widetilde{M}_n^{(k)} - b_n \right) > x \right\} \\ &\geq \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0) b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{a_n} > x \right\}. \end{aligned}$$

Next, similarly to (22) we can derive, for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, that

$$\begin{aligned} \mathbb{P} \left\{ a_n^{-1} \left(\widetilde{M}_n^{(k)} - b_n \right) > x \right\} &\leq \mathbb{P} \left\{ \max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0) b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{a_n} > x \right\} \\ &\quad + \mathbb{P} \left\{ \max_{i \leq n} \sup_{0 \leq t \leq (t_0 - \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{a_n} > x \right\} \\ &\quad + \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{a_n} > x \right\}, \end{aligned}$$

where the last two probabilities on the right-hand side tend to 0 as $n \rightarrow \infty$, as discussed in Remark 4.6. Thus, we obtain

$$(36) \quad \liminf_{n \rightarrow \infty} \mathbb{P} \left\{ \max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0) b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{a_n} > x \right\} \geq \mathbb{P} \left\{ \Lambda^{(k)} > x \right\}.$$

Therefore, (34) follows from (35) and (36), and the proof is complete. \square

4.1.2. *Proof for (ii) and (iii).* First, similarly to (22) we can derive, for any $\varepsilon_0 \in (0, t_0)$ and any $x \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P} \left\{ a_n^{-1} (M_n^{(k)} - b_n) > x \right\} &\leq \mathbb{P} \left\{ \max_{i \leq n} \sup_{0 \leq t \leq (t_0 - \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{a_n} > x \right\} \\ &\quad + \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{a_n} > x \right\} \\ &\quad + \mathbb{P} \left\{ \max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0) b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{a_n} > x \right\} \\ &=: I_1(\varepsilon_0, n, x) + I_2(\varepsilon_0, n, x) + I_3(\varepsilon_0, n, x). \end{aligned}$$

From Lemma 4.5 we know

$$\lim_{n \rightarrow \infty} I_1(\varepsilon_0, n, x) = \lim_{n \rightarrow \infty} I_2(\varepsilon_0, n, x) = 0.$$

For the remaining $I_3(\varepsilon_0, n, x)$, we note that

$$I_3(\varepsilon_0, n, x) \leq \mathbb{P} \left\{ \max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0) b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{a_n} + \sup_{(t_0 - \varepsilon_0) \leq t \leq (t_0 + \varepsilon_0)} \sigma_0 X(t) \frac{b_n^{H_0/\beta}}{a_n} > x \right\}.$$

Then, by (17), Lemma 4.7, and the independence of the Gaussian processes X and X_i 's, we obtain

$$(37) \quad \begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ a_n^{-1} (M_n^{(k)} - b_n) > x \right\} &\leq \lim_{\varepsilon_0 \rightarrow 0} \limsup_{n \rightarrow \infty} I_3(\varepsilon_0, n, x) \\ &\leq \mathbb{P} \left\{ \Lambda^{(k)} + \frac{\sigma_0 c \beta}{H} 1_{\{\beta = 2H - H_0\}} \mathcal{N} > x \right\}, \end{aligned}$$

where in the last inequality we used that, for $\beta = 2H - H_0$,

$$\begin{aligned} t_0^{H_0} \tau(2A^2)^{\frac{H_0 - \beta}{2(\beta - H)}} &= t_0^{H_0} \tau(2A^2)^{-1} \\ &= \left(\frac{H}{c(\beta - H)} \right)^{H_0/\beta} \left(\frac{2(\beta - H)}{\beta} \right) \frac{1}{2} \left(\frac{\beta}{\beta - H} \right)^2 \left(\frac{H}{c(\beta - H)} \right)^{-2H/\beta} \\ &= c\beta/H. \end{aligned}$$

Next, since

$$\begin{aligned} & \mathbb{P} \left\{ a_n^{-1}(M_n^{(k)} - b_n) > x \right\} \geq I_3(\varepsilon_0, n, x) \\ & \geq \mathbb{P} \left\{ \max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0)b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0)b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{a_n} - \frac{b_n^{H_0/\beta}}{a_n} \sup_{(t_0 - \varepsilon_0) \leq t \leq (t_0 + \varepsilon_0)} (-\sigma_0 X(t)) > x \right\}, \end{aligned}$$

and thus by the same reason as above we have

$$\begin{aligned} (38) \quad \liminf_{n \rightarrow \infty} \mathbb{P} \left\{ a_n^{-1}(M_n^{(k)} - b_n) > x \right\} & \geq \lim_{\varepsilon_0 \rightarrow 0} \liminf_{n \rightarrow \infty} I_3(\varepsilon_0, n, x) \\ & = \mathbb{P} \left\{ \Lambda^{(k)} + \frac{\sigma_0 c \beta}{H} 1_{\{\beta = 2H - H_0\}} \mathcal{N} > x \right\}. \end{aligned}$$

Consequently, combining (37) and (38) yields

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ a_n^{-1}(M_n^{(k)} - b_n) > x \right\} = \mathbb{P} \left\{ \Lambda^{(k)} + \frac{\sigma_0 c \beta}{H} 1_{\{\beta = 2H - H_0\}} \mathcal{N} > x \right\}.$$

This completes the proof for both (ii) and (iii).

4.2. Proof of Remarks 3.2 (d). We first consider the scenarios (i) and (iii) where $\beta \geq 2H - H_0$. The claim of non-existence of a phantom distribution function can be proved by a contradiction. If a phantom distribution function G exists, then by definition we know for a sequence $u_n(x) = e_n x + b_n$, $n \in \mathbb{N}$ with $x \in \mathbb{R}$ (here $e_n = \sigma_0 t_0^{H_0} b_n^{H_0/\beta}$ under scenario (i) and $e_n = a_n$ under scenario (iii)),

$$\mathbb{P} \{M_n \leq u_n(x)\} - G^n(u_n(x)) \rightarrow 0, \quad n \rightarrow \infty,$$

which, by Theorem 3.1, implies that

$$G^n(u_n(x)) \rightarrow \begin{cases} \mathbb{P} \{ \mathcal{N} \leq x \}, & \text{if } \beta > 2H - H_0, \\ \mathbb{P} \left\{ \Lambda^{(1)} + \frac{\sigma_0 c \beta}{H} \mathcal{N} \leq x \right\}, & \text{if } \beta = 2H - H_0, \end{cases} \quad n \rightarrow \infty.$$

The above result is not possible because these limiting distributions are not members of the only three possible non-degenerate extreme value distribution families for IID sequence. Thus, there is no phantom distribution function for the stationary sequence $\{Q_i\}_{i \geq 1}$ under scenarios (i) and (iii).

The claim of existence of a continuous phantom distribution function under scenario (ii) follows by applying Theorem 2 of [17]. Indeed, it can be shown by Theorem 3.1 that

$$\mathbb{P} \{M_{[nt]} \leq b_n\} = \mathbb{P} \left\{ a_{[nt]}^{-1}(M_{[nt]} - b_{[nt]}) \leq a_{[nt]}^{-1}(b_n - b_{[nt]}) \right\} \rightarrow e^{-t}, \quad \forall t > 0.$$

4.3. Proof of Theorem 3.3. In the following two subsections, we present the proof for scenario (i) and scenarios (ii)-(iii), respectively.

