

This is a repository copy of *Quantum Berezinian for a strange Lie superalgebra*.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/189938/>

Version: Published Version

Article:

Nazarov, Maxim (2022) Quantum Berezinian for a strange Lie superalgebra. *Journal of Mathematical Physics*. 081702. ISSN 1089-7658

<https://doi.org/10.1063/5.0102653>

Reuse

This article is distributed under the terms of the Creative Commons Attribution (CC BY) licence. This licence allows you to distribute, remix, tweak, and build upon the work, even commercially, as long as you credit the authors for the original work. More information and the full terms of the licence here:

<https://creativecommons.org/licenses/>

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

Quantum Berezinian for a strange Lie superalgebra ^{EP}

Cite as: J. Math. Phys. **63**, 081702 (2022); <https://doi.org/10.1063/5.0102653>

Submitted: 11 June 2022 • Accepted: 28 June 2022 • Published Online: 08 August 2022

Published open access through an agreement with JISC Collections

Maxim Nazarov

COLLECTIONS

^{EP} This paper was selected as an Editor's Pick



View Online



Export Citation



CrossMark

ARTICLES YOU MAY BE INTERESTED IN

[Ground state energy of the low density Bose gas with three-body interactions](#)

Journal of Mathematical Physics **63**, 071903 (2022); <https://doi.org/10.1063/5.0087026>

[Contextuality and the fundamental theorems of quantum mechanics](#)

Journal of Mathematical Physics **63**, 072103 (2022); <https://doi.org/10.1063/5.0012855>

[A non-trivial PT-symmetric continuum Hamiltonian and its eigenstates and eigenvalues](#)

Journal of Mathematical Physics **63**, 072104 (2022); <https://doi.org/10.1063/5.0096250>

Journal of
Mathematical Physics

Young Researcher Award

Recognizing the outstanding work of early career researchers

LEARN
MORE >>>

Quantum Berezinian for a strange Lie superalgebra

Cite as: J. Math. Phys. 63, 081702 (2022); doi: 10.1063/5.0102653

Submitted: 11 June 2022 • Accepted: 28 June 2022 •

Published Online: 8 August 2022



Maxim Nazarov^{a)}

AFFILIATIONS

Department of Mathematics, University of York, York YO10 5DD, United Kingdom

^{a)} Author to whom correspondence should be addressed: maxim.nazarov@york.ac.uk

ABSTRACT

We introduce a new family of central elements of the Yangian of the queer Lie superalgebra q_1 .

© 2022 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>). <https://doi.org/10.1063/5.0102653>

I. INTRODUCTION

In this article, we work over the complex field $\mathbb{C}$. The family of *strange* Lie superalgebras consists of the queer Lie superalgebras q_N and periplectic Lie superalgebras p_N , where N is any positive integer. Both q_N and p_N are fixed point subalgebras of the general linear Lie superalgebra $gl_{N|N}$ relative to certain involutive automorphisms. For q_N , this automorphism is denoted by π (see Sec. III).

Take the *twisted* polynomial current Lie superalgebra

$$\mathfrak{g} = \{X(u) \in gl_{N|N}[u] : \pi(X(u)) = X(-u)\}. \quad (1)$$

Then, the Yangian $Y(q_N)$ is a deformation of the universal enveloping algebra of $\mathfrak{g}$ in the class of Hopf superalgebras.

The Yangian $Y(q_N)$ has been discovered by the present author by extending to q_N the centralizer construction¹ of the Yangian of the general linear Lie algebra gl_N . The resulting definition of $Y(q_N)$ was published in Ref. 2 where the Yangian of the Lie superalgebra p_N was also defined. The Yangian $Y(q_N)$ from Ref. 2 was further studied in Ref. 3. Details of the original centralizer construction of $Y(q_N)$ involving the invariant theory of Lie superalgebras were later published in Ref. 4. There is no alternative definition of the Yangian of p_N , however, other than that given in Ref. 2.

Due to the centralizer construction of $Y(q_N)$, it appears in the theory of W -algebras. For any positive integer M , take the finite W -algebra of q_{MN} defined by a non-regular nilpotent odd element with the Jordan blocks of size M each. This W -algebra⁵ is a quotient of $Y(q_N)$.

The definition of the Yangian $Y(q_N)$ is stated in Sec. V. It is based on a new solution of the quantum Yang–Baxter equation (9) given in Ref. 2. This solution is a rational function (8) of two variables u, v with values in the supercommutant in $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$ of the image of q_N . Unlike all other rational solutions of Eq. (9) known before,² it is *not* a function of only the difference $u - v$ of the variables. In Sec. IV, we prove that (8) satisfies (9). The proof was not published previously.

In Secs. VII and VIII, we study in more detail the Yangian $Y(q_1)$. Take the finite W -algebra of q_N defined by any regular nilpotent odd element. This W -algebra⁶ is a quotient of $Y(q_1)$. Here, we introduce a new family of central elements of $Y(q_1)$. Its generating function $C(u)$ is a match to the generating function of central elements of the Yangian of $gl_{1|1}$, called the quantum Berezinian.⁷ We relate $C(u)$ to another generating function of central elements of $Y(q_1)$. The latter is just the specialization to $N = 1$ of the generating function $Z(u)$ of central elements of $Y(q_N)$ given in Ref. 2 and reviewed in our Sec. IV.

II. GENERAL CONVENTIONS

We will use the following general conventions. Let A and B be any two associative $\mathbb{Z}_2$ -graded algebras. Their tensor product $A \otimes B$ is also an associative $\mathbb{Z}_2$ -graded algebra such that for any homogeneous elements X, X' of A and Y, Y' of B , we have

$$(X \otimes Y)(X' \otimes Y') = XX' \otimes YY' (-1)^{\deg X' \deg Y},$$

$$\deg(X \otimes Y) = \deg X + \deg Y.$$

Furthermore, for any two $\mathbb{Z}_2$ -graded modules U and V over A and B , respectively, the vector space $U \otimes V$ is also a $\mathbb{Z}_2$ -graded module over $A \otimes B$ such that for any homogeneous elements $x \in U$ and $y \in V$,

$$(X \otimes Y)(x \otimes y) = Xx \otimes Yy (-1)^{\deg x \deg Y}, \quad (2)$$

$$\deg(x \otimes y) = \deg x + \deg y. \quad (3)$$

As usual, a homomorphism $\alpha : A \rightarrow B$ is a linear map such that $\alpha(XX') = \alpha(X)\alpha(X')$ for all $X, X' \in A$. However, an antihomomorphism $\beta : A \rightarrow B$ is a linear map such that for all homogeneous elements $X, X' \in A$,

$$\beta(XX') = \beta(X')\beta(X) (-1)^{\deg X \deg X'}.$$

Let n be any positive integer. If the algebra A is unital, let ι_p be its embedding into the tensor product $A^{\otimes n}$ as the p th tensor factor,

$$\iota_p(X) = 1^{\otimes(p-1)} \otimes X \otimes 1^{\otimes(n-p)} \quad \text{for } p = 1, \dots, n.$$

We will also employ various embeddings of $A^{\otimes m}$ to $A^{\otimes n}$ for $m = 1, \dots, n$. For any choice of m pairwise distinct indices $p_1, \dots, p_m \in \{1, \dots, n\}$ and of an element W of $A^{\otimes m}$ of the form $W = X^{(1)} \otimes \dots \otimes X^{(m)}$, we will denote

$$W_{p_1 \dots p_m} = \iota_{p_1}(X^{(1)}) \dots \iota_{p_m}(X^{(m)}) \in A^{\otimes n}. \quad (4)$$

We will extend the notation $W_{p_1 \dots p_m}$ to all elements W of $A^{\otimes m}$ by linearity.

III. THE QUEER LIE SUPERALGEBRA

Let the indices i, j run through $\pm 1, \dots, \pm N$. We will write $\bar{i} = 0$ if $i > 0$ and $\bar{i} = 1$ if $i < 0$. Hence, $\bar{i}$ will take values in $\mathbb{Z}_2$. Consider the $\mathbb{Z}_2$ -graded vector space $\mathbb{C}^{N|N}$. Let $e_i \in \mathbb{C}^{N|N}$ be an element of the standard basis. The $\mathbb{Z}_2$ -grading on $\mathbb{C}^{N|N}$ is defined by $\deg e_i = \bar{i}$.

