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The (partially) massless spin-3/2 and spin-5/2 fields in de Sitter spacetime as unitary and non-unitary representations of the de Sitter algebra

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Abstract: The divergence-free and gamma-traceless vector-spinor eigenfunctions, as well as the divergence-free and gamma-traceless rank-2 symmetric tensor-spinor eigenfunctions, of the Dirac operator on the N-sphere (S^N) are written down explicitly for $N \geq 3$. The spin-3/2 and spin-5/2 eigenmodes of the Dirac operator with arbitrary imaginary mass parameter on N-dimensional ($N \geq 3$) de Sitter spacetime (dS_N) are obtained by analytic continuation. Their transformation properties under the de Sitter algebra spin$(N,1)$ are studied. For N odd, we show that there is no de Sitter (dS) invariant scalar product for these eigenmodes. For N even, although dS invariant scalar products exist, positive-definiteness of the norm occurs only for the strictly and partially massless theories in $N = 4$ dimensions. For $N = 4$, the way in which the eigenmodes form unitary strictly and partially massless representations of spin$(4,1)$ is emphasised. The analysis presented in this paper reveals previously unknown features of the gauge-invariant theories with spin 3/2 and 5/2 on dS_N ($N \geq 3$): the strictly massless spin-3/2 field theory, as well as the strictly and partially massless spin-5/2 field theories, are unitary only for $N = 4$. In particular, a unitary theory for the gravitino field on dS_N does not exist unless $N = 4$.

Contents

1 Introduction

1.1 Background

The de Sitter spacetime, apart from its relevance to inflationary cosmology, is also thought to be a good model for the asymptotic future of our Universe, as suggested by current experimental evidence in favour of a positive cosmological constant [1–3]. The N-dimensional de Sitter spacetime (dS_N) is the maximally symmetric solution of the vacuum Einstein field equations with positive cosmological constant Λ [4]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0, \tag{1.1}$$

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar and

$$\Lambda = \frac{(N-2)(N-1)}{2\,\mathscr{R}^2}, \tag{1.2}$$

while $\mathscr{R}$ is the de Sitter radius. Throughout this paper we use units in which $\mathscr{R} = 1$.

Unlike Minkowskian field theories, possible field theories of spin s on dS_N are not restricted to the two usual cases of massive and strictly massless theories, where for $N = 4$ the former has $2s + 1$ propagating degrees of freedom (DoF), while the latter has only 2 helicity DoF ($\pm s$) due to the gauge invariance of the theory [5]. On dS_N there also exist intermediate gauge-invariant theories for $s \geq 2$, known as **partially massless** [1] theories [6–10]. For a given spin $s \geq 1$, there exists one strictly massless theory and $[s] - 1$ different partially massless theories, where $[s] = s$ if the spin s is an integer and $[s] = s - 1/2$ if s is a half-odd integer. Partial masslessness was first observed for the spin-2 field by Deser and Nepomechie [11, 12] and for higher integer-spin fields by Higuchi [13]. Partially massless theories with various spins have been discussed further in a series of papers by Deser and Waldron [6–10, 14]. From now on, we use the term 'massless theory' in order to refer to a gauge-invariant theory that is either strictly or partially massless.

Each massless theory is conveniently labelled by a distinct value of the 'depth' $\tau = 1, 2, ..., [s]$ (where the value $\tau = 1$ corresponds to strict masslessness) and in 4 dimensions there are 2τ propagating helicities, namely: $(\pm s, \pm(s-1), ..., \pm(s-\tau+1))$ [6, 8, 9]. For given spin s and depth τ, each of these gauge-invariant theories corresponds to a distinct tuning of the mass parameter to the cosmological constant Λ [6, 8, 9, 13, 14]. Higuchi classified the tunings of the mass parameter for massless theories with arbitrary integer spin by studying the group-theoretic properties of the eigenmodes of the Laplace-Beltrami (LB) operator on dS_N [13, 15]. Deser and Waldron gave an analogous classification for arbitrary integer and half-odd-integer spins by using group representation methods based on the dS/CFT correspondence [14].

A field theory on dS_N is unitary only if it corresponds to a unitary representation of the de Sitter algebra spin$(N, 1)$. Unitarity is very important for physical problems since it ensures the positivity of probabilities. The unitarity of massless totally symmetric field

[1] Partially massless theories exist also in anti-de Sitter spacetime. Partially and strictly massless theories on both de Sitter and anti-de Sitter spacetimes are discussed in Ref. [8].

representations with arbitrary integer spin on dS_N has been studied in detail by Higuchi [13, 15]. More specifically, by applying analytic continuation to the totally symmetric, traceless and divergence-free tensor-eigenfunctions of the LB operator on the N-sphere (S^N), he obtained the totally symmetric, traceless and divergence-free tensor eigenmodes of the LB operator on dS_N. Then, by calculating the norm of these eigenmodes explicitly, he showed that all massless theories with arbitrary integer spin s on dS_N are unitary (due to the positivity of the norm). However, such an analysis for half-odd-integer spins is still absent from the literature. It is the purpose of the present article to start filling this gap in the literature for the vector-spinor and symmetric rank-2 tensor-spinor fields on dS_N, by working along the lines of Higuchi's analysis [13, 15].

Particles with arbitrary half-odd-integer spin $s \equiv r + 1/2$ on dS_N can be described by totally symmetric [2] tensor-spinors $\Psi_{\mu_1 \dots \mu_r}$ satisfying [6, 14]

$$\left(\nabla\!\!\!\!/ + M \right) \Psi_{\mu_1 \dots \mu_r} = 0 \tag{1.3}$$

$$\nabla^{\alpha} \Psi_{\alpha \mu_2 \dots \mu_r} = 0, \quad \gamma^{\alpha} \Psi_{\alpha \mu_2 \dots \mu_r} = 0, \tag{1.4}$$

where $\nabla\!\!\!\!/ = \gamma^{\nu} \nabla_{\nu}$ is the Dirac operator on dS_N. (See Subsection 2.2 for our convention for the gamma matrices.) From now on, we will refer to the divergence-free and gamma-tracelessness conditions in eq. (1.4) as the TT conditions. For each value of the mass parameter M, the TT eigenmodes $\Psi_{\mu_1 \dots \mu_r}$ in eq. (1.3) form a representation of the de Sitter algebra spin$(N, 1)$. The half-odd-integer-spin theories described by eqs. (1.3) and (1.4) become gauge-invariant (i.e. massless) for the following imaginary values [3] of the mass parameter $M = i\tilde{M}$ [14]:

$$\tilde{M}^2 = -M^2 = \left(r - \tau + \frac{N-2}{2} \right)^2 \qquad (\tau = 1, ..., r) \tag{1.5}$$

for $r \geq 1$ [4]. Real values of M (including $M = 0$) correspond to non-gauge-invariant theories and they are discussed in Appendix A.

1.2 Main aim and strategy

The main aim of this paper is to study the unitarity of three different representations of spin$(N, 1)$, corresponding to: the strictly massless spin-3/2 field (i.e. the gravitino field), the strictly massless spin-5/2 field and the partially massless spin-5/2 field on dS_N ($N \geq 3$). Since all these massless theories occur for the imaginary values (1.5) of the mass parameter,

[2] Note that not all possible half-odd-integer-spin representations of spin$(N, 1)$ can be formed by totally symmetric tensor-spinors. Similarly, not all possible integer-spin representations of spin$(N, 1)$ can be formed by totally symmetric tensors. Mixed-symmetry fields on dS_N and the corresponding representations of the de Sitter algebra have been discussed in Ref. [16].

[3] The imaginary values of M in eq. (1.5) imply that the action functional for massless half-odd-integer-spin theories on dS_N is not hermitian. The fact that the gauge-invariant spin-3/2 field theory in de Sitter spacetime has an imaginary mass parameter had been already observed in cosmological supergravity [17].

[4] In the case of the spin-1/2 field, where $r = 0$, there is no gauge-invariance for any value of M. However, for N even, the spin-1/2 theory with $M = 0$ behaves as a massless theory since the spin$(N, 1)$ representation formed by the eigenmodes is reducible. Note also that for N even, the non-gauge-invariant $M = 0$ theories with $r \geq 1$ are "massless" in the sense of reducibility of the representation - see Appendix A.

we will focus our group-theoretic analysis on the case where M is an arbitrary imaginary number $M = i\tilde{M}$ ($\tilde{M} \neq 0$) and we will specialise to the massless values (1.5) when necessary. The basic steps of our analysis are as follows:

- We obtain the TT vector-spinor eigenmodes Ψ_{μ_1} (spin-3/2 modes) and the TT symmetric tensor-spinor eigenmodes $\Psi_{\mu_1\mu_2}$ (spin-5/2 modes) of eq. (1.3) with arbitrary imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$). (The case with $M = 0$, as well as the cases with any real mass parameter M, are discussed in Appendix A.)
- We study the transformation properties of the eigenmodes under a spin$(N, 1)$ boost.
- By exploiting the transformation properties of the eigenmodes under the spin$(N, 1)$ boost, we examine when their norm with respect to a de Sitter (dS) invariant scalar product is positive-definite.

As in the relevant study of integer-spin fields [13, 15], in order to obtain the TT eigenmodes of eq. (1.3) we will take advantage of the well-known fact that S^N can be analytically continued to dS_N (see Section 7). Motivated by this, we will write down explicitly the mode solutions of the following eigenvalue equation on S^N:

$$\nabla\!\!\!\!/\, \psi_{\mu_1\ldots\mu_r} = i\zeta\psi_{\mu_1\ldots\mu_r} \tag{1.6}$$

$$\nabla^{\alpha}\psi_{\alpha\mu_2\ldots\mu_r} = 0, \quad \gamma^{\alpha}\psi_{\alpha\mu_2\ldots\mu_r} = 0, \tag{1.7}$$

where $\psi_{\mu_1\ldots\mu_r}$ is a totally symmetric tensor-spinor of rank r on S^N which also satisfies the TT conditions (1.7) and $\nabla\!\!\!\!/$ is the Dirac operator on S^N. The eigenvalue in eq. (1.6) is imaginary, i.e. $\zeta \in \mathbb{R}$, since - as is well known - $\nabla\!\!\!\!/^{\,2}$ is negative semidefinite on compact spin manifolds. We call the eigenmodes satisfying eqs. (1.6) and (1.7) the **symmetric tensor-spinor spherical harmonics (STSSH's)**. In the present work we study only the STSSH's with ranks $r = 1$ and $r = 2$ on S^N ($N \geq 3$), where we are also going to normalise them, as well as study their transformation properties under a spin$(N + 1)$ transformation (spin$(N + 1)$ is the Lie algebra of the isometry group of S^N). Note that the unnormalised STSSH's of rank $r = 1$ - i.e. the TT vector-spinor eigenmodes of the Dirac operator $\nabla\!\!\!\!/$ on S^N - have been already constructed in Ref. [18], but no emphasis was given on their group-theoretic properties. To our knowledge, the STSSH's of rank $r = 2$ are constructed in the present paper for the first time (see Section 5 and Appendix E). By applying analytic continuation techniques to eqs. (1.6) and (1.7), one can obtain eqs. (1.3) and (1.4), respectively, on dS_N.

1.3 Main result

Our main result is:

- The strictly massless spin-3/2 field and the strictly and partially massless spin-5/2 fields on dS_N ($N \geq 3$) are unitary only for $N = 4$.

(In this paper we do not discuss the vector-spinor field and the symmetric tensor-spinor field on dS_2.) In particular, a unitary theory for the gravitino (corresponding to the strictly massless spin-3/2 field) on dS_N ($N \geq 3$) does not exist unless $N = 4$. In order to arrive at our main result, we study the group-theoretic properties of the spin-3/2 and spin-5/2 TT eigenmodes of eq. (1.3) with arbitrary imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$) and we show:

1. **For even $N > 4$**: all dS invariant scalar products for these eigenmodes must be indefinite for all imaginary $M = i\tilde{M}$ ($\tilde{M} \neq 0$). This is demonstrated by showing that both positive-norm and negative-norm mode solutions exist and they mix with each other under spin$(N, 1)$ for all $\tilde{M} \neq 0$ [including the strictly and partially massless values (1.5)].

2. **For $N = 4$**: all dS invariant scalar products for these eigenmodes must be indefinite unless $\tilde{M}$ is tuned to the massless values (1.5). The solution space of the massless theories is divided into two spin$(4, 1)$ invariant subspaces, denoted as $\mathscr{H}_-$ and $\mathscr{H}_+$, where all mode solutions in $\mathscr{H}_-$ have 'negative helicity', while all mode solutions in $\mathscr{H}_+$ have 'positive helicity'. Then, we introduce a specific dS invariant scalar product [eq. (8.19)] in $\mathscr{H}_-$ and $\mathscr{H}_+$. For this choice of scalar product, it happens that the norm is positive-definite in $\mathscr{H}_-$ and negative-definite in $\mathscr{H}_+$. However, group-theoretically, we are allowed to have a different scalar product for each invariant subspace (since they correspond to different irreducible representations). Thus, by a redefinition of the scalar product in $\mathscr{H}_+$, we can change the sign of the associated norm and make it positive-definite. This shows that $\mathscr{H}_-$ and $\mathscr{H}_+$ form a direct sum of unitary irreducible representations of spin$(4, 1)$.

3. **For N odd**: For all $M = i\tilde{M} \neq 0$ [including the strictly and partially massless values (1.5)], there does not exist any dS invariant scalar product for these eigenmodes. Thus, by definition, the corresponding spin$(N, 1)$ representations are not unitary.

To our knowledge, all these features of the strictly massless spin-3/2 field and of the strictly and partially massless spin-5/2 fields on dS_N are unknown in the mathematical physics community. In Appendix A, we verify our results by making use of the known classification of the Unitary Irreducible Representations (UIR's) of spin$(N, 1)$ [19, 20]. Our analysis in Appendix A suggests that partially or strictly massless unitary theories with any half-odd-integer spin exist only in 4 dimensions. Also, for the sake of completeness, in Appendix A we provide a detailed correspondence between all unitary spin $s \in \{ 3/2, 5/2 \}$ theories with real and imaginary mass parameter on dS_N ($N \geq 3$) and UIR's of spin$(N, 1)$, where we also give the explicit values for the labels of the UIR's for each unitary theory. We believe that the exact correspondence between half-odd-integer-spin theories on dS_N ($N \geq 3$) and UIR's given in Appendix A is absent from the literature.

Our main result stands in contrast to the claims made in Refs. [14, 16]. The non-unitarity of the massless spin-3/2 and spin-5/2 fields on dS_N for $N \neq 4$ was missed in Refs. [14, 16], apparently because the norm of the corresponding eigenmodes was not examined. We note that the positivity of the norm in the Hilbert space of the massless theories for $N = 4$ had been emphasised in the relevant works by Deser and Waldron - see e.g. Ref. [6].

1.4 Outline of the paper, notation and conventions

The paper is organised as follows. In Section 2, we begin by presenting the Christoffel symbols, vielbein fields and spin connection components on S^N in geodesic polar coordinates. Then, we present the basics about gamma-matrices and tensor-spinor fields on S^N. We also review the eigenspinors of the Dirac operator on S^{N-1}. In Section 3, we present the functions that describe the dependence of the STSSH's on the geodesic distance (θ_N) from the North Pole of S^N. In Section 4, we write down explicitly the unnormalised STSSH's of rank 1 on S^N (which have been constructed in Ref. [18]). In Section 5, we write down explicitly the unnormalised STSSH's of rank 2 on S^N (which we construct in Appendix E). In Section 6, we use the Lie-Lorentz derivative [27] in order to study the transformation properties of the STSSH's of rank r ($r \in \{1, 2\}$) on S^N under a spin$(N+1)$ transformation and we give their normalisation factors. In Section 7, we begin by obtaining the vector-spinor and rank-2 symmetric tensor-spinor TT eigenmodes of the Dirac operator with arbitrary imaginary mass parameter on dS_N by analytically continuing the STSSH's of rank 1 and rank 2, respectively, on S^N. Then, we identify the 'pure gauge' modes of the massless spin-3/2 and spin-5/2 theories on dS_N. In Section 8, we derive the main result of this paper (i.e. we prove statements 1, 2 and 3 listed above), by studying the transformation properties of the TT eigenmodes of eq. (1.3) with arbitrary imaginary mass parameter under a spin$(N, 1)$ boost. More specifically, in Subsection 8.1, we show that all dS invariant scalar products must be indefinite for even $N > 4$ (i.e. we prove statement 1). Also, for even $N \geq 4$, we show that the 'pure gauge' modes in the massless theories with spin $s \in \{3/2, 5/2\}$ have zero norm with respect to any dS invariant scalar product. Then, for $N = 4$, we show that the requirement for dS invariance of the scalar product does not imply the indefiniteness of the norm if and only if the imaginary mass parameter $M = i\tilde{M}$ (with $\tilde{M} \neq 0$) takes the massless values (1.5). We also find that for the massless theories with spin $s \in \{3/2, 5/2\}$ on dS_4, the eigenmodes with negative helicity and the ones with positive helicity separately form irreducible representations of spin$(4, 1)$ (the unitarity of these irreducible representations is proved in Subsection 8.2). In Subsection 8.2, we calculate the norms of the eigenmodes on dS_N (for even $N \geq 4$) with respect to a specific dS invariant scalar product and we verify statement 1 (which was proved in the previous Subsection) and we also prove statement 2. Subsection 8.3 concerns the case with N odd and we prove statement 3. Finally, in Section 9, we give a summary of our results. We also discuss the possible generalisation of our results to higher half-odd-integer spins, as well as to other vacuum spacetimes with positive cosmological constant.

There are seven Appendices. In Appendix A, we first verify our main result (presented in Subsection 1.3 of the Introduction) by using the known classification of the UIR's of spin$(N, 1)$. Then, for the sake of completeness, we identify the unitary field theories with spin $s \in \{3/2, 5/2\}$ and real mass parameter on dS_N with known UIR's of spin$(N, 1)$. In Appendix E, we construct the STSSH's of rank 2 on S^N by making use of the method of separation of variables. In this method, the STSSH's of rank 2 on S^N are expressed in terms of STSSH's of rank $\tilde{r}$ ($0 \leq \tilde{r} \leq 2$) on S^{N-1}. In Appendix F, we present technical details omitted in Section 6. To be specific, we first give a detailed derivation of the formulae for the spin$(N+1)$ transformation of the rank-1 STSSH's and we determine their normalisation factors. Then, we discuss briefly the derivation of the transformation formulae and the normalisation factors for the rank-2 STSSH's on S^N. The rest of the Appendices concern other technical details that were omitted in the main text.

Notation and conventions. We use the mostly plus metric sign convention for dS_N. Lowercase Greek tensor indices refer to components with respect to the "coordinate basis". Lowercase Latin tensor indices refer to components with respect to the vielbein basis. Summation over repeated indices is understood. We denote the symmetrisation of a pair of indices as $A_{(\mu\nu)} \equiv (A_{\mu\nu} + A_{\nu\mu})/2$ and the anti-symmetrisation as $A_{[\mu\nu]} \equiv (A_{\mu\nu} - A_{\nu\mu})/2$. Spinor indices are always suppressed throughout this paper. We use the term 'massless field' of spin $s \in \{3/2, 5/2\}$ to refer to either one of the following three cases (unless otherwise stated): the strictly massless spin-3/2 field ($r = \tau = 1$), the strictly massless spin-5/2 field ($r = \tau + 1 = 2$), the partially massless spin-5/2 field ($r = \tau = 2$). The complex conjugate of the complex number z is denoted as z^*.

2 Geometry of the N-sphere and tensor-spinor fields

2.1 Coordinate system, Christoffel symbols and spin connection

The N-sphere (S^N) embedded in the Euclidean space $\mathbb{R}^{N+1}$ is described by

$$\delta_{ab} X^a X^b = 1, \tag{2.1}$$

($a, b = 1, 2, ..., N+1$) where δ_{ab} is the Kronecker delta symbol and $X^1, X^2, ..., X^{N+1}$ are the standard coordinates for $\mathbb{R}^{N+1}$. The geodesic polar coordinates are given by

$$\begin{aligned} X^{N+1} &= X^{N+1}(\theta_N) = \cos\theta_N \\ X^i &= X^i(\theta_N, \boldsymbol{\theta}_{N-1}) = \sin\theta_N \ \tilde{X}^i(\boldsymbol{\theta}_{N-1}), \quad i = 1, ..., N, \end{aligned} \tag{2.2}$$

where $0 \le \theta_N \le \pi$ is the geodesic distance from the North Pole and $\boldsymbol{\theta}_{N-1} = (\theta_{N-1}, ..., \theta_1)$ (where $0 \le \theta_1 < 2\pi$ and $0 \le \theta_i \le \pi$ for $i = 2, 3, ..., N-1$). The $\tilde{X}^i$'s in eq. (2.2) are the geodesic polar coordinates for S^{N-1} in N-dimensional Euclidean space.

The line element for S^N is expressed in coordinates (2.2) as

$$ds_N^2 = d\theta_N^2 + \sin^2\theta_N ds_{N-1}^2, \tag{2.3}$$

where ds_{N-1}^2 is the line element for S^{N-1}. (Note that we define $ds_1^2 \equiv d\theta_1^2$.) The non-zero Christoffel symbols in geodesic polar coordinates are

$$\begin{aligned} &\Gamma^{\theta_N}_{\theta_i\theta_j} = -\sin\theta_N \cos\theta_N \ \tilde{g}_{\theta_i\theta_j}, \ \ \Gamma^{\theta_i}_{\theta_j\theta_N} = \cot\theta_N \ \tilde{g}^{\theta_i}_{\theta_j}, \\ &\Gamma^{\theta_k}_{\theta_i\theta_j} = \tilde{\Gamma}^{\theta_k}_{\theta_i\theta_j}, \end{aligned} \tag{2.4}$$

where $\tilde{g}_{\theta_i\theta_j}$ and $\tilde{\Gamma}^{\theta_k}_{\theta_i\theta_j}$ are the metric tensor and the Christoffel symbols, respectively, on S^{N-1}. The vielbein fields $\boldsymbol{e}_a = e^\mu{}_a \partial_\mu$ (where $a = 1, ..., N$ and $\mu = \theta_1, ..., \theta_N$), determining an orthonormal frame, satisfy

$$e_\mu{}^a \, e_\nu{}^b \delta_{ab} = g_{\mu\nu}, \quad e^\mu{}_a \, e_\mu{}^b = \delta^b_a, \tag{2.5}$$

where the co-vielbein fields $\boldsymbol{e}^a = e_\mu{}^a\, dx^\mu$ define the dual coframe. The co-vielbein transforms under local rotations $\Lambda : S^N \to \mathrm{SO}(N)$ as

$$\boldsymbol{e}^a \to \Lambda(x)^a{}_b\, \boldsymbol{e}^b. \tag{2.6}$$

In geodesic polar coordinates the non-zero components of the vielbein fields are given by

$$e^{\theta_N}{}_N = 1, \qquad e^{\theta_i}{}_i = \frac{1}{\sin\theta_N}\tilde{e}^{\theta_i}{}_i, \qquad i = 1, ..., N-1, \tag{2.7}$$

where $\tilde{e}^{\theta_i}{}_i$ are the vielbein fields on S^{N-1}. The spin connection $\omega_{abc} = \omega_{a[bc]} \equiv (\omega_{abc} - \omega_{acb})/2$ is given by

$$\omega_{abc} = -e^\mu{}_a \Big(\partial_\mu e^\lambda{}_b + \Gamma^\lambda_{\mu\nu} e^\nu{}_b\Big) e_{\lambda c} \tag{2.8}$$

and its only non-zero components are

$$\omega_{ijk} = \frac{\tilde{\omega}_{ijk}}{\sin\theta_N}, \qquad \omega_{iNk} = -\omega_{ikN} = -\cot\theta_N\, \delta_{ik}, \qquad i, j, k = 1, ..., N-1, \tag{2.9}$$

where $\tilde{\omega}_{ijk}$ are the spin connection components on S^{N-1}. (Note that the sign convention we use for the spin connection is the opposite of the one used in Refs. [21, 22].)