4.3.1. Proof for (i). Due to the weak convergence result in scenario (i) of Theorem 3.1 and the arguments as in the proof of Proposition 2.1 in [37], it is sufficient to show that

$$(39) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \left| b_n^{-H_0/\beta} (M_n^{(k)} - b_n) \right| > s \right\} ds = 0.$$

Note that

$$\begin{aligned} & \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \left| b_n^{-H_0/\beta} (M_n^{(k)} - b_n) \right| > s \right\} ds \\ & = \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ b_n^{-H_0/\beta} (M_n^{(k)} - b_n) > s \right\} ds + \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ b_n^{-H_0/\beta} (M_n^{(k)} - b_n) < -s \right\} ds \\ & \leq \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ b_n^{-H_0/\beta} (M_n^{(1)} - b_n) > s \right\} ds + \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ b_n^{-H_0/\beta} (M_n^{(k)} - b_n) < -s \right\} ds \\ & =: H_1(n, L) + H_2(n, L). \end{aligned}$$

Below we discuss $H_1(n, L)$ and $H_2(n, L)$ for large n and L , and aim to find uniform (for large n and large s) integrable upper bounds for the probability terms in their integrands so that (39) holds.

Consider $H_1(n, L)$. Fix a small $\hat{c} \in (0, c)$, we can choose a large enough G such that

$$(40) \quad \frac{(t_0(1+G))^H}{1+(c-\hat{c})(t_0(1+G))^\beta} < \frac{t_0^H}{1+ct_0^\beta} = A,$$

$$(41) \quad \delta_G := 2((1+G)^\beta(c-\hat{c})/c-1) > 0,$$

and

$$(42) \quad \left(\frac{c}{c-\hat{c}}\right)^{-2H/\beta} (1+\delta_G/2)^{2(1-H/\beta)} = \left(\frac{c}{c-\hat{c}}\right)^{-2} (1+G)^{2(\beta-H)} \geq 4.$$

It follows that

$$\begin{aligned} \mathbb{P}\left\{b_n^{-H_0/\beta}(M_n^{(1)} - b_n) > s\right\} &\leq \mathbb{P}\left\{\max_{i \leq n} \sup_{0 \leq t \leq (1+G)t_0 b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} > s\right\} \\ &\quad + \mathbb{P}\left\{\max_{i \leq n} \sup_{t \geq (1+G)t_0 b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} > s\right\} \\ &=: I_{11}(n, s) + I_{12}(n, s). \end{aligned}$$

By self-similarity, we have

$$(43) \quad \begin{aligned} I_{11}(n, s) &\leq \mathbb{P}\left\{\max_{i \leq n} \sup_{0 \leq t \leq (1+G)t_0 b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{b_n^{H_0/\beta}} > s/2\right\} + \mathbb{P}\left\{\sup_{0 \leq t \leq (1+G)t_0 b_n^{1/\beta}} \frac{\sigma_0 X(t)}{b_n^{H_0/\beta}} > s/2\right\} \\ &\leq n\mathbb{P}\left\{\sup_{0 \leq t \leq (1+G)t_0} \frac{X_i(t)}{1+ct^\beta} > f(n, s)\right\} + \mathbb{P}\left\{\sup_{0 \leq t \leq (1+G)t_0} \sigma_0 X(t) > \frac{s}{2}\right\}, \end{aligned}$$

where

$$f(n, s) := \frac{sb_n^{(H_0-H)/\beta}}{2(1+c(1+G)^\beta t_0^\beta)} + b_n^{1-H/\beta}, \quad n \in \mathbb{N}, s \geq L.$$

We have, from Proposition 2.1, that for all large n and s

$$(44) \quad \begin{aligned} n\mathbb{P}\left\{\sup_{0 \leq t \leq (1+G)t_0} \frac{X_i(t)}{1+ct^\beta} > f(n, s)\right\} &\leq 2 \frac{A^{\frac{3}{2}-\frac{2}{\alpha}} U^{\frac{1}{\alpha}} \mathcal{H}_\alpha}{2^{\frac{1}{\alpha}} B^{\frac{1}{2}}} \frac{f(n, s)^{-2}}{\overleftarrow{K}(f(n, s)^{-1})} \exp\left(-\left(\frac{f(n, s)^2}{2A^2} - \log n\right)\right) \\ &\leq \frac{A^{\frac{3}{2}-\frac{2}{\alpha}} U^{\frac{1}{\alpha}} \mathcal{H}_\alpha}{2^{\frac{1}{\alpha}-1} B^{\frac{1}{2}}} f(n, s)^{\gamma_0} \exp\left(-\left(\frac{f(n, s)^2}{2A^2} - \log n\right)\right), \end{aligned}$$

with some $\gamma_0 > 1$ large enough, where the second inequality follows since $(v^2 \overleftarrow{K}(v^{-1}))^{-1}, v > 0$ is a regularly varying function at infinity. By (18) and the assumption $\beta > 2H - H_0$, we have

$$\lim_{n \rightarrow \infty} \frac{b_n^{2(1-H/\beta)}/(2A^2) - \log n}{b_n^{\frac{\beta+H_0-2H}{\beta}}} = 0,$$

and thus

$$(45) \quad \begin{aligned} \frac{f(n, s)^2}{2A^2} - \log n &\geq \left(\frac{b_n^{2(1-H/\beta)}}{2A^2} - \log n\right) + \frac{sb_n^{\frac{\beta+H_0-2H}{\beta}}}{2A^2(1+c(1+G)^\beta t_0^\beta)} \\ &\geq \frac{Lb_n^{\frac{\beta+H_0-2H}{\beta}}}{4A^2(1+c(1+G)^\beta t_0^\beta)} + \frac{s-L}{2A^2(1+c(1+G)^\beta t_0^\beta)} \end{aligned}$$

holds for all large s and large n . Using the C_r inequality (see Lemma A in Appendix) we know

$$(46) \quad f(n, s)^{\gamma_0} \leq \frac{s^{\gamma_0} b_n^{(H_0-H)\gamma_0/\beta}}{2^{\gamma_0}(1+c(1+G)^\beta t_0^\beta)^{\gamma_0}} + b_n^{(1-H/\beta)\gamma_0}.$$

Then, substituting (45) and (46) into (44), we obtain that, for all large enough n ,

$$(47) \quad n\mathbb{P} \left\{ \sup_{0 \leq t \leq (1+G)t_0} \frac{X_i(t)}{1+ct^\beta} > f(n, s) \right\} \leq s^{\gamma_0} \exp \left(-\frac{s}{2A^2(1+c(1+G)^\beta t_0^\beta)} \right).$$

Furthermore, in the light of the Borell-TIS inequality, we get

$$(48) \quad \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \sup_{0 \leq t \leq (1+G)t_0} \sigma_0 X(t) > \frac{s}{2} \right\} ds \leq \int_L^\infty \lambda s^{\lambda-1} \exp \left(-\frac{(s/(2\sigma_0) - K_3)^2}{2(1+G)^{2H} t_0^{2H}} \right) ds \rightarrow 0$$

as $L \rightarrow \infty$, where $K_3 := \mathbb{E} \left\{ \sup_{0 \leq t \leq (1+G)t_0} X(t) \right\} < \infty$. Consequently, it follows from (43) and (47)-(48) that