Let $E_{ij} \in \text{End } \mathbb{C}^{N|N}$ be the standard matrix unit. It is defined by setting $E_{ij}e_k = \delta_{jk}e_i$. Then, the associative algebra $\text{End } \mathbb{C}^{N|N}$ is $\mathbb{Z}_2$ -graded so that $\deg E_{ij} = \bar{i} + \bar{j}$. Hence, $\mathbb{C}^{N|N}$ is a $\mathbb{Z}_2$ -graded module over $\text{End } \mathbb{C}^{N|N}$. For any n , we can identify the tensor product $(\text{End } \mathbb{C}^{N|N})^{\otimes n}$ with the algebra $\text{End } ((\mathbb{C}^{N|N})^{\otimes n})$ acting on the vector space $(\mathbb{C}^{N|N})^{\otimes n}$ by using conventions (2) and (3). An involutive automorphism π of $\text{End } \mathbb{C}^{N|N}$ is defined by

$$\pi : E_{ij} \mapsto E_{-i, -j}.$$

Consider the general linear Lie superalgebra $\mathfrak{gl}_{N|N}$. To avoid confusion, denote by e_{ij} the element of $\mathfrak{gl}_{N|N}$ corresponding to $E_{ij} \in \text{End } \mathbb{C}^{N|N}$. Then, $\deg e_{ij} = \bar{i} + \bar{j}$,

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj} (-1)^{(\bar{i} + \bar{j})(\bar{k} + \bar{l})}.$$

Therefore, π is also an involutive automorphism of $\mathfrak{gl}_{N|N}$.

Now, the *queer* Lie superalgebra $\mathfrak{q}_N$ is the fixed point subalgebra of $\mathfrak{gl}_{N|N}$ relative to the automorphism π . This subalgebra is spanned by the elements

$$f_{ij} = e_{ij} + \pi(e_{ij}) = e_{ij} + e_{-i, -j}.$$

In the Lie superalgebra $\mathfrak{q}_N$, we have $f_{-i, -j} = f_{ij}$ and

$$[f_{ij}, f_{kl}] = \delta_{jk} f_{il} + \delta_{j, -k} f_{-i, l} - (\delta_{li} f_{kj} + \delta_{-l, i} f_{k, -j}) (-1)^{(\bar{i} + \bar{j})(\bar{k} + \bar{l})}.$$

Note that the elements f_{ij} with $i > 0$ form a basis of $\mathfrak{q}_N$.

We will also work with the universal enveloping algebra $U(\mathfrak{q}_N)$. This is a $\mathbb{Z}_2$ -graded associative algebra generated by the elements f_{ij} with $\deg f_{ij} = \bar{i} + \bar{j}$ and the same relations as above, where the square brackets now stand for the supercommutator, however.

Note that $U(q_N)$ is a $\mathbb{Z}_2$ -graded Hopf algebra where the counit homomorphism $U(q_N) \rightarrow \mathbb{C}$, comultiplication homomorphism $U(q_N) \rightarrow U(q_N) \otimes U(q_N)$ and antipodal antihomomorphism $U(q_N) \rightarrow U(q_N)$ are defined by

$$f_{ij} \mapsto \delta_{ij}, \quad f_{ij} \mapsto f_{ij} \otimes 1 + 1 \otimes f_{ij} \quad \text{and} \quad f_{ij} \mapsto -f_{ij}.$$

Let us consider in more detail the Lie superalgebra q_1 . We will choose the basis of q_1 consisting of two elements,

$$\begin{aligned} a &= f_{11} = f_{-1,-1} = e_{11} + e_{-1,-1}, \\ b &= f_{1,-1} = f_{-1,1} = e_{1,-1} + e_{-1,1}. \end{aligned}$$

By the above general relations, we get Lie brackets in q_1 ,

$$[a, a] = [a, b] = 0 \quad \text{and} \quad [b, b] = 2a. \quad (5)$$

Hence, $U(q_1)$ is the $\mathbb{Z}_2$ -graded associative algebra generated by the elements a and b , where $\deg a = 0$ and $\deg b = 1$. The defining relations in $U(q_1)$ are the same (5) but with the square brackets now meaning the supercommutator. Hence, element a of $U(q_1)$ is central and $b^2 = a$. Element a generates the center of the superalgebra $U(q_1)$; see Ref. 4 for the corresponding general result about $U(q_N)$.

IV. THE R -MATRIX

Take the element of the algebra $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$,

$$P = \sum_{ij} E_{ij} \otimes E_{ji} (-1)^j.$$

It acts on the vector space $(\mathbb{C}^{N|N})^{\otimes 2}$ so that

$$e_i \otimes e_j \mapsto e_j \otimes e_i (-1)^{i \cdot j}.$$

We identify the algebra $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$ with the algebra $\text{End } ((\mathbb{C}^{N|N})^{\otimes 2})$ by using (2). Note that $P^2 = 1$ and

$$(\pi \otimes \pi)(P) = -P.$$

Furthermore, take the element of the algebra $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$,

$$Q = \sum_{ij} E_{-i,-j} \otimes E_{ji} (-1)^j.$$

Then, we have the equalities

$$Q = (\pi \otimes \text{id})(P) = (-\text{id} \otimes \pi)(P). \quad (6)$$

Note that by the above definitions of P and Q , we have

$$PQ + QP = 0 \quad \text{and} \quad Q^2 = 1. \quad (7)$$

Now consider a function of complex variables u, v with values in the algebra $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$,

$$R(u, v) = 1 - \frac{P}{u-v} + \frac{Q}{u+v}. \quad (8)$$

By (6), we have

$$\begin{aligned} (\pi \otimes 1)(R(u, v)) &= R(-u, v), \\ (1 \otimes \pi)(R(u, v)) &= R(u, -v). \end{aligned}$$

Furthermore, we have

$$R(u, v) R(-u, -v) = 1 - \frac{1}{(u-v)^2} - \frac{1}{(u+v)^2}.$$

Indeed, due to the relation $P^2 = 1$ and to (7),

$$\left(1 - \frac{P}{u-v} + \frac{Q}{u+v}\right) \left(1 + \frac{P}{u-v} - \frac{Q}{u+v}\right) = 1 - \frac{P^2}{(u-v)^2} + \frac{PQ + QP}{(u-v)(u+v)} - \frac{Q^2}{(u+v)^2} = 1 - \frac{1}{(u-v)^2} - \frac{1}{(u+v)^2}.$$

Let us now verify that the function $R(u, v)$ obeys the *Yang-Baxter equation* in $(\text{End } \mathbb{C}^{N|N})^{\otimes 3}(u, v, w)$,

$$R_{12}(u, v)R_{13}(u, w)R_{23}(v, w) = R_{23}(v, w)R_{13}(u, w)R_{12}(u, v). \quad (9)$$

Using definition (8), the equality in (9) will follow from the following relations in the algebra $(\text{End } \mathbb{C}^{N|N})^{\otimes 3}$:

$$P_{12} P_{13} = P_{23} P_{12} = P_{13} P_{23}, \quad (10)$$

$$P_{13} P_{12} = P_{12} P_{23} = P_{23} P_{13}, \quad (11)$$

$$Q_{12} Q_{13} = P_{23} Q_{12} = Q_{13} P_{23}, \quad (12)$$

$$Q_{13} Q_{12} = Q_{12} P_{23} = P_{23} Q_{13}, \quad (13)$$

$$Q_{12} P_{13} = Q_{23} Q_{12} = -P_{13} Q_{23}, \quad (14)$$

$$P_{13} Q_{12} = Q_{12} Q_{23} = -Q_{23} P_{13}, \quad (15)$$

$$P_{12} Q_{13} = Q_{23} P_{12} = -Q_{13} Q_{23}, \quad (16)$$

$$Q_{13} P_{12} = P_{12} Q_{23} = -Q_{23} Q_{13}, \quad (17)$$

$$P_{12} P_{13} P_{23} = P_{23} P_{13} P_{12}, \quad (18)$$

$$P_{12} Q_{13} Q_{23} = Q_{23} Q_{13} P_{12}, \quad (19)$$

$$Q_{12} Q_{13} P_{23} = P_{23} Q_{13} Q_{12}, \quad (20)$$

$$Q_{12} P_{13} Q_{23} = Q_{23} P_{13} Q_{12}, \quad (21)$$

$$P_{12} P_{13} Q_{23} = P_{23} Q_{13} P_{12} = Q_{12} P_{13} P_{23} = Q_{23} Q_{13} Q_{12}, \quad (22)$$

$$Q_{23} P_{13} P_{12} = P_{12} Q_{13} P_{23} = P_{23} P_{13} Q_{12} = Q_{12} Q_{13} Q_{23}. \quad (23)$$