2.2 Gamma matrices and tensor-spinor fields on the N-sphere

A Clifford algebra representation in N dimensions is generated by N gamma matrices. These are matrices of dimension $2^{[N/2]}$ - where $[N/2] = N/2$ if N is even and $[N/2] = (N-1)/2$ if N is odd - satisfying the anti-commutation relations

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}\mathbf{1}, \qquad a, b = 1, 2, ..., N, \tag{2.10}$$

where $\mathbf{1}$ is the identity matrix. We adopt the representation of gamma matrices used in Ref. [21], where gamma matrices in N dimensions are expressed in terms of gamma matrices in $N-1$ dimensions ($\widetilde{\gamma}^i$) as follows:

- For N even

  $$\gamma^N = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & i\widetilde{\gamma}^j \\ -i\widetilde{\gamma}^j & 0 \end{pmatrix}, \tag{2.11}$$

  ($j = 1, ..., N-1$) where the lower-dimensional gamma matrices satisfy the Euclidean Clifford algebra anti-commutation relations

  $$\{\widetilde{\gamma}^j, \widetilde{\gamma}^k\} = 2\delta^{jk}\mathbf{1}, \qquad j, k = 1, ..., N-1. \tag{2.12}$$

  By using the vielbein fields (2.7) we can express the gamma matrices (2.11) in the "coordinate basis" as $\gamma^\mu(x) = e^\mu{}_a(x)\, \gamma^a$. Note that one can construct the extra gamma matrix γ^{N+1}, which is given by the product $\gamma^{N+1} \equiv \epsilon\, \gamma^1\gamma^2 ... \gamma^N$, where ϵ is a phase factor. The matrix γ^{N+1} anti-commutes with each of the γ^a's in eq. (2.11). As in Ref. [21], we choose the phase factor ϵ such that

  $$\gamma^{N+1} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{2.13}$$

- For N odd

$$\gamma^N = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad \gamma^{N-1} = \tilde{\gamma}^{N-1} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

$$\gamma^j = \widetilde{\gamma}^j = \begin{pmatrix} 0 & i\widetilde{\widetilde{\gamma}}^j \\ -i\widetilde{\widetilde{\gamma}}^j & 0 \end{pmatrix}, \qquad j = 1, ..., N-2. \tag{2.14}$$

The double-tilde is used to denote gamma matrices in $N-2$ dimensions. In $N=1$ dimension the only (one-dimensional) gamma matrix is equal to 1. The gamma matrices (2.14) are expressed in the "coordinate basis" by using the vielbein fields (2.7), as in the case with N even.

Note that all gamma matrices in eqs. (2.11)-(2.14) are hermitian.

The tensor-spinor fields $\psi_{\mu_1...\mu_r}$ of rank r are defined as r^{th}-rank tensors where each one of the tensorial components transforms as a $2^{[N/2]}$-dimensional spinor under Spin(N) (double cover of SO(N)). Tensor-spinors transform under the local rotation of the co-vielbein in eq. (2.6) as

$$\psi_{\mu_1...\mu_r}(x) \to \Lambda(x)_{\mu_1}^{\ \nu_1} ... \Lambda(x)_{\mu_r}^{\ \nu_r}\, S(\Lambda(x))\, \psi_{\nu_1...\nu_r}(x), \tag{2.15}$$

where the matrix $\Lambda(x) \in \mathrm{SO}(N)$ acts on the tensor indices of $\psi_{\mu_1...\mu_r}$, while the matrix $S(\Lambda(x)) \in \mathrm{Spin}(N)$ acts on the spinor indices of $\psi_{\mu_1...\mu_r}$ (the spinor indices have been suppressed for convenience). For any $\Lambda(x) \in \mathrm{SO}(N)$ we have [23]

$$S(\Lambda(x))^{-1}\, \gamma^a\, S(\Lambda(x)) = \Lambda(x)^a{}_b \gamma^b, \tag{2.16}$$

where $S(\Lambda(x))$ is either one of the two matrices in Spin(N) that correspond to $\Lambda(x)$. (See, e.g., Ref. [21] and Appendix D of Ref. [23] for more detailed discussions on spinor representations of orthogonal groups.)

The covariant derivative for a vector-spinor field is given by

$$\nabla_\nu \psi_\mu = \partial_\nu \psi_\mu + \frac{1}{2}\omega_{\nu bc}\Sigma^{bc}\psi_\mu - \Gamma^\lambda{}_{\nu\mu}\psi_\lambda, \tag{2.17}$$

while the covariant derivative for a rank-2 tensor-spinor field is given by

$$\nabla_\nu \psi_{\mu_1\mu_2} = \partial_\nu \psi_{\mu_1\mu_2} + \frac{1}{2}\omega_{\nu bc}\Sigma^{bc}\psi_{\mu_1\mu_2} - \Gamma^\lambda{}_{\nu\mu_1}\psi_{\lambda\mu_2} - \Gamma^\lambda{}_{\nu\mu_2}\psi_{\mu_1\lambda}, \tag{2.18}$$

where $\omega_{\nu bc} = e_\nu{}^d \omega_{dbc}$ [see eq. (2.9)]. The matrices Σ^{ab} are the generators of the $2^{[N/2]}$-dimensional spinor representation of Spin(N) and they are given by

$$\Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] \tag{2.19}$$

$$= \frac{1}{2}\gamma^a \gamma^b - \frac{1}{2}\delta^{ab}, \qquad a, b = 1, ..., N. \tag{2.20}$$

They satisfy the Spin(N) algebra commutation relations

$$[\Sigma^{ab}, \Sigma^{cd}] = \delta^{bc}\Sigma^{ad} - \delta^{ac}\Sigma^{bd} + \delta^{ad}\Sigma^{bc} - \delta^{bd}\Sigma^{ac}. \tag{2.21}$$

(The gamma matrices are covariantly constant, i.e. $\nabla_a \gamma^b = 0$ - see e.g. Appendix D of Ref. [23].)

For later convenience, let us introduce the spinor eigenmodes $\chi_{\pm\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})$ of the Dirac operator on S^{N-1} (see also Ref. [21] and Appendix C of the present paper). These spinor eigenmodes satisfy [21]

$$\tilde{\nabla\!\!\!/} \chi_{\pm\ell\tilde{\rho}} = \pm i \left(\ell + \frac{N-1}{2} \right) \chi_{\pm\ell\tilde{\rho}}, \tag{2.22}$$

where $\tilde{\nabla\!\!\!/} = \gamma^a \tilde{\nabla}_a$ is the Dirac operator on S^{N-1}, $\tilde{\nabla}_a$ is the spinor covariant derivative on S^{N-1} and ℓ is the angular momentum quantum number on S^{N-1}. The symbol $\tilde{\rho}$ represents labels other than ℓ. The requirement for regularity of the spinor eigenmodes (2.22) on S^{N-1} restricts ℓ to take the values $\ell = 0, 1, 2, \ldots$ [21]. We suppose that the spinor eigenmodes (2.22) are normalised as

$$\int_{S^{N-1}} \sqrt{\tilde{g}}\, d\boldsymbol{\theta}_{N-1}\, \chi_{\pm\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})^{\dagger}\, \chi_{\pm\ell'\tilde{\rho}'}(\boldsymbol{\theta}_{N-1}) = \delta_{\ell\ell'} \delta_{\tilde{\rho}\tilde{\rho}'}, \tag{2.23}$$

where $d\boldsymbol{\theta}_{N-1} = d\theta_{N-1}\, d\theta_{N-2} \ldots d\theta_1$. The square root of the determinant of the metric on S^{N-1} is

$$\sqrt{\tilde{g}} = \sin^{N-2}\theta_{N-1}\, \sin^{N-3}\theta_{N-2} \ldots \sin\theta_2 \tag{2.24}$$

$$= \sin^{N-2}\theta_{N-1} \sqrt{\tilde{\tilde{g}}}, \tag{2.25}$$

where $\tilde{\tilde{g}}$ is the determinant of the metric on S^{N-2}. All the χ_+ eigenspinors are orthogonal to all the χ_- eigenspinors in eq. (2.23) [21].

3 The functions describing the dependence of STSSH's on θ_N

Before writing down the explicit form of the STSSH's of rank r $(= 1, 2)$ on S^N, it is useful to introduce the functions $\phi^{(a)}_{n\ell}(\theta_N)$ [eq. (3.1)] and $\psi^{(a)}_{n\ell}(\theta_N)$ [eq. (3.2)] that describe the dependence of the STSSH's on the geodesic distance from the North Pole, θ_N, since they are going to be used extensively in the rest of the paper. The properties of these functions play a crucial role in the normalisation of the STSSH's and in the derivation of the formulae for the spin($N+1$) transformation of the STSSH's (see Section 6 and Appendix F). Most importantly, in view of the analytic continuation of our STSSH's to dS_N, the properties of the functions $\phi^{(a)}_{n\ell}(\theta_N)$ and $\psi^{(a)}_{n\ell}(\theta_N)$ will play a very important role in studying the unitarity/non-unitarity of the spin($N, 1$) representations formed by the analytically continued STSSH's.

As we will see in Sections 4 and 5, the θ_N-dependence of the STSSH's on S^N is described by functions of the following form:

$$\begin{aligned}\phi_{n\ell}^{(a)}(\theta_N) = \kappa_\phi(n,\ell)\left(\cos\frac{\theta_N}{2}\right)^{\ell+1-a}\left(\sin\frac{\theta_N}{2}\right)^{\ell-a} \\ \times F\left(-n+\ell, n+\ell+N; \ell+\frac{N}{2}; \sin^2\frac{\theta_N}{2}\right),\end{aligned} \tag{3.1}$$

$$\begin{aligned}\psi_{n\ell}^{(a)}(\theta_N) = \kappa_\phi(n,\ell)\frac{n+\frac{N}{2}}{\ell+\frac{N}{2}}\left(\cos\frac{\theta_N}{2}\right)^{\ell-a}\left(\sin\frac{\theta_N}{2}\right)^{\ell+1-a} \\ \times F\left(-n+\ell, n+\ell+N; \ell+\frac{N+2}{2}; \sin^2\frac{\theta_N}{2}\right),\end{aligned} \tag{3.2}$$

where the normalisation factor $\kappa_\phi(n,\ell)$ is given by

$$\kappa_\phi(n,\ell) = \frac{\Gamma(n+N/2)}{\Gamma(n-\ell+1)\Gamma(\ell+N/2)}, \tag{3.3}$$

while $F(A,B;C;z)$ is the Gauss hypergeometric function [24]. The number a in eqs. (3.1) and (3.2) is taken to be an integer for the purposes of this paper. The functions in eqs. (3.1) and (3.2) can be expressed in terms of the Jacobi polynomials [24], where $\kappa_\phi(n,\ell)$ plays the role of the conventional normalisation factor for the Jacobi polynomials [24]. (These functions with $a=0$ were used to describe spinors on S^N [21].) As we will discuss in Section 4 and 5, the integer n is the angular momentum quantum number of the STSSH's on S^N and it labels the representation of spin$(N+1)$ formed by the STSSH's. The angular momentum quantum number on S^{N-1}, ℓ, is initially assumed to be a positive integer or zero (this requirement is motivated naturally in the recursive construction of the STSSH's on S^N in terms of STSSH's on S^{N-1} - see Appendix E). Furthermore, the requirement for absence of singularity in the STSSH's on S^N will give rise to the condition

$$n-\ell \in \mathbb{N}_0 \tag{3.4}$$

or equivalently $n \geq \ell$, where $\mathbb{N}_0$ is the set of positive integers including zero. Equation (3.4) is obtained in Appendix E, by requiring the regularity of $\phi_{n\ell}^{(a)}(\theta_N)$ and $\psi_{n\ell}^{(a)}(\theta_N)$ in the limit $\theta_N \to \pi$.

The functions $\phi_{n\ell}^{(a)}(\theta_N)$ and $\psi_{n\ell}^{(a)}(\theta_N)$ are related to each other by the following formulae:

$$\left(\frac{d}{d\theta_N}+\frac{N+2a-1}{2}\cot\theta_N+\frac{\ell+(N-1)/2}{\sin\theta_N}\right)\psi_{n\ell}^{(a)}(\theta_N) = \left(n+\frac{N}{2}\right)\phi_{n\ell}^{(a)}(\theta_N) \tag{3.5}$$

$$\left(\frac{d}{d\theta_N}+\frac{N+2a-1}{2}\cot\theta_N-\frac{\ell+(N-1)/2}{\sin\theta_N}\right)\phi_{n\ell}^{(a)}(\theta_N) = -\left(n+\frac{N}{2}\right)\psi_{n\ell}^{(a)}(\theta_N). \tag{3.6}$$

Equations (3.5) and (3.6) are proved using the raising and lowering operators for the Gauss hypergeometric function in Appendix B. Note also the relation

$$\psi_{n\ell}^{(a)}(\theta_N) = (-1)^{n-\ell}\phi_{n\ell}^{(a)}(\pi-\theta_N). \tag{3.7}$$

4 The STSSH's of rank 1 on the N-sphere

In this Section we write down explicitly the unnormalised STSSH's of rank 1 [i.e. the TT vector-spinor eigenmodes of eq. (1.6)], by following Ref. [18] where these eigenmodes have been constructed. However, we will present the results of Ref. [18] in a slightly modified manner that is more suitable for studying the group-theoretic properties of the eigenmodes.

4.1 STSSH's of rank 1 for N even

The equations (1.6) and (1.7) for the TT vector-spinor eigenmodes on S^N ($N \geq 4$) are written as

$$\nabla\!\!\!/ \psi^{(A;\sigma;n\ell;\tilde{\rho})}_{\pm\mu} = \pm i \left(n + \frac{N}{2}\right) \psi^{(A;\sigma;n\ell;\tilde{\rho})}_{\pm\mu}, \tag{4.1}$$

$$\nabla^{\alpha} \psi^{(A;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha} = \gamma^{\alpha} \psi^{(A;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha} = 0. \tag{4.2}$$

We have denoted the TT vector-spinor eigenmodes with eigenvalue $\pm i(n+\frac{N}{2})$ as $\psi^{(A;\sigma;n\ell;\tilde{\rho})}_{\pm\mu}$, where $n = 1, 2, ...$ and $\ell = 1, ..., n$ are the angular momentum quantum numbers on S^N and S^{N-1}, respectively. (The angular momentum quantum numbers for our STSSH's of rank $r \in \{1, 2\}$ on S^N satisfy $n \geq \ell \geq r$. The condition $n \geq \ell$ was discussed in the previous Section - see eq. (3.4). However, as we will see below, the condition $\ell \geq r$ is obtained by using the explicit expressions of the STSSH's.) The index σ takes the values "+" or "−" and is called the spin projection index on S^N. The symbol $\tilde{\rho}$ stands for angular momentum quantum numbers on $S^{N-2}, S^{N-3}, ..., S^2, S^1$ and spin projection indices on the even-dimensional spheres $S^{N-2}, S^{N-4}, ..., S^2$.[5]

Equations (4.1) and (4.2) have two different types of mode solutions, namely, the **type-*I* modes** and the **type-*II* modes** [18]. We assign to the label A the value 'I' in order to indicate the type-I modes ($\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\mu}$) and the value '$II$-$\tilde{A}$' in order to indicate the type-II modes ($\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\mu}$), where the label $\tilde{A}$ on S^{N-1} corresponds to A on S^N (the label $\tilde{A}$ is discussed further in the passage after eq. (4.15)).

For each value of n we have a representation of spin$(N+1)$ (i.e. algebra of Spin$(N+1)$) acting on the space of the eigenmodes $\psi^{(A;\sigma;n\ell;\tilde{\rho})}_{+\mu}$ (or $\psi^{(A;\sigma;n\ell;\tilde{\rho})}_{-\mu}$). The highest weight $\lambda = (\lambda_1, ..., \lambda_{N/2})$ for this representation is given by

$$\lambda = \left(n + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}\right), \qquad (n = 1, 2, ...) \tag{4.3}$$

which can be determined using the branching rules for spin$(N+1) \supset$ spin(N) [6]. Note that for $N = 4$ we have $\lambda = (n + 1/2, 3/2)$ ($n = 1, 2, ...$). The two sets of eigenmodes, $\{\psi^{(A;\sigma;n\ell;\tilde{\rho})}_{+\mu}\}$ and $\{\psi^{(A;\sigma;n\ell;\tilde{\rho})}_{-\mu}\}$, form equivalent representations and they are related to each other by $\psi^{(A;\sigma;n\ell;\tilde{\rho})}_{+\mu} = \gamma^{N+1}\psi^{(A;\sigma;n\ell;\tilde{\rho})}_{-\mu}$.

Type-*I* modes. The type-I modes are expressed in their vector components as

$$\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\mu} = \left(\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N}, \psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j}\right) \tag{4.4}$$

[5] Note that spin projection indices exist only on even-dimensional spheres - see e.g. Ref. [22].

[6] See, e.g., Refs. [21, 25, 26], as well as Appendix A.

($j=1,...,N-1$), where $\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N}$ is a spinor on S^{N-1}, while $\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j}$ is a vector-spinor on S^{N-1} [18]. The type-I modes with negative spin projection ($\sigma=-$) on S^N are given by [18]

$$\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1})=\begin{pmatrix}\phi^{(1)}_{n\ell}(\theta_N)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \pm i\psi^{(1)}_{n\ell}(\theta_N)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix} \tag{4.5}$$

$$\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1})=\begin{pmatrix}C^{(\uparrow)(1)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})+D^{(\uparrow)(1)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \\ \pm iC^{(\downarrow)(1)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\pm iD^{(\downarrow)(1)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix}. \tag{4.6}$$

The type-I modes with positive spin projection ($\sigma=+$) on S^N are given by [18]

$$\psi^{(I;+;n\ell;\rho)}_{\pm\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1})=\begin{pmatrix}i\psi^{(1)}_{n\ell}(\theta_N)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \pm\phi^{(1)}_{n\ell}(\theta_N)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix} \tag{4.7}$$

$$\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1})=\begin{pmatrix}iC^{(\downarrow)(1)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})-iD^{(\downarrow)(1)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \\ \pm C^{(\uparrow)(1)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\mp iD^{(\uparrow)(1)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix}. \tag{4.8}$$

The eigenspinors on S^{N-1}, $\chi_{\pm\ell\tilde{\rho}}$, satisfy eq. (2.22) and they are written down explicitly in Appendix C. The functions $\phi^{(1)}_{n\ell}$ and $\psi^{(1)}_{n\ell}$ are given by eqs. (3.1) and (3.2), respectively. The functions $C^{(\uparrow)(a)}_{n\ell}, C^{(\downarrow)(a)}_{n\ell}$ are expressed in terms of $\phi^{(a)}_{n\ell}$ and $\psi^{(a)}_{n\ell}$ as follows [18]:

$$\begin{aligned}C^{(\uparrow)(a)}_{n\ell}(\theta_N)=\frac{1}{\ell(\ell+N-1)}\Bigg\{&\sin\theta_N\left[\frac{N-1}{2}\cos\theta_N+\ell+\frac{N-1}{2}\right]\phi^{(a)}_{n\ell}(\theta_N)\\ &-\frac{N-1}{N-2}(n+\frac{N}{2})\sin^2\theta_N\,\psi^{(a)}_{n\ell}(\theta_N)\Bigg\},\end{aligned} \tag{4.9}$$

$$\begin{aligned}C^{(\downarrow)(a)}_{n\ell}(\theta_N)=\frac{1}{\ell(\ell+N-1)}\times\Bigg\{&\sin\theta_N\left[\frac{N-1}{2}\cos\theta_N-\ell-\frac{N-1}{2}\right]\psi^{(a)}_{n\ell}(\theta_N)\\ &+\frac{N-1}{N-2}(n+\frac{N}{2})\sin^2\theta_N\phi^{(a)}_{n\ell}(\theta_N)\Bigg\},\end{aligned} \tag{4.10}$$

while the functions $D^{(\uparrow)(a)}_{n\ell}$ and $D^{(\downarrow)(a)}_{n\ell}$ are given by:

$$D^{(\uparrow)(a)}_{n\ell}(\theta_N)=\frac{-i}{N-1}\left[-\left(\ell+\frac{N-1}{2}\right)C^{(\uparrow)(a)}_{n\ell}(\theta_N)+\sin\theta_N\,\phi^{(a)}_{n\ell}(\theta_N)\right] \tag{4.11}$$

and

$$D^{(\downarrow)(a)}_{n\ell}(\theta_N)=\frac{-i}{N-1}\left[-\left(\ell+\frac{N-1}{2}\right)C^{(\downarrow)(a)}_{n\ell}(\theta_N)-\sin\theta_N\,\psi^{(a)}_{n\ell}(\theta_N)\right], \tag{4.12}$$

respectively. The appearance of ℓ in the denominator in eqs. (4.9) and (4.10) reflects the fact that there is no type-I eigenmode if the θ_N-component (4.5) [or (4.7)] has $\ell = 0$ (i.e. ℓ has to satisfy $\ell \geq r = 1$). The condition $n \geq \ell$ and the quantisation of the eigenvalue in eq. (4.1) follow from the requirement of regularity of the functions $\phi^{(a)}_{n\ell}(\theta_N)$ and $\psi^{(a)}_{n\ell}(\theta_N)$ (see Appendix E). Thus, we have verified that the allowed values for the angular momentum quantum numbers are $n = 1, 2, ...$ and $\ell = 1, ..., n$.