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} I_{11}(n, s) ds = 0.$$

Next, we show

$$(49) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} I_{12}(n, s) ds = 0,$$

which, together with the above equation, will give the desired result that

$$(50) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} H_1(n, L) = 0.$$

Now, we focus on $I_{12}(n, s)$. By self-similarity, we have, for large n ,

$$(51) \quad \begin{aligned} I_{12}(n, s) &\leq \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq (1+G)t_0 b_n^{1/\beta}} \frac{X_i(t) - (c - \hat{c})t^\beta - b_n}{b_n^{H_0/\beta}} > s/2 \right\} \\ &\quad + \mathbb{P} \left\{ \sup_{t \geq (1+G)t_0 b_n^{1/\beta}} \frac{\sigma_0 X(t) - \hat{c}t^\beta}{b_n^{H_0/\beta}} > s/2 \right\} \\ &\leq n\mathbb{P} \left\{ \sup_{t \geq (1+G)t_0} X_1(t) - (1 + (c - \hat{c})t^\beta) b_n^{1-H/\beta} > b_n^{(H_0-H)/\beta} s/2 \right\} \\ &\quad + \mathbb{P} \left\{ \sup_{t \geq 0} \sigma_0 X(t) - \hat{c}t^\beta > b_n^{H_0/\beta} s/2 \right\} \\ &\leq n\mathbb{P} \left\{ \sup_{t \geq (1+G)t_0} \frac{X_1(t)}{d_{n,s} + (c - \hat{c})t^\beta} > b_n^{1-H/\beta} \right\} + \mathbb{P} \left\{ \sup_{t \geq 0} \sigma_0 X(t) - \hat{c}t^\beta > s/2 \right\}, \end{aligned}$$

with

$$d_{n,s} := 1 + \frac{s}{2} b_n^{H_0/\beta-1}.$$

From Proposition 2.1, we see

$$(52) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \sup_{t \geq 0} \sigma_0 X(t) - \hat{c}t^\beta > s/2 \right\} ds = 0.$$

Furthermore, define

$$g_{n,s}(t) := \frac{t^H}{d_{n,s} + (c - \hat{c})t^\beta}, \quad t \geq 0.$$

By (20), we know the unique maximum point of $g_{n,s}(t)$, $t \geq 0$ is given by

$$t_{n,s}^* = t_0 \left(\frac{c}{c - \hat{c}} d_{n,s} \right)^{1/\beta}.$$

Recalling δ_G defined in (41), we have

$$s \leq \delta_G b_n^{1-H_0/\beta} \Leftrightarrow t_{n,s}^* \leq (1+G)t_0.$$

Therefore, we can divide the following integral into two parts,

$$\begin{aligned}
& \int_L^\infty n\lambda s^{\lambda-1} \mathbb{P} \left\{ \sup_{t \geq (1+G)t_0} \frac{X_1(t)}{d_{n,s} + (c - \hat{c})t^\beta} > b_n^{1-H/\beta} \right\} ds \\
&= \left(\int_L^{\delta_G b_n^{1-H_0/\beta}} + \int_{\delta_G b_n^{1-H_0/\beta}}^\infty \right) n\lambda s^{\lambda-1} \mathbb{P} \left\{ \sup_{t \geq (1+G)t_0} \frac{X_1(t)}{d_{n,s} + (c - \hat{c})t^\beta} > b_n^{1-H/\beta} \right\} ds \\
(53) \quad &=: J_{11}(n, L) + J_{12}(n, L).
\end{aligned}$$

For the first integral $J_{11}(n, L)$, since $s \leq \delta_G b_n^{1-H_0/\beta}$, we obtain

$$\sup_{t \geq (1+G)t_0} g_{n,s}(t) = g_{n,s}((1+G)t_0) = \frac{(t_0(1+G))^H}{d_{n,s} + (c - \hat{c})(t_0(1+G))^\beta},$$

and thus by the Borell-TIS inequality,

$$n\mathbb{P} \left\{ \sup_{t \geq (1+G)t_0} \frac{X_1(t)}{d_{n,s} + (c - \hat{c})t^\beta} > b_n^{1-H/\beta} \right\} \leq n \exp \left(- \frac{(d_{n,s} + (c - \hat{c})(t_0(1+G))^\beta)^2}{2(t_0(1+G))^{2H}} (b_n^{1-H/\beta} - K_4)^2 \right)$$

holds for all large n such that $b_n^{1-H/\beta} > K_4$, where $K_4 := \mathbb{E} \left\{ \sup_{t \geq (1+G)t_0} \frac{X_1(t)}{1+(c-\hat{c})t^\beta} \right\} < \infty$. Since

$$(d_{n,s} + (c - \hat{c})(t_0(1+G))^\beta)^2 \geq (1 + (c - \hat{c})(t_0(1+G))^\beta)^2 + (1 + (c - \hat{c})(t_0(1+G))^\beta) s b_n^{H_0/\beta-1},$$

it follows, by (18), (40) and the assumption $\beta > 2H - H_0$, that, for all large n ,

$$\begin{aligned}
n\mathbb{P} \left\{ \sup_{t \geq (1+G)t_0} \frac{X_1(t)}{d_{n,s} + (c - \hat{c})t^\beta} > b_n^{1-H/\beta} \right\} &\leq n \exp \left(- \frac{(1 + (c - \hat{c})(t_0(1+G))^\beta)^2}{2(t_0(1+G))^{2H}} (b_n^{1-H/\beta} - K_4)^2 \right) \\
&\quad \times \exp \left(- \frac{(1 + (c - \hat{c})(t_0(1+G))^\beta) s}{4(t_0(1+G))^{2H}} b_n^{(\beta+H_0-2H)/\beta} \right) \\
&\leq \exp(-K_0 s),
\end{aligned}$$

holds, with some constant $K_0 > 0$. Thus,

$$(54) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} J_{11}(n, L) \leq \lim_{L \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} e^{-K_0 s} ds = 0.$$

For the second integral $J_{12}(n, L)$, since $s \geq \delta_G b_n^{1-H_0/\beta}$, we have

$$\sup_{t \geq (1+G)t_0} g_{n,s}(t) = g_{n,s}(t_{n,s}^*) = \left(\frac{c}{c - \hat{c}} \right)^{H/\beta} d_{n,s}^{H/\beta-1} A,$$

and thus by Borell-TIS inequality, for large enough n ,

$$n\mathbb{P} \left\{ \sup_{t \geq (1+G)t_0} \frac{X_1(t)}{d_{n,s} + (c - \hat{c})t^\beta} > b_n^{1-H/\beta} \right\} \leq n \exp \left(- \frac{1}{2A^2} \left(\frac{c}{c - \hat{c}} \right)^{-2H/\beta} d_{n,s}^{2(1-H/\beta)} (b_n^{1-H/\beta} - K_4)^2 \right).$$

Using a change of variable

$$v = \frac{s - \delta_G b_n^{1-H_0/\beta}}{b_n^{1-H_0/\beta}},$$

and the C_r inequality, we get

$$\begin{aligned}
d_{n,s}^{2(1-H/\beta)} &= (1 + \delta_G/2)^{2(1-H/\beta)} (1 + v/(2 + \delta_G))^{2(1-H/\beta)} \\
&\geq \frac{1}{2} (1 + \delta_G/2)^{2(1-H/\beta)} \left(1 + v^{2(1-H/\beta)}/(2 + \delta_G)^{2(1-H/\beta)} \right).
\end{aligned}$$