Relations (10) and (11) are used with the identity

$$\frac{1}{u-v} \frac{1}{u-w} - \frac{1}{u-v} \frac{1}{v-w} + \frac{1}{u-w} \frac{1}{v-w} = 0,$$

which is easy to verify. Relations (12) and (13) are used with the identity

$$\frac{1}{u+v} \frac{1}{u+w} + \frac{1}{u+v} \frac{1}{v-w} - \frac{1}{u+w} \frac{1}{v-w} = 0$$

obtained from the previous one by changing the sign of u . Relations (14) and (15) are used with the identity

$$\frac{1}{u+v} \frac{1}{u-w} + \frac{1}{u+v} \frac{1}{v+w} - \frac{1}{u-w} \frac{1}{v+w} = 0$$

obtained from the previous one by changing the sign of w . Relations (16) and (17) are used with the identity

$$\frac{1}{u-v} \frac{1}{u+w} - \frac{1}{u-v} \frac{1}{v+w} + \frac{1}{u+w} \frac{1}{v+w} = 0$$

obtained from the previous one by changing the sign of u once again. Finally, relations (22) and (23) are used along with another identity, which is easy to verify,

$$\frac{1}{u-v} \frac{1}{u-w} \frac{1}{v+w} - \frac{1}{u-v} \frac{1}{u+w} \frac{1}{v-w} + \frac{1}{u+v} \frac{1}{u-w} \frac{1}{v-w} - \frac{1}{u+v} \frac{1}{u+w} \frac{1}{v+w} = 0.$$

Let us verify relations (10)–(17). Relations (10) and (11) follow from the description of the action of P on the vector space $(\mathbb{C}^{N|N})^{\otimes 2}$ given in the beginning of this section. In turn, relations (12)–(17) follow from (10) and (11) by using the observation below. Let

$$J = \sum_i E_{i,-i} (-1)^i \in \text{End } \mathbb{C}^{N|N}.$$

Then, $Q = P(J \otimes J)$ by the definitions of P and Q . Note the equality $J^2 = -1$. Because $\deg J = 1$, in the algebra $(\text{End } \mathbb{C}^{N|N})^{\otimes 3}$ for any $p \neq q$, we also have the equality

$$J_p J_q = -J_q J_p,$$

where we use the notation (4) for $n = 3$ and $m = 1$. We omit the details of verifying (12)–(17) in this fashion.

Next, let us verify relations (18)–(23). It follows from the above-mentioned description of the action of P on $(\mathbb{C}^{N|N})^{\otimes 2}$ that either side of (18) is equal to P_{13} . In turn, relations (19)–(23) follow from (18). Here, we again use our observation involving the element J . In particular, either side of (21) is equal to $-P_{13}$. Thus, the function (8) obeys Eq. (9).

V. THE YANGIAN

The *Yangian* of the Lie superalgebra $\mathfrak{q}_N$ is a complex associative unital algebra $Y(\mathfrak{q}_N)$ with a set of generators

$$T_{ij}^{(r)} \text{ where } r = 1, 2, \dots \text{ and } i, j = \pm 1, \dots, \pm N.$$

The algebra $Y(\mathfrak{q}_N)$ is $\mathbb{Z}_2$ -graded so that $\deg T_{ij}^{(r)} = \bar{i} + \bar{j}$ for all indices r . To write down defining relations for these generators of $Y(\mathfrak{q}_N)$, we will use the formal power series in u^{-1} with coefficients from $Y(\mathfrak{q}_N)$,

$$T_{ij}(u) = \delta_{ij} \cdot 1 + T_{ij}^{(1)} u^{-1} + T_{ij}^{(2)} u^{-2} + \dots.$$

Let us combine all these series into a single element

$$T(u) = \sum_{i,j} E_{ij} \otimes T_{ij}(u)$$

of the algebra $(\text{End } \mathbb{C}^{N|N}) \otimes Y(\mathfrak{q}_N)[[u^{-1}]]$. Then, we will impose the relation

$$(\pi \otimes \text{id})(T(u)) = T(-u).$$

In terms of the series $T_{ij}(u)$, it means that for all i and j ,

$$T_{-i,-j}(u) = T_{ij}(-u). \quad (24)$$

In terms of the generators $T_{ij}^{(r)}$, it simply means that

$$T_{-i,-j}^{(r)} = (-1)^r T_{ij}^{(r)}. \quad (25)$$

For any n and any $p = 1, \dots, n$, we will denote

$$T_p(u) = (\iota_p \otimes \text{id})(T(u))$$

in the algebra $(\text{End } \mathbb{C}^{N|N})^{\otimes n} \otimes Y(q_N)[[u^{-1}]]$. By using this notation for $n = 2$, the remaining defining relations of the algebra $Y(q_N)$ can be written as the single equation

$$(R(u, v) \otimes 1) T_1(u) T_2(v) = T_2(v) T_1(u) (R(u, v) \otimes 1). \quad (26)$$

By using definition (8), expanding Eq. (26) in the basis of $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$ consisting of the vectors

$$E_{ij} \otimes E_{kl} (-1)^{\bar{j}\bar{k} + \bar{j}\bar{l} + \bar{k}\bar{l}}$$

with $i, j, k, l = \pm 1, \dots, \pm N$ yields the relations

$$[T_{ij}(u), T_{kl}(v)] (-1)^{\bar{i}\bar{k} + \bar{i}\bar{l} + \bar{k}\bar{l}} = \frac{T_{kj}(u) T_{il}(v) - T_{kj}(v) T_{il}(u)}{u - v} - \frac{T_{-kj}(u) T_{-il}(v) - T_{-kj}(v) T_{-il}(u)}{u + v} (-1)^{\bar{k} + \bar{i}} \quad (27)$$

in $Y(q_N)[[u^{-1}, v^{-1}]]$. The square brackets above stand for the supercommutator. The first fraction in (27) belongs to $Y(q_N)[[u^{-1}, v^{-1}]]$ because its numerator vanishes at $u - v = 0$. The second fraction in (27) belongs to $Y(q_N)[[u^{-1}, v^{-1}]]$ because its numerator vanishes at $u + v = 0$ by relations (24).

By comparing this definition of $Y(q_N)$ with the above relations in the algebra $U(q_N)$, it is direct to verify that a homomorphism $U(q_N) \rightarrow Y(q_N)$ can be defined by

$$f_{ij} \mapsto -T_{ji}^{(1)} (-1)^{\bar{i}}. \quad (28)$$

It is also straightforward to verify that a homomorphism $Y(q_N) \rightarrow U(q_N)$ can be defined by

$$T_{ij}(u) \mapsto \delta_{ij} - f_{ji} u^{-1} (-1)^{\bar{j}}. \quad (29)$$

The homomorphism (29) is clearly surjective. Note that the composition of (28) with (29) is just the identity map $U(q_N) \rightarrow U(q_N)$. This implies that the homomorphism (28) is injective.

Furthermore, it follows from our definition of $Y(q_N)$ that an antiautomorphism of $Y(q_N)$ can be defined by mapping

$$T_{ij}(u) \mapsto \tilde{T}_{ij}(u), \quad (30)$$

where the series $\tilde{T}_{ij}(u) \in Y(q_N)[[u^{-1}]]$ is defined by

$$T(u)^{-1} = \sum_{i,j} E_{ij} \otimes \tilde{T}_{ij}(u). \quad (31)$$

Indeed, by dividing (26) on the left and right by $T_2(u)$ and then by $T_1(u)$, we get the relation

$$(R(u, v) \otimes 1) T_2(v)^{-1} T_1(u)^{-1} = T_1(u)^{-1} T_2(v)^{-1} (R(u, v) \otimes 1).$$

Comparing this with (26) verifies the antiautomorphism property of the map (30); see also the relation (24).