Type-*II* modes. The vector components of the type-II modes are expressed as [18]

$$\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\mu} = \left(0, \psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j}\right), \tag{4.13}$$

($j = 1, ..., N-1$) where $\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N} = 0$. The type-$II$ modes (4.13) are TT vector-spinors on S^{N-1}. Thus, they can be constructed in terms of TT vector-spinor eigenmodes $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}(\boldsymbol{\theta}_{N-1})$ on S^{N-1} that satisfy

$$\tilde{\nabla\!\!\!/}\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j} = \pm i\left(\ell + \frac{N-1}{2}\right)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j} \tag{4.14}$$

$$\tilde{\gamma}^{\theta_i}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_i} = \tilde{\nabla}^{\theta_i}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_i} = 0, \tag{4.15}$$

where the label $\tilde{A}$ indicates the type of the eigenmode $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$. (The TT vector-spinor eigenmodes and the corresponding types of modes on odd-dimensional spheres are presented in Subsection 4.2.) The requirement for regularity of $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ on S^{N-1} gives the allowed values for ℓ, i.e. $\ell = 1, 2, ...$. This requirement for ℓ follows naturally from the recursive construction of the STSSH's of rank 1 in Ref. [18]. We suppose that the eigenmodes $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ are normalised on S^{N-1} as

$$\int_{S^{N-1}} \sqrt{\tilde{g}}\, d\boldsymbol{\theta}_{N-1}\, \tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_i}(\boldsymbol{\theta}_{N-1})^\dagger\, \tilde{\psi}^{(\tilde{A}';\ell'\tilde{\rho}')\theta_i}_{\pm}(\boldsymbol{\theta}_{N-1}) = \delta_{\ell\ell'}\delta_{\tilde{\rho}\tilde{\rho}'}\delta_{\tilde{A}\tilde{A}'}, \tag{4.16}$$

where $\sqrt{\tilde{g}}$ is given by eq. (2.24). Any two TT eigenmodes, $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\sigma\theta_i}$ and $\tilde{\psi}^{(\tilde{A}';\ell'\tilde{\rho}')}_{\sigma'\theta_i}$ ($\sigma, \sigma' = \pm$), with different signs for the eigenvalue and/or with different labels are orthogonal to each other since they are eigenmodes of the hermitian operator $i\tilde{\nabla\!\!\!/}$. The type-II modes $\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\mu}$ on S^N with negative ($\sigma = -$) and positive ($\sigma = +$) spin projections are given by [18]

$$\begin{aligned}
\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) &= 0 \\
\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_j}(\theta_N, \boldsymbol{\theta}_{N-1}) &= \begin{pmatrix} \phi^{(-1)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \\ \pm i\psi^{(-1)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) &= 0 \\
\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_j}(\theta_N, \boldsymbol{\theta}_{N-1}) &= \begin{pmatrix} i\psi^{(-1)}_{n\ell}(\theta_N)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}(\boldsymbol{\theta}_{N-1}) \\ \pm\phi^{(-1)}_{n\ell}(\theta_N)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}(\boldsymbol{\theta}_{N-1}) \end{pmatrix},
\end{aligned} \tag{4.18}$$

($j=1,...,N-1$) respectively. The functions $\phi^{(-1)}_{n\ell}$ and $\psi^{(-1)}_{n\ell}$ are given by eqs. (3.1) and (3.2), respectively. As in the case of type-I modes, we find the allowed values $n=1,2,...$ and $\ell=1,...,n$.

4.2 STSSH's of rank 1 for N odd

The eigenvalue equation and the TT conditions are given again by eqs. (4.1) and (4.2), respectively, while the gamma matrices are now given by eq. (2.14). The TT eigenmodes on S^N are denoted as $\psi^{(A;n\ell;\tilde{\rho})}_{\pm\mu}$. As in the even-dimensional case, the label A denotes the type of the mode, where the type-I modes ($\psi^{(I;n\ell;\tilde{\rho})}_{\pm\mu}$) on S^N are constructed in terms of eigenspinors on S^{N-1}, while the type-II modes ($\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\mu}$) on S^N are constructed in terms of TT eigenvector-spinors of type-$\tilde{A}$ on S^{N-1}. The allowed values for the angular momentum quantum numbers are $n=1,2,...$ and $\ell=1,...,n$. However, for N odd there is no spin projection index on S^N.[7] Here, the label $\tilde{\rho}$ stands for angular momentum quantum numbers on all lower-dimensional spheres $S^{N-2},...,S^2,S^1$ and spin projection indices on the even-dimensional spheres $S^{N-1},S^{N-3},...,S^2$. Note that TT eigenvector-spinor modes of any type on S^N (with arbitrary N) exist only for $N\geq 3$, while type-II modes exist only for $N\geq 4$ - see Ref. [18].

For each value of n we have a representation of spin$(N+1)$ acting on the space of the eigenmodes $\psi^{(A;n\ell;\tilde{\rho})}_{\pm\mu}$. The highest weights $\lambda^{\pm}=(\lambda^{\pm}_1,...,\lambda^{\pm}_{(N+1)/2})$ for these representations are given by

$$\lambda^{\pm}=\left(n+\frac{1}{2},\frac{3}{2},\frac{1}{2},...,\frac{1}{2},\pm\frac{1}{2}\right),\qquad (n=1,2,...) \tag{4.19}$$

which can be determined using the branching rules for spin$(N+1)\supset$ spin(N). [8] Unlike the case with N even, for N odd there does not exist any spinorial matrix that relates $\psi^{(A;n\ell;\tilde{\rho})}_{+\mu}$ and $\psi^{(A;n\ell;\tilde{\rho})}_{-\mu}$, since the two sets of modes form inequivalent representations of spin$(N+1)$. (In general, for N odd there does not exist any spinorial matrix that relates two STSSH's of arbitrary rank r with different sign for the eigenvalue.) Note that for $N=3$ we have $\lambda^{\pm}=(n+1/2,\ \pm 3/2)$ ($n=1,2,...$).

Type-*I* modes. The type-I modes are given by [18]

$$\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1})=\frac{1}{\sqrt{2}}(1+i\gamma^N)\left\{\phi^{(1)}_{n\ell}(\theta_N)\pm i\psi^{(1)}_{n\ell}(\theta_N)\gamma^N\right\}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \tag{4.20}$$

$$\begin{aligned}\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1})=\frac{1}{\sqrt{2}}(1+i\gamma^N)\Bigg\{&\left(C^{(\uparrow)(1)}_{n\ell}(\theta_N)\pm iC^{(\downarrow)(1)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\&+\left(D^{(\uparrow)(1)}_{n\ell}(\theta_N)\pm iD^{(\downarrow)(1)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\Bigg\},\end{aligned} \tag{4.21}$$

[7] See also Ref. [22].

[8] The branching rules for spin$(N+1)\supset$ spin(N) with N odd are different from the branching rules with N even. See, e.g., Refs. [21, 25, 26], as well as Appendix A.

($j = 1, ..., N-1$) where $\chi_{-\ell\tilde{\rho}}$ are the eigenspinors on S^{N-1} satisfying eq. (2.22). (Since γ^N anti-commutes with $\tilde{\nabla\!\!\!/}$ we have $\gamma^N \chi_{-\ell\tilde{\rho}} = \chi_{+\ell\tilde{\rho}}$ [21].) As in the case with N even, the functions $\phi^{(1)}_{n\ell}$ and $\psi^{(1)}_{n\ell}$ are given by eqs. (3.1) and (3.2), respectively, while the functions $C^{(\uparrow)(1)}_{n\ell}, C^{(\downarrow)(1)}_{n\ell}, D^{(\uparrow)(1)}_{n\ell}$ and $D^{(\downarrow)(1)}_{n\ell}$ are given by eqs. (4.9), (4.10), (4.11) and (4.12), respectively. As in the even-dimensional case, one finds that the angular momentum quantum numbers are allowed to take the values $n = 1, 2, ...$ and $\ell = 1, ..., n$.

Type-*II* modes. The type-II modes are given by [18]

$$\begin{aligned}
\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) &= 0 \\
\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_j}(\theta_N, \boldsymbol{\theta}_{N-1}) &= \frac{1}{\sqrt{2}}(1 + i\gamma^N)\left\{\phi^{(-1)}_{n\ell}(\theta_N) \pm i\psi^{(-1)}_{n\ell}(\theta_N)\gamma^N\right\}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}),
\end{aligned} \tag{4.22}$$

where the functions $\phi^{(-1)}_{n\ell}$ and $\psi^{(-1)}_{n\ell}$ are given by eqs. (3.1) and (3.2), respectively, while the rank-1 STSSH's of type-$\tilde{A}$ on S^{N-1}, $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}$, satisfy eqs. (4.14)-(4.16) (where $\gamma^N \tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j} = \tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}$). As in the case with N even, we find that the angular momentum quantum numbers are allowed to take the values: $n = 1, 2, ...$ and $\ell = 1, ..., n$.

5 The STSSH's of rank 2 on the N-sphere

In this Section we write down explicitly the STSSH's of rank 2 on S^N by using the method of separation of variables. In this method the STSSH's of rank 2 on S^N are expressed in terms of STSSH's of rank $\tilde{r}$ (where $\tilde{r} \leq r$) on S^{N-1}. (The 0th rank STSSH's are the eigenspinors of the Dirac operator constructed in Ref. [21].) We present the details of the calculations in Appendix E.

5.1 STSSH's of rank 2 for N even

The equations for the STSSH's of rank 2 are given by:

$$\nabla\!\!\!/ \psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu\nu} = \pm i|\zeta_{n,N}|\, \psi^{(B;\sigma;\ell_N\ell;\tilde{\rho})}_{\pm\mu\nu}, \tag{5.1}$$

$$\nabla^\alpha \psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha\nu} = \gamma^\alpha \psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha\nu} = 0, \tag{5.2}$$

$$g^{\alpha\beta}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha\beta} = 0, \tag{5.3}$$

[see eqs. (1.6) and (1.7)] where the labels $\sigma, n, \ell, \tilde{\rho}$ have the same meaning as in the case of STSSH's of rank 1 [see the passage after eqs. (4.1) and (4.2)]. Note that eq. (5.3) arises just by contracting the gamma-tracelessness condition in (5.2) with γ^ν. As demonstrated in Appendix E, by requiring our eigenmodes to be non-singular, we find the quantisation condition for the eigenvalue in eq. (5.1),

$$|\zeta_{n,N}| = n + \frac{N}{2}, \qquad n \in \mathbb{N}_0, \tag{5.4}$$

($\mathbb{N}_0$ is the set of positive integers including zero), while the allowed values for the angular momentum quantum numbers are found to be $n = 2, 3, ...$ and $\ell = 2, ..., n$. As we will discuss below, eqs. (5.1)-(5.3) have three different types of mode solutions, namely, the

type-*I* modes, the **type-*II* modes** and the **type-*III* modes**. The label B is used in order to indicate the type of the STSSH $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu\nu}$ on S^N.

For each value of n we have a representation of spin$(N+1)$ acting on the space of the eigenmodes $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{+\mu\nu}$ (or $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{-\mu\nu}$). The highest weight $\lambda = (\lambda_1, ..., \lambda_{N/2})$ for this representation is given by

$$\lambda = \left(n+\frac{1}{2}, \frac{5}{2}, \frac{1}{2}, ..., \frac{1}{2}\right), \qquad (n = 2, 3, ...), \tag{5.5}$$

which can be determined using the branching rules for spin$(N+1) \supset$ spin(N) [21, 25, 26]. Note that for $N = 4$ we have $\lambda = (n + 1/2, 5/2)$. As in the case of STSSH's of rank 1, the two sets of eigenmodes, $\{\, \psi^{(B;\sigma;n\ell;\tilde{\rho})}_{+\mu\nu} \,\}$ and $\{\, \psi^{(B;\sigma;n\ell;\tilde{\rho})}_{-\mu\nu} \,\}$, form equivalent representations and they are related to each other by $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{+\mu\nu} = \gamma^{N+1}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{-\mu\nu}$.

In analogy with the rank-1 STSSH's discussed in Section 4, the rank-2 type-*I* modes are constructed using the eigenspinors $\chi_{\pm\ell\tilde{\rho}}$ on S^{N-1} [eq. (2.22)], while the type-*II* modes are constructed using the TT eigenvector-spinors $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_i}$ on S^{N-1} [eqs. (4.14) and (4.15)]. The rank-2 type-*III* modes are constructed using the STSSH's of rank 2 on S^{N-1} ($\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}$), satisfying

$$\tilde{\nabla\!\!\!/}\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j} = \pm i\left(\ell + \frac{N-1}{2}\right)\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j} \tag{5.6}$$

$$\tilde{\gamma}^{\theta_i}\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j} = \tilde{\nabla}^{\theta_i}\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j} = 0, \tag{5.7}$$

$$\tilde{g}^{\theta_i\theta_j}\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j} = 0, \tag{5.8}$$

where the label $\tilde{B}$ indicates the type of the STSSH $\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}$ on S^{N-1}. (The rank-2 STSSH's on odd-dimensional spheres are presented in Subsection 5.2.) We require $\ell = 2, 3, ...$ in order for $\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}$ to be non-singular on S^{N-1}. This requirement for ℓ is motivated naturally in the recursive construction of the STSSH's of rank 2 in Appendix E. We suppose that the STSSH's on S^{N-1}, $\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}$, are normalised as

$$\int_{S^{N-1}} \sqrt{\tilde{g}}\, d\boldsymbol{\theta}_{N-1}\, \tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}(\boldsymbol{\theta}_{N-1})^{\dagger}\, \tilde{\psi}^{(\tilde{B}';\ell'\tilde{\rho}')\theta_i\theta_j}_{\pm}(\boldsymbol{\theta}_{N-1}) = \delta_{\ell\ell'}\delta_{\tilde{\rho}\tilde{\rho}'}\delta_{\tilde{B}\tilde{B}'}, \tag{5.9}$$

where all the $\tilde{\psi}_{+\theta_i\theta_j}$ modes are orthogonal to all the $\tilde{\psi}_{-\theta_i\theta_j}$ modes (see also the passage after eq. (4.16)). Now let us present the explicit form of the STSSH's of rank 2 on S^N (see Appendix E for the derivation).

Type-*I* modes. The type-*I* modes with negative spin projection ($\sigma = -$) on S^N are given by

$$\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) = \begin{pmatrix} \phi^{(2)}_{n\ell}(\theta_N)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \pm i\psi^{(2)}_{n\ell}(\theta_N)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \tag{5.10}$$

$$\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N, \boldsymbol{\theta}_{N-1}) = \begin{pmatrix} C^{(\uparrow)(2)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) + D^{(\uparrow)(2)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \pm iC^{(\downarrow)(2)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \pm iD^{(\downarrow)(2)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \tag{5.11}$$

$$
\begin{aligned}
&\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) \\
&\quad = \begin{pmatrix} K^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{g}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \pm i K^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{g}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \\
&\qquad + \begin{pmatrix} W^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{H}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) + T^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{H}'_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \pm i W^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{H}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \pm i T^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{H}'_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}, \qquad (5.12)
\end{aligned}
$$

$(j,k=1,...,N-1)$ where $\chi_{\pm\ell\tilde{\rho}}$ are the eigenspinors on S^{N-1} [see eq. (5.8)] and we have defined

$$
\tilde{H}_{\theta_j\theta_k} \equiv \tilde{\nabla}_{(\theta_j}\tilde{\nabla}_{\theta_k)} - \tilde{g}_{\theta_j\theta_k}\frac{\tilde{\Box}}{N-1}, \qquad (5.13)
$$

$$
\tilde{H}'_{\theta_j\theta_k} \equiv \tilde{\gamma}_{(\theta_j}\tilde{\nabla}_{\theta_k)} - \tilde{g}_{\theta_j\theta_k}\frac{\tilde{\nabla\!\!\!/}}{N-1}. \qquad (5.14)
$$

These differential operators satisfy $\tilde{g}^{\theta_j\theta_k}\tilde{H}_{\theta_j\theta_k} = \tilde{g}^{\theta_j\theta_k}\tilde{H}'_{\theta_j\theta_k} = 0$. Note that $\tilde{\nabla\!\!\!/}\chi_{\pm\ell\tilde{\rho}} = \pm i\left(\ell+\frac{N-1}{2}\right)\chi_{\pm\ell\tilde{\rho}}$ [eq. (2.22)], while $\tilde{\Box}\chi_{\pm\ell\tilde{\rho}} \equiv \tilde{\nabla}^{\theta_k}\tilde{\nabla}_{\theta_k}\chi_{\pm\ell\tilde{\rho}}$ is given by eq. (D.7). The function $\phi^{(2)}_{n\ell}$ is given by eq. (3.1), the function $\psi^{(2)}_{n\ell}$ is given by eq. (3.2), the functions $C^{(\uparrow)(2)}_{n\ell}$ and $C^{(\downarrow)(2)}_{n\ell}$ are given by eqs. (4.9) and (4.10), respectively, while the functions $D^{(\uparrow)(2)}_{n\ell}$ and $D^{(\downarrow)(2)}_{n\ell}$ are given by eqs. (4.11) and (4.12), respectively. The functions describing the dependence on θ_N in eq. (5.12) are given by

$$
K^{(\uparrow)}_{n\ell}(\theta_N) = -\frac{\sin^2\theta_N}{N-1}\phi^{(2)}_{n\ell}(\theta_N), \qquad (5.15)
$$

$$
K^{(\downarrow)}_{n\ell}(\theta_N) = -\frac{\sin^2\theta_N}{N-1}\psi^{(2)}_{n\ell}(\theta_N), \qquad (5.16)
$$

$$
T^{(\uparrow)}_{n\ell}(\theta_N) = \frac{-2i}{N+1}\left\{\sin\theta_N\,C^{(\uparrow)(2)}_{n\ell}(\theta_N) - \left(\ell+\frac{N-1}{2}\right)W^{(\uparrow)}_{n\ell}(\theta_N)\right\}, \qquad (5.17)
$$

$$
T^{(\downarrow)}_{n\ell}(\theta_N) = \frac{-2i}{N+1}\left\{-\sin\theta_N\,C^{(\downarrow)(2)}_{n\ell}(\theta_N) - \left(\ell+\frac{N-1}{2}\right)W^{(\downarrow)}_{n\ell}(\theta_N)\right\}, \qquad (5.18)
$$

$$
\begin{aligned}
W^{(\uparrow)}_{n\ell}(\theta_N) = &\frac{\sin\theta_N}{(\ell-1)(\ell+N)(N-1)} \\
&\times\Bigg\{\left[\frac{N(N-3)\left(\ell+\frac{N-1}{2}\right)}{N-1}+\frac{N(N+1)}{2}\cos\theta_N\right]C^{(\uparrow)(2)}_{n\ell}(\theta_N) \\
&- (n+\frac{N}{2})(N+1)\sin\theta_N\,C^{(\downarrow)(2)}_{n\ell}(\theta_N) + \frac{N+1}{N-1}\sin\theta_N\,\phi^{(2)}_{n\ell}(\theta_N)\Bigg\} \qquad (5.19)
\end{aligned}
$$

and

$$W^{(\downarrow)}_{n\ell}(\theta_N) = \frac{\sin\theta_N}{(\ell-1)(\ell+N)(N-1)}$$
$$\times\Bigg\{\Big[-\frac{N(N-3)\left(\ell+\frac{N-1}{2}\right)}{N-1}+\frac{N(N+1)}{2}\cos\theta_N\Big]\, C^{(\downarrow)(2)}_{n\ell}(\theta_N)$$
$$+\,(n+\frac{N}{2})(N+1)\sin\theta_N\, C^{(\uparrow)(2)}_{n\ell}(\theta_N)+\frac{N+1}{N-1}\sin\theta_N\,\psi^{(2)}_{n\ell}(\theta_N)\Bigg\}. \tag{5.20}$$

The type-I modes with positive spin projection, $\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\mu\nu}$, are given by expressions analogous to the expressions for $\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\mu\nu}$. To be specific, the expression for $\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}$ is found by exchanging $\phi^{(2)}_{n\ell}$ and $i\psi^{(2)}_{n\ell}$ and replacing $\chi_{-\ell\tilde{\rho}}$ by $\chi_{+\ell\tilde{\rho}}$ in eq. (5.10) and the expression for the component $\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}$ is found using eq. (5.11) as follows: we exchange $C^{(\uparrow)(2)}_{n\ell}$ and $iC^{(\downarrow)(2)}_{n\ell}$; we also exchange $D^{(\uparrow)(2)}_{n\ell}$ and $iD^{(\downarrow)(2)}_{n\ell}$ and we make the replacements $\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}} \to \tilde{\nabla}_{\theta_j}\chi_{+\ell\tilde{\rho}}$ and $\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}} \to -\tilde{\gamma}_{\theta_j}\chi_{+\ell\tilde{\rho}}$. Similarly, $\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ is found using the expression for $\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ [eq. (5.12)] as follows: we exchange the functions with superscript '(↑)' and the functions with superscript '(↓)', i.e., $K^{(\uparrow)}_{n\ell} \leftrightarrow iK^{(\downarrow)}_{n\ell}$, $W^{(\uparrow)}_{n\ell} \leftrightarrow iW^{(\downarrow)}_{n\ell}$ and $T^{(\uparrow)}_{n\ell} \leftrightarrow iT^{(\downarrow)}_{n\ell}$ (the symbol $\leftrightarrow$ denotes the exchange of the functions appearing in the two sides of the 'left-right' arrow) and we also make the replacements $\chi_{-\ell\tilde{\rho}} \to \chi_{+\ell\tilde{\rho}}$ and $\tilde{H}'_{\theta_j\theta_k} \to -\tilde{H}'_{\theta_j\theta_k}$ in eq. (5.12).

Let us now verify that the allowed values for the angular momentum quantum numbers n and ℓ for the type-I modes satisfy $n \geq \ell \geq r = 2$. As in the case of STSSH's of rank 1 (see Subsection 4.1), the appearance of ℓ in the denominator in eqs. (4.9) and (4.10) implies that there is no type-I mode if the $\theta_N\theta_N$-component (5.10) has $\ell = 0$. Similarly, as eqs. (5.19) and (5.20) indicate, there is no type-I mode with $\theta_N\theta_N$-component given by eq. (5.10) with $\ell = 1$. Also, as demonstrated in Appendix E, the quantisation condition (5.4) for the eigenvalue, as well as the condition $n-\ell \geq 0$, arise as the requirement for the absence of singularity in the functions $\phi^{(2)}_{n\ell}$ and $\psi^{(2)}_{n\ell}$. Thus, the allowed values for n and ℓ are $n = 2, 3, ...$ and $\ell = 2, ..., n$, respectively.

