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Elliptic Solutions of Boussinesq Type Lattice Equations and the Elliptic *N*th Root of Unity

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Abstract: We establish an infinite family of solutions in terms of elliptic functions of the lattice Boussinesq systems by setting up a direct linearisation scheme, which provides the solution structure for those equations in the elliptic case. The latter, which contains as main structural element a Cauchy kernel on the torus, is obtained from a dimensional reduction of the elliptic direct linearisation scheme of the lattice Kadomtsev–Petviashvili equation, which requires the introduction of a novel technical concept, namely the 'elliptic cube root of unity'. Thus, in order to implement the reduction we define, more generally, the notion of *elliptic N*th *root of unity*, and discuss some of its properties in connection with a special class of elliptic addition formulae. As a particular concrete application we present the class of elliptic multi-soliton solutions of the lattice Boussinesq systems.

Contents

1. Introduction

Many integrable equations, be it partial differential or difference equations, or autonomous ordinary differential equations or the difference equations describing integrable dynamical mappings, admit special solutions in terms of elliptic functions (or in terms of the associated quasi-periodic functions). A prototypical example is the Hirota equation

$$A\tau_{n+1,m,l}\tau_{n,m+1,l+1} + B\tau_{n,m+1,l}\tau_{n+1,m,l+1} + C\tau_{n,m,l+1}\tau_{n+1,m+1,l} = 0 \,, \tag{1.1a}$$

which is an integrable 3-dimensional lattice equation, i.e., a partial difference equation (PΔE), with arbitrary fixed coefficients A, B, C for the function $\tau_{n,m,l}$ of three discrete variables $n, m, l \in \mathbb{Z}$. This equation allows for elliptic (type) solutions in the form

$$\tau_{n,m,l} = \sigma(\xi_0 - n\delta - m\varepsilon - l\nu) \,,$$

provided the coefficients take the form:

$$A = \frac{\sigma(\varepsilon-\nu)}{\sigma(\varepsilon)\,\sigma(\nu)} \,, \quad B = \frac{\sigma(\nu-\delta)}{\sigma(\delta)\,\sigma(\nu)} \,, \quad C = \frac{\sigma(\delta-\varepsilon)}{\sigma(\delta)\,\sigma(\varepsilon)} \,, \tag{1.1b}$$

where $\sigma(x)$ is the Weierstrass σ-function, and where δ, ε and ν are fixed parameters, called lattice parameters, while ξ_0 is an initial value. The fact that the Weierstrass σ function solves the Hirota bilinear equation is a direct consequence of the three-term addition formula for the latter function, (cf. Eq. (A.6) of Appendix A). Typically, this is not the only elliptic type solution of (1.1a), but a special solution from which infinite families of such solutions can be constructed involving determinants with elliptic entries, cf. e.g. [39]. In the continuous case of integrable PDEs, e.g. Korteweg–de Vries (KdV) or Boussinesq (BSQ) type equations, such elliptic solutions exist as well, e.g. the well-known cnoidal wave solution (in terms of the Jacobi cn-function) of the KdV equation, or in the case of ODEs the Lamé-Baker function solution of the Lamé equation.

In recent years the theory of integrable difference equations has grown into a substantive body of new insights, cf. e.g. [12]. In this context, certain classes of integrable partial difference equations (otherwise known as *lattice equations*) have gained prominence, e.g. the class of quadrilateral lattice equations integrable through the so-called multidimensional consistency (MDC) property. The scalar affine-linear family of such equations was classified by Adler, Bobenko and Suris (ABS) in [2] and subsequently their elliptic solutions were established in [5,26]. Whereas the ABS equations are for single-component functions, elliptic solutions for higher rank equations, such as the lattice equations in the Gel'fand-Dikii (GD) hierarchy of [29] remained outstanding. A particular case of such equations is the (rank 3) lattice Boussinesq (BSQ) system, [29], which reads:

$$\begin{aligned}\frac{P-Q}{u_{n+1,m+1}-u_{n+2,m}} &- \frac{P-Q}{u_{n,m+2}-u_{n+1,m+1}} \\ &= (u_{n+1,m+2}-u_{n+2,m+1})(u_{n,m+1}-u_{n+2,m+2}) \\ &\quad -(u_{n,m+1}-u_{n+1,m})(u_{n,m}-u_{n+2,m+1}) \,,\end{aligned} \tag{1.2}$$

with P and Q being lattice parameters, and which can be visualised as a relation between the values of the solution function on the vertices on a 9-point stencil, or can be reformulated as a 2-component quadrilateral lattice system. The latter type of systems have yet to be classified, and our knowledge about both the structure of the equations (e.g. in terms of the polynomial structure of the function defining the equation, which is no longer affine-linear, as in the case of [2]) as well as the solution structure is as yet incomplete. In [29], cf. also [24,25,37], Eq. (1.2) and its companion equations (lattice modified BSQ equation and lattice Schwarzian BSQ equation), and their extensions (in the sense of the unfolding of the dispersion curves, cf. [40]) were constructed, and the integrability aspects (Lax pair, reductions, Poisson structure) were established, on the basis of a framework called *direct linearisation* (DL). The latter framework goes back to early papers by Fokas et al., [9,33], and is based on singular linear integral equations over arbitrary contours in the complex space of the spectral parameter, and allows to formulate the solution of spectral problems and corresponding nonlinear evolution equations in a compact way. The DL idea was subsequently developed in [31] into a flexible, albeit formal, machinery involving infinite matrix structures, having the advantage that several (Miura-related) equations can be treated within one framework. Furthermore, it allows the treatment of discrete as well as continuous equations on one and the same footing and within one formalism. Furthermore, elliptic type solutions can be treated within the formalism by a suitable choice of elliptic Cauchy kernel. Although the DL approach deviates from the inverse scattering formalism in that it doesn't address the issue of initial value problems, it nonetheless covers large classes of solutions, including soliton solutions and solutions can be otherwise obtained through the inverse scattering technique. However, to obtain algebra-geometric solutions the DL would have to be extended in a nontrivial way.

In the present paper we perform a first step towards the latter goal, namely to set up the framework for elliptic type solutions of the lattice BSQ systems. The reason why the BSQ system is of particular interest, is that, in spite of the interest that has been raised about the scalar quad-lattice equations of the ABS list, [2], they are still rather special and the corresponding techniques to study their integrability give little insight into what happens with multicomponent systems or lattice systems of higher order. Thus, is of interest to study elliptic solutions of the BSQ system, as it is the first case in the lattice GD hierarchy beyond the scalar quad-lattice situation, and quite representative of the generic case. The elliptic solutions we are interested in include both 'elementary' elliptic solutions, as well as multi-solitonic towers of elliptic solutions and in principle also the inverse scattering type solutions based on elliptic asymptotic behaviour. Thus, the paper extends to the higher rank case the results obtained in [26] for the ABS list (excluding the case of the so-called Q4 equation, which was covered in [5], but an analogue of which has not yet been established in the higher-rank case[1]). The strategy in the present paper, to set up the DL scheme is to realise the structure as a reduction of the one for the lattice Kadomtsev–Petviashvili (KP) system, whose elliptic type solutions were constructed in [39]. However, a technical problem in the BSQ case (which is one instance demonstrating that the quad-lattice is not representative of the generic situation) is the need for a new concept in the reduction. In fact, in the rational case, the reduction from KP to BSQ (or to other systems in the GD family) involves cube- or higher roots of unity. In order to perform the reduction from KP to such a system in the elliptic case, it turns out that an elliptic analogue of the roots of unity are needed. Such objects were considered only in the case $N = 3$ in [36] without explicitly referring to them as elliptic analogues of cube roots. Here, we introduce such objects for general N relying on higher-order elliptic identities which are given in Appendix B, (cf. also [7]). With the help of these elliptic roots of unity the reduction from the KP system to the BSQ and the higher-rank systems can be implemented via conditions on combinations of lattice shifts. Thus, from the DL structure for the lattice KP system we readily obtain the solution structure of the lattice GD systems, [29], within the DL framework.

[1] We note that in [8] a higher rank elliptic system was proposed generalising the 3-leg form of Q4, but it is not clear yet what is the rational form of that system.

The organisation of the paper is as follows. In Sect. 2 we develop the DL scheme for elliptic type solutions of the lattice KP class with the aim to make dimensional reductions to the KdV and BSQ cases. For the sake of the latter reduction we need an elliptic analogue of the cube root of unity. Thus, we introduce in Sect. 3 the notion of elliptic Nth root of unity, and examine some of its properties. In Sect. 4 we employ this novel concept to set up the DL system for the elliptic type solutions for the BSQ lattice equations, while in Sect. 5 we derive the actual nonlinear equations in closed form. In Sect. 6 we present explicit examples of elliptic solutions, namely elliptic seed and elliptic multi-soliton solutions, for the equations in their 'standard form'. Some conclusions follow in Sect. 7. For completeness we have given the elliptic parametrisation for the Lax pairs in Appendix D , although we do not need them for the purpose of presenting the elliptic solutions in this paper.

Warning Some standard facts on elliptic functions are introduced in Appendix A and we make heavy use of the various addition formulae throughout the paper, in particular some higher order addition formulae as given in Appendix C. The equations as they emerge from the structure are in our opinion best represented using suggestive notations, distinguishing variables from lattice parameters, but in the elliptic case both variables and parameters are defined through elliptic functions. Thus, as the story unfolds, we find it convenient to introduce several layers of somewhat ad-hoc notation in order to write the equations and the relations between them in what we feel is the most lucid way.

2. Elliptic DL Scheme for the Lattice KP Equations

In this section we develop the DL structure for elliptic type solutions of KP lattice equations. The structure is based on a unifying framework of formal linear integral equations, from which the various lattice KP equations can be derived. The set-up is that we start from a large solution class in terms of which a multitude of functions on the lattice can be defined, each of which solves an equation (or rather a parameter-family of equations) in the KP class. Thus, the DL structure not only provides solutions of the individual KP equations, but also supplies us with the (Miura/Bäcklund) relations between the various equations.

2.1. DL framework. In an early paper [27] we derived a system of integrable lattice equations (i.e. partial difference equations) associated with the lattice KP equation from a system of linear integral equations. In the present paper we generalise that integral equation to the elliptic case. The essential difference with the rational case is that the rational Cauchy kernel $(k+l')^{-1}$ in the integral equation of [27] is replaced by an elliptic kernel of the form $\Phi_\xi(\kappa+\ell')$ where Φ is the Lamé type function

$$\Phi_\xi(x) := \frac{\sigma(x+\xi)}{\sigma(x)\sigma(\xi)}\,, \tag{2.1}$$

in which $\sigma(x)$ is the Weierstrass sigma-function, [4]. This function has a number of remarkable properties, notably an elliptic analogue of the partial fraction expansion formula, namely Eq. (A.7) of Appendix A. Further properties of this and associated functions are given in Appendix A. An important difference between the elliptic and rational case is the introduction of the auxiliary complex variable ξ which plays the role of a background variable in all the formulae. In fact, when we characterise solutions from this scheme as being 'elliptic', it means that they are elliptic or quasi-elliptic functions of this variable ξ. (Note that in what follows this variable is so far free and undetermined, but will be later specified as being *non-autonomous*, i.e., dependent on the discrete independent variables of the system of partial difference equations that we will derive.)

We will now define the DL framework by writing down a set of linear integral equations in which the elliptic Cauchy kernel given above is the main ingredient. They are linear integral equations for a vector-valued function $\boldsymbol{u}_\kappa$, depending on a complex valued spectral parameter κ, of the following form[2]

$$\boldsymbol{u}_\kappa + \rho_\kappa \iint_D d\mu(\ell,\ell')\sigma_{\ell'}\boldsymbol{u}_\ell \Phi_\xi(\kappa+\ell') = \rho_\kappa \Phi_\xi(\boldsymbol{\Lambda})\boldsymbol{c}_\kappa \ , \tag{2.2a}$$

and one for its adjoint vector ${}^t\boldsymbol{u}_{\kappa'}$, depending on another spectral variable κ', of the form

$${}^t\boldsymbol{u}_{\kappa'} + \sigma_{\kappa'} \iint_D d\mu(\ell,\ell')\rho_\ell\, {}^t\boldsymbol{u}_{\ell'} \Phi_\xi(\kappa'+\ell) = \sigma_{\kappa'}\, {}^t\boldsymbol{c}_{\kappa'} \Phi_\xi({}^t\boldsymbol{\Lambda}) \ . \tag{2.2b}$$

The explanation of the notations in Eq. (2.2) requires some space. To start with the integrations, in both Eqs. (2.2a) and (2.2b) they are the same, and involve a general measure $d\mu(\ell,\ell')$ and the integration domain is any suitably chosen subset $D \subset \mathbb{C}^2$ in the space of both spectral variables ℓ and ℓ'. At this point it is not necessary to specify the measure and integration domain, as we treat the integral equations on a purely formal level. We merely need to rely on some general assumptions (such as the assumption that for given measure and integration domain D the solution of the integral equation is unique). We note that the integral equation (2.2a), in the rational case where the kernel $\Phi_\xi(\kappa+\ell')$ is replaced by $(\kappa+\ell')^{-1}$, can be viewed as a generalisation of the singular integral equations appearing in the nonlocal Riemann-Hilbert (RH) problem or $\bar\partial$-problem appearing in the inverse scattering approaches to KP type equations, cf. e.g. [10, 16]. However, those approaches start from a given multidimensional spectral problem (usually the time-dependent Schrödinger equation) and aim at solving initial value problems for (2+1)-dimensional KP type equations. Here our starting point is the integral equation itself, viewed as a formal structure that we aim to explore.

This brings us to the inhomogeneous term on the right hand sides of Eq. (2.2). These contain the (infinite) column and row vectors $\Phi_\xi(\boldsymbol{\Lambda})\boldsymbol{c}_\kappa$ and ${}^t\boldsymbol{c}_{\kappa'}\Phi_\xi({}^t\boldsymbol{\Lambda})$, as well as the plane wave factors ρ_κ and $\sigma_{\kappa'}$ (where the σ in $\sigma_{\kappa'}$ should not be confused with the Weierstrass sigma-function) respectively which will need some introduction.

First, the plane wave factors ρ_κ and $\sigma_{\kappa'}$ determine the dependence of the solutions $\boldsymbol{u}_\kappa$ and ${}^t\boldsymbol{u}_{\kappa'}$ on the main dynamical variables n, m, l, which can be assumed to take values in the integers, (i.e. $n, m, l \in \mathbb{Z}$), but in fact only need to increase in value by integer steps. In fact, we only need to assume that there is a set of elementary shift operations T_δ which act on the plane wave factors as follows:

$$\rho_\kappa \ \rightarrow \ \widetilde{\rho}_\kappa = T_\delta \rho_\kappa = \Phi_\delta(\kappa)\rho_\kappa \ \ , \ \ \sigma_{\kappa'} \ \rightarrow \ \widetilde{\sigma}_{\kappa'} = T_\delta \sigma_{\kappa'} = (\Phi_\delta(-\kappa'))^{-1}\sigma_{\kappa'} \ , \tag{2.3a}$$

where the tilde is shorthand for the shift operation, and where the shift is characterised by a complex 'lattice' parameter δ. This lattice can be seen as the grid formed by multiple operations of the shift operator, and hence we can identify

$$\rho_\kappa(n) = T_\delta^n \rho_\kappa(0) \ , \quad \sigma_{\kappa'}(n) = T_\delta^n \sigma_{\kappa'}(0) \ .$$

[2] For clarity of notation we indicate the spectral parameter dependence by a suffix rather than as an argument of a function.

Changing the lattice parameter δ means adding another degree of shift freedom to the lattice, and effectively adding another independent variable to the system. Thus, in addition to (2.3), we can consider the simultaneous shift relations

$$\rho_\kappa \;\to\; \widehat{\rho}_\kappa = T_\varepsilon \rho_\kappa = \Phi_\varepsilon(\kappa)\rho_\kappa \;,\;\; \sigma_{\kappa'} \;\to\; \widehat{\sigma}_{\kappa'} = T_\varepsilon \sigma_{\kappa'} = (\Phi_\varepsilon(-\kappa'))^{-1}\sigma_{\kappa'} \;, \tag{2.3b}$$

$$\rho_\kappa \;\to\; \overline{\rho}_\kappa = T_\nu \rho_\kappa = \Phi_\nu(\kappa)\rho_\kappa \;,\;\; \sigma_{\kappa'} \;\to\; \overline{\sigma}_{\kappa'} = T_\nu \sigma_{\kappa'} = (\Phi_\nu(-\kappa'))^{-1}\sigma_{\kappa'} \;, \tag{2.3c}$$

characterised by lattice parameters ε and ν respectively (the hat and bar again being shorthand for the operation of the shift). Since these shifts act on ρ_κ and $\sigma_{\kappa'}$ by multiplicative factors, it is evident that these shift operations commute, i.e. $T_\delta(T_\varepsilon\rho_\kappa) = T_\varepsilon(T_\delta\rho_\kappa)$, etc. (and similarly for their action on $\sigma_{\kappa'}$) and hence we can consider the simultaneous iteration of these shifts, creating a three-dimensional lattice of multiple shifts labelled by discrete variables (n, m, l). Thus, we create plane wave factors $\rho_\kappa(n, m, l)$ and $\sigma_{\kappa'}(n, m, l)$ which by construction become functions of the three-dimensional lattice, i.e.

$$\rho_\kappa(n, m, l) = (\Phi_\delta(\kappa))^n(\Phi_\varepsilon(\kappa))^m(\Phi_\nu(\kappa))^l\rho_\kappa(0, 0, 0) \;, \tag{2.4a}$$

$$\sigma_{\kappa'}(n, m, l) = (\Phi_\delta(-\kappa'))^{-n}(\Phi_\varepsilon(-\kappa'))^{-m}(\Phi_\nu(-\kappa'))^{-l}\sigma_{\kappa'}(0, 0, 0) \;, \tag{2.4b}$$

relative to some initial values $\rho_\kappa(0, 0, 0)$, $\sigma_{\kappa'}(0, 0, 0)$. Thus, by construction we have

$$(T_\delta\rho_\kappa)(n, m, l) = \rho_\kappa(n + 1, m, l) \;, \quad (T_\delta\sigma_{\kappa'})(n, m, l) = \sigma_{\kappa'}(n + 1, m, l) \;,$$

and similarly for the other lattice shifts:

$$(T_\varepsilon\rho_\kappa)(n, m, l) = \rho_\kappa(n, m + 1, l) \;, \quad (T_\varepsilon\sigma_{\kappa'})(n, m, l) = \sigma_{\kappa'}(n, m + 1, l) \;,$$
$$(T_\nu\rho_\kappa)(n, m, l) = \rho_\kappa(n, m, l + 1) \;, \quad (T_\nu\sigma_{\kappa'})(n, m, l) = \sigma_{\kappa'}(n, m, l + 1) \;.$$

Remark. Although n, m, l will play the role of the main independent variables for the partial difference equations which we will derive in due course, they are just parameters as far as the solutions of the integral equations (2.2) is concerned, and hence we assume that the measure $d\mu(\ell, \ell')$ and integration domain D do not depend on these variables. Clearly, as a consequence of the dependence on the variables n, m, l of the plane wave factors, the solutions $\boldsymbol{u}_\kappa$ and ${}^t\boldsymbol{u}_{\kappa'}$ of the integral equations will acquire a dependence on these variables as well, and an aim of the DL approach is to unravel how that dependence will manifest itself in terms of shift relations for these objects.

Second, the (infinite) column and row vectors $\Phi_\xi(\boldsymbol{\Lambda})\boldsymbol{c}_\kappa$ and ${}^t\boldsymbol{c}_{\kappa'}\Phi_\xi({}^t\boldsymbol{\Lambda})$ require some explanation. It was a finding going back to [31] that the dependence of the solutions of the integral equations on (continuous or discrete) dynamical variables is best controlled by considering an infinite set of integral equations with inhomogeneous terms given by different powers of the spectral parameter. This can be encoded in terms of the infinite (column resp. row) vectors $\boldsymbol{c}_\kappa = (\kappa^i)_{i\in\mathbb{Z}}$ and ${}^t\boldsymbol{c}_{\kappa'} = (\kappa'^j)_{j\in\mathbb{Z}}$ of monomials in κ and κ' respectively, and the action of index-raising operators $\boldsymbol{\Lambda}$ and ${}^t\boldsymbol{\Lambda}$, acting from the left and right respectively, as follows

$$(\boldsymbol{\Lambda}\boldsymbol{c}_\kappa)_i = (\boldsymbol{c}_\kappa)_{i+1} = \kappa^{i+1} \;, \quad ({}^t\boldsymbol{c}_{\kappa'}\,{}^t\boldsymbol{\Lambda})_{i'} = ({}^t\boldsymbol{c}_{\kappa'})_{i'+1} = (\kappa')^{i'+1} \;. \tag{2.5}$$

In the rational case, from the integral equations like (2.2) one can then derive a set of fundamental shift relations in terms of these operators representing a set of infinite recurrence relations between the components of the vectors $\boldsymbol{u}_\kappa$ and ${}^t\boldsymbol{u}_{\kappa'}$, from which subsequently (by eliminating the operators $\boldsymbol{\Lambda}$ and ${}^t\boldsymbol{\Lambda}$ combining different lattice shifts) the partial difference equations are derived in a systematic way, cf. e.g. [27,29,30].

In the elliptic case, however, due to the appearance of the elliptic Cauchy kernel, it is convenient to dress the inhomogenous terms with the factors $\Phi_\xi(\boldsymbol{\Lambda})$ and $\Phi_\xi({}^t\boldsymbol{\Lambda})$, as is made evident from the analysis presented in Appendix B. This leads then to the notion of what one could call *elliptic matrices*: these are obtained by the formal substitutions of the index-raising operators in the arguments of the relevant elliptic functions, such as

$$\Phi_\xi(\boldsymbol{\Lambda}), \quad \Phi_\xi({}^t\boldsymbol{\Lambda}), \quad \wp(\boldsymbol{\Lambda}), \quad \wp({}^t\boldsymbol{\Lambda}), \quad \zeta(\boldsymbol{\Lambda}), \quad \zeta({}^t\boldsymbol{\Lambda}) ,$$

where $\wp(x)$ and $\zeta(x)$ are the well known Weierstrass functions, with properties summarised in Appendix A. The introduction of such formal operators is justified by thinking of them in terms of their Fourier type symbols, which is the way in which they appear inside the integrals when performing computations. It is tempting to think of these operators as elliptic analogues of raising/lowering operators, defined through the formal power series expansions of the Weierstrass σ, ζ and $\wp$ functions. Consequently, they are assumed to obey the same addition formulae as the usual elliptic functions (see Appendix A), but in terms of arguments containing the operators $\boldsymbol{\Lambda}$ and ${}^t\boldsymbol{\Lambda}$. In terms of these elliptic matrices it is now possible to set up a formal algebraic structure, based on the integral equations (2.2), which allows us to derive the relevant nonlinear lattice equations as well as the associated Lax pairs.

2.2. Formal structure. Having explained the notations in Sect. 2.1, we can now set up and exploit the structure encoded by means of the index raising operators $\boldsymbol{\Lambda}$ and ${}^t\boldsymbol{\Lambda}$, and reformulate the integral equations in a more formal way for the convenience of deriving the lattice systems hidden in the structure.

In fact, the formal DL scheme can be summarised as follows. It contains three types of ingredients:

– An infinite ($\mathbb{Z}\times\mathbb{Z}$) matrix $\boldsymbol{C}$ which we can take of the form

$$\boldsymbol{C} = \iint_D d\mu(\ell,\ell')\, \rho_\ell\sigma_{\ell'}\boldsymbol{c}_\ell\, {}^t\boldsymbol{c}_{\ell'} \ , \tag{2.6}$$

in which $\boldsymbol{c}_\ell$ and ${}^t\boldsymbol{c}_\ell$ are infinite vectors with components $(\boldsymbol{c}_\ell)_j = ({}^t\boldsymbol{c}_\ell)_j = \ell^j$, and ρ_ℓ depends on additional variables that are to be determined later. The integrations over region D and measure $d\mu$ need not be specified at this point but we will loosely assume that they can be chosen such that the objects to be introduced below are well-defined.

– The matrices $\boldsymbol{\Lambda}$ and ${}^t\boldsymbol{\Lambda}$, as explained above, define the operations of index-raising when multiplied at the left respectively the right, acting on $\boldsymbol{c}_\kappa$ respectively ${}^t\boldsymbol{c}_{\kappa'}$ by $\boldsymbol{\Lambda}\boldsymbol{c}_\kappa = \kappa\boldsymbol{c}_\kappa$, resp. ${}^t\boldsymbol{c}_{\kappa'}\,{}^t\boldsymbol{\Lambda} = \kappa'\,{}^t\boldsymbol{c}_{\kappa'}$. Hence, these are right and left eigenvectors of the index raising operators.

– A formal *elliptic Cauchy kernel*, i.e., an infinite matrix $\boldsymbol{\Omega}_\xi$ obeying the equations:

$$\boldsymbol{\Omega}_\xi\Phi_\gamma(\boldsymbol{\Lambda}) - \Phi_\gamma(-{}^t\boldsymbol{\Lambda})\boldsymbol{\Omega}_{\xi+\gamma} = \Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\gamma}(\boldsymbol{\Lambda}) \ , \tag{2.7}$$

in which $\boldsymbol{O}$ is the projection matrix on the central element, i.e. $(\boldsymbol{O}\boldsymbol{C})_{i,j} = \delta_{i,0}\boldsymbol{C}_{0,j}$, etc.

As a consequence of the shift relations (2.3) of the plane wave factors ρ_k and $\sigma_{\kappa'}$ under translations along the lattice, we have the following linear relations for $\boldsymbol{C}$:

$$\widetilde{\boldsymbol{C}}\Phi_\delta(-{}^t\boldsymbol{\Lambda}) = \Phi_\delta(\boldsymbol{\Lambda})\boldsymbol{C}\ , \tag{2.8a}$$
$$\widehat{\boldsymbol{C}}\Phi_\varepsilon(-{}^t\boldsymbol{\Lambda}) = \Phi_\varepsilon(\boldsymbol{\Lambda})\boldsymbol{C}\ , \tag{2.8b}$$
$$\overline{\boldsymbol{C}}\Phi_\nu(-{}^t\boldsymbol{\Lambda}) = \Phi_\nu(\boldsymbol{\Lambda})\boldsymbol{C}\ , \tag{2.8c}$$

for the various lattice directions associated with the lattice parameters $p = \zeta(\delta)$, $q = \zeta(\varepsilon)$, resp. $r = \zeta(\nu)$. In principle, since these relations are linear, we can select an arbitrary set of lattice parameters p, q or r (or equivalently δ, ε and ν) each associated with its own lattice shift, and in view of the linearity of the corresponding equations for $\boldsymbol{C}$ we can impose all the discrete evolutions for all chosen values of these lattice parameters simultaneously.

The main objects of interest, however, are the ones obeying nonlinear equations, and these objects are the following:

– The infinite matrix

$$\boldsymbol{U}_\xi := \Phi_\xi(\boldsymbol{\Lambda})\boldsymbol{C}\left(\boldsymbol{1} + \boldsymbol{\Omega}_\xi\boldsymbol{C}\right)^{-1}\Phi_\xi({}^t\boldsymbol{\Lambda})\ . \tag{2.9}$$

– The τ-function given by the infinite determinant[3]

$$\tau_\xi := \det{}_{\mathbb{Z}\times\mathbb{Z}}\left(\boldsymbol{1} + \boldsymbol{\Omega}_\xi\cdot\boldsymbol{C}\right)\ . \tag{2.10}$$

– The ‘wave vectors’ $\boldsymbol{u}_\kappa$ and ${}^t\boldsymbol{u}_{\kappa'}$ defined by

$$\boldsymbol{u}_\kappa(\xi) = \left(\Phi_\xi(\boldsymbol{\Lambda}) - \boldsymbol{U}_\xi\Phi_\xi^{-1}({}^t\boldsymbol{\Lambda})\boldsymbol{\Omega}_\xi\right)\boldsymbol{c}_\kappa\rho_\kappa\ , \tag{2.11a}$$
$${}^t\boldsymbol{u}_{\kappa'}(\xi) = \sigma_{\kappa'}\,{}^t\boldsymbol{c}_{\kappa'}\left(\Phi_\xi({}^t\boldsymbol{\Lambda}) - \boldsymbol{\Omega}_\xi\Phi_\xi^{-1}(\boldsymbol{\Lambda})\boldsymbol{U}_\xi\right)\ , \tag{2.11b}$$

which can be identified with the solutions of the integral equations (2.2a) resp. (2.2b).

We note that the infinite-component matrix $\boldsymbol{U}_\xi$ (2.9), with entries $u_{i,j}(\xi)$, $(i, j \in \mathbb{Z})$, which are the main quantities of interest in what follows, also has the integral representation

$$\boldsymbol{U}_\xi\ :=\ \iint_D d\mu(\ell,\ell')\,\boldsymbol{u}_\ell(\xi)\,{}^t\boldsymbol{c}_{\ell'}\sigma_{\ell'}\Phi_\xi({}^t\boldsymbol{\Lambda})\ . \tag{2.12}$$

(For notational convenience, here and elsewhere we indicate the dependence on the variable ξ by a suffix, rather than writing it as an argument, e.g. rather than writing $\boldsymbol{U}(\xi)$.)

Let us now present the basic equations resulting from this scheme. From the above ingredients it is elementary to derive the basic relations describing the behaviour of τ_ξ,

[3] To make sense of the infinite determinant in (2.10) we can use the expansion formula

$$\det(\boldsymbol{1}+A) = 1 + \sum_i A_{ii} + \sum_{i<j}\begin{vmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj}\end{vmatrix} + \dots\ ,$$

which is valid if $A = \boldsymbol{\Omega}_\xi\cdot\boldsymbol{C}$ is of finite rank (which is the case for the solutions given in Sect. 6). In general the requirement of these factors to be of finite rank imposes some conditions on the integrations in (2.6), in view of the fact that all terms in the expansion are of the form $\mathrm{tr}_{\mathbb{Z}}\left((\boldsymbol{\Omega}_\xi\cdot\boldsymbol{C})^n\right)$ for some integer power n, and where $\mathrm{tr}_{\mathbb{Z}}$ means and infinite trace over all integer-labeled diagonal entries.

$\boldsymbol{U}_\xi$ and $\boldsymbol{u}_\kappa$ and ${}^t\boldsymbol{u}_{\kappa'}$ under lattice translations. Thus, for the τ-function we have (see Appendix B for the derivation):

$$\frac{\widetilde{\tau}_\xi}{\tau_{\xi+\delta}} = 1 - \left(\boldsymbol{U}_{\xi+\delta}\left[\zeta(\xi+\delta)\mathbf{1} - \zeta(\delta)\mathbf{1} + \zeta({}^t\boldsymbol{\Lambda}) - \zeta(\xi + {}^t\boldsymbol{\Lambda})\right]^{-1}\right)_{0,0} , \tag{2.13}$$

and similarly for the other lattice directions. Subsequently we will adopt the shorthand notation where the infinite unit matrix symbol $\mathbf{1}$ will be omitted when it multiplies a scalar function.