Therefore, for all large n ,

$$\begin{aligned}
J_{12}(n, L) &\leq \int_{\delta_G b_n^{1-H_0/\beta}}^{\infty} n \lambda s^{\lambda-1} \exp\left(-\frac{1}{2A^2} \left(\frac{c}{c-\hat{c}}\right)^{-2H/\beta} d_{n,s}^{2(1-H/\beta)} (b_n^{1-H/\beta} - K_4)^2\right) ds \\
&\leq \lambda n b_n^{\lambda(1-H_0/\beta)} \exp\left(-\frac{1}{4A^2} \left(\frac{c}{c-\hat{c}}\right)^{-2H/\beta} (1+\delta_G/2)^{2(1-H/\beta)} (b_n^{1-H/\beta} - K_4)^2\right) \\
&\quad \times \int_0^{\infty} (v+\delta_G)^{\lambda-1} \exp\left(-\frac{1}{4A^2} \left(\frac{c}{c-\hat{c}}\right)^{-2H/\beta} \left(\frac{v}{2}\right)^{2(1-H/\beta)} (b_n^{1-H/\beta} - K_4)^2\right) dv \\
&\leq \lambda n b_n^{\lambda(1-H_0/\beta)} \exp\left(-\frac{1}{A^2} (b_n^{1-H/\beta} - K_4)^2\right) \times \int_0^{\infty} (v+\delta_G)^{\lambda-1} \exp\left(-v^{2(1-H/\beta)}\right) dv,
\end{aligned}$$

where in the last inequality we have used (42). This, together with (18), implies

$$(55) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} J_{12}(n, L) = 0.$$

Consequently, substituting (54) and (55) into (53), and recalling (51)-(52), we prove the claim in (49). This gives us the desired result presented in (50).

Now consider $H_2(n, L)$. We have, by self-similarity and symmetry of Normal distribution, that

$$\begin{aligned}
\mathbb{P}\left\{b_n^{-H_0/\beta}(M_n^{(k)} - b_n) < -s\right\} &\leq \mathbb{P}\left\{\frac{\max_{i \leq n}^{(k)} X_i(t_0 b_n^{1/\beta}) + \sigma_0 X(t_0 b_n^{1/\beta}) - (1+ct_0^\beta)b_n}{b_n^{H_0/\beta}} < -s\right\} \\
&= \mathbb{P}\left\{\sigma_0 X(t_0) + \left(\max_{i \leq n}^{(k)} X_i(t_0) - (1+ct_0^\beta)b_n^{1-H/\beta}\right) b_n^{(H-H_0)/\beta} < -s\right\} \\
&\leq \mathbb{P}\left\{X(1) < -\frac{s}{2t_0^{H_0}\sigma_0}\right\} + \mathbb{P}\left\{\left(\max_{i \leq n}^{(k)} X_i(1) - \frac{1+ct_0^\beta}{t_0^H} b_n^{1-H/\beta}\right) b_n^{(H-H_0)/\beta} < -\frac{s}{2t_0^H}\right\} \\
&=: I_{21}(s) + I_{22}(n, s).
\end{aligned}$$

Obviously,

$$(56) \quad \lim_{L \rightarrow \infty} \lim_{n \rightarrow \infty} \int_L^{\infty} \lambda s^{\lambda-1} I_{21}(s) ds \leq \lim_{L \rightarrow \infty} \int_L^{\infty} \lambda s^{\lambda-1} \frac{2t_0^{H_0}\sigma_0}{\sqrt{2\pi}s} \exp\left(-\frac{s^2}{8t_0^{2H_0}\sigma_0^2}\right) ds = 0.$$

Recalling $Z_n^{(k)}$ and r_n as defined in (24)-(25), we obtain, by (26) and (27), that

$$\begin{aligned}
I_{22}(n, s) &= \mathbb{P}\left\{\frac{Z_n^{(k)}}{\sqrt{2 \log n b_n^{(H_0-H)/\beta}}} - r_n < -\frac{s}{2t_0^H}\right\} \\
&= \mathbb{P}\left\{Z_n^{(k)} < -\sqrt{2 \log n b_n^{(H_0-H)/\beta}} \left(\frac{s}{2t_0^H} - r_n\right)\right\} \\
&\leq \mathbb{P}\left\{Z_n^{(k)} < -s\right\} \leq \mathbb{P}\left\{|Z_n^{(k)}| > s\right\} \\
&\leq s^{-\kappa} \mathbb{E}\left\{|Z_n^{(k)}|^\kappa\right\}
\end{aligned}$$

holds for any $\kappa > 0$ and all large n and L , where the last inequality follows from Markov inequality. Choosing $\kappa > \lambda$ and then by Proposition 2.3, we conclude that

$$(57) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^{\infty} \lambda s^{\lambda-1} I_{22}(n, s) ds \leq \lim_{L \rightarrow \infty} \frac{\lambda}{\kappa - \lambda} \mathbb{E}\left\{|\Lambda^{(k)}|^\kappa\right\} L^{\lambda-\kappa} = 0.$$

Therefore, combining (56)-(57) yields

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} H_2(n, L) = 0,$$

which together with (50) establishes (39). This completes the proof for scenario (i).

4.3.2. *Proof for (ii) and (iii).* The idea of proof for these two scenarios is similar to that for scenario (i), thus we shall highlight the differences and omit some of the details when similar arguments in the proof for scenario (i) are applicable here. It is sufficient to show

$$(58) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \left| a_n^{-1} \left(M_n^{(k)} - b_n \right) \right| > s \right\} ds = 0.$$

Note that

$$\begin{aligned} & \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \left| a_n^{-1} \left(M_n^{(k)} - b_n \right) \right| > s \right\} ds \\ & \leq \int_L^\infty \lambda s^{\lambda-1} \left(\mathbb{P} \left\{ a_n^{-1} \left(M_n^{(1)} - b_n \right) > s \right\} + \mathbb{P} \left\{ a_n^{-1} \left(M_n^{(k)} - b_n \right) < -s \right\} \right) ds \\ & =: H_1(n, L) + H_2(n, L). \end{aligned}$$

Below we shall deal with $H_1(n, s)$ and $H_2(n, s)$, separately.

Consider $H_1(n, L)$. As before, we can choose a large $G > 0$ and some small $\hat{c} \in (0, c)$ such that (40)-(42) hold.

It follows that

$$\begin{aligned} \mathbb{P} \left\{ a_n^{-1} \left(M_n^{(1)} - b_n \right) > s \right\} & \leq \mathbb{P} \left\{ \max_{i \leq n} \sup_{0 \leq t \leq (1+G)t_0 b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{a_n} > s \right\} \\ & \quad + \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq (1+G)t_0 b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{a_n} > s \right\} \\ & =: I_{11}(n, s) + I_{12}(n, s). \end{aligned}$$

By self-similarity, we have

$$(59) \quad \begin{aligned} I_{11}(n, s) & \leq \mathbb{P} \left\{ \max_{i \leq n} \sup_{t \geq 0} \frac{X_i(t) - ct^\beta - b_n}{a_n} > s/2 \right\} + \mathbb{P} \left\{ \sup_{0 \leq t \leq (1+G)t_0 b_n^{1/\beta}} \frac{\sigma_0 X(t)}{a_n} > s/2 \right\} \\ & = \mathbb{P} \left\{ a_n^{-1} \left(\widetilde{M}_n^{(1)} - b_n \right) > s/2 \right\} + \mathbb{P} \left\{ \sup_{0 \leq t \leq (1+G)t_0} \sigma_0 X(t) > a_n b_n^{-H_0/\beta} s/2 \right\}. \end{aligned}$$