It also follows from (24) and (31) that for all i and j ,

$$\tilde{T}_{-i,-j}(u) = \tilde{T}_{ij}(-u). \quad (32)$$

There is a natural Hopf algebra structure on $Y(q_N)$. A coassociative comultiplication homomorphism,

$$\Delta : Y(q_N) \rightarrow Y(q_N) \otimes Y(q_N),$$

is defined by

$$\Delta : T_{ij}(u) \mapsto \sum_k T_{ik}(u) \otimes T_{kj}(u) (-1)^{(\bar{i} + \bar{k})(\bar{j} + \bar{k})}$$

where the tensor product is over the subalgebra $\mathbb{C}[[u^{-1}]]$ of $Y(q_N)[[u^{-1}]]$. Furthermore, the counit homomorphism $Y(q_N) \rightarrow \mathbb{C}$ is defined by mapping $T_{ij}(u) \mapsto \delta_{ij}$. The antipodal map $Y(q_N) \rightarrow Y(q_N)$ is the antiautomorphism (30). Justification of all these definitions is standard.³

The antipodal map of any Hopf algebra is a coalgebra antihomomorphism as well. Hence, for any indices i and j ,

$$\Delta : \tilde{T}_{ij}(u) \mapsto \sum_k \tilde{T}_{kj}(u) \otimes \tilde{T}_{ik}(u). \quad (33)$$

Note that (28) is a homomorphism of $\mathbb{Z}_2$ -graded Hopf algebras $U(q_N) \rightarrow Y(q_N)$. However, (29) is a homomorphism of $\mathbb{Z}_2$ -graded associative algebras $Y(q_N) \rightarrow U(q_N)$ only, *not* a homomorphism of Hopf algebras.

We can naturally define two ascending $\mathbb{Z}$ -filtrations on the algebra $Y(q_N)$. The first $\mathbb{Z}$ -filtration is defined by setting to r the degree of $T_{ij}^{(r)}$ for every i and j . Consider the corresponding $\mathbb{Z}$ -graded algebra $\text{gr } Y(q_N)$. It is also $\mathbb{Z}_2$ -graded. It follows from (27) that the algebra $\text{gr } Y(q_N)$ is supercommutative. By Ref. 3 (Corollary 2.4), the elements corresponding to $T_{ij}^{(r)}$ with $i > 0$ are free generators of this supercommutative algebra. Their freeness will also follow from the argument at the end of this section.

The second $\mathbb{Z}$ -filtration on the algebra $Y(q_N)$ is defined by setting the degree of $T_{ij}^{(r)}$ to $r - 1$. Let $\text{gr}' Y(q_N)$ be the corresponding $\mathbb{Z}$ -graded algebra. It is $\mathbb{Z}_2$ -graded too. Let $t_{ij}^{(r)}$ be the element of $\text{gr}' Y(q_N)$ defined by $T_{ij}^{(r)}$. Hence,

$$t_{-i,-j}^{(r)} = (-1)^r t_{ij}^{(r)}$$

by (25). For $r, s \geq 1$, by taking coefficients at $u^{-r} v^{-s}$ in (27), we get the supercommutation relations in $\text{gr}' Y(q_N)$,

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] (-1)^{\bar{i}\bar{k} + \bar{i}\bar{l} + \bar{k}\bar{l}} = \delta_{kj} t_{il}^{(r+s-1)} - t_{kj}^{(r+s-1)} \delta_{il} + (\delta_{-kj} t_{-i,l}^{(r+s-1)} - t_{k,-j}^{(r+s-1)} \delta_{i,-l}) (-1)^{\bar{k} + \bar{l} + r}.$$

These imply that for the Lie superalgebra (1), a surjective homomorphism $U(\mathfrak{g}) \rightarrow \text{gr}' Y(q_N)$ is defined by mapping

$$e_{ij} u^{r-1} + e_{-i,-j} (-u)^{r-1} \mapsto -t_{ji}^{(r)} (-1)^{\bar{i}}. \quad (34)$$

By Ref. 3 (Theorem 2.3), this homomorphism is injective too. The injectivity will also follow from the argument below.

Denote by γ_n the homomorphism $Y(q_N) \rightarrow U(q_N)^{\otimes n}$ defined by using the comultiplication $Y(q_N) \rightarrow Y(q_N)^{\otimes n}$ first and then applying the homomorphism (29) to each tensor factor of $Y(q_N)^{\otimes n}$. Let us prove that the kernels of all the homomorphisms γ_n with $n = 1, 2, \dots$ have zero intersection. We will follow Ref. 8 where the Yangian of the general linear Lie superalgebra $\mathfrak{gl}_{M|N}$ was considered.

The algebra $U(q_N)^{\otimes n}$ is generated by the elements

$$t_p(f_{ij}) \text{ where } p = 1, \dots, n \text{ and } i, j = \pm 1, \dots, \pm N.$$

Here, we use the notation of Sec. II with $A = U(q_N)$. Define an ascending $\mathbb{Z}$ -filtration on the algebra $U(q_N)^{\otimes n}$ in the standard way, that is, by setting to 1 the degrees of all the above generators. Consider the corresponding $\mathbb{Z}$ -graded algebra $\text{gr } U(q_N)^{\otimes n}$. It is also $\mathbb{Z}_2$ -graded and then supercommutative. Let $x_{ij}^{(p)}$ be the elements of this algebra, corresponding to the above displayed generators of $U(q_N)^{\otimes n}$. Note that for any indices i and j , we have

$$x_{-i,-j}^{(p)} = x_{ij}^{(p)}. \quad (35)$$

Because of the Poincaré–Birkhoff–Witt theorem for the Lie superalgebra q_N , the elements $x_{ij}^{(p)}$ with $i > 0$ are free generators of the supercommutative algebra $\text{gr } U(q_N)^{\otimes n}$.

By the definition of comultiplication on $Y(q_N)$, our γ_n maps the series $T_{ij}(u)$ to $(-1)^{\bar{i}\bar{j} + \bar{j}}$ times the sum over $k_1, \dots, k_{n-1} = \pm 1, \dots, \pm N$ of the tensor products,

$$\begin{aligned} & (\delta_{ik_1} - f_{k_1 i} u^{-1} (-1)^{\bar{i}\bar{k}_1}) \otimes \\ & (\delta_{k_1 k_2} - f_{k_2 k_1} u^{-1} (-1)^{\bar{k}_1 \bar{k}_2}) \otimes \\ & \vdots \\ & (\delta_{k_{n-2} k_{n-1}} - f_{k_{n-1} k_{n-2}} u^{-1} (-1)^{\bar{k}_{n-2} \bar{k}_{n-1}}) \otimes \\ & (\delta_{k_{n-1} j} - f_{j k_{n-1}} u^{-1} (-1)^{\bar{k}_{n-1} \bar{j}}), \end{aligned}$$

where we also used the definition (29). Hence, $T_{ij}^{(r)}$ with $r \leq n$ obtained by γ_n to $(-1)^{\bar{i}\bar{j}+\bar{j}+r}$ times the sum over $k_1, \dots, k_{r-1} = \pm 1, \dots, \pm N$ and $1 \leq p_1 < \dots < p_r \leq n$ of the products in $U(q_N)^{\otimes n}$,

$$t_{p_1}(f_{k_1 i}) t_{p_2}(f_{k_2 k_1}) \dots t_{p_{r-1}}(f_{k_{r-1} k_{r-2}}) t_{p_r}(f_{j k_{r-1}}) (-1)^{\bar{i} \bar{k}_1 + \bar{k}_1 \bar{k}_2 + \dots + \bar{k}_{r-2} \bar{k}_{r-1} + \bar{k}_{r-1} \bar{j}}.$$

If $r > n$, then $T_{ij}^{(r)}$ is annihilated by γ_n . Note that γ_n is also a homomorphism of $\mathbb{Z}$ -filtered algebras relative to the first filtration on $Y(q_N)$.

