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QUINARY FORMS AND PARAMODULAR FORMS

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ABSTRACT. We work out the exact relationship between algebraic modular forms for a two-by-two general unitary group over a definite quaternion algebra, and those arising from genera of positive-definite quinary lattices, relating stabilisers of local lattices with specific open compact subgroups, paramodular at split places, and with Atkin-Lehner operators. Combining this with the recent work of Rösner and Weissauer, proving conjectures of Ibukiyama on Jacquet-Langlands type correspondences (mildly generalised here), provides an effective tool for computing Hecke eigenvalues for Siegel modular forms of degree two and paramodular level. It also enables us to prove examples of congruences of Hecke eigenvalues connecting Siegel modular forms of degrees two and one. These include some of a type conjectured by Harder at level one, supported by computations of Fretwell at higher levels, and a subtly different congruence discovered experimentally by Buzzard and Golyshev.

1. INTRODUCTION

Modular forms play a central role in modern mathematics, leading to the development of very fruitful areas of mathematics (such as automorphic forms, Galois representations and many applications to diophantine problems). An instance of the interaction between modular forms and geometry is the modularity of rational elliptic curves as conjectured by Shimura and Taniyama, and proved by Wiles et al (in [Wil95] and [BCDT01]). A natural generalisation in this direction is understanding the relation between analytic objects and higher dimensional abelian varieties (a particular case of the Langlands program). In [Yos80] (§8, last paragraph of Example 2) Yoshida suggested that an abelian surface should be related to a Siegel modular form of degree 2.

Let \mathfrak{H}_2 be the Siegel upper half-plane of degree 2 consisting of 2×2 complex symmetric matrices whose imaginary part is positive-definite (a natural generalisation of Poincaré’s upper half plane). Siegel modular forms are holomorphic functions on \mathfrak{H}_2 that satisfy a transformation property similar to classical modular forms. More concretely, let V be a finite dimensional \mathbb{C} -vector space and let $\rho : \mathrm{GL}_2(\mathbb{C}) \rightarrow \mathrm{Aut}(V)$ be a representation. A Siegel modular form of weight ρ is an holomorphic map $f : \mathfrak{H}_2 \rightarrow V$ such that

$$F((AZ + B)(CZ + D)^{-1}) = \rho(CZ + D)(F(Z)),$$

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for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in a subgroup of the symplectic group $\mathrm{Sp}_2(\mathbb{Q})$ (see [vdG08] for a nice exposition).

In [BK14] (see also [BK19]) Brumer and Kramer made the following precise conjecture (known as the “paramodular conjecture”): abelian surfaces whose endomorphism ring over \mathbb{Q} equals \mathbb{Z} should be related to weight 2 (i.e. $V = \mathbb{C}$ and $\rho(CZ + D)w = \det(CZ + D)^2 w$) Siegel modular forms, transforming as above under the paramodular group of level N (the conductor of the surface) given by

$$P(N) := \left[\begin{array}{cccc} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \frac{1}{N}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{array} \right] \cap \mathrm{Sp}_2(\mathbb{Q}).$$

Some genuine cases of the paramodular conjecture were proven in [BPP⁺19] (see also [BCGP21]).

A related problem is that of constructing tables of paramodular forms. There are nowadays several different algorithms for computing classical modular forms. The most well-known are the modular symbol approach (as in [Cre97]), quaternion algebras and Brandt matrices (as in [Piz80]), or the use of ternary quadratic forms (as in [Bir91], [Tor05], [Ram14], [Hei16] and [HTV]).

There are some tables of paramodular forms, based on Fourier series expansions, due mostly to Poor, Yuen and some coauthors (see [PY15], [PSY17], [KPSY18] and [BPP⁺19]). A different approach using quinary forms, analogous to Birch’s use of ternary quadratic forms, can be used to compute Hecke eigenvalues more easily. This builds on the lattice-neighbour method for algebraic modular forms on orthogonal groups, using an algorithm of Plesken and Souvignier [PS97] to test for lattice isometry, as introduced by Greenberg and Voight [GV14]. Following earlier computations by Hein [Hei16] and Ladd [Lad18], this approach was developed in [RT20] and [Ram20a]. One of the main achievements of the present article is to extend their method to more general values of N (not just square-free ones) and weights. Conjecture 15 of [RT20] is a special case of results proved here. Note that what we call “ N ” here will generally be “ D ” later in the paper.

Our result is in the spirit of Eichler’s basis problem (as in [Eic73]). Eichler’s statement of the basis problem is the following: “the basis problem, is to give bases of linearly independent forms of these spaces which are arithmetically distinguished and whose Fourier series are known or easy to obtain”. Eichler’s solution, given a positive integer N (under the assumption that N is square-free, which was later relaxed by Hijikata in [Hij74]), takes a prime p dividing it. Then the space of quaternionic modular forms for the quaternion algebra ramified at $\{p, \infty\}$, of level given by an Eichler order (of level N), provides a solution to the basis problem. Furthermore, such a space can be computed easily (as Fourier expansions, corresponding to theta functions of positive-definite quadratic forms in four variables).

The main idea of Birch was to relate the arithmetic of quaternion algebras to ternary quadratic forms (instead of quaternary ones), making computations more efficient. In the present article, we present a partial solution to the basis problem for paramodular forms. The word partial refers to two main obstacles of our method. The first one is related to the possible weights we can compute. Unfortunately, paramodular forms of weight 2 (related to abelian surfaces) are not cohomological (as happens for classical weight 1 modular forms), hence they cannot be computed with our approach, whereas all weights with scalar part 3 or more can. The second

issue has to do with a big difference between classical and Siegel modular forms (of degree greater than one). For classical modular forms, a newform satisfies that its Fourier expansion is trivially determined by the eigenvalues of Hecke operators. This is no longer the case for Siegel modular forms. Our approach only allows to compute a basis for the space of algebraic modular forms (for the orthogonal group of a positive-definite quinary quadratic form) and to compute Hecke operators acting on them (as in [RT20]).

More concretely, let N be a positive integer, and assume that there exists a prime p such that $p \mid N$ but $p^2 \nmid N$. In Section 5 we prove that there is a (unique up to semi-equivalence) quinary positive-definite integral quadratic form Q , of determinant $2N$, with the following properties:

- The Hasse-Witt invariant of Q is -1 at p and ∞ , and $+1$ at all other primes.
- The quadratic form Q is special, with Eichler invariant $e(Q_q) = +1$ for all primes different from p (see §5).

Then, neglecting Yoshida lifts (cf. Proposition 9.1) and any Saito-Kurokawa lifts (cf. Proposition 9.4), the space of algebraic modular forms for the orthogonal group of Q , with values in a certain representation $W_{j+k-3, k-3}$, is isomorphic (as a Hecke module) to the space of p -new paramodular forms of weight $\det^k \otimes \text{Sym}^j$, with $k \geq 3$. This is Theorem 9.7, with $D^- = p$, $D^+ = N/p$.

Let B denote a definite quaternion algebra over \mathbb{Q} . The proof of our result exploits the relation between the algebraic group GSp_2 and its compact twist $\text{GU}(2, B)$. In a series of articles, Ibukiyama and some coauthors (see [IK17] and also [Ibu19, Ibu18]) stated conjectures relating automorphic forms on $\text{GU}(2, B)$ and GSp_2 , in the case of square-free levels (see the articles [Dem14, CD09] on computations of automorphic forms on $\text{GU}(2, B)$). The conjectures were proven by Rösner and Weissauer in a recent article [RW21], using the trace formula. A somewhat less general result was obtained independently by van Hoften [vH21] using very different, algebro-geometric tools. Although in [RW21] the result is proven for groups whose level involves only primes ramified in the quaternion algebra B , we extend their result to our more general setting (see Theorem 9.7).

A main contribution of the present article is to relate algebraic modular forms for $\text{GU}(2, B)$ with those for $\text{SO}(Q)$ for a suitable integral quadratic form Q (see Theorem 8.2). A partial result in this direction was obtained by Ladd ([Lad18]) in his doctoral thesis in the case $N = p$, though our approach was influenced more by a paper of Ibukiyama [Ibu19]. Our strategy is to construct a six-dimensional space U in $M_2(B)$ invariant under conjugation by $\text{GU}(2, B)$, and a quadratic form on it that is invariant under the action of $\text{GU}(2, B)$, and also under translation by scalar matrices. This induces a quadratic form on the five-dimensional quotient $V := U/\mathbb{Q}I$. We construct a rank 6 lattice inside U (defined in §4.1) and consider its quotient by $\mathbb{Z}I$. The dual of this produces the rank 5 integral lattice L . One is left to relate the compact level in $\text{GU}(2, B)$ studied by Ibukiyama with the stabiliser of L (up to the centre of $\text{GU}(2, B)$). These two groups are not exactly the same, as the Atkin-Lehner operator on $\text{GU}(2, B)$ stabilises the lattice L . To get the right subgroup, we define a character on the stabiliser of $L_p := L \otimes \mathbb{Z}_p$ for each prime p (see Definition 7.6), whose kernel does match the open compact subgroup of $\text{GU}(2, B_p)$ corresponding to a paramodular form. This allows us to transfer automorphic forms from one algebraic group to the other one.

It is important to mention that we can prove not only that the correspondence preserves Hecke operators, but also a precise relation between the action of the Atkin-Lehner operators (see Theorem 10.1). In particular, for genuine forms (those that cannot be constructed from forms for GL_2), the Atkin-Lehner sign changes sign for the ramified primes, while it stays the same for the non-ramified ones (as happens with the classical Jacquet-Langlands correspondence between GL_2 and B).

The above is directly applicable to the efficient computation of Hecke eigenvalues for Siegel modular forms of degree two and paramodular level, at least if the vector part of the weight is small, and the scalar part at least 3. But actually looking at the eigenvectors within spaces of algebraic modular forms also allows us to prove various instances of congruences of Hecke eigenvalues. This is the subject of §11. We warm up by re-proving a congruence originally obtained by Poor and Yuen [PY15, §8, Example 1]. This is of the form

$$\lambda_F(p) \equiv a_p(g) + p + p^2 \pmod{\lambda}.$$

On the left, F is a cuspidal Hecke eigenform of degree 2, weight 3 and paramodular level 61, and $\lambda_F(p)$ its eigenvalue for $T(p)$, with p any prime number different from 61. On the right, g is a newform of degree 1, of weight 4 for $\Gamma_0(61)$, with Hecke eigenvalues $a_p(g)$ in a field of degree 6, in which the modulus λ is a divisor of the rational prime 43. The right-hand-side can be interpreted as the eigenvalue of $T(p)$ on the Saito-Kurokawa lift $\mathrm{SK}(g)$ of g . Both F and $\mathrm{SK}(g)$ have corresponding eigenforms inside a space of algebraic modular forms arising from a certain genus of quinary lattices of determinant $2 \cdot 61$. We can prove the congruence of Hecke eigenvalues by observing that these eigenvectors are the same modulo λ . The modulus comes from the algebraic part of the critical L -value $L(3, g)$.

Such congruences can be extended to F of weight (k, j) with $k \geq 3$ and even $j > 0$, with g of weight $j + 2k - 2$ and $\lambda \mid \ell$ coming from $L(j + k, g)$. For F and g of level 1 this is a conjecture of Harder [Har08]. Computational evidence for some examples of levels 2, 3, 5, 7 was obtained by Fretwell [Fre18]. Congruences involving Saito-Kurokawa lifts are a degenerate case $j = 0$, but for $j > 0$ the problem is that the right hand side of the congruence, $a_p(g) + p^{k-2} + p^{j+k-1}$, is not the Hecke eigenvalue of $T(p)$ on any Siegel modular form. We address this by observing that $p^{k-2} + p^{j+k-1} = p^{k-2}(1 + p^{j+1})$, that $(1 + p^{j+1})$ is a Hecke eigenvalue for an Eisenstein series of level 1 and weight $j + 2$, and that modulo λ this can be replaced by a cuspidal eigenform of level q and weight $j + 2$, where q is an auxiliary prime such that $q^{j+2} \equiv 1 \pmod{\ell}$. Thus, modulo λ , $a_p(g) + p^{k-2} + p^{j+k-1}$ becomes the eigenvalue of $T(p)$ on some Yoshida lift, which does not exist as a holomorphic Siegel modular form of paramodular level, but does exist in one of our spaces of algebraic modular forms, allowing us to proceed almost as before to prove several congruences of this type. Actually the target F is represented by an eigenvector in a different space of algebraic modular forms, coming from a different genus of quinary lattices with the same determinant. But it is linked to the Yoshida lift by their mutual congruence with a third form, of paramodular level qN , represented by eigenvectors in both spaces.

The same idea using Yoshida lifts allows us to prove a congruence discovered experimentally by Buzzard and Golyshev (see Theorem 11.8). This involves the same F as in the example of Poor and Yuen, but the right hand side is now $1 + p^3 + pa_p(g)$, where g is now weight 2 and level 61, with Hecke eigenvalues in a cubic field, and λ is a divisor of 19 in this field. We use an auxiliary weight 4 form of level 37. This

congruence, and its proof, is more subtle in two ways. First, the modulus is not observable in a critical value of $L(s, g)$. Second, in the proof we see two eigenvectors that are not the same modulo λ , but they are forced nonetheless to lie in the same mod- λ Hecke eigenspace, thanks to the intervention of a newform of level $61 \cdot 37$, with Hecke eigenvalues congruent to both.

Acknowledgements. This project has its roots in a visit of Jeffery Hein and Watson Ladd to Uruguay in 2014 for a small research workshop on the subject of quinary orthogonal modular forms. The workshop topic was suggested by John Voight to whom we are grateful for many conversations regarding orthogonal modular forms. A main motivation for the present article was to prove the conjectures stated in [RT20].

The project benefitted from communications with Vasily Golyshev (on congruences) and Rainer Weissauer (on the relation between Siegel modular forms and automorphic forms for $\mathrm{GU}(2, B)$); indeed another main motivation for this paper was to prove the mod 19 congruence brought to our attention by Golyshev.

The first and fourth-named authors met during the workshop ‘‘Picard-Fuchs Equations and Hypergeometric Motives’’ at the Hausdorff Research Institute for Mathematics, Bonn, in March 2018, and are also grateful for the hospitality of the Max Planck Institute for Mathematics, Bonn, during a short visit in April 2019.

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2. THE GENERAL SPIN GROUP

Let k be a field of characteristic different from 2 and let (V, Q) be a quadratic space over k . Let $\mathrm{Cliff}(V)$ be the Clifford algebra attached to (V, Q) . Recall that the Clifford algebra $\mathrm{Cliff}(V)$ has a natural $\mathbb{Z}/2$ -graduation, so let $\mathrm{Cliff}_0(V)$ denote its even part. The subspace $\mathrm{Cliff}_0(V)$ is a subalgebra of $\mathrm{Cliff}(V)$ which is central if (V, Q) is regular (i.e. non-degenerate) of odd dimension, so from now on we assume this is the case.

The Clifford algebra $\mathrm{Cliff}(V)$ has two natural anti-involutions $*$: $\mathrm{Cliff}(V) \rightarrow \mathrm{Cliff}(V)$ (see [SP20]) which agree on the even part $\mathrm{Cliff}_0(V)$ so we do not need to make any particular choice.

Definition 2.1. The General Spin group $\mathrm{GSpin}(V)$ is the subgroup of $\mathrm{Cliff}_0(V)^\times$ given by

$$\mathrm{GSpin}(V) = \{g \in \mathrm{Cliff}_0(V) : g^*g \in k^\times \text{ and } g^{-1}Vg = V\}.$$

The spinor norm $\nu : \mathrm{GSpin}(V) \rightarrow k^\times$ is given by $\nu(g) = g^*g$.

There is a natural homomorphism

$$(1) \quad \phi : \mathrm{GSpin}(V) \rightarrow \mathrm{O}(V)$$

given by $\phi(g)(v) = gvg^{-1}$ (see [Cas78, Chapter 10, Lemma 3.1]).

Theorem 2.2. *The image of ϕ equals $\mathrm{SO}(V)$ and its kernel equals k^\times .*

Proof. See [Cas78, Chapter 10, Theorem 3.1]. □

When V has odd dimension, the natural copy of V in $\mathrm{Cliff}(V)$ lies in the odd part. However, in such a case, the centre of $\mathrm{Cliff}(V)$ has a one dimensional odd part. For example, if $\{e_1, \dots, e_n\}$ is an orthogonal basis, then the vector $c = e_1 \cdots e_n$

generates such subspace and $c^2 \in k$. In particular, the subspace $c \cdot V$ does lie in $\text{Cliff}_0(V)$ and furthermore, the elements of $\text{Cliff}_0(V)$ normalising V are the same as the ones normalising $c \cdot V$. Note also that the involution acts on $c \cdot V$ by $(-1)^{\lfloor n/2 \rfloor}$.

2.1. The 5-dimensional case. From now on we restrict to quadratic forms over a number field k . Let (V, Q) be a regular quinary quadratic space over k .

Lemma 2.3. *The subspace $U = \{v \in \text{Cliff}_0(V) : v^* = v\}$ equals the subspace $k \oplus c \cdot V$.*

Proof. Clearly k is invariant under the involution. Let $\{e_1, \dots, e_5\}$ be an orthogonal basis for V , so $\text{Cliff}_0(V) = k \oplus c \cdot V \oplus \langle e_i e_j : 1 \leq i < j \leq 5 \rangle$. The involution sends $e_i e_j$ to $e_j e_i = -e_i e_j$ and fixes $c \cdot V$, hence $U = k \oplus c \cdot V$. \square

In the five dimensional case, the condition $g^{-1}Vg = V$ on the definition of the General Spin group is superfluous provided $g^*g \in k^\times$.

Lemma 2.4. *If $\dim(V) = 5$, then*

$$\text{GSpin}(V) = \{g \in \text{Cliff}_0(V) : g^*g \in k^\times\}.$$

Proof. We recall the proof from [Eic52, §5.5]. Let $g \in \text{Cliff}_0(V)$ such that $g^*g \in k^\times$ and let $v \in V$. Then $g^*(cv)g$ is fixed by the involution so by Lemma 2.3 we have $g^*(cv)g = \alpha + cw$ with $\alpha \in k$ and $w \in V$. Squaring this equality gives $\nu(g)c^2Q(v) = \alpha^2 + c^2Q(w) + 2\alpha cw \in k$. If $v \neq 0$ then clearly $w \neq 0$ so $\alpha = 0$ and $g^*(cv)g = cw$. Since c is central, we conclude $g^{-1}vg = \frac{1}{\nu(g)}w \in V$. \square

For each place v of k the Hasse-Witt invariant $\text{HW}_v(Q) \in \{\pm 1\}$ is an invariant of the quadratic space V_v given by the class of $\text{Cliff}_0(V_v)$ in the Brauer group (see [Lam05, (3.12) in p.117]). If $\{e_1, \dots, e_5\}$ is an orthogonal basis with $Q(e_i) = a_i$ then

$$(2) \quad \text{HW}_v(Q) = (-1, -1)_v \prod_{i < j} (a_i, a_j)_v,$$

where the quadratic Hilbert symbol $(a, b)_v$ is $+1$ or -1 , according as $ax^2 + by^2 = z^2$ has, or has not (respectively), a solution $(x, y, z) \neq (0, 0, 0)$ in k_v^3 (see [Lam05] Proposition 3.20).

Remark 2.5. The definition of the Hasse-Witt invariant for quinary forms coincides with the classical Hasse invariant for odd primes, but it differs by $(-1, -1)_v$ for even primes and real places.

Let

$$(3) \quad S = \{v : \text{HW}_v(Q) = -1\},$$

the set of places where the quadratic form Q has Hasse-Witt invariant -1 . By Hilbert's reciprocity S has even cardinality.

Remark 2.6. For computational purposes, we assume k is totally real and the quinary quadratic form Q is totally positive definite, so all the archimedean places are in S .

Remark 2.7. Let B be the quaternion algebra over k ramified precisely at the places of S and denote $b \mapsto \bar{b}$ its standard involution. By definition, the central simple algebras $\text{Cliff}_0(V)$ and B correspond to the same class in the Brauer group. It follows that $\text{Cliff}_0(V) \simeq M_2(B)$ since $\dim \text{Cliff}_0(V) = 4 \dim B$.

Lemma 2.8. *The isomorphism $\text{Cliff}_0(V) \simeq M_2(B)$ can be chosen so that the involution of $\text{Cliff}_0(V)$ corresponds to the involution of $M_2(B)$ given by $m^* = \overline{m}^t$.*

Proof. Let $D = \det V$ and choose α and β such that $(-\alpha D, -\beta D)_v = \text{HW}_v(Q)$. Without loss of generality we can assume V has an orthogonal basis $\{e_1, e_2, e_3, e_4, e_5\}$ with $Q(e_1) = \alpha$, $Q(e_2) = \beta$, $Q(e_3) = \alpha\beta D$ and $Q(e_4) = Q(e_5) = D$.

Consider the representation of $\text{Cliff}_0(V)$ as a tensor product of quaternion algebras given in [Eic52, (5.18)]:

$$\text{Cliff}_0(V) \simeq [1, e_1 e_2, e_2 e_3, e_3 e_1] \otimes [1, e_1 e_2 e_3 e_4, e_1 e_2 e_3 e_5, e_4 e_5].$$

By our choice of α and β we have $[1, e_1 e_2, e_2 e_3, e_3 e_1] \simeq B$ with the involution on the left side corresponding to the standard involution of B .

On the other hand we have $[1, e_1 e_2 e_3 e_4, e_1 e_2 e_3 e_5, e_4 e_5] \simeq M_2(k)$ as follows:

$$e_1 e_2 e_3 e_4 \mapsto \begin{pmatrix} \alpha\beta D & 0 \\ 0 & -\alpha\beta D \end{pmatrix}, \quad e_1 e_2 e_3 e_5 \mapsto \begin{pmatrix} 0 & \alpha\beta D \\ \alpha\beta D & 0 \end{pmatrix}, \quad e_4 e_5 \mapsto \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

and the involution on the left side corresponds to the transpose in $M_2(k)$. Thus $\text{Cliff}_0(V) \simeq B \otimes M_2(k) \simeq M_2(B)$ and the involution of $\text{Cliff}_0(V)$ corresponds to the involution on $M_2(B)$ given by $m^* = \overline{m}^t$. \square

By the lemma we can (and will) identify $\text{Cliff}_0(V)$ with $M_2(B)$ with the involution given by $m^* = \overline{m}^t$. The group $\text{GSpin}(V)$ is then isomorphic to the group

$$(4) \quad \text{GU}(2, B) := \{g \in M_2(B) : g^* g = \nu(g)I, \nu(g) \in k^\times\}.$$

The group $\text{GU}(2, B)$ consists of the invertible elements in $M_2(B)$ preserving the hermitian form $\langle (x, y), (r, s) \rangle = \bar{x}r + \bar{y}s$ on B^2 (via left multiplication) up to scale.

3. QUATERNIONIC UNITARY GROUPS AND SOME LOCAL SUBGROUPS

Keep the notation of the previous section. Let B be a definite quaternion algebra over \mathbb{Q} ramified at a finite set of primes S (containing the infinity place), with main involution $\alpha \mapsto \bar{\alpha}$ and denote $\text{GU}(2, B) := \{g \in M_2(B) : g^* g = \nu(g)I, \nu(g) \in \mathbb{Q}^\times\}$.

3.1. Local subgroups of $\text{GU}(2, B)$. For v a rational place, let $B_v := B \otimes \mathbb{Q}_v$ denote the completion of B at v . We define for each place v an open compact subgroup $U_v \subset \text{GU}(2, B_v)$ as follows.

3.1.1. Archimedean place. Our assumption that B is definite implies that $\text{GU}(2, B_\infty)$ is the compact group $\text{Sp}(2)$ (up to center) of rank 2 (the compact form of $\text{Sp}(2, \mathbb{R})$). Hence we take $K_\infty := \text{GU}(2, B_\infty)$ as our compact open subgroup.

3.1.2. Non-archimedean places in S . Let R_p denote the unique maximal order of B_p [Vig80, Chapitre II, Lemme 1.5], and \mathfrak{p} its maximal ideal (given by the elements of norm divisible by p). Let $R_p^0 := \{r \in R_p : r + \bar{r} = 0\}$. Following [IK17], let $\xi \in M_2(B_p)$ be such that $\xi^* \xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and consider the \mathbb{Z}_p -lattice

$$(5) \quad L_p := \xi \cdot \begin{pmatrix} \mathfrak{p} \\ R_p \end{pmatrix} \subset B_p^2.$$

Let K_p^- be the subgroup of $\text{GU}(2, B_p)$ given by the stabiliser of L_p (column vectors) under the natural left action of $\text{GU}(2, B_p)$ on B_p^2 .

Let $H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and denote by $\text{GU}(1, 1, B_p)$ the group

$$\text{GU}(1, 1, B_p) := \{g \in M_2(B_p) : g^* H g = \nu(g)H, \nu(g) \in \mathbb{Q}_p^\times\}.$$

Conjugation by ξ gives an isomorphism between $\mathrm{GU}(2, B_p)$ and $\mathrm{GU}(1, 1, B_p)$. Under this isomorphism, the group K_p^- maps to the stabiliser $K^-(p)$ of the lattice $\mathfrak{p} \oplus R_p$ (see [Ibu19, §2] and [vH21, §4.3]), the intersection of $\mathrm{GU}(1, 1, B_p)$ with the order $\begin{pmatrix} R_p & \mathfrak{p} \\ \mathfrak{p}^{-1} & R_p \end{pmatrix}$ of $M_2(B_p)$.