For $\boldsymbol{U}_\xi$ we can derive the discrete matrix Riccati type of relations

$$\begin{aligned}&-\widetilde{\boldsymbol{U}}_\xi\left[\zeta(\xi+\delta) - \zeta(\delta) + \zeta({}^t\boldsymbol{\Lambda}) - \zeta(\xi + {}^t\boldsymbol{\Lambda})\right]\\ &\quad = \left[\zeta(\xi) + \zeta(\delta) + \zeta(\boldsymbol{\Lambda}) - \zeta(\xi+\delta+\boldsymbol{\Lambda}) - \widetilde{\boldsymbol{U}}_\xi\boldsymbol{O}\right]\boldsymbol{U}_{\xi+\delta} .\end{aligned} \tag{2.14}$$

Equation (2.14) together with its counterparts for the other lattice directions forms the starting point for the construction of a number of integrable three-dimensional lattice equations. By combining the different lattice translations associated with the different lattice parameters δ, ε, ν one can actually derive all relevant discrete equations within the KP family, as we shall show in the next section. Let us finish here by giving the linear equations for $\boldsymbol{u}_\kappa$ and ${}^t\boldsymbol{u}_{\kappa'}$ that form the basis for the derivation of the Lax pairs for the above-mentioned equations derived from (2.14). These relations read

$$\widetilde{\boldsymbol{u}}_\kappa(\xi) = \left[\zeta(\xi) + \zeta(\delta) + \zeta(\boldsymbol{\Lambda}) - \zeta(\xi+\delta+\boldsymbol{\Lambda}) - \widetilde{\boldsymbol{U}}_\xi\boldsymbol{O}\right]\boldsymbol{u}_\kappa(\xi+\delta) , \tag{2.15a}$$

$${}^t\boldsymbol{u}_{\kappa'}(\xi+\delta) = -{}^t\widetilde{\boldsymbol{u}}_{\kappa'}(\xi)\left[\zeta(\xi+\delta) - \zeta(\delta) + \zeta({}^t\boldsymbol{\Lambda}) - \zeta(\xi + {}^t\boldsymbol{\Lambda}) - \boldsymbol{O}\boldsymbol{U}_{\xi+\delta}\right] . \tag{2.15b}$$

Equations (2.13), (2.14) and (2.15), the derivation of which is given in Appendix B, form the starting point for the subsequent derivation of nonlinear equations in closed form. For this purpose we have to specify special elements (or combinations of elements) of the infinite matrix $\boldsymbol{U}_\xi$ and to combine the different lattice shifts in such a way that closed-form equations for such these preferred elements are obtained. This we will do in the next subsection.

2.3. Lattice KP systems from elliptic DL. We will now derive closed-form difference equations for special quantities defined in terms of the entries of the matrix $\boldsymbol{U}_\xi$. First, combining the three relations of the form (2.14) for the three lattice directions and their shifted counterparts we can eliminate all the elliptic index raising matrices $\zeta(\boldsymbol{\Lambda})$ and $\zeta({}^t\boldsymbol{\Lambda})$ and by taking the $(0, 0)$ entry of the matrix, i.e. setting $u(\xi) := (\boldsymbol{U}_\xi)_{0,0}$, we get a closed form relation for the function $u(\xi)$ which reads

$$\begin{aligned}&[\zeta(\delta) - \zeta(\varepsilon) + \zeta(\xi-\delta) - \zeta(\xi-\varepsilon)]\,\widehat{\widetilde{u}}(\xi-\delta-\varepsilon)\\ &\quad + [\zeta(\nu) - \zeta(\delta) + \zeta(\xi-\varepsilon-\nu) - \zeta(\xi-\delta-\varepsilon)]\,\widehat{u}(\xi-\varepsilon)\\ &\quad + \left(\widehat{\widetilde{u}}(\xi-\delta-\varepsilon) - \widehat{\overline{u}}(\xi-\varepsilon-\nu)\right)\widehat{u}(\xi-\varepsilon) \ + \text{ cycl.} \ = 0 ,\end{aligned} \tag{2.16}$$

in which '+ cycl.' means the addition of similar terms obtained by cyclic permutation of the action of the shifts $u \mapsto \widetilde{u}, u \mapsto \widehat{u}, u \mapsto \overline{u}$ together with corresponding replacements of the three parameters δ, ε and ν respectively. Below we will identify (2.16) with the lattice potential KP equation of [27].

Further equations can be derived from (2.14) and its counterpart in the other lattice directions for the parameter-dependent quantities

$$v_\alpha(\xi) := 1 - \Big([\zeta(\xi) + \zeta(\alpha) + \zeta(\boldsymbol{\Lambda}) - \zeta(\xi + \alpha + \boldsymbol{\Lambda})]^{-1}\, \boldsymbol{U}_\xi \Big)_{0,0} \,, \tag{2.17a}$$

$$w_\alpha(\xi) := 1 - \Big(\boldsymbol{U}_\xi \left[\zeta(\xi) + \zeta(\alpha) + \zeta({}^t\boldsymbol{\Lambda}) - \zeta(\xi + \alpha + {}^t\boldsymbol{\Lambda})\right]^{-1} \Big)_{0,0} \,, \tag{2.17b}$$

for which from (2.14) we immediately have

$$\widetilde{v}_\delta(\xi - \delta) w_{-\delta}(\xi) = 1 \,, \tag{2.18}$$

which are motivated by the τ-function relation (2.13) replacing the lattice parameter δ by an arbitrary fixed parameter α. By eliminating the terms with $\zeta({}^t\boldsymbol{\Lambda})$ in (2.14) and its counterpart with lattice parameter ε we can derive, for any fixed parameter α, the following relation

$$\begin{aligned}
&\zeta(\delta) - \zeta(\varepsilon) + \zeta(\xi - \delta) - \zeta(\xi - \varepsilon) + \widehat{u}(\xi - \varepsilon) - \widetilde{u}(\xi - \delta) \\
&\quad = [\zeta(\delta) - \zeta(\xi - \varepsilon) - \zeta(\alpha) + \zeta(\xi + \alpha - \delta - \varepsilon)]\, \frac{\widehat{v}_\alpha(\xi - \varepsilon)}{\widehat{\widetilde{v}}_\alpha(\xi - \delta - \varepsilon)} \\
&\qquad - [\zeta(\varepsilon) - \zeta(\xi - \delta) - \zeta(\alpha) + \zeta(\xi + \alpha - \delta - \varepsilon)]\, \frac{\widetilde{v}_\alpha(\xi - \delta)}{\widehat{\widetilde{v}}_\alpha(\xi - \delta - \varepsilon)} \,,
\end{aligned} \tag{2.19}$$

relating the lattice potential KP variable $u(\xi)$ to the variable $v_\alpha(\xi)$. To derive Eq. (2.19) use has been made of a special elliptic relation, namely Eq. (A.13) given in Appendix A. Using Eq. (2.19) for each pair of three lattice directions we can eliminate either $v_\alpha(\xi)$, in which case we recover the lattice equation (2.16) for $u(\xi)$, or we eliminate $u(\xi)$ and obtain a lattice equation for $v_\alpha(\xi)$, namely

$$\begin{aligned}
&[\zeta(\delta) - \zeta(\xi - \varepsilon) - \zeta(\alpha) + \zeta(\xi + \alpha - \delta - \varepsilon)]\, \frac{\widehat{v}_\alpha(\xi - \varepsilon)}{\widehat{\widetilde{v}}_\alpha(\xi - \delta - \varepsilon)} \\
&\quad - [\zeta(\varepsilon) - \zeta(\xi - \delta) - \zeta(\alpha) + \zeta(\xi + \alpha - \delta - \varepsilon)]\, \frac{\widetilde{v}_\alpha(\xi - \delta)}{\widehat{\widetilde{v}}_\alpha(\xi - \delta - \varepsilon)} \; + \; \text{cycl.} \; = 0 \,.
\end{aligned} \tag{2.20}$$

This equation can be identified with the lattice (potential) modified KP (mKP) equation, cf [27].

By setting α equal to either one of the three parameters δ, ε or ν, (2.20) reduces to a four-term equation and using the identifications (which follow from (2.13))

$$\widetilde{v}_\delta(\xi - \delta) = \frac{\tau_\xi}{\widetilde{\tau}_{\xi-\delta}} \quad \Leftrightarrow \quad w_{-\delta}(\xi) = \frac{\widetilde{\tau}_{\xi-\delta}}{\tau_\xi} \,, \tag{2.21}$$

and similarly for the other lattice directions, we then obtain the Hirota bilinear KP equation in the following form

$$\begin{aligned}
&[\zeta(\delta) - \zeta(\varepsilon) + \zeta(\xi - \delta) - \zeta(\xi - \varepsilon)]\, \widehat{\widetilde{\tau}}_{\xi-\delta-\varepsilon} \overline{\tau}_{\xi-\nu} \\
&\quad + [\zeta(\varepsilon) - \zeta(\nu) + \zeta(\xi - \varepsilon) - \zeta(\xi - \nu)]\, \widehat{\overline{\tau}}_{\xi-\varepsilon-\nu} \widetilde{\tau}_{\xi-\delta} \\
&\quad + [\zeta(\nu) - \zeta(\delta) + \zeta(\xi - \nu) - \zeta(\xi - \delta)]\, \widetilde{\overline{\tau}}_{\xi-\delta-\nu} \widehat{\tau}_{\xi-\varepsilon} \; = 0 \,.
\end{aligned} \tag{2.22}$$

The latter coincides with (1.1a) with the coefficients given by (1.1b) using the relation (A.4) of Appendix A.

Another object of interest within the scheme is the following

$$s_{\alpha,\beta}(\xi) := \Big([\zeta(\xi)+\zeta(\alpha)+\zeta(\boldsymbol{\Lambda})-\zeta(\xi+\alpha+\boldsymbol{\Lambda})]^{-1}\cdot \boldsymbol{U}_\xi \cdot \left[\zeta(\xi)+\zeta(\beta)+\zeta({}^t\boldsymbol{\Lambda})-\zeta(\xi+\beta+{}^t\boldsymbol{\Lambda})\right]^{-1}\Big)_{0,0} , \tag{2.23}$$

for which we can derive from (2.14) the following important identity

$$\widetilde{v}_\alpha(\xi)\ w_\beta(\xi+\delta) = 1-[\zeta(\xi+\delta)+\zeta(\alpha)-\zeta(\delta)-\zeta(\xi+\alpha)]\, s_{\alpha,\beta}(\xi+\delta) - [\zeta(\xi)+\zeta(\beta)+\zeta(\delta)-\zeta(\xi+\delta+\beta)]\,\widetilde{s}_{\alpha,\beta}(\xi) , \tag{2.24}$$

and similar relations for the other lattice shifts $\widehat{\ }$ and $\bar{\ }$ involving the parameters ε and ν instead of δ. In the derivation of (2.24) use has been made again of the special relation (A.13) of Appendix A. Eliminating the variables v_α and w_β by combining three lattice shifts, we arrive at a closed-form equation for $s_{\alpha,\beta}$ which reads

$$\frac{1-\chi^{(1)}_{\alpha,-\delta}(\xi-\nu)\overline{s}_{\alpha,\beta}(\xi-\nu)-\chi^{(1)}_{\beta,\delta}(\xi-\delta-\nu)\widetilde{\overline{s}}_{\alpha,\beta}(\xi-\delta-\nu)}{1-\chi^{(1)}_{\alpha,-\varepsilon}(\xi-\nu)\overline{s}_{\alpha,\beta}(\xi-\nu)-\chi^{(1)}_{\beta,\varepsilon}(\xi-\varepsilon-\nu)\widehat{\overline{s}}_{\alpha,\beta}(\xi-\varepsilon-\nu)}$$
$$=\frac{1-\chi^{(1)}_{\alpha,-\delta}(\xi-\varepsilon)\widehat{s}_{\alpha,\beta}(\xi-\varepsilon)-\chi^{(1)}_{\beta,\delta}(\xi-\delta-\varepsilon)\widehat{\widetilde{s}}_{\alpha,\beta}(\xi-\delta-\varepsilon)}{1-\chi^{(1)}_{\alpha,-\nu}(\xi-\varepsilon)\widehat{s}_{\alpha,\beta}(\xi-\varepsilon)-\chi^{(1)}_{\beta,\nu}(\xi-\varepsilon-\nu)\widehat{\overline{s}}_{\alpha,\beta}(\xi-\varepsilon-\nu)}$$
$$\times\frac{1-\chi^{(1)}_{\alpha,-\nu}(\xi-\delta)\widetilde{s}_{\alpha,\beta}(\xi-\delta)-\chi^{(1)}_{\beta,\nu}(\xi-\delta-\nu)\widetilde{\overline{s}}_{\alpha,\beta}(\xi-\delta-\nu)}{1-\chi^{(1)}_{\alpha,-\varepsilon}(\xi-\delta)\widetilde{s}_{\alpha,\beta}(\xi-\delta)-\chi^{(1)}_{\beta,\varepsilon}(\xi-\delta-\varepsilon)\widehat{\widetilde{s}}_{\alpha,\beta}(\xi-\delta-\varepsilon)} , \tag{2.25}$$

in which we have used the abbreviation

$$\chi^{(1)}_{\alpha,\beta}(x) := \zeta(\alpha)+\zeta(\beta)+\zeta(x)-\zeta(\alpha+\beta+x) = \frac{\Phi_\alpha(x)\,\Phi_\beta(x)}{\Phi_{\alpha+\beta}(x)} . \tag{2.26}$$

For arbitrary ξ the Eqs. (2.16), (2.20) and (2.25) form a complicated system of non-autonomous difference equations for effectively a set of four-variable functions, depending on the discrete variables n, m, l and a continuous variable ξ, as partial delay-difference equations. However, if we allow the variable ξ to depend on the discrete variables as follows:

$$\xi = \xi_0 - n\delta - m\varepsilon - l\nu , \tag{2.27}$$

where ξ_0 is fixed, i.e., the ξ becomes dynamical, then those equations are nothing else than the lattice KP, lattice modified KP and lattice Schwarzian KP equations, cf. [27], respectively, and can be transformed into their standard forms by an appropriate gauge transformation, thereby removing the elliptic coefficients. We will not do so here, as they are not the main objects of study in the present paper, but explicit soliton type solutions were given along that line in [39]. We now consider the problem of their dimensional reduction to equations in the GD hierarchy, which in the elliptic case, requires the notion of the elliptic Nth root of unity.

3. Elliptic *N*th Root of Unity

In the rational case, to perform the dimensional reduction from three-dimensional KP systems to two-dimensional lattice equations of GD type (including KdV or BSQ type equations), cf. [11,29], we need to choose integration measures of the integral equations (2.2) where the spectral variable ℓ' is identified with $-\omega^j\ell$, where ω is a primitive Nth root of unity, and j $(0 < j < N)$ are integers coprime to N. However, this prescription no longer works, for $N > 2$, in the elliptic case, and in fact we need to revisit our notion of what is meant by the Nth root of unity to get a sensible reduction. The main problem is that the multiplication by roots of unity *within the arguments of the elliptic functions* is not a natural operation and doesn't work through naturally in the formulae for the reduction.

For $N = 2$, however, the usual square root of unity '-1' can still be implemented within the arguments of the elliptic functions without a problem, but we need to reexamine what we actually mean by this elliptic root. In this context, we redefine it as follows: The elliptic square roots of unity are given by the condition

$$\Phi_\kappa(\delta)\Phi_\kappa(\omega(\delta)) = \wp(\kappa) - \wp(\delta)\ , \quad \forall\,\kappa\ , \tag{3.1}$$

where $\omega(\delta)$ is required to be independent of κ. The condition (3.1) obviously has the simple solution $\omega(\delta) = -\delta$, which coincides with what one has in the usual case of a square root of unity. Note that in this case the elliptic square root $-\delta$ can be 'normalised' by dividing by δ, and hence we can extract a notion of square root (namely -1) which is independent of the parameter δ. However, this is no longer the case when $N > 2$, and in those cases the elliptic roots will essentially depend on a parameter.

Thus already for $N = 3$ we see an essential change of this notion, when we define the elliptic cube root of unity as follows:

Definition 3.1. We call the parameter-dependent quantities $\omega_j(\delta)$ $(j = 0, 1, 2)$, where $\omega_0(\delta) = \delta$, the elliptic cube roots of unity, if they obey the following relation

$$\Phi_\kappa(\delta)\Phi_\kappa(\omega_1(\delta))\Phi_\kappa(\omega_2(\delta)) = -\tfrac{1}{2}\left(\wp'(\kappa) + \wp'(\delta)\right)\ , \quad \forall\,\kappa\ , \tag{3.2}$$

(for all κ in a fundamental domain of the argument of the elliptic functions) subject to the condition that the $\omega_j(\delta)$, $j = 0, 1, 2$, are independent of κ.

Assuming the existence of these elliptic cube roots,[4] which should not be confused with the half periods of the elliptic functions as in Appendix A, and which are determined modulo the period lattice, the following assertions can be made about the elliptic cube roots.

Lemma 3.1. *The condition* (3.2) *is equivalent to the following set of relations:*

$$\delta + \omega_1(\delta) + \omega_2(\delta) \equiv 0 (\textit{mod period lattice})\ , \tag{3.3a}$$

$$\zeta(\delta) + \zeta(\omega_1(\delta)) + \zeta(\omega_2(\delta)) = 0\ , \tag{3.3b}$$

$$\wp'(\delta) = \wp'(\omega_1(\delta)) = \wp'(\omega_2(\delta))\ . \tag{3.3c}$$

Furthermore, we also have

$$\wp(\delta) + \wp(\omega_1(\delta)) + \wp(\omega_2(\delta)) = 0\ . \tag{3.3d}$$

[4] In [36] these quantities were already mentioned (in Remark 7, on p. 1037 of that paper), but without pushing the notion of 'elliptic cube roots'. Here we argue that, more generally, a notion of 'elliptic Nth root of unity' makes perfect sense.

Proof. Using (A.5) of Appendix A, we have that the left-hand side of (3.2) can be written as

$$\begin{aligned}&\Phi_\kappa(\delta)\Phi_\kappa(\omega_1(\delta))\Phi_\kappa(\omega_2(\delta))\\&\quad=\Phi_\kappa(\delta)\Phi_\kappa(\omega_1(\delta)+\omega_2(\delta))\left[\zeta(\kappa)+\zeta(\omega_1(\delta))+\zeta(\omega_2(\delta))-\zeta(\kappa+\omega_1(\delta)+\omega_2(\delta))\right]\ ,\end{aligned}$$

whereas the right-hand side of (3.2), using (A.8) and (A.9), can be written as

$$\begin{aligned}-\tfrac{1}{2}\left(\wp'(\kappa)+\wp'(\delta)\right)&=-\left(\wp(\kappa)-\wp(\delta)\right)\,\tfrac{1}{2}\frac{\wp'(\kappa)+\wp'(\delta)}{\wp(\kappa)-\wp(\delta)}\\&=\Phi_\kappa(\delta)\Phi_\kappa(-\delta)\left[\zeta(\kappa)-\zeta(\delta)-\zeta(\kappa-\delta)\right].\end{aligned}$$

Since, by definition, the left-hand side equals the right-hand side for all κ in a fundamental domain of the period lattice, we must have that $\omega_1(\delta)+\omega_2(\delta)\equiv-\delta(\text{mod period lattice})$, i.e. Eq. (3.3a) must hold, and furthermore we have that (3.3b) is satisfied. Then from the latter we have from (A.9) that (3.3c) holds. Finally, from (A.10a) we subsequently have (3.3d). Conversely, from (3.3a) and (3.3b), using (A.5) we have

$$\begin{aligned}&\Phi_\kappa(\delta)\Phi_\kappa(\omega_1(\delta))\Phi_\kappa(\omega_2(\delta))\\&\quad=\Phi_\kappa(\delta)\Phi_\kappa(-\delta)\left[\zeta(\kappa)+\zeta(\omega_1(\delta))+\zeta(\omega_2(\delta))-\zeta(\kappa-\delta)\right]\\&\quad=\Phi_\kappa(\delta)\Phi_\kappa(-\delta)\left[\zeta(\kappa)-\zeta(\delta)-\zeta(\kappa-\delta)\right]\\&\quad=-\left(\wp(\kappa)-\wp(\delta)\right)\,\tfrac{1}{2}\frac{\wp'(\kappa)+\wp'(\delta)}{\wp(\kappa)-\wp(\delta)}\\&\quad=-\tfrac{1}{2}\left(\wp'(\kappa)+\wp'(\delta)\right)\ ,\end{aligned}$$

which is (3.2) holding for all κ in a fundamental domain. □

Remark. From Lemma 3.1 it follows that we could have defined the elliptic root of unity by the relation (3.3c) subject to (3.3a). In fact, this point of view asserts the existence of the elliptic cube root of unity as follows:

Lemma 3.2. *There exist solutions $\omega_j(\delta)$ ($j=0,1,2$) of the relations* (3.3c) *subject to the relation* (3.3a)*.*

Proof. Since $\wp'$ is an elliptic function with a pole of order three at zero, the equation $\wp'(\omega)-\wp'(\delta)=0$, for any given δ, has, apart from the trivial solution $\omega\equiv\delta(\text{modperiodlattice})$, two independent solutions ω_1,ω_2 modulo the period lattice of the elliptic function, i.e. solutions such that $\wp(\omega_1)\neq\wp(\omega_2)\neq\wp(\delta)$. Furthermore, from the Frobenius-Stickelberger formula (A.11) it follows that

$$\begin{vmatrix}1&\wp(\delta)&\wp'(\delta)\\1&\wp(\omega_1)&\wp'(\omega_1)\\1&\wp(\omega_2)&\wp'(\omega_2)\end{vmatrix}=-2\frac{\sigma(\delta-\omega_1)\,\sigma(\delta-\omega_2)\,\sigma(\omega_1-\omega_2)}{\sigma^3(\delta)\,\sigma^3(\omega_1)\,\sigma^3(\omega_2)}\,\sigma(\delta+\omega_1+\omega_2)=0\ ,$$

since the determinant on the left-hand side vanishes if (3.3c) holds. Hence, these roots must satisfy (3.3a) since $\sigma(\delta+\omega_1+\omega_2)=0$ if $\omega_j=\omega_j(\delta)$, ($j=1,2$). □

Remark. The rational analogue of the relation (3.2) is the requirement that the third order polynomial of an indeterminate k

$$(k-p_0)(k-p_1)(k-p_2)=k^3-p^3\ ,$$

is a pure cubic in k, which leads to the solution that the roots p_0, p_1, p_2 are related through cube roots of unity: $p_j = \omega^j p$ ($j = 0, 1, 2$) with $\omega = \exp(2\pi \mathrm{i}/3)$ (up to permutations of the roots) and i is the imaginary unit. Thus, the solutions of (3.2) as the elliptic analogue of a pure cubic and hence we can think of $\omega_1(\delta)$ and $\omega_2(\delta)$ as generalisations of $e^{2\pi\mathrm{i}/3}\delta$ and $e^{4\pi\mathrm{i}/3}\delta$, but here they are defined by the implicit equation (3.2). (In fact, when the invariant of the elliptic curve $g_2 = 0$ the elliptic cube roots correspond exactly to these expressions in terms of the usual cube roots.) Note also that when ω_1 and ω_2 correspond to the half periods of the period lattice, i.e. when $\omega_1(\delta) = \omega$ and $\omega_2(\delta) = \omega'$, and $\delta = -\omega - \omega'$ then the above relations (3.3) are all automatically satisfied by means of elliptic identities, since $\wp'$ evaluated at the half periods vanishes: $\wp'(\omega) = \wp'(\omega') = 0$.

In the case of $N = 4$ the elliptic quartic root of unity can be defined using the identity (C.2) of Appendix C, noting that here we need higher-order elliptic identities.

Definition 3.2. We call the parameter-dependent quantities $\omega_j(\delta)$ ($j = 0, 1, 2, 3$), where $\omega_0(\delta) = \delta$, the elliptic quartic roots of unity, if they obey the following relation

$$\Phi_\kappa(\delta)\Phi_\kappa(\omega_1(\delta))\Phi_\kappa(\omega_2(\delta))\Phi_\kappa(\omega_3(\delta)) = \frac{1}{6}\left(\wp''(\kappa) - \wp''(\delta)\right) , \quad \forall\kappa , \tag{3.4}$$

(for all κ in a fundamental domain of the argument of the elliptic functions) subject to the condition that the $\omega_j(\delta)$, $j = 0, 1, 2, 3$, are independent of κ.

Once again the relation (3.4) gives rise to some properties of the elliptic quartic roots of unity, namely

Lemma 3.3. *The condition* (3.4) *is equivalent to the following set of relations:*

$$\delta + \omega_1(\delta) + \omega_2(\delta) + \omega_3(\delta) \equiv 0 (\textit{mod period lattice}) , \tag{3.5a}$$

$$\zeta(\delta) + \zeta(\omega_1(\delta)) + \zeta(\omega_2(\delta)) + \zeta(\omega_3(\delta)) = 0 , \tag{3.5b}$$

$$\wp(\delta) + \wp(\omega_1(\delta)) + \wp(\omega_2(\delta)) + \wp(\omega_3(\delta)) = 0 , \tag{3.5c}$$

$$\wp''(\delta) = \wp''(\omega_1(\delta)) = \wp''(\omega_2(\delta))) = \wp''(\omega_3(\delta)) . \tag{3.5d}$$

Furthermore, we also have

$$\wp'(\delta) + \wp'(\omega_1(\delta)) + \wp'(\omega_2(\delta)) + \wp'(\omega_3(\delta)) = 0 . \tag{3.6}$$

Proof. On the one hand, using the relation (C.2) of Appendix C the left-hand side of (3.4) can be written as

$$\begin{aligned}
&\Phi_\kappa(\delta)\Phi_\kappa(\omega_1(\delta))\Phi_\kappa(\omega_2(\delta))\Phi_\kappa(\omega_3(\delta)) \\
&\quad = \Phi_\kappa(\delta)\tfrac{1}{2}\Phi_\kappa(\omega_1+\omega_2+\omega_3)\Big[\big(\zeta(\kappa)+\zeta(\omega_1)+\zeta(\omega_2)+\zeta(\omega_3)-\zeta(\kappa+\omega_1+\omega_2+\omega_3)\big)^2 \\
&\qquad +\wp(\kappa)-\big(\wp(\omega_1)+\wp(\omega_2)+\wp(\omega_3)+\wp(\kappa+\omega_1+\omega_2+\omega_3)\big)\Big] ,
\end{aligned}$$

on the other hand, rewriting the right-hand side of (3.4), and using the relation $\wp'' = 6\wp^2 - g_2/2$, we have

$$\begin{aligned}
&\tfrac{1}{6}\left(\wp''(\kappa)-\wp''(\delta)\right) = \wp^2(\kappa)-\wp^2(\delta) = \tfrac{1}{2}\left(\wp(\kappa)-\wp(\delta)\right)\left[2\wp(\kappa)+2\wp(\delta)\right] \\
&\quad = \tfrac{1}{2}\Phi_\kappa(\delta)\Phi_\kappa(-\delta)\Big[\big(\zeta(\kappa)-\zeta(\delta)-\zeta(\kappa-\delta)\big)^2+\wp(\kappa)-\big(-\wp(\delta)+\wp(\kappa-\delta)\big)\Big] ,
\end{aligned}$$

which by comparing with the result of the left-hand side, given that κ is arbitrary, leads to the relations (3.5a)–(3.5c). Furthermore, (3.5d) follows from the substitution $\kappa = -\omega_j$ into (3.4). Finally, Eq. (3.6) follows from (C.7) of Appendix C, setting the arguments equal to the elliptic quartic roots of unity. ◻

As in the case of $N = 3$ we can use Eq. (3.5d) to define the quartic roots of unity, with the existence of the following an analogous argument as for the cube roots as in Lemma 3.2, while the condition (3.5a) follows from the 4×4 Frobenius-Stickelberger detereminant (A.12), setting the arguments of the elliptic functions equal to the four elliptic quartic roots of unity.

Similarly, proceeding along the same line, we can, from the higher order identity (C.7) of Appendix C, derive the equation for the elliptic quintic root of unity. Thus, setting

$$\zeta(\delta) + \zeta(\omega_1(\delta)) + \zeta(\omega_2(\delta)) + \zeta(\omega_3(\delta)) + \zeta(\omega_4(\delta)) = 0\,, \tag{3.7a}$$

$$\wp(\delta) + \wp(\omega_1(\delta)) + \wp(\omega_2(\delta)) + \wp(\omega_3(\delta)) + \wp(\omega_4(\delta)) = 0\,, \tag{3.7b}$$

$$\wp'(\delta) + \wp'(\omega_1(\delta)) + \wp'(\omega_2(\delta)) + \wp'(\omega_3(\delta)) + \wp'(\omega_4(\delta)) = 0\,, \tag{3.7c}$$

together with $\delta + \omega_1(\delta) + \omega_2(\delta) + \omega_3(\delta) + \omega_4(\delta) \equiv 0(\text{mod period lattice})$, we derive from (C.7)

$$\begin{aligned}
&\Phi_\kappa(\delta)\Phi_\kappa(\omega_1(\delta))\Phi_\kappa(\omega_2(\delta))\Phi_\kappa(\omega_3(\delta))\Phi_\kappa(\omega_4(\delta)) \\
&\quad = \Phi_\kappa(\delta)\tfrac{1}{6}\Phi_\kappa(-\delta)\Big[\Big(\zeta(\kappa) - \zeta(\delta) - \zeta(\kappa-\delta)\Big)^3 + \wp'(\kappa-\delta) - \wp'(\kappa) + \wp'(\delta) \\
&\qquad + 3\Big(\zeta(\kappa) - \zeta(\delta) - \zeta(\kappa-\delta)\Big)\Big(\wp(\kappa) + \wp(\delta) - \wp(\kappa-\delta)\Big)\Big] \\
&\quad = \tfrac{1}{6}\Big(\wp(\kappa) - \wp(\delta)\Big)\Big[\Big(\zeta(\kappa) - \zeta(\delta) - \zeta(\kappa-\delta)\Big)\Big(4\wp(\kappa) + 4\wp(\delta) - 2\wp(\kappa-\delta)\Big) \\
&\qquad + \wp'(\kappa-\delta) - \wp'(\kappa) + \wp'(\delta)\Big] \\
&\quad = \tfrac{1}{6}\Big[-\tfrac{1}{2}\left(\wp'(\kappa) + \wp'(\delta)\right)\left(4\wp(\kappa) + 4\wp(\delta) - 2\wp(\kappa-\delta)\right) \\
&\qquad + \left(\wp(\kappa) - \wp(\delta)\right)\left(\wp'(\kappa-\delta) - \wp'(\kappa) + \wp'(\delta)\right)\Big] \\
&\quad = -\tfrac{1}{12}\left(6\wp(\kappa)\wp'(\kappa) + 6\wp(\delta)\wp'(\delta)\right) = -\tfrac{1}{24}\left(\wp'''(\kappa) + \wp'''(\delta)\right)\,,
\end{aligned}$$

where use has been made of the addition formula (A.10a), the relation (A.9), which implies also

$$\left(\wp'(\kappa) + \wp'(\delta)\right)\left(\wp(\kappa-\delta) - \wp(\delta)\right) = -\left(\wp(\kappa) - \wp(\delta)\right)\left(\wp'(\kappa-\delta) - \wp'(\delta)\right)\,,$$

and the well-known relation $\wp''' = 12\wp\wp'$.