For the first term, we have from the Markov inequality and (15) with $m_n = n$ (choosing $\kappa > \lambda$) that

$$(60) \quad \begin{aligned} & \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ a_n^{-1} \left(\widetilde{M}_n^{(1)} - b_n \right) > s/2 \right\} ds \\ & \leq \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{2^\kappa \lambda L^{\lambda-\kappa}}{\kappa - \lambda} \mathbb{E} \left\{ \left| a_n^{-1} \left(\widetilde{M}_n^{(1)} - b_n \right) \right|^\kappa \right\} = 0. \end{aligned}$$

Next, recalling (17) and using a similar argument as in (48), we obtain

$$(61) \quad \begin{aligned} & \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \sup_{0 \leq t \leq (1+G)t_0} \sigma_0 X(t) > a_n b_n^{-H_0/\beta} s/2 \right\} ds \\ & \leq \lim_{L \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \sup_{0 \leq t \leq (1+G)t_0} \sigma_0 X(t) > sA^2/(2\tau) \right\} ds = 0. \end{aligned}$$

Consequently, by (59)-(61) we have

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} I_{11}(n, s) ds = 0.$$

In order to obtain the desired result that

$$(62) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} H_1(n, L) = 0,$$

it remains to show

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} I_{12}(n, s) ds = 0,$$

which results from drawing the same arguments as in the proof of scenario (i), by replacing $b_n^{H_0/\beta}$ with a_n and noting that $\lim_{n \rightarrow \infty} a_n = \infty$ under the assumption of scenarios (ii) and (iii). The details are omitted. Now consider $H_2(n, L)$. It is worth mentioning that we cannot get useful upper bounds by simply taking a single point $t_0 b_n^{1/\beta}$ as in scenario (i). Instead, we shall use a suitable interval around $t_0 b_n^{1/\beta}$ as follows. By self-similarity, we have, for any $\varepsilon_0 \in (0, t_0)$,

$$\begin{aligned}
& \mathbb{P} \left\{ a_n^{-1} \left(M_n^{(k)} - b_n \right) < -s \right\} \leq \mathbb{P} \left\{ \max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0) b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) + \sigma_0 X(t) - ct^\beta - b_n}{a_n} < -s \right\} \\
& \leq \mathbb{P} \left\{ \max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0) b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{a_n} - \sup_{(t_0 - \varepsilon_0) b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{-\sigma_0 X(t)}{a_n} < -s \right\} \\
& \leq \mathbb{P} \left\{ - \sup_{(t_0 - \varepsilon_0) b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{-\sigma_0 X(t)}{a_n} < -\frac{s}{2} \right\} \\
& \quad + \mathbb{P} \left\{ \max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0) b_n^{1/\beta} \leq t \leq (t_0 + \varepsilon_0) b_n^{1/\beta}} \frac{X_i(t) - ct^\beta - b_n}{a_n} < -\frac{s}{2} \right\} \\
& \leq \mathbb{P} \left\{ \sup_{t_0 - \varepsilon_0 \leq t \leq t_0 + \varepsilon_0} \sigma_0 X(t) > \frac{a_n}{2 b_n^{H_0/\beta}} s \right\} \\
& \quad + \mathbb{P} \left\{ (1 + c(t_0 + \varepsilon_0)^\beta) \frac{b_n^{H/\beta}}{a_n} \left(\max_{i \leq n}^{(k)} \sup_{(t_0 - \varepsilon_0) \leq t \leq (t_0 + \varepsilon_0)} \frac{X_i(t)}{1 + ct^\beta} - b_n^{1-H/\beta} \right) < -\frac{s}{2} \right\} \\
& =: I_{21}(n, s) + I_{22}(n, s).
\end{aligned}$$

As shown in (61) we can obtain

$$(63) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} I_{21}(n, s) ds = 0.$$

In order to analyse $I_{22}(n, s)$, we shall introduce some further notation. Denote

$$\tilde{Y}_i = \sup_{(t_0 - \varepsilon_0) \leq t \leq (t_0 + \varepsilon_0)} \frac{X_i(t)}{1 + ct^\beta}, \quad i = 1, 2, \dots$$

We obtain, from Proposition 2.1, that

$$\mathbb{P} \left\{ \tilde{Y}_i > v \right\} = \tilde{R}(v) \exp \left(-\frac{v^2}{2A^2} \right) (1 + o(1)), \quad v \rightarrow \infty,$$

where

$$\tilde{R}(v) = \frac{A^{\frac{3}{2} - \frac{2}{\alpha}} \mathcal{H}_\alpha}{2^{\frac{1}{\alpha}} B^{\frac{1}{2}}} \frac{v^{-2}}{K(v^{-1})}, \quad v > 0.$$

Define

$$\begin{aligned}
\tilde{b}_n & := A(2 \log n)^{1/2} + A(2 \log n)^{-1/2} \log(\tilde{R}((2A^2 \log n)^{1/2})), \quad n \in \mathbb{N}, \\
\tilde{a}_n & := A(2 \log n)^{-1/2}, \quad n \in \mathbb{N}.
\end{aligned}$$

By Proposition 2.2, we have

$$\tilde{a}_n^{-1} \left(\max_{i \leq n}^{(k)} \tilde{Y}_i - \tilde{b}_n \right) \xrightarrow{d} \Lambda^{(k)}, \quad n \rightarrow \infty.$$

Next, it can be checked that

$$\frac{b_n^{H/\beta}}{a_n} = \frac{\tau}{2A^2} (2A^2 \log n)^{\frac{H}{\beta\tau} - \frac{1}{\tau} + 1} (1 + o(1)) = \frac{\tau}{2A^2} (2A^2 \log n)^{1/2} (1 + o(1)), \quad n \rightarrow \infty,$$

and thus

$$\lim_{n \rightarrow \infty} (1 + c(t_0 \pm \varepsilon_0)^\beta) \frac{b_n^{H/\beta}}{a_n} \tilde{a}_n = A \frac{1 + c(t_0 \pm \varepsilon_0)^\beta}{t_0^H} =: A_{\varepsilon_0}.$$

Further, by using second-order Taylor expansion, as $n \rightarrow \infty$,

$$b_n^{1-H/\beta} = A\sqrt{2\log n} \left(1 + \frac{1}{2}(\log n)^{-1} \log(R((2A^2 \log n)^{\frac{\beta}{2(\beta-H)}})) + O\left((\log n)^{-2}(\log(R((2A^2 \log n)^{\frac{\beta}{2(\beta-H)}})))^2\right) \right).$$

Moreover, by definition

$$R((2A^2 \log n)^{\frac{\beta}{2(\beta-H)}}) = \tilde{R}((2A^2 \log n)^{1/2}) = \frac{A^{\frac{3}{2}-\frac{2}{\alpha}} \mathcal{H}_\alpha}{2^{\frac{1}{\alpha}} B^{\frac{1}{2}}} \frac{(2A^2 \log n)^{-1}}{K((2A^2 \log n)^{-1/2})}.$$