Note that the subgroup $K^-(p)$ of $\mathrm{GU}(1, 1, B_p)$ is normalised by an Atkin-Lehner element $\omega'_p := \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} \in \mathrm{GU}(1, 1, B_p)$, which satisfies $(\omega'_p)^2 = pI$, and $(\omega'_p)^* H \omega'_p = pH$. So we define $\omega_p := \xi \omega'_p \xi^{-1} \in \mathrm{GU}(2, B_p)$, which normalises K_p^- , with $\omega_p^2 = pI$ and $\omega_p^* \omega_p = pI$, so $\nu(\omega_p) = p$.

3.1.3. Non-archimedean places not in S . At any prime number $p \notin S$ we fix an isomorphism $B_p \simeq M_2(\mathbb{Q}_p)$, then let R_p denote the matrix order $M_2(\mathbb{Z}_p)$. For n a non-negative integer, consider the \mathbb{Z}_p -lattice

$$(6) \quad L_{p^n} := \begin{pmatrix} M_2(\mathbb{Z}_p) \\ \pi^n \cdot M_2(\mathbb{Z}_p) \end{pmatrix} \subset B_p^2,$$

where $\pi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Similar to the definition of K_p^- above, we define $K_{p^n}^+$ as the subgroup of $G(\mathbb{Q}_p)$ of elements preserving the lattice L_{p^n} under left multiplication.

The main involution is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Consider the isomorphism $\Psi : M_2(B_p) \xrightarrow{\sim} M_4(\mathbb{Q}_p)$ given by

$$(7) \quad \Psi \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right) = \begin{pmatrix} a_{11} & b_{11} & a_{12} & b_{12} \\ c_{11} & d_{11} & c_{12} & d_{12} \\ a_{21} & b_{21} & a_{22} & b_{22} \\ c_{21} & d_{21} & c_{22} & d_{22} \end{pmatrix}.$$

Note that Ψ swaps second and third rows, then also second and third columns. Let $\mathrm{GSp}_2(\mathbb{Q}_p) = \{g \in M_4(\mathbb{Q}_p) : g^t J g = \nu J, \nu \in \mathbb{Q}^\times\}$, where

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Lemma 3.1. *The isomorphism Ψ induces an isomorphism between the groups $\mathrm{GU}(2, B_p)$ and $\mathrm{GSp}_2(\mathbb{Q}_p)$.*

Proof. If we let $J' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$, then $\Psi(J') = J$ and $J'^{-1} = -J'$. Since

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} {}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, we see that $B^* = J'^{-1} {}^t B J'$, where the transpose is as a 4-by-4 matrix. Hence for $g \in M_2(B_p)$,

$$g^* g = \nu I \iff J'^{-1} {}^t g J' g = \nu I \iff {}^t g J' g = \nu J' \iff {}^t \Psi(g) J \Psi(g) = \nu J.$$

□

The paramodular group of level p^n is given by $K(p^n) := \{k \in \mathrm{GSp}_2(\mathbb{Q}_p) : h^{-n} k h^n \in \mathrm{GL}_4(\mathbb{Z}_p)\}$, where $h := \mathrm{diag}(1, 1, 1, p)$.

Lemma 3.2. *The group $K_{p^n}^+$ maps onto the paramodular group $K(p^n)$ under the isomorphism (7).*

Proof. The isomorphism (7) sends the lattice $L = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p/p^n \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p/p^n \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p/p^n \\ p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ to itself, and $K(p^n)$ is the intersection of L with $\mathrm{GSp}_2(\mathbb{Q}_p)$, so it suffices to show that K_p^+ is the intersection of L with $\mathrm{GU}(2, B_p)$. If we describe L with the notation $\begin{pmatrix} M_2(\mathbb{Z}_p) & M_2(\mathbb{Z}_p) \cdot \pi^{-n} \\ \pi^n \cdot M_2(\mathbb{Z}_p) & \pi^n \cdot M_2(\mathbb{Z}_p) \cdot \pi^{-n} \end{pmatrix}$, it is easy to verify this, recalling that $L_{p^n} := \begin{pmatrix} M_2(\mathbb{Z}_p) \\ \pi^n \cdot M_2(\mathbb{Z}_p) \end{pmatrix}$. \square

It is well known that the integral condition on elements of $K(p^n)$ plus the fact that it preserves the symplectic form imply that $K(p^n)$ is the intersection of $\mathrm{GSp}_2(\mathbb{Q}_p)$ with the order

$$(8) \quad R := \begin{pmatrix} \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & p^{-n} \mathbb{Z}_p \\ \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \subset M_4(\mathbb{Q}_p).$$

Note that the subgroup $K(p^n)$ is normalised by the Atkin-Lehner element of $\mathrm{GSp}_2(\mathbb{Q}_p)$

$$(9) \quad W_{p^n} := \begin{pmatrix} 0 & p^n & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & p^n & 0 \end{pmatrix}.$$

The Atkin-Lehner involution satisfies $W_{p^n}^2 = p^n I$, and $W_{p^n}^* W_{p^n} = p^n I$ hence $\nu(W_{p^n}) = p^n$. The preimage of (8) under (7) can be expressed as the block matrices

$$(10) \quad \begin{pmatrix} M_2(\mathbb{Z}_p) & M_2(\mathbb{Z}_p) \cdot \bar{\pi}^n \\ \pi^n \cdot M_2(\mathbb{Z}_p) & \pi^n \cdot M_2(\mathbb{Z}_p) \cdot \bar{\pi}^{-n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{\pi}^{-n} \end{pmatrix} \begin{pmatrix} M_2(\mathbb{Z}_p) & M_2(\mathbb{Z}_p) \\ p^n M_2(\mathbb{Z}_p) & M_2(\mathbb{Z}_p) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \bar{\pi}^n \end{pmatrix}.$$

The preimage under (7) of the Atkin-Lehner involution equals $W_{p^n}^+ := \begin{pmatrix} 0 & 0 & p^n & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & p^n & 0 & 0 \end{pmatrix}$.

The following is [Vig80, Chapitre II, Theoreme 2.3(1)].

Lemma 3.3. *Let R_1, R_2 be two maximal orders of $M_n(\mathbb{Q}_p)$, then there exists $\alpha \in \mathrm{GL}_n(\mathbb{Q}_p)$ such that $\alpha R_1 \alpha^{-1} = R_2$.*

Lemma 3.4. *The only maximal orders in $M_4(\mathbb{Q}_p)$ containing R are $\alpha_m^{-1} M_4(\mathbb{Z}_p) \alpha_m$, for $0 \leq m \leq n$, where $\alpha_m := \mathrm{diag}(1, p^m, 1, p^{m-n})$.*

Proof. Recall the definition of R given in (8). By the previous lemma, any maximal order is of the form $\alpha^{-1} M_4(\mathbb{Z}_p) \alpha$, for some $\alpha \in \mathrm{GL}_4(\mathbb{Q}_p)$. Suppose that $R \subset \alpha^{-1} M_4(\mathbb{Z}_p) \alpha$. By the Iwasawa decomposition, we may left multiply α by an element

of $\mathrm{GL}_4(\mathbb{Z}_p)$ to put it in upper triangular form $\alpha = \begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{pmatrix}$. Applying

$\alpha R \alpha^{-1} \subset M_4(\mathbb{Z}_p)$ to the elements $\mathrm{diag}(0, 1, 0, 0)$, $\mathrm{diag}(0, 0, 1, 0)$ and $\mathrm{diag}(0, 0, 0, 1)$ of R , and inspecting the above-diagonal elements of the second, third and fourth columns respectively, we find that $\frac{b}{e}, \frac{c}{h}, \frac{f}{h}, \frac{d}{j}, \frac{g}{j}, \frac{i}{j} \in \mathbb{Z}_p$. The factorisation

$$\alpha = \begin{pmatrix} 1 & \frac{b}{e} & \frac{c}{h} & \frac{d}{j} \\ 0 & 1 & \frac{f}{h} & \frac{g}{j} \\ 0 & 0 & 1 & \frac{i}{j} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & j \end{pmatrix},$$

and the fact that the first factor is in $\mathrm{GL}_4(\mathbb{Z}_p)$ implies that we can assume α equals the second factor. Then

$$\alpha R\alpha^{-1} = \begin{pmatrix} \mathbb{Z}_p & p^n \frac{a}{e} \mathbb{Z}_p & \frac{a}{h} \mathbb{Z}_p & \frac{a}{j} \mathbb{Z}_p \\ \frac{e}{a} \mathbb{Z}_p & \mathbb{Z}_p & \frac{e}{h} \mathbb{Z}_p & p^{-n} \frac{e}{j} \mathbb{Z}_p \\ \frac{h}{a} \mathbb{Z}_p & p^n \frac{h}{e} \mathbb{Z}_p & \mathbb{Z}_p & \frac{h}{j} \mathbb{Z}_p \\ p^n \frac{j}{a} \mathbb{Z}_p & p^n \frac{j}{e} \mathbb{Z}_p & p^n \frac{j}{h} \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

For this to be contained in $M_4(\mathbb{Z}_p)$, we see that $-n \leq \mathrm{ord}_p(\frac{a}{e}) \leq 0$, $-n \leq \mathrm{ord}_p(\frac{h}{e}) \leq 0$, $\mathrm{ord}_p(\frac{a}{h}) = 0$, $0 \leq \mathrm{ord}_p(\frac{a}{j}) \leq n$, $\mathrm{ord}_p(\frac{e}{j}) = n$ and $0 \leq \mathrm{ord}_p(\frac{h}{j}) \leq n$. Hence (removing a scalar power of p and a diagonal unit matrix) we may assume that $\alpha = \alpha_m := \mathrm{diag}(1, p^m, 1, p^{m-n})$ for $0 \leq m \leq n$. \square

Lemma 3.5. *The unique way of writing R as an intersection of maximal orders in $M_4(\mathbb{Q}_p)$ is*

$$R = h^n M_4(\mathbb{Z}_p) h^{-n} \cap W_{p^n}^{-1} h^n M_4(\mathbb{Z}_p) h^{-n} W_{p^n}.$$

Proof. Recall that

$$R = \begin{pmatrix} \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & p^{-n} \mathbb{Z}_p \\ \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix},$$

while

$$\alpha_m^{-1} M_4(\mathbb{Z}_p) \alpha_m = \begin{pmatrix} \mathbb{Z}_p & p^m \mathbb{Z}_p & \mathbb{Z}_p & p^{m-n} \mathbb{Z}_p \\ p^{-m} \mathbb{Z}_p & \mathbb{Z}_p & p^{-m} \mathbb{Z}_p & p^{-n} \mathbb{Z}_p \\ \mathbb{Z}_p & p^m \mathbb{Z}_p & \mathbb{Z}_p & p^{m-n} \mathbb{Z}_p \\ p^{n-m} \mathbb{Z}_p & p^n \mathbb{Z}_p & p^{n-m} \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

If $R = \alpha_{m_1}^{-1} M_4(\mathbb{Z}_p) \alpha_{m_1} \cap \alpha_{m_2}^{-1} M_4(\mathbb{Z}_p) \alpha_{m_2}$, with $0 \leq m_1, m_2 \leq n$ then to avoid non-integral elements in the left entry of the second row we must have some $m_i = 0$, and to avoid non-integral elements in the top entry of the fourth column we must have some $m_i = n$, say $m_1 = 0$, $m_2 = n$.

Clearly

$$\alpha_0^{-1} M_4(\mathbb{Z}_p) \alpha_0 = h^n M_4(\mathbb{Z}_p) h^{-n},$$

since $\alpha_0 = \mathrm{diag}(1, 1, 1, p^{-n}) = h^{-n}$, and

$$\alpha_n^{-1} M_4(\mathbb{Z}_p) \alpha_n = W_{p^n}^{-1} h^n M_4(\mathbb{Z}_p) h^{-n} W_{p^n},$$

since

$$h^{-n} W_{p^n} = \begin{pmatrix} 0 & p^n & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \alpha_n.$$

\square

Lemma 3.6. *The normaliser of R , i.e. the set $\{g \in \mathrm{GL}_4(\mathbb{Q}_p) : g^{-1} R g = R\}$, is the union of $p^{\mathbb{Z}} W_{p^n}^{\mu} R^{\times}$ for $\mu = 0, 1$.*

Proof. By Lemma 3.5, conjugation by g either fixes or swaps the maximal orders $h^n M_4(\mathbb{Z}_p) h^{-n}$ and $W_{p^n}^{-1} h^n M_4(\mathbb{Z}_p) h^{-n} W_{p^n}$. Clearly it suffices to show that the normaliser of $M_4(\mathbb{Z}_p)$ is $p^{\mathbb{Z}} M_4(\mathbb{Z}_p)^{\times}$, but this can be done by reducing to diagonal elements, as in the proof of Lemma 3.4, but easier. \square

4. LOCAL LATTICES IN A SIX-DIMENSIONAL $\mathrm{GU}(2, B)$ -SPACE

Consider an action of $\mathrm{GU}(2, B)$ on the \mathbb{Q} -vector space (of Lemma 2.3)

$$(11) \quad U := \{A \in M_2(B) : A^* = A\}.$$

This space is very related to the one considered by Ibukiyama in [Ibu19], but we remove the trace zero hypothesis.

Lemma 4.1. *The vector space U is of dimension 6. Moreover, it is given by*

$$U = \left\{ \begin{pmatrix} s & r \\ \bar{r} & t \end{pmatrix} : s, t \in \mathbb{Q}, r \in B \right\}.$$

Proof. It is clear that if $\overline{M^t} = M$ then the entries (1, 1) and (2, 2) of the matrix are fixed by the involution, hence rational. The second hypothesis is also clear. \square

Note that the space U contains the centre of $M_2(B)$ (i.e. the rational scalar matrices). In Section 6 we will define a quadratic form on U invariant under translation by the centre, hence the quotient space becomes a quadratic space.

Proposition 4.2. *The group $\mathrm{GU}(2, B)$ acts on U via conjugation.*

Proof. If $g \in \mathrm{GU}(2, B)$ it satisfies that $g^{-1} = \frac{g^*}{\nu(g)}$. Then $(gvg^{-1})^* = (gv\frac{g^*}{\nu(g)})^* = \frac{g}{\nu(g)}v^*g^* = gvg^{-1}$ because $\nu(g)$ is in the centre of $M_2(B)$. \square

4.1. Local lattices in U at split primes. Let p be an unramified prime (i.e. $p \notin S$). Given $n \geq 0$, define a \mathbb{Z}_p -lattice $\mathfrak{U}_{p^n}^+ \subseteq U_p := U \otimes \mathbb{Q}_p$ by

$$(12) \quad \mathfrak{U}_{p^n}^+ := \left\{ \begin{pmatrix} s & \begin{pmatrix} p^n a & b \\ p^n c & d \end{pmatrix} \\ \begin{pmatrix} d & -b \\ -p^n c & p^n a \end{pmatrix} & t \end{pmatrix} : a, b, c, d, s, t \in \mathbb{Z}_p \right\}.$$

Using the previous notation (i.e. $\pi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$), the lattice can be written in the compact form $\mathfrak{U}_{p^n}^+ = \left\{ \begin{pmatrix} s & r \\ \bar{r} & t \end{pmatrix} : s, t \in \mathbb{Z}_p, r \in M_2(\mathbb{Z}_p) \cdot \bar{\pi}^n \right\}$.

Lemma 4.3. *Suppose that p is prime. The subring R' of $M_4(\mathbb{Q}_p)$ generated by $\mathfrak{U}_{p^n}^+$ equals*

$$\begin{pmatrix} \mathbb{Z}_p I_2 & 0_2 \\ 0_2 & \mathbb{Z}_p I_2 \end{pmatrix} \oplus \begin{pmatrix} p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & p^{2n} \mathbb{Z}_p & p^n \mathbb{Z}_p \end{pmatrix}.$$

Proof. Since $\begin{pmatrix} I_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix} \in R'$ and $\begin{pmatrix} 0_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix} \in R'$, if we multiply these elements by

elements in $\mathfrak{U}_{p^n}^+$ of the form $\begin{pmatrix} 0_2 & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} & 0_2 \end{pmatrix}$, with $b, d \in \mathbb{Z}_p$ and $a, c \in p^n \mathbb{Z}_p$, we find that $\begin{pmatrix} 0_2 & \begin{pmatrix} p^n \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \\ 0_2 & 0_2 \end{pmatrix} \subset R'$ and $\begin{pmatrix} 0_2 & 0_2 \\ \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & p^n \mathbb{Z}_p \end{pmatrix} & 0_2 \end{pmatrix} \subset R'$.

Multiplying elements of these subsets together, one way round or the other, we find that

$$\begin{pmatrix} \mathbb{Z}_p I_2 & 0_2 \\ 0_2 & \mathbb{Z}_p I_2 \end{pmatrix} \oplus \begin{pmatrix} p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & p^n \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & p^n \mathbb{Z}_p & p^{2n} \mathbb{Z}_p & p^n \mathbb{Z}_p \end{pmatrix} \subseteq R'.$$

It is easy to see that the left hand side contains \mathfrak{U}_p^+ and is closed under multiplication, hence the inclusion must be equality. \square

Remark 4.4. If one takes the second summand, divides the top left and bottom right 2×2 blocks by p^n and applies the map $\Psi : M_2(B_p) \rightarrow M_4(\mathbb{Q}_p)$ (cf. (7)) then one obtains R from the previous section (cf. (8)).

Remark 4.5. Anticipating Remark 6.4 below, when $p = 2$, had we replaced \mathfrak{U}_p^+ by its trace zero sublattice, we would only have been able to show that $\begin{pmatrix} 2I_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix}$ lies in R' .

4.2. Local lattices in U at non-split primes. Let p now be a ramified prime. Recall that ξ was chosen so that $\xi^* \xi = H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, giving $\xi^{-1} \text{GU}(2, B_p) \xi = \text{GU}(1, 1, B_p)$. Following [Ibu19, §4], define $\tilde{U}_p = \xi^{-1} U_p \xi$.

Lemma 4.6. *The vector space \tilde{U}_p equals the space $\left\{ \begin{pmatrix} r & s \\ t & \bar{r} \end{pmatrix} : s, t \in \mathbb{Q}_p, r \in B_p \right\}$.*

Proof. The proof is given in [Ibu19] (page 212), although the author considers only the subspace of U_p of trace zero elements. Recall that over \mathbb{Q}_p the two spaces differ by the identity matrix, which maps to itself under conjugation. \square

For any p , define $\mathfrak{U}_p^- \subseteq U_p$ by $\mathfrak{U}_p^- := \xi \tilde{\mathfrak{U}}_p^- \xi^{-1}$, where $\tilde{\mathfrak{U}}_p^- \subseteq \tilde{U}_p$ is defined by

$$(13) \quad \tilde{\mathfrak{U}}_p^- := \left\{ \begin{pmatrix} r & ps \\ t & \bar{r} \end{pmatrix} : r \in \mathcal{O}_p, s, t \in \mathbb{Z}_p \right\}.$$

Lemma 4.7. *Suppose that p is prime. The subring R' of $M_2(B_p)$ generated by $\tilde{\mathfrak{U}}_p^-$ equals*

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(B_p) : a, c, d \in R_p, b \in pR_p \text{ and } a \equiv \bar{d} \pmod{\pi} \right\}.$$

Proof. The proof is similar to [Ibu19, Lemma 4.2]. Multiplying the elements

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & \bar{r} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix},$$

proves that the element in the place (2, 1) of the matrix can be arbitrary. Considering the element $\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}$ we get the same result for the entry (1, 2) but with multiples of p . To get the diagonal elements, note that

$$\begin{pmatrix} r & 0 \\ 0 & \bar{r} \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & \bar{s} \end{pmatrix} = \begin{pmatrix} rs & 0 \\ 0 & \bar{r}\bar{s} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \bar{r}\bar{s} - \bar{s}\bar{r} \end{pmatrix}.$$

In particular, we can add to an element of $\tilde{\mathfrak{U}}_p^-$ a matrix with any element of the form $rs - sr$ to the place (2, 2). Note that any such difference lies in the unique maximal ideal (as the quotient of R_p by π is the finite field of p^2 elements, in particular is abelian) and in fact, they generate the maximal ideal, hence the statement. \square

5. SPECIAL LATTICES

Let R be a Dedekind domain with field of fractions k , of characteristic different from 2. Let (V, Q) be a regular quadratic space over k with associated symmetric bilinear form $\langle v, w \rangle = Q(v + w) - Q(v) - Q(w)$, so that $\langle v, v \rangle = 2Q(v)$.

An R -lattice on V is a finitely generated R -submodule $L \subset V$ such that $kL = V$. If L is an R -lattice, its *dual lattice* L^\vee is defined by

$$L^\vee := \{v \in V : \langle v, w \rangle \in R \ \forall w \in L\}.$$

A lattice L is *integral* if $Q(L) \subset R$; this implies $L \subseteq L^\vee$. If L is integral and $L = L^\vee$ it is called *even unimodular*. More generally L is called *modular* if $L = IL^\vee$ for some ideal $I \triangleleft R$. The (*signed*) *determinant* of a free lattice L is defined as $\det L := (-1)^{\lfloor n/2 \rfloor} \det(\langle v_i, v_j \rangle)$ where $\{v_1, \dots, v_n\}$ is a basis of L . Note that $\det L$ is, up to squares of units, independent of the choice of basis.

It is clear that if L is integral then $\det L \in R$. If, in addition, the rank is odd we have $\det L \in 2R$: considering the expression for $\det L$ as an alternating sum of products, transposition gives us pairs of products, equal because the matrix is symmetric, and a product can be paired with itself only when either the rank is even or the product includes an even factor from the diagonal.

A lattice is called *maximal* if it is maximal among all integral lattices in its ambient quadratic space. Note that an integral lattice with unit determinant is necessarily maximal since any proper super-lattice would have non-integral determinant; for a similar reason an integral lattice of odd rank whose determinant is twice a unit is maximal.

Definition 5.1. A *special lattice* is an integral R -lattice L of odd rank such that L^\vee/L is cyclic as an R -module.

5.1. Local classification of special lattices. In this section R is the ring of integers of a local field with valuation v and maximal ideal p . Since R is a principal ideal domain all R -lattices are free.

Theorem 5.2. *If L is a special lattice then $L = A \perp Rw$, where A is even unimodular and $w \in L$.*

Proof. The lattice L can be written as the orthogonal sum of modular lattices of rank 1 or 2 [O'M73, §91C]. Say

$$L = A_1 \perp \dots \perp A_s$$

where $A_i = I_i A_i^\vee$ for some $I_i \triangleleft R$. Then

$$L^\vee/L = \bigoplus_i A_i^\vee/I_i A_i^\vee \simeq \bigoplus_i (R/I_i)^{\dim A_i}.$$

Since L is special, at most one $I_{i_0} \neq R$ and necessarily $\dim A_{i_0} = 1$. If all $I_i = R$ choose any i_0 with $\dim A_{i_0} = 1$, which exists since $\dim L$ is odd. In any case $L = A \perp Rw$ where $A = \bigoplus_{i \neq i_0} A_i$ is even unimodular and $A_{i_0} = Rw$. \square

We now recall some useful facts about even unimodular lattices. In what follows we will say a unit u of R is *unramified* if it is a square modulo 4.

Lemma 5.3. *Let A be an even unimodular lattice. Then $\det A$ is an unramified unit and $\text{HW}_v(A) = 1$.*

Proof. When $p \nmid 2$ the lattice A has an orthogonal basis of vectors with unit norm so $\det A$ is a unit. The Hilbert symbol is trivial on units, hence $\text{HW}_v(A) = 1$.

When $p \mid 2$ the lattice A is the orthogonal sum of even unimodular lattices of rank 2 (since there are no even unimodular lattices of rank 1). An even unimodular lattice of rank 2 has matrix $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ with b a unit, hence its (signed) determinant $b^2 - 4ac$ is the square of a unit modulo 4. Since the determinant for lattices of even rank is multiplicative, it follows that $\det A$ is the square of a unit modulo 4.

For the Hasse-Witt invariant use the fact that for lattices A_1 and A_2 of even rank we have $\text{HW}_v(A_1 \perp A_2) = \text{HW}_v(A_1) \cdot \text{HW}_v(A_2) \cdot (\det A_1, \det A_2)_v$ (see [Lam05, (3.13) in p.117]); the Hilbert symbol is trivial on unramified units so it suffices to prove $\text{HW}_v(A) = 1$ when A has rank 2. Since A is even unimodular there is some $w_1 \in A$ with $Q(w_1) = \alpha \in R^\times$. Let $w_2 \in A$ such that $\langle w_1, w_2 \rangle = 0$ and let $Q(w_2) = \beta$. The vectors w_1 and w_2 span a lattice $B \subseteq A$ with $\det(B) = -\alpha\beta = \det A \cdot s^2$ for some $s \in R$ and $\text{HW}_v(A) = \text{HW}_v(B) = (\alpha, \beta)_v = (\alpha, -\alpha\beta)_v = (\alpha, \det A)_v = 1$ where in the last equality we have used that α is a unit and that $\det A$ is an unramified unit. \square

In view of this lemma we define an invariant for even unimodular lattices as follows.