Extrapolating from the latter cases of $N = 4$ and $N = 5$ cases, the general relation for the elliptic Nth root of unity can be conjectured, but the proof requires increasingly complicated elliptic identities along the lines given in Appendix C. Thus, we can define:

Definition 3.3. The elliptic Nth roots of unity are, up to the periodicity of the period lattice, defined as the roots of the following equation

$$\prod_{j=0}^{N-1} \Phi_\kappa(\omega_j(\delta)) = \tfrac{1}{(N-1)!}\left(\wp^{(N-2)}(-\kappa) - \wp^{(N-2)}(\delta)\right)\,, \quad \forall\,\kappa\,, \tag{3.8}$$

with $\omega_0(\delta) := \delta$, and subject to the condition that they are independent of κ.

That these elliptic roots $\omega_i(\delta)$, $(i = 0, \ldots, N-1)$ exist follows from the fact that the function $\wp^{(N-2)}$ is an elliptic function of order N, and hence that the equation $\wp^{(N-2)}(\omega) - \wp^{(N-2)}(\delta)$ has N roots (modulo the period lattice). Furthermore, from the Frobenius-Stickelberger determinant formula (A.12) of rank N it follows, as before, that we have

$$\sum_{j=0}^{N-1} \omega_j(\delta) \equiv 0(\text{mod period lattice}) , \tag{3.9}$$

and from the higher order elliptic identities, and by inserting $\kappa = -\omega_j, j = 0, 1, \ldots, N-1$ we also find that $\wp^{(N-2)}(\delta) = \wp^{(N-2)}(\omega_j)$, $\quad j = 0, 1, \ldots N-1$. The latter, can be taken as an alternative definition of the elliptic Nth root of unity. Then from elliptic identities such as the ones presented in Appendix C, it can be shown that the following relations hold:

$$\sum_{j=0}^{N-1} \zeta^{(l)}(\omega_j(\delta)) = 0 , \quad l = 0, \ldots, N-2 , \tag{3.10}$$

where $\zeta^{(l)}(x) = d^l\zeta(x)/dx^l$. We do not present here a formal proof of the above assertions, but merely indicate how this can be proven. In fact, the general form of identities of the type (C.1) and (C.7) which we need for the Nth root of unity can be described as follows:

$$\prod_{j=0}^{N-1} \Phi_{\kappa_j}(x) = \tfrac{1}{(N-1)!}\mathcal{F}(\kappa_0, \ldots, \kappa_{N-1}; x) ,$$

where the function $\mathcal{F}$ takes exactly the form of the expansion of the $(N-1)$th derivative of the Weierstrass σ-function:

$$\frac{\sigma'(x)}{\sigma(x)} = \zeta(x), \quad \frac{\sigma''(x)}{\sigma(x)} = \zeta^2(x) - \wp(x), \quad \frac{\sigma'''(x)}{\sigma(x)} = \zeta^3(x) - 3\zeta(x)\wp(x) - \wp'(x), \ldots$$

when expanded in ζ, $\wp$ and $\wp'$. The expression $\mathcal{F}$ corresponds to this expansion, where whenever we have an odd function in this expansion (like ζ, $\wp'$, etc.) we substitute the combination

$$\zeta(x) + \sum_{j=0}^{N-1} \zeta(\kappa_j) - \zeta\left(\sum_{j=0}^{N-1} \kappa_j + x\right) ,$$

(and similar for $\wp'$), and whenever we encounter $\wp$ (we have to replace everywhere the higher derivatives of $\wp$ by their expressions in terms of $\wp$ and $\wp'$ exclusively) we substitute the combination:

$$\sum_{j=0}^{N-1} \wp(\kappa_j) + \wp\left(\sum_{j=0}^{N-1} \kappa_j + x\right) - \wp(x) .$$

Subsequently, by substituting $\kappa_j = \omega_j(\delta)$ and $x = \kappa$ we obtain the identity (3.8) subject to (3.9). Conversely, we can revert the argument as in the above specific cases, and show that in return this is a sufficient condition for the set of relations (3.10) to hold.

4. Reductions from the KP System to the KdV and BSQ Systems

The fundamental KP relations can be summarised as follows. As a consequence of (2.3) the plane wave factors $\rho_\kappa(n,m,l)$ and $\sigma_{\kappa'}(n,m,l)$ are of the form

$$\rho_\kappa = \Phi_\delta(\kappa)^n \Phi_\varepsilon(\kappa)^m \Phi_\nu(\kappa)^l \rho_0 \,, \tag{4.1a}$$

$$\sigma_{\kappa'} = \Phi_\delta(-\kappa')^{-n} \Phi_\varepsilon(-\kappa')^{-m} \Phi_\nu(-\kappa')^{-l} \sigma_0 \,, \tag{4.1b}$$

and we set as in (2.27) the quantity $\xi = \xi(n,m,l)$ in the non-autonomous form $\xi = \xi_0 - n\delta - m\varepsilon - l\nu$, then the fundamental KP system from the relations in Sect. 2.1 takes the form:

$$-\widetilde{\boldsymbol{U}}_\xi\, \chi^{(1)}_{-\delta,\xi}({}^t\boldsymbol{\Lambda}) = \chi^{(1)}_{\delta,\widetilde{\xi}}(\boldsymbol{\Lambda})\,\boldsymbol{U}_\xi - \widetilde{\boldsymbol{U}}_\xi\,\boldsymbol{O}\,\boldsymbol{U}_\xi \,, \tag{4.2a}$$

$$\widetilde{\boldsymbol{u}_\kappa(\xi)} = \left(\chi^{(1)}_{\delta,\widetilde{\xi}}(\boldsymbol{\Lambda}) - \widetilde{\boldsymbol{U}}_\xi\,\boldsymbol{O}\right)\boldsymbol{u}_\kappa(\xi) \,, \tag{4.2b}$$

$${}^t\boldsymbol{u}_{\kappa'}(\xi) = -\widetilde{{}^t\boldsymbol{u}_{\kappa'}(\xi)}\,\left(\chi^{(1)}_{-\delta,\xi}({}^t\boldsymbol{\Lambda}) - \boldsymbol{O}\,\boldsymbol{U}_\xi\right) \,, \tag{4.2c}$$

with (2.26). (Note that here the shift on the object $\widetilde{\boldsymbol{U}}_\xi$ also implies a shift the variable ξ, i.e., $\widetilde{\boldsymbol{U}}_\xi := \widetilde{\boldsymbol{U}}_{\widetilde{\xi}}$, and $\widetilde{\boldsymbol{u}_\kappa(\xi)} = \widetilde{\boldsymbol{u}}_\kappa(\widetilde{\xi})$, and similarly for the other quantities depending on ξ, and for the other lattice shifts.) Similar relations to (4.2) hold for the other shifts on the multidimensional lattice, replacing $\widetilde{}$ by $\widehat{}$ or $\bar{}$ and δ by ε or ν respectively.

Applying multiple shifts we get higher order relations, e.g. the second order relation

$$\begin{aligned}\widehat{\widetilde{\boldsymbol{U}}}_\xi\, \chi^{(2)}_{-\delta,-\varepsilon,\xi}({}^t\boldsymbol{\Lambda}) &= \chi^{(2)}_{\delta,\varepsilon,\widehat{\widetilde{\xi}}}(\boldsymbol{\Lambda})\cdot\boldsymbol{U}_\xi - \chi^{(1)}_{\delta,\varepsilon}(\widehat{\widetilde{\xi}})\widehat{\widetilde{\boldsymbol{U}}}_\xi\,\boldsymbol{O}\,\boldsymbol{U}_\xi \\ &\quad + \widehat{\widetilde{\boldsymbol{U}}}_\xi\left(\boldsymbol{O}\,\eta_\xi(\boldsymbol{\Lambda}) - \eta_{\widehat{\widetilde{\xi}}}({}^t\boldsymbol{\Lambda})\,\boldsymbol{O}\right)\boldsymbol{U}_\xi \,,\end{aligned} \tag{4.3a}$$

$$\widehat{\widetilde{\boldsymbol{u}_\kappa(\xi)}} = \left[\chi^{(2)}_{\delta,\varepsilon,\widehat{\widetilde{\xi}}}(\boldsymbol{\Lambda}) - \chi^{(1)}_{\delta,\varepsilon}(\widehat{\widetilde{\xi}})\widehat{\widetilde{\boldsymbol{U}}}_\xi\cdot\boldsymbol{O} + \widehat{\widetilde{\boldsymbol{U}}}_\xi\left(\boldsymbol{O}\,\eta_\xi(\boldsymbol{\Lambda}) - \eta_{\widehat{\widetilde{\xi}}}({}^t\boldsymbol{\Lambda})\,\boldsymbol{O}\right)\right]\boldsymbol{u}_\kappa(\xi) \,, \tag{4.3b}$$

$$\begin{aligned}{}^t\boldsymbol{u}_{\kappa'}(\xi) &= \widehat{\widetilde{{}^t\boldsymbol{u}_{\kappa'}(\xi)}}\cdot\left[\chi^{(2)}_{-\delta,-\varepsilon,\xi}({}^t\boldsymbol{\Lambda}) - \chi^{(1)}_{-\delta,-\varepsilon}(\xi)\boldsymbol{O}\cdot\boldsymbol{U}_\xi \right.\\ &\quad \left. - \left(\boldsymbol{O}\,\eta_\xi(\boldsymbol{\Lambda}) - \eta_{\widehat{\widetilde{\xi}}}({}^t\boldsymbol{\Lambda})\,\boldsymbol{O}\right)\cdot\boldsymbol{U}_\xi\right] \,,\end{aligned} \tag{4.3c}$$

where, according to (C.1), we have the coefficients

$$\begin{aligned}\chi^{(2)}_{\alpha,\beta,\gamma}(x) := \frac{1}{2}\Big[&(\zeta(\alpha)+\zeta(\beta)+\zeta(\gamma)+\zeta(x)-\zeta(\alpha+\beta+\gamma+x))^2 \\ &+\wp(x) - (\wp(\alpha)+\wp(\beta)+\wp(\gamma)+\wp(\alpha+\beta+\gamma+x))\Big] \,,\end{aligned}$$

and taking into account the quantity (A.9)

$$\eta_\alpha(x) := \zeta(\alpha+x) - \zeta(\alpha) - \zeta(x) \,.$$

Note that the relations (4.3) do not depend on the intermediary steps (i.e., on single-shifted objects).

Remark. In general the following functions appear in the coefficients of the fundamental relations:

$$\chi^{(\mathcal{N})}_{\alpha_0,\alpha_1,\cdots,\alpha_{\mathcal{N}}}(x) = \frac{\prod_{j=0}^{\mathcal{N}} \Phi_{\alpha_j}(x)}{\Phi_{\alpha_0+\alpha_1+\cdots+\alpha_{\mathcal{N}}}(x)}\,, \quad (\mathcal{N} \in \mathbb{N}_+)$$

and they can be expressed in terms of ζ-functions using the relations of Appendix C.

Similarly the third-order shift relations read:

$$\begin{aligned}
-\overline{\widehat{\widetilde{\boldsymbol{U}}}}_\xi\,\chi^{(3)}_{-\delta,-\varepsilon,-\nu,\xi}({}^t\boldsymbol{\Lambda}) &= \chi^{(3)}_{\delta,\varepsilon,\nu,\widehat{\widetilde{\overline{\xi}}}}(\boldsymbol{\Lambda})\,\boldsymbol{U}_\xi - \chi^{[2]}_{\delta,\varepsilon,\nu}(\widetilde{\widehat{\overline{\xi}}})\overline{\widehat{\widetilde{\boldsymbol{U}}}}_\xi\,\boldsymbol{O}\boldsymbol{U}_\xi \\
&+ \chi^{[1]}_{\delta,\varepsilon,\nu}(\widetilde{\widehat{\overline{\xi}}})\overline{\widehat{\widetilde{\boldsymbol{U}}}}_\xi\left(\boldsymbol{O}\eta_\xi(\boldsymbol{\Lambda}) - \eta_{\widehat{\widetilde{\overline{\xi}}}}({}^t\boldsymbol{\Lambda})\boldsymbol{O}\right)\boldsymbol{U}_\xi \\
&- \overline{\widehat{\widetilde{\boldsymbol{U}}}}_\xi\left(\boldsymbol{O}\wp(\boldsymbol{\Lambda}) + \wp({}^t\boldsymbol{\Lambda})\boldsymbol{O} - \eta_{\widehat{\widetilde{\overline{\xi}}}}({}^t\boldsymbol{\Lambda})\boldsymbol{O}\eta_\xi(\boldsymbol{\Lambda})\right)\boldsymbol{U}_\xi\,, \qquad (4.4a)
\end{aligned}$$

$$\begin{aligned}
\overline{\widehat{\widetilde{\boldsymbol{u}_\kappa(\xi)}}} &= \chi^{(3)}_{\delta,\varepsilon,\nu,\widehat{\widetilde{\overline{\xi}}}}(\boldsymbol{\Lambda})\,\boldsymbol{u}_\kappa(\xi) - \chi^{[2]}_{\delta,\varepsilon,\nu}(\widetilde{\widehat{\overline{\xi}}})\overline{\widehat{\widetilde{\boldsymbol{U}}}}_\xi\,\boldsymbol{O}\boldsymbol{u}_\kappa(\xi) \\
&+ \chi^{[1]}_{\delta,\varepsilon,\nu}(\widetilde{\widehat{\overline{\xi}}})\overline{\widehat{\widetilde{\boldsymbol{U}}}}_\xi\left(\boldsymbol{O}\eta_\xi(\boldsymbol{\Lambda}) - \eta_{\widehat{\widetilde{\overline{\xi}}}}({}^t\boldsymbol{\Lambda})\boldsymbol{O}\right)\boldsymbol{u}_\kappa(\xi) \\
&- \overline{\widehat{\widetilde{\boldsymbol{U}}}}_\xi\left(\boldsymbol{O}\wp(\boldsymbol{\Lambda}) + \wp({}^t\boldsymbol{\Lambda})\boldsymbol{O} - \eta_{\widehat{\widetilde{\overline{\xi}}}}({}^t\boldsymbol{\Lambda})\boldsymbol{O}\eta_\xi(\boldsymbol{\Lambda})\right)\boldsymbol{u}_\kappa(\xi)\,, \qquad (4.4b)
\end{aligned}$$

$$\begin{aligned}
{}^t\boldsymbol{u}_{\kappa'} &= -\overline{\widehat{\widetilde{{}^t\boldsymbol{u}_{\kappa'}}}}\Big[\chi^{(3)}_{-\delta,-\varepsilon,-\nu,\xi}({}^t\boldsymbol{\Lambda}) - \chi^{[2]}_{-\delta,-\varepsilon,-\nu}(\xi)\boldsymbol{O}\boldsymbol{U}_\xi \\
&- \chi^{[1]}_{-\delta,-\varepsilon,-\nu}(\xi)\left(\boldsymbol{O}\eta_\xi(\boldsymbol{\Lambda}) - \eta_{\widehat{\widetilde{\overline{\xi}}}}({}^t\boldsymbol{\Lambda})\boldsymbol{O}\right)\boldsymbol{U}_\xi \\
&- \left(\boldsymbol{O}\wp(\boldsymbol{\Lambda}) + \wp({}^t\boldsymbol{\Lambda})\boldsymbol{O} - \eta_{\widehat{\widetilde{\overline{\xi}}}}({}^t\boldsymbol{\Lambda})\boldsymbol{O}\eta_\xi(\boldsymbol{\Lambda})\right)\boldsymbol{U}_\xi\Big], \qquad (4.4c)
\end{aligned}$$

where

$$\chi^{[1]}_{\delta,\varepsilon,\nu}(\widetilde{\widehat{\overline{\xi}}}) := \zeta(\delta) + \zeta(\varepsilon) + \zeta(\nu) + \zeta(\widehat{\widetilde{\overline{\xi}}}) - \zeta(\xi)\,,$$

and

$$\begin{aligned}
\chi^{[2]}_{\delta,\varepsilon,\nu}(\widetilde{\widehat{\overline{\xi}}}) &= \frac{1}{2}\Bigg[\left(\zeta(\delta) + \zeta(\varepsilon) + \zeta(\widetilde{\widehat{\overline{\xi}}}) - \zeta(\overline{\xi})\right)^2 - \wp(\delta) - \wp(\varepsilon) - \wp(\overline{\xi}) \\
&+ \left(\zeta(\varepsilon) + \zeta(\nu) + \zeta(\widehat{\overline{\xi}}) - \zeta(\xi)\right)^2 - \wp(\varepsilon) - \wp(\nu) - \wp(\widehat{\overline{\xi}}) \\
&-2\left(\zeta(\widehat{\overline{\xi}}) - \zeta(\widehat{\widetilde{\overline{\xi}}}) - \zeta(\delta)\right)\big(\zeta(\nu) + \zeta(\overline{\xi}) - \zeta(\xi)\big) + \wp(\widehat{\widetilde{\overline{\xi}}}) + \wp(\xi)\Bigg] \\
&= \left(\zeta(\widehat{\widetilde{\overline{\xi}}}) - \zeta(\xi) + \zeta(\delta) + \zeta(\varepsilon+\nu)\right)\big(\zeta(\delta) + \zeta(\varepsilon) + \zeta(\nu) - \zeta(\delta+\varepsilon+\nu)\big) \\
&+ \wp(\widehat{\widetilde{\overline{\xi}}}) + \wp(\xi) - \wp(\delta) = \chi^{[2]}_{-\delta,-\varepsilon,-\nu}(\xi)\,,
\end{aligned}$$

(which remarkably does not depend on the intermediate values $\widehat{\overline{\xi}}$ or $\overline{\xi}$). Note that the latter expression does not seem manifestly symmetric w.r.t. permutation of δ, ε, ν, but nevertheless it is, as can be shown using (A.8) from the Appendix C. Furthermore, we recall

$$\chi^{(3)}_{\alpha,\beta,\gamma,\delta}(x) := \frac{\Phi_\alpha(x)\,\Phi_\beta(x)\Phi_\gamma(x)\,\Phi_\delta(x)}{\Phi_{\alpha+\beta+\gamma+\delta}(x)}\ ,$$

which can be expressed in terms of ζ functions using the identities in Appendix C.

Finally, the relevant basic relations for the τ-function read:

$$\frac{\widetilde{\tau_\xi}}{\tau_\xi} = 1 - \left(U_\xi\, \frac{1}{\chi^{(1)}_{-\delta,\xi}({}^t\boldsymbol{\Lambda})} \right)_{0,0} \quad \Leftrightarrow \quad \frac{\tau_\xi}{\widetilde{\tau_\xi}} = 1 - \left(\frac{1}{\chi^{(1)}_{\delta,\widetilde{\xi}}(\boldsymbol{\Lambda})}\, \widetilde{\boldsymbol{U}_\xi} \right)_{0,0}\ , \tag{4.5}$$

as derived in Appendix B.

We can now use these higher order shift relations to obtain the higher order dimensional reductions of the lattice KP system, by imposing periodicity w.r.t. the combined shifts where the shift parameters are related through the elliptic roots of unity. In this way we get for $N = 2$ (elliptic square root of unity) the reduction to the lattice KdV system, while for $N = 3$ (elliptic cube root of unity) we attain the reduction to the lattice BSQ system, and hence to the elliptic solutions of the latter.

4.1. Reduction to the lattice KdV system. The reduction from KP to KdV type systems is imposed by requiring that the following condition holds on the plane wave factors:

$$T_{-\delta} \circ T_\delta(\rho_\kappa\, \sigma_{\kappa'}) = \rho_\kappa\, \sigma_{\kappa'}\ . \tag{4.6}$$

A remark is in order here as to the meaning of this relation. In fact, in the KP system each lattice parameter, say δ, can be associated with a direction in a lattice of arbitrary dimension. However, when acting on KP solutions in general position, the shifts T_δ and $T_{-\delta}$ are associated with two different directions in that lattice, δ with a lattice variables $n =: n_\delta$ and $-\delta$ with another discrete variable $n_{-\delta}$ independent of n_δ. However, identifying the variable $n_{-\delta}$ with the reverse direction of the variable n_δ through the relation (4.6) we force a dimensional reduction on the system, which essentially reduces the 3-dimensional KP lattice system to a 2-dimensional system which is the lattice KdV system. Obviously we will still have the KP equation valid between any three shifts, but by using (4.6) any KP type equation between shifts in the n_δ and $n_{-\delta}$ directions will trivialise.

Note, furthermore, that in (4.6) we have effectively employed the elliptic square roots of unity $\omega_0(\delta) = \delta$ and $\omega_1(\delta) = -\delta$ as explained in Sect. 3, and (4.6) amounts to the assertion that the reverse operation to any shift in the multidimensional lattice generated by shifts T_δ (for all δ) corresponding to a parameter δ acts in the same way on the dependent variables as the shift corresponding to the parameter $-\delta$. Hence in the KP integral equations (2.2) we need to restrict ourselves to integration over suitable measure and contour, say $d\mu(\ell)$ and Γ, where this identification is implemented. Thus, the integral equation for the KdV class, reads:

$$\boldsymbol{u}_\kappa + \rho_\kappa \int_\Gamma d\mu(\ell)\, \sigma_{-\omega_1(\ell)} \boldsymbol{u}_\ell \Phi_\xi(\kappa - \omega_1(\ell)) = \rho_\kappa \Phi_\xi(\boldsymbol{\Lambda}) \boldsymbol{c}_\kappa\ , \tag{4.7}$$

with the corresponding expression for the matrix $\boldsymbol{U}_\xi$:

$$\boldsymbol{U}_\xi = \int_\Gamma d\mu(\ell)\, \sigma_{-\omega_1(\ell)} \boldsymbol{u}_\ell\, {}^t\boldsymbol{c}_{-\omega_1(\ell)} \boldsymbol{\Phi}_\xi({}^t\boldsymbol{\Lambda})\ . \tag{4.8}$$

Thus, the reduction to the lattice KdV system is obtained by implementing the relations

$$T_\delta \circ T_{-\delta} \boldsymbol{U}_\xi = \boldsymbol{U}_\xi\ , \quad T_\delta \circ T_{-\delta} \boldsymbol{u}_\kappa(\xi) = (\wp(\kappa) - \wp(\delta))\, \boldsymbol{u}_\kappa(\xi)\ , \tag{4.9}$$

which follow as additional conditions on the fundamental system of KP equations given in Sect. 2. The relations for the KdV class can be obtained in this way, and comprise:

$$-\widetilde{\boldsymbol{U}_\xi}\, \chi^{(1)}_{-\delta,\xi}({}^t\boldsymbol{\Lambda}) = \chi^{(1)}_{\delta,\widetilde{\xi}}(\boldsymbol{\Lambda})\, \boldsymbol{U}_\xi - \widetilde{\boldsymbol{U}_\xi}\, \boldsymbol{O}\, \boldsymbol{U}_\xi\ , \tag{4.10a}$$

$$\widetilde{\boldsymbol{u}_\kappa(\xi)} = \left(\chi^{(1)}_{\delta,\widetilde{\xi}}(\boldsymbol{\Lambda}) - \widetilde{\boldsymbol{U}_\xi}\, \boldsymbol{O}\right) \boldsymbol{u}_\kappa(\xi)\ , \tag{4.10b}$$

$$(\wp(\kappa) - \wp(\delta))\, \boldsymbol{u}_\kappa(\xi) = \left(\chi^{(1)}_{-\delta,\xi}(\boldsymbol{\Lambda}) - \boldsymbol{U}_\xi \boldsymbol{O}\right) \widetilde{\boldsymbol{u}_\kappa(\xi)}\ , \tag{4.10c}$$

where the first two relations are exactly the same as in the KP case, but where the last additional relation is obtained from the condition (4.9). We note that the reduction to the KdV case can also be interpreted as the condition that the infinite matrix $\boldsymbol{U}_\xi$ is symmetric under transposition, i.e. $(\boldsymbol{U}_\xi)^T = \boldsymbol{U}_\xi$. Furthermore, in this reduction we have purely algebraic relations (i.e., without involving any lattice shifts), which are obtained from the double-shift relations (4.3) by taking $\varepsilon = \omega(\delta)$ and taking into account the condition (4.9). Thus, we get

$$\boldsymbol{U}_\xi \cdot \wp({}^t\boldsymbol{\Lambda}) = \wp(\boldsymbol{\Lambda}) \cdot \boldsymbol{U}_\xi + \boldsymbol{U}_\xi \left[\boldsymbol{O}\eta_\xi(\boldsymbol{\Lambda}) - \eta_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\right] \boldsymbol{U}_\xi\ , \tag{4.11a}$$

$$\wp(\kappa)\boldsymbol{u}_\kappa(\xi) = \wp(\boldsymbol{\Lambda}) \cdot \boldsymbol{u}_\kappa(\xi) + \boldsymbol{U}_\xi \left[\boldsymbol{O}\eta_\xi(\boldsymbol{\Lambda}) - \eta_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\right] \boldsymbol{u}_\kappa(\xi)\ . \tag{4.11b}$$

The resulting equations in this case are the equations in the KdV class of lattice equations, namely two-dimensional lattice equations involving only two lattice directions. Thus, as an example, we mention here only the lattice (potential) KdV equation, cf. [30], in the form

$$\begin{aligned}&[\zeta(\alpha) - \zeta(\beta) + \zeta(\xi - \alpha) - \zeta(\xi - \beta) + \widehat{u}(\xi - \beta) - \widetilde{u}(\xi - \alpha)] \\ &\quad \times \left[\zeta(\alpha) + \zeta(\beta) - \zeta(\xi) + \zeta(\xi - \alpha - \beta) + u(\xi) - \widehat{\widetilde{u}}(\xi - \alpha - \beta)\right] = \wp(\alpha) - \wp(\beta).\end{aligned} \tag{4.12}$$

In [26] it was shown how to obtain, from an elliptic Cauchy matrix scheme, elliptic multi-soliton solutions for all ABS equations [2], apart from Q4 (solitons of the latter were obtained by a different approach in [5]). In principle the DL scheme covers an even wider class of solutions. Since in this paper we concentrate on the higher-rank elliptic reductions, we will refrain here from deriving all the lattice equations from the above scheme and refer to [26] for details.

4.2. Reduction to the lattice BSQ system. The dimensional reduction to the BSQ case (i.e., the case $N=3$) from the lattice KP case is obtained by imposing the condition

$$T_\delta \circ T_{\omega_1(\delta)} \circ T_{\omega_2(\delta)} \left(\rho_\kappa\, \sigma_{\kappa'}\right) = \rho_\kappa\, \sigma_{\kappa'} \ , \tag{4.13}$$

which leads to the identifications $\kappa' = -\omega_j(\kappa)$ $(j=0,1,2)$ in the integral equation (2.2), where we have set for convenience $\omega_0(\kappa) := \kappa$. A similar remark as in the KdV reduction case applies to the relation (4.13), namely that for generic solutions of the KP system the lattice directions associated with lattice parameters δ, $\omega_1(\delta)$ and $\omega_2(\delta)$ should be viewed as distinct, and act on three independent lattice variables n_δ, $n_{\omega_1(\delta)}$ and $n_{\omega_2(\delta)}$ respectively. However, in special solutions of the KP system, on which the three-fold condition (4.13) holds, there is an implicit identification between these lattice variables and their corresponding elementary shifts.

It is here where the elliptic cube root of unity is seen to play the key role. In fact, from the defining relation (3.2) we have

$$T_\delta \circ T_{\omega_1(\delta)} \circ T_{\omega_2(\delta)}\, \rho_\kappa = \Phi_\delta(\kappa)\, \Phi_{\omega_1(\delta)}(\kappa)\, \Phi_{\omega_2(\delta)}(\kappa)\, \rho_\kappa = -\frac{1}{2}\left(\wp'(\kappa)+\wp'(\delta)\right) \rho_\kappa \ ,$$

and similarly, we have that

$$T_\delta \circ T_{\omega_1(\delta)} \circ T_{\omega_2(\delta)} \sigma_{\kappa'} = \left(\frac{1}{2}\left(\wp'(\kappa')-\wp'(\delta)\right)\right)^{-1} \sigma_{\kappa'} \ .$$

Hence, we get the condition (4.13) to hold provided that $\kappa' = -\omega_j(\kappa)$ $(j=0,1,2)$. The latter is the condition on the spectral variable which reduces the integral equations (2.2) to the ones for the BSQ case.