Type-*II* modes. The type-II modes with negative spin projection ($\sigma = -$) on S^N are given by

$$\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) = 0 \tag{5.21}$$

$$\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) = \begin{pmatrix} \phi^{(0)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \\ \pm i\psi^{(0)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \tag{5.22}$$

$$\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = \begin{pmatrix} \Gamma^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) + \Delta^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) \\ \\ \pm i\Gamma^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) \pm i\Delta^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \tag{5.23}$$

($j,k = 1,...,N-1$), where $\phi^{(0)}_{n\ell}$ is given by eq. (3.1) and $\psi^{(0)}_{n\ell}$ is given by eq. (3.2). The type-$\tilde{A}$ TT vector-spinor eigenmodes $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_k}$ on S^{N-1} satisfy eqs. (4.14)-(4.16) and they are non-singular on S^{N-1} for $\ell = 1, 2, ...$ (see Section 4). The functions describing the dependence on θ_N in eq. (5.23) are given by

$$\frac{\Delta^{(\uparrow)}_{n\ell}(\theta_N)}{2} = \frac{-i}{N+1}\left[-\frac{\ell+\frac{N-1}{2}}{2}\Gamma^{(\uparrow)}_{n\ell}(\theta_N) + \sin\theta_N\,\phi^{(0)}_{n\ell}(\theta_N)\right], \tag{5.24}$$

$$\frac{\Delta^{(\downarrow)}_{n\ell}(\theta_N)}{2} = \frac{-i}{N+1}\left[-\frac{\ell+\frac{N-1}{2}}{2}\Gamma^{(\downarrow)}_{n\ell}(\theta_N) - \sin\theta_N\,\psi^{(0)}_{n\ell}(\theta_N)\right] \tag{5.25}$$

and

$$\begin{aligned}\frac{\Gamma^{(\uparrow)}_{n\ell}(\theta_N)}{2} = \frac{1}{(\ell-1)(\ell+N)}\Bigg\{&\sin\theta_N\left[\frac{N+1}{2}\cos\theta_N + \ell + \frac{N-1}{2}\right]\phi^{(0)}_{n\ell}(\theta_N)\\ &-\frac{N+1}{N}(n+\frac{N}{2})\sin^2\theta_N\psi^{(0)}_{n\ell}(\theta_N)\Bigg\},\end{aligned} \tag{5.26}$$

$$\begin{aligned}\frac{\Gamma^{(\downarrow)}_{n\ell}(\theta_N)}{2} = \frac{1}{(\ell-1)(\ell+N)}\Bigg\{&\sin\theta_N\left[\frac{N+1}{2}\cos\theta_N - \ell - \frac{N-1}{2}\right]\psi^{(0)}_{n\ell}(\theta_N)\\ &+\frac{N+1}{N}(n+\frac{N}{2})\sin^2\theta_N\phi^{(0)}_{n\ell}(\theta_N)\Bigg\}.\end{aligned} \tag{5.27}$$

The expressions for the type-*II* modes with positive spin projection, $\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\mu\nu}$, are analogous to the expressions for $\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\mu\nu}$ presented above. More specifically, the expression for $\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}$ is found by exchanging $\phi^{(0)}_{n\ell}$ and $i\psi^{(0)}_{n\ell}$ and making the replacement $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j} \to \tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}$ in eq. (5.22). The steps required in order to find the expression for $\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ by using eq. (5.23) are: we exchange $\Gamma^{(\uparrow)}_{n\ell}$ and $i\Gamma^{(\downarrow)}_{n\ell}$, as well as $\Delta^{(\uparrow)}_{n\ell}$ and $i\Delta^{(\downarrow)}_{n\ell}$, and we make the replacements $\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)} \to \tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}$ and $\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)} \to -\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}$ in eq. (5.23).

Let us now verify that the allowed values for the angular momentum quantum numbers n and ℓ for the type-*II* modes satisfy $n \geq \ell \geq r = 2$. As mentioned in Section 4, the eigenvector-spinors on S^{N-1} ($\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}$) are non-singular for $\ell \geq 1$. Also, since $\ell - 1$ appears in the denominator in eqs. (5.26) and (5.27), there is no type-*II* mode with $\theta_N\theta_j$-component given by eq. (5.22) with $\ell = 1$. As in the case of the type-*I* modes, the quantisation condition (5.4) and the condition $n - \ell \geq 0$ arise by demanding $\phi^{(0)}_{n\ell}$ and $\psi^{(0)}_{n\ell}$ to be non-singular. Hence, the allowed values for the angular momentum quantum numbers are $n = 2, 3, ...$ and $\ell = 2, ..., n$.

Type-*III* modes. The type-*III* modes with negative ($\sigma = -$) and positive ($\sigma = +$) spin projections on S^N are given by

$$\psi^{(III\text{-}\tilde{B};-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) = 0 \tag{5.28}$$

$$\psi^{(III\text{-}\tilde{B};-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) = 0 \tag{5.29}$$

$$\psi^{(III\text{-}\tilde{B};-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = \begin{pmatrix} \phi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \\ \pm i\psi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \tag{5.30}$$

and

$$\psi^{(III\text{-}\tilde{B};+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) = 0 \tag{5.31}$$

$$\psi^{(III\text{-}\tilde{B};+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) = 0 \tag{5.32}$$

$$\psi^{(III\text{-}\tilde{B};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = \begin{pmatrix} i\psi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \\ \pm\phi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}, \tag{5.33}$$

$(j,k=1,...,N-1)$ respectively, where $\phi^{(-2)}_{n\ell}$ is given by eq. (3.1) and $\psi^{(-2)}_{n\ell}$ is given by eq. (3.2). The STSSH's of rank 2 on S^{N-1}, $\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}$, satisfy eqs. (5.6)-(5.9) and they are non-singular for $\ell = 2,3,...$ (see the next Subsection). By working as in the case of type-I and type-II modes discussed above, we find again that the allowed values for the angular momentum quantum numbers are $n = 2,3,...$ and $\ell = 2,...,n$.

5.2 STSSH's of rank 2 for N odd

The equations for the STSSH's of rank 2 are given by eqs. (5.1)-(5.3), where the gamma matrices are given by eq. (2.14). We denote the STSSH's of rank 2 as $\psi^{(B;n\ell;\tilde{\rho})}_{\pm\mu\nu}$ (with $n = 2,...$ and $\ell = 2,...,n$), where the label B denotes the type of the mode. Note that for N odd there is no spin projection index on S^N [see also the passage before eq. (4.19)]. The labels n,ℓ and $\tilde{\rho}$ have the same meaning as in the case of the STSSH's of rank 1 in Subsection 4.2.

For each value of n we have a representation of spin$(N+1)$ acting on the space of the eigenmodes $\psi^{(B;n\ell;\tilde{\rho})}_{\pm\mu\nu}$. The highest weights $\lambda^{\pm} = (\lambda^{\pm}_1,...,\lambda^{\pm}_{(N+1)/2})$ for these representations are

$$\lambda^{\pm} = \left(n+\frac{1}{2},\frac{5}{2},\frac{1}{2},...,\frac{1}{2},\pm\frac{1}{2}\right), \qquad (n = 2,3,...), \tag{5.34}$$

which can be determined using the branching rules for spin$(N+1) \supset$ spin(N) [21, 25, 26]. Note that for $N = 3$ we have $\lambda^{\pm} = (n+1/2, \pm 5/2)$.

Type-*I* modes. The type-I modes on S^N are given by

$$\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) =\frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\Big\{\phi^{(2)}_{n\ell}(\theta_N)\pm i\psi^{(2)}_{n\ell}(\theta_N)\gamma^N\Big\}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \tag{5.35}$$

$$\begin{aligned}\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) =&\frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\Bigg\{\left(C^{(\uparrow)(2)}_{n\ell}(\theta_N)\pm iC^{(\downarrow)(2)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ &+\left(D^{(\uparrow)(2)}_{n\ell}(\theta_N)\pm iD^{(\downarrow)(2)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\Bigg\}\end{aligned} \tag{5.36}$$

$$\begin{aligned}\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =&\frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\Bigg\{\left(K^{(\uparrow)}_{n\ell}(\theta_N)\pm iK^{(\downarrow)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{g}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ &+\left(W^{(\uparrow)}_{n\ell}(\theta_N)\pm iW^{(\downarrow)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{H}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ &+\left(T^{(\uparrow)}_{n\ell}(\theta_N)\pm iT^{(\downarrow)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{H}'_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\Bigg\}\end{aligned} \tag{5.37}$$

($j,k=1,...,N-1$) where the eigenspinors $\chi_{-\ell\tilde{\rho}}$ on S^{N-1} satisfy eq. (2.22). The functions $\phi^{(2)}_{n\ell},\psi^{(2)}_{n\ell},C^{(b)(2)}_{n\ell},D^{(b)(2)}_{n\ell},K^{(b)}_{n\ell},W^{(b)}_{n\ell}$ and $T^{(b)}_{n\ell}$ (where $b=\uparrow,\downarrow$), describing the dependence on θ_N, are the same as in the even-dimensional case [see eqs. (5.10)-(5.12)], while $\tilde{H}_{\theta_j\theta_k}$ and $\tilde{H}'_{\theta_j\theta_k}$ are given again by eqs. (5.13) and (5.14), respectively.

Type-*II* modes. The type-II modes on S^N are given by

$$\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1})=0 \tag{5.38}$$

$$\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1})=\frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\Big\{\phi^{(0)}_{n\ell}(\theta_N)\pm i\psi^{(0)}_{n\ell}(\theta_N)\gamma^N\Big\}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \tag{5.39}$$

$$\begin{aligned}\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1})=&\frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\Bigg\{\left(\Gamma^{(\uparrow)}_{n\ell}(\theta_N)\pm i\Gamma^{(\downarrow)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1})\\ &+\left(\Delta^{(\uparrow)}_{n\ell}(\theta_N)\pm i\Delta^{(\downarrow)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1})\Bigg\},\end{aligned} \tag{5.40}$$

($j,k=1,...,N-1$) where the TT eigenvector-spinors $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k}$ on S^{N-1} satisfy eqs. (4.14)-(4.16). As in the even-dimensional case, the functions $\phi^{(0)}_{n\ell}$ and $\psi^{(0)}_{n\ell}$ are given by eqs. (3.1) and (3.2), respectively. The functions $\Delta^{(\uparrow)}_{n\ell},\Delta^{(\downarrow)}_{n\ell},\Gamma^{(\uparrow)}_{n\ell}$ and $\Gamma^{(\downarrow)}_{n\ell}$ are given by eqs. (5.24) , (5.25), (5.26) and (5.27), respectively.

Type-*III* modes. The type-III modes on S^N are given by

$$\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1})=0 \tag{5.41}$$

$$\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1})=0 \tag{5.42}$$

$$\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1})=\frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\Big\{\phi^{(-2)}_{n\ell}(\theta_N)\pm i\psi^{(-2)}_{n\ell}(\theta_N)\gamma^N\Big\}\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}), \tag{5.43}$$

($j,k=1,...,N-1$) where the rank-2 STSSH's on S^{N-1} ($\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}$) satisfy eqs. (5.6)-(5.9), while the functions $\phi^{(-2)}_{n\ell}$ and $\psi^{(-2)}_{n\ell}$ are given by eqs. (3.1) and (3.2), respectively.

As in the case with N even, by requiring that the rank-2 STSSH's of all types (i.e. type-*I*, type-*II* and type-*III*) on S^N are non-singular, we obtain the quantisation condition (5.4) for the eigenvalue, while the allowed values for the angular momentum quantum numbers are found to be $n = 2, 3, ...$ and $\ell = 2, ..., n$.

6 Normalisation factors and transformation properties under spin($N+1$) of rank-1 and rank-2 STSSH's

In this Section, we study the transformation properties of a specific class of STSSH's of ranks 1 and 2 on S^N under a spin($N+1$) transformation. We also write down explicitly the normalisation factors for all STSSH's of ranks 1 and 2 and we make a conjecture for the normalisation factors for STSSH's of arbitrary rank r.

In order to derive the transformation formulae and determine the normalisation factors for STSSH's of ranks 1 and 2, we introduce an inner product on the solution space of eqs. (1.6) and (1.7) and we also exploit the spin($N+1$) invariance of this inner product. The transformation properties and the normalisation factors that we present in this Section have been obtained after long and tedious calculations. For this reason, in this Section, we simply present the results of our lengthy calculations and provide the necessary mathematical background (for example, we discuss the Lie-Lorentz derivative (6.1) [27]). We refer the reader to Appendix F for details of the calculations.

6.1 Lie-Lorentz derivative and spin($N+1$) invariant inner product

Let $\psi_{\mu_1...\mu_r}$ be any tensor-spinor of rank r and ξ be any Killing vector on S^N. The infinitesimal change $\delta_\xi \psi_{\mu_1...\mu_r}$ due to the spin($N+1$) transformation generated by ξ is conveniently described by the **Lie-Lorentz derivative** [27]

$$\begin{aligned} \mathbb{L}_\xi \; \psi_{\mu_1...\mu_r} = \; & \xi^\nu \nabla_\nu \psi_{\mu_1...\mu_r} + \psi_{\nu\mu_2...\mu_r} \nabla_{\mu_1}\xi^\nu + \psi_{\mu_1\nu\mu_3...\mu_r}\nabla_{\mu_2}\xi^\nu + ... + \psi_{\mu_1...\mu_{r-1}\nu}\nabla_{\mu_r}\xi^\nu \\ & + \frac{1}{4}\nabla_\kappa \xi_\lambda \gamma^\kappa \gamma^\lambda \psi_{\mu_1...\mu_r}. \end{aligned} \tag{6.1}$$

The Lie-Lorentz derivative satisfies [27]

$$\mathbb{L}_\xi \; {e_\mu}^a = 0, \tag{6.2a}$$

$$\mathbb{L}_\xi \; \gamma^a = 0 \tag{6.2b}$$

and - after a straightforward calculation - one can verify that

$$\left(\mathbb{L}_\xi \nabla_\mu - \nabla_\mu \mathbb{L}_\xi\right) \; \psi_{\mu_1...\mu_r} = 0. \tag{6.3}$$

Thus, if $\psi_{\mu_1...\mu_r}$ satisfies eqs. (1.6) and (1.7) (i.e., if $\psi_{\mu_1...\mu_r}$ is a STSSH of rank r), then $\mathbb{L}_\xi \; \psi_{\mu_1...\mu_r}$ also satisfies eqs. (1.6) and (1.7).

Let us introduce the following inner product on the solution space of eqs. (1.6) and (1.7):

$$\left(\psi^{(1)}, \psi^{(2)}\right)_{(r)} = \int_{S^N} \sqrt{g}\, d\boldsymbol{\theta}_N \, \psi^{(1)\dagger}_{\mu_1...\mu_r} \; \psi^{(2)\mu_1...\mu_r}, \tag{6.4}$$

where $d\boldsymbol{\theta}_N$ stands for $d\theta_N ... d\theta_2\, d\theta_1$, while $\psi^{(1)}_{\mu_1...\mu_r}$ and $\psi^{(2)}_{\mu_1...\mu_r}$ are any two STSSH's of rank r with the same angular momentum n on S^N.[9] Since the inner product (6.4) is invariant under spin$(N+1)$, we have

$$\left(\mathbb{L}_\xi \psi^{(1)}, \psi^{(2)}\right)_{(r)} + \left(\psi^{(1)}, \mathbb{L}_\xi \psi^{(2)}\right)_{(r)} = 0 \tag{6.5}$$

for any Killing vector ξ on S^N. We will study the transformation properties of a certain class of STSSH's of ranks 1 and 2 under spin$(N+1)$, by specialising to the case where the Killing vector in eq. (6.1) is given by $\xi = \mathscr{S}$, where

$$\mathscr{S} = \mathscr{S}^\mu \partial_\mu = \cos\theta_{N-1} \frac{\partial}{\partial\theta_N} - \cot\theta_N \, \sin\theta_{N-1} \frac{\partial}{\partial\theta_{N-1}}. \tag{6.6}$$

Now, let us discuss the certain class of STSSH's of ranks 1 and 2 on S^N $(N \geq 3)$, the transformation properties of which we are interested in.

- In the case of STSSH's of rank $r = 1$, we will study the transformation properties of the class of STSSH's which comprises: the type-*I* modes and a certain kind of type-*II* modes, called type-*II*-*I* modes. The type-*II*-*I* modes on S^N are defined for $N \geq 4$ and they are constructed in terms of type-*I* eigenvector-spinors on S^{N-1}. Thus, the type-*II*-*I* modes on S^N are given by letting $\tilde{A} = I$ in eqs. (4.17) and (4.18) (for N even) and in eq. (4.22) (for N odd).

- In the case of STSSH's of rank $r = 2$, we will study the class of STSSH's which comprises: the type-*I* modes, the type-*II*-*I* modes and the type-*III*-*I* modes. As in the case of rank-1 STSSH's, the type-*II*-*I* modes on S^N are defined for $N \geq 4$ and they are constructed in terms of type-*I* eigenvector-spinors on S^{N-1}. Thus, these modes are given by letting $\tilde{A} = I$ in eqs. (5.21)-(5.23) (for N even) and in eqs. (5.38)-(5.40) (for N odd). The type-*III*-*I* modes on S^N are defined for $N \geq 4$ and they are constructed in terms of type-*I* STSSH's of rank 2 on S^{N-1}. Thus, the type-*III*-*I* modes on S^N are given by letting $\tilde{B} = I$ in eqs. (5.30) and (5.33) (for N even) and in eq. (5.43) (for N odd).

6.2 Normalisation factors and transformation properties under spin$(N+1)$ of STSSH's of ranks 1 and 2

Case 1: N even. Using the inner product (6.4), we define the normalisation factors $c_N^{(B;r)}(n,\ell)$ for the STSSH's of arbitrary rank r and type B on S^N, $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1...\mu_r}$, as

$$\left(\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm}, \psi^{(B';\sigma';n\ell';\tilde{\rho}')}_{\pm}\right)_{(r)} \equiv \left|\frac{c_N^{(B;r)}(n,\ell)}{\sqrt{2}}\right|^{-2} \delta_{BB'}\delta_{\sigma\sigma'}\delta_{\ell\ell'}\delta_{\tilde{\rho}\tilde{\rho}'}. \tag{6.7}$$

(The normalised STSSH's are $c_N^{(B;r)}(n,\ell)/\sqrt{2}\; \psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1...\mu_r}$.) As discussed in Sections 4 and 5 (for $r=1$ and $r=2$, respectively), the STSSH's of rank r on S^N, $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1...\mu_r}$, are constructed

[9] Any two STSSH's with different signs for the eigenvalue in eq. (1.6) and/or with different n are orthogonal to each other, since $i\nabla\!\!\!/$ is hermitian with respect to the inner product (6.4).

in terms of STSSH's of rank $\tilde{r} \leq r$ on S^{N-1}, using the method of separation of variables. The type of the mode $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1\ldots\mu_r}$ (i.e. the value assigned to the label B) depends on the choice of $\tilde{r}$. For convenience, instead of using the symbol $\tilde{r}$, let us denote the rank of the STSSH's on S^{N-1} as $\tilde{r}_{(B)}$, where the type-I STSSH's ($\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1\ldots\mu_r}$) have $\tilde{r}_{(I)} = 0$, the type-II STSSH's ($\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1\ldots\mu_r}$) have $\tilde{r}_{(II)} = 1$, the type-III STSSH's ($\psi^{(III\text{-}\tilde{B};\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1\ldots\mu_r}$) have $\tilde{r}_{(III)} = 2$ and so forth. As shown in Appendix F, the normalisation factors for STSSH's of rank $r \in \{1, 2\}$ are given by

$$\left|\frac{c_N^{(B;r)}(n,\ell)}{\sqrt{2}}\right|^2 = \frac{2^{-N-2r+1+4\tilde{r}_{(B)}}}{\binom{r}{\tilde{r}_{(B)}}} \frac{\Gamma(n-\ell+1)\Gamma(n+\ell+N)}{|\Gamma(n+\frac{N}{2})|^2} \times \left(\prod_{j=\tilde{r}_{(B)}}^{r-1} \frac{N+j+\tilde{r}_{(B)}-2}{N+2j-1}\right)\left(\prod_{j=\tilde{r}_{(B)}}^{r-1}(\ell-j)(\ell+N-1+j)\right) \times \prod_{j=1}^{r-\tilde{r}_{(B)}} \frac{1}{\left(n+\frac{N}{2}\right)^2 - \left(r-j+\frac{N-2}{2}\right)^2} \tag{6.8}$$

($\tilde{r}_{(B)} \leq r$) where $\binom{r}{\tilde{r}_{(B)}}$ is the binomial coefficient. Here, if $\nu_1 > \nu_2$, then $\prod_{j=\nu_1}^{\nu_2} = 1$. We have proved eq. (6.8) only for $r = 1$ (where $B = I, II$) and for $r = 2$ (where $B = I, II, III$). We make the following conjecture, which is true for $r = 1$ and $r = 2$:

Conjecture: The normalisation factors for all types of STSSH's (i.e. STSSH's with all possible values of B) of arbitrary rank $r \geq 1$ on S^N are given by eq. (6.8), where $n \geq \ell \geq r \geq \tilde{r}_{(B)}$ and $\tilde{r}_{(B)} \in \{0, 1, ..., r\}$.

Before presenting the transformation properties of our STSSH's of rank r ($r = 1, 2$) under spin$(N+1)$, let us introduce the shorthand notation $\psi^{(B;\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_r}$ for the STSSH's of ranks 1 and 2, defined as follows:

$$\psi^{(B;\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_1} = \psi^{(B;\sigma;n\ell m;\rho)}_{\pm\mu_1} \quad (B = I,\ II\text{-}I), \tag{6.9a}$$

$$\psi^{(III\text{-}I;\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_1} = 0, \tag{6.9b}$$

$$\psi^{(B;\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_2} = \psi^{(B;\sigma;n\ell m;\rho)}_{\pm\mu_1\mu_2} \quad (B = I, II\text{-}I, III\text{-}I), \tag{6.9c}$$

where we have also written out explicitly the dependence on the angular momentum quantum number on S^{N-2}, m, which corresponds to ℓ on S^{N-1}. The symbol ρ represents labels other than σ, n, ℓ and m. For the type-I modes we have $m = 0, 1, ..., \ell$, for the type-II modes we have $m = 1, 2, ..., \ell$ and for the type-III modes we have $m = 2, 3, ..., \ell$. (In other words $\ell \geq m \geq \tilde{r}_{(B)}$.)

As demonstrated in Appendix F, the spin$(N+1)$ transformation of the type-I modes is expressed as

$$\mathbb{L}_{\mathscr{S}}\psi^{(I;\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_r} = \mathscr{A}^{(I)}\,\psi^{(I;\sigma;n\,(\ell+1)\,m;\rho)}_{\pm\boldsymbol{N}_r} + \mathscr{B}^{(I)}\,\psi^{(I;\sigma;n\,(\ell-1)\,m;\rho)}_{\pm\boldsymbol{N}_r} - i\varkappa^{(I)}\,\psi^{(I;-\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_r} + \mathscr{K}^{(I\to II)}\,\psi^{(II\text{-}I;\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_r}, \tag{6.10}$$

where the coefficients on the right-hand side of eq. (6.10) are

$$\mathscr{A}^{(I)} = -\frac{(n+\ell+N)(\ell+N+r-1)}{2(\ell+\frac{N}{2})(\ell+N-1)} \times \sqrt{(\ell-m+1)(\ell+N-1+m)}, \tag{6.11}$$

$$\mathscr{B}^{(I)} = \frac{(n-\ell+1)(\ell-r)}{2(\ell+\frac{N-2}{2})\ell} \times \sqrt{(\ell-m)(\ell+m+N-2)}, \tag{6.12}$$

$$\varkappa^{(I)} = -\frac{(n+\frac{N}{2})(m+\frac{N-2}{2})(N+2r-2)}{2(\ell+\frac{N-2}{2})(\ell+\frac{N}{2})(N-2)}, \tag{6.13}$$

and

$$\mathscr{K}^{(I\to II)} = -\frac{4\left[(n+\frac{N}{2})^2-(N-2)^2/4\right](N+r-2)}{\ell(\ell+N-1)(N-2)} \times \sqrt{\frac{N-3}{N-2}\frac{m(m+N-2)}{(\ell+1)(\ell+N-2)}}. \tag{6.14}$$

Equations (6.10)-(6.14) hold for $r=1,2$. Note that the sign of the spin projection index σ is flipped in the third term of the linear combination in eq. (6.10), while $i\varkappa^{(I)}$ is the only imaginary coefficient on the right-hand side of this equation. Also, note that $\mathscr{K}^{(I\to II)}$ vanishes for $m=0$, i.e. for $m=0$ there is no mixing between type-*I* and type-*II*-*I* modes in eq. (6.10). This is consistent with the fact that type-*II* modes are defined only for $m=1,2,...,\ell$.