Definition 5.4. If A is an even unimodular lattice, we let

$$d(A) = (\tilde{p}, \det A)_v.$$

where \tilde{p} is a local uniformizer.

Note that $d(A)$ is independent of the choice of uniformizer. Indeed, since $\det A$ is an unramified unit the Hilbert symbol $(u, \det A)_v$ equals 1 for any unit u .

Lemma 5.5. *Let A be an even unimodular lattice. Then the invariant $d(A)$, together with the rank, determines the isometry class of A .*

Proof. Suppose A and A' are unimodular lattices of the same rank such that $d(A) = d(A')$. The latter means that $\det A / \det A'$ is a square modulo $4p$ and by the local square theorem (see [O'M73, 63:1]) this implies that $\det A / \det A'$ is a square in R . We conclude A and A' have the same rank, determinant and Hasse-Witt invariant so they lie in isometric quadratic spaces ([O'M73, 63:20]). Finally, A and A' are maximal lattices in isometric quadratic spaces so they are themselves isometric ([O'M73, 91:2]). \square

Lemma 5.6. *Let $n \geq 1$ and $d \in \{\pm 1\}$. There is an even unimodular lattice A of rank $2n$ and $d(A) = d$.*

Proof. Pick $u \in R^\times$ such that $u \equiv 1 \pmod{4}$ and $(p, u)_v = d$, and write $u = 1 - 4\alpha$ with $\alpha \in R$. Let J be the binary lattice with matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2\alpha \end{pmatrix}$ which is even unimodular of determinant u , and let H be the hyperbolic plane which is even unimodular of determinant 1. Then $A = J \perp H^{n-1}$ has rank $2n$, determinant u and invariant $d(A) = d$. \square

Proposition 5.7. *Let L be a special lattice of determinant $2N$ with $p \nmid N$. Then $\text{HW}_v(L) = 1$, L is maximal, and the class of N modulo squares, together with the rank, determines the isometry class of L . In particular $L = A \perp R w$ where A is even unimodular of determinant 1 and $Q(w) = N$.*

Proof. Write $L = A \perp Rw$ as in Theorem 5.2. From [Lam05, (3.13) in p.117] we have $\text{HW}_v(L) = \text{HW}_v(A) \cdot \text{HW}_v(Rw) \cdot (-Q(w), \det A)_v$. By Lemma 5.3, we have $\text{HW}_v(A) = 1$ and $\det A$ is an unramified unit; the hypothesis implies $Q(w)$ is a unit and so the Hilbert symbol $(-Q(w), \det A)_v$ equals 1. Finally $\text{HW}_v(Rw) = 1$ because it has rank 1, and it follows that $\text{HW}_v(L) = 1$.

Since L is of odd rank with $\det L \in 2R^\times$ we have that L is maximal. From this it follows, as in the proof of Lemma 5.5, that N and the rank determine the isometry class of L .

For the last claim let A be an orthogonal sum of hyperbolic planes so it is even unimodular with $\det A = 1$ and consider a unary lattice Rw with $Q(w) = N$. Then $A \perp Rw$ has the same determinant and rank as L . \square

We aim to classify the special lattices of a given rank and determinant. For this purpose we introduce an invariant which, in the case of rank 3, is related to the Eichler invariant of quaternion orders.

Definition 5.8. The *Eichler invariant* of a special lattice L of determinant $2N$ is given by

$$e(L) = \begin{cases} 1 & \text{if } 4N Q(v) = 1 \text{ for some } v \in L^\vee; \\ -1 & \text{otherwise.} \end{cases}$$

Proposition 5.9. *Let L be a special lattice of determinant $2N$ and write $L = A \perp Rw$ as in Theorem 5.2. If $p \nmid N$ then $e(L) = 1$ and if $p \mid N$ then $e(L) = d(A)$.*

Proof. We have $L^\vee = A \perp \frac{1}{2a}Rw$ where $a = Q(w)$ so any vector $v \in L^\vee$ can be written as $v = \frac{x}{2a}w + u$ with $x \in R$ and $u \in A$. We compute $4N Q(v) = tx^2 + 4N Q(u)$ where $t = \frac{N}{a} = \det A$. When $p \nmid N$ we can assume, by Proposition 5.7, that $t = \det A = 1$ and so $4N Q(v)$ represents 1. When $p \mid N$ we have $4N Q(v) \equiv tx^2 \pmod{4p}$ represents 1 if and only if t is a square modulo $4p$, i.e. if and only if $d(A) = 1$. \square

In any case we can always assume $e(L) = d(A)$; when $p \nmid N$ using the last part of Proposition 5.7.

Corollary 5.10. *For a special lattice of determinant $2N$ the Hasse-Witt invariant satisfies $\text{HW}_v(L) = e(L)^{v(N)}$.*

Proof. Write $L = A \perp Rw$ as in Theorem 5.2. We know $\text{HW}_v(A) = 1$ by Lemma 5.3; also $\text{HW}_v(Rw) = 1$ since Rw has rank 1. Applying [Lam05, (3.13) in p.117] as before we conclude $\text{HW}_v(A \perp Rw) = (-Q(w), \det A)_v$. Since $\det A$ is an unramified unit and $N/Q(w)$ is a unit we have $(-Q(w), \det A)_v = (N, \det A)_v = d(A)^{v(N)} = e(L)^{v(N)}$. \square

Corollary 5.11. *Two special lattices of the same rank, determinant and Eichler invariant are isometric.*

Proof. Let L and L' be the two lattices of determinant $2N$. When $p \nmid N$ the claim follows from Proposition 5.7. When $p \mid N$ write $L = A \perp Rw$ and $L' = A' \perp Rw'$. The hypothesis $e(L) = e(L')$ implies, by Proposition 5.9, that $d(A) = d(A')$ and hence, by Lemma 5.5 that A and A' are isometric. Moreover $Q(w)/Q(w') = \det A'/\det A$ is the square of a unit so that Rw and Rw' are isometric. We conclude that L and L' are isometric. \square

Theorem 5.12. *For a given $n \geq 1$, $N \in R$ and $e \in \{\pm 1\}$ there exists a unique isometry class of special lattices of rank $2n+1$, determinant $2N$ and Eichler invariant e , provided $e = 1$ when $N \in R^\times$.*

Proof. Let A be an even unimodular lattice of rank $2n$ with invariant $d(A) = e$ which exists by Lemma 5.6. Then $L = A \perp R w$ where $Q(w) = N/\det A$ has rank $2n+1$, determinant $2N$ and $e(L) = d(A) = e$. Uniqueness follows from Corollary 5.11. \square

5.2. Global classification. In this section we let R be the ring of integers of a number field k with r_1 real places. If p is a prime ideal of R , by R_p we denote the completion of R at p . If L is an R -lattice, then $L_p = L \otimes R_p$.

Proposition 5.13. *Let L be an R -lattice of odd rank. Then L is special if and only if L_p is special for all non-archimedean places p of R .*

Proof. If L is special then $L^\vee/L \simeq R/I$ for some ideal $I \triangleleft R$. Then $L_p^\vee/L_p \simeq R_p/I_p$ is cyclic.

For the converse let S be a finite set of primes such that L_p is even unimodular for $p \notin S$. For $p \in S$ we have $L_p^\vee/L_p \simeq R_p/I_p = R/(I_p \cap R)$ for some ideal $I_p \triangleleft R_p$. Then, using the Chinese remainder theorem, we have $L^\vee/L \simeq R/I$ where $I = \prod_{p \in S} (R \cap I_p)$. \square

Theorem 5.14. *For a given $n \geq 3$ odd, $N \in R$ and $e_p \in \{\pm 1\}$ for each $p \mid N$, there is a unique genus of totally positive definite special lattices of rank n , determinant $2N$ and local Eichler invariants e_p , provided*

$$\prod_{p \mid N} e_p^{v_p(N)} = (-1)^{[k:\mathbb{Q}]}.$$

Proof. This follows immediately from the local classification Theorem 5.12, together with the fact [O'M73, 72:1] that the only global obstruction is $\prod \text{HW}_v(L) = 1$. By Corollary 5.10 the left hand side is the product of $\text{HW}_v(L)$ for the non-archimedean places. Note that $\text{HW}_v(L) = 1$ for complex places and $\text{HW}_v(L) = -1$ for real places; hence the product of $\text{HW}_v(L)$ for the archimedean places is $(-1)^{r_1}$. Since $[k:\mathbb{Q}] \equiv r_1 \pmod{2}$ this equals the right hand side. \square

6. A SPECIAL GLOBAL QUINARY LATTICE FOR $\text{GU}(2, B)$

Let $D^- = \prod_{p \in S} p$ and let D^+ be a positive integer, not necessarily square-free, but coprime to D^- , and let $D := D^- D^+$. Let Q be the quadratic form on U given by

$$(14) \quad Q\left(\begin{pmatrix} s & r \\ r & t \end{pmatrix}\right) = \frac{1}{4D}((s-t)^2 + 4N(r)),$$

where $N(r) = r\bar{r}$ is the norm of r as an element of the quaternion algebra.

Remark 6.1. It follows from its definition that the quadratic form Q is invariant under translation by centre elements, i.e. $Q(v + \lambda I) = Q(v)$ if $\lambda \in \mathbb{Q}$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(B)$, let $\text{Adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ denote its usual adjoint matrix.

Lemma 6.2. *If $v \in U$ the following relation holds*

$$(15) \quad \frac{1}{4D}(v - \text{Adj}(v))^2 = \begin{pmatrix} Q(v) & 0 \\ 0 & Q(v) \end{pmatrix}.$$

Proof. Follows from an elementary computation. \square

In particular, we can alternatively define the quadratic form via

$$(16) \quad Q(v) = \frac{1}{8D} \operatorname{Tr}((v - \operatorname{Adj}(v))^2).$$

Definition 6.3. Let (V, Q) be the quadratic space $V = U/\mathbb{Q}I$, with the quadratic form Q in the quotient space.

Remark 6.4. Over \mathbb{Q} we can take an orthogonal complement for the scalar matrices subspace (given by the elements in U whose trace is zero), and the quadratic space $\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)^\perp, Q\right)$ (isometric to the space (V, Q)) is isometric to the space considered by Ibukiyama in [Ibu19]. In this case the quadratic forms becomes $Q\left(\begin{smallmatrix} t & \bar{r} \\ r & -t \end{smallmatrix}\right) = \frac{1}{D}(t^2 + \mathcal{N}(r))$ (i.e. $Q(A) = \frac{1}{2D} \operatorname{Tr}(A^2)$). The advantage of working with the 6-dimensional space will prove crucial while working over \mathbb{Z}_2 , cf. Remark 4.5.

Since $B \otimes \mathbb{R}$ splits over \mathbb{C} , B can be embedded in $M_2(\mathbb{C})$, hence $M_2(B)$ in $M_4(\mathbb{C})$, so the following is immediate.

Lemma 6.5. *Let $g \in \operatorname{GU}(2, B)$ and $v \in U$, then $\operatorname{Tr}(gvg^{-1}) = \operatorname{Tr}(v)$.*

Note that v , considered as an element of $M_2(B)$, has rational scalar entries on the leading diagonal, so its trace as a 2×2 matrix and its trace as an element of $M_4(\mathbb{C})$ differ only by a factor of 2.

Proposition 6.6. *The action of $\operatorname{GU}(2, B)$ on U (and on V) preserves the quadratic form Q , up to a factor $\nu(g)$.*

Proof. Note that $v - \operatorname{Adj}(v) = 2v - \operatorname{Tr}(v)$, hence by (16), $8D Q(gvg^{-1}) = \operatorname{Tr}(2gvg^{-1} - \operatorname{Tr}(gvg^{-1}))^2$. But $(2gvg^{-1} - \operatorname{Tr}(gvg^{-1}))^2 = g(2v - \operatorname{Tr}(v))^2 g^{-1} = g(v - \operatorname{Adj}(v))^2 g^{-1} = 4Dg \begin{pmatrix} Q(v) & 0 \\ 0 & Q(v) \end{pmatrix} g^{-1} = 4D Q(v)I$ and the result follows. \square

6.0.1. *The quadratic form at split primes.* If p is a split prime, i.e. $B_p \simeq M_2(\mathbb{Q}_p)$, with $r \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the quadratic form (on trace zero elements) $\frac{1}{D}(t^2 + r\bar{r})$ becomes $\frac{1}{D}(t^2 + ad - bc)$, giving an isomorphism $\operatorname{GU}(2, B_p)/\mathbb{Q}_p^\times \simeq \operatorname{SO}_5(\mathbb{Q}_p)$, the split special orthogonal group.

6.0.2. *The quadratic form at non-split primes.* As mentioned earlier, we can identify the quotient $U_p/\mathbb{Q}_p I$ with the subspace of trace zero matrices. If $A \in U_p$ and $\tilde{A} = \xi^{-1} A \xi \in \tilde{U}_p$, $\operatorname{tr}(A) = 0 \iff \operatorname{tr}(\tilde{A}) = 0$, hence the trace zero elements can be described by

$$\tilde{V}_p = \left\{ \begin{pmatrix} y & s \\ t & \bar{y} \end{pmatrix} : t, s \in \mathbb{Q}_p, y \in B_p, y + \bar{y} = 0 \right\}.$$

and the quadratic form becomes

$$Q(A) = \frac{1}{2D} \operatorname{tr}(A^2) = \frac{1}{2D} \operatorname{tr}(\tilde{A}^2) = \frac{1}{D} (st + y^2).$$

This is the quadratic form associated with the non-split special orthogonal group $\operatorname{SO}_5^*(\mathbb{Q}_p)$ (cf. [GR06, §3]), and we get isomorphisms

$$\operatorname{GU}(2, B_p)/\mathbb{Q}_p^\times \simeq \operatorname{GU}(1, 1, B_p)/\mathbb{Q}_p^\times \simeq \operatorname{SO}_5^*(\mathbb{Q}_p).$$

6.1. A global lattice in V . Let $D = D^-D^+$ as before, and consider the 5-dimensional \mathbb{Q} -vector space $V = U/\mathbb{Q}I$ with the quadratic form Q defined in (14).

Recall that by the local-to-global principle, to give a lattice in a \mathbb{Q} -vector space is equivalent to giving it locally for each finite place p .

Definition 6.7. Let L be the \mathbb{Z} -lattice in V whose local completion $L_p := L \otimes \mathbb{Z}_p$ at a prime p is as follows.

- (1) For $p \nmid D$, $L_p := (\mathfrak{U}_{p^0}^+/\mathbb{Z}_p I)^\vee$.
- (2) For $p \mid D^+$, $L_p := (\mathfrak{U}_{p^n}^+/\mathbb{Z}_p I)^\vee$, where $n = v_p(D^+)$.
- (3) For $p \mid D^-$, $L_p := (\mathfrak{U}_p^-/\mathbb{Z}_p I)^\vee$.

Proposition 6.8. *The lattice L is integral with respect to the quadratic form Q . Furthermore, there exists a basis such that the Hessian matrix of the quinary form is as follows:*

- If $p \nmid D^-$, let $n = v_p(D)$, then

$$H(Q) = 2D \perp \frac{D}{p^n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \frac{D}{p^n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- If $p \mid D^-$ is odd, let ε be a non-square modulo p . Then

$$H(Q) = 2D\varepsilon \perp \frac{2D}{p} \perp \frac{-2D\varepsilon}{p} \perp \frac{D}{p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- If $2 \mid D^-$, then

$$H(Q) = \frac{-2D}{3} \perp \frac{-D}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \frac{D}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular, the determinant of $H(Q)$ equals $2D$.

Proof. We can check this locally prime-by-prime. For the determinant statement, since Q is positive-definite and $D > 0$, it is enough to check that the valuation is correct at each prime p .

- (1) If $p \nmid D$ the quaternion algebra B is unramified at p . In the canonical basis $\mathcal{B} := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right\}$ the quadratic norm form has Hessian matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let $n = v_p(D)$. Consider the basis for $\mathfrak{U}_{p^n}^+$

$$\left\{ \begin{pmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix}, \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix}, \begin{pmatrix} 0_2 & v\bar{\pi}^n \\ \pi^n \bar{v} & 0_2 \end{pmatrix} : v \in \mathcal{B} \right\}.$$

In particular, a basis for the quotient $\mathfrak{U}_{p^n}^+/\mathbb{Z}_p I$ is given by the last five elements. It is easy to check that in such a basis, the quadratic form Q has Hessian matrix

$$\frac{1}{4D} \left(2 \perp 4p^n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp 4p^n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

For $L_p = (\mathfrak{U}_{p^n}^+/\mathbb{Z}_p I)^\vee$ a Hessian matrix is then

$$(17) \quad 2D \perp \frac{D}{p^n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \frac{D}{p^n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Integrality at p follows from the fact that $D/p^n \in \mathbb{Z}_p^\times$.

- (2) If $p \mid D^-$, the quaternion algebra is ramified at p . Recall from Lemma 6.2 that if $v \in U$, $(v - \text{Adj}(v))^2$ is a diagonal matrix, hence its trace (which gives the quadratic form) is invariant under conjugation by ξ . In particular, it is enough to understand the lattice

$$\left\{ \begin{pmatrix} r & ps \\ t & \bar{r} \end{pmatrix} : s, t \in \mathbb{Z}_p, r \in \mathcal{O}_p \right\}.$$

with the quadratic form $\frac{1}{4D}((r - \bar{r})^2 + 4pst)$. If p is odd, then $\mathcal{O}_p = \langle 1, \mu, \pi, \mu\pi \rangle_{\mathbb{Z}_p}$, where the last three elements have trace zero and satisfy $\mu^2 = \varepsilon \in \mathbb{Z}_p^\times$ (a non-square), $\pi^2 = -p$ and $\pi\mu = -\mu\pi$. Then the Hessian matrix of the quadratic form $\frac{1}{4D}(r - \bar{r})^2$ has diagonal entries (see [Brz83] and also Section 5 of [Lem11])

$$\frac{1}{4D}(0 \perp 8\varepsilon \perp 8p \perp -8p\varepsilon).$$

Then in the basis $\left\{ \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix}, \begin{pmatrix} \mu\pi & 0 \\ 0 & -\mu\pi \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} \right\}$ (a basis for $\mathfrak{A}_p^-/\mathbb{Z}_p I$) the quadratic form has matrix

$$\frac{4}{4D} \left(2\varepsilon \perp 2p \perp -2p\varepsilon \perp \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix} \right).$$

In particular, its dual lattice has Hessian matrix

$$(18) \quad 2D\varepsilon \perp \frac{2D}{p} \perp \frac{-2D\varepsilon}{p} \perp \frac{D}{p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This implies both the integrality and the determinant statement (recall that $v_p(D) = 1$ hence $D/p \in \mathbb{Z}_p^\times$).

If $p = 2$, B_2 is the Hamilton 2-adic quaternion algebra ($i^2 = j^2 = -1$), a basis for \mathcal{O}_2 is $\langle 1, i, j, \frac{1+i+j+k}{2} \rangle$. A better basis for the quadratic form $(r - \bar{r})^2/2$ (over \mathbb{Z}_2) is $\langle 1, \frac{1+i+j+k}{2}, \frac{-1+2i-j-k}{3}, \frac{-1-i+2j-k}{3} \rangle$, where the Gram matrix becomes

$$\frac{4}{4D} \left(0 \perp -3/2 \perp \frac{2}{3} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \right)$$

Then, the quinary form in the quotient equals

$$\frac{-3}{2D} \perp, -\frac{2}{3D} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \perp \frac{1}{D} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

and its dual lattice has Gram matrix

$$\frac{-2D}{3} \perp \frac{-D}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \frac{D}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is an integral quadratic form, whose Hessian matrix has determinant valuation 2 (since $D/2$ is a unit in \mathbb{Z}_2).

□

Proposition 6.9. *The lattice L is special. Furthermore, its Eichler and Hasse-Witt invariants are the following:*

- (1) $\text{HW}_\infty(Q) = -1$.
- (2) $\text{HW}_p(Q) = e(L_p) = 1$ if $p \nmid D^-$.
- (3) $\text{HW}_p(Q) = e(L_p) = -1$ if $p \mid D^-$.

Proof. By Corollary 5.10 it is enough to compute the Eichler invariant at each finite place and the Hasse-Witt invariant at the infinite place. Note that Proposition 6.8 expresses each completion of the lattice at a finite place as an orthogonal sum of a rank one lattice and a unimodular one, hence L is special by Proposition 5.13.

- (1) At the infinity place, (2) gives $\text{HW}_\infty(Q) = (-1, -1)_\infty(1, 1)_\infty^{10} = -1$.
- (2) If $p \nmid D^-$, Proposition 6.8 implies that $e(L_p) = (p, 1)_p = 1$.
- (3) If $p \mid D^-$ is odd, Proposition 6.8 implies that $e(L_p) = (p, \varepsilon)_p = -1$ while the case $p = 2$ gives $e(L_2) = (2, -3)_2 = -1$.

□

Remark 6.10. Let (\tilde{V}, \tilde{Q}) be a quinary quadratic space, whose quadratic form is positive definite. Then by Remark 2.7 the even Clifford algebra $\text{Cliff}_0(\tilde{V})$ is isomorphic to $M_2(B)$, where B is a quaternion algebra ramified precisely at infinity and the finite primes where \tilde{Q} has Hasse-Witt invariant -1 . The quadratic space (V, Q) constructed from $M_2(B)$ then has the same Hasse-Witt invariants as (\tilde{V}, \tilde{Q}) by the last proposition, in particular they are isometric, providing an isomorphism $\text{GU}(2, B)/\mathbb{Q}^\times \simeq \text{SO}(\tilde{V})$.

Remark 6.11. Let (\tilde{L}, \tilde{Q}) be a quinary lattice, whose quadratic form is positive definite, and is special of determinant $2D$. (Note that if D is square-free, (\tilde{L}, \tilde{Q}) is automatically special.) Let $S = \{p : e(\tilde{L})_p = -1\}$ and suppose that $v_p(D) = 1$ for all $p \in S$. Let (L, Q) be the quinary lattice of Definition 6.7. Then (L, Q) and (\tilde{L}, \tilde{Q}) are in the same genus, in particular $\text{SO}(Q) \simeq \text{SO}(\tilde{Q})$.

6.2. Radicals. Let (q, Λ_p) be an integral quadratic form, where Λ_p is a \mathbb{Z}_p -lattice.

Definition 6.12. The *radical* of the form (q, Λ_p) equals

$$\text{Rad}(q, \Lambda_p) := \{v \in \Lambda_p \otimes (\mathbb{Z}/2p) : \langle v, w \rangle \equiv 0 \pmod{2p} \ \forall w \in \Lambda_p\}.$$

In particular, if $p \neq 2$, $\text{Rad}(q, \Lambda_p)$ is an \mathbb{F}_p -vector space, while for $p = 2$ it is a $\mathbb{Z}/4$ -module.

Lemma 6.13. *Let p be a prime number and (Q, L_p) be as in Definition 6.7. If $p \mid D$ then $\text{Rad}(Q, L_p)$ is a $\mathbb{Z}/2p$ lattice of rank 1.*

Proof. Recall from Proposition 6.8 that the quadratic form Q is equivalent to

$$H(Q) = \begin{cases} 2D \oplus \frac{D}{p^n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \frac{D}{p^n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } p \mid D^+, \\ 2D\varepsilon \oplus \frac{2D}{p} \oplus \frac{-2D\varepsilon}{p} \oplus \frac{D}{p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } p \mid D^-, p \neq 2, \\ \frac{-2D}{3} \oplus \frac{-D}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus \frac{D}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } 2 \mid D^-. \end{cases}$$

In all cases, the first element of the basis clearly spans the radical. □

7. STABILISERS OF THE LOCAL LATTICES L_p

Let us compute for each prime p the stabiliser (under conjugation) of the lattice L_p of Definition 6.7. For ease of notation, let us denote by \mathfrak{U}_p either $\mathfrak{U}_{p^0}^+$, $\mathfrak{U}_{p^n}^+$ or \mathfrak{U}_p^- according to the case.

Lemma 7.1. *The stabiliser in $\mathrm{GU}(2, B_p)$ of the rank five lattice L_p equals that of the rank six lattice \mathfrak{U}_p .*

Proof. Since the quadratic form is invariant under the conjugation action of $\mathrm{GU}(2, B_p)$ (by a local version of Proposition 6.6) the stabiliser of L_p is the same as that of its dual lattice $\mathfrak{U}_p/\mathbb{Z}_p I$, which is what we shall actually look at.