Thus, Eq. (4.13) when implemented on the KP integral equations (2.2), e.g. by restricting the measure $d\mu(\ell,\ell')$ to contain δ-functions of the form $\delta(\ell'+\omega_j(\ell))$ $(j=1,2)$, leads to the following integral equation for the BSQ elliptic solutions:

$$\boldsymbol{u}_\kappa(\xi) + \rho_\kappa \sum_{j=1}^{2} \int_{\Gamma_j} d\mu_j(\ell)\, \sigma_{-\omega_j(\ell)} \Phi_\xi(\kappa - \omega_j(\ell)) \boldsymbol{u}_\ell(\xi) = \rho_\kappa \Phi_\xi(\boldsymbol{\Lambda}) \boldsymbol{c}_\kappa \ , \tag{4.14}$$

where ξ is given as in (2.27)—which guarantees that

$$T_\delta \circ T_{\omega_1(\delta)} \circ T_{\omega_2(\delta)} \xi \equiv \xi (\text{mod period lattice}) \ ,$$

and where Γ_j and $d\mu_j(\ell)$ are contours and measures that need to be suitably chosen (see Sect. 6 for particular cases of choices for Γ_j and $d\mu_j(\ell)$). Furthermore, as a consequence of the reduction the eigenvector $\boldsymbol{u}_\kappa$ obeys

$$T_\delta \circ T_{\omega_1(\delta)} \circ T_{\omega_2(\delta)} \boldsymbol{u}_\kappa = -\frac{1}{2}\left(\wp'(\delta)+\wp'(\kappa)\right) \boldsymbol{u}_\kappa \ . \tag{4.15}$$

Together with the formal integral equation (4.14) we have in this reduced case the following formula for the infinite matrix $\boldsymbol{U}_\xi$:

$$\boldsymbol{U}_\xi = \sum_{j=1}^{2} \int_{\Gamma_j} d\mu_j(\ell)\, \sigma_{-\omega_j(\ell)} \boldsymbol{u}_\ell(\xi)\, {}^t\boldsymbol{c}_{-\omega_j(\ell)} \Phi_\xi({}^t\boldsymbol{\Lambda}) \ . \tag{4.16}$$

As a consequence of the analysis of the KP system, which fully goes through modulo the reduction, we can now readily inherit all the fundamental relations for the infinite-matrix scheme for the BSQ system by implementing the constraint (4.15) on the KP system (4.2) in the multidimensional space of lattice variables. By assuming that this multidimensional grid includes the lattice directions associated with the parameters $\omega_1(\delta)$ and $\omega_2(\delta)$, subject to the condition (3.2), we get the following fundamental infinite matrix system for the elliptic BSQ solution. First, the discrete dynamics for any direction indicated by the parameter δ, is given by the relations:

$$-\widetilde{\boldsymbol{U}}_\xi\, \chi^{(1)}_{-\delta,\xi}({}^t\boldsymbol{\Lambda}) = \chi^{(1)}_{\delta,\widetilde{\xi}}(\boldsymbol{\Lambda})\,\boldsymbol{U}_\xi - \widetilde{\boldsymbol{U}}_\xi\, \boldsymbol{O}\,\boldsymbol{U}_\xi\ , \tag{4.17a}$$

$$\widetilde{\boldsymbol{u}_\kappa(\xi)} = \left(\chi^{(1)}_{\delta,\widetilde{\xi}}(\boldsymbol{\Lambda}) - \widetilde{\boldsymbol{U}}_\xi\, \boldsymbol{O}\right)\boldsymbol{u}_\kappa(\xi)\ , \tag{4.17b}$$

for the forward shift (which is the same relation as for the KP system), while for the reverse shift we have additional relations:

$$\begin{aligned}
&\boldsymbol{U}_\xi\left[\wp(\xi) + \wp({}^t\boldsymbol{\Lambda}) + \wp(\delta) - \eta_{-\delta}(\xi)\eta_\xi({}^t\boldsymbol{\Lambda})\right] \\
&\quad = \left[\wp(\widetilde{\xi}) + \wp(\boldsymbol{\Lambda}) + \wp(\delta) - \eta_\delta(\widetilde{\xi})\eta_{\widetilde{\xi}}(\boldsymbol{\Lambda})\right]\widetilde{\boldsymbol{U}}_\xi \\
&\qquad + \boldsymbol{U}_\xi\left[\eta_{-\delta}(\xi)\boldsymbol{O} + \boldsymbol{O}\eta_{\widetilde{\xi}}(\boldsymbol{\Lambda}) - \eta_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\right]\widetilde{\boldsymbol{U}}_\xi\ ,
\end{aligned} \tag{4.17c}$$

$$\begin{aligned}
&-\tfrac{1}{2}(\wp'(\delta) + \wp'(\kappa))\boldsymbol{u}_\kappa(\xi) = \left[\wp(\widetilde{\xi}) + \wp(\boldsymbol{\Lambda}) + \wp(\delta) - \eta_\delta(\widetilde{\xi})\eta_{\widetilde{\xi}}(\boldsymbol{\Lambda})\right]\widetilde{\boldsymbol{u}_\kappa(\xi)} \\
&\quad + \boldsymbol{U}_\xi\left[\eta_{-\delta}(\xi)\boldsymbol{O} + \boldsymbol{O}\eta_{\widetilde{\xi}}(\boldsymbol{\Lambda}) - \eta_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\right]\widetilde{\boldsymbol{u}_\kappa(\xi)}\ ,
\end{aligned} \tag{4.17d}$$

and similar relations for the other lattice directions. These follow directly from the double-shift KP relations (4.3) implementing the triple shift conditions on the reduction (4.15). Second, the condition (4.15) gives rise to the following algebraic constraints (which involve no lattice shifts):

$$\begin{aligned}
&-\tfrac{1}{2}\boldsymbol{U}_\xi\,\wp'({}^t\boldsymbol{\Lambda}) \\
&\quad = \tfrac{1}{2}\wp'(\boldsymbol{\Lambda})\boldsymbol{U}_\xi + \boldsymbol{U}_\xi\left[\boldsymbol{O}\wp(\boldsymbol{\Lambda}) + \wp({}^t\boldsymbol{\Lambda})\boldsymbol{O} - \eta_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\eta_\xi(\boldsymbol{\Lambda})\right]\boldsymbol{U}_\xi + \wp(\xi)\boldsymbol{U}_\xi\boldsymbol{O}\boldsymbol{U}_\xi\ ,
\end{aligned} \tag{4.18a}$$

$$\begin{aligned}
&\tfrac{1}{2}\wp'(\kappa)\boldsymbol{u}_\kappa(\xi) = \tfrac{1}{2}\wp'(\boldsymbol{\Lambda})\boldsymbol{u}_\kappa(\xi) \\
&\quad + \boldsymbol{U}_\xi\left[\boldsymbol{O}\wp(\boldsymbol{\Lambda}) + \wp({}^t\boldsymbol{\Lambda})\boldsymbol{O} - \eta_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\eta_\xi(\boldsymbol{\Lambda})\right]\boldsymbol{u}_\kappa + \wp(\xi)\boldsymbol{U}_\xi\boldsymbol{O}\boldsymbol{u}_\kappa(\xi)\ ,
\end{aligned} \tag{4.18b}$$

which follow from (4.4) implementing the condition (4.15), and noting that

$$\chi^{[2]}_{\delta,\omega_1(\delta),\omega_2(\delta)}(\xi) = \wp(\xi)\ .$$

The system of equations (4.17) in all relevant lattice directions form the fundamental system from which the BSQ lattice system, including the corresponding Lax pairs, can be derived, thus implying that the quantities we will extract from the infinite matrix $\boldsymbol{U}_\xi$ of (4.16) the elliptic class of solutions of those lattice systems.

The next step, in order to derive from the set of relations (4.17) closed-form equations in terms of single elements of the matrix $\boldsymbol{U}_\xi$ or combinations thereof, is to identify the relevant entries and to specify the relations involving those entries. Then closed-form equations are obtained by combining those relations for two different lattice shifts associated with two different lattice directions. For the sake of transparency of the structure, we will introduce yet another set of notations, which will make the relations as close as possible to the ones in the rational case. Thus, we introduce the notations:

$$\boldsymbol{\Lambda}_\xi := \eta_\xi(\boldsymbol{\Lambda}) \,, \quad {}^t\boldsymbol{\Lambda}_\xi := \eta_\xi({}^t\boldsymbol{\Lambda}) \,,$$

i.e. ξ-dependent raising operators acting from the left and right, and

$$p_\xi := \eta_{-\delta}(\xi) = -\eta_\delta(\widetilde{\xi}) \,, \quad q_\xi := \eta_{-\varepsilon}(\xi) = -\eta_\varepsilon(\widehat{\xi}) \,,$$

ξ-dependent lattice 'parameters' (replacing in a sense the lattice parameters p and q of [29]). In terms of these the fundamental relations (4.17) can be rewritten as

$$\widetilde{\boldsymbol{U}}_\xi(p_\xi + {}^t\boldsymbol{\Lambda}_{\widetilde{\xi}}) = (p_\xi - \boldsymbol{\Lambda}_\xi)\boldsymbol{U}_\xi - \widetilde{\boldsymbol{U}}_\xi \boldsymbol{O} \boldsymbol{U}_\xi \,, \tag{4.19a}$$

$$\widetilde{\boldsymbol{u}_\kappa(\xi)} = \left[p_\xi - \boldsymbol{\Lambda}_\xi - \widetilde{\boldsymbol{U}}_\xi \boldsymbol{O}\right]\boldsymbol{u}_\kappa(\xi) \,, \tag{4.19b}$$

$$\begin{aligned}\frac{1}{2}\boldsymbol{U}_\xi \frac{\wp'(\delta) - \wp'({}^t\boldsymbol{\Lambda})}{p_\xi + {}^t\boldsymbol{\Lambda}_{\widetilde{\xi}}} &= \frac{1}{2}\frac{\wp'(\delta) + \wp'(\boldsymbol{\Lambda})}{p_\xi - \boldsymbol{\Lambda}_\xi}\widetilde{\boldsymbol{U}}_\xi \\ &\quad - \boldsymbol{U}_\xi\left(p_\xi \boldsymbol{O} + \boldsymbol{O}\boldsymbol{\Lambda}_{\widetilde{\xi}} - {}^t\boldsymbol{\Lambda}_\xi \boldsymbol{O}\right)\widetilde{\boldsymbol{U}}_\xi \,,\end{aligned} \tag{4.19c}$$

$$\begin{aligned}&\tfrac{1}{2}(\wp'(\delta) + \wp'(\kappa))\boldsymbol{u}_\kappa(\xi) \\ &= \left[\frac{1}{2}\frac{\wp'(\delta) + \wp'(\boldsymbol{\Lambda})}{p_\xi - \boldsymbol{\Lambda}_\xi} - \boldsymbol{U}_\xi\left(p_\xi \boldsymbol{O} + \boldsymbol{O}\boldsymbol{\Lambda}_{\widetilde{\xi}} - {}^t\boldsymbol{\Lambda}_\xi \boldsymbol{O}\right)\right]\widetilde{\boldsymbol{u}_\kappa(\xi)} \,,\end{aligned} \tag{4.19d}$$

(and a similar set of relations with $p_\xi \to q_\xi$ and $\widetilde{\ } \to \widehat{\ }$), with

$$\chi^{(1)}_{-\delta,\xi}({}^t\boldsymbol{\Lambda}) = -(p_\xi + {}^t\boldsymbol{\Lambda}_{\widetilde{\xi}}) \,, \quad \chi^{(1)}_{\delta,\widetilde{\xi}}(\boldsymbol{\Lambda}) = p_\xi - \boldsymbol{\Lambda}_\xi \,,$$

and similarly for the other lattice directions. Some useful properties of these quantities are given by

$$\chi^{(1)}_{\delta,\widetilde{\xi}}(\boldsymbol{\Lambda})\,\chi^{(1)}_{-\delta,\xi}(\boldsymbol{\Lambda}) = -(p_\xi - \boldsymbol{\Lambda}_\xi)(p_\xi + \boldsymbol{\Lambda}_{\widetilde{\xi}}) = \wp(\boldsymbol{\Lambda}) - \wp(\delta) \,,$$

and similarly for the quantities q_ξ with $\boldsymbol{\Lambda}_{\widetilde{\xi}}$ replaced by $\boldsymbol{\Lambda}_{\widehat{\xi}}$ and δ by ε.

5. Lattice BSQ Systems

In the previous section we have obtained the set of fundamental shift relations, in particular (4.19), in terms of one particular lattice shift. However, these relations are still not in closed form, since they involve the elliptic raising operators $\boldsymbol{\Lambda}_\xi$ and ${}^t\boldsymbol{\Lambda}_\xi$. In this section we will endeavour to eliminate these operators by combining the relations for different shifts on a two-dimensional lattice associated with two lattice parameters δ and ε, and thus obtain some concrete closed-form lattice equations within the BSQ class, from the fundamental relations in terms of the infinite matrix $\boldsymbol{U}_\xi$. The corresponding Lax pairs for those equations are constructed in Appendix D.

5.1. Derivation of the regular lattice BSQ equation. We proceed by constructing a system of closed-form equations for specific elements of the matrix $\boldsymbol{U}_\xi$ by combining the relations from the system (4.17) for two different lattice shifts with parameters δ and ε. Setting by definition:

$$u_{0,0} := u_{0,0}(\xi) = \big(\boldsymbol{U}_\xi\big)_{0,0}\ , \quad u_{1,0} := u_{1,0}(\xi) = \big(\boldsymbol{\Lambda}_\xi \boldsymbol{U}_\xi\big)_{0,0}\ , \tag{5.1a}$$

$$u_{0,1} := u_{0,1}(\xi) = \big(\boldsymbol{U}_\xi\, {}^t\boldsymbol{\Lambda}_\xi\big)_{0,0}\ , \quad u_{1,1} := u_{1,1}(\xi) = \big(\boldsymbol{\Lambda}_\xi \boldsymbol{U}_\xi\, {}^t\boldsymbol{\Lambda}_\xi\big)_{0,0}\ , \tag{5.1b}$$

$$u_{2,0} := u_{2,0}(\xi) = \big(\wp(\boldsymbol{\Lambda})\boldsymbol{U}_\xi\big)_{0,0}\ , \quad u_{0,2} := u_{0,2}(\xi) = \big(\boldsymbol{U}_\xi \wp({}^t\boldsymbol{\Lambda})\big)_{0,0}\ , \tag{5.1c}$$

we can derive various relations from the system (4.19) by taking the $(\ )_{0,0}$ element, either directly or after multiplication by factors of the form $p_\xi + {}^t\boldsymbol{\Lambda}_{\widetilde{\xi}}$ (from the right) or $p_\xi - \boldsymbol{\Lambda}_\xi$ (from the left). Thus, from (4.19a) and (4.17c) as well as (4.19c) we obtain the following set of relations (suppressing the arguments ξ of the functions):

$$p_\xi \widetilde{u_{0,0}} + \widetilde{u_{0,1}} = p_\xi u_{0,0} - u_{1,0} - \widetilde{u_{0,0}} u_{0,0}\ , \tag{5.2a}$$

$$p_\xi^2 \widetilde{u_{0,0}} + p_\xi (\widetilde{u_{1,0}} + \widetilde{u_{0,1}}) + \widetilde{u_{1,1}} = \wp(\delta) u_{0,0} - u_{2,0} - (p_\xi \widetilde{u_{0,0}} + \widetilde{u_{1,0}}) u_{0,0}\ , \tag{5.2b}$$

$$\wp(\delta)\widetilde{u_{0,0}} - \widetilde{u_{0,2}} = p_\xi^2 u_{0,0} - p_\xi (u_{0,1} + u_{1,0}) + u_{1,1} - \widetilde{u_{0,0}}(p_\xi u_{0,0} - u_{0,1})\ , \tag{5.2c}$$

$$\begin{aligned}&(\wp(\delta) + \wp(\xi)) u_{0,0} + u_{0,2} - p_\xi u_{0,1} \\ &\quad = (\wp(\delta) + \wp(\widetilde{\xi}))\widetilde{u_{0,0}} + \widetilde{u_{2,0}} + (p_\xi + u_{0,0})\widetilde{u_{1,0}} + (p_\xi u_{0,0} - u_{0,1})\widetilde{u_{0,0}}\ .\end{aligned} \tag{5.2d}$$

Similarly from their counterparts for the other shift, i.e. replacing the shift $\widetilde{\ }$ by the shift $\widehat{\ }$ and p_ξ by q_ξ, we obtain:

$$q_\xi \widehat{u_{0,0}} + \widehat{u_{0,1}} = q_\xi u_{0,0} - u_{1,0} - \widehat{u_{0,0}} u_{0,0}\ , \tag{5.3a}$$

$$q_\xi^2 \widehat{u_{0,0}} + q_\xi (\widehat{u_{1,0}} + \widehat{u_{0,1}}) + \widehat{u_{1,1}} = \wp(\varepsilon) u_{0,0} - u_{2,0} - (q_\xi \widehat{u_{0,0}} + \widehat{u_{1,0}}) u_{0,0}\ , \tag{5.3b}$$

$$\wp(\varepsilon)\widehat{u_{0,0}} - \widehat{u_{0,2}} = q_\xi^2 u_{0,0} - q_\xi (u_{0,1} + u_{1,0}) + u_{1,1} - \widehat{u_{0,0}}(q_\xi u_{0,0} - u_{0,1})\ , \tag{5.3c}$$

$$\begin{aligned}&(\wp(\varepsilon) + \wp(\xi)) u_{0,0} + u_{0,2} - q_\xi u_{0,1} \\ &\quad = (\wp(\varepsilon) + \wp(\widehat{\xi}))\widehat{u_{0,0}} + \widehat{u_{2,0}} + (q_\xi + u_{0,0})\widehat{u_{1,0}} + (q_\xi u_{0,0} - u_{0,1})\widehat{u_{0,0}}\ .\end{aligned} \tag{5.3d}$$

There exist several consistencies among these relations. For instance, subtracting (5.3b) from (5.2b) and using (5.2c) and (5.3c) to eliminate $u_{1,1}$ we find a triviality provided the following relation

$$p_{\widehat{\xi}} + q_\xi = q_{\widetilde{\xi}} + p_\xi\ , \tag{5.4a}$$

which follows from the definition of these quantities, is being used. Another relation that is frequently needed in these computations is

$$p_{\widehat{\xi}} q_\xi + \wp(\widehat{\xi}) = q_{\widetilde{\xi}} p_\xi + \wp(\widetilde{\xi})\ , \tag{5.4b}$$

which is a consequence of

$$\wp(\delta) + \wp(\varepsilon) + \wp(\widetilde{\xi}) + p_\xi q_{\widetilde{\xi}} = -\frac{1}{2}\frac{\wp'(\delta) - \wp'(\varepsilon)}{p_\xi - q_\xi}\ , \tag{5.4c}$$

which in turn follows from the general relation (A.14) of Appendix A.

Combining the above relations in a nontrivial way, while eliminating $u_{0,2}$ and $u_{2,0}$ and $u_{1,1}$, is obtained by subtracting (5.3c) from (5.2c) and using the $\widetilde{\ }$-shift of (5.3d) and the $\widehat{\ }$-shift of (5.2d), which leads to the following relation:

$$\begin{aligned}&\frac{1}{2}\frac{\wp'(\delta)-\wp'(\varepsilon)}{p_\xi - q_\xi + \widehat{u_{0,0}} - \widetilde{u_{0,0}}}\\ &\quad = \frac{1}{2}\frac{\wp'(\delta)-\wp'(\varepsilon)}{p_\xi - q_\xi} + \widehat{\widetilde{u_{1,0}}} + u_{0,1} + u_{0,0}\widehat{\widetilde{u_{0,0}}} + (p_{\hat{\xi}} + q_\xi)(\widehat{\widetilde{u_{0,0}}} - u_{0,0})\,, \end{aligned} \tag{5.5}$$

which together with (5.2a) and (5.3a) form a coupled 3-component system for the functions $u_{0,0}$, $u_{1,0}$ and $u_{0,1}$ which is equivalent to the lattice BSQ system. In fact, by eliminating the quantities $u_{1,0}$ and $u_{0,1}$ in this three-component lattice system, by shifting (5.5) in both lattice directions, we obtain the 9-point equation:

$$\begin{aligned}&\frac{1}{2}\frac{\wp'(\delta)-\wp'(\varepsilon)}{p_{\hat{\xi}} - q_{\hat{\xi}} + \widehat{\widehat{u_{0,0}}} - \widehat{\widetilde{u_{0,0}}}} - \frac{1}{2}\frac{\wp'(\delta)-\wp'(\varepsilon)}{p_{\tilde{\xi}} - q_{\tilde{\xi}} + \widehat{\widetilde{u_{0,0}}} - \widetilde{\widetilde{u_{0,0}}}}\\ &\quad = \left(p_{\hat{\tilde{\xi}}} - q_{\hat{\tilde{\xi}}} + \widehat{\widehat{\widetilde{u_{0,0}}}} - \widehat{\widetilde{\widetilde{u_{0,0}}}}\right)\left(p_{\hat{\hat{\tilde{\xi}}}} + p_{\hat{\hat{\xi}}} + +q_{\hat{\xi}} + \widehat{u_{0,0}} - \widehat{\widehat{\widetilde{\widetilde{u_{0,0}}}}}\right)\\ &\qquad - \left(p_\xi - q_\xi + \widehat{u_{0,0}} - \widetilde{u_{0,0}}\right)\left(p_{\hat{\tilde{\xi}}} + p_{\hat{\xi}} + q_\xi + u_{0,0} - \widehat{\widetilde{\widetilde{u_{0,0}}}}\right)\,. \end{aligned} \tag{5.6}$$

In Sect. 6, we will relate (5.6) to the 'normalised' form of the lattice BSQ equation (1.2) and show that the DL scheme leads to explicit solutions, namely of elliptic seed and soliton type.

5.2. Parameter-dependent quantities and the lattice modified BSQ equation. In order to obtain closed-form equations from the shift relations (4.17a) and (4.17c), we introduce the following objects:

$$s_\alpha(\xi) = \eta_\alpha(\xi) - \left(\left(\chi^{(1)}_{\alpha,\xi}(\boldsymbol{\Lambda})\right)^{-1}\boldsymbol{U}_\xi\,\eta_\xi({}^t\boldsymbol{\Lambda})\right)_{0,0}\,, \tag{5.7a}$$

$$t_\beta(\xi) = \eta_\beta(\xi) - \left(\eta_\xi(\boldsymbol{\Lambda})\boldsymbol{U}_\xi\left(\chi^{(1)}_{\beta,\xi}({}^t\boldsymbol{\Lambda})\right)^{-1}\right)_{0,0}\,, \tag{5.7b}$$

$$r_\alpha(\xi) = \wp(\alpha) - \left(\left(\chi^{(1)}_{\alpha,\xi}(\boldsymbol{\Lambda})\right)^{-1}\boldsymbol{U}_\xi\,\wp({}^t\boldsymbol{\Lambda})\right)_{0,0}\,, \tag{5.7c}$$

$$z_\beta(\xi) = \wp(\beta) - \left(\wp(\boldsymbol{\Lambda})\boldsymbol{U}_\xi\left(\chi^{(1)}_{\beta,\xi}({}^t\boldsymbol{\Lambda})\right)^{-1}\right)_{0,0}\,, \tag{5.7d}$$

and recall $v_\alpha(\xi)$, $w_\beta(\xi)$, $s_{\alpha,\beta}(\xi)$ defined in Sect. 2.3 :

$$v_\alpha(\xi) = 1 - \left(\left(\chi^{(1)}_{\alpha,\xi}(\boldsymbol{\Lambda})\right)^{-1}\boldsymbol{U}_\xi\right)_{0,0}\,, \tag{5.7e}$$

$$w_\beta(\xi) = 1 - \left(\boldsymbol{U}_\xi\left(\chi^{(1)}_{\beta,\xi}({}^t\boldsymbol{\Lambda})\right)^{-1}\right)_{0,0}\,, \tag{5.7f}$$

$$s_{\alpha,\beta}(\xi) = \left(\left(\chi^{(1)}_{\alpha,\xi}(\boldsymbol{\Lambda})\right)^{-1}\boldsymbol{U}_\xi\left(\chi^{(1)}_{\beta,\xi}({}^t\boldsymbol{\Lambda})\right)^{-1}\right)_{0,0}\,. \tag{5.7g}$$

Several identities are needed to derive the relevant single-shift relations for these quantities, namely

$$\frac{\chi^{(1)}_{\delta,\widetilde{\xi}}(\boldsymbol{\Lambda})}{\chi^{(1)}_{\alpha,\widetilde{\xi}}(\boldsymbol{\Lambda})}=1-\frac{\eta_\delta(\widetilde{\xi})-\eta_\alpha(\widetilde{\xi})}{\chi^{(1)}_{\alpha,\xi}(\boldsymbol{\Lambda})}\,,\quad \frac{\chi^{(1)}_{-\delta,\xi}({}^t\boldsymbol{\Lambda})}{\chi^{(1)}_{\beta,\xi}({}^t\boldsymbol{\Lambda})}=1-\frac{\eta_{-\delta}(\xi)-\eta_\beta(\xi)}{\chi^{(1)}_{\beta,\widetilde{\xi}}({}^t\boldsymbol{\Lambda})}\,, \tag{5.8}$$

which follow from (A.13), as well as

$$\frac{\wp(\widetilde{\xi})+\wp(\boldsymbol{\Lambda})+\wp(\delta)-\eta_\delta(\widetilde{\xi})\eta_{\widetilde{\xi}}(\boldsymbol{\Lambda})}{\chi^{(1)}_{\alpha,\xi}(\boldsymbol{\Lambda})}$$
$$=\zeta(\alpha+\xi)-\zeta(\delta)-\zeta(\widetilde{\xi})-\zeta(\alpha+\boldsymbol{\Lambda})+\zeta(\boldsymbol{\Lambda})-\eta_\delta(\widetilde{\xi})\frac{\eta_{-\delta}(\xi)-\eta_\alpha(\xi)}{\chi^{(1)}_{\alpha,\widetilde{\xi}}(\boldsymbol{\Lambda})}\,, \tag{5.9a}$$

$$\frac{\wp(\xi)+\wp({}^t\boldsymbol{\Lambda})+\wp(\delta)-\eta_{-\delta}(\xi)\eta_\xi({}^t\boldsymbol{\Lambda})}{\chi^{(1)}_{\beta,\widetilde{\xi}}({}^t\boldsymbol{\Lambda})}$$
$$=\zeta(\beta+\widetilde{\xi})+\zeta(\delta)-\zeta(\xi)-\zeta(\beta+{}^t\boldsymbol{\Lambda})+\zeta({}^t\boldsymbol{\Lambda})-\eta_{-\delta}(\xi)\frac{\eta_\delta(\widetilde{\xi})-\eta_\beta(\widetilde{\xi})}{\chi^{(1)}_{\beta,\xi}({}^t\boldsymbol{\Lambda})}\,, \tag{5.9b}$$

which follow from (A.10a) and (A.8) in combination with (5.8). Using (5.8) by multiplying (4.17a) by factors $\left(\chi^{(1)}_{\alpha,\widetilde{\xi}}(\boldsymbol{\Lambda})\right)^{-1}$ from the left and by $\left(\chi^{(1)}_{\beta,\xi}({}^t\boldsymbol{\Lambda})\right)^{-1}$ from the right and taking the $(\ )_{0,0}$ element, and using the relations (5.9) by multiplying (4.17c) by factors $\left(\chi^{(1)}_{\alpha,\xi}(\boldsymbol{\Lambda})\right)^{-1}$ from the left and by $\left(\chi^{(1)}_{\beta,\widetilde{\xi}}({}^t\boldsymbol{\Lambda})\right)^{-1}$ from the right and taking the $(\ )_{0,0}$ element, we obtain the following relations relating the 2-parameter quantity $s_{\alpha,\beta}(\xi)$ to the other quantities defined above:

$$1+(p_\xi+a_{\widetilde{\xi}})s_{\alpha,\beta}(\xi)-(p_\xi-b_\xi)\widetilde{s_{\alpha,\beta}(\xi)}=\widetilde{v_\alpha(\xi)}w_\beta(\xi)\,, \tag{5.10a}$$

$$p_\xi-a_\xi+b_{\widetilde{\xi}}+B_{-\delta}(\xi)s_{\alpha,\beta}(\xi)-A_\delta(\widetilde{\xi})\widetilde{s_{\alpha,\beta}(\xi)}$$
$$=p_\xi v_\alpha(\xi)\widetilde{w_\beta(\xi)}+v_\alpha(\xi)\widetilde{t_\beta(\xi)}-s_\alpha(\xi)\widetilde{w_\beta(\xi)}\,, \tag{5.10b}$$

respectively, in which, for convenience, we have introduced the notations:

$$a_\xi:=\eta_\alpha(\xi)\,,\qquad b_\xi:=\eta_\beta(\xi)\,, \tag{5.11a}$$

$$A_\nu(\xi):=\frac{1}{2}\frac{\wp'(\nu)-\wp'(\alpha)}{\eta_\nu(\xi)-\eta_\alpha(\xi)}\,,\quad B_\nu(\xi):=\frac{1}{2}\frac{\wp'(\nu)-\wp'(\beta)}{\eta_\nu(\xi)-\eta_\beta(\xi)}\,. \tag{5.11b}$$

The latter quantities appear in (5.10b) in the following form:

$$A_\delta(\widetilde{\xi})=p_\xi(p_\xi-a_\xi)-\wp(\widetilde{\xi})+\wp(\alpha)$$
$$=\wp(\alpha)+\wp(\xi)+\wp(\delta)-\eta_{-\delta}(\xi)\eta_\alpha(\xi)\,, \tag{5.12a}$$

$$B_{-\delta}(\xi)=p_\xi(p_\xi+b_{\widetilde{\xi}})-\wp(\xi)+\wp(\beta)$$
$$=\wp(\beta)+\wp(\widetilde{\xi})+\wp(\delta)-\eta_\delta(\widetilde{\xi})\eta_\beta(\widetilde{\xi})\,, \tag{5.12b}$$

(noting the curious interchange of the roles of ξ and $\widetilde{\xi}$ when δ changes into $-\delta$). Note that we also have the relations:

$$A_\delta(\widetilde{\xi}) = p_{-\alpha}(p_\xi - a_\xi)\ , \quad B_{-\delta}(\xi) = p_\beta(p_\xi + b_{\widetilde{\xi}})\ . \tag{5.13}$$

For the single-parameter quantities $s_\alpha, t_\beta, r_\alpha, z_\beta$ we can derive a system of shift relations in a similar way, using single-parameter multiplying factors from the left or the right before taking the $(\ )_{0,0}$ elements. Thus, from (4.17a) multiplying from the left by $\left(\chi^{(1)}_{\alpha,\widetilde{\xi}}(\boldsymbol{\Lambda})\right)^{-1}$ or by $\left(\chi^{(1)}_{\beta,\xi}({}^t\boldsymbol{\Lambda})\right)^{-1}$ from the right and projecting on the $(\ )_{0,0}$ element, we obtain:

$$\widetilde{s_\alpha(\xi)} = -(p_\xi + u_{0,0})\widetilde{v_\alpha(\xi)} + (p_\xi + a_{\widetilde{\xi}})v_\alpha(\xi)\ , \tag{5.14a}$$

$$t_\beta(\xi) = (p_\xi - \widetilde{u_{0,0}})w_\beta(\xi) - (p_\xi - b_\xi)\widetilde{w_\beta(\xi)}\ . \tag{5.14b}$$

Furthermore, by multiplying (4.17a) from the left by $\left(\chi^{(1)}_{\alpha,\widetilde{\xi}}(\boldsymbol{\Lambda})\right)^{-1}$ and from the right by $\chi^{(1)}_{\delta,\widetilde{\xi}}({}^t\boldsymbol{\Lambda})$ and taking the $(\ )_{0,0}$ element, respectively by multiplying from the right by $\left(\chi^{(1)}_{\beta,\xi}({}^t\boldsymbol{\Lambda})\right)^{-1}$ and from the left by $\chi^{(1)}_{-\delta,\xi}(\boldsymbol{\Lambda})$ and projecting, we obtain

$$\widetilde{r_\alpha(\xi)} = (\wp(\delta) + p_\xi u_{0,0} - u_{0,1})\widetilde{v_\alpha(\xi)} - (p_\xi + a_{\widetilde{\xi}})\left(p_\xi v_\alpha(\xi) - s_\alpha(\xi)\right)\ , \tag{5.15a}$$

$$z_\beta(\xi) = (\wp(\delta) - p_\xi\widetilde{u_{0,0}} - \widetilde{u_{1,0}})w_\beta(\xi) - (p_\xi - b_\xi)\left(p_\xi\widetilde{w_\beta(\xi)} + \widetilde{t_\beta(\xi)}\right)\ . \tag{5.15b}$$

Finally, from (4.17c) by multiplying from the left by $\left(\chi^{(1)}_{\alpha,\xi}(\boldsymbol{\Lambda})\right)^{-1}$, or from the right by $\left(\chi^{(1)}_{\beta,\widetilde{\xi}}({}^t\boldsymbol{\Lambda})\right)^{-1}$ and taking the $(\ )_{0,0}$ elements, while using the relations (5.9a) and (5.9b) respectively, we obtain

$$r_\alpha(\xi) = A_\delta(\widetilde{\xi})\widetilde{v_\alpha(\xi)} + (p_\xi - \widetilde{u_{0,0}})s_\alpha(\xi) - \left(\wp(\xi) + \wp(\delta) - p_\xi\widetilde{u_{0,0}} - \widetilde{u_{1,0}}\right)v_\alpha(\xi)\ , \tag{5.16a}$$

$$\widetilde{z_\beta(\xi)} = B_{-\delta}(\xi)w_\beta(\xi) - (p_\xi + u_{0,0})\widetilde{t_\beta(\xi)} - (\wp(\widetilde{\xi}) + \wp(\delta) + p_\xi u_{0,0} - u_{0,1})\widetilde{w_\beta(\xi)}\ . \tag{5.16b}$$

Henceforth we will, for brevity, suppress the arguments ξ in these parameter functions, and simply write v_α for $v_\alpha(\xi)$, etc., where it is understood that shifts $\widetilde{v_\alpha} = \widetilde{v_\alpha(\xi)}$ act on the function as well as its argument, unless explicitly stated.