The element of the algebra $\text{gr } U(q_N)^{\otimes n}$ corresponding to the last displayed product in $U(q_N)^{\otimes n}$ is by definition

$$x_{k_1 i}^{(p_1)} x_{k_2 k_1}^{(p_2)} \dots x_{k_{r-1} k_{r-2}}^{(p_{r-1})} x_{j k_{r-1}}^{(p_r)} (-1)^{\bar{i} \bar{k}_1 + \bar{k}_1 \bar{k}_2 + \dots + \bar{k}_{r-2} \bar{k}_{r-1} + \bar{k}_{r-1} \bar{j}}. \quad (36)$$

Let $y_{ij}^{(r)}$ be the sum over all $k_1, \dots, k_{r-1} = \pm 1, \dots, \pm N$ and all $1 \leq p_1 < \dots < p_r \leq n$ of the products (36) in the algebra $\text{gr } U(q_N)^{\otimes n}$ multiplied by $(-1)^{\bar{i}\bar{j}+\bar{j}}$. We have

$$y_{-i,-j}^{(r)} = (-1)^r y_{ij}^{(r)}.$$

We can also take the element of the $\mathbb{Z}$ -graded algebra $\text{gr } Y(q_N)$ corresponding to $(-1)^r T_{ij}^{(r)}$. Its image relative to the homomorphism $\text{gr } Y(q_N) \rightarrow \text{gr } U(q_N)^{\otimes n}$ defined by γ_n coincides with $y_{ij}^{(r)}$. However, we do not need this fact.

We will prove that the supercommutative monomials in the elements $y_{ij}^{(r)}$ with $i > 0$ and $r \leq n$ are all linearly independent. Hence, the kernels of the homomorphisms γ_n with $n = 1, 2, \dots$ will have zero intersection. Moreover, the freeness property of the supercommutative algebra $\text{gr } Y(q_N)$ stated above will then follow too. Furthermore, the injectivity of homomorphism (34) follows from linear independence of those monomials for every n .

Let $i = 1, \dots, N$ and $j = \pm 1, \dots, \pm N$. Fix any total ordering of the triples (i, j, r) where $r = 1, \dots, n$. Using this ordering, form a matrix of the left superderivatives,

$$\partial y_{ij}^{(r)} / \partial x_{kl}^{(p)}, \quad (37)$$

where the triples (k, l, p) range over the same ordered set as the triples (i, j, r) . Fix complex numbers $x_1, \dots, x_n$ so that $x_r \pm x_p \neq 0$ for all $r < p$. Due to freeness of the supercommutative algebra $\text{gr } U(q_N)^{\otimes n}$, we can specialize

$$x_{kl}^{(p)} = x_p \delta_{kl} \quad (38)$$

in the matrix of superderivatives (37). It suffices to show that the determinant of the specialized matrix is not zero.

For $r \geq 1$, take the elementary symmetric polynomial

$$\sigma_r(x_1, \dots, x_n) = \sum_{p_1 < \dots < p_r} x_{p_1} \dots x_{p_r}.$$

We will assume that $\sigma_0(x_1, \dots, x_n) = 1$. If $j > 0$, then the specialization of superderivative (37) and (38) equals

$$\sigma_{r-1}(x_1, \dots, x_{p-1}, x_{p+1}, \dots, x_n) \delta_{il} \delta_{kj}.$$

If $j < 0$, then the specialization of (37) and (38) equals

$$-\sigma_{r-1}(x_1, \dots, x_{p-1}, -x_{p+1}, \dots, -x_n) \delta_{i,-l} \delta_{k,-j}.$$

Here, we used (35) and the condition that $i, k > 0$ in (37).

A detailed calculation from the proof of Theorem 1 of Ref. 8 shows that the determinant of the matrix formed by

$$\sigma_{r-1}(x_1, \dots, x_{p-1}, x_{p+1}, \dots, x_n),$$

with $r, p = 1, \dots, n$, is equal to the product

$$\prod_{r < p} (x_r - x_p).$$

Similarly, the determinant of the matrix formed by

$$\sigma_{r-1}(x_1, \dots, x_{p-1}, -x_{p+1}, \dots, -x_n), \quad (39)$$

with $r, p = 1, \dots, n$, is equal to the product

$$\prod_{r < p} (x_r + x_p).$$

One reduces the latter calculation by taking differences of adjacent columns of the matrix of (39). Both products are not zero due to our choice of the numbers $x_1, \dots, x_n$. Hence, the determinant of the matrix of (37) is not zero.

VI. THE CENTER

There is a natural family of central elements of $Y(q_N)$. To define it, let us consider the antiautomorphism τ of the $\mathbb{Z}_2$ -graded associative algebra $\text{End } \mathbb{C}^{N|N}$ defined by

$$\tau : E_{ij} \mapsto E_{ji} (-1)^{\bar{i}\bar{j} + \bar{i}}.$$

Take the element of the algebra $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$,

$$K = \sum_{i,j} E_{ij} \otimes E_{ij} (-1)^{\bar{i}\bar{j}}.$$

Then, we have the equalities

$$K = (\tau \otimes \text{id})(P) = (\text{id} \otimes \tau)(P). \quad (40)$$

Note that the image of the action of K on the vector space $(\mathbb{C}^{N|N})^{\otimes 2}$ is one dimensional. This image is spanned by

$$\sum_i e_i \otimes e_i.$$

Here, we again identify the algebra $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$ with $\text{End } ((\mathbb{C}^{N|N})^{\otimes 2})$ by using (2). Furthermore, take the element

$$L = \sum_{i,j} E_{-i,-j} \otimes E_{ij} (-1)^{\bar{i}\bar{j}}.$$

Similar to (40), we have the equalities in $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$,

$$L = (\tau \otimes \text{id})(Q) = (\text{id} \otimes \tau)(Q). \quad (41)$$

The image of the action of L on $(\mathbb{C}^{N|N})^{\otimes 2}$ is again one dimensional and spanned by the vector

$$\sum_i e_i \otimes e_{-i} (-1)^{\bar{i}}.$$

Now, consider a function of complex variables u, v with values in the algebra $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$,

$$S(u, v) = 1 - \frac{K}{u - v} + \frac{L}{u + v}.$$

Then, due to (8) and to (40) and (41), we have the equalities

$$S(u, v) = (\tau \otimes \text{id})(R(u, v)) = (\text{id} \otimes \tau)(R(u, v)).$$

Note that by the above definitions of K and L , we have

$$K L = L K = 0 \quad \text{and} \quad K^2 = L^2 = 0.$$

These equalities imply that

$$S(u, v) S(-u, -v) = 1. \quad (42)$$

We will use the notation

$$T'(u) = (\tau \otimes \text{id})(T(u)^{-1}).$$

Let us now divide (26) on the left and right by $T_2(u)$ and apply to the resulting relation in the algebra,

$$(\text{End } \mathbb{C}^{N|N})^{\otimes 2} \otimes Y(q_N) [[u^{-1}, v^{-1}]],$$

the antiautomorphism τ relative to the second tensor factor $\text{End } \mathbb{C}^{N|N}$. We get the relation

$$(S(-u, -v) \otimes 1) T_1(u) T_2'(v) = T_2'(v) T_1(u) (S(-u, -v) \otimes 1) \quad (43)$$

where we also used equality (42). Next, let us multiply relation (43) by $v - u$ and then set $v = u$. We get

$$(K \otimes 1) T_1(u) T_2'(u) = T_2'(u) T_1(u) (K \otimes 1). \quad (44)$$

Because the image of the action of K on $(\mathbb{C}^{N|N})^{\otimes 2}$ is one dimensional, either side of relation (44) equals $K \otimes Z(u)$ for some series $Z(u) \in Y(q_N) [[u^{-1}]]$. By using the definition of K and expanding (44), we get

$$\begin{aligned} \sum_k T_{ki}(u) \tilde{T}_{jk}(u) &= Z(u) \delta_{ij}, \\ \sum_k \tilde{T}_{jk}(u) T_{ki}(u) &= Z(u) \delta_{ij}. \end{aligned} \quad (45)$$

Relations (24) now imply that $Z(u) = Z(-u)$. Hence,

$$Z(u) = 1 + Z_2 u^{-2} + Z_4 u^{-4} + \dots$$

for some elements $Z_2, Z_4, \dots \in Y(q_N)$ of $\mathbb{Z}_2$ -degree 0. By Proposition 3.1 of Ref. 3, all these elements are central in $Y(q_N)$. The centrality of any $X \in Y(q_N)$ means that the supercommutator $[X, Y] = 0$ for every $Y \in Y(q_N)$. By Proposition 3.5 of Ref. 3, for $r \geq 1$, the element of the algebra $\text{gr}' Y(q_N)$ corresponding to $Z_{2r} \in Y(q_N)$ is $(2r - 1)$ times

$$\sum_i t_{ii}^{(2r-1)} (-1)^{\bar{i}}. \quad (46)$$

Through the isomorphism $U(\mathfrak{g}) \rightarrow \text{gr}' Y(q_N)$ defined by (34), the element (46) corresponds to

$$- \sum_i (e_{ii} + e_{-i, -i}) u^{2r-2} \in \mathfrak{g}. \quad (47)$$

By Proposition 3.6 of Ref. 3, the elements (47) with $r = 1, 2, \dots$ freely generate the center of $U(\mathfrak{g})$. Hence, the elements $Z_2, Z_4, \dots$ freely generate the center of $Y(q_N)$.