The action of $\mathrm{GU}(2, B_p)$ is trivial at the identity matrix, hence if an element stabilises the rank 6 lattice \mathfrak{U}_p it also stabilises the quotient $\mathfrak{U}_p/\mathbb{Z}_p I$. To prove the converse, let g be an element stabilising the quotient lattice $\mathfrak{U}_p/\mathbb{Z}_p I$. Let $v \in \mathfrak{U}_p$ be any vector, so $g\bar{v}g^{-1} = \bar{w}$ for some w in L_p . In particular, there exists $\lambda \in \mathbb{Q}_p$ such that

$$(19) \quad gvg^{-1} = w + \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

for some element $w \in \mathfrak{U}_p$ in the preimage of \bar{w} . Since $v, w \in \mathfrak{U}_p$, their traces are integral and since $\mathrm{tr}(gvg^{-1}) = \mathrm{tr}(v)$ (by Lemma 6.5), $2\lambda \in \mathbb{Z}_p$. This gives the statement when $p \neq 2$. Suppose that $p = 2$ and $\lambda \notin \mathbb{Z}_2$. We can look at the ‘‘determinants’’ of equation (19). For that purpose, take a quadratic extension of \mathbb{Q}_2 that splits the quaternion algebra, and take the determinant (as 4×4 matrices with coefficients in such an extension). Since all elements of \mathfrak{U}_2 have integral entries, their determinants are integral. Since $\det(AB) = \det(BA)$, $\det(gvg^{-1})$ is integral, which is not the case for $w + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ (as it corresponds to a 4×4 matrix with integral entries outside the diagonal, but with negative valuation at all diagonal elements). \square

Proposition 7.2. *Let $p \notin S$ be an unramified prime.*

- (1) *The subgroup $K_{0,p}$ of $\mathrm{GU}(2, B_p)$ preserves the \mathbb{Z}_p -lattice $\mathfrak{U}_{p^0}^+ \subseteq U_p$, which was defined by*

$$\mathfrak{U}_{p^0}^+ = \left\{ \begin{pmatrix} s & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} & t \end{pmatrix} : s, t, a, b, c, d \in \mathbb{Z}_p \right\}$$

- (2) *In fact, the image of $K_{p^0}^+$ is the full stabiliser of $\mathfrak{U}_{p^0}^+$ in $\mathrm{GU}(2, B_p)/\mathbb{Q}_p^\times$.*

Proof. (1) Immediate.

- (2) Suppose $g \in \mathrm{GU}(2, B_p) \simeq \mathrm{GSp}_2(\mathbb{Q}_p)$ is such that $g\mathfrak{U}_{p^0}^+g^{-1} = \mathfrak{U}_{p^0}^+$. By [Ibu19, Lemma 4.1], $gM_4(\mathbb{Z}_p)g^{-1} = M_4(\mathbb{Z}_p)$, hence we are led to compute the normaliser of $M_4(\mathbb{Z}_p)$. Although the same proof given by Eichler to prove the ‘‘Lemma’’ ([Eic73], page 93 for $M_2(\mathbb{Z}_p)$) applies *mutatis mutandis*, we recall the one given by Ibukiyama. For some sufficiently large n , $p^n g \in M_4(\mathbb{Z}_p)$, and then $p^n g M_4(\mathbb{Z}_p) = p^n M_4(\mathbb{Z}_p) g$ is a two-sided ideal of $M_4(\mathbb{Z}_p)$. As in [Ibu18, Lemma 3.1], necessarily $p^n g M_4(\mathbb{Z}_p) = p^e M_4(\mathbb{Z}_p)$ for some $e \geq 0$. Equating sets of determinants, $p^{4n} \det(g)_{\mathbb{Z}_p} = p^{4e} \mathbb{Z}_p$, so $\det(g) \in p^{4(e-n)} \mathbb{Z}_p^\times$, and $p^{n-e} g \in \mathrm{GSp}_2(\mathbb{Q}_p) \cap \mathrm{GL}_4(\mathbb{Z}_p) = K_{p^0}^+$, as required. \square

Let us state an elementary result.

Lemma 7.3. *If $A \in \frac{1}{p^n}\mathbb{Z}_p \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + M_2(\mathbb{Z}_p)$ has integral determinant, then $A \in M_2(\mathbb{Z}_p)$. Similarly, if $A \in \frac{1}{p^n}\mathbb{Z}_p \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \mathbb{Z}_p & p^{-n}\mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ has integral determinant, then $A \in \begin{pmatrix} \mathbb{Z}_p & p^{-n}\mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$.*

Proof. Suppose on the contrary that $A = \begin{pmatrix} \frac{a}{p^r} & b \\ c & \frac{d}{p^r} \end{pmatrix}$ with $a, d \in \mathbb{Z}_p^\times$, $0 < r \leq n$. The hypothesis $\det(A) = \frac{ad}{p^{2r}} - bc \in \mathbb{Z}_p$ implies $2r \leq 0$ getting a contradiction. The other case is similar. \square

Proposition 7.4. *Let $p \notin S$ be an unramified prime.*

- (1) *The subgroup $K_{p^n}^+$ of $\mathrm{GU}(2, B_p)$ preserves the \mathbb{Z}_p -lattice $\mathfrak{U}_{p^n}^+ \subseteq U_p$, which was defined by*

$$\mathfrak{U}_{p^n}^+ := \left\{ \begin{pmatrix} s & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} & t \end{pmatrix} : s, t, b, d \in \mathbb{Z}_p, a, c \in p^n\mathbb{Z}_p \right\}.$$

So does the Atkin-Lehner element $W_{p^n}^+$.

- (2) *In fact, the subgroup of $\mathrm{GU}(2, B_p)/\mathbb{Q}_p^\times$ generated by the images of $K_{p^n}^+$ and $W_{p^n}^+$ is the full stabiliser of $\mathfrak{U}_{p^n}^+$.*

Proof. (1) Recall that

$$K_{p^n}^+ := \{k \in \mathrm{GU}(2, B_p) : h^{-n}kh^n \in \mathrm{GL}_4(\mathbb{Z}_p)\},$$

where $h := \mathrm{diag}(1, 1, 1, p)$. Given $A = \begin{pmatrix} s & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} & t \end{pmatrix} \in U_p$,

$$(20) \quad h^{-n}Ah^n \in M_4(\mathbb{Z}_p) \iff \begin{pmatrix} s & \begin{pmatrix} a & p^nb \\ c & p^nd \end{pmatrix} \\ \begin{pmatrix} d & -b \\ -p^{-n}c & p^{-n}a \end{pmatrix} & t \end{pmatrix} \in M_4(\mathbb{Z}_p) \\ \iff b, d \in \mathbb{Z}_p, a, c \in p^n\mathbb{Z}_p \iff A \in \mathfrak{U}_{p^n}^+.$$

If $k \in K_{p^n}^+$ and $A \in \mathfrak{U}_{p^n}^+$, by Proposition 4.2, $kAk^{-1} \in U_p$. Let $k = h^nmh^{-n} \in K_{p^n}^+$, with $m \in \mathrm{GL}_4(\mathbb{Z}_p)$, and $A = h^nm'h^{-n} \in \mathfrak{U}_{p^n}^+$, with $m' \in M_4(\mathbb{Z}_p)$, then

$$kAk^{-1} = h^nm(h^{-n}(h^nm'h^{-n})h^n)m^{-1}h^{-n} = h^n(mm'm^{-1})h^{-n},$$

and the latter is an element in U_p which is in $\mathfrak{U}_{p^n}^+$ by (20).

Recall that $W_{p^n}^+ := \begin{pmatrix} 0 & 0 & p^n & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & p^n & 0 & 0 \end{pmatrix}$. Then if $A \in \mathfrak{U}_{p^n}^+$ is as before,

$$W_{p^n}^+AW_{p^n}^{+^{-1}} = p^{-n}W_{p^n}^+AW_{p^n}^+ = \begin{pmatrix} t & \begin{pmatrix} p^nd & -b \\ -p^nc & a \end{pmatrix} \\ \begin{pmatrix} a & b \\ p^nc & p^nd \end{pmatrix} & s \end{pmatrix} \in \mathfrak{U}_{p^n}^+.$$

(2) Note that $K_{p^n}^+ = \mathrm{GU}(2, B_p) \cap \tilde{R}^\times$, where $\tilde{R} = \Psi^{-1}(R)$ (cf. (8)) given by

$$\tilde{R} = \left(\begin{array}{cc} \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} & \begin{pmatrix} p^n \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \\ \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & p^n \mathbb{Z}_p \end{pmatrix} & \begin{pmatrix} \mathbb{Z}_p & p^{-n} \mathbb{Z}_p \\ p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \end{array} \right).$$

By Lemma 3.6 it suffices to show that if $g \in \mathrm{GU}(2, B_p)$ satisfies $g^{-1} \mathfrak{U}_{p^n}^+ g = \mathfrak{U}_{p^n}^+$ then $g^{-1} \tilde{R} g = \tilde{R}$. Switching g and g^{-1} to get the reverse inclusion, it suffices to show that $g^{-1} \tilde{R} g \subseteq \tilde{R}$.

By Lemma 4.3, the minimal order containing $\mathfrak{U}_{p^n}^+$ equals

$$R' = \left(\begin{array}{cc} \mathbb{Z}_p I_2 & 0_2 \\ 0_2 & \mathbb{Z}_p I_2 \end{array} \right) \oplus \left(\begin{array}{cc} \begin{pmatrix} p^n \mathbb{Z}_p & p^n \mathbb{Z}_p \\ p^n \mathbb{Z}_p & p^n \mathbb{Z}_p \end{pmatrix} & \begin{pmatrix} p^n \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \\ \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & p^n \mathbb{Z}_p \end{pmatrix} & \begin{pmatrix} p^n \mathbb{Z}_p & \mathbb{Z}_p \\ p^{2n} \mathbb{Z}_p & p^n \mathbb{Z}_p \end{pmatrix} \end{array} \right).$$

In particular, if $g^{-1} \mathfrak{U}_{p^n}^+ g = \mathfrak{U}_{p^n}^+$ then $g^{-1} R' g = R'$. \tilde{R} is generated (as a \mathbb{Z}_p -module) by R' and by elements of the form $\begin{pmatrix} M & 0_2 \\ 0_2 & 0_2 \end{pmatrix}$ or $\begin{pmatrix} 0_2 & 0_2 \\ 0_2 & M' \end{pmatrix}$, with $M \in M_2(\mathbb{Z}_p)$ and $M' \in \begin{pmatrix} \mathbb{Z}_p & p^{-n} \mathbb{Z}_p \\ p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$. The Atkin-Lehner operator also fixes R' , and since conjugating by $W_{p^n}^+$ a general element of the first form gives one of the second, it suffices to show that if $M \in M_2(\mathbb{Z}_p)$, $g^{-1} \begin{pmatrix} M & 0_2 \\ 0_2 & 0_2 \end{pmatrix} g \in \tilde{R}$.

Write $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with $A, B, C, D \in M_2(\mathbb{Q}_p)$. From $g^* g = \nu(g) I$, we get

$$g^{-1} = \frac{1}{\nu(g)} g^* = \frac{1}{\nu(g)} \begin{pmatrix} \bar{A} & \bar{C} \\ \bar{B} & \bar{D} \end{pmatrix}.$$

Recall that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\bar{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and $A\bar{A} = \bar{A}A = \det A$. Now,

$$(21) \quad g^{-1} \begin{pmatrix} M & 0_2 \\ 0_2 & 0_2 \end{pmatrix} g = \frac{1}{\nu(g)} \begin{pmatrix} \bar{A}MA & \bar{A}MB \\ \bar{B}MA & \bar{B}MB \end{pmatrix}.$$

In the particular case $M = I_2$, since $\begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \in R'$, looking at the top left and bottom right blocks we find that $\frac{\det A}{\nu(g)}, \frac{\det B}{\nu(g)} \in \mathbb{Z}_p$.

Since $\begin{pmatrix} p^n M & 0_2 \\ 0_2 & 0_2 \end{pmatrix} \in R'$, its conjugate is also in R' , hence the relation (21) implies that $\frac{1}{\nu(g)} \bar{A} p^n M A \in \mathbb{Z}_p + p^n M_2(\mathbb{Z}_p)$, so $\frac{1}{\nu(g)} \bar{A} M A \in \frac{1}{p^n} \mathbb{Z}_p + M_2(\mathbb{Z}_p)$. On the other hand, $\det \left(\frac{1}{\nu(g)} \bar{A} M A \right) = \frac{\det A}{\nu(g)} \det(M) \in \mathbb{Z}_p$, hence Lemma 7.3 implies that $\frac{1}{\nu(g)} \bar{A} M A \in M_2(\mathbb{Z}_p)$.

Similarly, $\frac{1}{\nu(g)} \bar{B} M B \in \frac{1}{p^n} \mathbb{Z}_p + \begin{pmatrix} \mathbb{Z}_p & p^{-n} \mathbb{Z}_p \\ p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$. The same determinant computation combined with Lemma 7.3 implies that $\frac{1}{\nu(g)} \bar{B} M B \in \begin{pmatrix} \mathbb{Z}_p & p^{-n} \mathbb{Z}_p \\ p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$.

Finally,

$$\frac{1}{\nu(g)} \bar{A} M B = \frac{\bar{A} M A}{\nu(g)} \frac{\nu(g)}{\det A} \frac{\bar{A} B}{\nu(g)}.$$

We have already shown that the first factor is in $M_2(\mathbb{Z}_p)$, the second factor is in \mathbb{Z}_p since $\frac{\det A}{\nu(g)} \in \mathbb{Z}_p^\times$, and the third factor is in $\begin{pmatrix} p^n \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$, using the special case $M = I_2$. Hence $\frac{1}{\nu(g)} \bar{A} M B \in \begin{pmatrix} p^n \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$, and similarly $\frac{1}{\nu(g)} \bar{B} M A \in \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & p^n \mathbb{Z}_p \end{pmatrix}$. \square

Proposition 7.5. *Let $p \in S$ be a ramified prime.*

- (1) *The subgroup K_p^- of $\mathrm{GU}(2, B_p)$ preserves the \mathbb{Z}_p -lattice $\mathfrak{U}_p^- \subseteq U_p$, which was defined by $\mathfrak{U}_p^- = \xi \tilde{\mathfrak{U}}_p^- \xi^{-1}$, where*

$$\tilde{\mathfrak{U}}_p^- := \left\{ \begin{pmatrix} r & ps \\ t & \bar{r} \end{pmatrix} : r \in \mathcal{O}_p, s, t \in \mathbb{Z}_p \right\}.$$

So does the Atkin-Lehner element ω_p .

- (2) *The subgroup of $\mathrm{GU}(2, B_p)/\mathbb{Q}_p^\times$ generated by the images of K_p^- and ω_p is the full stabiliser of \mathfrak{U}_p^- .*

Proof. (1) Let $\tilde{h} := \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$, where $\pi^2 = -p$. Recall that $K^-(p)$ is the stabiliser (via left multiplication) of the lattice $\mathfrak{p} \oplus \mathcal{O}_p = \tilde{h}\mathcal{O}_p^2$. Then $k \in K^-(p) \iff \tilde{h}^{-1}k\tilde{h} \in M_2(\mathcal{O}_p)$. Given $\tilde{A} = \begin{pmatrix} r & s \\ t & \bar{r} \end{pmatrix} \in \xi U_p \xi^{-1}$,

$$(22) \quad \begin{aligned} \tilde{h}^{-1}\tilde{A}\tilde{h} \in M_2(\mathcal{O}_p) &\iff \begin{pmatrix} \pi^{-1}r\pi & \pi^{-1}s \\ t\pi & \bar{r} \end{pmatrix} \in M_2(\mathcal{O}_p) \\ &\iff r \in \mathcal{O}_p, s \in p\mathbb{Z}_p, t \in \mathbb{Z}_p \iff \tilde{A} \in \tilde{\mathfrak{U}}_p^-. \end{aligned}$$

Now given $k \in K^-(p)$, $k = \tilde{h}m\tilde{h}^{-1}$ with $m \in \mathrm{GL}_2(\mathcal{O}_p)$, hence

$$k\tilde{A}k^{-1} = \tilde{h}m\tilde{h}^{-1}\tilde{A}\tilde{h}m^{-1}\tilde{h}^{-1}.$$

If $\tilde{A} \in \tilde{\mathfrak{U}}_p^-$ then $\tilde{h}^{-1}\tilde{A}\tilde{h} = m' \in M_2(\mathcal{O}_p)$, and $\tilde{h}^{-1}(k\tilde{A}k^{-1})\tilde{h} = mm'm^{-1} \in M_2(\mathcal{O}_p)$, so $k\tilde{A}k^{-1} \in \tilde{\mathfrak{U}}_p^-$, as required. Also,

$$\omega_p \tilde{A} \omega_p^{-1} = p^{-1} \omega_p \tilde{A} \omega_p = \begin{pmatrix} \bar{r} & pt \\ s/p & r \end{pmatrix} \in \tilde{\mathfrak{U}}_p^-.$$

- (2) Suppose that $g \in \mathrm{GU}(1, 1, B_p)$ is such that $g\tilde{\mathfrak{U}}_p^-g^{-1} = \tilde{\mathfrak{U}}_p^-$. By Lemma 4.7, the element g also normalises the order

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} R_p & pR_p \\ R_p & R_p \end{pmatrix} : a \equiv \bar{d} \pmod{\pi} \right\}.$$

By [Ibu19, Lemma 4.3], then $gRg^{-1} = R$, where $R = \begin{pmatrix} \mathcal{O}_p & \mathfrak{p} \\ \mathfrak{p}^{-1} & \mathcal{O}_p \end{pmatrix}$ is the left order of $\mathfrak{p} \oplus \mathcal{O}_p$ (this is precisely the statement of [Ibu19, Corollary 4.4], which is only stated for odd primes).

Looking at the form of R , clearly for some sufficiently large n , $p^n g \in R$, and then $p^n g R = p^n R g$ is a two-sided ideal of R . As in [Ibu18, Proposition 3.2], $p^n g R = \omega_p^e R$ for some $e \geq 0$. Recalling that $\omega_p^2 = pI$, $\omega_p^{2n-e} g R = R$, so $\omega_p^{2n-e} g \in R^\times$. Letting $m = \lfloor \frac{2n-e+1}{2} \rfloor$, either $p^m g \in R^\times \cap \mathrm{GU}(1, 1, B_p) = K^-(p)$, or $p^m \omega_p^{-1} g \in R^\times \cap \mathrm{GU}(1, 1, B_p) = K^-(p)$, as required. \square

7.1. Paramodular subgroups as kernels of sign characters. Let $\mathrm{Stab}(L_p) := \{g \in \mathrm{GU}(2, B_p) : gvg^{-1} = v \ \forall v \in L_p\}$ be the stabiliser of L_p . If $g \in \mathrm{Stab}(L_p)$ then the action of g preserves $\mathrm{Rad}(Q, L_p)$. Suppose that $p \mid D$, and let v_0 denote a generator of $\mathrm{Rad}(Q, L_p)$ as $\mathbb{Z}/2p$ -module (it has rank 1 by Lemma 6.13). Since $Q(gv_0g^{-1}) = Q(v_0)$, we must have $gv_0g^{-1} \equiv \pm v_0 \pmod{2p}$.

Definition 7.6. Let p be a prime dividing D . Define a homomorphism $\theta_p : \text{Stab}(L_p) \rightarrow \{\pm 1\}$ by

$$gv_0g^{-1} \equiv \theta_p(g)v_0 \pmod{2p}.$$

By Propositions 7.5 and 7.4, if $p \mid D$ then

$$\text{Stab}(L_p) = \begin{cases} \langle p^{\mathbb{Z}}, \omega_p, K_p^- \rangle & \text{if } p \mid D^-, \\ \langle p^{\mathbb{Z}}, W_{p^n}^+, K_{p^n}^+ \rangle & \text{if } p^n \parallel D^+. \end{cases}$$

Proposition 7.7. *Let p be an odd prime such that $p \mid D$. Then:*

- (1) $\theta_p(p^{\mathbb{Z}}) = \{1\}$.
- (2) If $p \mid D^-$, $\theta_p(K_p^-) = \{1\}$ and $\theta_p(\omega_p) = -1$.
- (3) If $p^n \parallel D^+$, $\theta_p(K_{p^n}^+) = \{1\}$ and $\theta_p(W_{p^n}^+) = -1$.

Proof. (1) Immediate.

(2) From the proof of Proposition 6.8, it follows that if $p \neq 2$, then the generator of $\text{Rad}(Q, L_p)$ is the element $\begin{pmatrix} p\mu & 0 \\ 0 & -p\mu \end{pmatrix}$ (we are taking the dual of the basis described in the proof of Proposition 6.8), where $\text{tr}(\mu) = 0$ and $\mu^2 = \varepsilon$. Recall that $\mathcal{O}_p = \langle 1, \mu, \pi, \pi\mu \rangle$, where $\pi\mu = -\mu\pi$, $\pi^2 = -p$. If $g = \begin{pmatrix} a & \pi^{-1}b \\ \pi c & d \end{pmatrix} \in K_p^-$, then $g^{-1} = \frac{1}{\nu(g)} \begin{pmatrix} \bar{d} & \bar{b}\bar{\pi}^{-1} \\ \bar{c}\bar{\pi} & \bar{a} \end{pmatrix}$ (where $\nu(g)$ is a unit). In particular, $\bar{d}a - \bar{b}c = \nu(g)$ (as $\bar{\pi}^{-1} = -\pi^{-1}$). Then

$$(23) \quad \frac{p}{\nu(g)} \begin{pmatrix} \bar{d} & \bar{b}\bar{\pi}^{-1} \\ \bar{c}\bar{\pi} & \bar{a} \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix} \begin{pmatrix} a & \pi^{-1}b \\ \pi c & d \end{pmatrix} = \frac{p}{\nu(g)} \begin{pmatrix} \bar{d}\mu a + \bar{b}\bar{\pi}^{-1}\bar{\mu}\pi c & * \\ * & * \end{pmatrix}.$$

The (1, 1) entry equals then $\frac{p}{\nu(g)}(\bar{d}\mu a - \bar{b}\mu c)$ (recall that $\bar{\mu} = -\mu$ and that μ and π anti-commute), and since $\mathcal{O}_p/(\pi)$ is commutative (generated by $\{1, \mu\}$), it is congruent to $p\mu \frac{\bar{d}a - \bar{b}c}{\nu(g)} = p\mu$ modulo πp . Looking at this one entry modulo πp is enough to distinguish the cases $gv_0g^{-1} \equiv \pm v_0 \pmod{2pL_p}$.

For the Atkin-Lehner statement,

$$\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \begin{pmatrix} p\mu & 0 \\ 0 & p\bar{\mu} \end{pmatrix} \begin{pmatrix} 0 & 1/p \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p\bar{\mu} & 0 \\ 0 & p\mu \end{pmatrix} = - \begin{pmatrix} p\mu & 0 \\ 0 & -p\mu \end{pmatrix}.$$

For $p = 2$ (the $(-1, -1)$ algebra), let $\mu = 2D \left(\frac{1+i+j+k}{2} \right)$ and let $\pi = i + k$, so that $R_2 = \langle 1, \mu, \pi, \pi\mu \rangle$ and π generates the maximal ideal. From the proof of Proposition 6.8 it follows that once again the generator of $\text{Rad}(Q, L_2)$ is the element $\begin{pmatrix} 4\mu & 0 \\ 0 & -4\mu \end{pmatrix}$. Looking at the (1, 1) entry of (23), we are left to prove that

$$(24) \quad \frac{\bar{d}\mu a + \bar{b}\bar{\pi}^{-1}\bar{\mu}\pi c}{\nu(g)} \equiv \mu \pmod{4}.$$

Recall that the maximal order of B_2 equals $R_2 = \langle 1, \frac{1+i+j+k}{2}, \frac{-1+2i-j-k}{3}, \frac{-1-i+2j-k}{3} \rangle$ (so μ equals $2D$ times the second element). Note that the left hand side of (24) lies in $4R_2$ so in particular, it can be written as $4\alpha + \beta\mu + 4\gamma e_3 + 4\delta e_4$, for some $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_2$, where e_3, e_4 denote the third and fourth elements of the generators of R_2 . In particular, it is congruent to $\beta\mu$ modulo $4L_2$ (as expected).

It is easy to check that $\bar{\pi}^{-1}\bar{\mu}\pi = -2D + \frac{\mu}{3} - 2De_4$, hence the left hand side of (24) is congruent to $\frac{\bar{d}\mu a - \bar{b}\mu c}{\nu(g)}$ modulo $4L_2$.

On the other hand, since $R_2/2$ is commutative (as can easily be verified), $\frac{\bar{d}\mu a - \bar{b}\mu c}{\nu(g)}$ is congruent to μ modulo $8R_2$ (recall that $g^{-1}g = 1$ together with $\bar{\pi} = -\pi$ implies that $\bar{d}a + \bar{b}c = \nu(g)$). In particular,

$$\frac{\bar{d}\mu a - \bar{b}\mu c}{\nu(g)} = 8\tilde{\alpha} + \tilde{\beta}\mu + 8\tilde{\gamma}e_3 + 8\tilde{\delta}e_4.$$

Since $\text{tr}(1) \equiv \text{tr}(e_3) \equiv \text{tr}(e_4) \equiv 0 \pmod{2}$, the right hand side of the previous line has trace congruent to $2D\tilde{\beta}$ (mod 16). Using the fact that in a quaternion algebra, $\text{tr}(\alpha\beta) = \text{tr}(\beta\alpha)$, the left hand side has trace equal to

$$\text{tr}\left(\frac{(\bar{d}a - \bar{b}c)}{\nu(g)}\mu\right) = \text{tr}(\mu) = 2D.$$

Since $v_2(D) = 1$, we get that $\tilde{\beta} \equiv 1 \pmod{4}$ as stated. The Atkin-Lehner statement follows from the same proof of the odd prime case, noting that (once again) $\bar{\mu} \equiv -\mu$ modulo $4L_2$.