Obviously the relations (5.14)–(5.16) have their counterpart for the other lattice direction, replacing $\widetilde{\ }$ by $\widehat{\ }$ and δ by ε. By combining relations of this type for the two lattice directions, and thereby eliminating some of the parameter quantities, various significant additional relations of Miura type can be derived. Here the relations (5.2a) and (5.3a) are needed as well to eliminate the quantities $u_{0,0}$, $u_{1,0}$ and $u_{0,1}$ when necessary. For instance, from (5.16) by eliminating the quantities r_α and z_β and using (5.2a) and (5.3a) we find the 2-shift relations

$$A_\delta(\widetilde{\xi})\frac{\widetilde{v_\alpha}}{v_\alpha} - A_\varepsilon(\widehat{\xi})\frac{\widehat{v_\alpha}}{v_\alpha} = (p_\xi - q_\xi + \widehat{u_{0,0}} - \widetilde{u_{0,0}}) \left(p_{\widehat{\xi}} + q_\xi - \frac{s_\alpha}{v_\alpha} - \widehat{\widetilde{u_{0,0}}}\right) , \tag{5.17a}$$

$$B_{-\delta}(\widehat{\xi})\frac{\widehat{w_\beta}}{\widehat{\widetilde{w_\beta}}} - B_{-\varepsilon}(\widetilde{\xi})\frac{\widetilde{w_\beta}}{\widehat{\widetilde{w_\beta}}} = (p_\xi - q_\xi + \widehat{u_{0,0}} - \widetilde{u_{0,0}}) \left(p_{\widehat{\xi}} + q_\xi + u_{0,0} + \frac{\widehat{\widetilde{t_\beta}}}{\widehat{\widetilde{w_\beta}}}\right), \tag{5.17b}$$

where we have also used (5.4a) and the relation

$$(p_\xi - q_\xi)(p_{\widehat{\xi}} + q_\xi) = \wp(\delta) - \wp(\varepsilon) , \tag{5.18}$$

which follows from the definitions of these quantities and the relation (A.8). At the same time, from (5.14) and its counterparts with shifts in the other lattice direction, we can derive the relations:

$$p_\xi - q_\xi + \widehat{u_{0,0}} - \widetilde{u_{0,0}} = (p_{\widehat{\xi}} + a_{\widehat{\widetilde{\xi}}})\frac{\widehat{v_\alpha}}{\widehat{\widetilde{v_\alpha}}} - (q_{\widetilde{\xi}} + a_{\widehat{\widetilde{\xi}}})\frac{\widetilde{v_\alpha}}{\widehat{\widetilde{v_\alpha}}} \tag{5.19a}$$

$$= (p_\xi - b_\xi)\frac{\widetilde{w_\beta}}{w_\beta} - (q_\xi - b_\xi)\frac{\widehat{w_\beta}}{w_\beta} , \tag{5.19b}$$

as well as the relations

$$p_\xi - q_\xi - \frac{\widehat{s_\alpha}}{\widehat{v_\alpha}} + \frac{\widetilde{s_\alpha}}{\widetilde{v_\alpha}} = (p_\xi + a_{\widetilde{\xi}})\frac{v_\alpha}{\widetilde{v_\alpha}} - (q_\xi + a_{\widehat{\xi}})\frac{v_\alpha}{\widehat{v_\alpha}} , \tag{5.20a}$$

$$p_{\widehat{\xi}} - q_{\widetilde{\xi}} - \frac{\widehat{t_\beta}}{\widehat{w_\beta}} + \frac{\widetilde{t_\beta}}{\widetilde{w_\beta}} = (p_{\widehat{\xi}} - b_{\widehat{\xi}})\frac{\widehat{\widetilde{w_\beta}}}{\widehat{w_\beta}} - (q_{\widetilde{\xi}} - b_{\widetilde{\xi}})\frac{\widehat{\widetilde{w_\beta}}}{\widetilde{w_\beta}} . \tag{5.20b}$$

We note at this point that we have now already two closed systems of equations, namely one for the quantities $u_{0,0}$, v_α and s_α given by (5.17a), and the two relations given by (5.14a) and its counterpart in the other lattice shift direction, and the other for $u_{0,0}$, w_β and t_β (5.17b), and the two relations given by (5.14b) with its counterpart in the other lattice direction. However, one can also derive at this juncture a closed-form 9-point equation for v_α, or alternatively for w_β, alone. In fact, by solving s_α/v_α from (5.17a), and inserting the result into (5.14a), we get

$$p_\xi + p_{\widehat{\widetilde{\xi}}} + q_{\widetilde{\xi}} + u_{0,0} - \widehat{\widetilde{\widetilde{u_{0,0}}}} - \frac{\widehat{\widetilde{\widetilde{v_\alpha}}}}{\widetilde{v_\alpha}}\, \frac{A_\delta(\widetilde{\widetilde{\xi}})\widetilde{\widetilde{v_\alpha}} - A_\varepsilon(\widehat{\widetilde{\xi}})\widehat{\widetilde{v_\alpha}}}{(p_{\widehat{\widetilde{\xi}}} + a_{\widehat{\widetilde{\widetilde{\xi}}}})\widehat{\widetilde{v_\alpha}} - (q_{\widetilde{\widetilde{\xi}}} + a_{\widehat{\widetilde{\widetilde{\xi}}}})\widetilde{\widetilde{v_\alpha}}} = (p_\xi + a_{\widetilde{\xi}})\frac{v_\alpha}{\widetilde{v_\alpha}} ,$$

and by subtracting a second copy of this relation with δ and ε interchanged (interchanging also p_ξ and q_ξ and the $\widetilde{\ }$ and $\widehat{\ }$ lattice shifts) we can use (5.19a) to eliminate all the quantities $u_{0,0}$. The result is a 9-point equation for v_α, namely

$$
\begin{aligned}
&\frac{-A_\delta(\widehat{\widetilde{\xi}})\widehat{\widetilde{v_\alpha(\xi)}} + A_\varepsilon(\widehat{\widehat{\xi}})\widehat{\widehat{v_\alpha(\xi)}}}{(q_{\widehat{\widetilde{\xi}}} + a_{\widehat{\widehat{\widetilde{\xi}}}})\widehat{\widetilde{v_\alpha(\xi)}} - (p_{\widehat{\widehat{\xi}}} + a_{\widehat{\widehat{\widetilde{\xi}}}})\widehat{\widehat{v_\alpha(\xi)}}}\,\frac{\widehat{\widehat{\widetilde{v_\alpha(\xi)}}}}{\widehat{v_\alpha(\xi)}} - \frac{-A_\varepsilon(\widehat{\widetilde{\xi}})\widehat{\widetilde{v_\alpha(\xi)}} + A_\delta(\widetilde{\widetilde{\xi}})\widetilde{\widetilde{v_\alpha(\xi)}}}{(p_{\widehat{\widetilde{\xi}}} + a_{\widehat{\widetilde{\widetilde{\xi}}}})\widehat{\widetilde{v_\alpha(\xi)}} - (q_{\widetilde{\widetilde{\xi}}} + a_{\widehat{\widetilde{\widetilde{\xi}}}})\widetilde{\widetilde{v_\alpha(\xi)}}}\,\frac{\widehat{\widetilde{\widetilde{v_\alpha(\xi)}}}}{\widetilde{v_\alpha(\xi)}} \\
&= \left(p_\xi + a_{\widetilde{\xi}}\right)\frac{v_\alpha(\xi)}{\widetilde{v_\alpha(\xi)}} - \left(q_\xi + a_{\widehat{\xi}}\right)\frac{v_\alpha(\xi)}{\widehat{v_\alpha(\xi)}} \\
&\quad -\left(p_{\widehat{\widetilde{\xi}}} + a_{\widehat{\widehat{\widetilde{\xi}}}}\right)\frac{\widehat{\widehat{\widetilde{v_\alpha(\xi)}}}}{\widehat{\widehat{\widetilde{\widetilde{v_\alpha(\xi)}}}}} + \left(q_{\widehat{\widetilde{\xi}}} + a_{\widehat{\widetilde{\widetilde{\xi}}}}\right)\frac{\widehat{\widetilde{\widetilde{v_\alpha(\xi)}}}}{\widehat{\widehat{\widetilde{\widetilde{v_\alpha(\xi)}}}}}\,,
\end{aligned}
\tag{5.21}
$$

which can be brought into the form of the lattice modified BSQ equation of [29].

Similarly, solving $\widehat{\widetilde{t_\beta/w_\beta}}$ from (5.17b) and inserting this into (5.14b) we get

$$
\frac{w_\beta}{\widehat{\widetilde{w_\beta}}}\,\frac{B_{-\delta}(\widehat{\xi})\widehat{w_\beta} - B_{-\varepsilon}(\widetilde{\xi})\widetilde{w_\beta}}{(p_\xi - b_\xi)\widetilde{w_\beta} - (q_\xi - b_\xi)\widehat{w_\beta}} + (p_{\widehat{\widetilde{\xi}}} - b_{\widehat{\widetilde{\xi}}})\frac{\widehat{\widetilde{\widetilde{w_\beta}}}}{\widehat{\widetilde{w_\beta}}} = q_\xi + p_{\widehat{\xi}} + p_{\widehat{\widetilde{\xi}}} + u_{0,0} - \widehat{\widetilde{\widetilde{u_{0,0}}}}\,,
$$

and subtracting the shifted versions of this relation in both lattice directions, we can eliminate the quantities $u_{0,0}$ by using (5.19b) to get a closed-form 9-point relation for w_β. The result is:

$$
\begin{aligned}
&\frac{\widehat{w_\beta}}{\widehat{\widehat{\widetilde{w_\beta}}}}\,\frac{B_{-\delta}(\widehat{\widehat{\xi}})\widehat{\widehat{w_\beta}} - B_{-\varepsilon}(\widehat{\widetilde{\xi}})\widehat{\widetilde{w_\beta}}}{(p_{\widehat{\xi}} - b_{\widehat{\xi}})\widehat{\widetilde{w_\beta}} - (q_{\widehat{\xi}} - b_{\widehat{\xi}})\widehat{\widehat{w_\beta}}} - (p_{\widehat{\widetilde{\widetilde{\xi}}}} - b_{\widehat{\widetilde{\widetilde{\xi}}}})\frac{\widehat{\widetilde{\widetilde{\widetilde{w_\beta}}}}}{\widehat{\widetilde{\widetilde{w_\beta}}}} \\
&\quad + (p_{\widehat{\widehat{\widetilde{\xi}}}} - b_{\widehat{\widehat{\widetilde{\xi}}}})\frac{\widehat{\widehat{\widehat{\widetilde{w_\beta}}}}}{\widehat{\widehat{\widetilde{w_\beta}}}} - \frac{\widetilde{w_\beta}}{\widehat{\widetilde{\widetilde{w_\beta}}}}\,\frac{B_{-\delta}(\widehat{\widetilde{\xi}})\widehat{\widetilde{w_\beta}} - B_{-\varepsilon}(\widetilde{\widetilde{\xi}})\widetilde{\widetilde{w_\beta}}}{(p_{\widetilde{\xi}} - b_{\widetilde{\xi}})\widetilde{\widetilde{w_\beta}} - (q_{\widetilde{\xi}} - b_{\widetilde{\xi}})\widehat{\widetilde{w_\beta}}}
\end{aligned}
\tag{5.22}
$$

$$
= (p_\xi - b_\xi)\frac{\widetilde{w_\beta}}{w_\beta} - (q_\xi - b_\xi)\frac{\widehat{w_\beta}}{w_\beta} - (p_{\widehat{\widetilde{\widetilde{\xi}}}} - b_{\widehat{\widetilde{\widetilde{\xi}}}})\frac{\widehat{\widetilde{\widetilde{\widetilde{w_\beta}}}}}{\widehat{\widetilde{\widetilde{w_\beta}}}} + (q_{\widehat{\widetilde{\widetilde{\xi}}}} - b_{\widehat{\widetilde{\widetilde{\xi}}}})\frac{\widehat{\widehat{\widetilde{\widetilde{w_\beta}}}}}{\widehat{\widetilde{\widetilde{w_\beta}}}}\,. \tag{5.23}
$$

In Sect. 6 we shall bring this equation in the standard form of lattice modified BSQ and present some explicit elliptic type solutions.

5.3. Trilinear equation of the τ-function for the lattice BSQ system. The τ-function associated with the lattice BSQ system is exactly the same as given for the lattice KP system, introduced in Sect. 2.2, namely given by (2.10), but of course taking into account the BSQ reduction discussed in Sect. 4.2. The principal relations this τ-function obeys were derived in Appendix B, and are given by te shift relations

$$
\frac{\underset{\sim}{\tau_\xi}}{\tau_\xi} = v_\delta(\xi)\,, \quad \frac{\widetilde{\tau_\xi}}{\tau_\xi} = w_{-\delta}(\xi)\,, \quad \frac{\underset{\frown}{\tau_\xi}}{\tau_\xi} = v_\varepsilon(\xi), \quad \frac{\widehat{\tau_\xi}}{\tau_\xi} = w_{-\varepsilon}(\xi)\,, \tag{5.24}
$$

where the overtilde $\widetilde{\tau_\xi}$ and the undertilde $\underset{\sim}{\tau_\xi}$ denotes the forward and backward shift T_δ and T_δ^{-1} respectively, implemented also on the argument ξ, and similarly for the overhat $\widehat{\tau_\xi}$ and underhat $\underset{\frown}{\tau}_\xi$, associated with the forward and backward shifts T_ε and T_ε^{-1} respectively, acting also on the argument ξ. As a consequence of (5.24), together with the relations

$$1-\chi^{(1)}_{\alpha,-\delta}(\xi)s_{\alpha,\beta}(\xi)-\chi^{(1)}_{\beta,\delta}(\widetilde{\xi})\widetilde{s_{\alpha,\beta}(\xi)}=\widetilde{v_\alpha(\xi)}w_\beta(\xi)\ , \tag{5.25a}$$

$$\begin{aligned}&\zeta(\xi+\alpha)-\zeta(\alpha)-\zeta(\delta)+\zeta(\beta)-\zeta(\widetilde{\xi}+\beta)\\&-\frac{1}{2}\frac{\wp'(\beta)+\wp'(\delta)}{\eta_\beta(\xi)-\eta_{-\delta}(\xi)}s_{\alpha,\beta}(\xi)-\frac{1}{2}\frac{\wp'(\delta)-\wp'(\alpha)}{\eta_\alpha(\widetilde{\xi})-\eta_\delta(\widetilde{\xi})}\widetilde{s_{\alpha,\beta}(\xi)}\\&\quad=\ s_\alpha(\xi)\widetilde{w_\beta(\xi)}-v_\alpha(\xi)\widetilde{t_\beta(\xi)}+\eta_\delta(\widetilde{\xi})v_\alpha(\xi)\widetilde{w_\beta(\xi)}\ ,\end{aligned} \tag{5.25b}$$

(which are equivalent to (5.10)), setting $\alpha=\delta$, $\beta=-\varepsilon$, we also have

$$1-\chi^{(1)}_{\delta,-\varepsilon}(\xi)s_{\delta,-\varepsilon}(\xi)=\frac{\underset{\sim}{\widehat{\tau_\xi}}}{\tau_\xi}\ . \tag{5.26}$$

Furthermore, from (5.19b), setting $\beta=-\delta$, together with (5.24), we get

$$p_\xi-q_\xi+\widehat{u_{0,0}}-\widetilde{u_{0,0}}=(p_\xi-q_\xi)\frac{\tau_\xi\ \widehat{\widetilde{\tau_\xi}}}{\widehat{\tau_\xi}\ \widetilde{\tau_\xi}}\ , \tag{5.27}$$

while from (5.14a) setting $\alpha=\delta$, which implies $a_{\widetilde{\xi}}=-p_\xi$, we see that

$$\frac{\widetilde{s_\delta}}{\widetilde{v_\delta}}=-(p_\xi+u_{0,0})\ .$$

The latter inserted into (5.17a), with $\alpha=\delta$, implying $A_\delta(\widetilde{\xi})=P_\delta(\xi)$, $A_\varepsilon(\widehat{\xi})=P_\varepsilon(\widehat{\xi})$, gives us:

$$(p_\xi-q_\xi)\left(p_{\widetilde{\xi}}+q_\xi+p_{\underset{\sim}{\xi}}+\underset{\sim}{u}_{0,0}-\widehat{\widetilde{u_{0,0}}}\right)\frac{\widehat{\widetilde{\tau_\xi}}}{\widehat{\tau_\xi}\ \widetilde{\tau_\xi}}=P_\delta(\xi)\frac{\tau_\xi}{\widetilde{\tau_\xi}\ \underset{\sim}{\tau_\xi}}-P_\varepsilon(\widehat{\xi})\frac{\underset{\sim}{\widehat{\tau_\xi}}}{\widehat{\tau_\xi}\ \underset{\sim}{\tau_\xi}}\ ,$$

where now the coefficients $A_\delta(\widetilde{\xi})$ and $A_\varepsilon(\widehat{\xi})$ when $\alpha=\delta$ are given by

$$P_\delta(\xi):=\ A_\delta(\widetilde{\xi})\big|_{\alpha=\delta}=\frac{1}{2}\frac{\wp''(\delta)}{\wp(\delta)-\wp(\xi)}\ ,\quad P_\varepsilon(\widehat{\xi}):=\ A_\varepsilon(\widehat{\xi})\big|_{\alpha=\delta}=\frac{1}{2}\frac{\wp'(\varepsilon)-\wp'(\delta)}{\eta_\varepsilon(\widehat{\xi})-\eta_\delta(\widehat{\xi})}\ .$$

Adding to the latter relation the one in which the $\widetilde{\ }$ and $\widehat{\ }$ shifts and the δ and ε parameters are interchanged, thereby introducing the coefficients

$$Q_\varepsilon(\xi):=\ A_\varepsilon(\widetilde{\xi})\big|_{\alpha=\varepsilon}=\frac{1}{2}\frac{\wp''(\varepsilon)}{\wp(\varepsilon)-\wp(\xi)}\ ,\quad Q_\delta(\widetilde{\xi}):=\ A_\delta(\widetilde{\xi})\big|_{\alpha=\varepsilon}=\frac{1}{2}\frac{\wp'(\delta)-\wp'(\varepsilon)}{\eta_\delta(\widetilde{\xi})-\eta_\varepsilon(\widetilde{\xi})}\ ,$$

and eliminating the $u_{0,0}$ using the relation (5.27), we arrive at a *trilinear* equation for the τ-function, namely

$$\begin{aligned}(p_\xi-q_\xi)(p_{\underset{\sim}{\xi}}-q_{\underset{\frown}{\xi}})\underset{\frown\sim}{\tau_\xi}\tau_\xi\widehat{\widetilde{\tau_\xi}}=&\ P_\delta(\xi)\underset{\frown}{\tau_\xi}\tau_\xi\widehat{\tau_\xi}+Q_\varepsilon(\xi)\ \underset{\sim}{\tau_\xi}\ \tau_\xi\widetilde{\tau_\xi}\\&-Q_\delta(\widetilde{\xi})\ \underset{\sim}{\tau_\xi}\ \underset{\frown}{\widetilde{\tau_\xi}}\widehat{\tau_\xi}-P_\varepsilon(\widehat{\xi})\underset{\frown}{\tau_\xi}\ \underset{\sim}{\widehat{\tau_\xi}}\ \widetilde{\tau_\xi}.\end{aligned} \tag{5.28}$$

Taking $\tau_\xi = \frac{f}{\sigma(\xi)}$ in (5.28), we derive an autonomous trilinear equation

$$\left(\Phi_\delta(-\varepsilon)\right)^2 \underset{\sim}{\underset{\frown}{f}}\, f\, \widehat{\widetilde{f}} + \frac{1}{2}\left(\wp''(\delta)\sigma^2(\delta)\underset{\frown}{f}\, f\, \widehat{f} + \wp''(\varepsilon)\sigma^2(\varepsilon)\, \underset{\sim}{f}\, f\, \widetilde{f}\right) \\ = \Phi_{-\varepsilon}(\omega_1(\delta))\Phi_{-\varepsilon}(\omega_2(\delta))(\underset{\sim}{f}\, \underset{\frown}{\widetilde{f}}\widehat{f} + \underset{\frown}{f}\, \underset{\sim}{\widehat{f}}\, \widetilde{f}). \tag{5.29}$$

A similar trilinear equation like (5.29) was found in [40] for the extended lattice BSQ system in the case of the rational parametrisation of the equation.

5.4. 2-parameter variables and generalised lattice Schwarzian BSQ equation. Now we will focus on the relations (5.10) for the quantity $s_{\alpha,\beta}(\xi)$ and derive a closed-form equation for that quantity.

First, we want to rewrite the relations in a form that is more suitable for a comparison with the standard ABS class of quad-lattice equations, cf. [2]. To achieve this we introduce the quantities[5]

$$S_{\alpha,\beta}(\xi) = s_{\alpha,\beta}(\xi) - \left(\chi^{(1)}_{\alpha,\beta}(\xi)\right)^{-1} . \tag{5.30}$$

The corresponding shift relations following from (5.10), or equivalently (5.10b), read

$$(p_\xi + a_{\tilde{\xi}})S_{\alpha,\beta} - (p_\xi - b_\xi)\widetilde{S_{\alpha,\beta}} = \widetilde{v_\alpha} w_\beta \,, \tag{5.31a}$$

$$B_{-\delta}(\xi)S_{\alpha,\beta} - A_\delta(\widetilde{\xi})\widetilde{S_{\alpha,\beta}} = p_\xi v_\alpha \widetilde{w_\beta} + v_\alpha \widetilde{t_\beta} - s_\alpha \widetilde{w_\beta} \,, \tag{5.31b}$$

(we omit the arguments ξ in the main functions, while keeping them in the coefficients, where it is understood that all the shifts acting on those functions also act on that argument ξ of the functions), and similarly the corresponding counterparts in the other lattice direction, namely

$$(q_\xi + a_{\hat{\xi}})S_{\alpha,\beta} - (q_\xi - b_\xi)\widehat{S_{\alpha,\beta}} = \widehat{v_\alpha} w_\beta \,, \tag{5.32a}$$

$$B_{-\varepsilon}(\xi)S_{\alpha,\beta} - A_\varepsilon(\widehat{\xi})\widehat{S_{\alpha,\beta}} = q_\xi v_\alpha \widehat{w_\beta} + v_\alpha \widehat{t_\beta} - s_\alpha \widehat{w_\beta} \,. \tag{5.32b}$$

Next, taking the $\widehat{\ }$ shift of (5.31b) and using (5.17b) and the $\{\widehat{\ }, \varepsilon\}$ version of (5.14a) to replace $\widehat{\widetilde{t_\beta(\xi)}}$ and $\widehat{s_\alpha(\xi)}$, respectively, and using (5.19b) to replace $p - q + \widehat{u_{0,0}} - \widetilde{u_{0,0}}$, we obtain

$$(q_\xi + a_{\hat{\xi}})v_\alpha\, \widehat{\widetilde{w_\beta}} = A_\delta(\widehat{\widetilde{\xi}})\widehat{\widetilde{S_{\alpha,\beta}}} - B_{-\delta}(\widehat{\xi})\widehat{S_{\alpha,\beta}} \\ + \widehat{v_\alpha}\, w_\beta\, \frac{B_{-\delta}(\widehat{\xi})\widehat{w_\beta} - B_{-\varepsilon}(\widetilde{\xi})\widetilde{w_\beta}}{(p_\xi - b_\xi)\widetilde{w_\beta} - (q_\xi - b_\xi)\widehat{w_\beta}} \,. \tag{5.33}$$

[5] From (5.26) we see that the quantity $S_{\alpha,\beta}$ could readily expressed in the τ-function if we allow there to lattice shifts associated with the parameters α and β in lieu of δ and ε. This would require, by MDC, that we have to extend the plane wave factors ρ_κ and $\sigma_{\kappa'}$ in the DL scheme to contain factors with α and β as lattice parameters, and hence directions in the lattice with discrete variables associated with these additional lattice directions.

Multiplying numerator and denominator in the fraction on the right-hand side by $\widehat{\widetilde{v}}_\alpha$ we can use (5.32a) to express the entire right-hand side in terms of $S_{\alpha,\beta}$, which leads to

$$v_\alpha\, \widehat{\widetilde{w}}_\beta = \frac{\mathcal{Q}(S_{\alpha,\beta}, \widetilde{S_{\alpha,\beta}}, \widehat{S_{\alpha,\beta}}, \widehat{\widetilde{S_{\alpha,\beta}}})}{(p_\xi - b_\xi)(q_{\widetilde{\xi}} + a_{\widehat{\widetilde{\xi}}})\widetilde{S_{\alpha,\beta}} - (q_\xi - b_\xi)(p_{\widehat{\xi}} + a_{\widehat{\widetilde{\xi}}})\widehat{S_{\alpha,\beta}}} \tag{5.34}$$

where in the denominator some of the terms have disappeared by virtue of the relation

$$(p_\xi - b_\xi)(q_{\widetilde{\xi}} - b_{\widetilde{\xi}}) - (q_\xi - b_\xi)(p_{\widehat{\xi}} - b_{\widehat{\xi}}) = \chi^{(1)}_{\beta,\delta}(\widetilde{\xi})\chi^{(1)}_{\beta,\varepsilon}(\widehat{\widetilde{\xi}}) - \chi^{(1)}_{\beta,\varepsilon}(\widehat{\xi})\chi^{(1)}_{\beta,\delta}(\widehat{\widetilde{\xi}}) = 0\,,$$

and where the quad function in the numerator is given by

$$\begin{aligned}\mathcal{Q}(S_{\alpha,\beta}, \widetilde{S_{\alpha,\beta}}, \widehat{S_{\alpha,\beta}}, \widehat{\widetilde{S_{\alpha,\beta}}}) = {}& p_\beta(p_{\widehat{\xi}} + a_{\widehat{\widetilde{\xi}}})(p_{\widehat{\xi}} + b_{\widehat{\widetilde{\xi}}})\left(S_{\alpha,\beta}\widehat{S_{\alpha,\beta}} + \gamma\,\widetilde{S_{\alpha,\beta}}\widehat{\widetilde{S_{\alpha,\beta}}}\right) \\ & - q_\beta(q_{\widetilde{\xi}} + a_{\widehat{\widetilde{\xi}}})(q_{\widetilde{\xi}} + b_{\widehat{\widetilde{\xi}}})\left(S_{\alpha,\beta}\widetilde{S_{\alpha,\beta}} + \gamma'\,\widehat{S_{\alpha,\beta}}\widehat{\widetilde{S_{\alpha,\beta}}}\right) \\ & + \frac{1}{2}\left(\wp'(\delta) - \wp'(\varepsilon)\right)\left(S_{\alpha,\beta}\widehat{\widetilde{S_{\alpha,\beta}}} + \gamma''\,\widetilde{S_{\alpha,\beta}}\widehat{S_{\alpha,\beta}}\right)\,,\end{aligned} \tag{5.35a}$$

where we have used (5.13) and where the coefficients γ, γ' and γ'' are given by

$$\gamma = \frac{p_{-\alpha}}{p_\beta}\,\frac{(p_\xi - a_\xi)(p_\xi - b_\xi)}{(p_{\widehat{\xi}} + a_{\widehat{\widetilde{\xi}}})(p_{\widehat{\xi}} + b_{\widehat{\widetilde{\xi}}})}\,, \tag{5.35b}$$

$$\gamma' = \frac{q_{-\alpha}}{q_\beta}\,\frac{(q_\xi - a_\xi)(q_\xi - b_\xi)}{(q_{\widetilde{\xi}} + a_{\widehat{\widetilde{\xi}}})(q_{\widetilde{\xi}} + b_{\widehat{\widetilde{\xi}}})}\,, \tag{5.35c}$$

$$\begin{aligned}\gamma'' = {}& \frac{\wp'(\beta) + \wp'(\delta)}{\wp'(\delta) - \wp'(\varepsilon)}\,\frac{(p_\xi - a_\xi)(p_\xi - b_\xi)}{(p_{\widehat{\xi}} - a_{\widehat{\xi}})(p_{\widehat{\xi}} - b_{\widehat{\xi}})} \\ & - \frac{\wp'(\beta) + \wp'(\varepsilon)}{\wp'(\delta) - \wp'(\varepsilon)}\,\frac{(q_\xi - a_\xi)(q_\xi - b_\xi)}{(q_{\widetilde{\xi}} - a_{\widetilde{\xi}})(q_{\widetilde{\xi}} - b_{\widetilde{\xi}})}\,,\end{aligned} \tag{5.35d}$$

where we have used the identities[6]

$$\frac{p_{\widehat{\xi}} - a_{\widehat{\xi}}}{p_\xi - a_\xi} = \frac{q_{\widetilde{\xi}} - a_{\widetilde{\xi}}}{q_\xi - a_\xi} = \frac{p_\xi + a_{\widetilde{\xi}}}{p_{\widehat{\xi}} + a_{\widehat{\widetilde{\xi}}}} = \frac{q_\xi + a_{\widehat{\xi}}}{q_{\widetilde{\xi}} + a_{\widehat{\widetilde{\xi}}}}\,, \tag{5.36a}$$

---

[6] These identities are also consequences of the chain of relations

$$\frac{\chi^{(1)}_{\alpha,-\varepsilon}(\widetilde{\xi})}{\chi^{(1)}_{\alpha,-\varepsilon}(\xi)} = \frac{\chi^{(1)}_{\alpha,\delta}(\widetilde{\xi})}{\chi^{(1)}_{\alpha,\delta}(\widehat{\widetilde{\xi}})} = \frac{\chi^{(1)}_{\alpha,-\delta}(\widehat{\xi})}{\chi^{(1)}_{\alpha,-\delta}(\xi)} = \frac{\chi^{(1)}_{\alpha,\varepsilon}(\widehat{\xi})}{\chi^{(1)}_{\alpha,\varepsilon}(\widehat{\widetilde{\xi}})},$$

which in turn follow from the expressions for the $\chi^{(1)}$ quantities in terms of the quantities Φ as in (2.26), and using the identifications

$$\chi^{(1)}_{\delta,\alpha}(\widetilde{\xi}) = p_\xi - a_\xi\,,\quad \chi^{(1)}_{-\delta,\alpha}(\xi) = -(p_\xi + a_{\widetilde{\xi}})\,,\quad \chi^{(1)}_{\delta,\beta}(\widetilde{\xi}) = p_\xi - b_\xi\,,\quad \chi^{(1)}_{-\delta,\beta}(\xi) = -(p_\xi + b_{\widetilde{\xi}})\,,$$

and similarly in the other direction

$$\chi^{(1)}_{\varepsilon,\alpha}(\widetilde{\xi}) = q_\xi - a_\xi\,,\quad \chi^{(1)}_{-\varepsilon,\alpha}(\xi) = -(q_\xi + a_{\widehat{\xi}})\,,\quad \chi^{(1)}_{\varepsilon,\beta}(\widetilde{\xi}) = q_\xi - b_\xi\,,\quad \chi^{(1)}_{-\varepsilon,\beta}(\xi) = -(q_\xi + b_{\widehat{\xi}})\,.$$

(and similarly with a's replaced by b's). The latter are a consequence of the relations

$$(p_\xi - a_\xi)(p_\xi + a_{\widetilde{\xi}}) = \wp(\delta) - \wp(\alpha)\ ,\ \ (q_\xi - b_\xi)(q_\xi + b_{\widehat{\xi}}) = \wp(\varepsilon) - \wp(\beta)\ , \tag{5.36b}$$

and similar relations with a and b and δ and ε interchanged.