The spin$(N+1)$ transformation of the type-*II*-*I* modes is expressed as

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi^{(II\text{-}I;\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_r} = {} & \mathscr{A}^{(II)}\,\psi^{(II\text{-}I;\sigma;n\,(\ell+1)\,m;\rho)}_{\pm\boldsymbol{N}_r} + \mathscr{B}^{(II)}\,\psi^{(II\text{-}I;\sigma;n\,(\ell-1)\,m;\rho)}_{\pm\boldsymbol{N}_r}\\ & - i\varkappa^{(II)}\psi^{(II\text{-}I;-\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_r}\\ & + \mathscr{K}^{(II\to I)}\,\psi^{(I;\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_r} + \mathscr{K}^{(II\to III)}\,\psi^{(III\text{-}I;\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_r}\end{aligned} \tag{6.15}$$

where

$$\begin{aligned}\mathscr{A}^{(II)} = {} & -\frac{(n+\ell+N)(\ell+N+r-1)}{2(\ell+\frac{N}{2})(\ell+N)}\\ & \times\sqrt{\frac{(\ell+2)(\ell+N-2)}{(\ell+1)(\ell+N-1)}(\ell-m+1)(\ell+m+N-1)},\end{aligned} \tag{6.16}$$

$$\mathscr{B}^{(II)} = \frac{(n-\ell+1)(\ell-r)}{2(\ell+\frac{N-2}{2})(\ell-1)} \times \sqrt{\frac{(\ell+1)(\ell+N-3)}{\ell(\ell+N-2)}(\ell-m)(\ell+m+N-2)}, \tag{6.17}$$

$$\varkappa^{(II)} = \frac{-(n+\frac{N}{2})(m+\frac{N-2}{2})(N-4)}{2(\ell+\frac{N-2}{2})(\ell+\frac{N}{2})(N-2)} \times \left(\frac{N+2}{N}\right)^{r-1} \tag{6.18}$$

$$\tag{6.19}$$

$$\mathscr{K}^{(II\to I)} = \frac{r}{4} \times \sqrt{\frac{(N-3)m(m+N-2)}{(N-2)(\ell+1)(\ell+N-2)}}, \tag{6.20}$$

where $r = 1, 2$ and

$$\mathscr{K}^{(II\to III)} = -\,2^3 \frac{\left[\left(n+\frac{N}{2}\right)^2 - N^2/4\right](N+1)}{(\ell-1)(\ell+N)N} \times \sqrt{\frac{N-2}{N}\,\frac{(m-1)(m+N-1)}{\ell(\ell+N-1)}} \tag{6.21}$$

[eq. (6.21) is defined only for $r = 2$]. The sign of the spin projection index is flipped in the third term of the linear combination in eq. (6.15), while $i\varkappa^{(II)}$ is the only imaginary coefficient on the right-hand side of this equation. Note that $\varkappa^{(II)}$ vanishes for $N = 4$ and thus type-*II-I* modes with different spin projections on S^4 do not mix with each other under the transformation (6.15).

The spin$(N+1)$ transformation of the rank-2 type-*III-I* modes is expressed as a linear combination of other STSSH's of rank 2, as follows:

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi^{(\textit{III-I};\sigma;n\ell m;\rho)}_{\pm\mu_1\mu_2} = {} & \mathscr{A}^{(III)}\,\psi^{(\textit{III-I};\sigma;n\,(\ell+1)\,m;\rho)}_{\pm\mu_1\mu_2} + \mathscr{B}^{(III)}\,\psi^{(\textit{III-I};\sigma;n\,(\ell-1)\,m;\rho)}_{\pm\mu_1\mu_2} \\ & - i\varkappa^{(III)}\,\psi^{(\textit{III-I};-\sigma;n\ell m;\rho)}_{\pm\mu_1\mu_2} + \mathscr{K}^{(III\to II)}\,\psi^{(\textit{II-I};\sigma;n\ell m;\rho)}_{\pm\mu_1\mu_2},\end{aligned} \tag{6.22}$$

where

$$\mathscr{A}^{(III)} = -\frac{(n+\ell+N)}{2(\ell+\frac{N}{2})} \times \sqrt{\frac{(\ell+2)(\ell+N-2)}{\ell(\ell+N)}(\ell-m+1)(\ell+m+N-1)}, \tag{6.23}$$

$$\mathscr{B}^{(III)} = \frac{(n-\ell+1)}{2(\ell+\frac{N-2}{2})} \times \sqrt{\frac{(\ell+1)(\ell+N-3)}{(\ell-1)(\ell+N-1)}(\ell-m)(\ell+m+N-2)}, \tag{6.24}$$

$$\varkappa^{(III)} = -\frac{(n+\frac{N}{2})(m+\frac{N-2}{2})(N-4)}{2(\ell+\frac{N-2}{2})(\ell+\frac{N}{2})N} \tag{6.25}$$

and

$$\mathscr{K}^{(III\to II)} = \frac{1}{4}\sqrt{\frac{(N-2)(m-1)(m+N-1)}{N\,\ell(\ell+N-1)}}. \tag{6.26}$$

As in eqs. (6.10) and (6.15), the spin projection index σ has flipped sign in the third term of the linear combination in eq. (6.22). The STSSH's $\psi^{(\textit{III-I};-;n\ell m;\rho)}_{\pm\mu\nu}$ and $\psi^{(\textit{III-I};+;n\ell m;\rho)}_{\pm\mu\nu}$ do not mix with each other for $N = 4$ since the coefficient $\varkappa^{(III)}$ [eq. (6.25)] vanishes for this value of N.

Case 2: N odd. As in the case with N even, the normalisation factors for the STSSH's $\psi^{(B;n\ell;\tilde\rho)}_{\pm\mu_1\ldots\mu_r}$ are defined using the inner product (6.4), as[10]

$$\left(\psi^{(B;n\ell;\tilde\rho)}_{\pm}, \psi^{(B';n\ell';\tilde\rho')}_{\pm}\right)_{(r)} \equiv \left|\frac{c_N^{(B;r)}(n,\ell)}{\sqrt{2}}\right|^{-2} \delta_{BB'}\delta_{\ell\ell'}\delta_{\tilde\rho\tilde\rho'}. \tag{6.27}$$

[10]Recall that for N odd the STSSH's $\psi^{(B;n\ell;\tilde\rho)}_{\pm\mu_1\ldots\mu_r}$ do not have a spin projection index on S^N. They are just labelled by the angular momentum quantum numbers n and ℓ, while the angular momentum quantum numbers on $S^{N-2}, S^{N-3}, ..., S^2, S^1$ and the spin projection indices on the even-dimensional spheres $S^{N-1}, S^{N-3}, ..., S^2$ are represented by $\tilde\rho$.

As demonstrated in Appendix F, the normalisation factors for N odd are given again by eq. (6.8). The conjecture for the normalisation factors of the STSSH's in the passage below eq. (6.8) is made for both N odd and N even.

As in the case with N even, we introduce the shorthand notation $\psi^{(B;n\ell;\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_r}$ for the STSSH's of ranks 1 and 2, as

$$\psi^{(B;n\ell;\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_1} = \psi^{(B;n\ell;\sigma_{N-1};m;\rho)}_{\pm\mu_1} \quad (B = I,\ \mathit{II}\text{-}I), \tag{6.28a}$$

$$\psi^{(\mathit{III}\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_1} = 0, \tag{6.28b}$$

$$\psi^{(B;n\ell;\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_2} = \psi^{(B;n\ell;\sigma_{N-1};m;\rho)}_{\pm\mu_1\mu_2} \quad (B = I, \mathit{II}\text{-}I, \mathit{III}\text{-}I), \tag{6.28c}$$

where we have also written out explicitly the dependence on the angular momentum quantum number on S^{N-2}, m, which corresponds to ℓ on S^{N-1}, as well as the dependence on the spin projection index on S^{N-1} ($\sigma_{N-1} = \pm$). The symbol ρ represents labels other than n, ℓ, σ_{N-1} and m.

As shown in Appendix F, the spin$(N+1)$ transformation of the type-I, type-II-I and type-III-I modes are expressed as

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi^{(I;n\ell;\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_r} &= \mathscr{A}^{(I)}\,\psi^{(I;n\,(\ell+1);\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_r} + \mathscr{B}^{(I)}\,\psi^{(I;n\,(\ell-1);\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_r}\\ &\quad \pm i\,\sigma_{N-1}\,\varkappa^{(I)}\,\psi^{(I;n\ell;\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_r} + \mathscr{K}^{(I\to \mathit{II})}\,\psi^{(\mathit{II}\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_r},\end{aligned} \tag{6.29}$$

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi^{(\mathit{II}\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_r} &= \mathscr{A}^{(\mathit{II})}\,\psi^{(\mathit{II}\text{-}I;n\,(\ell+1);\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_r} + \mathscr{B}^{(\mathit{II})}\,\psi^{(\mathit{II}\text{-}I;n\,(\ell-1);\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_r}\\ &\quad \pm i\,\sigma_{N-1}\,\varkappa^{(\mathit{II})}\,\psi^{(\mathit{II}\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_r} + \mathscr{K}^{(\mathit{II}\to I)}\,\psi^{(I;\sigma;n\ell;\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_r}\\ &\quad + \mathscr{K}^{(\mathit{II}\to \mathit{III})}\,\psi^{(\mathit{III}\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm\boldsymbol{N}_r},\end{aligned} \tag{6.30}$$

and

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi^{(\mathit{III}\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm\mu_1\mu_2} &= \mathscr{A}^{(\mathit{III})}\,\psi^{(\mathit{III}\text{-}I;n\,(\ell+1);\sigma_{N-1};m;\rho)}_{\pm\mu_1\mu_2} + \mathscr{B}^{(\mathit{III})}\,\psi^{(\mathit{III}\text{-}I;n\,(\ell-1);\sigma_{N-1};m;\rho)}_{\pm\mu_1\mu_2}\\ &\quad \pm i\,\sigma_{N-1}\,\varkappa^{(\mathit{III})}\,\psi^{(\mathit{III}\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm\mu_1\mu_2} + \mathscr{K}^{(\mathit{III}\to \mathit{II})}\,\psi^{(\mathit{II}\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm\mu_1\mu_2},\end{aligned} \tag{6.31}$$

respectively. [In eqs. (6.29) and (6.30) we have $r \in \{1,2\}$, while eq. (6.31) is relevant only for $r = 2$.] All coefficients in eqs. (6.29)-(6.31) are given by the same expressions as the coefficients in the case with N even [see eqs. (6.10), (6.15) and (6.22)]. Unlike the even-dimensional case, the two spin projections $\sigma_{N-1} = \pm$ do not mix with each other in eqs. (6.29)-(6.31). However, the two spin projections $\sigma_{N-1} = \pm$ mix with each other under spin(N) transformations. Note that the transformation formulae (6.30) and (6.31) are defined only for $N \geq 5$ (N odd), since type-II and type-III modes on S^N do not exist[11] for $N = 3$.

We are now ready to analytically continue our rank-1 and rank-2 STSSH's to dS_N and study the group representation properties of the analytically continued STSSH's.

[11] This is consistent with the fact that the coefficient $\mathscr{K}^{(I\to \mathit{II})}$, given by eq. (6.14), vanishes for $N = 3$.

7 Obtaining spin-3/2 and spin-5/2 mode solutions on N-dimensional de Sitter spacetime by the analytic continuation of STSSH's

7.1 Analytic continuation techniques

In this Section, we begin by discussing our analytic continuation techniques for STSSH's of arbitrary rank r and then we specialise to the cases with $r = 1$ and $r = 2$.

It is well known that dS_N can be obtained by an "analytic continuation" of S^N (see, e.g., Ref. [13]). By replacing the angle θ_N in the line element of S^N (2.3) as:

$$\theta_N \to x(t) \equiv \frac{\pi}{2} - it, \tag{7.1}$$

($t \in \mathbb{R}$) we find the line element for global dS_N:

$$ds^2 = -dt^2 + \cosh^2 t \, ds^2_{N-1}. \tag{7.2}$$

Motivated by this observation, we can obtain the field equations (1.3) and (1.4) on dS_N by analytically continuing eqs. (1.6) and (1.7), respectively, for the STSSH's on S^N. For convenience, let us give here again eqs. (1.6) and (1.7) for STSSH's on S^N:

$$\nabla\!\!\!\!/ \, \psi_{\pm\mu_1\dots\mu_r} = \pm i \left(n + \frac{N}{2} \right) \psi_{\pm\mu_1\dots\mu_r}, \qquad (n = r, r+1, \dots) \tag{7.3}$$

$$\nabla^\alpha \psi_{\pm\alpha\mu_2\dots\mu_r} = 0, \quad \gamma^\alpha \psi_{\pm\alpha\mu_2\dots\mu_r} = 0. \tag{7.4}$$

Without loss of generality, we can choose to analytically continue the STSSH's with either one of the two signs for the eigenvalue in eq. (7.3), since each of the two sets of modes, $\{\psi_{+\mu_1\dots\mu_r}\}$ and $\{\psi_{-\mu_1\dots\mu_r}\}$, forms independently a unitary representation of spin$(N+1)$ labelled by n (see the beginning of Sections 4 and 5). Here we choose to analytically continue the STSSH's $\psi_{-\mu_1\dots\mu_r}$. By making the following replacements in eqs. (7.3) and (7.4):

$$\theta_N \to x(t) \equiv \frac{\pi}{2} - it, \qquad n \to \tilde{M} - \frac{N}{2} \qquad (t \in \mathbb{R}, \ \tilde{M} \in \mathbb{R} \setminus \{0\}) \tag{7.5}$$

we obtain eqs. (1.3) and (1.4), respectively, with imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$) on dS_N. Recall that we are mainly interested in field equations with imaginary mass parameter because our aim is to study strictly and partially massless representations of spin$(N, 1)$, where the mass parameter takes the imaginary values (1.5). Note that the gamma matrices on S^N [eqs. (2.11) and (2.14)] transform under the replacement (7.1) as: $\gamma^N \to i\gamma^N = \gamma^0$, while the γ^j's ($j = 1, \dots, N-1$) remain unchanged.[12]

Let us now give a prescription for obtaining the explicit form of the spin-3/2 and spin-5/2 TT mode functions with mass parameter $M = i\tilde{M}$ on dS_N by analytically continuing the STSSH's of rank 1 and 2, respectively. The functions describing the time-dependence

[12] Alternatively, we could analytically continue the STSSH's on S^N by making the replacement $\theta_N \to \pi/2 + it$ instead of the replacement (7.1). The analytically continued STSSH's with $\theta_N \to \pi/2 - it$ and the ones with $\theta_N \to \pi/2 + it$ are related to each other by charge conjugation. However, these two cases of analytically continued STSSH's form equivalent representations of spin$(N, 1)$.

of the analytically continued STSSH's are found by making the replacements (7.5) in the (unnormalised) functions $\phi^{(a)}_{n\ell}(\theta_N)$ [eq. (3.1)] and $\psi^{(a)}_{n\ell}(\theta_N)$ [eq. (3.2)], as

$$\hat{\phi}^{(a)}_{\tilde{M}\ell}(t) \equiv \left[\kappa_\phi\left(\tilde{M}-\frac{N}{2},\ell\right)\right]^{-1} \phi^{(a)}_{(\tilde{M}-\frac{N}{2})\,\ell}(x(t)) \tag{7.6}$$

$$= \left(\cos\frac{x(t)}{2}\right)^{\ell+1-a}\left(\sin\frac{x(t)}{2}\right)^{\ell-a} \times F\left(-\tilde{M}+\frac{N}{2}+\ell, \tilde{M}+\ell+\frac{N}{2}; \ell+\frac{N}{2}; \sin^2\frac{x(t)}{2}\right), \tag{7.7}$$

$$\hat{\psi}^{(a)}_{\tilde{M}\ell}(t) \equiv \left[\kappa_\phi\left(\tilde{M}-\frac{N}{2},\ell\right)\right]^{-1} \psi^{(a)}_{(\tilde{M}-\frac{N}{2})\,\ell}(x(t)) \tag{7.8}$$

$$= \frac{\tilde{M}}{\ell+\frac{N}{2}}\left(\cos\frac{x(t)}{2}\right)^{\ell-a}\left(\sin\frac{x(t)}{2}\right)^{\ell+1-a} \times F\left(-\tilde{M}+\frac{N}{2}+\ell, \tilde{M}+\ell+\frac{N}{2}; \ell+\frac{N+2}{2}; \sin^2\frac{x(t)}{2}\right), \tag{7.9}$$

where $\kappa_\phi(\tilde{M}-\frac{N}{2},\ell)$ is given by eq. (3.3) with n replaced by $\tilde{M}-\frac{N}{2}$, while

$$\cos\frac{x(t)}{2} = \left(\sin\frac{x(t)}{2}\right)^* = \frac{\sqrt{2}}{2}\left(\cosh\frac{t}{2}+i\sinh\frac{t}{2}\right). \tag{7.10}$$

Note that $\hat{\phi}^{(a)}_{(-\tilde{M})\ell} = \hat{\phi}^{(a)}_{\tilde{M}\ell}$ and $\hat{\psi}^{(a)}_{(-\tilde{M})\ell} = -\hat{\psi}^{(a)}_{\tilde{M}\ell}$. The condition $\ell \leq n$ does not hold for dS_N. Now ℓ can be any positive integer with $\ell \geq r$.

For brevity, let us use again the shorthand notation introduced in eqs. (6.9) (for N even) and (6.28) (for N odd). For N even, we denote the analytically continued STSSH's as $\Psi^{(B;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}(t,\boldsymbol{\theta}_{N-1})$ (where $\sigma=\pm$ is the spin projection index on dS_N, while $m\leq\ell$ and $\ell = r,\, r+1,\dots$). We define the modes $\Psi^{(B;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}$ by making the replacements (7.5) in the STSSH's $\psi^{(B;\sigma;n\ell m;\rho)}_{-\boldsymbol{N}_r}$ on S^N, as

$$\Psi^{(B;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}(t,\boldsymbol{\theta}_{N-1}) = \left[\kappa_\phi\left(\tilde{M}-\frac{N}{2},\ell\right)\right]^{-1} \psi^{\left(B;\sigma;(\tilde{M}-N/2)\,\ell m;\rho\right)}_{-\boldsymbol{N}_r}(\pi/2-it,\boldsymbol{\theta}_{N-1}) \tag{7.11}$$

where $\left[\kappa_\phi\left(\tilde{M}-\frac{N}{2},\ell\right)\right]^{-1}$ is essentially the factor used in eqs. (7.6) and (7.8) [it is used in order to cancel the normalisation factor (3.3) of the Jacobi polynomials]. Note that, by viewing the replacement $\theta_N \to \frac{\pi}{2}-it$ as a coordinate change, we find that $\psi^{(B;\sigma;n\ell m;\rho)}_{-\theta_N}$ transforms as

$$\psi^{(B;\sigma;n\ell m;\rho)}_{-\theta_N} \to i\,\psi^{\left(B;\sigma;(\tilde{M}-N/2)\,\ell m;\rho\right)}_{-t}.$$

Similarly, $\psi^{(B;\sigma;n\ell m;\rho)}_{-\theta_N\theta_N}$ and $\psi^{(B;\sigma;n\ell m;\rho)}_{-\theta_N\theta_j}$ transform as

$$\psi^{(B;\sigma;n\ell m;\rho)}_{-\theta_N\theta_N} \to -\psi^{\left(B;\sigma;(\tilde{M}-N/2)\,\ell m;\rho\right)}_{-t\,t}$$

and

$$\psi_{-\theta_N \theta_j}^{(B;\sigma;n\ell m;\rho)} \to i\, \psi_{-t\, \theta_j}^{\left(B;\sigma;(\tilde{M}-N/2)\, \ell m;\rho\right)},$$

respectively.

For N odd, the analytically continued STSSH's are denoted as $\Psi_{\boldsymbol{N}_r}^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)}$ (where $\sigma_{N-1} = \pm$, $m \leq \ell$ and $\ell = r,\, r+1, \ldots$). They are obtained by analytically continuing the STSSH's $\psi_{-\boldsymbol{N}_r}^{(B;n\ell;\sigma_{N-1};m;\rho)}(\theta_N, \boldsymbol{\theta}_{N-1})$ on S^N, as

$$\Psi_{\boldsymbol{N}_r}^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)}(t, \boldsymbol{\theta}_{N-1}) = \left[\kappa_\phi \left(\tilde{M} - \frac{N}{2}, \ell\right)\right]^{-1} \psi_{-\boldsymbol{N}_r}^{\left(B;(\tilde{M}-N/2)\, \ell;\sigma_{N-1};m;\rho\right)}(\pi/2 - it, \boldsymbol{\theta}_{N-1}). \tag{7.12}$$

Note that, unlike the case with N even [eq. (7.11)], the analytically continued STSSH's (7.12) have a spin projection index (σ_{N-1}) on S^{N-1} instead of a spin projection index on dS_N.

7.2 Pure gauge modes for the massless spin-3/2 and spin-5/2 theories

As in Minkowski spacetime, (strictly and partially) massless field theories in dS_N are gauge invariant [8]. In terms of mode solutions of the corresponding field equations, gauge invariance manifests itself through the appearance of 'pure gauge' modes in the set of mode solutions. The 'pure gauge' modes do not describe propagating DoF of the field theory and - assuming that there exists an invariant inner product for the mode solutions - these modes have zero norm (see, e.g. Ref. [13]).