(3) Arguing as in the previous case, $v_0 := 2D \cdot \begin{pmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix}$ is a generator for the radical. Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{p^n}^+ \subset \text{GU}(2, B_p) \simeq \text{GSp}_2(\mathbb{Q}_p)$,

$$gv_0g^{-1} = \frac{2D}{\nu(g)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \frac{2D}{\nu(g)} \begin{pmatrix} a\bar{a} & a\bar{c} \\ c\bar{a} & c\bar{c} \end{pmatrix}.$$

The condition

$$\frac{1}{\nu(g)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = I_4,$$

implies that $a\bar{a} = \nu(g) - b\bar{b}$ and $c\bar{c} = \nu(g) - d\bar{d}$. Then

$$\begin{aligned} gv_0g^{-1} - v_0 &= \frac{2D}{\nu(g)} \begin{pmatrix} \nu(g) - b\bar{b} - \nu(g) & a\bar{c} \\ c\bar{a} & \nu(g) - d\bar{d} \end{pmatrix} = \\ &= \frac{2D}{\nu(g)} (\nu(g) - d\bar{d}) \cdot \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} - \frac{2D}{\nu(g)} \begin{pmatrix} (\nu(g) + b\bar{b} - d\bar{d})I_2 & a\bar{c} \\ c\bar{a} & 0_2 \end{pmatrix}. \end{aligned}$$

The first term is zero in $U_p/\mathbb{Q}_p I$, hence it is enough to check that $\nu(g) + b\bar{b} - d\bar{d} \equiv 0 \pmod{2p}$. The equivalence $g^{-1}g = I_4$ implies that $\nu(g) - b\bar{b} - d\bar{d} = 0$, hence it is enough to prove that $2b\bar{b} \equiv 0 \pmod{2p}$. The fact $g \in K_{p^n}^+ \implies b \in \begin{pmatrix} p^n\mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$, so $b\bar{b} = \det b \in p^n\mathbb{Z}_p$ hence the statement. Regarding the Atkin-Lehner statement,

$$\begin{aligned} W_{p^n}^+ \frac{v_0}{2D} W_{p^n}^{+ -1} &= \begin{pmatrix} 0 & 0 & p^n & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & p^n & 0 & 0 \end{pmatrix} \begin{pmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p^{-n} \\ p^{-n} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & p^n & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p^{-n} \\ p^{-n} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix} \simeq -\frac{v_0}{2D}, \end{aligned}$$

where the last statement comes from the fact that our lattice is a quotient by scalar matrices. Then $\theta_{p^n}(W_{p^n}^+) = -1$. \square

Remark 7.8. Our lattice L_p with quadratic form Q is equivalent (up to unit scaling of the form) to Gross’s $\Lambda(N)$ with his $\langle, \rangle/N$ [Gro16, §5], to Tsai’s \mathbb{L}_m with \langle, \rangle_m [Tsa13, Definition 7.1.1], and to Lachausée’s L with quadratic form q' [Lac20, Definition 3.4.1, §3.5]. Our v_0 is Gross’s Nc , Tsai’s $\mathfrak{p}^m v_0$ and Lachausée’s v'_0 . Remarque 2 after [Lac20, Proposition-Définition 3.5.1] shows that our θ_{p^n} (when descended to $\mathrm{GU}(2, B_p)/\mathbb{Q}_p^\times \simeq \mathrm{SO}_5(\mathbb{Q}_p)$) is the same as his α , with kernel his J^+ (case $n = 1$), or the character considered for any $n \geq 1$ by Tsai [Tsa13, Definition 7.1.2]. All these authors work more generally with $\mathrm{SO}_{2m+1}(\mathbb{Q}_p)$ for any $m \geq 1$, where the kernel of θ_{p^n} on $\mathrm{SO}_{2m+1}(\mathbb{Q}_p)$ is Brumer and Kramer’s generalisation of $\Gamma_0(p^n)$ for $m = 1$ and paramodular subgroups for $m = 2$ [BK14].

Remark 7.9. A clear statement of Proposition 7.4(2) is given in [Tsa13, §6.2, p.81], and some combination of Propositions 7.4(2) and 7.7(3) is stated in [Gro16, §5 “When $n = 2$ ”]. As far as we know, the detailed proofs we present here are the first in the literature.

Remark 7.10. As explained in § 2, we may reconstruct $M_2(B_p)$ as the even part of the Clifford algebra of the quadratic space V_p . The multiplier character $\nu : \mathrm{GU}(2, B_p) \rightarrow \mathbb{Q}_p^\times$ (well-defined modulo squares on $\mathrm{GU}(2, B_p)/\mathbb{Q}_p^\times$) becomes the spinor norm $\nu : \mathrm{SO}_5(\mathbb{Q}_p) \rightarrow \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ or $\nu : \mathrm{SO}_5^*(\mathbb{Q}_p) \rightarrow \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$. Let $\chi : \mathbb{Q}_p^\times \rightarrow \{\pm 1\}$ be the unramified character. Then $\chi \circ \nu$ acts as $+1$ on $K_{p^n}^+$ ($p^n \parallel D^+$) or $K_{p^n}^-$ ($p \mid D^-$), on which ν takes unit values. Its acts as -1 on the Atkin-Lehner elements W_{p^n} (when n is odd) and ω_p , on which we have seen ν takes values p^n, p respectively. So when $p^n \parallel D^+$ with n odd, θ_p agrees with $\chi \circ \nu$. But beware that when $p^n \parallel D^+$ with n even, θ_p and $\chi \circ \nu$ are not the same, as the latter is the trivial character.

8. AN ISOMORPHISM OF ALGEBRAIC MODULAR FORMS FOR $\mathrm{GU}(2, B)$ AND $\mathrm{SO}(V)$.

8.1. Finite-dimensional complex representations of $\mathrm{GSp}_2(\mathbb{C})$ and $\mathrm{SO}_5(\mathbb{C})$.

Whereas $B \otimes \mathbb{R} \not\simeq M_2(\mathbb{R})$, $B \otimes \mathbb{C} \simeq M_2(\mathbb{C})$, so the situation is like for $B \otimes \mathbb{Q}_p$ with $p \nmid D^-$. Thus $\mathrm{GU}(2, B \otimes \mathbb{C}) \simeq \mathrm{GSp}_2(\mathbb{C})$ and

$$V \otimes \mathbb{C} \simeq \left\{ \left(\begin{array}{cc|cc} t & & d & -b \\ & & -c & a \\ \hline a & b & & \\ c & d & & -t \end{array} \right) : t, a, b, c, d \in \mathbb{C} \right\},$$

with $Q(A) = \frac{1}{D}(t^2 + ad - bc)$. Let W be the natural 4-dimensional complex representation of $\mathrm{GU}(2, B \otimes \mathbb{C}) \simeq \mathrm{GSp}_2(\mathbb{C})$. Given positive integers $a \geq b \geq 0$, let $n = a + b$. Consider the representation $W^{\otimes(n)}$ (the tensor product representation). The permutation group S_n acts on $W^{\otimes(n)}$ and its action commutes with that of $\mathrm{GSp}_2(\mathbb{C})$. The Young diagram

(25)

1	2	...	b	...	a
1	2	...	b		

gives rise to a idempotent on the complex group algebra of S_n (as explained for example in [FH91, Section 4.1]). More concretely, enumerate the squares of (25) (from 1 to a in the first row and from $a + 1$ to n in the second one). The diagram (25) corresponds to the partition $\lambda : n = a + b$. Let P be the subgroup of S_n preserving

the rows of (25) (via our enumeration) and Q that preserving the columns. Define two elements:

$$a_\lambda = \sum_{\sigma \in P} e_\sigma \quad \text{and} \quad b_\lambda = \sum_{\tau \in Q} \text{sgn}(\tau) e_\tau.$$

Then define $c_\lambda := a_\lambda \cdot b_\lambda$, and let \mathbb{S}_λ denote the subspace $c_\lambda W^{\otimes(n)}$. Such a representation is not in general irreducible. For each pair of indices $I = \{p, q\}$ with $1 \leq p < q \leq n$, let $\Phi_I : W^{\otimes(n)} \rightarrow W^{\otimes(n-2)}$ be the linear map

$$(26) \quad \Phi_I(v_1 \otimes \cdots \otimes v_n) = E(v_p, v_q) v_1 \otimes \cdots \otimes \widehat{v}_p \otimes \cdots \otimes \widehat{v}_q \otimes \cdots \otimes v_n,$$

where \widehat{v}_i means remove such term from the tensor product, and $E(v, w)$ is the symplectic form preserved by Sp_2 . Let $W^{(n)}$ denote the intersection of the kernels of all the Φ_I and $\mathbb{S}_{\langle \lambda \rangle} = W^{(n)} \cap \mathbb{S}_\lambda$.

Theorem 8.1. *The representation $\mathbb{S}_{\langle \lambda \rangle}$ of $\text{GSp}_2(\mathbb{C})$ is irreducible. The standard maximal torus acts on a highest weight vector via the character*

$$\text{diag}(t_1, t_0 t_1^{-1}, t_2, t_0 t_2^{-1}) \mapsto t_1^a t_2^b.$$

Furthermore, its dimension equals

$$\dim(\mathbb{S}_{\langle \lambda \rangle}) = \frac{(a+b+3)(a-b+1)(a+2)(b+1)}{6}.$$

Proof. See Theorem 17.11 and Exercise 24.17 of [FH91], which shows that in fact the restriction to $\text{Sp}_2(\mathbb{C})$ is irreducible. \square

Note that the central character of $\mathbb{S}_{\langle \lambda \rangle}$ is $z \mapsto z^{a+b} = z^n$, as it must be, inside $W^{\otimes(n)}$. The similitude character restricts to the torus as

$$\text{diag}(t_1, t_0 t_1^{-1}, t_2, t_0 t_2^{-1}) \mapsto t_0.$$

For $z = zI_4$ (with $z \in \mathbb{C}^\times$), $t_1 = t_2 = z$ and $t_0 = z^2$, so $\nu(zI_4) = z^2$.

From now on we restrict to the case $a \equiv b \pmod{2}$. Then the representation of $\text{GSp}_2(\mathbb{C})$ on

$$W_{a,b} := \mathbb{S}_{\langle \lambda \rangle} \otimes \nu^{-\frac{a+b}{2}}$$

has trivial central character, so descends to an irreducible representation of

$$\text{GSp}_2(\mathbb{C})/\mathbb{C}^\times \simeq \text{SO}_5(\mathbb{C}),$$

which may be extended to $\text{O}_5(\mathbb{C})$ by letting $-I_5$ act trivially. (We shall not consider the other extension, where $-I_5$ acts by -1 .)

The trivial representation is $W_{0,0}$, the original W is $W_{1,0}$, while $W_{1,1}$ is the irreducible 5-dimensional representation of $\text{O}_5(\mathbb{C})$. We constructed this as V (rather $V \otimes \mathbb{C}$) inside the matrix space $\text{Hom}(W, W) \simeq W^* \otimes W$, on which the natural action of $\text{GSp}_2(\mathbb{C})$ is via g^{-1} on the domain, g on the codomain, i.e. our conjugation. Via the symplectic form, $W^* \simeq W \otimes \nu^{-1}$, so we see $V \otimes \mathbb{C}$ inside $W^{\otimes(2)} \otimes \nu^{-1}$. In fact the 6-dimensional anti-symmetric part $(\bigwedge^2 W) \otimes \nu^{-1}$ is a direct sum of $\mathbb{C}I_4$ and $V \otimes \mathbb{C}$.

All representations of $\text{SO}_5(\mathbb{C})$ come from Young diagrams and tensor powers of the 5-dimensional representation $W_{1,1}$, with $W_{a,b}$ associated to the partition $\frac{a+b}{2} + \frac{a-b}{2}$

of a . To make sense of this, note that conjugation by $\text{diag}(t_1, t_0 t_1^{-1}, t_2, t_0 t_2^{-1})$ acts

$$\text{on } \begin{pmatrix} t & \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & -t \end{pmatrix} \text{ by}$$

$$(t, a, b, c, d) \mapsto (t, s_2^{-1}a, s_1 b, s_1^{-1}c, s_2 d),$$

where $(s_1, s_2) = (t_1 t_2 t_0^{-1}, t_1 t_2^{-1})$. The highest weight character on $W_{a,b}$ is now

$$\text{diag}(t_1, t_0 t_1^{-1}, t_2, t_0 t_2^{-1}) \mapsto t_1^a t_2^b t_0^{-(a+b)/2} = s_1^{(a+b)/2} s_2^{(a-b)/2}.$$

8.2. Spaces of algebraic modular forms. Following the conventions of [GV14, Introduction], let G/\mathbb{Q} be a reductive group with $G(\mathbb{R})$ compact. Let W' be a finite-dimensional complex representation of $G(\mathbb{Q})$, and let \widehat{K} be an open compact subgroup of $G(\mathbb{A}_f)$, where $\mathbb{A}_f \simeq \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ is the ring of finite adèles. We define

$$M(W', \widehat{K}) := \{f : G(\mathbb{A}_f) \rightarrow W' : f(\gamma g k) = \gamma \cdot f(g) \quad \forall \gamma \in G(\mathbb{Q}), g \in G(\mathbb{A}_f), k \in \widehat{K}\}.$$

If $\{g_i : 1 \leq i \leq h\}$ is a set of representatives of $G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / \widehat{K}$ then

$$(27) \quad M(W', \widehat{K}) \simeq \bigoplus_{i=1}^h (W')^{\Gamma_i},$$

where $\Gamma_i := g_i \widehat{K} g_i^{-1} \cap G(\mathbb{Q})$.

Two special cases will be of particular interest to us.

- (1) G such that $G(\mathbb{Q}) = \text{GU}(2, B)$, with B/\mathbb{Q} a definite quaternion algebra as above, $W' = W_{a,b}$ with $a \geq b \geq 0$, as in the previous section, and $\widehat{K} = \widehat{K}(D)$ defined by its local components

$$\widehat{K}(D)_p := \begin{cases} K_{0,p} & \text{if } p \nmid D; \\ K_{p^n}^+ & \text{if } p^n \parallel D^+; \\ K_p^- & \text{if } p \mid D^-. \end{cases}$$

Then

$$M_{a,b}(\widehat{K}(D)) := M(W_{a,b}, \widehat{K}(D)).$$

Although strictly speaking $G(\mathbb{R})$ is only compact modulo its centre, since $W_{a,b}$ has trivial central character (recall our assumption $a \equiv b \pmod{2}$) we could just as well be working with $\text{GU}(2, B)/\mathbb{Q}^\times$, which does satisfy the condition.

- (2) $G = \text{SO}(\tilde{V})$, where \tilde{V} is as in Remark 6.10. Since the lattice $\tilde{L} \subset \tilde{V}$ from Remark 6.11 is in the same genus as $L \subset V$, there is an isomorphism Φ of \mathbb{Q} -quadratic spaces from (\tilde{V}, \tilde{Q}) to (V, Q) , such that for each prime p there is $h_p \in \text{SO}(V_p)$ with

$$\Phi(\tilde{L}_p) = h_p L_p.$$

(Though at first $h_p \in \text{O}(V_p)$, we may ensure it is in $\text{SO}(V_p)$ by multiplying by $-I_5$ if necessary.) Via Φ , $\text{SO}(\tilde{V}) \simeq \text{SO}(V) \simeq \text{GU}(2, B)/\mathbb{Q}^\times$ (with slight abuse of notation), so the representations $W_{a,b}$ may be viewed also as representations of $\text{SO}(\tilde{V})$.

It follows that there is essentially no difference between $(\tilde{V}, \tilde{L}, \tilde{Q})$ and (V, L, Q) , as far as algebraic modular forms are concerned. (In (27), replacing

\widehat{K} by $h\widehat{K}h^{-1}$ and g_i by g_ih^{-1} , where $h := (h_p)$, leaves Γ_i the same.) So from this point onwards we neglect the distinction.

We define an open compact subgroup $\widehat{K(L)}$ of $G(\mathbb{A}_f)$ by

$$\widehat{K(L)}_p := \text{Stab}_{G(\mathbb{Q}_p)}(L_p) \quad \forall \text{ primes } p,$$

and let

$$\widetilde{M}_{a,b}(\widehat{K(L)}) := M(W_{a,b}, \widehat{K(L)}).$$

We also define a slightly smaller subgroup $\widehat{K(L)}^+$ by

$$\widehat{K(L)}_p^+ := \begin{cases} \text{Stab}_{G(\mathbb{Q}_p)}(L_p) & \text{for } p \nmid D; \\ \text{Stab}_{G(\mathbb{Q}_p)}(L_p)^+ & \text{for } p \mid D, \end{cases}$$

where $\text{Stab}_{G(\mathbb{Q}_p)}(L_p)^+$ is the subgroup of index 2 in $\text{Stab}_{G(\mathbb{Q}_p)}(L_p)$ that is the kernel of θ_p , with θ_p as in Definition 7.6. Then we define

$$\widetilde{M}_{a,b}(\widehat{K(L)}^+) := M(W_{a,b}, \widehat{K(L)}^+).$$

Theorem 8.2. *There is an isomorphism*

$$\widetilde{M}_{a,b}(\widehat{K(L)}^+) \simeq M_{a,b}(\widehat{K(D)}),$$

equivariant for the right-translation action of $SO(V \otimes \mathbb{A}_f)$ on the left-hand-side and the isomorphic $\text{GU}(2, B \otimes \mathbb{A}_f)/\mathbb{A}_f^\times$ on the right-hand-side. For $p \mid D$, the action of the involution $\text{Stab}_{G(\mathbb{Q}_p)}(L_p)/\text{Stab}_{G(\mathbb{Q}_p)}(L_p)^+$ on the left matches that on the right of $W_{p^n}^+$ (for $p^n \parallel D^+$) or ω_p (for $p \mid D^-$).

Proof. This is a direct consequence of Propositions 7.4(2), 7.5(2) and 7.7. \square

Remark 8.3. Ladd [Lad18, Theorem 1] addressed the case $D = p$ by a different approach, proving that the left-hand-side injects into the right-hand-side.

Remark 8.4. We extended the representations $W_{a,b}$ of $SO(V)$ to representations of $O(V)$ by making the element $-I_5 \in O(V) - SO(V)$ act trivially. Extending $\tilde{f} \in \widetilde{M}_{a,b}(\widehat{K(L)})$ to $O(V \otimes \mathbb{A}_f)$ by $\tilde{f}(-I_5g) := \tilde{f}(g)$, and noting that $-I_5 \in \text{Stab}_{O(V_p)}(L_p)$, we see that $\widetilde{M}_{a,b}(\widehat{K(L)})$ remains the same if everywhere in the definition we substitute $O(V)$ (and $O(V \otimes \mathbb{A}_f)$ etc.) for $SO(V)$ (and $SO(V \otimes \mathbb{A}_f)$ etc.).

Proposition 8.5. *For $p \nmid D$, the Hecke operators $T(p)$ and $T_1(p^2)$ on $M_{a,b}(\widehat{K(D)})$ correspond to the p -neighbour and p^2 -neighbour Hecke operators on $\widetilde{M}_{a,b}(\widehat{K(L)}^+)$, in the sense of [GV14, Theorem 5.11].*

Proof. (1) The Hecke operator usually denoted $T(p)$ or $T_1(p)$ is associated to the double coset $K_{0,p} \text{diag}(p, 1, p, 1) K_{0,p}$. Note that $\Psi : \text{GU}(2, B_p) \simeq \text{GSp}_2(\mathbb{Q}_p)$ maps $\text{diag}(p, 1, p, 1)$ to the usual $\text{diag}(p, p, 1, 1)$. Acting by conjugation on U_p ,

$$\text{diag}(p, 1, p, 1) : \begin{pmatrix} t & \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & s \end{pmatrix} \mapsto \begin{pmatrix} t & \begin{pmatrix} d & -pb \\ -c/p & a \end{pmatrix} \\ \begin{pmatrix} a & pb \\ c/p & d \end{pmatrix} & s \end{pmatrix}.$$

In the language of §8.1, $(t_0, t_1, t_2) = (p, p, p)$, $(s_1, s_2) = (t_1 t_2 t_0^{-1}, t_1 t_2^{-1}) = (p, 1)$, and

$$\text{diag}(p, 1, p, 1) : (t, s, a, b, c, d) \mapsto (t, s, a, pb, p^{-1}c, d).$$

Thus (passing to the 5-dimensional quotient) $T(p)$ on $M_{a,b}(\widehat{K}(D))$ corresponds to

$$\text{Stab}_{\text{O}(V_p)}(L_p) \text{diag}(1, 1, p, p^{-1}, 1) \text{Stab}_{\text{O}(V_p)}(L_p)$$

on $\widetilde{M}_{a,b}(\widehat{K(L)}^+)$, which is the p -neighbour operator as in [GV14, Theorem 5.11].

- (2) Similarly if $(t_0, t_1, t_2) = (p^2, 1, p)$ then $(s_1, s_2) = (p^{-1}, p^{-1})$, and we see that the operator $T_1(p^2)$ on $M_{a,b}(\widehat{K}(D))$ corresponds to the p^2 -neighbour Hecke operator on $\widetilde{M}_{a,b}(\widehat{K(L)}^+)$, in the sense of [GV14, Theorem 5.11] (which might also reasonably be called “ (p, p) -neighbour operator”). \square

Given a positive divisor $d \mid D$ with $\gcd(d, D/d) = 1$, we may define a character $\theta_d : \widehat{K(L)} \rightarrow \{\pm 1\}$ by

$$\theta_d|_{\widehat{K(L)}_p} = \begin{cases} \text{id.} & \text{if } p \nmid d; \\ \theta_p & \text{if } p \mid d. \end{cases}$$

We define a subspace $\widetilde{M}_{a,b}(\widehat{K(L)}^+)^{\theta_d}$ of $\widetilde{M}_{a,b}(\widehat{K(L)}^+)$ by

$$\begin{aligned} \widetilde{M}_{a,b}(\widehat{K(L)}^+)^{\theta_d} := \{ \tilde{f} \in \widetilde{M}_{a,b}(\widehat{K(L)}^+) : f(gk) = \theta_d(k)f(g) \\ \forall g \in \text{SO}(V \otimes \mathbb{A}_f), k \in \widehat{K(L)} \}. \end{aligned}$$

Under the isomorphism of Theorem 8.2, clearly this corresponds to the subspace of $M_{a,b}(\widehat{K}(D))$ on which $W_{p^n}^+$ or ω_p acts as -1 precisely for $p \mid d$.

Corollary 8.6. *Keeping the previous notation, there is a natural isomorphism*

$$M_{a,b}(\widehat{K}(D)) \simeq \bigoplus_{\substack{d \mid D \\ \gcd(d, D/d)=1}} \widetilde{M}_{a,b}(\widehat{K(L)}^+)^{\theta_d}.$$

If there is no p such that $\text{ord}_p(d)$ is even, in particular if d is square-free, then θ_d may be extended to $\text{SO}(V \otimes \mathbb{A}_f)$, since locally at $p \mid d$ it is $\chi \circ \nu$, as in Remark 7.10. Thus we may define a representation $W_{a,b} \otimes \theta_d$ of $\text{SO}(V \otimes \mathbb{A})$, with $\text{SO}(V \otimes \mathbb{R}) \simeq \text{SO}_5(\mathbb{R})$ acting on the $W_{a,b}$ factor. This restricts to give a representation of the diagonally embedded $\text{SO}(V)$.

Proposition 8.7. *Suppose that d is a positive divisor of D with $\gcd(d, D/d) = 1$, and that there is no p such that $\text{ord}_p(d)$ is even. Then*

$$\widetilde{M}_{a,b}(\widehat{K(L)}^+)^{\theta_d} \simeq M(W_{a,b} \otimes \theta_d, \widehat{K(L)}).$$

Proof. An element of $M(W_{a,b} \otimes \theta_d, \widehat{K(L)})$ is a function $f : \text{SO}(V \otimes \mathbb{A}_f) \rightarrow W_{a,b}$ such that for $\gamma \in \text{SO}(V)$, $g \in \text{SO}(V \otimes \mathbb{A}_f)$ and $k \in \widehat{K(L)}$,

$$f(\gamma g k) = \theta_d(\gamma) \gamma \cdot f(g).$$

An element of $\widetilde{M}_{a,b}(\widehat{K(L)}^+)^{\theta_d}$ is a function $\tilde{f} : \mathrm{SO}(V \otimes \mathbb{A}_f) \rightarrow W_{a,b}$ such that for $\gamma \in \mathrm{SO}(V)$, $g \in \mathrm{SO}(V \otimes \mathbb{A}_f)$ and $k \in \widehat{K(L)}$,

$$\tilde{f}(\gamma g k) = \theta_d(k) \gamma \cdot f(g).$$

$\tilde{f}(g) := \theta_d(g) f(g)$ gives the isomorphism we seek. \square

9. FROM ALGEBRAIC MODULAR FORMS FOR $\mathrm{GU}(2, B)$ TO SIEGEL MODULAR FORMS OF PARAMODULAR LEVEL

For any positive integer N , let

$$P(N) := \left[\begin{array}{cccc} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \frac{1}{N}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{array} \right] \cap \mathrm{Sp}_2(\mathbb{Q})$$

be the paramodular group of level N . For integers $k \geq 1$, $j \geq 0$, let $\rho_{k,j} : \mathrm{GL}_2(\mathbb{C}) \rightarrow \mathrm{Aut}(V_{k,j})$ be the $\det^k \otimes \mathrm{Sym}^j$ representation. Let $S_{k,j}(P(N))$ be the space of holomorphic functions $F : \mathfrak{H}_2 \rightarrow V_{k,j}$, satisfying a cuspidality condition, and

$$F((AZ + B)(CZ + D)^{-1}) = \rho_{k,j}(CZ + D)(F(Z)) \quad \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in P(N).$$

Note that such space is trivially zero if j is odd, since $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in P(N)$ for all N . For this reason, we restrict to the case j even.