We can bring the quadrilateral $\mathcal{Q}$ into a more standard form by performing a further (gauge) transformation, namely

$$S_{\alpha,\beta}(\xi) = \phi_{\alpha,\beta}(\xi)\mathfrak{u}_{\alpha,\beta}(\xi)\ , \tag{5.37a}$$

where the function $\phi_{\alpha,\beta}(\xi)$ solves the compatible system of first order ordinary difference equations

$$\widetilde{\phi_{\alpha,\beta}(\xi)} = \left(\frac{p_\beta}{p_{-\alpha}} \frac{(p_\xi + a_{\widetilde{\xi}})(p_\xi + b_{\widetilde{\xi}})}{(p_\xi - a_\xi)(p_\xi - b_\xi)} \right)^{1/2} \phi_{\alpha,\beta}(\xi)\ , \tag{5.37b}$$

$$\widehat{\phi_{\alpha,\beta}(\xi)} = \left(\frac{q_\beta}{q_{-\alpha}} \frac{(q_\xi + a_{\widehat{\xi}})(q_\xi + b_{\widehat{\xi}})}{(q_\xi - a_\xi)(q_\xi - b_\xi)} \right)^{1/2} \phi_{\alpha,\beta}(\xi)\ . \tag{5.37c}$$

Implementing the change of variables (5.37) the relation (5.34) changes into

$$2\frac{(p_\xi + a_{\widetilde{\xi}})(q_{\widetilde{\xi}} + a_{\widehat{\widetilde{\xi}}})}{P_\alpha^- \, Q_\alpha^- \, \widehat{\widetilde{\phi_{\alpha,\beta}(\xi)}}}\, v_\alpha\, \widehat{\widetilde{w_\beta}} = \frac{\bar{\mathcal{Q}}(\mathfrak{u}_{\alpha,\beta}, \widetilde{\mathfrak{u}_{\alpha,\beta}}, \widehat{\mathfrak{u}_{\alpha,\beta}}, \widehat{\widetilde{\mathfrak{u}_{\alpha,\beta}}})}{Q_\alpha^- \, P_\beta^+ \widetilde{\mathfrak{u}_{\alpha,\beta}} - P_\alpha^- \, Q_\beta^+ \widehat{\mathfrak{u}_{\alpha,\beta}}} \tag{5.38a}$$

where now

$$\begin{aligned} \bar{\mathcal{Q}}(\mathfrak{u}_{\alpha,\beta}, \widetilde{\mathfrak{u}_{\alpha,\beta}}, \widehat{\mathfrak{u}_{\alpha,\beta}}, \widehat{\widetilde{\mathfrak{u}_{\alpha,\beta}}}) = {} & P_\alpha^- \, P_\beta^+ \left(\mathfrak{u}_{\alpha,\beta} \widehat{\mathfrak{u}_{\alpha,\beta}} + \widetilde{\mathfrak{u}_{\alpha,\beta}} \widehat{\widetilde{\mathfrak{u}_{\alpha,\beta}}} \right) \\ & - Q_\alpha^- \, Q_\beta^+ \left(\mathfrak{u}_{\alpha,\beta} \widetilde{\mathfrak{u}_{\alpha,\beta}} + \widehat{\mathfrak{u}_{\alpha,\beta}} \widehat{\widetilde{\mathfrak{u}_{\alpha,\beta}}} \right) \\ & + \left(\wp'(\delta) - \wp'(\varepsilon) \right) \left(\mathfrak{u}_{\alpha,\beta} \widehat{\widetilde{\mathfrak{u}_{\alpha,\beta}}} + \widetilde{\mathfrak{u}_{\alpha,\beta}} \widehat{\mathfrak{u}_{\alpha,\beta}} \right)\ , \end{aligned} \tag{5.38b}$$

in which

$$(P_\alpha^\pm)^2 = \wp'(\delta) \pm \wp'(\alpha)\ , \quad (P_\beta^\pm)^2 = \wp'(\delta) \pm \wp'(\beta)\ , \tag{5.39a}$$

$$(Q_\alpha^\pm)^2 = \wp'(\varepsilon) \pm \wp'(\alpha)\ , \quad (Q_\beta^\pm)^2 = \wp'(\varepsilon) \pm \wp'(\beta)\ . \tag{5.39b}$$

Furthermore, rewriting (5.31a) in terms of $\mathfrak{u}_{\alpha,\beta}$ we have

$$P_\alpha^- \mathfrak{u}_{\alpha,\beta} - P_\beta^+ \widetilde{\mathfrak{u}_{\alpha,\beta}} = \frac{P_\alpha^-}{(p_\xi + a_{\widetilde{\xi}})\phi_{\alpha,\beta}(\xi)} \widetilde{v_\alpha}\, w_\beta\ . \tag{5.40}$$

Equation (5.38), together with (5.40) and its counterpart in the other direction (with δ replaced by ε and $\widetilde{\ }$ replaced by $\widehat{\ }$) forms a closed quadrilateral system for the three variables $\mathfrak{u}_{\alpha,\beta}$, v_α and w_β, which is the most general form of the lattice BSQ system. However, by using the identity

$$\frac{\widehat{(v_\alpha \widehat{\widetilde{w_\beta}})}}{\widetilde{(v_\alpha \widehat{\widetilde{w_\beta}})}} = \frac{\widehat{\widehat{\widetilde{(\widetilde{v_\alpha} w_\beta)}}}}{\widehat{\widetilde{\widetilde{(\widehat{v_\alpha} w_\beta)}}}} \, \frac{\widehat{v_\alpha} w_\beta}{\widetilde{v_\alpha} w_\beta}\ ,$$

and inserting the expressions for the quantities in brackets in terms of $\mathfrak{u}_{\alpha,\beta}$ one can write down a closed form 9-point equation in terms of $\mathfrak{u}_{\alpha,\beta}$ alone, see [40]. In the elliptic case that equation takes exactly the same form, except for the parametrisation of the lattice parameters involved, namely

$$
\begin{aligned}
&\frac{\bar{\mathfrak{Q}}(\widehat{\mathfrak{u}_{\alpha,\beta}}, \widehat{\widetilde{\mathfrak{u}_{\alpha,\beta}}}, \widehat{\widehat{\mathfrak{u}_{\alpha,\beta}}}, \widehat{\widehat{\widetilde{\mathfrak{u}_{\alpha,\beta}}}})}{\bar{\mathfrak{Q}}(\widetilde{\mathfrak{u}_{\alpha,\beta}}, \widetilde{\widetilde{\mathfrak{u}_{\alpha,\beta}}}, \widehat{\widetilde{\mathfrak{u}_{\alpha,\beta}}}, \widehat{\widetilde{\widetilde{\mathfrak{u}_{\alpha,\beta}}}})} \\
&= \frac{\left(Q_\alpha^- \mathfrak{u}_{\alpha,\beta} - Q_\beta^+ \widehat{\mathfrak{u}_{\alpha,\beta}}\right)\left(Q_\alpha^- P_\beta^+ \widehat{\widetilde{\mathfrak{u}_{\alpha,\beta}}} - P_\alpha^- Q_\beta^+ \widehat{\widehat{\mathfrak{u}_{\alpha,\beta}}}\right)\left(P_\alpha^- \widehat{\widehat{\widetilde{\mathfrak{u}_{\alpha,\beta}}}} - P_\beta^+ \widehat{\widehat{\widetilde{\widetilde{\mathfrak{u}_{\alpha,\beta}}}}}\right)}{\left(P_\alpha^- \mathfrak{u}_{\alpha,\beta} - P_\beta^+ \widetilde{\mathfrak{u}_{\alpha,\beta}}\right)\left(Q_\alpha^- P_\beta^+ \widetilde{\widetilde{\mathfrak{u}_{\alpha,\beta}}} - P_\alpha^- Q_\beta^+ \widetilde{\widehat{\mathfrak{u}_{\alpha,\beta}}}\right)\left(Q_\alpha^- \widehat{\widetilde{\widetilde{\mathfrak{u}_{\alpha,\beta}}}} - Q_\beta^+ \widehat{\widehat{\widetilde{\widetilde{\mathfrak{u}_{\alpha,\beta}}}}}\right)} .
\end{aligned}
\tag{5.41}
$$

The latter equation, which can be thought of as a generalisation of the lattice Schwarzian BSQ equation, [24], contains both the ‘regular’ lattice BSQ equation, (1.2) as well as the lattice modified BSQ equation as special coalescence limits on the parameters α and β. Thus, the 9-point equation is the most general lattice BSQ system known so far.

6. Elliptic Seed and Soliton Solutions of the Lattice BSQ Systems

In the previous section we have obtained various lattice equations in the BSQ class, like the regular lattice BSQ equation (5.6), the lattice modified BSQ equations (5.21) or (5.22) and the generalised Schwarzian BSQ equation (5.41), as well as the trilinear equation for the τ-function (5.28). From the DL structure we automatically have a large class of elliptic solutions for those equations, including soliton solutions and inverse scattering type solutions. However, the equations we obtained are not yet in ‘standard form’, cf. [12], in that they contain elliptic coefficients. In this section we will perform point transformations to bring the equations in the required standard form, and in doing so we get the elementary elliptic seed solutions of the standard lattice BSQ systems. Furthermore, we present some explicit one- and two-elliptic soliton solutions, and demonstrate the general structure of the elliptic multi-soliton solutions in terms of the corresponding Cauchy matrix scheme.

6.1. Seed and elliptic 1-soliton solutions of the lattice BSQ system. Consider the following point transformation

$$u_{0,0} = x_0 - u \ , \tag{6.1a}$$
$$u_{1,0} = y_0 - v - x_0 u_{0,0} \ , \tag{6.1b}$$
$$u_{0,1} = z_0 - w - x_0 u_{0,0} \ , \tag{6.1c}$$

with

$$x_0 = \zeta(\xi) + n\zeta(\delta) + m\zeta(\varepsilon) - \zeta(\xi_0)\,, \quad \xi = \xi_0 - n\delta - m\varepsilon, \tag{6.2a}$$

$$y_0 = \frac{1}{2}x_0^2 - \frac{1}{2}\wp(\xi) + \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0))\,, \tag{6.2b}$$

$$z_0 = \frac{1}{2}x_0^2 - \frac{1}{2}\wp(\xi) - \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0))\,, \tag{6.2c}$$

and $u = u(n,m),\ v = v(n,m),\ w = w(n,m)$. Imposing this transformation on (5.2a), (5.3a) and (5.5), i.e.,

$$p_\xi \widetilde{u_{0,0}} + \widetilde{u_{0,1}} = p_\xi u_{0,0} - u_{1,0} - \widetilde{u_{0,0}} u_{0,0}\,, \tag{6.3a}$$

$$q_\xi \widehat{u_{0,0}} + \widehat{u_{0,1}} = q_\xi u_{0,0} - u_{1,0} - \widehat{u_{0,0}} u_{0,0}\,, \tag{6.3b}$$

$$\begin{aligned}\frac{1}{2}\frac{\wp'(\delta) - \wp'(\varepsilon)}{p_\xi - q_\xi + \widehat{u_{0,0}} - \widetilde{u_{0,0}}} &= \frac{1}{2}\frac{\wp'(\delta) - \wp'(\varepsilon)}{p_\xi - q_\xi} + \widehat{\widetilde{u_{1,0}}} + u_{0,1} + u_{0,0}\widehat{\widetilde{u_{0,0}}} \\ &\quad + (p_{\widehat{\xi}} + q_\xi)(\widehat{\widetilde{u_{0,0}}} - u_{0,0})\,,\end{aligned} \tag{6.3c}$$

we readily obtain the following three-component lattice BSQ system

$$\widetilde{w} - u\widetilde{u} + v = 0\,, \tag{6.4a}$$

$$\widehat{w} - u\widehat{u} + v = 0\,, \tag{6.4b}$$

$$\frac{1}{2}\frac{\wp'(\delta) - \wp'(\varepsilon)}{\widehat{u} - \widetilde{u}} = w - u\widehat{\widetilde{u}} + \widehat{\widetilde{v}}\,, \tag{6.4c}$$

which was referred to as the (B-2) system in Ref. [13], but which appeared also in [35,37]. Thus, we attain the following result on the 0-soliton solution ($0SS$) of the lattice equations.

Proposition 6.1. *The system (6.4) admits the following elliptic 'seed' solution*

$$u^{0SS}(n,m) = x_0 = \zeta(\xi) + n\zeta(\delta) + m\zeta(\varepsilon) - \zeta(\xi_0)\,, \tag{6.5a}$$

$$v^{0SS}(n,m) = y_0 = \frac{1}{2}x_0^2 - \frac{1}{2}\wp(\xi) + \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0))\,, \tag{6.5b}$$

$$w^{0SS}(n,m) = z_0 = \frac{1}{2}x_0^2 - \frac{1}{2}\wp(\xi) - \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0))\,, \tag{6.5c}$$

where $\xi = \xi_0 - n\delta - m\varepsilon$ and ξ_0 is an arbitrary initial value.

By eliminating v and w in the lattice BSQ system (6.4), we also get the lattice BSQ equation (1.2), rewritten as

$$\frac{\frac{1}{2}(\wp'(\delta) - \wp'(\varepsilon))}{\widehat{\widetilde{u}} - \widetilde{\widetilde{u}}} - \frac{\frac{1}{2}(\wp'(\delta) - \wp'(\varepsilon))}{\widehat{\widehat{u}} - \widehat{\widetilde{u}}} = (\widehat{\widehat{\widetilde{u}}} - \widehat{\widetilde{\widetilde{u}}})(\widehat{u} - \widehat{\widehat{\widetilde{\widetilde{u}}}}) - (\widehat{u} - \widetilde{u})(u - \widehat{\widetilde{\widetilde{u}}})\,, \tag{6.6}$$

together with its elliptic seed solution (6.5a).

Meanwhile, it is noted that (6.6) can be transformed into (5.6) by the transformation (6.1a) and (6.2a). Moreover, making use of the seed solution (6.5) and the Bäcklund transformations (see [34] and [40]), we can construct the elliptic 1-soliton solution ($1SS$) as follows.

Proposition 6.2. *The system* (6.4) *admits the following elliptic 1-soliton solution (with soliton parameter κ)*

$$u^{1SS} = x_0 + \frac{\eta_\xi(\kappa)\Phi_\xi(\kappa) + \eta_\xi(\omega_1(\kappa))\Phi_\xi(\omega_1(\kappa))\rho_1 + \eta_\xi(\omega_2(\kappa))\Phi_\xi(\omega_2(\kappa))\rho_2}{\Phi_\xi(\kappa) + \Phi_\xi(\omega_1(\kappa))\rho_1 + \Phi_\xi(\omega_2(\kappa))\rho_2}, \tag{6.7a}$$

$$\begin{aligned} v^{1SS} &= y_0 + x_0\frac{\eta_\xi(\kappa)\Phi_\xi(\kappa) + \eta_\xi(\omega_1(\kappa))\Phi_\xi(\omega_1(\kappa))\rho_1 + \eta_\xi(\omega_2(\kappa))\Phi_\xi(\omega_2(\kappa))\rho_2}{\Phi_\xi(\kappa) + \Phi_\xi(\omega_1(\kappa))\rho_1 + \Phi_\xi(\omega_2(\kappa))\rho_2} \\ &+ \frac{(\wp(\kappa)+\wp(\xi))\Phi_\xi(\kappa) + (\wp(\omega_1(\kappa))+\wp(\xi))\Phi_\xi(\omega_1(\kappa))\rho_1 + (\wp(\omega_2(\kappa))+\wp(\xi))\Phi_\xi(\omega_2(\kappa))\rho_2}{\Phi_\xi(\kappa) + \Phi_\xi(\omega_1(\kappa))\rho_1 + \Phi_\xi(\omega_2(\kappa))\rho_2}, \end{aligned} \tag{6.7b}$$

$$w^{1SS} = z_0 + \wp(\xi) + x_0\frac{\eta_\xi(\kappa)\Phi_\xi(\kappa) + \eta_\xi(\omega_1(\kappa))\Phi_\xi(\omega_1(\kappa))\rho_1 + \eta_\xi(\omega_2(\kappa))\Phi_\xi(\omega_2(\kappa))\rho_2}{\Phi_\xi(\kappa) + \Phi_\xi(\omega_1(\kappa))\rho_1 + \Phi_\xi(\omega_2(\kappa))\rho_2}, \tag{6.7c}$$

where ω_1 and ω_2 are the elliptic cube roots of unity (*see Sect.* 3) *subject to $\kappa + \omega_1(\kappa) + \omega_2(\kappa) \equiv 0$*(*mod period lattice*) *and where ρ_i $(i = 1, 2)$ are plane wave factors given by*

$$\rho_i = \rho_i(n, m; \kappa) = \left(\frac{\Phi_\delta(-\omega_i(\kappa))}{\Phi_\delta(-\kappa)}\right)^n \left(\frac{\Phi_\varepsilon(-\omega_i(\kappa))}{\Phi_\varepsilon(-\kappa)}\right)^m \cdot \frac{\rho_i^0}{\rho_0^0}, \quad (i = 1, 2). \tag{6.8}$$

In particular the 9-point equation (6.6) *admits the solution given by* (6.7a)*.*

6.2. Elliptic seed solution for the lattice modified BSQ equation. To derive the 3-component form of the lattice modified BSQ equation as in [13], we consider the following point transformation

$$u_{0,0} = x_0 - U, \tag{6.9a}$$

$$v_\alpha(\xi) = \Big(\Phi_\alpha(-\delta)\Big)^n \Big(\Phi_\alpha(-\varepsilon)\Big)^m \frac{1}{\Phi_\alpha(\xi)} V, \tag{6.9b}$$

$$s_\alpha(\xi) = \Big(\Phi_\alpha(-\delta)\Big)^n \Big(\Phi_\alpha(-\varepsilon)\Big)^m \frac{1}{\Phi_\alpha(\xi)} \Big(Y - (x_0 + 2C)V\Big), \tag{6.9c}$$

with new functions $U = U(n, m),\ V = V(n, m),\ Y = Y(n, m)$, and where x_0 is given by

$$x_0 = \zeta(\xi) + n\zeta(\delta) + m\zeta(\varepsilon) - \zeta(\xi_0) + C(n + m), \tag{6.10}$$

and C is an arbitrary complex constant. By using this transformation, (5.14a) and its $\{\widehat{\ }, \varepsilon\}$ counterparts and (5.17a) are turned to the following system corresponding to the (A-2) system of [13]

$$\widetilde{Y} = U\widetilde{V} - V, \quad \widehat{Y} = U\widehat{V} - V, \tag{6.11a}$$

$$Y = V\widehat{\widetilde{U}} + \frac{\frac{1}{2}(P_\alpha^-)^2\widetilde{V} - \frac{1}{2}(Q_\alpha^-)^2\widehat{V}}{\widehat{U} - \widetilde{U}}, \tag{6.11b}$$

in which we have used the notation in (5.39). Eliminating U and Y in the above system, we obtain the lattice modified BSQ equation, written as the 9-point equation, [29],

$$
\left(\frac{\frac{1}{2}(P_\alpha^-)^2\widehat{\widetilde{V}}-\frac{1}{2}(Q_\alpha^-)^2\widehat{\widehat{V}}}{\widehat{\widehat{V}}-\widehat{\widetilde{V}}}\right)\frac{\widehat{\widehat{\widetilde{V}}}}{\widehat{V}}-\left(\frac{\frac{1}{2}(P_\alpha^-)^2\widetilde{\widetilde{V}}-\frac{1}{2}(Q_\alpha^-)^2\widehat{\widetilde{V}}}{\widehat{\widetilde{V}}-\widetilde{\widetilde{V}}}\right)\frac{\widehat{\widetilde{\widetilde{V}}}}{\widetilde{V}}
$$
$$
=\frac{V}{\widetilde{V}}-\frac{V}{\widehat{V}}-\frac{\widehat{\widehat{\widetilde{V}}}}{\widehat{\widetilde{\widetilde{V}}}}+\frac{\widehat{\widetilde{\widetilde{V}}}}{\widehat{\widehat{\widetilde{V}}}}\,. \tag{6.12}
$$

Alternatively, for the system (5.17b) and its $\{\widehat{\ }, \varepsilon\}$ counterparts and (5.14b), we employ the following transformation

$$u_{0,0} = -U + x_0\,, \tag{6.13a}$$
$$w_\beta(\xi) = \left(\frac{1}{\Phi_\beta(\delta)}\right)^n\left(\frac{1}{\Phi_\beta(\varepsilon)}\right)^m\frac{1}{\Phi_\beta(\xi)}W\,, \tag{6.13b}$$
$$t_\beta(\xi) = \left(\frac{1}{\Phi_\beta(\delta)}\right)^n\left(\frac{1}{\Phi_\beta(\varepsilon)}\right)^m\frac{1}{\Phi_\beta(\xi)}\Big(Z-(x_0-2C)V\Big)\,, \tag{6.13c}$$

and obtain

$$Z=\widetilde{U}W-\widetilde{W}\,,\quad Z=\widehat{U}W-\widehat{W}\,, \tag{6.14a}$$
$$\widehat{\widetilde{Z}}=U\widehat{\widetilde{W}}+\frac{\frac{1}{2}(P_\beta^+)^2\widehat{W}-\frac{1}{2}(Q_\beta^+)^2\widetilde{W}}{\widehat{U}-\widetilde{U}}\,, \tag{6.14b}$$

with $U = U(n,m)$, $W = W(n,m)$ and $Z = Z(n,m)$ functions of n, m and C a constant, whereas x_0 was defined in (6.10) appearing before in (6.9). Eliminating U and Z in (6.14) or noticing the reversal symmetry (see [13]) of the system (6.11) we have an alternate form of the lattice modified BSQ equation, written as

$$
\left(\frac{\frac{1}{2}(P_\beta^+)^2\widehat{\widehat{W}}-\frac{1}{2}(Q_\beta^+)^2\widehat{\widetilde{W}}}{\widehat{\widehat{W}}-\widehat{\widetilde{W}}}\right)\frac{\widehat{W}}{\widehat{\widehat{\widetilde{W}}}}-\left(\frac{\frac{1}{2}(Q_\beta^+)^2\widetilde{\widetilde{W}}-\frac{1}{2}(P_\beta^+)^2\widehat{\widetilde{W}}}{\widetilde{\widetilde{W}}-\widehat{\widetilde{W}}}\right)\frac{\widetilde{W}}{\widehat{\widetilde{\widetilde{W}}}}
$$
$$
=\frac{\widetilde{W}}{W}-\frac{\widehat{W}}{W}-\frac{\widehat{\widehat{\widetilde{W}}}}{\widehat{\widetilde{\widetilde{W}}}}+\frac{\widehat{\widetilde{\widetilde{W}}}}{\widehat{\widehat{\widetilde{W}}}}\,, \tag{6.15}
$$

which is related to the equation (5.22) through the transformation (6.13b).

The seed solutions (i.e., the 0-soliton solutions of the lattice equations) can be read off from the change of dependent variables (6.9) and (6.13), by removing $\boldsymbol{U}_\xi$ in the definitions (5.7). Thus, we arrive at the following result concerning the seed solutions.

Proposition 6.3. *The lattice modified BSQ equation* (6.12) *admits the following elliptic seed solution*

$$V^{0SS}(n,m)=\Big(\Phi_\alpha(-\delta)\Big)^{-n}\Big(\Phi_\alpha(-\varepsilon)\Big)^{-m}\Phi_\alpha(\xi)\,. \tag{6.16}$$

Similarly, the alternative lattice modified BSQ (6.15) admits the elliptic seed solution

$$W^{0SS}(n,m)=\Big(\Phi_\beta(\delta)\Big)^{n}\Big(\Phi_\beta(\varepsilon)\Big)^{m}\Phi_\beta(\xi)\,. \tag{6.17}$$

The elliptic 1-soliton solutions can be readily constructed as well from the DL framework, but we will omit the explicit formulae here, as we will present the general elliptic Cauchy matrix forms in Sect. 6.4 below.

6.3. Elliptic seed solution for the lattice Schwarzian BSQ equation. Now we want to present the lattice Schwarzian BSQ equation from the three-component lattice system, i.e, (5.31a), (5.32a) and (5.33). Employing the following point transformation to this system

$$v_\alpha(\xi) = \Big(\varPhi_{-\alpha}(\delta)\Big)^n \Big(\varPhi_{-\alpha}(\varepsilon)\Big)^m \frac{1}{\varPhi_\alpha(\xi)} V \,, \tag{6.18a}$$

$$w_\alpha(\xi) = \Big(\varPhi_{\beta}(\delta)\Big)^{-n} \Big(\varPhi_{\beta}(\varepsilon)\Big)^{-m} \frac{1}{\varPhi_\beta(\xi)} W \,, \tag{6.18b}$$

$$S_{\alpha,\beta}(\xi) = \left(\frac{\varPhi_{-\alpha}(\delta)}{\varPhi_\beta(\delta)}\right)^n \left(\frac{\varPhi_{-\alpha}(\varepsilon)}{\varPhi_\beta(\varepsilon)}\right)^m \frac{1}{\varPhi_\alpha(\xi)\varPhi_\beta(\xi)} H \,, \tag{6.18c}$$

where $V = V(n,m),\ W = W(n,m)$ and $H = H(n,m)$ are functions of n, m, we get the following system corresponding to the (C-3) system in Ref. [13]

$$H - \widetilde{H} = \widetilde{V} W \,, \quad H - \widehat{H} = \widehat{V} W \,, \tag{6.19a}$$

$$V \widehat{\widetilde{W}} = \frac{\frac{1}{2}(Q^+_\beta)^2 \widehat{V}\widetilde{W} - \frac{1}{2}(P^+_\beta)^2 \widetilde{V}\widehat{W}}{\widetilde{W} - \widehat{W}} W + \frac{1}{2}(\wp'(\alpha) + \wp'(\beta))\widehat{\widetilde{H}} \,, \tag{6.19b}$$

which leads to the lattice Schwarzian BSQ equation, written as

$$\begin{aligned} &\frac{(\widehat{H} - \widehat{\widetilde{H}})(\widehat{\widehat{H}} - \widehat{\widehat{\widetilde{H}}})(P^+_\beta)^2 - (\widehat{H} - \widehat{\widehat{H}})(\widehat{\widetilde{H}} - \widehat{\widehat{\widetilde{H}}})(Q^+_\beta)^2 - (\wp'(\alpha) + \wp'(\beta))\widehat{\widehat{\widetilde{H}}}(\widehat{\widetilde{H}} - \widehat{\widehat{H}})}{(\widetilde{H} - \widetilde{\widetilde{H}})(\widehat{\widetilde{H}} - \widehat{\widetilde{\widetilde{H}}})(P^+_\beta)^2 - (\widetilde{H} - \widehat{\widetilde{H}})(\widetilde{\widetilde{H}} - \widehat{\widetilde{\widetilde{H}}})(Q^+_\beta)^2 - (\wp'(\alpha) + \wp'(\beta))\widehat{\widetilde{\widetilde{H}}}(\widetilde{\widetilde{H}} - \widehat{\widetilde{H}})} \\ &= \frac{(H - \widehat{H})(\widehat{\widetilde{H}} - \widehat{\widehat{H}})(\widehat{\widehat{\widetilde{H}}} - \widehat{\widehat{\widetilde{\widetilde{H}}}})}{(H - \widetilde{H})(\widetilde{\widetilde{H}} - \widehat{\widetilde{H}})(\widehat{\widetilde{\widetilde{H}}} - \widehat{\widehat{\widetilde{\widetilde{H}}}})} \,. \end{aligned} \tag{6.20}$$

From the transformations (6.18c) and (5.30) and the definition of $s_{\alpha,\beta}(\xi)$ given by (5.7g), we deduce the following

Proposition 6.4. *The lattice Schwarzian BSQ equation* (6.20) *admits the following elliptic seed solution*

$$H = -\left(\frac{\varPhi_\beta(\delta)}{\varPhi_{-\alpha}(\delta)}\right)^n \left(\frac{\varPhi_\beta(\varepsilon)}{\varPhi_{-\alpha}(\varepsilon)}\right)^m \varPhi_{\alpha+\beta}(\xi) \,. \tag{6.21}$$

In the next subsection we will present the general elliptic Cauchy matrix framework from which the elliptic multi-soliton solutions can be readily found in a closed form.