Our $Z(u)$ is also comultiplicative for $Y(q_N)$, that is,

$$\Delta : Z(u) \mapsto Z(u) \otimes Z(u). \quad (48)$$

Indeed, by setting $j = i$ in (45) and then employing (33), the comultiplication maps $Z(u)$ to

$$\begin{aligned} \sum_{h,k,l} (T_{kl}(u) \otimes T_{li}(u)) (\tilde{T}_{hk}(u) \otimes \tilde{T}_{ih}(u)) (-1)^{(\bar{i} + \bar{l})(\bar{k} + \bar{l})} &= \sum_{h,k,l} (T_{kl}(u) \tilde{T}_{hk}(u)) \otimes (T_{li}(u) \tilde{T}_{ih}(u)) (-1)^{(\bar{i} + \bar{l})(\bar{k} + \bar{l})} \\ &= \sum_{h,l} (Z(u) \delta_{hl}) \otimes (T_{li}(u) \tilde{T}_{ih}(u)) (-1)^{(\bar{i} + \bar{l})(\bar{k} + \bar{l})}, \end{aligned}$$

which is equal to the right-hand side of (48) as we stated. Here, we used relations (45) again.

Note that $Z(u) \mapsto 1$ by the counit map $Y(q_N) \rightarrow \mathbb{C}$. Due to the axioms of a Hopf algebra, it now follows from (48) that $Z(u) \mapsto Z(u)^{-1}$ under the antipodal map. The square of the antipodal map is always a homomorphism of associative algebras. By Proposition 3.2 of Ref. 3, under this homomorphism of $Y(q_N)$, for any indices i and j ,

$$T_{ij}(u) \mapsto Z(u)^{-1} T_{ij}(u).$$

VII. THE CASE OF $N = 1$

Let us now consider in more detail the Yangian $Y(q_1)$. For short, we will denote

$$A(u) = T_{11}(u) \quad \text{and} \quad B(u) = T_{1,-1}(u).$$

The coefficients of the series $A(u)$ and $B(u)$ in $Y(q_1)$ are of $\mathbb{Z}_2$ -degrees 0 and 1, respectively. It follows from (24) that the algebra $Y(q_1)$ is generated by these coefficients.

By (27), the supercommutation relations below define the Yangian $Y(q_1)$ as an associative $\mathbb{Z}_2$ -graded algebra,

$$[A(u), A(v)] = \frac{A(u)A(v) - A(v)A(u)}{u-v} - \frac{B(-u)B(-v) - B(v)B(u)}{u+v}, \quad (49)$$

$$[A(u), B(v)] = \frac{A(u)B(v) - A(v)B(u)}{u-v} + \frac{B(-u)A(-v) - B(v)A(u)}{u+v}, \quad (50)$$

$$[B(u), A(v)] = \frac{B(u)A(v) - B(v)A(u)}{u-v} - \frac{A(-u)B(-v) - A(v)B(u)}{u+v}, \quad (51)$$

$$[B(u), B(v)] = \frac{B(u)B(v) - B(v)B(u)}{u-v} + \frac{A(-u)A(-v) - A(v)A(u)}{u+v}. \quad (52)$$

By our definitions, the comultiplication on $Y(q_1)$ maps

$$A(u) \mapsto A(u) \otimes A(u) - B(u) \otimes B(-u), \quad (53)$$

$$B(u) \mapsto A(u) \otimes B(u) + B(u) \otimes A(-u). \quad (54)$$

Furthermore, let us denote

$$\tilde{A}(u) = \tilde{T}_{11}(u) \quad \text{and} \quad \tilde{B}(u) = \tilde{T}_{1,-1}(u).$$

Hence, $\tilde{A}(u)$ and $\tilde{B}(u)$ are, respectively, the images of $A(u)$ and $B(u)$ by the antiautomorphism (30) of $Y(q_1)$. Due to (32) and (33), the comultiplication on $Y(q_1)$ maps

$$\tilde{A}(u) \mapsto \tilde{A}(u) \otimes \tilde{A}(u) + \tilde{B}(-u) \otimes \tilde{B}(u), \quad (55)$$

$$\tilde{B}(u) \mapsto \tilde{B}(u) \otimes \tilde{A}(u) + \tilde{A}(-u) \otimes \tilde{B}(u). \quad (56)$$

Consider the homomorphism $Y(q_1) \rightarrow U(q_1)$ defined by (29). It maps

$$A(u) \mapsto \frac{u-a}{u} \quad \text{and} \quad B(u) \mapsto \frac{b}{u}. \quad (57)$$

By definition (31), we have the following two equations:

$$A(u)\tilde{A}(u) - B(u)\tilde{B}(-u) = 1, \quad (58)$$

$$B(-u)\tilde{A}(u) + A(-u)\tilde{B}(-u) = 0. \quad (59)$$

These two equations determine $\tilde{A}(u)$ and $\tilde{B}(-u)$ uniquely by $A(u), B(u)$ and $A(-u), B(-u)$. Using these equations along with (57), the homomorphism (29) for $N = 1$ maps

$$\tilde{A}(u) \mapsto \frac{(u+a)u}{u^2 - a^2 - a}, \quad (60)$$

$$\tilde{B}(-u) \mapsto \frac{bu}{u^2 - a^2 - a}. \quad (61)$$

We also employ the centrality of a and the relation $b^2 = a$ in $U(q_1)$ but omit the details of this direct calculation.

For $N = 1$, by setting $i = j = 1$ in (45), we obtain

$$Z(u) = A(u)\tilde{A}(u) + B(-u)\tilde{B}(u).$$

By (57), (60), and (61), the homomorphism (29) maps $Z(u)$ to

$$\frac{u-a}{u} \frac{(u+a)u}{u^2-a^2-a} - \frac{b}{u} \frac{-bu}{u^2-a^2-a} = \frac{u^2-a^2}{u^2-a^2-a} + \frac{b^2}{u^2-a^2-a} = \frac{u^2-a^2+a}{u^2-a^2-a}.$$

VIII. THE QUANTUM BEREZINIAN

In this section, we will introduce a family of generators of the center of $Y(q_1)$, different from the family provided for $N = 1$ by the coefficients of the series $Z(u)$. Denote

$$C(u) = A(u)\tilde{A}(-u) \quad \text{and} \quad D(u) = B(u)\tilde{B}(u). \quad (62)$$

The coefficients of these two series are of $\mathbb{Z}_2$ -degree zero. The series $C(u)$ will be called the *quantum Berezinian* for the Yangian $Y(q_1)$. To justify this terminology, consider the $\mathbb{Z}$ -graded algebra $\text{gr } Y(q_1)$. Take the image of $C(u)$ in the supercommutative algebra $(\text{gr } Y(q_1))[[u^{-1}]]$. Relations (58) and (59) imply that the matrix

$$\begin{bmatrix} T_{11}(u) & -T_{1,-1}(u) \\ T_{-1,1}(u) & T_{-1,-1}(u) \end{bmatrix} = \begin{bmatrix} A(u) & -B(u) \\ B(-u) & A(-u) \end{bmatrix} \quad (63)$$

with entries from $Y(q_1)[[u^{-1}]]$ has the inverse matrix

$$\begin{bmatrix} \tilde{T}_{11}(u) & -\tilde{T}_{1,-1}(u) \\ \tilde{T}_{-1,1}(u) & \tilde{T}_{-1,-1}(u) \end{bmatrix} = \begin{bmatrix} \tilde{A}(u) & -\tilde{B}(u) \\ \tilde{B}(-u) & \tilde{A}(-u) \end{bmatrix}.$$

Hence, the image of $C(u)$ is the Berezinian of the matrix with the entries from $(\text{gr } Y(q_1))[[u^{-1}]]$ corresponding to the entries of (63). It is also called the *superdeterminant*.