Hence, we derive that

$$(64) \quad \lim_{n \rightarrow \infty} \tilde{a}_n^{-1} (\tilde{b}_n - b_n^{1-H/\beta}) = 0,$$

and thus for all large n ,

$$\begin{aligned} I_{22}(n, s) &= \mathbb{P} \left\{ (1 + c(t_0 + \varepsilon_0)^\beta) \frac{b_n^{H/\beta}}{a_n} \tilde{a}_n \left(\frac{\max_{i \leq n}^{(k)} \tilde{Y}_i - \tilde{b}_n}{\tilde{a}_n} + \frac{\tilde{b}_n - b_n^{1-H/\beta}}{\tilde{a}_n} \right) < -\frac{s}{2} \right\} \\ &\leq \mathbb{P} \left\{ \tilde{a}_n^{-1} \left(\max_{i \leq n}^{(k)} \tilde{Y}_i - \tilde{b}_n \right) < -\frac{s}{4A_{\varepsilon_0}} \right\} \\ &\leq (4A_{\varepsilon_0})^\kappa s^{-\kappa} \mathbb{E} \left\{ \left| \tilde{a}_n^{-1} \left(\max_{i \leq n}^{(k)} \tilde{Y}_i - \tilde{b}_n \right) \right|^\kappa \right\} \end{aligned}$$

holds for any $\kappa > 0$. In view of the definition of \tilde{Y}_1 , it follows that

$$\mathbb{P} \left\{ \tilde{Y}_1 \leq -x \right\} \leq \mathbb{P} \left\{ \frac{X_1(t_0)}{1 + ct_0^\beta} \leq -x \right\} \leq \frac{A}{\sqrt{2\pi}x} e^{-\frac{x^2}{2A^2}}, \quad \forall x > 0,$$

fulfilling (9), and thus we conclude from Proposition 2.3 that, for a chosen $\kappa > \lambda$,

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} I_{22}(n, s) ds = 0.$$

This, together with (63), implies

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} H_2(n, L) = 0.$$

Consequently, from the above equation and (62) we establish (58), and thus the proof for scenarios (ii) and (iii) is complete.

4.4. Proof of Theorem 3.5. The proof follows from the same lines as the proofs of Theorem 3.1 and Theorem 3.3, by applying Proposition 2.4 and utilising two types of inequalities for some of the bounds therein. These two types of inequalities are akin to the following:

- A lower bound using $\max_{i \leq n}^{(k)} X_i(t_0 b_n^{1/\beta}) \geq \max_{i \leq m_n}^{(k)} X_i(t_0 b_n^{1/\beta})$ in (21).
- An upper bound using $\max_{i \leq n}^{(k)} \sup_{0 \leq t \leq (1-\varepsilon_0)b_n^{1/\beta}} (X_i(t) - c_i t^\beta) \leq \max_{i \leq n}^{(k)} \sup_{0 \leq t \leq (1-\varepsilon_0)b_n^{1/\beta}} (X_i(t) - ct^\beta)$ in (30).

Thus, we omit the details. The proof is complete.

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APPENDIX

In this appendix, we present proofs for the propositions displayed in Section 2.2. We also include the C_r inequalities that have been frequently used in our proofs.

Proof of Proposition 2.2: The proof follows closely from some existing results. First, thanks to the closure property of maximum domain of attraction of the Gumbel distribution under tail equivalence (see, e.g., Proposition 3.3.28 in [18]), the weak limit result for $k = 1$ (i.e., the Gumbel limit theorem for the maximum) follows similarly to Theorem 1.5.3 in [27] by noting that $\lim_{n \rightarrow \infty} n(1 - F(\mu_n + \nu_n x)) = e^{-x}$, $\forall x \in \mathbb{R}$. Secondly, for general fixed $k > 1$ the result follows by an application of Theorem 2.2.2 in [27] where it is shown that for an IID sequence the convergence for maxima is equivalent to the convergence for order statistics. \square

Proof of Proposition 2.3: By Proposition 2.2 and the same arguments as those used in the proof of Proposition 2.1 of [37], we only need to show that

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \left| \nu_n^{-1} \left(Y_n^{(k)} - \mu_n \right) \right| > s \right\} ds = 0.$$

Further, note that

$$\begin{aligned} \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \left| \nu_n^{-1} \left(Y_n^{(k)} - \mu_n \right) \right| > s \right\} ds &\leq \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \nu_n^{-1} \left(Y_n^{(k)} - \mu_n \right) < -s \right\} ds \\ (65) \quad &+ \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \nu_n^{-1} \left(Y_n^{(k)} - \mu_n \right) > s \right\} ds \\ &=: I_1(n, L) + I_2(n, L). \end{aligned}$$

We shall first focus on $I_1(n, L)$. It can be checked that (cf. Proposition 4.1.2 in [18])

$$\mathbb{P} \left\{ \nu_n^{-1} \left(Y_n^{(k)} - \mu_n \right) < -s \right\} = \sum_{j=0}^{k-1} \binom{n}{j} (\mathbb{P} \{ Y_1 \geq \mu_n - \nu_n s \})^j (\mathbb{P} \{ Y_1 \leq \mu_n - \nu_n s \})^{n-j}.$$

By Stirling's approximation, we see that, to verify

$$(66) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} I_1(n, L) = 0,$$

it suffices to show, for any fixed $j = 0, \dots, k-1$,

$$(67) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_L^\infty s^{\lambda-1} n^j (\mathbb{P} \{ Y_1 \geq \mu_n - \nu_n s \})^j (\mathbb{P} \{ Y_1 \leq \mu_n - \nu_n s \})^{n-j} ds = 0.$$

We prove this equality by dividing the above integral into three parts as follows (with $\omega_n = \sqrt{C^{-1} \log n}$):

$$\begin{aligned} &\int_L^\infty s^{\lambda-1} n^j (\mathbb{P} \{ Y_1 \geq \mu_n - \nu_n s \})^j (\mathbb{P} \{ Y_1 \leq \mu_n - \nu_n s \})^{n-j} ds \\ &= \left(\int_L^{\omega_n} + \int_{\omega_n}^{(\mu_n+L)/\nu_n} + \int_{(\mu_n+L)/\nu_n}^\infty \right) s^{\lambda-1} n^j (\mathbb{P} \{ Y_1 \geq \mu_n - \nu_n s \})^j (\mathbb{P} \{ Y_1 \leq \mu_n - \nu_n s \})^{n-j} ds \\ (68) \quad &=: Q_1(n, L) + Q_2(n, L) + Q_3(n, L), \end{aligned}$$

which is always valid for sufficiently large n .

We first consider $Q_1(n, L)$. By the fact that $\log(1-x) \leq -x, x \in (0, 1)$, we obtain

$$\begin{aligned} & n^j (\mathbb{P}\{Y_1 > \mu_n - \nu_n s\})^j (\mathbb{P}\{Y_1 \leq \mu_n - \nu_n s\})^{n-j} \\ & \leq n^j (\mathbb{P}\{Y_1 > \mu_n - \nu_n s\})^j \exp(-(n-j)\mathbb{P}\{Y_1 > \mu_n - \nu_n s\}). \end{aligned}$$

Below, we consider uniform bounds for $\mathbb{P}\{Y_1 > \mu_n - \nu_n s\}$, for all large enough n and all $s \in [L, \omega_n]$. Note

$$(69) \quad \sup_{s \in [L, \omega_n]} \nu_n s / \mu_n \rightarrow 0$$

as $n \rightarrow \infty$, then by the tail asymptotics of Y_1 in (7) we have

$$\frac{1}{2}\rho(\mu_n - \nu_n s) \exp(-C(\mu_n - \nu_n s)^\tau) \leq \mathbb{P}\{Y_1 > \mu_n - \nu_n s\} \leq 2\rho(\mu_n - \nu_n s) \exp(-C(\mu_n - \nu_n s)^\tau)$$

holds for all large enough n and all $s \in [L, \omega_n]$, and thus

$$\begin{aligned} & n^j (\mathbb{P}\{Y_1 > \mu_n - \nu_n s\})^j (\mathbb{P}\{Y_1 \leq \mu_n - \nu_n s\})^{n-j} \\ & \leq (2n\rho(\mu_n - \nu_n s))^j \exp(-jC(\mu_n - \nu_n s)^\tau) \\ & \quad \times \exp\left(-\frac{n-j}{2}\rho(\mu_n - \nu_n s) \exp(-C(\mu_n - \nu_n s)^\tau)\right). \end{aligned}$$