Although it is not true that $P(p^n)$ is contained in $P(p^m)$ for $n > m$, there is still a theory of oldforms and newforms for automorphic forms on GSp_2 , as studied in [RS07] (in particular, the formula for the number of oldforms given in Theorem 7.5.6).

In a series of articles, Ibukiyama and some coauthors (see [IK17] and also [Ibu19, Ibu18]) stated a series of conjectures relating automorphic forms on $\mathrm{GU}(2, B)$ and GSp_2 . The conjectures were proven by Rösner and Weissauer in a recent article [RW21], using the trace formula. A somewhat less general result was obtained independently by van Hoften [vH21] using very different, algebro-geometric tools. Although the original conjecture (and the proof) was made for square-free levels $D = D^-$, a more general version holds. In [IK17] the authors consider a quaternion algebra ramified at all primes dividing D and at infinity, (taking as open compact subgroup the one corresponding to $D^+ = 1$), and relate the spaces $M_{a,b}(\widehat{K}(D))$ and $S_{b+3, a-b}(P(D))$.

9.1. Generalisation of Ibukiyama-Kitayama conjecture. For $M \mid N$ with $\mathrm{gcd}(M, N/M) = 1$, let $S_k^{M\text{-new}}(N)$ denote the space of cusp forms for $\Gamma_0(N)$ (genus-1) that are new at M . We keep the notation of previous sections, i.e. $D = D^+ D^-$ where D^- is square-free. The contribution of the Yoshida and Saito-Kurokawa lifts to the algebraic modular forms for $\mathrm{GU}(2, B)$ is given in Propositions 9.1 and 9.4 below. To cover the general case, the statements are necessarily complicated. To warm up, the reader may wish to read [RW21, Propositions 12.1, 12.2], for the simpler statements in the case $D^+ = 1$.

Proposition 9.1 (Yoshida lifts). *Fix a pair of positive integers d_+, c_+ with $d_+ c_+ \mid D^+$. There is an injective embedding, sending eigenforms to eigenforms (in both*

cases, for Hecke operators at primes not dividing the level, Atkin-Lehner involutions at primes dividing the level, where the level depends on the form),

$$\iota : \bigoplus_{\substack{d_- | D^- \\ \omega(d_-) \text{ odd}}} S_{2+a-b}^{new}(d_- d_+) \times S_{4+a+b}^{new} \left(\frac{D^-}{d_-} c_+ \right) \hookrightarrow M_{a,b}(\widehat{K}(D^- d_+ c_+))$$

The oldforms in $M_{a,b}(\widehat{K}(D))$ generated from the $\iota(g, h)$ by applying Roberts and Schmidt's level-raising operators θ, θ' and η , at primes dividing $\frac{D^+}{d_+ c_+}$, (or the newforms $\iota(g, h)$ if $\frac{D^+}{d_+ c_+} = 1$), span (as we vary d_+ and c_+) the subspace $M_{a,b}(\widehat{K}(D))_Y$ of all forms of Yoshida type, i.e. the span of $\widehat{K}(D)$ -fixed vectors in endoscopic automorphic representations.

Furthermore, for eigenforms g and h the spinor L -function $L^{D^- d_+ c_+}(s, \iota(g, h), \text{spin})$ of $\iota(g, h)$ is given by $L^{D^- d_+ c_+}(s - b - 1, g) L^{D^- d_+ c_+}(s, h)$, where the superscript indicates that Euler factors at primes dividing $D^- d_+ c_+$ are omitted.

Proof. The proof of [RW21, Proposition 12.1] (based on results of Chan and Gan ([CG15])) applies to our more general context, but there $D^+ = 1$ so there is no d_+ , c_+ or oldforms. Let $\pi^{\iota(g, h)}$ be the associated automorphic representation of $\text{GU}(2, B \otimes \mathbb{A})$.

In the notation of [CG15, 2.2, 2.4], $\pi_p^{\iota(g, h)}$ is π_ϕ^{-+} for $p \mid d_-$, π_ϕ^{+-} for $p \mid \frac{D^-}{d_-}$, and $\pi_\infty^{\iota(g, h)}$ is π^{-+} . Since $\omega(d_-)$ is odd and $\omega\left(\frac{D^-}{d_-}\right)$ is even, the product of the $\pm\pm$ pairs over all places is trivial, hence [CG15, Theorem 3.1] (Arthur's multiplicity formula) gives the existence of $\pi^{\iota(g, h)}$. Note that for primes $p \nmid D^-$, $\pi_p^{\iota(g, h)}$ is π_ϕ^+ in the notation of [CG15, 2.1], or even π_ϕ^{++} in the notation of [CG15, 2.3], and does not contribute to the product. (When there are two elements in the L -packet, we must choose the generic one, to have paramodular fixed vectors, cf. [SS13, Remark 3.5].)

We need to know that the paramodular level of $\pi_p^{\iota(g, h)}$ (i.e. the least n such that it has non-zero K_p^+ -invariants, necessarily 1-dimensional) is $p^{\text{ord}_p(d_+ c_+)}$, for primes $p \nmid D^-$, so that $\iota(g, h) \in M_{a,b}(\widehat{K}(D^- d_+ c_+))$. This follows from the fact that the L -parameter of $\pi_p^{\iota(g, h)}$ is a kind of direct sum of those of π_p^g and π_p^h (representations of $\text{GL}_2(\mathbb{Q}_p)$), from the behaviour of ϵ -factors under the local Langlands correspondence, and the relation between paramodular level and ϵ -factors, for generic representations, which is [RS07, Corollary 7.5.5]. \square

Example 9.2. Let $D^- = 5$, $D^+ = 77$ and $(a, b) = (0, 0)$. Then the only non-zero contributions come from taking $d_- = 5$, $d_+ \in \{1, 7, 11, 77\}$ and $c_+ = D^+/d_+$. The algorithm described in [RT20] computes the space $M_{0,0}(385)$ corresponding to the genus of quinary forms with Hasse-Witt invariant -1 only at the prime 5. There are four rational Yoshida lifts, corresponding to the pairs of modular forms: (35.2.a.b, 11.4.a.a), (35.2.a.a, 11.4.a.a), (55.2.a.a, 7.4.a.a) and (55.2.a.b, 7.4.a.a).

There are precisely two newforms in $S_2(\Gamma_0(35))$, labelled 35.2.a.a and 35.2.a.b and two newforms in $S_2(\Gamma_0(55))$, labelled 55.2.a.a and 55.2.a.b. There is a unique form in $S_4(\Gamma_0(7))$ labelled 7.4.a.a and one form in $S_4(\Gamma_0(11))$ labelled 11.4.a.a. All other combinations of spaces involve one that is trivial.

Lemma 9.3. *Consider an eigenform $f \in M_{b,b}(\widehat{K}(D))$ whose associated automorphic representation π^f of $\mathrm{GU}(2, B \otimes \mathbb{A})$ is CAP associated to the Siegel parabolic. In the notation of [Gan08], π^f belongs to a global A-packet $A_{\tau, \chi}$, where τ is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character, and χ is a quadratic character of $\mathbb{A}^\times / \mathbb{Q}^\times$. Then χ is trivial.*

Proof. This is modelled on the proof (in the case $\mathrm{GSp}_2(\mathbb{A})$) of [Sch20, Proposition 5.2(i)], where τ is μ and χ is σ . (In [RS07, §5.5], τ is π and χ is σ .)

At any split prime p , π_p^f is the unique irreducible quotient of a representation denoted $\nu^{1/2}\tau_p \rtimes \nu^{-1/2}\chi_p$, induced from a Siegel parabolic. At a non-split prime $p \mid D^-$, π_p^f is the unique irreducible quotient of a representation denoted $\nu^{1/2}\mathrm{JL}(\tau_p) \rtimes \nu^{-1/2}\chi_p$, induced from a parabolic with Levi subgroup $B_p^\times \times \mathbb{Q}_p^\times$. Considering the action of an element (id, u) of the Levi subgroup, where $u \in \mathbb{Z}_p^\times$, it is easy to show that π_p^f cannot have a non-zero $\widehat{K}(D)_p$ -fixed vector unless χ_p is unramified, and the only χ such that all χ_p are unramified is the trivial character. \square

Let $\mathfrak{S}_{2b+4}^{\frac{D^-}{d_-}\text{-new}, -(-1)^{\omega(d_-)}}$ ($\Gamma_0(D/d_-)$) be the subspace of $S_{2b+4}^{\frac{D^-}{d_-}\text{-new}, -(-1)^{\omega(d_-)}}$ ($\Gamma_0(D/d_-)$) spanned by eigenforms with the same Atkin-Lehner eigenvalues, at all primes dividing D^+ , as the newforms they come from. For more, see around [RS06, (23)].

Proposition 9.4 (Saito-Kurokawa lifts). *For each positive divisor d_- of D^- there is an injective embedding*

$$\iota_{d_-} : \mathfrak{S}_{2b+4}^{\frac{D^-}{d_-}\text{-new}, -(-1)^{\omega(d_-)}} (\Gamma_0(D/d_-)) \hookrightarrow M_{b,b}(\widehat{K}(D))$$

Furthermore, the span of the images of the ι_{d_-} is precisely the subspace $M_{b,b}(\widehat{K}(D))_{\mathrm{SK}}$ of forms of Saito-Kurokawa type, i.e. spanned by $\widehat{K}(D)$ -fixed vectors in automorphic representations that are CAP for the Siegel parabolic subgroup.

If $h \in \mathfrak{S}_{2b+4}^{\frac{D^-}{d_-}\text{-new}, -(-1)^{\omega(d_-)}} (\Gamma_0(D/d_-))$ is an eigenform (of Hecke operators for $p \nmid \frac{D}{d_-}$, Atkin-Lehner involutions for $p \mid \frac{D}{d_-}$), then $\iota_{d_-}(h)$ is an eigenform (for Hecke operators at $p \nmid D$, Atkin-Lehner involutions at $p \mid D$). The spinor L -function $L^D(s, \iota_{d_-}(h), \mathrm{spin})$ of $\iota_{d_-}(h)$ is given by $\zeta^D(s-b-2)\zeta^D(s-b-1)L^D(s, h)$, where the superscript D indicates that Euler factors at primes $p \mid D$ are omitted.

Proof. This may be proved in very much the same way as [RW21, Proposition 12.2]. (The part about the span of the images follows from a theorem of E. Sayag [Gan08, Theorem 6.9].) The difference is that on both left and right, forms need not be D^+ -new, whereas in [RW21, Proposition 12.2], $D^+ = 1$.

Any eigenform on the left comes from some newform $g \in S_{2b+4}^{\mathrm{new}, -(-1)^{\omega(d_-)}} \left(\Gamma_0 \left(\frac{D^-}{d_-} \frac{D^+}{d_+} \right) \right)$, for some positive divisor d_+ of D^+ . The sign $-(-1)^{\omega(d_-)}$ is by definition required to be the global sign in the functional equation of the L -function associated to g . Using Gan's Saito-Kurokawa lifting [Gan08] as in [RW21, Proposition 12.2], we obtain an eigenform $G \in M_{b,b}(\widehat{K}(D/d_+))$. In fact D^+/d_+ is precisely the paramodular level of the cuspidal automorphic representation π^G of $\mathrm{GU}(2, B \otimes \mathbb{A})$ attached to G . This means that for all split primes, $p \nmid D^-$, the least n such that π_p^G has non-zero $K_{p^n}^+$ -invariants (necessarily 1-dimensional) is $\mathrm{ord}_p(D^+/d_+)$. This is precisely [RS07, Proposition 5.5.5(i)], bearing in mind that at split places Gan's Saito-Kurokawa lifting is locally the local Saito-Kurokawa lifting of [RS07, §5.5].

On the left we can obtain, from g , oldforms in $S_{2b+4}^{D_-^- \text{-new}, -(-1)^{\omega(d_-)}}(\Gamma_0(D/d_-))$. Recall that we are taking only those with the same Atkin-Lehner eigenvalues as g for primes dividing D^+ . On the right we can obtain from G oldforms in $M_{b,b}(\widehat{K}(D))$, by applying Roberts and Schmidt's operators θ and η at primes dividing d_+ . Including their third level-raising operator θ' would not produce anything linearly independent of what one obtains using θ and η , since by [RS07, Theorem 5.5.9], θ and θ' act the same (up to \pm) on paramodular-fixed vectors in local representations of Saito-Kurokawa type.

In fact, by Lemma 9.3 and [RS07, Theorem 5.5.9(iv)], the sign is $+$, i.e. θ and θ' act the same, and the oldforms coming from G as above have the same Atkin-Lehner eigenvalues as G . As explained in the discussion preceding [RS06, Theorem 6.3], the oldforms coming from g can be exactly paired with those coming from G . Although the argument there was applied to Siegel modular forms, it applies just the same, being local at split primes. \square

Example 9.5. Let us continue the previous example. Let $D^- = 5$ and $D^+ = 77$. Taking $d_- = 1$ implies computing the space $S_4^{5\text{-new}, -}(385)$. The newforms of the different spaces are $S_4^{\text{new}, -}(5) = \{5.4.a.a\}$, $S_4^{\text{new}, -}(35) = \{35.4.a.a\}$, $S_4^{\text{new}, -}(55) = \emptyset$ and $S_4^{\text{new}, -}(385) = \{385.4.a.b, 385.4.a.e, 385.4.a.h, 385.4.a.i, 385.4.a.j, 385.4.a.l\}$.

Taking $d_- = 5$ implies computing spaces of forms of level not divisible by 5 and sign $+1$ in their functional equation. The newforms of the different spaces are $S_4^{\text{new}, +}(1) = \emptyset$, $S_4^{\text{new}, +}(7) = \{7.4.a.a\}$, $S_4^{\text{new}, +}(11) = \{11.4.a.a\}$ and $S_4^{\text{new}, +}(77) = \{77.4.a.a, 77.4.a.d, 77.4.a.e\}$. All such forms contribute to the space $M_{0,0}(\widehat{K}(385))$.

Recall that S denotes the finite set of finite places at which the definite quaternion algebra is non-split. For a finite $p \notin S$ we have $\text{GU}(2, B_p) \simeq \text{GSp}_2(\mathbb{Q}_p)$. Let $f \in M_{(a,b)}(\widehat{K}(D))$ be a Hecke eigenform, generating an automorphic representation π_f of $\text{GU}(2, B \otimes \mathbb{A})$. An automorphic representation π' of $\text{GSp}_2(\mathbb{A})$ is said to be *weakly equivalent* to π_f if the representations π'_p and $\pi_{f,p}$ of $\text{GSp}_2(\mathbb{Q}_p)$ are isomorphic for all finite p outside some finite set $T \supseteq S$. If π_f is weakly equivalent to some π' that is an irreducible constituent of something parabolically induced from a proper Levi subgroup, but not to a *cuspidal* such π' , then we shall say that f is of Eisenstein type. We may call the subspace spanned by such f the Eisenstein subspace $M_{(a,b)}(\widehat{K}(D))_{\text{Eis}}$.

Let $M_{(a,b)}(\widehat{K}(D))_G$ denote the subspace of $M_{(a,b)}(\widehat{K}(D))$ orthogonal to all Yoshida lifts, to all Saito-Kurokawa lifts, and to all forms of Eisenstein type. Here we are using a natural inner product, with respect to which Hecke operators are self-adjoint. Let $S_{k,j}^{D^- \text{-new}}(P(D))_G$ denote the subspace of forms (among those D^- -new in the sense of Roberts and Schmidt) that are orthogonal to all Saito-Kurokawa lifts. When $j > 0$ there are no Saito-Kurokawa lifts, and this is the whole space. Then we have the following natural generalisation of the conjecture of Ibukiyama-Kitayama proved in [RW21, Proposition 12.3]. The subscript “ G ” stands for “general type”, and will be justified in the course of the proof.

Theorem 9.6. *There is a linear isomorphism, sending eigenforms to eigenforms (for Hecke operators at $p \nmid D$, Atkin-Lehner involutions at $p \mid D$), and preserving L -functions (with Euler factors at primes dividing D omitted):*

$$S_{b+3, a-b}^{D^- \text{-new}}(P(D))_G \simeq M_{a,b}(\widehat{K}(D))_G.$$

Proof. This may be proved following [RW21, Proposition 12.3], though since for us $\omega(D^-)$ is odd, the large part of their proof dealing with their Conjecture 7.5 is not needed here.

There are no forms of Yoshida type on the left, as explained at the beginning of the proof of [RW21, Theorem 12.3], or on the right, by definition. By [Sch20, Proposition 5.1], there cannot be forms that are CAP with respect to the Borel or Klingen parabolic subgroups. This leaves only forms of general type, neither CAP nor endoscopic (nor Eisenstein).

For forms of general type we apply [RW21, Theorem 11.4] as in [RW21, Proposition 12.3]. Again, the difference is that on both sides there may be forms that are not D^+ -new. Consider an eigenform f in $M_{a,b}(\widehat{K}(D/d_+))_G$ (where d_+ is some positive divisor of D^+), with paramodular level exactly D^+/d_+ , and associated automorphic representation π^f of $\mathrm{GU}(2, B \otimes \mathbb{A})$. Applying [RW21, Theorem 11.4] gives us an eigenform $F \in S_{b+3, a-b}^{D^--\mathrm{new}}(P(D/d_+))_G$, with exact paramodular level D/d_+ , and associated cuspidal automorphic representation π^F of $\mathrm{GSp}_2(\mathbb{A})$. The arguments of [RW21, Proposition 12.3] prove this paramodular level for primes dividing D^- , but it also holds at other primes p , simply because $\pi_p^f \simeq \pi_p^F$ as representations of $\mathrm{GSp}_2(\mathbb{Q}_p)$, even for $p \mid D^+$. Note that although [RW21, Theorem 11.4(2)] only says that π_p^f and π_p^F are in the same L -packet, they have to be generic in order to have paramodular-fixed vectors, cf. [SS13, Remark 3.5], and then [RW21, Proposition 10.1] tells us that the L -packet contains a unique generic representation.

Then the oldforms in $M_{a,b}(\widehat{K}(D))_G$ and $S_{b+3, a-b}^{D^--\mathrm{new}}(P(D))_G$ generated from f and F respectively, by applying Roberts and Schmidt's level-raising operators θ, θ' and η at primes $p \mid d_+$, exactly correspond, again because $\pi_p^f \simeq \pi_p^F$. \square

Remark 9.7. Under a different convention, we could leave in the D^- -new Saito-Kurokawa lifts on both sides (thus having simply $S_{b+3, a-b}^{D^--\mathrm{new}}(P(D))$ on the left), and they would correspond to each other.

In order to understand the right-hand-side of the isomorphism better, we need to have a closer look at the subspace $M_{(a,b)}(\widehat{K}(D))_{\mathrm{Eis}}$ of $M_{(a,b)}(\widehat{K}(D))$, one of those "removed" in passing to $M_{(a,b)}(\widehat{K}(D))_G$.

- Proposition 9.8.** (1) *If $a = b = 0$ (so $k = 3, j = 0$) then the constant functions are of Eisenstein type.*
- (2) *If $D^- = p$, a prime number, then $M_{a,b}(\widehat{K}(D))_{\mathrm{Eis}}$ is trivial, for any $a > b > 0$ (i.e. $j > 0, k > 3$). If $D = D^- = p$ then $M_{a,b}(\widehat{K}(D))_{\mathrm{Eis}}$ is trivial, for any $a \geq b \geq 0$ except $(a, b) = (0, 0)$, (i.e. $j \geq 0, k \geq 3$, except $(k, j) = (3, 0)$).*
- (3) *If $D = D^-$ then the constant functions span $M_{(0,0)}(\widehat{K}(D))_{\mathrm{Eis}}$.*
- (4) *If $D = D^-$ and either $a = b > 0$ (i.e. $j = 0, k > 3$) or $a > b \geq 2$ (i.e. $j > 0, k \geq 5$) then $M_{(a,b)}(\widehat{K}(D))_{\mathrm{Eis}}$ is trivial.*

Proof. (1) Here π_f is trivial, and so is π' , which then occurs in the representation parabolically induced from the trivial character of a maximal torus.

(2) This is part of [vH21, Theorem 8.2.1 (2), (3)].

(3), (4) As observed by Weissauer, these follow from [IK17, Main Theorem 1.2], which shows that in Theorem 9.7 (i.e. [RW21, Proposition 12.3] when

$D = D^-$), the dimensions of the two sides match without having to have removed anything Eisenstein from $M_{(a,b)}(\widehat{K}(D))$. □

Conjecturally [IK17, Main Theorem 1.2], hence also $M_{a,b}(\widehat{K}(D))_{\text{Eis}} = \{0\}$ should hold for any $(a,b) \neq (0,0)$ (i.e. $k \geq 3$ and $(k,j) \neq (3,0)$). We expect also that both can be generalised to the case $D^+ > 1$ (i.e. $D \neq D^-$), without having to assume $D^- = p$.

Theorem 9.9. *Keeping the previous notation, let $D = D^+D^-$. Then*

$$S_{b+3,a-b}^{D^--new}(P(D))_G \simeq \bigoplus_{\substack{d|D \\ \gcd(d,D/d)=1}} \widetilde{M}_{a,b}(\widehat{K(L)}^+)^{\theta_d}.$$

Proof. Follows from Corollary 8.6 and Theorem 9.7. □

In particular, this proves Conjecture 15 in [RT20].

Remark 9.10. Theorem 9.10 provides the partial solution to the basis problem announced in the introduction when the level of the paramodular form is divisible by a prime to the first power, in the sense that there is an efficient method to compute bases for the spaces of orthogonal modular forms that appear on the right hand side of the isomorphism, as well as the action of Hecke operators on them. If N is a positive integer such that there exists p with $v_p(N) = 1$, then the space of paramodular newforms of level N can be computed in this sense using the decomposition $N = p(N/p)$, i.e. looking at a special positive definite quinary form of discriminant $2N$ with Hasse-Witt invariant -1 at p , and Eichler invariant $+1$ at all primes dividing N/p as described in Section 6.1.

10. ATKIN-LEHNER EIGENVALUES

Recall from Corollary 8.6 the decomposition

$$M_{a,b}(\widehat{K}(D)) \simeq \bigoplus_{\substack{d|D \\ \gcd(d,D/d)=1}} \widetilde{M}_{a,b}(\widehat{K(L)}^+)^{\theta_d}.$$

We know that the left hand side can contain eigenforms of Saito-Kurokawa type (if $a = b$), Yoshida type and general type. We now wish to identify, for each of these types, what we have in a given $\widetilde{M}_{a,b}(\widehat{K(L)}^+)^{\theta_d}$, in terms of the Atkin-Lehner eigenvalues of associated modular forms. In this section, as already in the previous section, we freely apply the isomorphism $\Psi : \text{GU}(2, B_p) \simeq \text{GSp}_2(\mathbb{Q}_p)$ at split p , and do not distinguish between $W_{p^n}^+$ and W_{p^n} .

Theorem 10.1. *Let $M_{a,b}(\widehat{K}(D))$ be a space of $\text{GU}(2, B)$ algebraic modular forms as in §8.2. In particular $D = D^-D^+$, with $\gcd(D^-, D^+) = 1$ and D^- square-free, $\omega(D^-)$ odd. Consider $f \in M_{a,b}(\widehat{K}(D))$, a Hecke eigenform for $T(p)$ and $T_1(p^2)$ (all $p \nmid D$), W_{p^n} ($p^n \parallel D^+$) and ω_p ($p \mid D^-$). For $p \mid D$ let e_p denote the eigenvalue of W_{p^n} or ω_p on f .*

- (1) If $f \in M_{a,b}(\widehat{K}(D))_G$, with corresponding $F \in S_{b+3,a-b}^{D^- \text{-new}}(P(D))_G$, with eigenvalues w_p of W_p^n on F (where n depends on p , with $p^n \parallel D$), then

$$e_p = \begin{cases} w_p & \text{if } p \mid D^+; \\ -w_p & \text{if } p \mid D^-. \end{cases}$$

- (2) If $f \in M_{a,b}(\widehat{K}(D))_{\text{SK}}$, say $f = \iota_{d^-}(h)$ for some $h \in \mathfrak{S}_{2b+4}^{\frac{D^-}{d^-} \text{-new}, -(-1)^{\omega(d^-)}}(\Gamma_0(D/d_-))$. Let ϵ_p be the local sign at p attached to h , in particular ϵ_p is the eigenvalue of a GL_2 Atkin-Lehner operator when $p \mid \frac{D}{d^-}$, and is 1 for other p . Then

$$\epsilon_p = \begin{cases} \epsilon_p & \text{if } p \mid d_- D^+; \\ -\epsilon_p & \text{if } p \mid \frac{D}{d^-}. \end{cases}$$

- (3) If $f' = \iota(g, h) \in M_{a,b}(\widehat{K}(D^- d_+ c_+))_Y$, as in Proposition 9.1, then the eigenvalue of W_p^n on f' , for $p \mid D^+$ is a product of local signs $\epsilon_p(g)\epsilon_p(h)$, while that of ω_p for $p \mid D^-$ is $-\epsilon_p(g)\epsilon_p(h)$.