We will show that coefficients of each of series $C(u)$ and $D(u)$ generate the center of $Y(q_1)$. We will also link the two series to each other and to $Z(u)$ at $N = 1$. We will use the fact that the comultiplication on $Y(q_1)$ maps

$$C(u) \mapsto C(u) \otimes C(u) + D(u) \otimes D(-u), \quad (64)$$

$$D(u) \mapsto C(u) \otimes D(u) + D(u) \otimes C(-u). \quad (65)$$

Indeed, by (53)–(56), the comultiplication on $Y(q_1)$ maps $C(u)$ to the product in $Y(q_1)^{\otimes 2}[[u^{-1}]]$,

$$\begin{aligned} & (A(u) \otimes A(u) - B(u) \otimes B(-u)) (\tilde{A}(-u) \otimes \tilde{A}(-u) + \tilde{B}(u) \otimes \tilde{B}(-u)) \\ &= (A(u)\tilde{A}(-u)) \otimes (A(u)\tilde{A}(-u)) + (B(u)\tilde{B}(u)) \otimes (B(-u)\tilde{B}(-u)) + (A(u)\tilde{B}(u)) \otimes (A(u)\tilde{B}(-u)) \\ & \quad - (B(u)\tilde{A}(-u)) \otimes (B(-u)\tilde{A}(-u)). \end{aligned}$$

The last two displayed tensor products cancel each other due to relation (59) and the relation

$$A(u)\tilde{B}(-u) + B(-u)\tilde{A}(-u) = 0 \quad (66)$$

obtained by setting $i = 1$ and $j = 2$ in (45) when $N = 1$. The sum of the preceding two tensor products is the right-hand side of (64) by definition. Similarly, the comultiplication maps $D(u)$ to

$$\begin{aligned} & (A(u) \otimes B(u) + B(u) \otimes A(-u)) (\tilde{B}(u) \otimes \tilde{A}(u) + \tilde{A}(-u) \otimes \tilde{B}(u)) \\ &= (B(u)\tilde{A}(-u)) \otimes (A(-u)\tilde{B}(u)) - (A(u)\tilde{B}(u)) \otimes (B(u)\tilde{A}(u)) + (A(u)\tilde{A}(-u)) \otimes (B(u)\tilde{B}(u)) \\ & \quad + (B(u)\tilde{B}(u)) \otimes (A(-u)\tilde{A}(u)). \end{aligned}$$

The first two tensor products at the right-hand side of the last display cancel each other by the relations (59) and (66). The sum of the next two tensor products is the right-hand side of (65). Hence, we have (64) and (65).

Now, consider the matrix with entries from $Y(q_1)[[u^{-1}]]$,

$$\begin{bmatrix} C(u) & D(u) \\ D(-u) & C(-u) \end{bmatrix}. \quad (67)$$

The two assignments (64) and (65) imply that applying the comultiplication on $Y(q_1)$ to this matrix amounts to multiplying (67) by itself as a matrix while taking tensor products of entries instead of usual multiplication. This means that the matrix (67) is *comultiplicative* for $Y(q_1)$.

Due to (62), the antiautomorphism (30) of $Y(q_1)$ maps

$$C(u) \mapsto (Z(-u)^{-1}A(-u))\tilde{A}(u) = Z(u)^{-1}C(-u),$$

$$D(u) \mapsto -(Z(u)^{-1}B(u))\tilde{B}(u) = -Z(u)^{-1}D(u).$$

Here, we used the relation $Z(-u) = Z(u)$. We also used the description of the square of the antiautomorphism (30) of $Y(q_1)$; see the very end of Sec. IV. Hence, the antiautomorphism (30) of $Y(q_1)$ maps the matrix (67) to

$$Z(u)^{-1} \begin{bmatrix} C(-u) & -D(u) \\ -D(-u) & C(u) \end{bmatrix}.$$

On the other hand, by the axioms of the anipodal, map the comultiplicativity of the matrix (67) for $Y(q_1)$ implies that (30) inverts this matrix. By equating to the identity matrix the product of (67) with the last displayed matrix, we obtain the relations $C(u)D(u) = D(u)C(u)$ and

$$C(u)C(-u) - D(u)D(-u) = Z(u). \quad (68)$$

Thus, $Z(u)$ is equal to the determinant of the matrix (67). This yields explicit expressions for the coefficients of the series $Z(u)$ in terms of those of the series $C(u)$ and $D(u)$. Pairwise commutativity of all entries of the matrix (67) will follow from centrality of their coefficients in $Y(q_1)$.

Due to (57), (60), and (61), the homomorphism (29) maps

$$C(u) \mapsto \frac{u-a}{u} \frac{(u-a)u}{u^2-a^2-a} = \frac{(u-a)^2}{u^2-a^2-a},$$

$$D(u) \mapsto \frac{b}{u} \frac{-bu}{u^2-a^2-a} = \frac{-a}{u^2-a^2-a}.$$

Hence, the matrix (67) gets mapped by (29) to the matrix

$$\frac{1}{u^2-a^2-a} \begin{bmatrix} (u-a)^2 & -a \\ -a & (u+a)^2 \end{bmatrix}.$$

According to general conventions of Sec. II for each $p = 1, \dots, n$, denote $a_p = \iota_p(a)$ in the algebra $U(q_1)^{\otimes n}$. Then, $\deg a_p = 0$ relative to the $\mathbb{Z}_2$ -grading on $U(q_1)^{\otimes n}$. Hence, the elements $a_1, \dots, a_n$ commute with each other.

Consider the homomorphism $\gamma_n : Y(q_1) \rightarrow U(q_1)^{\otimes n}$ defined as in Sec. V but for $N = 1$. The arguments above prove that γ_n maps the matrix (67) to the product over $p = 1, \dots, n$ of the matrices

$$\frac{1}{u^2-a_p^2-a_p} \begin{bmatrix} (u-a_p)^2 & -a_p \\ -a_p & (u+a_p)^2 \end{bmatrix}. \quad (69)$$

The matrices (69) commute, so the ordering of the factors in the product does not matter. Moreover, the entries of the product of all these n matrices are rational functions of u with values in the ring symmetric of polynomials in $a_1, \dots, a_n$ with complex coefficients.

Note that conjugating each matrix (69) by the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

amounts to changing the sign of u in it. Thus, the product of the n matrices (69) can be written as the matrix

$$\begin{bmatrix} \varphi_n(u) & \psi_n(u) \\ \psi_n(-u) & \varphi_n(-u) \end{bmatrix}$$

for certain $\varphi_n(u)$ and $\psi_n(u)$. Here, $\psi_n(-u) = \psi_n(u)$ as all n matrices (69) are symmetric and pairwise commute. Moreover, it is easy to verify by induction on n that

$$\varphi_n(u) - \varphi_n(-u) = 4u\psi_n(u).$$

Indeed, for $n = 1$, this relation is obvious. If $n > 1$ and this relation holds for $n - 1$ instead of n , then we have

$$\begin{aligned} (u^2 - a_p^2 - a_p)(\varphi_n(u) - \varphi_n(-u)) &= (u - a_p)^2 \varphi_{n-1}(u) - a_p \psi_n(u) - (u + a_p)^2 \varphi_{n-1}(-u) + a_p \psi_n(-u) \\ &= (u^2 + a_p^2)(\varphi_{n-1}(u) - \varphi_{n-1}(-u)) - 2u a_p (\varphi_{n-1}(u) + \varphi_{n-1}(-u)) \\ &= (u + a_p)^2 (\varphi_{n-1}(u) - \varphi_{n-1}(-u)) - 2u a_p (\varphi_{n-1}(u) - \varphi_{n-1}(-u)) - 2u a_p (\varphi_{n-1}(u) + \varphi_{n-1}(-u)) \\ &= 4u(u + a_p)^2 \psi_{n-1}(u) - 4u a_p \varphi_{n-1}(u) \\ &= 4u(u^2 - a_p^2 - a_p) \psi_n(u). \end{aligned}$$

The kernels of homomorphisms γ_n with $n = 1, 2, \dots$ have zero intersection in $Y(q_1)$, see the end of Sec. V. Hence we get the relations $D(-u) = D(u)$ and

$$C(u) - C(-u) = 4u D(u) \quad (70)$$

in $Y(q_1)[[u^{-1}]]$. Moreover, because for any n the images of the coefficients of the series $C(u)$ and $D(u)$ relative to the homomorphism γ_n belong to the centre of $U(q_1)^{\otimes n}$, the coefficients themselves belong to the centre of $Y(q_1)$.