For later convenience, let us present the 'pure gauge' modes that appear among the analytically continued STSSH's of rank r ($r = 1, 2$) when we tune the imaginary mass parameter ($M = i\tilde{M}$) to the massless values $\tilde{M} = \pm\, [r - \tau + (N-2)/2]$, where $\tau = 1, .., r$ [see eq. (1.5)]. For each massless value of $\tilde{M}$, the analytically continued STSSH's of rank r with $r - \tau \geq \tilde{r} \geq 0$ are 'pure gauge' modes, where $\tilde{r}$ is the rank of the STSSH on S^{N-1} used in the method of separation of variables (see Sections 4 and 5). In Section 8 we will verify that our 'pure gauge' modes have zero norm associated to a spin$(N,1)$ invariant scalar product for N even. We will also demonstrate that for N odd there does not exist any spin$(N,1)$ invariant scalar product for the analytically continued STSSH's with imaginary mass parameter. Thus, for N odd the norm of the 'pure gauge' modes cannot be calculated in a meaningful way, as there is no de Sitter invariant notion of norm.

Strictly massless spin-3/2 field. The mass parameter for the strictly massless spin-3/2 field is given by $M = i\tilde{M} = \pm i(N-2)/2$ [this is found by letting $r = \tau = 1$ in eq. (1.5)]. The analytically continued STSSH's of type-I ($\tilde{r} = 0$) are 'pure gauge' modes. As demonstrated in Appendix G, the analytically continued rank-1 STSSH's (7.11) of type-I with $\tilde{M} = \pm(N-2)/2$ are expressed in a 'pure gauge' form as follows:

$$\Psi_\mu^{(I;(\pm\frac{N-2}{2});\tilde{\ell})}(t, \boldsymbol{\theta}_{N-1}) = \left(\nabla_\mu \pm \frac{i}{2}\gamma_\mu\right) \Lambda_\pm^{(\tilde{\ell})}(t, \boldsymbol{\theta}_{N-1}), \tag{7.13}$$

where for brevity we use the symbol $\tilde{\ell}$ to represent all the labels of the analytically continued STSSH's which have not been written down explicitly. The Dirac spinors $\Lambda_\pm^{(\tilde{\ell})}(t, \boldsymbol{\theta}_{N-1})$

satisfy

$$\nabla\!\!\!/ \Lambda_{\pm}^{(\tilde{\ell})} = \mp i\,\frac{N}{2}\,\Lambda_{\pm}^{(\tilde{\ell})}. \tag{7.14}$$

The 'pure gauge' expression (7.13) for the type-I modes coincides with the form of the infinitesimal gauge transformation [8] (with a specific gauge condition) that leaves invariant the action for the strictly massless spin-3/2 field in dS_4. In Section 8 we show that the 'pure gauge' modes (7.13) have vanishing dS invariant norm for even $N \geq 4$.

Strictly massless spin-5/2 field. The mass parameter for the strictly massless spin-5/2 field is given by $M = i\tilde{M} = \pm iN/2$ [this is found by letting $r = 2$ and $\tau = 1$ in eq. (1.5)]. There are two types of 'pure gauge' modes, namely the analytically continued STSSH's of type-I ($\tilde{r} = 0$) and type-II ($\tilde{r} = 1$). As demonstrated in Appendix G, the analytically continued rank-2 STSSH's (7.11) of type-I and type-II with $\tilde{M} = \pm N/2$ are expressed in the following 'pure gauge' form:

$$\Psi_{\mu\nu}^{(B;(\pm\frac{N}{2});\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1}) = \left(\nabla_{(\mu} \pm \frac{i}{2}\gamma_{(\mu}\right)\lambda_{\pm\nu)}^{(B;\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1}), \qquad B = I, II, \tag{7.15}$$

where the gauge functions $\lambda_{\pm\mu}^{(B;\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1})$ $(B = I, II)$ are vector-spinor fields satisfying

$$\nabla\!\!\!/ \lambda_{\pm\mu}^{(B;\tilde{\ell})} = \mp i\,\frac{N+2}{2}\,\lambda_{\pm\mu}^{(B;\tilde{\ell})} \tag{7.16}$$

$$\gamma^{\mu}\lambda_{\pm\mu}^{(B;\tilde{\ell})} = \nabla^{\mu}\lambda_{\pm\mu}^{(B;\tilde{\ell})} = 0. \tag{7.17}$$

The vector-spinors $\lambda_{\pm\mu}^{(B;\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1})$ are given by the analytic continuation of rank-1 STSSH's of type-B $(B = I, II)$ - see Appendix G. Note that the 'pure gauge' expressions (7.15) for the type-I and type-II modes coincide with the form of the infinitesimal gauge transformation [8] (with a specific gauge condition) for the gauge-invariant action for the strictly massless spin-5/2 field in dS_4. In Section 8 we show that the 'pure gauge' modes (7.15) have zero (dS invariant) norm for even $N \geq 4$.

Partially massless spin-5/2 field. The mass parameter for the partially massless spin-5/2 field is given by $M = i\tilde{M} = \pm i(N-2)/2$ [this is found by letting $r = 2$ and $\tau = 2$ in eq. (1.5)]. The analytically continued STSSH's of type-I ($\tilde{r} = 0$) are 'pure gauge' modes. As demonstrated in Appendix G, the analytically continued rank-2 STSSH's (7.11) of type-I with $\tilde{M} = \pm(N-2)/2$ are expressed in a 'pure gauge' form as follows:

$$\Psi_{\mu\nu}^{(I;(\pm\frac{N-2}{2});\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1}) = \left(\nabla_{(\mu}\nabla_{\nu)} \pm i\gamma_{(\mu}\nabla_{\nu)} + \frac{3}{4}g_{\mu\nu}\right)\varphi_{\pm}^{(\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1}), \tag{7.18}$$

where the spinor modes $\varphi_{\pm}^{(\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1})$ satisfy

$$\nabla\!\!\!/ \varphi_{\pm}^{(\tilde{\ell})} = \mp i\,\frac{N+2}{2}\,\varphi_{\pm}^{(\tilde{\ell})}. \tag{7.19}$$

In Section 8 we show that the 'pure gauge' modes (7.18) have zero (dS invariant) norm for even $N \geq 4$. We note that we have not constructed a gauge-invariant action for the partially massless spin-$5/2$ field in dS_N with infinitesimal gauge transformation of the form (7.18). However, we call the modes (7.18) 'pure gauge' modes because we expect that such an action exists and that the expression (7.18) describes infinitesimal gauge transformations (satisfying a specific gauge condition) for this action.

In Appendix G, we discuss the relation between our 'pure gauge' modes (7.18) and the gauge transformation of the partially massless spin-$5/2$ field in dS_4 given in Ref. [8]. More specifically, we observe the following intriguing fact: for a specific choice for the spinor gauge function in the gauge transformation used in Ref. [8], the gamma-traceless part of this gauge transformation can be expressed in our 'pure gauge' form (7.18).

8 (Non)unitarity of the massless representations of spin$(N,1)$ formed by the analytically continued rank-1 and rank-2 STSSH's

For each value of the imaginary mass parameter $M = i\tilde{M}$ in eq. (1.3), the TT tensor-spinor mode solutions (i.e. the analytically continued STSSH's) form a representation of spin$(N,1)$. If one introduces a dS invariant scalar product among the analytically continued STSSH's, then the unitarity of the representation is equivalent to the positive-definiteness of the associated norm. If there is no dS invariant scalar product, then the corresponding representation of spin$(N,1)$ is, by definition, not unitary.

In this Section we prove statements 1, 2 and 3 presented in the Introduction, which give rise to the main result of our paper (which we mention here again for convenience): the strictly massless spin-$3/2$ field theory and the strictly and partially massless spin-$5/2$ field theories on dS_N ($N \geq 3$) are unitary only for $N = 4$.

8.1 The massless spin-3/2 and spin-5/2 representations of spin$(N,1)$ are non-unitary for even $N > 4$

In this Subsection, we show that the representations of spin$(N,1)$ with even $N > 4$ formed by the spin-$3/2$ and spin-$5/2$ TT mode solutions of eq. (1.3) with arbitrary imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$) are non-unitary (i.e. we prove statement 1). In order to arrive at this result we study the transformation properties of our analytically continued STSSH's under a spin$(N,1)$ boost and then we investigate the positive-definiteness (or indefiniteness) of the norm associated to a dS invariant scalar product for even $N > 4$. (In this Subsection we work without specifying the form of the dS invariant scalar product.) We also find that for $N = 4$ the requirement for dS invariance of the scalar product does not imply the indefiniteness of the norm if and only if the mass parameter $\tilde{M}$ is tuned to the massless values (1.5). Furthermore, for $N = 4$ and $\tilde{M}$ given by eq. (1.5), we show that the TT modes are divided into two spin$(4,1)$ invariant subspaces, denoted as $\mathscr{H}^-$ and $\mathscr{H}^+$ (where each subspace contains modes with definite helicity). The positivity of the norm in each of these subspaces is shown in Subsection 8.2 by calculating explicitly the norms of the eigenmodes with respect to a specific dS invariant scalar product. (In Subsection 8.2 we also verify the results obtained in the present Subsection for even $N > 4$ by explicit calculation of the norms of the eigenmodes with arbitrary imaginary mass parameter $M = i\tilde{M} \neq 0$.)

The analytic continuation techniques introduced in Section 7 can also be applied to the transformation properties of the STSSH's under spin$(N+1)$. By doing so, one obtains the transformation properties of the analytically continued STSSH's on dS_N under spin$(N,1)$. Let us make the replacement (7.1) in the Killing vector $\mathscr{S}^\mu$ [eq. (6.6)] on S^N. One finds that the analytically continued version of $\mathscr{S}^\mu$ is expressed as iX^μ, where X^μ is the following boost generator of spin$(N,1)$:

$$X^\mu \partial_\mu = \cos\theta_{N-1}\,\frac{\partial}{\partial t} - \tanh t\,\sin\theta_{N-1}\,\frac{\partial}{\partial \theta_{N-1}}. \tag{8.1}$$

The de Sitter algebra spin$(N,1)$ is generated by the de Sitter boost (8.1) and the generators of spin(N).

By making the replacements (7.5) in the spin$(N+1)$ transformation formulae (6.10), (6.15) and (6.22) [and using eq. (7.11)], we find

$$\begin{aligned}\mathbb{L}_X \Psi^{(I;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r} = &-i\,c_{(\ell)}\,\mathscr{A}^{(I)}\,\Psi^{(I;\sigma;\tilde{M}\,(\ell+1)\,m;\rho)}_{\boldsymbol{N}_r} - \frac{i}{c_{(\ell-1)}}\mathscr{B}^{(I)}\,\Psi^{(I;\sigma;\tilde{M}\,(\ell-1)\,m;\rho)}_{\boldsymbol{N}_r}\\ &-\varkappa^{(I)}\,\Psi^{(I;-\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r} - i\mathscr{K}^{(I\to II)}\,\Psi^{(II\text{-}I;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r},\end{aligned} \tag{8.2}$$

$$\begin{aligned}\mathbb{L}_X \Psi^{(II\text{-}I;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r} = &-i\,c_{(\ell)}\mathscr{A}^{(II)}\,\Psi^{(II\text{-}I;\sigma;\tilde{M}\,(\ell+1)\,m;\rho)}_{\boldsymbol{N}_r} - \frac{i}{c_{(\ell-1)}}\mathscr{B}^{(II)}\,\Psi^{(II\text{-}I;\sigma;\tilde{M}\,(\ell-1)\,m;\rho)}_{\boldsymbol{N}_r}\\ &-\varkappa^{(II)}\Psi^{(II\text{-}I;-\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r} - i\mathscr{K}^{(II\to I)}\,\Psi^{(I;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}\\ &-i\mathscr{K}^{(II\to III)}\,\Psi^{(III\text{-}I;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}\end{aligned} \tag{8.3}$$

$(r=1,2,)$ and

$$\begin{aligned}\mathbb{L}_X \Psi^{(III\text{-}I;\sigma;\tilde{M}\ell m;\rho)}_{\mu_1\mu_2} = &-i\,c_{(\ell)}\mathscr{A}^{(III)}\,\Psi^{(III\text{-}I;\sigma;\tilde{M}\,(\ell+1)\,m;\rho)}_{\mu_1\mu_2} - \frac{i}{c_{(\ell-1)}}\mathscr{B}^{(III)}\,\Psi^{(III\text{-}I;\sigma;\tilde{M}\,(\ell-1)\,m;\rho)}_{\mu_1\mu_2}\\ &-\varkappa^{(III)}\Psi^{(III\text{-}I;-\sigma;\tilde{M}\ell m;\rho)}_{\mu_1\mu_2} - i\mathscr{K}^{(III\to II)}\,\Psi^{(II\text{-}I;\sigma;\tilde{M}\ell m;\rho)}_{\mu_1\mu_2},\end{aligned} \tag{8.4}$$

respectively, with

$$c_{(\ell)} = \frac{\kappa_\phi(\tilde{M}-\frac{N}{2},\ell+1)}{\kappa_\phi(\tilde{M}-\frac{N}{2},\ell)} = \frac{\tilde{M}-\ell-\frac{N}{2}}{\ell+N/2}, \tag{8.5}$$

where $\kappa_\phi(\tilde{M}-N/2,\ell)$ is found by eq. (3.3) and $\mathbb{L}_X$ is the Lie-Lorentz derivative (6.1) on dS_N. The coefficients $\mathscr{A}^{(B)}, \mathscr{B}^{(B)}, \varkappa^{(B)}$ (with $B=I,\ II,\ III$), $\mathscr{K}^{(I\to II)}, \mathscr{K}^{(II\to I)}, \mathscr{K}^{(II\to III)}$ and $\mathscr{K}^{(III\to II)}$ are found by making the replacement $n \to \tilde{M}-N/2$ in the corresponding expressions for the coefficients of STSSH's on S^N [see eqs. (6.10), (6.15) and (6.22)]. Note that we use the same symbols to represent the coefficients in the transformation formulae on S^N and the analytically continued coefficients on dS_N.

Let $\langle \Psi^{(1)}, \Psi^{(2)}\rangle_{(r)}$ be a spin$(N,1)$ invariant scalar product for any two analytically continued rank-r STSSH's $\Psi^{(1)}_{\boldsymbol{N}_r}, \Psi^{(2)}_{\boldsymbol{N}_r}$ $(r=1,2)$ with imaginary mass parameter $M=i\tilde{M}$ $(\tilde{M}\neq 0)$. Due to the spin$(N,1)$ invariance of the scalar product we have

$$\langle \mathbb{L}_\xi \Psi^{(1)}, \Psi^{(2)}\rangle_{(r)} + \langle \Psi^{(1)}, \mathbb{L}_\xi \Psi^{(2)}\rangle_{(r)} = 0 \tag{8.6}$$

for any Killing vector ξ on dS_N. Then, by letting $\Psi^{(1)}_{\boldsymbol{N}_r} = \Psi^{(B;-;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}$ and $\Psi^{(2)}_{\boldsymbol{N}_r} = \Psi^{(B;+;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}$ (with $B = I, \mathit{II}\text{-}I, \mathit{III}\text{-}I$) in eq. (8.6) with $\xi = X$ and using the transformation formulae (8.2)-(8.4), we find that the norms of eigenmodes with opposite spin projections must satisfy:

$$\varkappa^{(I)} \times \left(\langle \Psi^{(I;-;\tilde{M}\ell m;\rho)}, \Psi^{(I;-;\tilde{M}\ell m;\rho)} \rangle_{(r)} + \langle \Psi^{(I;+;\tilde{M}\ell m;\rho)}, \Psi^{(I;+;\tilde{M}\ell m;\rho)} \rangle_{(r)} \right) = 0, \tag{8.7}$$

$$\begin{aligned} \varkappa^{(\mathit{II})} \times \Big(& \langle \Psi^{(\mathit{II}\text{-}I;-;\tilde{M}\ell m;\rho)}, \Psi^{(\mathit{II}\text{-}I;-;\tilde{M}\ell m;\rho)} \rangle_{(r)} \\ & + \langle \Psi^{(\mathit{II}\text{-}I;+;\tilde{M}\ell m;\rho)}, \Psi^{(\mathit{II}\text{-}I;+;\tilde{M}\ell m;\rho)} \rangle_{(r)} \Big) = 0, \end{aligned} \tag{8.8}$$

$$\begin{aligned} \varkappa^{(\mathit{III})} \times \Big(& \langle \Psi^{(\mathit{III}\text{-}I;-;\tilde{M}\ell m;\rho)}, \Psi^{(\mathit{III}\text{-}I;-;\tilde{M}\ell m;\rho)} \rangle_{(r)} \\ & + \langle \Psi^{(\mathit{III}\text{-}I;+;\tilde{M}\ell m;\rho)}, \Psi^{(\mathit{III}\text{-}I;+;\tilde{M}\ell m;\rho)} \rangle_{(r)} \Big) = 0. \end{aligned} \tag{8.9}$$

Note that, since the scalar product is also spin(N) invariant, analytically continued STSSH's of different type or/and with different values for ℓ are orthogonal to each other because they correspond to inequivalent irreducible representations of spin(N) in the decomposition spin$(N,1) \supset$ spin(N). For convenience, we give here the explicit form of the analytically continued coefficients $\varkappa^{(I)}$ [eq. (6.13)], $\varkappa^{(\mathit{II})}$ [eq. (6.18)] and $\varkappa^{(\mathit{III})}$ [eq. (6.25)]:

$$\varkappa^{(I)} = -\frac{\tilde{M}(m + \frac{N-2}{2})(N + 2r - 2)}{2(\ell + \frac{N-2}{2})(\ell + \frac{N}{2})(N-2)} \qquad (r = 1, 2), \tag{8.10}$$

$$\varkappa^{(\mathit{II})} = -\frac{\tilde{M}(m + \frac{N-2}{2})(N-4)}{2(\ell + \frac{N-2}{2})(\ell + \frac{N}{2})(N-2)} \times \left(\frac{N+2}{N} \right)^{r-1} \qquad (r = 1, 2), \tag{8.11}$$

$$\varkappa^{(\mathit{III})} = -\frac{\tilde{M}(m + \frac{N-2}{2})(N-4)}{2(\ell + \frac{N-2}{2})(\ell + \frac{N}{2})N} \tag{8.12}$$

[eq. (8.12) is relevant only for spin-5/2 modes, i.e. only for $r = 2$]. We also give the explicit form of the analytically continued coefficients $\mathscr{K}^{(I \to \mathit{II})}$ [eq. (6.14)] and $\mathscr{K}^{(\mathit{II} \to \mathit{III})}$ [eq. (6.21)]:

$$\mathscr{K}^{(I \to \mathit{II})} = -\frac{4\left(\tilde{M}^2 - (N-2)^2/4\right)(N + r - 2)}{\ell(\ell + N - 1)(N-2)} \times \sqrt{\frac{N-3}{N-2} \frac{m(m + N - 2)}{(\ell+1)(\ell + N - 2)}} \quad (r = 1, 2), \tag{8.13}$$

$$\mathscr{K}^{(\mathit{II} \to \mathit{III})} = -\,2^3 \frac{\left(\tilde{M}^2 - N^2/4\right)(N+1)}{(\ell - 1)(\ell + N)N} \times \sqrt{\frac{N-2}{N} \frac{(m-1)(m + N - 1)}{\ell(\ell + N - 1)}}, \tag{8.14}$$

where eq. (8.14) is relevant only for $r = 2$. The analytically continued coefficients $\mathscr{K}^{(II\to I)}$ and $\mathscr{K}^{(III\to II)}$ are given by the same expressions as the coefficients on S^N, i.e. eqs. (6.20) and (6.26), respectively.

Let us first discuss the case with even $N > 4$, where $\varkappa^{(I)}, \varkappa^{(II)}$ and $\varkappa^{(III)}$ are all non-zero (for all $\tilde{M} \neq 0$). The representation can be unitary only if eqs. (8.7)-(8.9) are consistent with the positive-definiteness of the norm. However, it is clear from eqs. (8.7)-(8.9) that the norm of the modes $\Psi^{(B;-;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}$ is opposite of the norm of the modes $\Psi^{(B;+;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}$ ($B = I, II\text{-}I, III\text{-}I$) for all $\tilde{M} \neq 0$. Hence, for even $N > 4$, there are negative-norm modes for all values of $\tilde{M} \neq 0$, unless all modes have zero norm. (Not all modes could have zero norm if the field were to describe a physical particle.) Thus, we have proved statement 1.

Before discussing the case with $N = 4$, we can show that the 'pure gauge' modes (discussed in Subsection 7.2), which appear among the TT mode solutions in the massless theories, have zero norm with respect to any dS invariant scalar product for even $N \geq 4$, as follows [28]. For the strictly massless spin-3/2 theory ($r = \tau = 1$), as well as for the partially massless spin-5/2 theory ($r = \tau = 2$), the mass parameter is $\tilde{M}^2 = (N-2)^2/4$ [see eq. (1.5)], while the type-I modes are 'pure gauge' modes. We observe that the coefficient $\mathscr{K}^{(I\to II)}$ [eq. (8.13)] vanishes for $\tilde{M}^2 = (N-2)^2/4$ (with $r = 1, 2$). Then, by letting $\Psi^{(1)}_{\boldsymbol{N}_r} = \Psi^{\left(I;\sigma;(\pm\frac{N-2}{2})\ell m;\rho\right)}_{\boldsymbol{N}_r}$ and $\Psi^{(2)}_{\boldsymbol{N}_r} = \Psi^{\left(II\text{-}I;\sigma;(\pm\frac{N-2}{2})\ell m;\rho\right)}_{\boldsymbol{N}_r}$ in eq. (8.6) with $\xi = X$ and using the transformation formulae (8.2) and (8.3), we straightforwardly find $\langle \Psi^{\left(I;\sigma;(\pm\frac{N-2}{2})\ell m;\rho\right)}, \Psi^{\left(I;\sigma;(\pm\frac{N-2}{2})\ell m;\rho\right)}\rangle_{(r)} = 0$ (with $r = 1, 2$), i.e. the type-I modes have zero norm for even $N \geq 4$. For the strictly massless spin-5/2 theory ($r = \tau + 1 = 2$) the mass parameter is $\tilde{M}^2 = N^2/4$ [see eq. (1.5)], while both type-I and type-II modes are 'pure gauge' modes. For this value of $\tilde{M}^2$ the coefficient $\mathscr{K}^{(II\to III)}$ [eq. (8.14)] vanishes. By letting $\Psi^{(1)}_{\boldsymbol{N}_r} = \Psi^{\left(II\text{-}I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)}_{\mu_1\mu_2}$ and $\Psi^{(2)}_{\boldsymbol{N}_r} = \Psi^{\left(III\text{-}I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)}_{\mu_1\mu_2}$ in eq. (8.6) with $\xi = X$ and using the transformation formulae (8.3) (with $r = 2$) and (8.4), we find $\langle \Psi^{\left(II\text{-}I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)}, \Psi^{\left(II\text{-}I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)}\rangle_{(r=2)} = 0$. Then, by letting $\Psi^{(1)}_{\boldsymbol{N}_r} = \Psi^{\left(I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)}_{\mu_1\mu_2}$ and $\Psi^{(2)}_{\boldsymbol{N}_r} = \Psi^{\left(II\text{-}I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)}_{\mu_1\mu_2}$ in eq. (8.6) with $\xi = X$ and using the transformation formulae (8.2) (with $r = 2$) and (8.3) (with $r = 2$), we find $\langle \Psi^{\left(I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)}, \Psi^{\left(I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)}\rangle_{(r=2)} = 0$. Thus, in the strictly massless spin-5/2 theory the 'pure gauge' modes have zero norm for even $N \geq 4$.