6.4. Cauchy matrix scheme for elliptic multi-soliton solutions. We now consider the elliptic multi-soliton solutions, which we obtain by choosing a particular measure, namely in (4.14)

$$d\mu_j(\ell) = \frac{1}{2\pi \mathrm{i}} \sum_{j'=1}^{N_j} \frac{\Lambda_{j,j'} d\ell}{\ell - \kappa_{j,j'}} , \tag{6.22}$$

where $\Lambda_{j,j'}$ are residues of the measures $d\mu_j(\ell)$, and choosing the Γ_j $(j = 1, 2)$ to be contours in the complex plane surrounding the singularities $\{\kappa_{j,j'}\}$. In this case the linear integral equation (4.14) is reduced to a matrix system of the form

$$\boldsymbol{u}_\kappa + \sum_{j=1}^{2} \sum_{j'=1}^{N_j} \Lambda_{j,j'}\, \rho_\kappa\, \Phi_\xi(\kappa - \omega_j(\kappa_{j,j'}))\, \sigma_{-\omega_j(\kappa_{j,j'})} \boldsymbol{u}_{\kappa_{j,j'}} = \rho_\kappa \Phi_\xi(\boldsymbol{\Lambda}) \boldsymbol{c}_\kappa \ . \tag{6.23}$$

Setting $\kappa = \kappa_{i,i'}$, where $i = 1, 2$, $i' = 1, \ldots, N_i$ in Eq. (6.23), we then have a linear system for the quantities $u_{\kappa_{i,i'}}$ from the equation (6.23), i.e.,

$$\begin{aligned}
&\left(\boldsymbol{u}_{\kappa_{1,1}}, \ldots, \boldsymbol{u}_{\kappa_{1,N_1}}, \boldsymbol{u}_{\kappa_{2,1}}, \ldots, \boldsymbol{u}_{\kappa_{2,N_2}}\right)(\boldsymbol{I}_{\mathcal{N}} + \boldsymbol{M}) \\
&= \Big(\rho_{\kappa_{1,1}} \Phi_\xi(\kappa_{1,1}) \boldsymbol{c}_{\kappa_{1,1}}, \ldots, \rho_{\kappa_{1,N_1}} \Phi_\xi(\kappa_{1,N_1}) \boldsymbol{c}_{\kappa_{1,N_1}}, \rho_{\kappa_{2,1}} \Phi_\xi(\kappa_{2,1}) \boldsymbol{c}_{\kappa_{2,1}}, \\
&\quad \ldots, \rho_{\kappa_{2,N_2}} \Phi_\xi(\kappa_{2,N_2}) \boldsymbol{c}_{\kappa_{2,N_2}}\Big),
\end{aligned} \tag{6.24}$$

which can be rewritten as

$$\begin{aligned}
&\left(\boldsymbol{u}_{\kappa_{1,1}}, \ldots, \boldsymbol{u}_{\kappa_{1,N_1}}, \boldsymbol{u}_{\kappa_{2,1}}, \ldots, \boldsymbol{u}_{\kappa_{2,N_2}}\right)(\boldsymbol{I}_{\mathcal{N}} + \boldsymbol{M}) \\
&= \left(\boldsymbol{c}_{\kappa_{1,1}}, \ldots, \boldsymbol{c}_{\kappa_{1,N_1}}, \boldsymbol{c}_{\kappa_{2,1}}, \ldots, \boldsymbol{c}_{\kappa_{2,N_2}}\right) \boldsymbol{R} ,
\end{aligned} \tag{6.25a}$$

where

$$\boldsymbol{R} = \mathrm{diag}\Big(\rho_{\kappa_{1,1}} \Phi_\xi(\kappa_{1,1}), \ldots, \rho_{\kappa_{1,N_1}} \Phi_\xi(\kappa_{1,N_1}), \rho_{\kappa_{2,1}} \Phi_\xi(\kappa_{2,1}), \ldots, \rho_{\kappa_{2,N_2}} \Phi_\xi(\kappa_{2,N_2})\Big) , \tag{6.25b}$$

$\boldsymbol{I}_{\mathcal{N}}$ is an identity matrix of size $\mathcal{N} = N_1 + N_2$ and $\boldsymbol{M}$ is a 2×2 block generalised Cauchy matrix with rectangular blocks of size $N_j \times N_i$ $(j, i = 1, 2)$ with elements

$$\begin{aligned}
&M_{(j,j'),(i,i')} = \rho_{\kappa_{i,i'}} \Lambda_{j,j'} \sigma_{-\omega_j(\kappa_{j,j'})} \Phi_\xi(\kappa_{i,i'} - \omega_j(\kappa_{j,j'})) \\
&\quad i, j = 1, 2, \quad i' = 1, \ldots, N_i, \quad j' = 1, \ldots, N_j \ .
\end{aligned} \tag{6.25c}$$

We can now make the soliton solutions explicit by assuming that the coefficients $\boldsymbol{\Lambda}_{j,j'}$ are chosen such that the matrix $\boldsymbol{I}_{\mathcal{N}} + \boldsymbol{M}$ is invertible, in which case solutions to (6.23) could be written out explicitly. Making use of (4.16), (6.22) and (6.25), we can also obtain the explicit expression for $\boldsymbol{U}_\xi$ written as

$$\begin{aligned}
\boldsymbol{U}_\xi = &\left(\boldsymbol{c}_{\kappa_{1,1}}, \ldots, \boldsymbol{c}_{\kappa_{1,N_1}}, \boldsymbol{c}_{\kappa_{2,1}}, \ldots, \boldsymbol{c}_{\kappa_{2,N_2}}\right) \boldsymbol{R}(\boldsymbol{I}_{\mathcal{N}} + \boldsymbol{M})^{-1} \\
&\cdot \boldsymbol{S}\left(\boldsymbol{c}_{-\omega_1(\kappa_{1,1})}, \ldots, \boldsymbol{c}_{-\omega_1(\kappa_{1,N_1})}, \boldsymbol{c}_{-\omega_2(\kappa_{2,1})}, \ldots, \boldsymbol{c}_{-\omega_2(\kappa_{2,N_2})}\right)^T ,
\end{aligned} \tag{6.26a}$$

with elements

$$(\boldsymbol{U}_\xi)_{i,j} = \boldsymbol{r}\boldsymbol{K}^i(\boldsymbol{I}_{\mathcal{N}}+\boldsymbol{M})^{-1}\boldsymbol{L}^j\boldsymbol{s}^T \ , \tag{6.26b}$$

in which

$$\begin{aligned}\boldsymbol{S} = \operatorname{diag}\Big(&\sigma_{-\omega_1(\kappa_{1,1})}\Phi_\xi(-\omega_1(\kappa_{1,1})),\dots,\sigma_{-\omega_1(\kappa_{1,N_1})}\Phi_\xi(-\omega_1(\kappa_{1,N_1})),\\ &\sigma_{-\omega_2(\kappa_{2,1})}\Phi_\xi(-\omega_2(\kappa_{2,1})),\dots,\sigma_{-\omega_2(\kappa_{2,N_2})}\Phi_\xi(-\omega_2(\kappa_{2,N_2}))\Big)\ ,\end{aligned} \tag{6.26c}$$

and $\boldsymbol{K}$ and $\boldsymbol{L}$ are the diagonal matrices given by

$$\boldsymbol{K} = \operatorname{diag}\left(\kappa_{1,1},\dots,\kappa_{1,N_1},\kappa_{2,1},\dots,\kappa_{2,N_2}\right)\ , \tag{6.26d}$$

$$\boldsymbol{L} = \operatorname{diag}\left(-\omega_1(\kappa_{1,1}),\dots,-\omega_1(\kappa_{1,N_1}),-\omega_2(\kappa_{2,1}),\dots,-\omega_2(\kappa_{2,N_2})\right), \tag{6.26e}$$

and $\boldsymbol{r}$, $\boldsymbol{s}$ are row vectors given by

$$\boldsymbol{r} = \left(\rho_{\kappa_{1,1}}\Phi_\xi(\kappa_{1,1}),\dots,\rho_{\kappa_{1,N_1}}\Phi_\xi(\kappa_{1,N_1}),\rho_{\kappa_{2,1}}\Phi_\xi(\kappa_{2,1}),\dots,\rho_{\kappa_{2,N_2}}\Phi_\xi(\kappa_{2,N_2})\right)\ , \tag{6.26f}$$

$$\begin{aligned}\boldsymbol{s} = \Big(&\sigma_{-\omega_1(\kappa_{1,1})}\Phi_\xi(-\omega_1(\kappa_{1,1})),\dots,\sigma_{-\omega_1(\kappa_{1,N_1})}\Phi_\xi(-\omega_1(\kappa_{1,N_1})),\\ &\sigma_{-\omega_2(\kappa_{2,1})}\Phi_\xi(-\omega_2(\kappa_{2,1})),\dots,\sigma_{-\omega_2(\kappa_{2,N_2})}\Phi_\xi(-\omega_2(\kappa_{2,N_2}))\Big)\ .\end{aligned} \tag{6.26g}$$

By using the relation (2.5) and the above expression for $\boldsymbol{U}_\xi$, we can write the quantities introduced in (5.7) in the following way:

$$s_\alpha(\xi) = \eta_\alpha(\xi) - \boldsymbol{r}\left(\chi^{(1)}_{\alpha,\xi}(\boldsymbol{K})\right)^{-1}(\boldsymbol{I}_{\mathcal{N}}+\boldsymbol{M})^{-1}\eta_\xi(\boldsymbol{L})\,\boldsymbol{s}^T\ , \tag{6.27a}$$

$$t_\beta(\xi) = \eta_\beta(\xi) - \boldsymbol{r}\,\eta_\xi(\boldsymbol{K})\,(\boldsymbol{I}_{\mathcal{N}}+\boldsymbol{M})^{-1}\left(\chi^{(1)}_{\beta,\xi}(\boldsymbol{L})\right)^{-1}\boldsymbol{s}^T\ , \tag{6.27b}$$

$$r_\alpha(\xi) = \wp(\alpha) - \boldsymbol{r}\left(\chi^{(1)}_{\alpha,\xi}(\boldsymbol{K})\right)^{-1}(\boldsymbol{I}_{\mathcal{N}}+\boldsymbol{M})^{-1}\wp(\boldsymbol{L})\,\boldsymbol{s}^T\ , \tag{6.27c}$$

$$z_\beta(\xi) = \wp(\beta) - \boldsymbol{r}\,\wp(\boldsymbol{K})\,(\boldsymbol{I}_{\mathcal{N}}+\boldsymbol{M})^{-1}\left(\chi^{(1)}_{\beta,\xi}(\boldsymbol{L})\right)^{-1}\boldsymbol{s}^T\ , \tag{6.27d}$$

$$v_\alpha(\xi) = 1 - \boldsymbol{r}\left(\chi^{(1)}_{\alpha,\xi}(\boldsymbol{K})\right)^{-1}(\boldsymbol{I}_{\mathcal{N}}+\boldsymbol{M})^{-1}\boldsymbol{s}^T\ , \tag{6.27e}$$

$$w_\beta(\xi) = 1 - \boldsymbol{r}\,(\boldsymbol{I}_{\mathcal{N}}+\boldsymbol{M})^{-1}\left(\chi^{(1)}_{\beta,\xi}(\boldsymbol{L})\right)^{-1}\boldsymbol{s}^T\ , \tag{6.27f}$$

$$s_{\alpha,\beta}(\xi) = \boldsymbol{r}\left(\chi^{(1)}_{\alpha,\xi}(\boldsymbol{K})\right)^{-1}(\boldsymbol{I}_{\mathcal{N}}+\boldsymbol{M})^{-1}\left(\chi^{(1)}_{\beta,\xi}(\boldsymbol{L})\right)^{-1}\boldsymbol{s}^T\ . \tag{6.27g}$$

Note that in Eq. (6.27) the matrices $\wp(\boldsymbol{K})$, $\eta_\xi(\boldsymbol{L})$, $\chi^{(1)}_{\alpha,\xi}(\boldsymbol{K})$ and $\chi^{(1)}_{\beta,\xi}(\boldsymbol{L})$ denote the diagonal matrices with as entries the corresponding elliptic functions evaluated at the entries of the corresponding matrices $\boldsymbol{K}$ or $\boldsymbol{L}$, e.g.

$$\wp(\boldsymbol{K}) = \operatorname{diag}\left(\wp(\kappa_{1,1}),\dots,\wp(\kappa_{1,N_1}),\wp(\kappa_{2,1}),\dots,\wp(\kappa_{2,N_2})\right)\ ,$$

etc. Note that all these expressions (6.27) can be written out explicitly after choosing the various parameters of the solution. In this sense, τ-function can also be expressed by

$$\tau_\xi = \det{}_{\mathcal{N}\times\mathcal{N}}(\boldsymbol{I}_{\mathcal{N}}+\boldsymbol{M})\ ,\quad \mathcal{N} = N_1+N_2\ , \tag{6.28}$$

where the explicit expression can be computed by means of the usual expansion methods for determinants.

7. Conclusions

In this paper we constructed a rich class of elliptic solutions of a family of BSQ type partial difference equations, in particular the lattice BSQ equation (1.2), the lattice modified BSQ equations (5.21) and (5.22), and the lattice Schwarzian BSQ equation (6.20), while the most general equation is the generalised lattice Schwarzian BSQ equation (5.41), and the underlying equation for the τ-function is given by trilinear equation (5.28). The various equations, which in their scalar form are difference equations on a 9-point stencil, can also be written in their multi-component forms, (see Sect. 6). The direct linearisation (DL) framework, equipped with an elliptic Cauchy kernel, produces in particular elliptic multi-soliton solutions (when the integration in the integral equations such as (4.14) together with (4.16) only involve the residues at simple poles in the spectral variable, see Sect. 6.4), but more generally, for more general integration measures $d\mu_j(\ell)$ and integration contours Γ_j in (4.14) it comprises also inverse scattering type solutions.

A salient feature of the DL approach is that it incorporates several integrable models within one framework through the infinite matrix (4.16), the distinct entries of which obey various Miura related equations. The structure of the BSQ system (or more generally of the Nth GD hierarchy) was obtained performing a dimensional reduction on the KP system, by imposing a constraint of the type (4.13). However, where in the case of rational Cauchy kernel this would involve the emergence of cube roots of unity, in the elliptic case one needs an elliptic analogue of the notion of cube root of unity for the reduction to work. These, or more generally the elliptic Nth root of unity, were defined in Sect. 3. Unlike the conventional roots of unity, these elliptic analogues depend intrinsically on a parameter - in the case at hand, the lattice parameter. This novel concept of *elliptic Nth root of unity* may well have a significance in other areas of mathematics, e.g. cyclotomic fields.

Whereas we restricted ourselves in this paper to solutions of the BSQ class of lattice systems, the main underlying structural relations were given for the entire lattice GD hierarchy, and thus the results obtained may be readily generalised to the higher order systems in this hierarchy, although the development of the explicit formulae and corresponding computations may fast become quite overwhelming. The present paper has made clear that the KdV case ($N = 2$) of the GD hierarchy is not really representative of what goes on in the general case, while the BSQ case ($N = 3$) provides a first good insight into the real intricacies of the general construction. Thus, in spite of the importance of the famous ABS classification results of [2], the present work demonstrates that there is ‘integrable life beyond ABS’ and that in many respects the results on the latter systems do not really provide the necessary insights for multicomponent and higher-order integrable lattice systems.

A particular generalisation that we have not considered (yet) in the present paper, is that of the so-called *extended BSQ systems*. These involve parameter-generalisations of the lattice BSQ systems of [29] found by Hietarinta in [13]. It was shown in [40], in the rational case, how these parameter generalisations fit into the DL framework, thereby providing a rich class of solutions of those extended BSQ systems, cf. also [14] for a recent review. In fact, those extensions arise from an unfolding of the dispersion curve from $G(k,\omega) = \omega^3 - k^3 = 0$ to $G(k,\omega) = k^3 - \omega^3 + \alpha_2(\omega^2 - k^2) + \alpha_1(\omega - k) = 0$ where the parameters α_1 and α_2 are the extension parameters, the inclusion of which effectively embeds the ($N = 2$) KdV system into the ($N = 3$) BSQ system. In the elliptic case, this extension would require an unfolding of the elliptic cube root of unity condition, to a more general condition on the roots $\omega_j(\delta)$, $(j = 0, 1, \cdots, N-1)$, namely

$$\prod_{j=0}^{N-1} \Phi_\kappa(\omega_j(\delta)) = \sum_{j=0}^{N-2} \alpha_j \left(\wp^{(j)}(-\kappa) - \wp^{(j)}(\delta) \right) ,$$

(possibly with $\alpha_{N-2} = 1$) but we leave the development of the relevant explicit formulae involving the extension parameters α_j corresponding to a future publication.

As a final remark we need to point out that in this paper we have considered elliptic solutions of equations that in themselves are rational expressions of the dependent variables and their shifts. There are, however, also equations that one could characterise as *elliptic integrable systems*, i.e. equations which are either parametrised through elliptic functions (i.e. where the dependent variable enters in the argument of elliptic functions) or where the equation, albeit of rational form, contains an algebraic curve in terms of the dependent variable. Prime examples of the latter are the Krichever–Novikov equation or the fully anisotropic Landau–Lifschitz equations, cf. e.g. [6,19,20,22,32] and their discrete analogues [1,8,28]. It is not clear at this stage how these two matters, elliptic equations versus elliptic solutions, are connected, but some hint as to this connection may be in the elliptic versions of integrable many-body systems [3,17,18,21], which can be viewed both as a class of elliptic solutions as well as elliptic integrable systems in their own right.

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Appendix A: Formulae for elliptic functions

Here, we collect some useful formulae for elliptic functions, see also the standard textbooks e.g. [38]. The Weierstrass sigma-function is defined by

$$\sigma(x) =: \sigma(x|2\omega, 2\omega') = x \prod_{(k,\ell)\neq(0,0)} (1 - \frac{x}{\omega_{k\ell}}) \exp\left[\frac{x}{\omega_{k\ell}} + \frac{1}{2}(\frac{x}{\omega_{k\ell}})^2 \right] , \quad \text{(A.1)}$$

with $\omega_{kl} = 2k\omega + 2\ell\omega'$ and 2ω, $2\omega'$ being a fixed pair of the primitive periods. The relations between the Weierstrass functions are given by

$$\zeta(x) = \frac{\sigma'(x)}{\sigma(x)} , \quad \wp(x) = -\zeta'(x) , \quad \text{(A.2)}$$

where $\sigma(x)$ and $\zeta(x)$ are odd functions and $\wp(x)$ is an even function of its argument. We recall also that the $\sigma(x)$ is an entire function, and $\zeta(x)$ is a meromorphic function having simple poles at ω_{kl}, both being quasi-periodic, i.e. obeying the periodicity conditions

$$\sigma(x+2\omega) = -\sigma(x)e^{2\zeta(\omega)(x+\omega)} , \quad \sigma(x+2\omega') = -\sigma(x)e^{2\zeta(\omega')(x+\omega')} , \tag{A.3a}$$

$$\zeta(x+2\omega) = \zeta(x) + 2\zeta(\omega) , \quad \zeta(x+2\omega') = \zeta(x) + 2\zeta(\omega') , \tag{A.3b}$$

in which the half periods satisfy the relation

$$\zeta(\omega)\omega' - \zeta(\omega')\omega = \frac{\pi \mathrm{i}}{2} ,$$

whereas $\wp(x)$ is doubly periodic. From an algebraic point of view, the most important property of these elliptic functions is the existence of a number of functional relations, the most fundamental being

$$\zeta(\alpha) + \zeta(\beta) + \zeta(\gamma) - \zeta(\alpha+\beta+\gamma) = \frac{\sigma(\alpha+\beta)\sigma(\beta+\gamma)\sigma(\gamma+\alpha)}{\sigma(\alpha)\sigma(\beta)\sigma(\gamma)\sigma(\alpha+\beta+\gamma)} , \tag{A.4}$$

which can also be cast into the following form

$$\Phi_\kappa(x)\Phi_\kappa(y) = \Phi_\kappa(x+y)\left[\zeta(\kappa) + \zeta(x) + \zeta(y) - \zeta(\kappa+x+y)\right] . \tag{A.5}$$

The well-known three-term relation for $\sigma(x)$ is a consequence of (A.4)

$$\begin{aligned}\sigma(x+y)\sigma(x-y)\sigma(a+b)\sigma(a-b) &= \sigma(x+a)\sigma(x-a)\sigma(y+b)\sigma(y-b)\\ &\quad -\sigma(x+b)\sigma(x-b)\sigma(y+a)\sigma(y-a),\end{aligned} \tag{A.6}$$

and this equation can be cast into the following convenient form

$$\Phi_\kappa(x)\Phi_\lambda(y) = \Phi_\kappa(x-y)\Phi_{\kappa+\lambda}(y) + \Phi_{\kappa+\lambda}(x)\Phi_\lambda(y-x) , \tag{A.7}$$

which is obtained from the elliptic analogue of the partial fraction expansion, i.e. Eq. (A.5). Another important relation is given by

$$\begin{aligned}&\left[\zeta(\alpha) + \zeta(\beta) + \zeta(\gamma) - \zeta(\alpha+\beta+\gamma)\right]\left[\zeta(\alpha+\beta) - \zeta(\beta) - \zeta(\alpha+\gamma) + \zeta(\gamma)\right]\\ &= -\Phi_\gamma(\beta)\Phi_{-\gamma}(\beta) = \wp(\gamma) - \wp(\beta) ,\end{aligned} \tag{A.8}$$

as well as

$$\eta_\alpha(\beta) := \zeta(\alpha+\beta) - \zeta(\alpha) - \zeta(\beta) = \frac{1}{2}\frac{\wp'(\alpha) - \wp'(\beta)}{\wp(\alpha) - \wp(\beta)} . \tag{A.9}$$

The well-known addition formula for the Weierstrass elliptic function can be written in the form

$$\begin{aligned}\eta_\alpha(\beta)^2 = \Big(\zeta(\alpha+\beta) - \zeta(\alpha) - \zeta(\beta)\Big)^2 &= \wp(\alpha) + \wp(\beta) + \wp(\alpha+\beta)\\ = \frac{1}{4}\left(\frac{\wp'(\alpha) - \wp'(\beta)}{\wp(\alpha) - \wp(\beta)}\right)^2 ,\end{aligned} \tag{A.10a}$$

which together with the relation

$$\wp'(\alpha+\beta)=\wp'(\alpha)+\wp'(\beta)+3\frac{\wp'(\alpha)\wp(\beta)-\wp'(\beta)\wp(\alpha)}{\wp(\alpha)-\wp(\beta)} \\ -\frac{1}{4}\left(\frac{\wp'(\alpha)-\wp'(\beta)}{\wp(\alpha)-\wp(\beta)}\right)^3 , \tag{A.10b}$$

characterises the addition rule on the elliptic curve parametrised by $(\wp(x),\wp'(x))$. Equation (A.10b) is a consequence of the determinant relation

$$\begin{vmatrix} 1 & \wp(\alpha) & \wp'(\alpha) \\ 1 & \wp(\beta) & \wp'(\beta) \\ 1 & \wp(\gamma) & \wp'(\gamma) \end{vmatrix} = -2\frac{\sigma(\alpha-\beta)\,\sigma(\alpha-\gamma)\,\sigma(\beta-\gamma)}{\sigma^3(\alpha)\,\sigma^3(\beta)\,\sigma^3(\gamma)}\,\sigma(\alpha+\beta+\gamma)\,, \tag{A.11}$$

which is a special case of the famous Frobenius-Stickelberger formula

$$\begin{vmatrix} 1 & \wp(x_0) & \wp'(x_0) & \cdots\cdots & \wp^{(n-1)}(x_0) \\ 1 & \wp(x_1) & \wp'(x_1) & \cdots\cdots & \wp^{(n-1)}(x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(x_n) & \wp'(x_n) & \cdots\cdots & \wp^{(n-1)}(x_n) \end{vmatrix} \\ =(-1)^{\frac{1}{2}n(n-1)}1!2!\cdots n!\,\frac{\sigma(x_0+x_1+\cdots+x_n)\prod_{i<j=0}^{n}\sigma(x_i-x_j)}{\sigma^{n+1}(x_0)\,\sigma^{n+1}(x_1)\cdots\sigma^{n+1}(x_n)}\,, \tag{A.12}$$

which is an elliptic type van der Monde determinant.
An important special relation used in the derivations of Sects. 2.2 and 5.2 is:

$$\frac{\zeta(\xi)+\zeta(\alpha)+\zeta(\lambda)-\zeta(\xi+\alpha+\lambda)}{\zeta(\xi)+\zeta(\beta)+\zeta(\lambda)-\zeta(\xi+\beta+\lambda)} \\ +\frac{\zeta(\xi+\alpha)-\zeta(\alpha)+\zeta(\beta)-\zeta(\xi+\beta)}{\zeta(\xi+\alpha)+\zeta(\lambda)+\zeta(\beta)-\zeta(\xi+\alpha+\beta+\lambda)}=1\,, \tag{A.13}$$

which is used in the derivation of the various versions of the discrete KP equation for elliptic solutions. Furthermore, the following identity used several times in the BSQ reduction case:

$$\wp(\alpha)+\wp(\beta)+\wp(\lambda+\alpha)-\eta_{-\alpha}(\lambda+\alpha)\eta_\beta(\lambda+\alpha)=\frac{1}{2}\frac{\wp'(\alpha)-\wp'(\beta)}{\eta_\alpha(\lambda)-\eta_\beta(\lambda)}\,, \tag{A.14}$$

which can be deduced by using (A.8)–(A.10a).

Appendix B: Derivation of the fundamental KP relations

Here we derive the fundamental relations of Sect. 2.2, Eqs. (2.10)–(2.15). Let $\boldsymbol{C}$ be an infinite matrix as in (2.6), obeying the linear relations (2.8). Let us introduce the infinite matrix $\boldsymbol{U}_\xi^0$ defined by the relation

$$\boldsymbol{U}_\xi^0=\left(\boldsymbol{1}-\boldsymbol{U}_\xi^0\,\boldsymbol{\Omega}_\xi\right)\boldsymbol{C}\,, \tag{B.1}$$

where the infinite matrix $\boldsymbol{\Omega}_\xi$ is given by

$$\boldsymbol{\Omega}_\xi = :\Phi_\xi(\boldsymbol{\Lambda} + {}^t\boldsymbol{\Lambda})\,\boldsymbol{O}: , \tag{B.2}$$

where $\Phi_\xi(x)$ is the function (2.1). Eq. (B.2) should be viewed as a formal power series in both the operators $\boldsymbol{\Lambda}$ and ${}^t\boldsymbol{\Lambda}$ where the normal ordering symbol $: \, :$ means that all operators ${}^t\boldsymbol{\Lambda}$ are ordered on the left of the projection matrix $\boldsymbol{O}$ and all operators $\boldsymbol{\Lambda}$ on the right of $\boldsymbol{O}$. Then, as a consequence of the addition formula (A.7), we have the operator identity (2.7).

Since in (B.1) the dependence of $\boldsymbol{U}_\xi^0$ only stems from the matrix $\boldsymbol{\Omega}_\xi$ and the dependence on the discrete variables n, m, l via the matrix $\boldsymbol{C}$, we can derive the elementary shift relation of $\boldsymbol{U}_\xi^0$ as follows:

$$\begin{aligned}
\widetilde{\boldsymbol{U}}_\xi^0\,\Phi_\delta(-{}^t\boldsymbol{\Lambda}) &= \left(\mathbf{1} - \widetilde{\boldsymbol{U}}_\xi^0\,\boldsymbol{\Omega}_\xi\right)\Phi_\delta(\boldsymbol{\Lambda})\boldsymbol{C} \\
&= \Phi_\delta(\boldsymbol{\Lambda})\,\boldsymbol{C} - \widetilde{\boldsymbol{U}}_\xi^0\,\left[\Phi_\delta(-{}^t\boldsymbol{\Lambda})\,\boldsymbol{\Omega}_{\xi+\delta} + \Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\delta}(\boldsymbol{\Lambda})\right]\boldsymbol{C} \\
&\Rightarrow \widetilde{\boldsymbol{U}}_\xi^0\,\Phi_\delta(-{}^t\boldsymbol{\Lambda})\left(\mathbf{1} + \boldsymbol{\Omega}_{\xi+\delta}\boldsymbol{C}\right) = \Phi_\delta(\boldsymbol{\Lambda})\,\boldsymbol{C} - \widetilde{\boldsymbol{U}}_\xi^0\Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\delta}(\boldsymbol{\Lambda})\boldsymbol{C}\ , \\
&\Rightarrow \widetilde{\boldsymbol{U}}_\xi^0\,\Phi_\delta(-{}^t\boldsymbol{\Lambda}) = \left[\Phi_\delta(\boldsymbol{\Lambda}) - \widetilde{\boldsymbol{U}}_\xi^0\Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\delta}(\boldsymbol{\Lambda})\right]\boldsymbol{C}\left(\mathbf{1} + \boldsymbol{\Omega}_{\xi+\delta}\boldsymbol{C}\right)^{-1}\ ,
\end{aligned}$$

and since from (B.1) we have

$$\boldsymbol{C}\left(\mathbf{1} + \boldsymbol{\Omega}_{\xi+\delta}\boldsymbol{C}\right)^{-1} = \boldsymbol{U}_{\xi+\delta}^0\ ,$$

we arrive at:

$$\widetilde{\boldsymbol{U}}_\xi^0\,\Phi_\delta(-{}^t\boldsymbol{\Lambda}) = \left[\Phi_\delta(\boldsymbol{\Lambda}) - \widetilde{\boldsymbol{U}}_\xi^0\Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\delta}(\boldsymbol{\Lambda})\right]\boldsymbol{U}_{\xi+\delta}^0\ . \tag{B.3}$$

Multiplying (B.3) from the left by $\Phi_\xi(\boldsymbol{\Lambda})$ and from the right by $\Phi_{\xi+\delta}({}^t\boldsymbol{\Lambda})$ and setting

$$\Phi_\xi(\boldsymbol{\Lambda})\boldsymbol{U}_\xi^0\Phi_\xi({}^t\boldsymbol{\Lambda}) =: \boldsymbol{U}_\xi$$

in accordance with (2.9), we get

$$\widetilde{\boldsymbol{U}}_\xi\,\frac{\Phi_\delta(-{}^t\boldsymbol{\Lambda})\Phi_{\xi+\delta}({}^t\boldsymbol{\Lambda})}{\Phi_\xi({}^t\boldsymbol{\Lambda})} = \frac{\Phi_\xi(\boldsymbol{\Lambda})\Phi_\delta(\boldsymbol{\Lambda})}{\Phi_{\xi+\delta}(\boldsymbol{\Lambda})}\boldsymbol{U}_{\xi+\delta} - \widetilde{\boldsymbol{U}}_\xi\,\boldsymbol{O}\boldsymbol{U}_{\xi+\delta}\ . \tag{B.4}$$

The coefficients in the first and second term of (B.4) can be cast in the form of the functions $-\chi^{(1)}_{-\delta,\xi+\delta}({}^t\boldsymbol{\Lambda})$ and $\chi^{(1)}_{\delta,\xi}(\boldsymbol{\Lambda})$ respectively.