Oldforms in $M_{a,b}(\widehat{K}(D))_Y$ may be produced from f' by (possibly repeated) application of Roberts and Schmidt's level-raising operators θ, θ' (at $p \mid \frac{D^+}{d_+ c_+}$) and η (at p when $p^2 \mid \frac{D^+}{d_+ c_+}$). Application of $\eta, \theta + \theta'$ or $\theta - \theta'$ takes (Atkin-Lehner) eigenforms to eigenforms, with a change of sign in the case of $\theta - \theta'$.

Proof. (1) First suppose that $f \in M_{a,b}(\widehat{K}(D))_G$. Let π^F be the cuspidal automorphic representation of $\text{GSp}_2(\mathbb{A})$ associated with F , and π^f the automorphic representation of $\text{GU}(2, B \otimes \mathbb{A})$ attached to f . For $p \mid D^+$, $\pi_p^f \simeq \pi_p^F$, so $e_p = w_p$. For $p \mid D^-$, we need to show that ω_p acts on f by $-w_p$.

As in the proof of [RW21, Proposition 12.3], π_p^F is of type IIa, in the notation (from [RS07]) used there. The local component π_p^F (for $p \mid D^-$) is of type IIa, i.e. $\chi \text{St}_{\text{GL}_2} \rtimes \sigma$, for some unramified characters σ, χ of \mathbb{Q}_p^\times , with $\chi^2 \neq \nu^{\pm 1}, \chi \neq \nu^{\pm 3/2}$, where ν is an unramified character of \mathbb{Q}_p^\times with $\nu(p) = p^{-1}$. (Here we switch temporarily to the notation of [RS07], [Sch17], so ν here is what would have been χ^{-1} in the notation of Remark 7.10, not the spinor norm.) The fact that σ and χ must be unramified follows from the fact that $p \parallel D$, and the 'N' column of [RS07, Table A.14]. By [RS07, Table A.12], the eigenvalue of W_p acting on F is $w_p = -(\chi\sigma)(p)$. Note that the ambiguous comment above [RS07, Table A.15] suggests this might not be correct, but it is referring to the \pm -signs under the dimensions of the fixed spaces, not the entries in the Atkin-Lehner eigenvalue column. This is clearer in [Sch05, §1.3].

The local component of π_p^f , viewed as a representation of $\text{GU}(1, 1, B_p)$, is of the type called IIa^G in [Sch17, table in A.4]. It is $\chi 1_{B^\times} \rtimes \sigma$, for the same χ and σ as above. For $\pi \in B_p$ with $\pi^2 = -p$, the Atkin-Lehner element in $\text{GU}(1, 1, B_p)$ is

$$\omega_p' = \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix} \begin{pmatrix} 0 & -\pi \\ \pi^{-1} & 0 \end{pmatrix},$$

with $\begin{pmatrix} 0 & -\pi \\ \pi^{-1} & 0 \end{pmatrix} \in K^-(p)$, so $\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$ is an acceptable alternative Atkin-Lehner element. It now follows from [Sch17, after Lemma A.2] that ω_p acts on the K_p^- -fixed line in π_p^f as multiplication by $(\chi\sigma)(p) = -w_p$.

- (2) Suppose that $f \in M_{a,b}(\widehat{K}(D))_{\text{SK}}$, with $f = \iota_{d_-}(h)$ for some eigenform $h \in \mathfrak{S}_{\frac{D_-}{2b+4}}^{\text{new}, -(-1)^{\omega(d_-)}}(\Gamma_0(D/d_-))$, ϵ_p the local sign at p attached to h , or rather to the newform g from which it comes.

Suppose that $p \mid D^-$. As in the proof of [RW21, Proposition 12.2], the representation π_p^f of $\text{GU}(1, 1, B_p)$ could be IIa^G (if $p \mid d_-$), or Vb^G or VIc^G (if $p \nmid \frac{D_-}{d_-}$).

We have already seen that in the case that $\pi_p^f \simeq \chi 1_{B^\times} \rtimes \sigma$, of type IIa^G , $e_p = (\chi\sigma)(p)$. As in [RS07, Proposition 5.5.1(i)], there is an associated representation of $\text{GSp}_2(\mathbb{Q}_p)$, of type IIb , which is $\chi 1_{\text{GL}_2} \rtimes \sigma$. The σ here is the same as the σ in the tables at the end of [RS07], but it corresponds to the $\chi^{-1}\sigma$ in [RS07, Proposition 5.5.1(i)], where χ is the same. If we call the σ in [RS07, Proposition 5.5.1(i)] “ σ' ” instead, then $e_p = \sigma'(p)$. In fact σ' is trivial, by Lemma 9.3. On the other hand, $\epsilon_p = 1$, since $p \nmid \frac{D_-}{d_-}$. Hence $e_p = \epsilon_p$.

Next consider the case that π_p^f is Vb^G , so it is $L(\nu^{1/2}\xi 1_{B^\times}, \nu^{-1/2}\sigma)$ in the notation of [Sch17, Proposition A.1], and is a quotient of the induced representation $\nu^{1/2}\xi 1_{B^\times} \rtimes \nu^{-1/2}\sigma$. This is for characters ξ, σ of \mathbb{Q}_p^\times , with ξ non-trivial quadratic. Arguing as in the proof of Lemma 9.3 (considering also elements of B_p^\times with unit norm), we see that ξ and σ are unramified. (Alternatively, in the next paragraph we can apply $p \parallel D$ and the ‘ N ’ column of [RS07, Table A.14].) Consider $\mathfrak{h} : \text{GU}(1, 1, B_p) \rightarrow \mathcal{V}$ a vector in the space of the induced representation $\nu^{1/2}\xi 1_{B^\times} \rtimes \nu^{-1/2}\sigma$, mapping to a K_p^- -fixed vector in $L(\nu^{1/2}\xi 1_{B^\times}, \nu^{-1/2}\sigma)$, and normalised so that $\mathfrak{h}(\text{id.}) = 1$. The eigenvalue of ω_p^f is $\mathfrak{h}(\begin{pmatrix} a & 0 \\ 0 & \pi \end{pmatrix})$, where $\pi \in B_p$ with $\pi^2 = -p$. Letting $a = \pi, \lambda = p$, this is

$$\mathfrak{h}\left(\begin{pmatrix} a & 0 \\ 0 & \lambda \bar{a}^{-1} \end{pmatrix}\right) = \nu^{1/2}\xi(a\bar{a})\nu^{-1/2}\sigma(\lambda) = \xi(p)\sigma(p) = -\sigma(p),$$

where $\xi(p) = -1$ because ξ is non-trivial, quadratic and unramified. This computation is of exactly the same type that may be used to prove that ω_p acts as $(\chi\sigma)(p)$ in the case IIa^G . In the case Vb^G we have simply substituted $\nu^{1/2}\xi$ for χ and $\nu^{-1/2}\sigma$ for σ .

There is a corresponding representation of $\text{GSp}_2(\mathbb{Q}_p)$ of type Vb , which is $L(\nu^{1/2}\xi \text{St}_{\text{GL}_2}, \nu^{-1/2}\sigma)$, and by [RS07, Table A.12], the eigenvalue of W_p on a K_p^+ -fixed vector is $\sigma(p)$. Since this local representation is of Saito-Kurokawa type (for the same newform g), this is ϵ_p , by [RS07, Proposition 5.5.8(i)]. Hence $e_p = -\sigma(p) = -\epsilon_p$, as required.

Finally, suppose that π_p^f is VIc^G , which is $\nu^{1/2} 1_{B^\times} \rtimes \nu^{-1/2}\sigma$, with σ unramified. Then similarly $e_p = \sigma(p)$. There is a corresponding representation of $\text{GSp}_2(\mathbb{Q}_p)$ of type VIc , which is $L(\nu^{1/2}\text{St}_{\text{GL}_2}, \nu^{-1/2}\sigma)$, and by [RS07, Table A.12], the eigenvalue of W_p on a K_p^+ -fixed vector is $-\sigma(p)$. Since this local representation is of Saito-Kurokawa type (for the same newform

g), this is ϵ_p , by [RS07, Proposition 5.5.8(i)]. Hence $e_p = \sigma(p) = -\epsilon_p$, as required.

- (3) The first part, in the case that $p \mid D^+$, follows from the fact that the L -parameter of $\pi_p^{\iota(g,h)}$ is a kind of direct sum of those of π_p^g and π_p^h (representations of $\mathrm{GL}_2(\mathbb{Q}_p)$), from the behaviour of ϵ -factors under the local Langlands correspondence, and the relation between Atkin-Lehner eigenvalues and ϵ -factors, for generic representations of $\mathrm{GSp}_2(\mathbb{Q}_p)$, which is [RS07, Corollary 7.5.5]. In the case $p \mid D^-$, the representation $\pi_p^{\iota(g,h)}$ of $\mathrm{GU}(1, 1, \mathbb{Q}_p)$ is of type IIa^G (as in the proof of [RW21, Proposition 12.3]). The corresponding representation of $\mathrm{GSp}_2(\mathbb{Q}_p)$ of type IIa has the same L -parameter (cf. [JLR12, Table 1]), and we may argue as in the proof of (1) to obtain the sign-change in the Atkin-Lehner eigenvalue.

The second part follows from the fact that (being a bit sloppy over the distinction between different Atkin-Lehner involutions at different levels), Atkin-Lehner involutions commute with η and intertwine θ and θ' with each other. (This is in [RS06, §4] or [RS07, §3.2].) That none of η , $\theta + \theta'$ or $\theta - \theta'$ will produce 0 follows from the linear independence in [RS07, Theorem 7.5.6]. (See also following [RS07, (5.49)].) □

Remark 10.2. Returning to the decomposition

$$M_{a,b}(\widehat{K}(D)) \simeq \bigoplus_{\substack{d \mid D \\ \gcd(d,D/d)=1}} \widetilde{M}_{a,b}(\widehat{K(L)}^+)^{\theta_d},$$

given an eigenform $f \in M_{a,b}(\widehat{K}(D))$ as in Theorem 10.1, we want to know for which d the corresponding \tilde{f} lies in $\widetilde{M}_{a,b}(\widehat{K(L)}^+)^{\theta_d}$. We already know that this is characterised by $p \mid d \iff e_p = -1$, and the theorem, which tells us the e_p for all primes $p \mid D$, thus determines d .

Example 10.3. (1) is illustrated by [RT20, Example 9], where $D = D^- = 61$. Here there is $F \in S_{3,0}(P(61))_G$ with $w_{61} = -1$ (as in [PY15, §8, Example 1]), while for the associated $f \in M_{0,0}(\widehat{K}(61))$, $e_{61} = +1$, hence for $\tilde{f} \in \widetilde{M}_{0,0}(\widehat{K(61)}^+)$, $d = 1$.

Remark 10.4. The second part of (3) applies equally to the production of oldforms from newforms in (1). For the linear independence, apply [RS07, Theorem 7.5.8], that π_p^f is generic because it is tempered. For $D^- = 61$, $D^+ = 5$, oldforms with both signs for e_5 , both coming from the newform in the previous remark, are D_2 and E_1 in [RT20, Table 3]. As expected, $d = 1$ when $e_5 = +1$ (for D_2), while $d = 5$ when $e_5 = -1$ (for E_1).

Example 10.5. For the squarefree $D = 5 \cdot 61$, (2) is illustrated by A_3 , C_4 , D_8 and G_2 in [RT20, Table 3], while (3) is illustrated by D_5 and F_1 in [RT20, Table 3]. In reading that table, beware that the Atkin-Lehner eigenvalues are not in the column “A-L”, but they can be deduced from the values of d .

Remark 10.6. In the case that $D = D^- = p$, not yet published work of T. Ibukiyama provides formulas for the dimensions of the subspaces of $M_{a,b}(\widehat{K}(p))$ on which $e_p = \pm 1$. He has very recently applied our theorem to deduce the answer to the corresponding question for spaces of paramodular forms [Ibu22].

11. APPLICATIONS TO CONGRUENCES

Theorem 11.1. *Let $F \in S_{k,j}(P(N))_G$ be a new Hecke eigenform, with $k \geq 3$. For any prime number ℓ , there exists a 4-dimensional $\overline{\mathbb{Q}}_\ell$ -vector space V , a continuous representation*

$$\rho_F : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}_{\overline{\mathbb{Q}}_\ell}(V),$$

and a Galois-equivariant symplectic pairing

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \overline{\mathbb{Q}}_\ell(3 - 2k - j),$$

such that for each prime $p \nmid \ell N$ it is unramified, with $\det(I - \rho_F(\text{Frob}_p^{-1})p^{-s})$ the reciprocal of the Euler factor at p in the spin L -function of F . In particular, for $p \nmid \ell N$, $\text{tr}(\rho_F(\text{Frob}_p^{-1}))$ is the Hecke eigenvalue $\lambda_F(p)$ of the Hecke operator $T(p)$ on F .

The Hodge-Tate weights of $\rho_F|_{\text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)}$ are $0, k - 2, j + k - 1$ and $j + 2k - 3$. If $\ell \nmid N$ then $\rho_F|_{\text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)}$ is crystalline, and the Artin conductor of ρ_F is N .

This comes from the work of many mathematicians, and is summarised in [Mok14, Theorem 3.1, Remark 3.3(3)]. (The part about the conductor follows from the local-global compatibility.) It is part of a more general theorem, about cohomological automorphic representations of $\text{GSp}_2(\mathbb{A}_F)$, with F a totally real field. This uses a lifting to $\text{GL}_4(\mathbb{A}_F)$, and the ℓ -adic cohomology of Shimura varieties for unitary groups. But in the case $F = \mathbb{Q}$, much of it had been proved by Weissauer [Wei05, Theorem I], [Wei08], using the ℓ -adic cohomology of Siegel modular three-folds. See the remarks at the ends of Sections 2.4 and 3.1 of [Wei21] on the status of the theorem for F totally real, and that the complete theorem is secure in the case $F = \mathbb{Q}$.

Since we assume that $k \geq 3$, $\ell \geq 5$, and since also $\rho_F|_{\text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)}$ is de Rham, it is known that ρ_F is irreducible, as in [Wei21, Theorem 1.1(i) and §2.4]. If we choose a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -invariant $\overline{\mathbb{Z}}_\ell$ -lattice then reduce modulo the maximal ideal, we get a representation $\overline{\rho}_F$ of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on a 4-dimensional $\overline{\mathbb{F}}_\ell$ -vector space. For some ℓ it may be reducible, in which case it depends on the choice of invariant lattice, but its irreducible composition factors are independent of the choice.

If θ on W is a composition factor of $\overline{\rho}_F$ then so is $\text{Hom}_{\overline{\mathbb{F}}_\ell[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]}(W, \overline{\mathbb{F}}_\ell(3 - 2k - j))$, i.e. $\theta^*(3 - 2k - j)$. If $\ell > j + 2k - 2$ and $\ell \nmid N$ then the Hodge-Tate weights can be detected on $\overline{\rho}_F$, via its associated Fontaine-Lafaille module [FL82]. They come in twisted dual pairs $\{0, j + 2k - 3\}$ and $\{k - 2, j + k - 1\}$. It follows that if N is square-free, then any 1-dimensional composition factor must be unramified away from ℓ , to avoid it and its twisted dual partner (which is different) contributing a square factor to the conductor.

Therefore, for $\ell > j + 2k - 2$, $\ell \nmid N$ and N square-free, the only possible pairs of 1-dimensional composition factors of $\overline{\rho}_F$ are $\{\overline{\mathbb{F}}_\ell, \overline{\mathbb{F}}_\ell(3 - 2k - j)\}$ and $\{\overline{\mathbb{F}}_\ell(2 - k), \overline{\mathbb{F}}_\ell(1 - k - j)\}$. It follows that if $\overline{\rho}_F$ is reducible, the possibilities for its composition factors are as follows.

- (1) $\{\overline{\rho}_g, \overline{\mathbb{F}}_\ell(2 - k), \overline{\mathbb{F}}_\ell(1 - k - j)\}$, where $\overline{\rho}_g$ is a 2-dimensional representation attached to a cuspidal Hecke eigenform for $\Gamma_0(N)$, of weight $j + 2k - 2$.
- (2) $\{\overline{\rho}_g(2 - k), \overline{\mathbb{F}}_\ell, \overline{\mathbb{F}}_\ell(3 - 2k - j)\}$, where $\overline{\rho}_g$ is a 2-dimensional representation attached to a cuspidal Hecke eigenform for $\Gamma_0(N)$, of weight $j + 2$.

- (3) $\{\bar{\rho}_g, \bar{\rho}_h(2-k)\}$, 2-dimensional representations attached to cuspidal Hecke eigenforms g of weight $j+2k-2$, h of weight $j+2$, of levels $\Gamma_0(M)$ and $\Gamma_0(N/M)$ for some $M \mid N$.
- (4) $\{\bar{\mathbb{F}}_\ell, \bar{\mathbb{F}}_\ell(2-k), \bar{\mathbb{F}}_\ell(1-k-j), \bar{\mathbb{F}}_\ell(3-2k-j)\}$.
- (5) $\{\bar{\rho}, \bar{\rho}^\vee(3-2k-j)\}$, with 2-dimensional $\bar{\rho}$, Hodge-Tate weights $\{0, j+k-1\}$ or $\{0, k-2\}$.

In cases (2) and (3) we have used the theorem of Khare and Wintenberger [KW09a, KW09b] and Kisin [Kis09], that an odd, irreducible representation of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ with coefficients in $\bar{\mathbb{F}}_\ell$ is modular, proved also by Dieulefait in the case that ℓN is odd [Die12]. The oddness of the two-dimensional factors in these cases is an easy consequence of the fact that j is even. Given that N is square-free, case (5) (“related 2-dimensional constituents” in [Die02, §4.2]) could only happen if $\bar{\rho}$ has Artin conductor 1, which is unlikely in our situation, where $N > 1$ because it is divisible by the discriminant of a definite quaternion algebra.

These reducibilities (in the first four cases) translate into congruences of Hecke eigenvalues as follows, for all primes $p \nmid \ell N$. The instances we prove are actually for all $p \nmid N$.

- (1) $\lambda_F(p) \equiv a_p(g) + p^{k-2} + p^{j+k-1} \pmod{\lambda}$, with g a cuspidal Hecke eigenform for $\Gamma_0(N)$, of weight $j+2k-2$.
- (2) $\lambda_F(p) \equiv p^{k-2}a_p(g) + 1 + p^{j+2k-3} \pmod{\lambda}$, with g a cuspidal Hecke eigenform for $\Gamma_0(N)$, of weight $j+2$.
- (3) $\lambda_F(p) \equiv a_p(g) + p^{k-2}a_p(h)$, with cuspidal Hecke eigenforms g of weight $j+2k-2$, h of weight $j+2$, of levels $\Gamma_0(M)$ and $\Gamma_0(N/M)$ for some $M \mid N$.
- (4) $\lambda_F(p) \equiv 1 + p^{k-2} + p^{j+k-1} + p^{j+2k-3} \pmod{\lambda}$.

Here, λ is a divisor of ℓ in a sufficiently large extension of \mathbb{Q} . Unless the conductor of $\bar{\rho}_F$ happens to be a proper divisor of the conductor of ρ_F , g in (1) and (2), and g, h in (3), will be newforms. This will certainly be the case in the instances of such congruences that we prove, which will be of types (1) and (2), though the proof relies on viewing them as like type (3), except with $\bar{\rho}_h$ or $\bar{\rho}_g$ (respectively) reducible.

The strategy for proving instances of such congruences is as follows. Assuming N is exactly divisible by at least one prime, we choose a square-free divisor D^- of N with $\omega(D^-)$ odd, and apply Theorem 9.7 for suitable $D = D^-D^+$ with $\gcd(D^-, D^+) = 1$ and $N \mid D$. So we take F (or some oldform derived from F if $N \neq D$) in $S_{k,j}^{D^- \text{-new}}(P(D))_G$, and corresponding $f \in M_{j+k-3, k-3}(\widehat{K}(D))_G$. For $p \nmid D$, the Hecke eigenvalue $\lambda_F(p)$, the left-hand-side of the congruence, is the same as the eigenvalue of $T(p)$ on f .

The aim is to arrange for there to be another Hecke eigenform $f_1 \in M_{j+k-3, k-3}(\widehat{K}(D))$ such that the right-hand-side is the eigenvalue of $T(p)$ on f_1 , or is at least congruent to it modulo λ . If we can then observe a congruence $\pmod{\lambda}$ between the vectors f and f_1 inside $M_{j+k-3, k-3}(\widehat{K}(D))$ (with respect to the natural integral structure on the coefficient module $W_{j+k-3, k-3}$) then the desired congruence of Hecke eigenvalues follows, at least for $p \nmid D$. (In fact this gives us a congruence of Hecke eigenvalues also for $T_1(p^2)$, not just for $T(p)$.)

We further arrange for f and f_1 to have matching Atkin-Lehner eigenvalues for $p \mid D$. Then in practice we actually prove the congruence between corresponding \tilde{f} and \tilde{f}_1 in $\widetilde{M}_{j+k-3, k-3}(\widehat{K}(L)^+)^{\theta_d}$, for some suitable $d \mid D$ with $\gcd(d, D/d) = 1$, determined by Theorem 10.1. Actually, except in the first following section, we

shall see some interesting departures from the simple strategy outlined above, with the involvement of additional eigenforms in $\widetilde{M}_{j+k-3,k-3}(\widehat{K(L)^+})^{\theta_d}$, or even of two spaces for different genera of lattices.

11.1. Congruences between forms of Saito-Kurokawa and general types.

First we look at case (1), in the sub-case $j = 0$, with a newform $g \in S_{2k-2}(\Gamma_0(N))$. If the sign in the functional equation of $L(g, s)$ is $(-1)^{\omega(N)}$, and q is an auxiliary prime when $\omega(N)$ is even, then for $p \nmid N$ (or $p \nmid Nq$), $a_p(g) + p^{k-2} + p^{k-1}$ is the Hecke eigenvalue of $T(p)$ on the Saito-Kurokawa lift

$$\hat{g} = \begin{cases} \iota_1(g) & \text{if } \omega(N) \text{ is odd;} \\ \iota_q(g) & \text{if } \omega(N) \text{ is even.} \end{cases}$$

In Proposition 9.4, $D = D^- = N$ or Nq , and $d_- = 1$ or q . We seek then a congruence between \tilde{f} and \tilde{g} in $M_{k-3,k-3}(\widehat{K(L)^+})^{\theta_d}$, for suitable d , where \tilde{f} comes from a newform $F \in S_k(P(N))_G := S_{k,0}(P(N))_G$.

11.1.1. *Example with $N = 61$, $k = 3$, $\ell = 43$.* The space $S_4(\Gamma_0(61))$ (61.4.a) is 15-dimensional, spanned by a newform g (61.4.a.a) with Hecke eigenvalues in a number field E of degree 6, and a newform whose coefficient field has degree 9. The sign in the functional equation of $L(g, s)$ is $\epsilon_{61} = -1$. Consequently there exists a 6-dimensional subspace of Saito-Kurokawa lifts, of g and its conjugates, inside $S_3(P(61))$. In [PY15, §8, Example 1], Poor and Yuen refer to this as the subspace of Gritsenko lifts of associated Jacobi forms. They show that $S_3(P(61))$ is 7-dimensional, with $S_3(P(61))_G$ spanned by a Hecke eigenform F with rational Fourier coefficients and Atkin-Lehner eigenvalue $w_{61} = -1$. They also prove (with appropriate scaling) a mod 43 congruence of Fourier coefficients between F and some Gritsenko lift (not a Hecke eigenform). It is easy to deduce from this a congruence of Hecke eigenvalues (or even of Fourier coefficients) between F and the Saito-Kurokawa lift $\text{SK}(g)$, modulo some prime divisor λ of 43 in E , as in [Dum22, Example 5.7]. It is (for all primes $p \neq 61$)

$$\lambda_F(p) \equiv a_p(g) + p + p^2 \pmod{\lambda}.$$

We can recover this as a congruence of Hecke eigenvalues between associated f and \hat{g} in $M_{0,0}(\widehat{K}(61))$, with $D = D^- = 61$, $d_- = 1$. (Note that, unlike the method of Poor and Yuen, ours does not lead to a congruence of Fourier coefficients.) By Theorem 10.1, ω_{61} has eigenvalue $e_{61} = 1$ on both f and \hat{g} . So both \tilde{f} and \tilde{g} find themselves inside $\widetilde{M}_{0,0}(\widehat{K(L)^+})^{\theta_1}$ (cf. Remark 10.2). In fact, as noted in Example 10.3, the space $\widetilde{M}_{0,0}(\widehat{K(L)^+})^{\theta_1}$ was computed in [RT20, Example 9]. Computing with integral coefficients, the congruence between the suitably scaled eigenvectors \tilde{f} and \tilde{g} may be observed directly.