Just by the definitions (62) we have the expansions

$$C(u) = 1 + C_1 u^{-1} + C_2 u^{-2} + \dots$$

for certain central elements $C_1, C_2, \dots \in Y(q_1)$ and

$$D(u) = D_2 u^{-2} + D_4 u^{-4} + \dots$$

for another central elements $D_2, D_4, \dots \in Y(q_1)$. Here, we also used the relation $D(-u) = D(u)$. By (70), we get

$$C_1 = 2D_2, \quad C_3 = 2D_4, \quad \dots$$

Now consider the $\mathbb{Z}$ -graded algebra $\text{gr}' Y(q_1)$ defined as in Sec. V but for $N = 1$. Take the twisted current Lie superalgebra (1) with $N = 1$. For $r \geq 1$, consider the element of $\text{gr}' Y(q_1)$ corresponding to $C_{2r-1} \in Y(q_1)$. Through the isomorphism $U(\mathfrak{g}) \rightarrow \text{gr}' Y(q_1)$ defined by (34), this element of $\text{gr}' Y(q_1)$ corresponds to

$$-2(e_{11} + e_{-1,-1})u^{2r-2} = -2au^{2r-2} \in \mathfrak{g}.$$

Due to Proposition 3.6 of Ref. 3, the latter elements of $\mathfrak{g}$ with $r = 1, 2, \dots$ freely generate the center of $U(\mathfrak{g})$. Therefore, $C_1, C_3, \dots$ freely generate the center of $Y(q_1)$. They get degrees $0, 2, \dots$ by the $\mathbb{Z}$ -filtration defining $\text{gr}' Y(q_1)$.

By Theorem 3.4 of Ref. 3, the coefficients $Z_2, Z_4, \dots$ of $Z(u)$ for $N = 1$ also freely generate the center of $Y(q_1)$. They have degrees $0, 2, \dots$ by the same $\mathbb{Z}$ -filtration on $Y(q_1)$. The left-hand side of (68) involves both $C_1, C_3, \dots$ and $C_2, C_4, \dots$. To express $Z_2, Z_4, \dots$ in $C_1, C_3, \dots$ only, we will use the homomorphisms $\gamma_n : Y(q_1) \rightarrow U(q_1)^{\otimes n}$.

By our argument using the matrix (67), the image of the series $C(u)$ by γ_n equals $\varphi_n(u)$. Consider $\varphi_n(u)$ as a formal power series in u^{-1} with coefficients being some polynomials in $a_1, \dots, a_n$. By taking only the top degree components of these coefficients, we obtain from $\varphi_n(u)$,

$$\prod_p \frac{(u - a_p)^2}{u^2 - a_p^2} = \prod_p \frac{u - a_p}{u + a_p} = \exp \left(- \sum_{r \geq 1} \frac{2a_1^{2r-1} + \dots + 2a_n^{2r-1}}{(2r-1)u^{2r-1}} \right).$$

The latter equality is obtained by taking the logarithm of the product and then exponentiating. The coefficients of the above series at $u^{-1}, u^{-3}, \dots, u^{1-2n}$ are algebraically independent. Consequently, the coefficients of $\varphi_n(u)$ at $u^{-1}, u^{-3}, \dots, u^{1-2n}$ are also algebraically independent. This provides another proof of algebraic independence of the central elements $C_1, C_3, \dots$ of the Yangian $Y(q_1)$.

Now denote by $\omega_n(u)$ the image of the series $Z(u)$ by the homomorphism γ_n . Due to (48), our $\omega_n(u)$ equals

$$\prod_p \frac{u^2 - a_p^2 + a_p}{u^2 - a_p^2 - a_p} = \prod_p \left(1 + \frac{2a_p}{u^2 - a_p^2 - a_p} \right) = \prod_p \left(1 + \sum_{r \geq 1} \frac{2a_p(a_p^2 + a_p)^{r-1}}{u^{2r}} \right)$$

where again $p = 1, \dots, n$. Also see the end of Sec. VII. Consider $\omega_n(u)$ as a formal power series in u^{-1} with the coefficients being polynomials in $a_1, \dots, a_n$. For $r \geq 1$ the top degree component of the coefficient at u^{-2r} is

$$2a_1^{2r-1} + \dots + 2a_n^{2r-1}.$$

Therefore, the coefficients of $\omega_n(u)$ at $u^{-2}, u^{-4}, \dots, u^{-2n}$ are algebraically independent polynomials in $a_1, \dots, a_n$. Without relying on Ref. 3, the latter fact implies the freeness of the generators $Z_2, Z_4, \dots$ of the center of $Y(q_1)$.

We can uniquely express the coefficients of $\omega_n(u)$ and $\varphi_n(u)$ at u^{-2n} in the coefficients of the same series $\varphi_n(u)$ at $u^{-1}, u^{-3}, \dots, u^{1-2n}$. Hence, we express Z_{2n} and C_{2n} in $C_1, C_3, \dots, C_{2n-1}$. We used the fact that C_{2n} is of degree $2n - 2$ relative to the second $\mathbb{Z}$ -filtration on $Y(q_1)$.

We can also uniquely express the coefficients of $\varphi_n(u)$ at u^{1-2n} and u^{-2n} in the coefficients of the series $\omega_n(u)$ at $u^{-2}, u^{-4}, \dots, u^{-2n}$. Thus, we express C_{2n-1} and C_{2n} in $Z_2, Z_4, \dots, Z_{2n}$.

It would be interesting to deduce relation (70) and the centrality of the coefficients of $C(u)$ directly from the defining relations of the algebra $Y(q_1)$, without invoking its representation theory. It would be also interesting to relate the coefficients of $C(u)$ to the central elements of $Y(q_1)$, which were recently introduced in Ref. 9.

Toward the end of our Introduction, we mentioned the quantum Berezinian for the Yangian of $\mathfrak{gl}_{1|1}$. This is the specialization to $M = N = 1$ of the quantum Berezinian⁷ for the Yangian of any general linear Lie superalgebra $\mathfrak{gl}_{M|N}$. It would be fascinating to extend the definition of the series $C(u)$ to the Yangian $Y(q_N)$ for any $N > 1$.

ACKNOWLEDGMENTS

Thanks are to Evgeny Sklyanin for his generous help.

AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

Author Contributions

Maxim Nazarov: Investigation (lead).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

REFERENCES

- ¹G. Olshanski, Dokl. Math. **36**, 569–573 (1988).
- ²M. L. Nazarov, *Quantum Groups*, Lecture Notes in Mathematics (Springer, 1992), Vol. 1510, pp. 90–97.
- ³M. Nazarov, Commun. Math. Phys. **208**, 195–223 (1999).
- ⁴M. Nazarov and A. Sergeev, *Studies in Lie Theory*, Progress in Mathematics (Birkhäuser, 2006), Vol. 243, pp. 417–441.
- ⁵E. Poletaeva and V. Serganova, J. Math. Phys. **58**, 111701 (2017).
- ⁶E. Poletaeva and V. Serganova, Adv. Math. **300**, 320–359 (2016).
- ⁷M. L. Nazarov, Lett. Math. Phys. **21**, 123–131 (1991).
- ⁸L. Gow, Commun. Math. Phys. **276**, 799–825 (2007).
- ⁹E. Poletaeva and V. Serganova, J. Algebra **570**, 140–163 (2021).