Next, we derive uniform bounds for $(\mu_n - \nu_n s)^\tau$, for all $s \in [L, \omega_n]$. It can be checked that

$$(70) \quad 1 - \tau_M x \leq (1-x)^\tau \leq 1 - \tau_m x, \quad \forall x \in [0, 1],$$

where $\tau_M = \max(\tau, 1), \tau_m = \min(\tau, 1)$. The Taylor's expansion yields

$$(71) \quad \mu_n^\tau = C^{-1} \log n \left(1 + \frac{\log(\rho((C^{-1} \log n)^{1/\tau}))}{\log n} + O\left(\frac{\log(\rho((C^{-1} \log n)^{1/\tau}))}{\log n}\right)^2 \right), \quad n \rightarrow \infty.$$

Thus, it follows from (69)-(71), that, for all large enough n and all $s \in [L, \omega_n]$,

$$(72) \quad \begin{aligned} (\mu_n - \nu_n s)^\tau & \leq \mu_n^\tau - \tau_m \nu_n \mu_n^{\tau-1} s \\ & \leq C^{-1} \log n + C^{-1} \log\left(\rho((C^{-1} \log n)^{1/\tau})\right) + 1 - \tau_m s / (2C\tau), \end{aligned}$$

and

$$\begin{aligned} (\mu_n - \nu_n s)^\tau & \geq \mu_n^\tau - \tau_M \nu_n \mu_n^{\tau-1} s \\ & \geq C^{-1} \log n + C^{-1} \log\left(\rho((C^{-1} \log n)^{1/\tau})\right) - 1 - 2\tau_M s / (C\tau). \end{aligned}$$

Therefore, by (69) and the Uniform Convergence Theorem (cf. Theorem 1.5.2 in [5]), we get

$$\lim_{n \rightarrow \infty} \sup_{s \in [L, \omega_n]} \left| \frac{\rho(\mu_n - \nu_n s)}{\rho((C^{-1} \log n)^{1/\tau})} - 1 \right| = 0,$$

and thus for all large enough n and all $s \in [L, \omega_n]$,

$$n^j (\mathbb{P}\{Y_1 > \mu_n - \nu_n s\})^j (\mathbb{P}\{Y_1 \leq \mu_n - \nu_n s\})^{n-j} \leq 4^j \exp\left(jC + \frac{2\tau_M j s}{\tau} - \frac{1}{4} \exp\left(-C + \frac{\tau_m s}{2\tau}\right)\right),$$

implying

$$(73) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} Q_1(n, L) \leq \lim_{L \rightarrow \infty} \int_L^\infty 4^j s^{\lambda-1} \exp\left(jC + \frac{2\tau_M j s}{\tau} - \frac{1}{4} \exp\left(-C + \frac{\tau_m s}{2\tau}\right)\right) ds = 0.$$

Now consider $Q_2(n, L)$. Similarly as before, we have, for all large enough n ,

$$\begin{aligned} Q_2(n, L) & \leq n^j ((\mu_n + L)/\nu_n)^{\lambda+1} (\mathbb{P}\{Y_1 \leq \mu_n - \nu_n \omega_n\})^{n-j} \\ & \leq n^j ((\mu_n + L)/\nu_n)^{\lambda+1} \exp\left(-\frac{n-j}{2}\rho(\mu_n - \nu_n \omega_n) \exp(-C(\mu_n - \nu_n \omega_n)^\tau)\right). \end{aligned}$$

Furthermore, it follows from an application of the upper bound in (72) with $s = \omega_n$ and the Uniform Convergence Theorem that

$$\exp\left(-\frac{n-j}{2}\rho(\mu_n - \nu_n\omega_n)\exp(-C(\mu_n - \nu_n\omega_n)^\tau)\right) \leq \exp\left(-\frac{1}{4}\exp\left(-C + \frac{\tau_m\omega_n}{2\tau}\right)\right).$$

Therefore,

$$(74) \quad \begin{aligned} & \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} Q_2(n, L) \\ & \leq \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} (2\tau \log n)^{\lambda+1} \exp\left(-\frac{1}{4}\exp\left(-C + \frac{\tau_m\sqrt{C^{-1}\log n}}{2\tau}\right) + j \log n\right) = 0. \end{aligned}$$

For $Q_3(n, L)$, we have, by assumption (9) that, for any large L ,

$$\begin{aligned} Q_3(n, L) & \leq \int_{(\mu_n+L)/\nu_n}^{\infty} s^{\lambda-1} n^j (\mathbb{P}\{Y_1 \leq \mu_n - \nu_n s\})^{n-j} ds \\ & \leq \text{Const.} \cdot \int_{(\mu_n+L)/\nu_n}^{\infty} s^{\lambda-1} n^j (\nu_n s - \mu_n)^{-(n-j)\eta} ds \\ & \leq \text{Const.} \cdot n^j \nu_n^{-\lambda} L^{-(n-j)\eta+1} \int_1^{\infty} (\mu_n^{\lambda-1} + (tL)^{\lambda-1}) t^{-(n-j)\eta} dt \\ & \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, where in the third inequality we used a change of variable $t = (\nu_n s - \mu_n)/L$ and the C_r inequality. Thus,

$$(75) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} Q_3(n, L) = 0.$$

Consequently, the claim (66) follows by combing (67)-(68) and (73)-(75).

Now, it remains to show

$$(76) \quad \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} I_2(n, L) = 0.$$

To this end, we shall look for suitable upper bounds of $I_2(n, L)$ for all large enough n, L . It follows that

$$\begin{aligned} I_2(n, L) & \leq \int_L^{\infty} \lambda s^{\lambda-1} \mathbb{P}\left\{\nu_n^{-1}(Y_n^{(1)} - \mu_n) > s\right\} ds \\ & \leq \lambda n \int_L^{\infty} s^{\lambda-1} \mathbb{P}\{Y_1 > \mu_n + \nu_n s\} ds \\ & = \lambda n \mathbb{P}\{Y_1 > \mu_n\} \int_L^{\infty} s^{\lambda-1} \frac{\mathbb{P}\{Y_1 > \mu_n + \nu_n s\}}{\mathbb{P}\{Y_1 > \mu_n\}} ds. \end{aligned}$$

It is easy to check that $\lim_{n \rightarrow \infty} n \mathbb{P}\{Y_1 > \mu_n\} = 1$. We now proceed to find suitable uniform integrable bounds for $\mathbb{P}\{Y_1 > \mu_n + \nu_n s\} / \mathbb{P}\{Y_1 > \mu_n\}$, for all large n, s . By (7) we have

$$\frac{\mathbb{P}\{Y_1 > \mu_n + \nu_n s\}}{\mathbb{P}\{Y_1 > \mu_n\}} \leq 2 \frac{\rho(\mu_n + \nu_n s)}{\rho(\mu_n)} e^{-C((\mu_n + \nu_n s)^\tau - \mu_n^\tau)}$$

holds for all large enough n, s . Using the Potter's bounds and the C_r inequality, we can show that

$$\frac{\rho(\mu_n + \nu_n s)}{\rho(\mu_n)} = \frac{\rho(\mu_n(1 + \nu_n s/\mu_n))}{\rho(\mu_n)} \leq D_0(1 + s^{2|\gamma|})$$

holds for all large enough n, s , with some constant $D_0 > 0$ independent of n, s , where we recall that γ is the regularly varying index of $\rho(\cdot)$. Thus, we obtain, for all large n, L ,

$$(77) \quad I_2(n, L) \leq 4\lambda D_0 \int_L^{\infty} s^{\lambda-1} (1 + s^{2|\gamma|}) e^{-C\mu_n^\tau((1 + \nu_n s/\mu_n)^\tau - 1)} ds.$$