Let us now discuss the case with $N = 4$. First, we show that if $N = 4$, then the dS invariance of the scalar product (8.6) (with $\xi = X$) for the analytically continued STSSH's with imaginary mass parameter $M = i\tilde{M} \neq 0$ does not require indefiniteness of the norm if and only if $\tilde{M}$ is tuned to the massless values (1.5). This can be shown as follows. For $N = 4$ eqs. (8.8) and (8.9) are trivial due to the vanishing of $\varkappa^{(II)}$ [eq. (8.11)] and $\varkappa^{(III)}$ [eq. (8.12)], respectively. It is clear that if eq. (8.7) is not trivial, then the indefiniteness of the norm can not be avoided. Equation (8.7) becomes trivial if we tune $\tilde{M}$ to the strictly/partially massless values (1.5) because for this value of $\tilde{M}$ the type-I modes are pure gauge (i.e. zero-norm modes). Hence, for $N = 4$ the dS invariance of the scalar product does not require the indefiniteness of the norm for the massless theories with spin $s \in \{3/2, 5/2\}$. Note that, since $\varkappa^{(II)}$ and $\varkappa^{(III)}$ are zero, the (non-zero-norm) eigenmodes with negative spin projection do not mix with the eigenmodes with positive spin projection under the spin(4, 1) boost in eqs. (8.3) and (8.4). We have also verified that (non-zero-norm) eigenmodes with different spin projections on dS_4 do not mix each other under spin(4).

According to our analysis in the previous paragraph, in the case of massless theories with spin $s = r + 1/2$ ($r \in \{1, 2\}$) on dS_4, we conclude the following:

- The set $\mathscr{H}^- = \{ \Psi_{\boldsymbol{N}_r}^{(B;-;\tilde{M}\ell;\tilde{\rho})} \}$ of (non-zero-norm) TT eigenmodes with negative spin projection forms an irreducible representation of spin(4, 1).
- The set $\mathscr{H}^+ = \{ \Psi_{\boldsymbol{N}_r}^{(B;+;\tilde{M}\ell;\tilde{\rho})} \}$ of (non-zero-norm) TT eigenmodes with positive spin projection forms separately an irreducible representation of spin(4, 1).[13]

The two sets of eigenmodes, $\mathscr{H}^+$ and $\mathscr{H}^-$, form a direct sum of irreducible representations of spin(4, 1). In Subsection 8.2 we are going to show that these irreducible representations are unitary by demonstrating the positivity of the norm in each subspace. [As we demonstrate in Appendix A, this is a direct sum of Discrete Series representations of spin(4, 1).] Note that zero-norm modes (i.e. 'pure gauge' modes) transform only into zero-norm modes under spin(4, 1) and they can be identified with zero, since, as we discussed above, the coefficient (8.13) (in the transformation formula (8.2) with $r \in \{1, 2\}$) vanishes for $\tilde{M}^2 = (N-2)^2/4$, while the coefficient (8.14) (in the transformation formula (8.3) with $r = 2$) vanishes for $\tilde{M}^2 = N^2/4$. For the strictly massless spin-3/2 theory ($r = \tau = 1$, $\tilde{M}^2 = (N-2)^2/4$) and the partially massless spin-5/2 theory ($r = \tau = 2$, $\tilde{M}^2 = (N-2)^2/4$), where the type-I modes have zero norm, the action of spin(4, 1) is defined on equivalence classes of the TT modes contained in $\mathscr{H}^\sigma$ ($\sigma = \pm$) with the equivalence relation

$$\Psi_{\boldsymbol{N}_r}^{\left(B;\sigma;(\pm\frac{N-2}{2})\ell;\tilde{\rho}\right)} \sim \Psi_{\boldsymbol{N}_r}^{\left(B;\sigma;(\pm\frac{N-2}{2})\ell;\tilde{\rho}\right)} + \Psi_{\boldsymbol{N}_r}^{\left(I;\sigma';(\pm\frac{N-2}{2})\ell';\tilde{\rho}'\right)}$$

(with $B = \mathit{II}\text{-}I$ for $r = 1$ and $B = \mathit{II}\text{-}I, \mathit{III}\text{-}I$ for $r = 2$), where $\Psi_{\boldsymbol{N}_r}^{\left(I;\sigma';(\pm\frac{N-2}{2})\ell';\tilde{\rho}'\right)}$ is any type-I mode, i.e. the labels σ', ℓ' and $\tilde{\rho}'$ are no necessarily equal to σ, ℓ and $\tilde{\rho}$, respectively. For the strictly massless spin-5/2 theory ($r = \tau + 1 = 2$, $\tilde{M}^2 = N^2/4$), where both type-I and type-II-I modes have zero norm, the action of spin(4, 1) is defined on equivalence classes of type-III-I modes in $\mathscr{H}^\sigma$ ($\sigma = \pm$) with the equivalence relation

$$\Psi_{\mu_1\mu_2}^{\left(\mathit{III}\text{-}I;\sigma;(\pm\frac{N}{2})\ell;\tilde{\rho}\right)} \sim \Psi_{\mu_1\mu_2}^{\left(\mathit{III}\text{-}I;\sigma;(\pm\frac{N}{2})\ell;\tilde{\rho}\right)} + \Psi_{\mu_1\mu_2}^{(PG)},$$

where $\Psi_{\mu_1\mu_2}^{(PG)}$ is any (finite or infinite) linear combination of type-I and type-II modes.

For the strictly massless theories with spin $s \in \{3/2, 5/2\}$ on dS_4, the set $\mathscr{H}^-$ is identified with the set of states with 'negative helicity' ($-s$), while the set $\mathscr{H}^+$ is identified with the set of states with 'positive helicity' ($+s$). This can be understood as follows. As in Ref. [29], let us introduce the helicity operator $\tilde{\epsilon}_{\theta_i}{}^{\theta_j\theta_k}\tilde{\nabla}_{\theta_j}$, where $\tilde{\epsilon}_{\theta_i\theta_j\theta_k}$ is the invariant 3-form on S^3 ($i, j, k \in \{1, 2, 3\}$). For the strictly massless spin-3/2 theory on dS_4, where

$$\mathscr{H}^\sigma = \{ \Psi_{\boldsymbol{N}_1}^{(B;\sigma;\tilde{M}\ell;\tilde{\rho})} \} = \{ \Psi_{\mu}^{(\mathit{II}\text{-}I;\sigma;(\pm 1)\ell;\tilde{\rho})} \},$$

[13] This situation is analogous to the case of the strictly massless spin-2 field in dS_4 [29], where self-dual and anti-self-dual modes correspond to different irreducible representations of SO(4).

it can readily be shown that eigenmodes with different spin projections belong to different eigenspaces of the helicity operator, as

$$\tilde{\epsilon}_{\theta_i}{}^{\theta_j\theta_k}\tilde{\nabla}_{\theta_j}\Psi^{(II\text{-}I;\sigma;(\pm1)\ell;\tilde{\rho})}_{\theta_k} \propto \tilde{\nabla\!\!\!/}\Psi^{(II\text{-}I;\sigma;(\pm1)\ell;\tilde{\rho})}_{\theta_i} = i\sigma\left(\ell+\frac{3}{2}\right)\Psi^{(II\text{-}I;\sigma;(\pm1)\ell;\tilde{\rho})}_{\theta_i}. \tag{8.15}$$

(This equation can be readily proved using the fact that $\tilde{\epsilon}_{\theta_i\theta_j\theta_k} \propto \tilde{\gamma}_{\theta_i\theta_j\theta_k}$, where $\tilde{\gamma}_{\theta_i\theta_j\theta_k}$ is the third-rank gamma matrix on S^3 which is given by the anti-symmetrised product of three gamma matrices $\tilde{\gamma}_{\theta_i\theta_j\theta_k} = \tilde{\gamma}_{[\theta_i}\tilde{\gamma}_{\theta_j}\tilde{\gamma}_{\theta_k]}$ - see e.g. Ref. [30].) Similarly, for the strictly massless spin-5/2 theory on dS_4, where

$$\mathscr{H}^{\sigma} = \{\,\Psi^{(B;\sigma;\tilde{M}\ell;\tilde{\rho})}_{\boldsymbol{N}_2}\,\} = \{\,\Psi^{(III\text{-}I;\sigma;(\pm2)\ell;\tilde{\rho})}_{\mu\nu}\,\},$$

it can readily be shown that

$$\tilde{\epsilon}_{\theta_i}{}^{\theta_j\theta_k}\tilde{\nabla}_{\theta_j}\Psi^{(III\text{-}I;\sigma;(\pm2)\ell;\tilde{\rho})}_{\theta_k\theta_l} \propto \tilde{\nabla\!\!\!/}\Psi^{(III\text{-}I;\sigma;(\pm2)\ell;\tilde{\rho})}_{\theta_i\theta_l} = i\sigma\left(\ell+\frac{3}{2}\right)\Psi^{(III\text{-}I;\sigma;(\pm2)\ell;\tilde{\rho})}_{\theta_i\theta_l}. \tag{8.16}$$

In the case of the partially massless spin-5/2 field on dS_4, where

$$\mathscr{H}^{\sigma} = \{\,\Psi^{(B;\sigma;\tilde{M}\ell;\tilde{\rho})}_{\boldsymbol{N}_2}\,\} = \{\,\Psi^{(II\text{-}I;\sigma;(\pm1)\ell;\tilde{\rho})}_{\mu\nu}, \Psi^{(III\text{-}I;\sigma;(\pm1)\ell;\tilde{\rho})}_{\mu\nu}\,\},$$

the helicity operator can not be defined in the same way. However, it is natural to identify $\mathscr{H}^{-}$ with the set of states with helicities $(-5/2, -3/2)$ and $\mathscr{H}^{+}$ with the set of states with helicities $(+5/2, +3/2)$.

Below we choose a specific dS invariant scalar product for the analytically continued STSSH's with imaginary mass parameter. By calculating the associated norms of the modes we will verify the non-unitarity of the spin$(N,1)$ representations for even $N > 4$ for arbitrary imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$). Also, in the case of massless theories on dS_4, we will show that each of the spin$(4,1)$ invariant subspaces, $\mathscr{H}^{-}$ and $\mathscr{H}^{+}$, separately forms a unitary representation of spin$(4,1)$ (and, thus, we have a direct sum of UIR's of spin$(4,1)$).

8.2 Massless spin-3/2 and spin-5/2 representations of spin$(N,1)$ for N even: norms of the eigenmodes

In this Subsection, by calculating the norms of the analytically continued STSSH's explicitly, we show that the representations of spin$(N,1)$ (even $N \geq 4$) formed by the spin-3/2 and spin-5/2 TT mode solutions of eq. (1.3) with arbitrary imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$) are non-unitary, unless the following two conditions hold at the same time: i) $N = 4$ and ii) $\tilde{M}$ is tuned to the massless values (1.5). For $N = 4$, we show that the TT modes in the massless theories form a direct sum of UIR's of spin$(4,1)$. In other words, in the present Subsection we verify the results of Subsection 8.1 for even $N > 4$ and we prove statement 2.

Let $\Psi^{(1)}_{\mu_1\ldots\mu_r}$ and $\Psi^{(2)}_{\mu_1\ldots\mu_r}$ be any two analytically continued STSSH's [satisfying eqs. (1.3) and (1.4)] with the same imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$) on dS_N (N even). The (axial) vector current

$$J^{\mu} = i\,\overline{\Psi}^{(1)}_{\mu_1\ldots\mu_r}\gamma^{\mu}\gamma^{N+1}\Psi^{(2)\mu_1\ldots\mu_r} \tag{8.17}$$

is covariantly conserved [28], where $\overline{\Psi}^{(1)}_{\mu_1\ldots\mu_r} = i\Psi^{(1)\dagger}_{\mu_1\ldots\mu_r}\gamma^0$ and we used the fact that gamma matrices are covariantly constant. Then, the scalar product

$$\langle\Psi^{(1)},\Psi^{(2)}\rangle_{(r)} = \int_{S^{N-1}} \sqrt{-g}\, d\boldsymbol{\theta}_{N-1}\, J^0 \tag{8.18}$$

is time independent, where $d\boldsymbol{\theta}_{N-1}$ stands for $d\theta_1\, d\theta_2 ... d\theta_{N-1}$, while g is the determinant of the de Sitter metric. This scalar product is equivalently written as

$$\langle\Psi^{(1)},\Psi^{(2)}\rangle_{(r)} = \cosh^{N-1} t \int_{S^{N-1}} \sqrt{\tilde{g}}\, d\boldsymbol{\theta}_{N-1}\, \Psi^{(1)\dagger}_{\mu_1\ldots\mu_r}\gamma^{N+1}\Psi^{(2)\mu_1\ldots\mu_r}, \tag{8.19}$$

where we used $(\gamma^0)^2 = -\mathbf{1}$, as well as

$$\sqrt{-g} = \cosh^{N-1} t\, \sqrt{\tilde{g}}, \tag{8.20}$$

while $\sqrt{\tilde{g}}$ is given by eq. (2.24).

Now let us show that the scalar product (8.19) is de Sitter invariant. Let ξ^μ be a Killing vector of dS_N satisfying

$$\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0. \tag{8.21}$$

The infinitesimal change $\delta_\xi J^\mu$ of the current (8.17) under the spin$(N,1)$ transformation generated by ξ^μ is described by the Lie derivative

$$\begin{aligned}\delta_\xi J^\mu = \mathscr{L}_\xi J^\mu &= \xi^\nu\nabla_\nu J^\mu - J^\nu\nabla_\nu\xi^\mu \\ &= \nabla_\nu(\xi^\nu J^\mu - J^\nu\xi^\mu),\end{aligned} \tag{8.22}$$

where we used $\nabla_\mu J^\mu = \nabla_\mu\xi^\mu = 0$. Then, it is straightforward to find

$$\delta_\xi J^0 = \frac{1}{\sqrt{-g}}\partial_{\theta_\kappa}\left[\sqrt{-g}(\xi^{\theta_\kappa}J^0 - J^{\theta_\kappa}\xi^0)\right], \tag{8.23}$$

where $\kappa = 1, ..., N-1$. By integrating eq. (8.23) over S^{N-1} we find that the scalar product (8.19) is de Sitter invariant, as

$$\delta_\xi\, \langle\Psi^{(1)},\Psi^{(2)}\rangle_{(r)} = \int_{S^{N-1}} d\boldsymbol{\theta}_{N-1}\, \sqrt{-g}\, \delta_\xi J^0 = 0. \tag{8.24}$$

It is possible to calculate the norms of the analytically continued STSSH's of ranks 1 and 2 [the analytically continued STSSH's are defined by eq. (7.11)] using the de Sitter invariant scalar product (8.19). We find in this manner

$$
\begin{aligned}
\langle \Psi^{(B;\sigma;\tilde{M}\ell;\tilde{\rho})}, \Psi^{(B';\sigma';\tilde{M}\ell';\tilde{\rho}')}\rangle_{(r)} = (-\sigma) \times & \binom{r}{\tilde{r}_{(B)}} 2^{N+2r-1-4\tilde{r}_{(B)}} \\
& \times \frac{|\Gamma(\ell+\frac{N}{2})|^2}{\Gamma(\ell+\frac{N}{2}+\tilde{M})\Gamma(\ell+\frac{N}{2}-\tilde{M})} \\
& \times \left(\prod_{j=\tilde{r}_{(B)}}^{r-1} \frac{N+2j-1}{N+j+\tilde{r}_{(B)}-2} \right) \\
& \times \left(\prod_{j=\tilde{r}_{(B)}}^{r-1} \frac{1}{(\ell-j)(\ell+N-1+j)} \right) \\
& \times \left(\prod_{j=1}^{r-\tilde{r}_{(B)}} \left\{ -\tilde{M}^2 + \left(r-j+\frac{N-2}{2} \right)^2 \right\} \right) \delta_{\sigma\sigma'}\delta_{\ell\ell'}\delta_{\tilde{\rho}\tilde{\rho}'}
\end{aligned}
\tag{8.25}
$$

for $r \in \{1,2\}$ and $B = I, II, III$ (where $\sigma = \pm$, $\tilde{M} \in \mathbb{R} \setminus \{0\}$, $\tilde{r}_{(B)} \leq r$, while $\tilde{r}_{(I)} = 0$, $\tilde{r}_{(II)} = 1$ and $\tilde{r}_{(III)} = 2$). The norms of type-*I* and type-*II* spin-3/2 modes, as well as the norms of type-*II* and type-*III* spin-5/2 modes, can be determined by direct calculation using the time-independence of the scalar product (8.19). The calculations are simplified by using

$$
\left| \hat{\phi}^{(a)}_{\tilde{M}\ell}(t=0) \right|^2 - \left| \hat{\psi}^{(a)}_{\tilde{M}\ell}(t=0) \right|^2 = \frac{2^{N+2a-1}\,|\Gamma(\ell+\frac{N}{2})|^2}{\Gamma(\ell+\frac{N}{2}+\tilde{M})\Gamma(\ell+\frac{N}{2}-\tilde{M})}. \tag{8.26}
$$

[This equation can readily be proved using eqs. (B.7) and (B.8).] Once the norms of type-*II* and type-*III* spin-5/2 modes have been calculated, the norm of the type-*I* spin-5/2 modes is readily found using the dS invariance (8.6) of the inner product between type-*I* and type-*II* modes (by making use of the transformation formulae (8.2) and (8.3)).

As a consistency check, by using our result for the norms (8.25) of the eigenmodes with spin $s \in \{3/2, 5/2\}$, we can reproduce the strictly/partially massless tunings (1.5) for the imaginary mass parameter as follows. For $r = 1$ (spin-3/2 field), we find that the norm (8.25) of type-*I* modes ($\tilde{r}_{(I)} = 0$) becomes zero if $\tilde{M}^2 = (N-2)^2/4$, corresponding to the strictly massless spin-3/2 theory. For $r = 2$ (spin-5/2 field), we find that both type-*I* ($\tilde{r}_{(I)} = 0$) and type-*II* ($\tilde{r}_{(II)} = 1$) modes have zero norm (8.25) for $\tilde{M}^2 = N^2/4$, corresponding to the strictly massless spin-5/2 theory. Finally, for $r = 2$, we find that type-*I* ($\tilde{r}_{(I)} = 0$) modes have zero norm (8.25) for $\tilde{M}^2 = (N-2)^2/4$, corresponding to the partially massless spin-5/2 theory.

We observe that the sign of the norm (8.25) depends on the sign of the spin projection index $\sigma = \pm$, as expected from the dS invariance of the scalar product (8.7)-(8.9). Thus, it is easy to understand that representations of spin$(N,1)$ with spin $s \in \{3/2, 5/2\}$ and arbitrary imaginary mass parameter $M = i\tilde{M} \neq 0$ are non-unitary for even $N > 4$, since positive-norm and negative-norm modes mix with each other under spin$(N,1)$ [see the transformation formulae (8.3) and (8.4)]. Similarly, we find that for $N = 4$ the representations of $\text{spin}(4,1)$ are not unitary if $\tilde{M}$ is not given by the massless values in eq. (1.5).

Now, let us suppose that the following two conditions are satisfied at the same time: i) $N = 4$ and ii) the imaginary mass parameter is tuned to the massless values (1.5). According to our discussion for the $N = 4$ case in Subsection 8.1, each of the solution subspaces, $\mathscr{H}^-$ and $\mathscr{H}^+$, forms separately an irreducible representation of $\text{spin}(4,1)$ with spin $s = r + 1/2$ ($r \in \{1, 2\}$). (The 'pure gauge' modes are identified with zero in each subspace.) We can show that the subspaces $\mathscr{H}^-$ and $\mathscr{H}^+$ form a direct sum of UIR's of $\text{spin}(4,1)$ as follows. By observing that the norms (8.25) of the eigenmodes depend on the spin projection, we have:

- For the set of eigenmodes with negative spin projection (or negative helicity) $\mathscr{H}^- = \{\, \Psi^{(B;-;\tilde{M}\ell;\tilde{\rho})}_{\boldsymbol{N}_r} \,\}$, the positive-definite inner product is
$$\begin{aligned}&\langle \Psi^{(B;-;\tilde{M}\ell;\tilde{\rho})}, \Psi^{(B';-;\tilde{M}\ell';\tilde{\rho}')} \rangle_{(r)} \\ &= \cosh^3 t \int_{S^3} \sqrt{\tilde{g}}\, d\boldsymbol{\theta}_3\, \Psi^{(B;-;\tilde{M}\ell;\tilde{\rho})\dagger}_{\mu_1 \dots \mu_r}\, \gamma^5\, \Psi^{(B';-;\tilde{M}\ell';\tilde{\rho}')\mu_1 \dots \mu_r}\end{aligned}$$
The explicit expression for the positive-definite norm is given by eq. (8.25).

- For the set of eigenmodes with positive spin projection (or positive helicity) $\mathscr{H}^+ = \{\, \Psi^{(B;+;\tilde{M}\ell;\tilde{\rho})}_{\boldsymbol{N}_r} \,\}$, the positive-definite inner product is
$$- \langle \Psi^{(B;+;\tilde{M}\ell;\tilde{\rho})}, \Psi^{(B';+;\tilde{M}\ell';\tilde{\rho}')} \rangle_{(r)} \,.$$
The explicit expression for the positive-definite norm is given by the negative of eq. (8.25).

8.3 The massless spin-3/2 and spin-5/2 representations of spin$(N,1)$ are non-unitary for N odd

In this Subsection, we show that the massless field theories with spin $s \in \{3/2, 5/2\}$ on dS_N (N odd) are not unitary (i.e. we prove statement 3).