We computed in Sage [Sag21], using the package `quinary_module_1.sage`, which may be found in [Ram20b]. The quadratic form q is associated to a special lattice of determinant $2D$, with $D = 61$. All the quinary forms used in this article, like those tabulated in <http://www.cmat.edu.uy/cnt/omf5/>, were obtained via a box search.

```
sage: q = QuadraticForm(ZZ, 5, [1, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 0,
1, 0, 8])
sage: qmod = quinary_module(q)
```



```

sage: T2 = qmod.Tp_d(2, 1)
sage: T2
[7 4 4 0 0 0 0 0]
[1 4 3 3 3 1 0 0]
[1 3 3 0 0 0 2 6]
[0 2 0 5 0 2 2 4]
[0 6 0 0 1 0 4 4]
[0 1 0 3 0 9 0 2]
[0 0 4 6 4 0 1 0]
[0 0 3 3 1 1 0 7]
sage: T2.fcp()
(x - 15) * (x + 7) * (x^6 - 29*x^5 + 322*x^4 - 1714*x^3 + 4471*x^2 -
5205*x + 2026)
sage: v61 = (T2 + 7).right_kernel().0
sage: v61
(0, 6, -6, -4, -12, 0, 12, 3)
sage: K.<a> = NumberField(T2.fcp()[2][0])
sage: vSK = (T2 - a).right_kernel().0
sage: vSK*= denominator(vSK)
sage: I = K.ideal([43, a + 7])
sage: R = K.residual_field(I)
sage: (v61 - 4*vSK).change_ring(R) == 0
True

```

Note that we are able to use the single Hecke operator $T(2)$ to decompose the space into simple Hecke submodules.

Generalising a construction of Ribet, as in [BL21, §11], the invariant $\overline{\mathbb{Z}}_{43}$ -lattice in V may be chosen in such a way that inside $\overline{\rho}_F$ we get a non-trivial extension of $\overline{\mathbb{F}}_{43}(-1)$ by $\overline{\rho}_g$, hence of $\overline{\mathbb{F}}_{43}$ by $\overline{\rho}_g(1)$. This leads to a non-zero element in a certain Selmer group, which by the Bloch-Kato conjecture should lead to divisibility by λ of a suitably normalised algebraic part $L_{\text{alg}}^N(3, g)$ (Euler factors at primes dividing N omitted). Note that 3 is paired with 1 by the functional equation with respect to s and $4 - s$. More generally, $j + k$ is paired with $k - 2$. Using the command `LRatio` in the Magma computer package [BCP97], one readily checks that in fact $43 \mid \text{Nm}_{E/\mathbb{Q}}(L_{\text{alg}}(3, g))$.

A general theorem of Brown and Li, Corollary 6.14 in [BL21], proves a mod λ congruence of Hecke eigenvalues between $\text{SK}(g)$ ($g \in S_{2k-2}(\Gamma_0(N))^-$ a newform) and a Hecke eigenform $F \in S_k(P(N))_G$, from divisibility by λ of $L_{\text{alg}}(k, g)$, under various conditions. These include that $k \geq 6$, so it does not apply here.

11.1.2. *Example with $N = 89$, $k = 3$, $\ell = 29$.* As in the previous example, we can prove a congruence between the Hecke eigenvalues of the classical modular form g with label 89.4.a.c (whose coefficient field has degree 6) and a paramodular form in $S_3(P(89))_G$. More precisely, the space $S_3(P(89))$ decomposes as the sum of 4 eigenspaces, 3 of degree 1 and 1 of degree 6. This can be proved as before using the Hecke operator $T(2)$.

Because the degree of E , the coefficient field of g , is 6 we conclude that the eigenspace of degree 6 in $S_3(P(89))$ must correspond to a Saito-Kurokawa lift of g . Of the other 3 eigenspaces, two correspond to Saito-Kurokawa lifts, which we can identify by looking at their eigenvalues. The third one, spanned by F , must be in

$S_3(P(89))_G$. Note that, whereas when $N = 61$ we already knew the existence of F , by work of Poor and Yuen, here, computing with algebraic modular forms only, we are guaranteed its existence by Theorem 9.7 combined with Proposition 9.9(3).

Using the same argument as in the previous example, we have proved the following.

Theorem 11.2. *The following congruence holds for all primes $p \neq 89$:*

$$\lambda_F(p) \equiv a_p(g) + p + p^2 \pmod{\lambda},$$

where λ is a prime divisor of 29 in E .

Note that once again, $29 \mid \text{Nm}_{E/\mathbb{Q}}(L_{\text{alg}}(3, g)) = 348/11$.

11.2. Harder’s conjecture for paramodular level: examples of Fretwell. Still in case (1), we suppose now that $j > 0$. Then the right hand side $a_p(g) + p^{k-2} + p^{j+k-1}$ is no longer the Hecke eigenvalue of $T(p)$ on some element of $S_{k,j}(P(N))$. Still, one may conjecture that if $\ell > j + 2k - 2$ and λ divides $L_{\text{alg}}(j + k, g)$ then there exists a Hecke eigenform $F \in S_{k,j}(P(N))_G$ satisfying the congruence

$$\lambda_F(p) \equiv a_p(g) + p^{k-2} + p^{j+k-1} \pmod{\lambda}.$$

This conjecture was made by Harder in the case $N = 1$ [Har08] and an instance $(k, j, \ell) = (10, 4, 41)$ was proved by Chenevier and Lannes [CL19], using algebraic modular forms, with constant coefficients, for orthogonal groups of even unimodular lattices of rank 24. Further examples in the case $N = 1$ have been proved by methods farther removed from those we use here. Ibukiyama [Ibu14] proved an example $(k, j, l) = (5, 18, 43)$, subject to his conjecture on a Shimura correspondence in degree 2, which was subsequently proved by Ishimoto [Ish22]. Recently Atobe, Chida, Ibukiyama, Katsurada and Yamauchi [ACI+21] have reproved the case $(k, j, l) = (10, 4, 41)$ and added two further instances, namely $(k, j, l) = (14, 4, 4289)$ and $(4, 24, 97)$. They deduce these from congruences between (vector-valued) degree-4 Klingen-Eisenstein series (of Saito-Kurokawa lifts) and cuspidal lifts, proved using differential operators and pullback formulas.

For $N = 2, 3, 5, 7$, Fretwell [Fre18] found experimental evidence for several instances of such congruences, in each case checking it for a few small values of p . He computed Hecke eigenvalues by computing traces of Hecke operators on 1-dimensional spaces of algebraic modular forms for $\text{GU}(2, B)$, with B a definite quaternion algebra over \mathbb{Q} , ramified at N . In Theorem 11.3 below, we prove an instance of a congruence of the same type, but for larger N . Three further examples we have proved, noted in §11.2.2, include one of Fretwell’s.

11.2.1. *Example with $N = 19, k = 3, j = 2, \ell = 7$.* Notice that $\ell > j + 2k - 2$, just. Consider the newform $g \in S_6(\Gamma_0(19))$, with rational Hecke eigenvalues and $\epsilon_{19} = -1$ (19.6.a.a)). Its Fourier expansion starts as

$$g = q - 6q^2 + 4q^3 + 4q^4 + 54q^5 - 24q^6 + \dots$$

Using the command LRatio in the Magma computer package [BCP97], one checks that $7 \mid L_{\text{alg}}(5, g)$, so we expect a Hecke eigenform $F \in S_{3,2}(P(19))_G$ with

$$(28) \quad \lambda_F(p) \equiv a_p(g) + p + p^4 \pmod{7},$$

for all primes $p \nmid 7 \cdot 19$.

Theorem 11.3. *The congruence (28) holds for all $p \neq 19$.*

Proof. We may view the right-hand-side as $a_p(g) + p(1 + p^3)$, which looks like the Hecke eigenvalue of a Yoshida lift of g and an Eisenstein series of weight 4 and level 1. Though there is no such thing, we can replace the Eisenstein series by a cuspidal Hecke eigenform h of weight 4 and prime level q , chosen to have the same Hecke eigenvalues mod 7. Such an h exists as long as $q^4 \equiv 1 \pmod{7}$. This is an instance of a general theorem on congruences “of local origin”, conjectured by Harder [Har13] and proved independently in [DF14, Theorem 1.1] and by Billerey and Menares [BM16]. In the case at hand, the smallest q that works is $q = 13$, and we can find the form h (13.4.a.a) in the LMFDB [LMF21], with $\epsilon_{13} = -1$,

$$h = q - 5q^2 - 7q^3 + 17q^4 - 7q^5 + 35q^6 + \dots$$

By Proposition 9.1, with $a = 2, b = 0, d_- = D^- = 13, c_+ = D^+ = 19$, we have $f_1 = \iota(h, g) \in M_{2,0}(\widehat{K}(13 \cdot 19))$. By Theorem 10.1, ω_{13} and W_{19} have eigenvalues $e_{13} = 1, e_{19} = -1$ on f_1 . Hence the corresponding \tilde{f}_1 lives in $\widetilde{M}_{2,0}(\widehat{K}(L_1)^+)^{\theta_{19}}$, where L_1 is the lattice associated to $D^- = 13$ and $D^+ = 19$.

With $D = D^- = 19$, one finds that the space $\widetilde{M}_{2,0}(\widehat{K}(L)^+)^{\theta_1}$ (with a different L for $D = 19$) is 1-dimensional, spanned by an eigenform corresponding by Theorem 9.7 to our target $F \in S_{3,2}(P(19))_G$, with $w_{19} = -1$ by Theorem 10.1. Following Remark 10.4 (and switching to $D^- = 19, D^+ = 13$), we can manufacture an associated oldform $f \in M_{2,0}(\widehat{K}(19 \cdot 13))$ with $e_{19} = 1, e_{13} = -1$, whose associated \tilde{f} is in $\widetilde{M}_{2,0}(\widehat{K}(L_2)^+)^{\theta_{13}}$, where L_2 corresponds to $D^- = 19, D^+ = 13$.

The space $\widetilde{M}_{2,0}(\widehat{K}(L_1)^+)^{\theta_{19}}$ (working with coefficients in a \mathbb{Q} -vector space) decomposes as $A_1 \oplus A_2 \oplus A_3$ where A_1 corresponds to the Yoshida lift f_1 , A_2 to the Yoshida lift of the forms 13.4.a.a and 19.6.a.d. We wish to show that the 29-dimensional \mathbb{Q} -vector space A_3 (irreducible as a Hecke module) corresponds to a paramodular newform of level $13 \cdot 19$ for $k = 3, j = 2$, and its Galois conjugates. This will follow from Theorem 9.7 if we can show that $M_{2,0}(\widehat{K}(13 \cdot 19))_{\text{Eis}}$ is trivial, so that A_3 survives into $M_{2,0}(\widehat{K}(13 \cdot 19))_G$. (Note also that it cannot be 19-old, since one finds that $\dim S_{3,2}(P(13)) = 0$.)

To show this, we cannot use Proposition 9.9(3), since $D^+ > 1$ (and $j > 0, k < 5$). But we can use a similar idea with the aid of some computation. Using [IK17, Main Theorem 1.1] we find that $\dim S_{3,2}(P(13 \cdot 19)) = 189$. We have to remove a 2-dimensional space of 13-old forms coming from F , and since $j > 0$ there are no Saito-Kurokawa lifts to remove, so $\dim S_{3,2}^{13\text{-new}}(P(13 \cdot 19))_G = 187$. By Theorem 9.7, this is a lower bound for $\dim \widetilde{M}_{2,0}(\widehat{K}(L_1)^+) - \dim \widetilde{M}_{2,0}(\widehat{K}(L_1)^+)_Y$ (where Y denotes Yoshida lifts), with equality if and only if $M_{2,0}(\widehat{K}(13 \cdot 19))_{\text{Eis}}$ is trivial. But by computation we find that $\dim \widetilde{M}_{2,0}(\widehat{K}(L_1)^+) - \dim \widetilde{M}_{2,0}(\widehat{K}(L_1)^+)_Y$ is indeed 187, with contributions 66, 30, 29 and 62 from $d = 1, 13, 19$ and $13 \cdot 19$, respectively.

The space $\widetilde{M}_{2,0}(\widehat{K}(L_2)^+)^{\theta_{13}}$ decomposes as $B_1 \oplus B_2 \oplus B_3$ where B_1 corresponds to the oldform \tilde{f} , B_2 to the Yoshida lift of the forms 19.4.a.a and 13.6.a.b. Lastly, the 29-dimensional space B_3 must be isomorphic (as a Hecke module) to A_3 , corresponding to the same paramodular newform of level $13 \cdot 19$ for $k = 3, j = 2$.

If we work with coefficients in \mathbb{Z} -modules rather than \mathbb{Q} -vector spaces, we do not get direct sum decompositions. The space A_1 is included in A_3 modulo 7, and B_1 is included in B_3 modulo 7. In fact, the isomorphism of A_3 to B_3 sends A_1 to B_1

modulo 7. We conclude that the eigenvalues of \tilde{f} and f_1 must be the same modulo 7. This would be for all $p \nmid 19 \cdot 13$, but we may check the congruence for $p = 13$ by hand, or by choosing a different q , as remarked below. \square

The computations for this theorem were done using a package similar to that used in the previous example, but implemented in Pari/GP [PAR18], where we implemented the representations in general. This also can be found in [Ram20b].

The next three primes q such that $q^4 \equiv 1 \pmod{7}$ are $q = 29, 41$ and 43 . We checked that the same congruence can be proved in the same fashion using any of these q in place of $q = 13$. Note in particular that in all such cases there exists a congruence modulo 7 between the paramodular form F and a paramodular newform of level $19 \cdot q$. This naturally raises the following question.

Question 11.4 (level-raising). *Consider a new Hecke eigenform $F \in S_{k,j}(P(N))_G$ such that (for some $\ell > j + 2k - 2$) $\bar{\rho}_F$ is reducible of type (1). Is it true that for any prime q such that $q^{j+2} \equiv 1 \pmod{\ell}$, there exists a new Hecke eigenform $H \in S_{k,j}(P(qN))_G$ such that $\bar{\rho}_H$ has the same composition factors as $\bar{\rho}_F$?*

Note that $\bar{\rho}_F$ is reducible, so although we call this “level-raising”, it should perhaps be viewed as more closely analogous to [BM16], [DF14] or [Yoo19] than to well-known results of Diamond and Ribet on level-raising for residually irreducible representations [Rib84], [Dia91].

11.2.2. *Further examples.* With the above approach we only succeeded in proving one of the examples of Fretwell. The reason is that our method is very efficient for small values of j , in which case the dimension of $W_{j+k-3,k-3}$ (especially for small k) can be manageably small, cf. Theorem 8.1.

The example we could handle is when $N = 5$, $k = 7$, $j = 2$, and $\ell = 61$. Let $g \in S_{14}(\Gamma_0(5))$ be the newform with LMFDB label 5.14.a.b and coefficient field E of degree 3. Computing the newform in Magma (via the use of modular symbols), one can verify that the algebraic value $L_{\text{alg}}(9, g)$ equals $58712500 = 2^2 \cdot 5^5 \cdot 7 \cdot 11 \cdot 61$. Let $F \in S_{7,2}(P(5))_G$.

Theorem 11.5. *The following congruence holds, for all primes $p \neq 5$:*

$$\lambda_F(p) \equiv a_p(g) + p^3 + p^8 \pmod{\lambda}$$

where λ is a prime in E that divides 61.

Proof. Take the form $h \in S_4(\Gamma_0(11))$ of label 11.4.a.a to construct the Yoshida lift, and proceed as in the proof of Theorem 11.3. Note that $11^4 \equiv 1 \pmod{61}$. \square

We also proved the following additional two examples.

Theorem 11.6. *In the previous notation, let $N = 42$, $k = 3$, $j = 2$, and $\ell = 13$. Let g be the newform 42.6.a.d and F the paramodular form in $S_{3,2}(P(42))$. Then the value $L_{\text{alg}}(g, 5) = 13$ and for all primes $p \nmid 42$,*

$$\lambda_F(p) \equiv a_p(g) + p + p^4 \pmod{13}.$$

Proof. The algebraic value can be computed with Magma. To prove the other claim, proceed as in the previous examples, using the form 5.4.a.a to construct the Yoshida lift. The only difference with the other examples is that since the primes 2, 3, 5 and 7 all divide qN , we have to decompose the space using $T(11)$, whereas in earlier examples we always used $T(2)$. \square

Theorem 11.7. *In the previous notation, let $N = 13$, $k = 3$, $j = 4$, and $\ell = 11$. Let g be the newform 13.6.a.a and F the paramodular form in $S_{3,4}(P(13))_G$. Then $L_{alg}(g, 7) = 220/119$ and for all primes $p \neq 13$,*

$$\lambda_F(p) \equiv a_p(g) + p + p^6 \pmod{11}.$$

Proof. Take the form 23.6.a.a to construct the Yoshida lift. \square

11.3. Proof of a congruence of Buzzard and Golyshev. We turn now to case (2). We already saw, in §11.1.1, a congruence, modulo a divisor of 43, involving $F \in S_3(P(61))_G$ and $g \in S_4(\Gamma_0(61))$. Around the end of 2010, V. Golyshev conjectured the existence of a second congruence for F , beyond the one involving 43, then K. Buzzard found it experimentally, having realised the possibility of it involving weight 2 rather than weight 4, and computations of A. Mellit provided further support. The congruence is

$$(29) \quad \lambda_F(p) \equiv 1 + p^3 + pa_p(g) \pmod{\lambda},$$

where $g \in S_2(\Gamma_0(61))$ (61.2.a.b) is a newform with cubic coefficient field E , $\epsilon_{61} = -1$, and λ is a prime divisor of 19 in E .

If this is true then inside $\bar{\rho}_F$ we get a non-trivial extension of $\bar{\mathbb{F}}_{19}$ by $\bar{\rho}_g(-1)$ ($\bar{\rho}_g(2-k)$ in general), which is connected by the Bloch-Kato conjecture to $L(3, g)$ ($L(j+k)$ in general), but since this is a non-critical value (not in the range $1 \leq s \leq (j+2) - 1$) we cannot detect the factor computationally.

Theorem 11.8. *The congruence (29) holds, for all primes $p \neq 61$.*

Proof. To prove the congruence, we interpret $1 + p^3 + pa_p(g)$ as congruent mod λ to $a_p(h) + pa_p(g)$, where $h \in S_4(\Gamma_0(q))$ is a newform congruent mod λ to the level 1 Eisenstein series of weight 4, with $q^4 \equiv 1 \pmod{19}$. The smallest q we can use is $q = 37$, and h is 37.4.a.a, with $\epsilon_{37} = -1$. Since h has coefficient field of degree 4, we need to replace E by its compositum with this field, and λ by a suitable divisor of the original.

By Proposition 9.1, with $a = 0, b = 0, d_- = D^- = 61, c_- = D^+ = 37$, we have $f_1 = \iota(g, h) \in M_{0,0}(\widehat{K}(61 \cdot 37))$. By Theorem 10.1, ω_{61} and W_{37} have eigenvalues $e_{61} = +1, e_{37} = -1$ on f_1 . Hence the corresponding \tilde{f}_1 lives in $\widetilde{M}_{0,0}(\widehat{K(L)}^+)^{\theta_{37}}$. Using Theorem 10.1 ($w_{61} = -1 \implies e_{61} = +1$) and Remark 10.4, we may produce an oldform f associated to F , with \tilde{f} in the same $\widetilde{M}_{0,0}(\widehat{K(L)}^+)^{\theta_{37}}$.

If we compute the Hecke operator $T(2)$ restricted to $\widetilde{M}_{0,0}(\widehat{K(L)}^+)^{\theta_{37}}$, its characteristic polynomial factors as $(x+7) \cdot p_2(x) \cdot p_3(x)$, where p_2 and p_3 are irreducible of degree 12 and 211 respectively. Working with \mathbb{Z} -coefficients, let C_1, C_2, C_3 be \mathbb{Z} -submodules of $\widetilde{M}_{0,0}(\widehat{K(L)}^+)^{\theta_{37}}$ killed by $T(2) + 7, p_2(T(2))$ and $p_3(T(2))$, respectively. The space C_1 corresponds to \tilde{f} , C_2 to \tilde{f}_1 (and its Galois conjugacy class) and C_3 likewise to a paramodular newform of level $37 \cdot 61$ for $k = 3, j = 0$. (Strictly speaking, we do not know that C_3 corresponds to a newform, since Proposition 9.9(3) applies only when $D^+ = 1$, but this proof of the congruence works anyway, without having to know it.)

The kernel of $T(2)+7$ on $\widetilde{M}_{0,0}(\widehat{K(L)}^+)^{\theta_{37}} \otimes \mathbb{F}_{19}$ has dimension 2. In $\widetilde{M}_{0,0}(\widehat{K(L)}^+)^{\theta_{37}} \otimes \mathbb{Z}_{19}$ we find four $T(2)$ -eigenspaces with eigenvalues congruent to -7 modulo 19, all rank-one and therefore common eigenspaces for all the $T(p)$ and $T_1(p^2)$ ($p \nmid 61 \cdot 37$).

The line $C_1 \otimes \mathbb{Z}_{19}$ has eigenvalue -7 , inside $C_2 \otimes \mathbb{Z}_{19}$ we have a line with eigenvalue $-7 + 10 \cdot 19 + 8 \cdot 19^2 + \dots$, inside $C_3 \otimes \mathbb{Z}_{19}$ eigenvalues $-7 + 15 \cdot 19 + 2 \cdot 19^2 + \dots$ and $-7 + 18 \cdot 19 + 10 \cdot 19^2 + \dots$.

We find that when we take four eigenvectors spanning these eigenspaces, and reduce them mod 19, say to $\{v_1, v_2, v_3, v_4\}$, which lie in the aforementioned 2-dimensional kernel of $T(2) + 7$ on $\widehat{M}_{0,0}(\widehat{K(L)}^+)^{\theta_{37}} \otimes \mathbb{F}_{19}$, no two of them is collinear. Now for any $p \nmid 61 \cdot 37$, consider the eigenvalues $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{F}_{19}$ of $T(p)$ acting on v_1, v_2, v_3, v_4 respectively. If $\mu_1 \neq \mu_2$ then v_3 , being in neither the μ_1 -eigenspace nor the μ_2 -eigenspace, would not be an eigenvector. This kind of contradiction shows that all four vectors lie in a single 2-dimensional simultaneous eigenspace for all the $T(p)$ and $T_1(p^2)$ ($p \nmid 61 \cdot 37$). This implies the congruence we were to prove. (It may be checked by hand for the auxiliary prime $p = 37$.) \square

Remark 11.9. The fact that the simultaneous eigenspace is not 1-dimensional may be viewed as a “multiplicity-one failure”, analogous to that discovered by Ribet and Yoo for certain Eisenstein ideals at composite level [Yoo16, Example 4.7, Remark 4.11].

The above computations were also performed using the Pari/GP package. Note that our paramodular form has reducible residual representation modulo 19, and is congruent to a newform of paramodular level $61 \cdot q$ (with $q = 37$). The next q such that $q^4 \equiv 1 \pmod{19}$ is $q = 113$. This is rather large to reprove the congruence.

11.3.1. *Example for $N = 89$, $\ell = 5$.* As in the case of $N = 61$, we also have a “second” congruence for $N = 89$, involving a modular form of weight 2. Let g_1, g_2 be the classical modular forms of level 89, weight 2 and sign -1 in their L -functions, with LMFDB labels [89.2.a.b](#) and [89.2.a.c](#) respectively. Their coefficient fields have degree 1 and 5 respectively, and we denote by E the second one. It is easy to prove in Sage that

$$a_p(g_1) \equiv a_p(g_2) \pmod{\lambda}$$

where λ is a prime dividing 5 in E , using the class `CuspForms` and the method `Hecke_matrix` to compute the Hecke matrix at 2 and prove the congruence for the corresponding eigenspaces.

Using the same method as before we have proved the following.

Theorem 11.10. *The following congruences hold, for all primes $p \neq 89$:*

$$\lambda_F(p) \equiv pa_p(g_1) + 1 + p^3 \pmod{5};$$

$$\lambda_F(p) \equiv pa_p(g_2) + 1 + p^3 \pmod{\lambda},$$

where λ is a prime divisor of 5 in E .

Proof. Take $q = 7$, $d_- = 1$, $D^- = 89$ and $D^+ = 7$, and proceed as with the previous example using the form $h \in S_4(\Gamma_0(7))$ of label [7.4.a.a](#) to construct the Yoshida lift. \square

In this case all primes $q \neq 5$ satisfy $q^4 \equiv 1 \pmod{5}$. We checked that there exists a congruence modulo 5 of the paramodular form F of level 89 and a paramodular newform of level $89 \cdot q$ for $q = 2, 3, 7, 11$. So once again, the following seems a very natural question.

Question 11.11. Consider a new Hecke eigenform $F \in S_{k,j}(P(N))_G$ such that (for some $\ell > j + 2k - 2$) $\bar{\rho}_F$ is reducible of type (2). Is it true that for any prime q such that $q^{j+2k-2} \equiv 1 \pmod{\ell}$, there exists a new Hecke eigenform $H \in S_{k,j}(P(qN))_G$ such that $\bar{\rho}_H$ has the same composition factors as $\bar{\rho}_F$?

Remark 11.12. It is actually only those q with $q^4 \equiv 1 \pmod{5^2}$ (such as $q = 7$) that are useful for proving the congruences in Theorem 11.10. This is because the general condition for existence of $h \in S_\kappa(\Gamma_0(q))$ congruent to $E_\kappa \pmod{\lambda}$ (with even $\kappa \geq 4$ and $\ell > 3$) is that $\text{ord}_\ell\left(\frac{B_\kappa}{2^\kappa}(q^\kappa - 1)\right) > 0$. We only have to worry about ℓ dividing the denominator of the first factor when $\ell \mid \kappa$ or (von Staudt-Clausen) $(\ell - 1) \mid \kappa$ (so not when $\ell > \kappa + 1$), but in the case at hand $\ell = 5$ and $\kappa = 4$, so $(\ell - 1) = \kappa$ and $\text{ord}_5 B_4 = -1$.

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