In order to obtain upper bounds for the exponential term in (77), we shall distinguish case $\tau \geq 1$ and case $\tau < 1$. For the case $\tau \geq 1$, it is obvious that $(1 + \nu_n s / \mu_n)^\tau \geq (1 + \nu_n s / \mu_n)$ and thus

$$I_2(n, L) \leq 4\lambda D_0 \int_L^\infty s^{\lambda-1} (1 + s^{2|\gamma|}) e^{-\frac{s}{2\tau}} ds$$

for all large n . This yields (76) for $\tau \geq 1$. For the case $\tau < 1$, we first fix some large $L_0 > 0$ such that

$$(1+x)^\tau \geq 1 + \frac{1}{2}x^\tau, \quad \forall x > L_0,$$

and then, we choose some $a \in (0, \tau(1 + L_0)^{\tau-1})$ such that

$$(1+x)^\tau \geq 1 + ax, \quad \forall x \in [0, L_0].$$

From the above two inequalities, we can obtain that

$$(1 + \nu_n s / \mu_n)^\tau \geq \begin{cases} 1 + \frac{1}{2}(\nu_n s / \mu_n)^\tau, & \text{if } s > L_0 \mu_n / \nu_n, \\ 1 + a \nu_n s / \mu_n, & \text{if } s \leq L_0 \mu_n / \nu_n. \end{cases}$$

Further, noting that $\lim_{n \rightarrow \infty} \nu_n = \infty$ for $\tau < 1$, we derive, for all large n, L ,

$$\begin{aligned} I_2(n, L) &\leq 4\lambda D_0 \int_L^{L_0 \mu_n / \nu_n} s^{\lambda-1} (1 + s^{2|\gamma|}) e^{-\frac{as}{2\tau}} ds \\ &\quad + 4\lambda D_0 \int_{L_0 \mu_n / \nu_n}^\infty s^{\lambda-1} (1 + s^{2|\gamma|}) e^{-s^\tau} ds. \end{aligned}$$

This implies (76) for $\tau < 1$. Therefore, (76) is established for all $\tau > 0$, and thus the proof is complete. \square

Proof of Proposition 2.4: If $m_n = n$ (i.e., $c = c_i, i \geq 1$) then the claim follows immediately from Propositions 2.2 and 2.3 for the IID sequence $\{\tilde{Q}_i\}_{i \geq 1}$. We now focus on the non-stationary case where $m_n < n$. To show (14) is equivalent to show that, for any $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ a_{m_n}^{-1} (\tilde{M}_n^{(k)} - b_{m_n}) > x \right\} = \mathbb{P} \left\{ \Lambda^{(k)} > x \right\}.$$

Clearly, we have

$$\mathbb{P} \left\{ a_{m_n}^{-1} (\tilde{M}_n^{(k)} - b_{m_n}) > x \right\} \geq \mathbb{P} \left\{ a_{m_n}^{-1} (\tilde{M}_{m_n}^{(k)} - b_{m_n}) > x \right\}$$

and

$$\mathbb{P} \left\{ a_{m_n}^{-1} (\tilde{M}_n^{(k)} - b_{m_n}) > x \right\} \leq \mathbb{P} \left\{ a_{m_n}^{-1} (\tilde{M}_{m_n}^{(k)} - b_{m_n}) > x \right\} + \mathbb{P} \left\{ \cup_{m_n < l \leq n} (\tilde{Q}_l > b_{m_n} + a_{m_n} x) \right\}.$$

We have already shown that $\lim_{n \rightarrow \infty} \mathbb{P} \left\{ a_{m_n}^{-1} (\tilde{M}_{m_n}^{(k)} - b_{m_n}) > x \right\} = \mathbb{P} \left\{ \Lambda^{(k)} > x \right\}$. Next, note that $A = A(c)$ defined in (3) as a function of c is strictly decreasing. It follows from Proposition 2.1 that, for any $l > m_n$,

$$\mathbb{P} \left\{ \tilde{Q}_l > b_{m_n} + a_{m_n} x \right\} = o \left(\mathbb{P} \left\{ \tilde{Q}_1 > b_{m_n} + a_{m_n} x \right\} \right), \quad n \rightarrow \infty.$$

Thus,

$$\mathbb{P} \left\{ \cup_{m_n < l \leq n} (\tilde{Q}_l > b_{m_n} + a_{m_n} x) \right\} \leq (n - m_n) o \left(\mathbb{P} \left\{ \tilde{Q}_1 > b_{m_n} + a_{m_n} x \right\} \right) \rightarrow 0, \quad n \rightarrow \infty,$$

where we use the fact that

$$(n - m_n) \mathbb{P} \left\{ \tilde{Q}_1 > b_{m_n} + a_{m_n} x \right\} \rightarrow (1 - p) e^{-x}, \quad n \rightarrow \infty.$$

Consequently, the claim in (14) follows. Next we show that (15) can be established similarly as Proposition 2.3. In fact, considering in formula (65) $Y_n^{(k)}$ to be $\tilde{M}_n^{(k)}$, μ_n to be b_n , and ν_n to be $\sigma_0 t_0^{H_0} b_n^{H_0/\beta}$ if $\beta > 2H - H_0$,

and $\nu_n = a_n$, otherwise, respectively, we have

$$\begin{aligned} I_1(n, L) &\leq \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \nu_{m_n}^{-1} \left(\widetilde{M}_{m_n}^{(k)} - \mu_{m_n} \right) < -s \right\} ds, \\ I_2(n, L) &\leq \int_L^\infty \lambda s^{\lambda-1} \mathbb{P} \left\{ \nu_{m_n}^{-1} \left(\max_{i \leq n} \sup_{t \geq 0} (X_i(t) - ct^\beta) - \mu_{m_n} \right) > s \right\} ds. \end{aligned}$$

The rest of the proof follows the same lines of arguments as those in the proof of Proposition 2.3. This completes the proof. \square

Lemma A. [*C_r inequalities*] *Let $a_i, i = 1, 2, \dots, n$, and α be positive constants, we have*

$$\begin{cases} n^{1-\alpha} (\sum_{i=1}^n a_i)^\alpha \leq \sum_{i=1}^n a_i^\alpha \leq (\sum_{i=1}^n a_i)^\alpha, & \text{if } \alpha > 1, \\ (\sum_{i=1}^n a_i)^\alpha \leq \sum_{i=1}^n a_i^\alpha \leq n^{1-\alpha} (\sum_{i=1}^n a_i)^\alpha, & \text{if } \alpha \leq 1. \end{cases}$$

In particular,

$$n^{-1} \sum_{i=1}^n a_i^\alpha \leq \left(\sum_{i=1}^n a_i \right)^\alpha \leq n^\alpha \sum_{i=1}^n a_i^\alpha.$$

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