Next, we derive the relations for infinite vectors

$$\boldsymbol{u}_\kappa^0(\xi) := \left(\mathbf{1} - \boldsymbol{U}_\xi^0\,\boldsymbol{\Omega}_\xi\right)\boldsymbol{c}_\kappa\rho_\kappa\ , \quad {}^t\boldsymbol{u}_{\kappa'}^0(\xi) := \sigma_{\kappa'}\,{}^t\boldsymbol{c}_{\kappa'}\left(\mathbf{1} - \boldsymbol{\Omega}_\xi\boldsymbol{U}_\xi^0\right)\ , \tag{B.5}$$

with ρ_κ and $\sigma_{\kappa'}$ obeying (2.3). Deriving the shift relation for $\boldsymbol{u}_\kappa^0(\xi)$ proceeds as follows

$$\begin{aligned}
\widetilde{\boldsymbol{u}}_\kappa^0(\xi) &= \left(\mathbf{1} - \widetilde{\boldsymbol{U}}_\xi^0\,\boldsymbol{\Omega}_\xi\right)\Phi_\delta(\boldsymbol{\Lambda})\boldsymbol{c}_\kappa\rho_\kappa \\
&= \Phi_\delta(\boldsymbol{\Lambda})\,\boldsymbol{c}_\kappa\rho_\kappa - \widetilde{\boldsymbol{U}}_\xi^0\,\left[\Phi_\delta(-{}^t\boldsymbol{\Lambda})\,\boldsymbol{\Omega}_{\xi+\delta} + \Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\delta}(\boldsymbol{\Lambda})\right]\boldsymbol{c}_\kappa\rho_\kappa \\
&= \Phi_\delta(\boldsymbol{\Lambda})\,\boldsymbol{c}_\kappa\rho_\kappa - \widetilde{\boldsymbol{U}}_\xi^0\,\Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\delta}(\boldsymbol{\Lambda})\boldsymbol{c}_\kappa\rho_\kappa \\
&\quad - \left[\Phi_\delta(\boldsymbol{\Lambda})\boldsymbol{U}_{\xi+\delta}^0 - \widetilde{\boldsymbol{U}}_\xi^0\Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\delta}(\boldsymbol{\Lambda})\boldsymbol{U}_{\xi+\delta}^0\right]\boldsymbol{\Omega}_{\xi+\delta}\boldsymbol{c}_\kappa\rho_\kappa \\
&= \Phi_\delta(\boldsymbol{\Lambda})\,\boldsymbol{u}_\kappa^0(\xi+\delta) - \widetilde{\boldsymbol{U}}_\xi^0\Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\delta}(\boldsymbol{\Lambda})\boldsymbol{u}_\kappa^0(\xi+\delta)\ ,
\end{aligned}$$

where in the third step we have made use of (B.3). Setting

$$\Phi_\xi(\boldsymbol{\Lambda})\boldsymbol{u}^0_\kappa(\xi) =: \boldsymbol{u}_\kappa(\xi)\ , \quad \text{and similarly} \quad {}^t\boldsymbol{u}^0_{\kappa'}(\xi)\,\Phi_\xi({}^t\boldsymbol{\Lambda}) =: {}^t\boldsymbol{u}_{\kappa'}(\xi)\ ,$$

we obtain from the latter result

$$\widetilde{\boldsymbol{u}}_\kappa(\xi) = \frac{\Phi_\xi(\boldsymbol{\Lambda})\Phi_\delta(\boldsymbol{\Lambda})}{\Phi_{\xi+\delta}(\boldsymbol{\Lambda})}\boldsymbol{u}_\kappa(\xi+\delta) - \widetilde{\boldsymbol{U}}_\xi\,\boldsymbol{O}\,\boldsymbol{u}_\kappa(\xi+\delta)\ , \tag{B.6}$$

where the coefficient in the first term on the r.h.s. equals $\chi^{(1)}_{\delta,\xi}(\boldsymbol{\Lambda})$. In a similar way the shift relation for the adjoint vector ${}^t\boldsymbol{u}^0_{\kappa'}(\xi)$ can be derived, namely

$$\begin{aligned}
{}^t\boldsymbol{u}^0_{\kappa'}(\xi+\delta) &= \widetilde{\sigma}_\kappa\,{}^t\boldsymbol{c}_{\kappa'}\Phi_\delta(-{}^t\boldsymbol{\Lambda})\left(\boldsymbol{1}-\boldsymbol{\Omega}_{\xi+\delta}\boldsymbol{U}^0_{\xi+\delta}\right)\\
&= \widetilde{\sigma}_{\kappa'}\,{}^t\boldsymbol{c}_{\kappa'}\Phi_\delta(-{}^t\boldsymbol{\Lambda}) - \widetilde{\sigma}_{\kappa'}\,{}^t\boldsymbol{c}_{\kappa'}\left[\boldsymbol{\Omega}_\xi\Phi_\delta(\boldsymbol{\Lambda}) - \Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\delta}(\boldsymbol{\Lambda})\right]\boldsymbol{U}^0_{\xi+\delta}\\
&= \widetilde{\sigma}_{\kappa'}\,{}^t\boldsymbol{c}_{\kappa'}\Phi_\delta(-{}^t\boldsymbol{\Lambda}) + \widetilde{\sigma}_{\kappa'}\,{}^t\boldsymbol{c}_{\kappa'}\Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\delta}(\boldsymbol{\Lambda})\boldsymbol{U}^0_{\xi+\delta}\\
&\quad -\widetilde{\sigma}_{\kappa'}\,{}^t\boldsymbol{c}_{\kappa'}\boldsymbol{\Omega}_\xi\left[\widetilde{\boldsymbol{U}}^0_\xi\Phi_\delta(-{}^t\boldsymbol{\Lambda}) + \widetilde{\boldsymbol{U}}^0_\xi\Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\delta}(\boldsymbol{\Lambda})\boldsymbol{U}^0_{\xi+\delta}\right]\\
&= \widetilde{{}^t\boldsymbol{u}^0_{\kappa'}}(\xi)\Phi_\delta(-{}^t\boldsymbol{\Lambda}) + \widetilde{{}^t\boldsymbol{u}^0_{\kappa'}}(\xi)\Phi_\xi({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_{\xi+\delta}(\boldsymbol{\Lambda})\boldsymbol{U}^0_{\xi+\delta}\ .
\end{aligned}$$

Multiplying the latter result from the right by $\Phi_{\xi+\delta}({}^t\boldsymbol{\Lambda})$ we get

$${}^t\boldsymbol{u}_{\kappa'}(\xi+\delta) = \widetilde{{}^t\boldsymbol{u}_{\kappa'}}(\xi)\frac{\Phi_\delta(-{}^t\boldsymbol{\Lambda})\Phi_{\xi+\delta}({}^t\boldsymbol{\Lambda})}{\Phi_\xi({}^t\boldsymbol{\Lambda})} + \widetilde{{}^t\boldsymbol{u}_{\kappa'}}(\xi)\boldsymbol{O}\boldsymbol{U}_{\xi+\delta}\ , \tag{B.7}$$

where the coefficient in the first terms on the r.h.s. is identified with $-\chi^{(1)}_{-\delta,\xi}({}^t\boldsymbol{\Lambda})$. Finally, we derive the relations from the τ-function defined by (2.10). Applying the $\widetilde{}$-shift we have

$$\begin{aligned}
\widetilde{\tau}_\xi &= \det\left(\boldsymbol{1}+\boldsymbol{\Omega}_{\widetilde{\xi}}\widetilde{\boldsymbol{C}}\right) = \det\left(\boldsymbol{1}+\boldsymbol{\Omega}_{\widetilde{\xi}}\Phi_\delta(\boldsymbol{\Lambda})\boldsymbol{C}(\Phi_\delta(-{}^t\boldsymbol{\Lambda}))^{-1}\right)\\
&= \det\Big(\boldsymbol{1}+\left[\boldsymbol{\Omega}_{\widetilde{\xi}}\Phi_\delta(\boldsymbol{\Lambda}) - \Phi_\delta(-{}^t\boldsymbol{\Lambda})\boldsymbol{\Omega}_\xi\right]\boldsymbol{C}(\Phi_\delta(-{}^t\boldsymbol{\Lambda}))^{-1}\\
&\qquad +\Phi_\delta(-{}^t\boldsymbol{\Lambda})\boldsymbol{\Omega}_\xi\boldsymbol{C}(\Phi_\delta(-{}^t\boldsymbol{\Lambda}))^{-1}\Big)\\
&= \det\left(\boldsymbol{1}+\Phi_{\widetilde{\xi}}({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_\xi(\boldsymbol{\Lambda})\boldsymbol{C}(\Phi_\delta(-{}^t\boldsymbol{\Lambda}))^{-1} + \Phi_\delta(-{}^t\boldsymbol{\Lambda})\boldsymbol{\Omega}_\xi\boldsymbol{C}(\Phi_\delta(-{}^t\boldsymbol{\Lambda}))^{-1}\right)\\
&= \det\left(\boldsymbol{1}+\boldsymbol{\Omega}_\xi\boldsymbol{C} + (\Phi_\delta(-{}^t\boldsymbol{\Lambda}))^{-1}\Phi_{\widetilde{\xi}}({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_\xi(\boldsymbol{\Lambda})\boldsymbol{C}\right)\\
&= \det\left(\boldsymbol{1}+\boldsymbol{\Omega}_\xi\boldsymbol{C}\right)\,\det\left(\boldsymbol{1}+(\Phi_\delta(-{}^t\boldsymbol{\Lambda}))^{-1}\Phi_{\widetilde{\xi}}({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_\xi(\boldsymbol{\Lambda})\boldsymbol{C}\left(\boldsymbol{1}+\boldsymbol{\Omega}_\xi\boldsymbol{C}\right)^{-1}\right)\ ,
\end{aligned}$$

where we have subsequently used the shift relation (2.8a), the relation (2.7) for the Cauchy kernel and the invariance of the determinant under similarity transformations, as well as the definition (B.1). Thus, we get

$$\begin{aligned}
\widetilde{\tau}_\xi/\tau_\xi &= \det\left(\boldsymbol{1}+(\Phi_\delta(-{}^t\boldsymbol{\Lambda}))^{-1}\Phi_{\widetilde{\xi}}({}^t\boldsymbol{\Lambda})\boldsymbol{O}\Phi_\xi(\boldsymbol{\Lambda})\boldsymbol{U}^0_\xi\right)\\
&= 1+\left(\Phi_\xi(\boldsymbol{\Lambda})\boldsymbol{U}^0_\xi(\Phi_\delta(-{}^t\boldsymbol{\Lambda}))^{-1}\Phi_{\widetilde{\xi}}({}^t\boldsymbol{\Lambda})\right)_{0,0}\ ,
\end{aligned}$$

where the latter step uses the fact that $\boldsymbol{O}$ is a rank 1 projector matrix and we apply the famous Weinstein-Aroszajn formula:

$$\det\left(\mathbf{1}+\boldsymbol{a}^T\,\boldsymbol{b}\right)=1+\boldsymbol{b}\cdot\boldsymbol{a}^T\ ,$$

for determinants involving a rank 1 perturbation from the unit matrix.

Appendix C: Some higher-order elliptic identities

The following higher addition rules were established in Appendix C of [7], which for the sake of self-containedness we reiterate here. They may be used, in future, for deriving the higher reductions from the lattice KP system to the lattice GD hierarchy, and they play a role in establishing the defining relations for the higher elliptic roots of unity. First, we have the following generalisation of (A.4)

$$\begin{aligned}
&\sigma(\kappa+x)\,\sigma(\lambda+x)\,\sigma(\mu+x)\sigma(\kappa+\lambda+\mu+y)\,\sigma^2(y)\\
&\quad-\sigma(\kappa+y)\,\sigma(\lambda+y)\,\sigma(\mu+y)\sigma(\kappa+\lambda+\mu+x)\,\sigma^2(x)\\
&=\ \sigma(\kappa)\,\sigma(\lambda)\,\sigma(\mu)\,\sigma(x)\,\sigma(y)\,\sigma(\kappa+\lambda+\mu+x+y)\,\sigma(y-x)\\
&\quad\times[\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(x)+\zeta(y)-\zeta(\kappa+\lambda+\mu+x+y)]
\end{aligned}$$

which derives from:

$$\begin{aligned}
&\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(x)+\zeta(y)-\zeta(\kappa+\lambda+\mu+x+y)\\
&\quad=\ \frac{\Phi_\kappa(x)\Phi_\lambda(x)\Phi_\mu(x)\Phi_{\kappa+\lambda+\mu}(y)-\Phi_\kappa(y)\Phi_\lambda(y)\Phi_\mu(y)\Phi_{\kappa+\lambda+\mu}(x)}{\Phi_{\kappa+\lambda+\mu}(x+y)\,(\wp(x)-\wp(y))}\ .
\end{aligned}$$

Furthermore, we have [7]

$$\begin{aligned}
&\Phi_\kappa(x)\Phi_\kappa(y)\Phi_\kappa(z)\\
&\quad=\tfrac{1}{2}\Phi_\kappa(x+y+z)\,\Big[(\zeta(\kappa)+\zeta(x)+\zeta(y)+\zeta(z)-\zeta(\kappa+x+y+z))^2\\
&\qquad+\wp(\kappa)-(\wp(x)+\wp(y)+\wp(z)+\wp(\kappa+x+y+z))\Big].
\end{aligned}\tag{C.1}$$

At the next level we get the following identity:

$$\begin{aligned}
\Phi_\kappa(x)\Phi_\lambda(y)\Phi_\mu(z)\Phi_\nu(w)=\ &\Phi_{\kappa+\lambda+\mu+\nu}(x)\Phi_\lambda(y-x)\Phi_\mu(z-x)\Phi_\nu(w-x)\\
&+\Phi_\kappa(x-y)\Phi_{\kappa+\lambda+\mu+\nu}(y)\Phi_\mu(z-y)\Phi_\nu(w-y)\\
&+\Phi_\kappa(x-z)\Phi_\lambda(y-z)\Phi_{\kappa+\lambda+\mu+\nu}(z)\Phi_\nu(w-z)\\
&+\Phi_\kappa(x-w)\Phi_\lambda(y-w)\Phi_\mu(z-w)\Phi_{\kappa+\lambda+\mu+\nu}(w).
\end{aligned}\tag{C.2}$$

Setting $w=z+\delta$ and taking the limit $\delta\to0$ in (C.2) we obtain:

$$\begin{aligned}
&\Phi_\kappa(x)\Phi_\lambda(y)\Phi_\mu(z)\Phi_\nu(z)-\Phi_{\kappa+\lambda+\mu+\nu}(x)\Phi_\lambda(y-x)\Phi_\mu(z-x)\Phi_\nu(z-x)\\
&\quad-\Phi_\kappa(x-y)\Phi_{\kappa+\lambda+\mu+\nu}(y)\Phi_\mu(z-y)\Phi_\nu(z-y)\\
&=\ \Phi_\kappa(x-z)\Phi_\lambda(y-z)\Phi_{\kappa+\lambda+\mu+\nu}(z)\\
&\quad\times[\zeta(\nu)+\zeta(\mu)+\zeta(z)+\zeta(\kappa+x-z)+\zeta(\lambda+y-z)-\zeta(x-z)-\zeta(y-z)\\
&\quad-\zeta(\kappa+\lambda+\mu+\nu+z)]\ ,
\end{aligned}\tag{C.3}$$

which, after a renaming of the arguments, can be cast into the following form:

$$\begin{aligned}
&\Phi_{\kappa+\lambda+\mu+\nu}(x+y+z)\,\frac{\sigma(x+y+z)\,\sigma(x-y)\,\sigma(x-z)\,\sigma(y-z)}{\sigma^3(x)\,\sigma^3(y)\,\sigma^3(z)}\\
&\quad\times[\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(\nu)+\zeta(x)+\zeta(y)+\zeta(z) \qquad\qquad (\mathrm{C}.4)\\
&\quad-\zeta(\kappa+\lambda+\mu+\nu+x+y+z)]\\
&=\ \Phi_\kappa(x)\Phi_\lambda(x)\Phi_\mu(x)\Phi_\nu(x)\,(\wp(z)-\wp(y))\,\Phi_{\kappa+\lambda+\mu+\nu}(y+z)\\
&\quad+\Phi_\kappa(y)\Phi_\lambda(y)\Phi_\mu(y)\Phi_\nu(y)\,(\wp(x)-\wp(z))\,\Phi_{\kappa+\lambda+\mu+\nu}(x+z)\\
&\quad+\Phi_\kappa(z)\Phi_\lambda(z)\Phi_\mu(z)\Phi_\nu(z)\,(\wp(y)-\wp(x))\,\Phi_{\kappa+\lambda+\mu+\nu}(x+y)\,. \qquad (\mathrm{C}.5)
\end{aligned}$$

In the next step we set $z = y + \varepsilon$ in (C.3), and take the limit $\varepsilon \to 0$, which yields the relation

$$\begin{aligned}
&\Phi_\kappa(x)\Phi_\lambda(y)\Phi_\mu(y)\Phi_\nu(y)-\Phi_{\kappa+\lambda+\mu+\nu}(x)\Phi_\lambda(y-x)\Phi_\mu(y-x)\Phi_\nu(y-x)\\
&=\ \tfrac{1}{2}\Phi_{\kappa+\lambda+\mu+\nu}(y)\Phi_\kappa(x-y)\Big[\wp(y)+\wp(x-y)-\wp(\lambda)-\wp(\mu)-\wp(\nu)\\
&\quad+\Big(\zeta(\lambda)+\zeta(\mu)+\zeta(\nu)+\zeta(\kappa+x-y)-\zeta(x-y)+\zeta(y)-\zeta(\kappa+\lambda+\mu+\nu+y)\Big)^2\\
&\quad-\wp(\kappa+x-y)-\wp(\kappa+\lambda+\mu+\nu+y)\Big], \qquad (\mathrm{C}.6)
\end{aligned}$$

which generalises (C.1). In the final step, however, setting $y = x + \gamma$ and letting $\gamma \to 0$ (which requires expansions up to third order in γ), we get the identity

$$\begin{aligned}
&\Phi_\kappa(x)\Phi_\lambda(x)\Phi_\mu(x)\Phi_\nu(x)\\
&=\tfrac{1}{6}\Phi_{\kappa+\lambda+\mu+\nu}(x)\Big\{\Big(\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(\nu)+\zeta(x)-\zeta(\kappa+\lambda+\mu+\nu+x)\Big)^3\\
&\quad-3\Big(\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(\nu)+\zeta(x)-\zeta(\kappa+\lambda+\mu+\nu+x)\Big)\\
&\quad\times\Big(\wp(\kappa)+\wp(\lambda)+\wp(\mu)+\wp(\nu)+\wp(\kappa+\lambda+\mu+\nu+x)-\wp(x)\Big)\\
&\quad-\Big(\wp'(\kappa)+\wp'(\lambda)+\wp'(\mu)+\wp'(\nu)+\wp'(x)-\wp'(\kappa+\lambda+\mu+\nu+x)\Big)\Big\}\,. \qquad (\mathrm{C}.7)
\end{aligned}$$

The previous relations can be obviously generalised to arbitrary order. Thus, the general form of the basic identity (A.7) (3-term relation for the σ-function, or the elliptic partial fraction expansion formula) is:

$$\prod_{i=1}^{n}\Phi_{\kappa_i}(x_i)=\sum_{i=1}^{n}\Phi_{\kappa_1+\cdots+\kappa_n}(x_i)\prod_{\substack{j=1\\ j\neq i}}^{n}\Phi_{\kappa_j}(x_j-x_i)\,. \qquad (\mathrm{C}.8)$$

Extending this identity to $n+1$ variables, including a κ_0 and x_0, and subsequently taking the limit $x_0 = x_1 + \varepsilon$, with $\varepsilon \to 0$, we obtain the following identity (after some obvious relabelling of parameters and changes of variables):

$$\begin{aligned}
&(-1)^{n-1}\Phi_{\kappa_0+\kappa_1+\cdots+\kappa_n}(x_1+\cdots+x_n)\,\frac{\sigma(x_1+\cdots+x_n)}{\prod_{j=1}^{n}\sigma(x_j)}\\
&\quad\times\left[\zeta(\kappa_0)+\sum_{j=1}^{n}\big(\zeta(\kappa_j)+\zeta(x_j)\big)-\zeta(\kappa_0+\kappa_1+\cdots+\kappa_n+x_1+\cdots+x_n)\right]\\
&=\sum_{i=1}^{n}\Phi_{\kappa_0+\kappa_1+\cdots+\kappa_n}(x_1+\cdots+\cancel{x_i}+\cdots+x_n)\,\frac{\sigma(x_1+\cdots+\cancel{x_i}+\cdots+x_n)\,\sigma^{n-1}(x_i)}{\prod_{\substack{j=1\\ j\neq i}}^{n}\sigma(x_i-x_j)}\prod_{j=0}^{n}\Phi_{\kappa_j}(x_i)\,.
\end{aligned}$$

These identities are associated with the Frobenius-Stickelberger determinantal formula (A.12).

Appendix D: Lax pairs. The Lax pairs for the BSQ type lattice equations are derived from the fundamental system of relations (4.17b) and (4.17d), or equivalently (4.19b) and (4.19d), for the vectors $\boldsymbol{u}_\kappa(\xi)$ together with (4.18b), and their counterparts in the other lattice direction. By choosing specific components of these infinite vectors we construct 3-component vectors which will yield the eigenvectors for the corresponding 3×3 matrix Lax pairs. There are two different choices of entries that we will consider, leading to the Lax pairs for the lattice BSQ system and lattice modified BSQ system respectively.

a) Lax pair for the lattice BSQ system

In this case we specify the entries:

$$(u_\kappa(\xi))_0 := (\boldsymbol{u}_\kappa(\xi))_0 \ , \quad (u_\kappa(\xi))_1 := \left(\boldsymbol{\Lambda}_\xi \boldsymbol{u}_\kappa(\xi)\right)_0 \ , \quad (u_\kappa(\xi))_2 := (\wp(\boldsymbol{\Lambda})\boldsymbol{u}_\kappa(\xi))_0 \ ,$$

and consider the relations (4.17), together with (4.18), to derive a Lax pair for the 3-component vector

$$\boldsymbol{\phi}(\xi) := \Big((u_\kappa(\xi))_0,\ (u_\kappa(\xi))_1,\ (u_\kappa(\xi))_2\Big) \ . \tag{D.1}$$

In fact, the dynamical relations (4.17), using the above notation, can be rewritten as:

$$\widetilde{\boldsymbol{u}_\kappa(\xi)} = \left(p_\xi - \boldsymbol{\Lambda}_\xi - \widetilde{\boldsymbol{U}}_\xi \boldsymbol{O}\right)\boldsymbol{u}_\kappa(\xi) \ , \tag{D.2a}$$

$$\boldsymbol{\Lambda}_{\widetilde{\xi}}\widetilde{\boldsymbol{u}_\kappa(\xi)} = \left(p_\xi \boldsymbol{\Lambda}_\xi - \wp(\widetilde{\xi}) - \wp(\xi) - \wp(\boldsymbol{\Lambda})\right)\boldsymbol{u}_\kappa(\xi) - \boldsymbol{\Lambda}_{\widetilde{\xi}}\widetilde{\boldsymbol{U}}_\xi\ \boldsymbol{O}\ \boldsymbol{u}_\kappa(\xi) \ , \tag{D.2b}$$

$$\begin{aligned} -\tfrac{1}{2}\left(\wp'(\delta) + \wp'(\kappa)\right)\boldsymbol{u}_\kappa(\xi) &= \left(\wp(\widetilde{\xi}) + \wp(\delta) + \wp(\boldsymbol{\Lambda}) + p_\xi \boldsymbol{\Lambda}_{\widetilde{\xi}}\right)\widetilde{\boldsymbol{u}_\kappa(\xi)} \\ &\quad + \boldsymbol{U}_\xi\left(p_\xi \boldsymbol{O} + \boldsymbol{O}\boldsymbol{\Lambda}_{\widetilde{\xi}} - {}^t\boldsymbol{\Lambda}_\xi \boldsymbol{O}\right)\widetilde{\boldsymbol{u}_\kappa(\xi)} \ , \end{aligned} \tag{D.2c}$$

and similar relations for the other lattice direction. Taking the 0-component of these relations and their counterparts in the other direction we constitute the Lax pair as the shift relations for the vector $\boldsymbol{\phi}(\xi)$, as given by

$$\widetilde{\boldsymbol{\phi}(\xi)} = \boldsymbol{L}_\kappa(\xi)\boldsymbol{\phi}(\xi) \ , \qquad \widehat{\boldsymbol{\phi}(\xi)} = \boldsymbol{M}_\kappa(\xi)\boldsymbol{\phi}(\xi) \ , \tag{D.3a}$$

with Lax matrices $\boldsymbol{L}_\kappa(\xi)$ and $\boldsymbol{M}_\kappa(\xi)$ given by

$$\boldsymbol{L}_\kappa(\xi) := \begin{pmatrix} p_\xi - \widetilde{u_{0,0}} & -1 & 0 \\ -\wp(\widetilde{\xi}) - \wp(\xi) - \widetilde{u_{1,0}} & p_\xi & -1 \\ \clubsuit & -\wp(\xi) - u_{0,1} & p_\xi + u_{0,0} \end{pmatrix} \ , \tag{D.3b}$$

in which

$$\begin{aligned} \clubsuit := -\tfrac{1}{2}\left(\wp'(\delta) + \wp'(\kappa)\right) + (p_\xi - \widetilde{u_{0,0}})\left(u_{0,1} - \wp(\widetilde{\xi}) - \wp(\delta) - p_\xi u_{0,0}\right) \\ + (p_\xi + u_{0,0})\left(\wp(\widetilde{\xi}) + \wp(\xi) + \widetilde{u_{1,0}}\right) \ , \end{aligned}$$

and $\boldsymbol{M}_\kappa(\xi)$ similarly replacing $\widetilde{\ }$ by $\widehat{\ }$ and δ by ε everywhere. The entry ♣ is such that $\det(\boldsymbol{L}_\kappa(\xi)) = -\frac{1}{2}(\wp'(\delta)+\wp'(\kappa))$. The compatibility of the Lax pair, $\widehat{\boldsymbol{L}_\kappa(\xi)}\boldsymbol{M}_\kappa(\xi) = \widetilde{\boldsymbol{M}_\kappa(\xi)}\boldsymbol{L}(\kappa(\xi)$, yields the following coupled set of relations

$$\widetilde{u_{1,0}} - \widehat{u_{1,0}} = \left(p_\xi - q_\xi + \widehat{u_{0,0}} - \widetilde{u_{0,0}}\right)\widehat{\widetilde{u_{0,0}}} - p_{\widehat{\xi}}\widehat{u_{0,0}} + q_{\widetilde{\xi}}\widetilde{u_{0,0}}\,, \tag{D.4a}$$

$$\widetilde{u_{0,1}} - \widehat{u_{0,1}} = \left(p_\xi - q_\xi + \widehat{u_{0,0}} - \widetilde{u_{0,0}}\right)u_{0,0} - p_\xi\widetilde{u_{0,0}} + q_\xi\widehat{u_{0,0}}\,, \tag{D.4b}$$

$$\begin{aligned}\frac{1}{2}\frac{\wp'(\delta)-\wp'(\varepsilon)}{p_\xi - q_\xi + \widehat{u_{0,0}} - \widetilde{u_{0,0}}} &= \widehat{\widetilde{u_{1,0}}} + u_{0,1} + (p_{\widehat{\xi}} + q_\xi + u_{0,0})(\widehat{\widetilde{u_{0,0}}} - p_\xi - q_{\widetilde{\xi}}) \\ &\quad + \tfrac{1}{2}\left(\wp(\xi) + \wp(\widehat{\widetilde{\xi}}) - \wp(\delta) - \wp(\varepsilon)\right) + \tfrac{1}{2}\left(p_\xi + q_{\widetilde{\xi}}\right)^2\,,\end{aligned} \tag{D.4c}$$

where a number of identities are used, such as (5.4a) and (5.4b), as well as

$$(p_{\widehat{\xi}} + q_\xi)\left(\wp(\widehat{\xi}) - \wp(\widetilde{\xi})\right) = (p_\xi - q_\xi)\left(\wp(\widehat{\widetilde{\xi}}) - \wp(\xi)\right) = (p_{\widetilde{\xi}} - p_\xi)\left(\wp(\delta) - \wp(\varepsilon)\right)\,.$$

This coupled set of equations is similar to the one appearing in [25,35] and by elimination of the quantities $u_{1,0}$ and $u_{0,1}$ yields the lattice BSQ system (5.6). In principle, the Lax pair can be used to investigate initial-boundary value problems for the lattice BSQ equation in a similar way as the inverse scattering scheme for the BSQ equation. That approach to solutions is beyond the scope of the present paper.

b) Lax pair for the lattice modified BSQ system

In the case of the lattice modified BSQ equation we can chose an eigenvector in the form

$$\psi(\xi) = \left(\left(\left(\chi^{(1)}_{\alpha,\boldsymbol{\Lambda}}(\xi)\right)^{-1}\boldsymbol{u}_\kappa(\xi)\right)_0, (\boldsymbol{u}_\kappa(\xi))_0, \left(\boldsymbol{\Lambda}_\xi\boldsymbol{u}_\kappa(\xi)\right)_0\right)^T\,, \tag{D.5}$$

depending on the additional parameter α. For this choice of eigenvector we can make use of the identities (5.8) and (5.9a) to derive the corresponding relations for the entries of $\psi_\kappa(\xi)$ from the fundamental relations (4.17b) and (4.17d). Thus, we can derive the Lax pairs

$$\widetilde{\psi(\xi)} = \mathcal{L}_\kappa(\xi)\psi(\xi),\quad \widehat{\psi(\xi)} = \mathcal{M}_\kappa(\xi)\psi(\xi) \tag{D.6a}$$

where the modified BSQ Lax matrices are given by

$$\mathcal{L}_\kappa(\xi) = \begin{pmatrix} p_\xi + a_{\widetilde{\xi}} & \widetilde{v_\alpha(\xi)} & 0 \\ 0 & p_\xi - \widetilde{u_{0,0}} & -1 \\ \frac{1}{2}\frac{\wp'(\alpha)+\wp'(\kappa)}{v_\alpha(\xi)} & \spadesuit & p_\xi - \frac{s_\alpha(\xi)}{v_\alpha(\xi)} \end{pmatrix}\,, \tag{D.6b}$$

in which

$$\spadesuit = -(p_\xi - \widetilde{u_{0,0}})\left(p_\xi - \frac{s_\alpha(\xi)}{v_\alpha(\xi)}\right) - \frac{1}{2}\frac{\wp'(\delta)-\wp'(\alpha)}{p_\xi + a_{\widetilde{\xi}}}\frac{\widetilde{v_\alpha(\xi)}}{v_\alpha(\xi)}\,, \tag{D.6c}$$

and similarly $\mathcal{M}_\kappa$ obtained from (D.6b) by replacing δ by ε (and hence p_ξ by q_ξ) and $\widetilde{\ }$ by $\widehat{\ }$. The quantity indicated by ♠ is such that $\det(\mathcal{L}_\kappa) = -\frac{1}{2}(\wp'(\delta)+\wp'(\kappa))\widetilde{v_\alpha}/v_\alpha$ and $\det(\mathcal{M}_\kappa) = -\frac{1}{2}(\wp'(\varepsilon)+\wp'(\kappa))\widehat{v_\alpha}/v_\alpha$.

The compatibility relation $\widehat{\mathcal{L}_\kappa}\,\mathcal{M}_\kappa = \widetilde{\mathcal{M}_\kappa}\,\mathcal{L}_\kappa$ leads to the relations (5.19a), (5.17a) and (5.20a), as well as to the relation

$$A_\delta(\widehat{\widetilde{\xi}})\frac{\widehat{\widetilde{v}}_\alpha}{\widehat{v}_\alpha} - A_\varepsilon(\widehat{\widetilde{\xi}})\frac{\widehat{\widetilde{v}}_\alpha}{\widetilde{v}_\alpha} = \left(p_{\widehat{\xi}} + q_\xi - \widehat{\widetilde{u}_{0,0}} - \frac{s_\alpha}{v_\alpha}\right)\left(p_{\widehat{\xi}} - q_{\widetilde{\xi}} + \frac{\widetilde{s}_\alpha}{\widetilde{v}_\alpha} - \frac{\widetilde{s}_\alpha}{\widetilde{v}_\alpha}\right), \quad \text{(D.7)}$$

which can be obtained from combining (5.16a) and (5.15a). The relations thus obtained from the Lax pair allow us to reconstruct the lattice modified BSQ equation (5.21). Similarly, a Lax pair can be set up for the modified BSQ equation in the form (5.22), starting from the relations in Sect. 4 for the adjoint vector ${}^t\boldsymbol{u}_{\kappa'}(\xi)$, and involving the quantities w_β. We will omit the details here.

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