As in the case with N even, we study the transformation properties of the analytically continued STSSH's under the de Sitter boost (8.1). By making the replacements (7.5) in the $\text{spin}(N+1)$ transformation formulae (6.29), (6.30) and (6.31) [and using eq. (7.12)], we find

$$\begin{aligned}\mathbb{L}_X \Psi^{(I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} = & - i\, c_{(\ell)}\, \mathscr{A}^{(I)}\, \Psi^{(I;\tilde{M}\,(\ell+1);\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} - \frac{i}{c_{(\ell-1)}} \mathscr{B}^{(I)}\, \Psi^{(I;\tilde{M}\,(\ell-1);\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} \\ & - \sigma_{N-1}\, \varkappa^{(I)}\, \Psi^{(I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} - i \mathscr{K}^{(I \to II)}\, \Psi^{(II\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r}, \quad (8.27)\end{aligned}$$

$$\begin{aligned}\mathbb{L}_X \Psi^{(II\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} = & - i\, c_{(\ell)} \mathscr{A}^{(II)}\, \Psi^{(II\text{-}I;\tilde{M}\,(\ell+1);\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} \\ & - \frac{i}{c_{(\ell-1)}} \mathscr{B}^{(II)}\, \Psi^{(II\text{-}I;\tilde{M}\,(\ell-1);\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} \\ & - \sigma_{N-1}\, \varkappa^{(II)} \Psi^{(II\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} - i \mathscr{K}^{(II \to I)}\, \Psi^{(I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} \\ & - i \mathscr{K}^{(II \to III)}\, \Psi^{(III\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} \quad (8.28)\end{aligned}$$

$(r = 1, 2,)$ and

$$\begin{aligned}
&\mathbb{L}_X \Psi^{(\mathit{III}\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\mu_1\mu_2} \\
&= -i\, c_{(\ell)} \mathscr{A}^{(\mathit{III})}\, \Psi^{(\mathit{III}\text{-}I;\tilde{M}\,(\ell+1);\sigma_{N-1};m;\rho)}_{\mu_1\mu_2} - \frac{i}{c_{(\ell-1)}} \mathscr{B}^{(\mathit{III})}\, \Psi^{(\mathit{III}\text{-}I;\tilde{M}\,(\ell-1);\sigma_{N-1};m;\rho)}_{\mu_1\mu_2} \\
&\quad - \sigma_{N-1}\, \varkappa^{(\mathit{III})} \Psi^{(\mathit{III}\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\mu_1\mu_2} - i \mathscr{K}^{(\mathit{III}\to \mathit{II})}\, \Psi^{(\mathit{II}\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\mu_1\mu_2},
\end{aligned} \tag{8.29}$$

respectively, where all the coefficients on the right-hand sides of eqs. (8.27)-(8.29) are the same as the coefficients used in the case with N even [see eqs. (8.2)-(8.4)].

Now, we will show that the representations of $\mathrm{spin}(N, 1)$ (N odd) formed by the spin-$3/2$ and spin-$5/2$ TT mode solutions of eq. (1.3) are non-unitary for all values of the imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$). Let $\langle \Psi^{(1)}, \Psi^{(2)} \rangle$ be a dS invariant scalar product for any two analytically continued STSSH's $\Psi^{(1)}, \Psi^{(2)}$ [satisfying eqs. (1.3) and (1.4)] with $M = i\tilde{M}$ and $\tilde{M} \neq 0$. We will show that this scalar product must vanish for all eigenmodes. First, let us make the following observation. The infinitesimal transformations $\mathbb{L}_X \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r}$ given by eqs. (8.27)-(8.29), always give rise to a term of the form $\varkappa^{(B)}\, \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r}$ in the linear combination on the right-hand sides of each of eqs. (8.27)-(8.29). The coefficients $\varkappa^{(I)}, \varkappa^{(\mathit{II})}$ and $\varkappa^{(\mathit{III})}$ are given by eqs. (8.10), (8.11) and (8.12), respectively, and they are all non-zero for N odd. Thus, by combining the dS invariance of the scalar product:

$$\langle \mathbb{L}_X \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)}, \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)} \rangle + \langle \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)}, \mathbb{L}_X \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)} \rangle = 0 \tag{8.30}$$

with the transformation formulae (8.27)-(8.29), we find

$$\langle \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)}, \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)} \rangle = 0 \tag{8.31}$$

for $B = I, \mathit{II}\text{-}I, \mathit{III}\text{-}I$ and for all $\tilde{M} \neq 0$. Then, since the eigenmodes with different labels are orthogonal, we conclude that there is no dS invariant scalar product (which is not identically zero).

9 Summary and discussions

In this paper, we showed that the strictly massless spin-$3/2$ field (i.e. gravitino field) theory, as well as the strictly and partially massless spin-$5/2$ field theories on dS_N ($N \geq 3$) are unitary only in $N = 4$ dimensions. In order to arrive at this result, we studied the group-theoretic properties of the eigenmodes for the following field theories with imaginary mass parameter on dS_N ($N \geq 3$): the vector-spinor field and the symmetric rank-2 tensor-spinor field. The corresponding eigenmodes satisfy eq. (1.3) with $M = i\tilde{M}$ ($\tilde{M} \neq 0$) and the TT conditions (1.4). These eigenmodes were obtained by analytically continuing STSSH's on S^N. The transformation properties of these eigenmodes under a $\mathrm{spin}(N, 1)$ boost were studied. By using these transformation properties, we showed that all dS invariant scalar products for even $N > 4$ are indefinite. We also showed that all dS invariant scalar products must vanish identically for odd N. It was found that dS invariant scalar products that are positive-definite are allowed only for strictly and partially massless theories in $N = 4$ dimensions (and, thus, these theories are unitary). Also, for these unitary spin-s ($s \in \{3/2, 5/2\}$) theories in dS_4, we showed that eigenmodes with positive helicity and the ones with negative helicity separately form UIR's of spin$(4,1)$. All the results mentioned in this paragraph are summarised as statements 1, 2 and 3 in the Introduction.

In Appendix A, we verify our main result by using the known classification of the UIR's of spin$(N,1)$. Also, our analysis in Appendix A suggests that the (strictly and partially) massless totally symmetric tensor-spinor fields with arbitrary half-odd-integer spin $s \geq 7/2$ on dS_N ($N \geq 3$) are unitary only for $N = 4$. It would be interesting to verify this by studying the group-theoretic properties of the corresponding eigenmodes, as we did in the present paper for the spin-3/2 and spin-5/2 fields.

It would also be interesting to investigate whether our result about the non-unitarity of the gauge-invariant spin-3/2 and spin-5/2 theories on dS_N for $N \neq 4$ could be extended to other N-dimensional vacuum spacetimes with positive cosmological constant. As an argument pointing towards the possible generalisation of our result, we would like to mention the forbidden mass range for the symmetric spin-2 field on dS_N [13, 31]. The forbidden mass range for the symmetric spin-2 field on dS_N was explained group-theoretically in Ref. [13] and it was first observed for dS_4 in Ref. [31]. However, it was later shown that the forbidden mass range exists in all 4-dimensional vacuum spacetimes with positive cosmological constant [32].

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A Interpretation of the main result in terms of the classification of the Unitary Irreducible Representations of spin$(N,1)$

In this Appendix, we verify the main result of this paper by using the classification of the Unitary Irreducible Representations (UIR's) of spin$(N,1)$ ($N \geq 3$) given by Ottoson [19] and Schwarz [20] (see also Refs. [33–35]). More specifically, we will demonstrate that there are no UIR's of spin$(N,1)$ that correspond to the strictly massless spin-3/2 field and to the strictly and partially massless spin-5/2 fields on dS_N for $N \neq 4$. Then, for $N = 4$, we will identify the UIR's of spin$(4,1)$ that correspond to the unitary strictly massless spin-3/2 field and to the unitary strictly and partially massless spin-5/2 fields on dS_4 (these UIR's have also been identified in Ref. [36]). For the sake of completeness, we also identify the UIR's of spin$(N,1)$ ($N \geq 3$) that correspond to spin-3/2 and spin-5/2 fields with real mass parameter M on dS_N. The identification of the UIR's of spin$(N,1)$ that correspond to massive and massless totally symmetric tensors of arbitrary rank on dS_N has been given in Ref. [15]. More recently, a field theoretic interpretation of the UIR's of spin$(N,1)$ for totally symmetric and mixed-symmetry tensor (and tensor-spinor) fields on dS_N was given in Ref. [16]. However, as mentioned in the Introduction, we disagree with the claims made in Ref. [16] about the unitarity of the gauge-invariant symmetric tensor-spinor fields for $N \neq 4$.

Below, we begin by reviewing the classification of the UIR's by Ottoson [19] and Schwarz [20]. Then, we will use this classification to verify the main result of the present paper: 'the strictly massless spin-3/2 field theory and the strictly and partially massless spin-5/2 field theories on dS_N are unitary only in $N = 4$ dimensions'.

Ottoson [19] and Schwarz [20] have obtained the UIR's of spin$(N,1)$ in the decomposition spin$(N,1) \supset$ spin(N).[14] Under this decomposition, an irreducible representation of spin(N) appears at most once in a UIR of spin$(N,1)$ [37]. The case with $N = 2p$ and the case with $N = 2p+1$, where p is a positive integer, are studied separately.

It is well known that a representation of spin$(2p)$ or spin$(2p+1)$ is labelled by the highest weight of the representation [25, 26], denoted here as $[f] = (f_1, f_2, ..., f_p)$, where

$$f_1 \geq f_2 \geq ... \geq f_{p-1} \geq |f_p| \qquad \text{for spin}(2p) \tag{A.1}$$

$$f_1 \geq f_2 \geq ... \geq f_{p-1} \geq f_p \geq 0 \qquad \text{for spin}(2p+1). \tag{A.2}$$

The label f_p can be negative for spin$(2p)$. The labels f_j $(j = 1, ..., p)$ in eqs. (A.1) and (A.2) are all integers or all half-odd integers.

A.1 Classification of the UIR's of spin$(N,1)$

We adopt the notation for the labels of the UIR's that was used by Higuchi in Ref. [15].
UIR's of spin$(2p,1)$. A UIR of spin$(2p,1)$ $(p = N/2 \geq 2)$ is labelled by the set of numbers $[F] = (F_0, F_1, ..., F_{p-1})$. The labels $F_1, ..., F_{p-1}$ satisfy

$$F_1 \geq F_2 \geq ... \geq F_{p-1} \geq 0 \tag{A.3}$$

and they are all integers or all half-odd integers at the same time. A representation $(f_1, ..., f_p)$ of spin$(2p)$ that is contained in the UIR $(F_0, F_1, ..., F_{p-1})$ satisfies

$$f_1 \geq F_1 \geq f_2 \geq F_2 \geq ... \geq f_{p-1} \geq F_{p-1} \geq |f_p|. \tag{A.4}$$

[14] As an alternative to the decomposition SO$(N,1) \supset$ SO(N) (or Spin$(N,1) \supset$ Spin(N)), the UIR's of the group SO$(N,1)$ can be obtained by making use of the theory of induced representations for the parabolic subgroup of SO$(N,1)$, as in Refs. [16, 26]. This approach is suitable for applications to Conformal Quantum Field theory in $\mathbb{R}^{N-1}$, where the conformal group is SO$(N,1)$ [26]. Also, this approach is related to realisations of the dS/CFT correspondence between field representations on dS_N (like the ones studied in the present paper) and conformal fields in $\mathbb{R}^{N-1}$ [14]. In this classification, each UIR of SO$(N,1)$ is labelled by the highest weight of SO$(N-1)$ (encoding the spin of the field in dS_N, as well as the spin of the corresponding conformal field in $\mathbb{R}^{N-1}$) and a "conformal weight" $\Delta_c \in \mathbb{C}$ which is the weight for SO$(1,1)$ (see Ref. [16] for more details). The conformal weight can be expressed in terms of the mass parameter of the field in dS_N [14, 16].

Ottoson's labels [19] and our labels are related to each other by [15]:

$$f_j = l_{2p-1,j} + j - p \quad (j = 1, ..., p), \tag{A.5a}$$
$$F_j = l_{2p,j} + j - p \quad (j = 1, ..., p-1), \tag{A.5b}$$
$$F_0 = l_{2p,p} - p. \tag{A.5c}$$

Schwarz's labels [20] and our labels are related to each other by:

$$f_j = m_{2p,p-j+1} \quad (j = 1, ..., p), \tag{A.6a}$$
$$F_j = m_{2p+1,p-j} \quad (j = 1, ..., p-1), \tag{A.6b}$$
$$F_0 = z_{2p+1,p}. \tag{A.6c}$$

The UIR's of spin$(2p,1)$ $(p = N/2)$ are classified as follows:

- **Principal Series $D_{\text{prin}}([F])$ (where $[F] = (F_0, F_1, ..., F_{p-1})$) :**

$$F_0 = -p + \frac{1}{2} + iy = -\frac{N-1}{2} + iy \quad (y > 0).$$

The labels $F_1, F_2, ..., F_{p-1}$ are all integers or half-odd integers.

- **Complementary Series $D_{\text{comp}}([F])$:**

$$-\frac{N-1}{2} = -p + \frac{1}{2} \leq F_0 < -\tilde{n} \quad (\tilde{n} \text{ is an integer and } 0 \leq \tilde{n} \leq p-1).$$

If $0 \leq \tilde{n} < p-1$, then $F_{\tilde{n}+1} = F_{\tilde{n}+2} = ... = F_{p-1} = 0$ and $F_1, F_2, ..., F_{\tilde{n}}$ are all positive integers, while for the spin$(2p)$ content we have $f_{\tilde{n}+2} = f_{\tilde{n}+3} = ... = f_p = 0$. If $\tilde{n} = p-1$, then $F_1, F_2, ..., F_{p-1}$ are all positive integers. Our Complementary Series are called Exceptional Series $D(e; l_{2p,1}, ..., l_{2p,p})$ in Ottoson's classification [15, 19]. (Our notation is related to Schwarz's notation [20] as follows. The case with $0 \leq \tilde{n} < p-1$ corresponds to $D^k(m_{2p+1,k+1} \ ... \ m_{2p+1,p-1}; x_{2p+1,p})$, where k is related to $\tilde{n}$ by $k = p - \tilde{n} - 1$, while the case with $\tilde{n} = p-1$ corresponds to $D^0(m_{2p+1,1} \ ... \ m_{2p+1,p-1}; x_{2p+1,p})$.)

- **Exceptional Series $D_{\text{ex}}([F])$:**

$$F_0 = -\tilde{n} \quad (\tilde{n} \text{ is an integer and } 1 \leq \tilde{n} \leq p-1).$$

If $1 \leq \tilde{n} < p-1$, then $F_{\tilde{n}+1} = F_{\tilde{n}+2} = ... = F_{p-1} = 0$ and $F_1, F_2, ..., F_{\tilde{n}}$ are all positive integers, while for the spin$(2p)$ content we have $f_{\tilde{n}+1} = f_{\tilde{n}+2} = ... = f_p = 0$. If $\tilde{n} = p-1$, then $F_1, F_2, ..., F_{p-1}$ are all positive integers, while $f_p = 0$. Our Exceptional Series is called Supplementary Series $D(s; l_{2p,1}, ..., l_{2p,p})$ in Ottoson's classification [15, 19]. (Our notation is related to Schwarz's notation [20] as follows. The case with $1 \leq \tilde{n} < p-1$ corresponds to $D^k(m_{2p+1,k+1} \ ... \ m_{2p+1,p-1}; m_{2p+1,p})$, where k is related to $\tilde{n}$ by $k = p - \tilde{n} - 1$, while the case with $\tilde{n} = p-1$ corresponds to $D^0(m_{2p+1,1} \ ... \ m_{2p+1,p-1}; m_{2p+1,p})$.)

- **Discrete Series $D^{\pm}([F])$:** F_0 is real and it is an integer or half-odd integer at the same time as the labels $F_1, F_2, ..., F_{p-1}$. Also, the following conditions have to be satisfied:

$$F_{p-1} \geq f_p \geq F_0 + p \geq \frac{1}{2} \qquad \text{for } D^+([F]), \tag{A.7}$$

$$-F_{p-1} \leq f_p \leq -(F_0 + p) \leq -\frac{1}{2} \qquad \text{for } D^-([F]). \tag{A.8}$$

  Our Discrete Series $D^{\pm}([F])$ are called Exceptional Series $D(\pm; l_{2p,1}, ..., l_{2p,p})$ in Ottoson's classification [15, 19]. Also, our Discrete Series $D^{\pm}([F])$ correspond to

$$D^{\pm}(m_{2p+1,1} \dots m_{2p+1,p-1}; m_{2p+1,p})$$

  in Schwarz's classification [20].

For a UIR of spin$(2p, 1)$ labelled by $[F] = (F_0, F_1, ..., F_{p-1})$ - or by $(l_{2p,1}, l_{2p,2}, ..., l_{2p,p})$ in Ottoson's notation [19] - the quadratic Casimir $C_2([F])$ is expressed as

$$C_2([F]) = \sum_{k=0}^{p-1} F_k \left(F_k + 2p - 2k - 1\right) = \sum_{j=1}^{p} l_{2p,j}(l_{2p,j} - 1) - \frac{p(p-1)(p+1)}{3}. \tag{A.9}$$

UIR's of spin$(2p+1,1)$. A UIR of spin$(2p+1,1)$ ($p = (N-1)/2$) is labelled by $[F] = (F_0, F_1, ..., F_p)$. The labels $F_1, ..., F_p$ satisfy

$$F_1 \geq F_2 \geq ... \geq F_p \geq 0 \tag{A.10}$$

and they are all integers or all half-odd integers. A representation $(f_1, ..., f_p)$ of spin$(2p+1)$ that is contained in the UIR $(F_0, F_1, ..., F_p)$ satisfies

$$f_1 \geq F_1 \geq f_2 \geq F_2 \geq ... \geq f_p \geq F_p \geq 0. \tag{A.11}$$

Ottoson's labels [19] and our labels are related to each other by [15]:

$$f_j = l_{2p,j} + j - p - 1 \qquad (j = 1, ..., p), \tag{A.12a}$$
$$F_j = l_{2p+1,j} + j - p \qquad (j = 1, ..., p), \tag{A.12b}$$
$$F_0 = l_{2p+1,p+1} - p, \tag{A.12c}$$

while Schwarz's labels [20] and our labels are related to each other by:

$$f_j = m_{2p+1,p-j+1} \qquad (j = 1, ..., p), \tag{A.13a}$$
$$F_j = m_{2p+2,p-j+1} \qquad (j = 1, ..., p), \tag{A.13b}$$
$$F_0 = z_{2p+2,p+1}. \tag{A.13c}$$

The UIR's of spin$(2p+1,1)$ (where $p = (N-1)/2 \geq 1$) are classified as follows:

- **Principal Series $D_{\text{prin}}([F])$ (where $[F] = (F_0, F_1, ..., F_p)$) :**

$$F_0 = -p + iy = -\frac{N-1}{2} + iy \qquad (y \in \mathbb{R}).$$

The labels $F_1, F_2, ..., F_p$ are all integers or all half-odd integers. If $F_p = 0$, then the UIR with $F_0 = -(N-1)/2 + iy$ and the UIR with $F_0 = -(N-1)/2 - iy$ are equivalent (and thus we can let $y \geq 0$).

- **Complementary Series $D_{\text{comp}}([F])$:**

$$-\frac{N-1}{2} = -p < F_0 < -\tilde{n} \qquad (\tilde{n} \text{ is an integer and } 0 \leq \tilde{n} \leq p-1),$$

while $F_{\tilde{n}+1} = F_{\tilde{n}+2} = ... = F_p = 0$ and $F_1, F_2, ..., F_{\tilde{n}}$ are all positive integers, where for the spin$(2p+1)$ content we have $f_{\tilde{n}+2} = f_{\tilde{n}+3} = ... = f_p = 0$. (Our Complementary Series corresponds to $D^k(m_{2p+2,k+1} \; ... \; m_{2p+2,p}; x_{2p+2,p+1})$ in Schwarz's classification [20], where k is related to $\tilde{n}$ by $k = p - \tilde{n}$.)

- **Exceptional Series $D_{\text{ex}}([F])$:**

$$F_0 = -\tilde{n} \qquad (\tilde{n} \text{ is an integer and } 1 \leq \tilde{n} \leq p-1),$$

where $F_{\tilde{n}+1} = F_{\tilde{n}+2} = ... = F_p = 0$ and $F_1, F_2, ..., F_{\tilde{n}}$ are all positive integers, where for the spin$(2p+1)$ content we have $f_{\tilde{n}+1} = f_{\tilde{n}+2} = ... = f_p = 0$. (Our Exceptional Series corresponds to $D^k(m_{2p+2,k+1} \; ... \; m_{2p+2,p}; m_{2p+2,p+1})$ in Schwarz's classification [20], where k is related to $\tilde{n}$ by $k = p - \tilde{n}$.)

For a UIR of spin$(2p+1,1)$ labelled by $[F] = (F_0, F_1, ..., F_p)$ - or by $(l_{2p+1,1}, ..., l_{2p+1,p+1})$ in Ottoson's notation [19] - the quadratic Casimir $C_2([F])$ is expressed as

$$C_2([F]) = \sum_{k=0}^{p} F_k \left(F_k + 2p - 2k\right) = \sum_{j=1}^{p+1} l^2_{2p+1,j} - \frac{p(p+1)(p+\frac{1}{2})}{3}. \tag{A.14}$$

A.2 The quadratic Casimir for analytically continued STSSH's and some useful information for massless theories with spin $s \in \{3/2, 5/2\}$

The quadratic Casimir for the spin$(N,1)$ representation formed by the analytically continued STSSH's with imaginary mass parameter on dS_N can be determined as follows.

***N* even.** The STSSH's of (arbitrary) rank r on S^N satisfy eqs. (7.3) and (7.4). The STSSH's form a unitary representation of spin$(N+1)$ labelled by the highest weight

$$\begin{aligned} \lambda &= \left(\lambda_1, ..., \lambda_{N/2}\right) \\ &= \left(n + \frac{1}{2}, r + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right) \qquad (n = r, \, r+1, ...). \end{aligned}$$

The quadratic Casimir $\mathscr{C}_2(\lambda)$ for any spin$(N+1)$ (N even) representation λ is given by [26]

$$\mathscr{C}_2(\lambda) = \sum_{j=1}^{N/2} \lambda_j \left(\lambda_j + N - 2j + 1\right). \tag{A.15}$$

By specialising to the spin$(N+1)$ representation $\lambda = \left(n+\frac{1}{2}, r+\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$ formed by STSSH's of rank r on S^N, we find the quadratic Casimir from eq. (A.15) as

$$\begin{aligned}\mathscr{C}_2(\lambda) &= \left(n+\frac{N}{2}\right)^2 - r - \frac{N(N-1)}{4} + \frac{(N-2)(N-3)}{8} + s(s+N-2) \\ &= -\nabla^\mu \nabla_\mu + \frac{(N-2)(N-3)}{8} + s(s+N-2) \quad (\text{where } s = r+1/2),\end{aligned} \tag{A.16}$$

where in the second line we used that $\nabla^\mu \nabla_\mu$ acts on STSSH's of rank r on S^N as

$$\nabla^\mu \nabla_\mu = \nabla\!\!\!/^2 + \frac{N(N-1)}{4} + r.$$

Using the analytic continuation techniques discussed in Section 7, we can find the quadratic Casimir for the representation of spin$(N,1)$ formed by the analytically continued STSSH's with imaginary mass parameter $M = i\tilde{M}$ on dS_N. More specifically, by replacing n by $\tilde{M} - N/2$ in eq. (A.16) we immediately find

$$\mathscr{C}_{2;dS_N} = \tilde{M}^2 - r - \frac{N(N-1)}{4} + \frac{(N-2)(N-3)}{8} + s(s+N-2) \tag{A.17}$$

($s = r+1/2$). If the analytically continued STSSH's form a UIR of spin$(N,1)$ labelled by $[F] = (F_0, F_1, ..., F_{p-1})$ ($p = N/2$), then the analytically continued Casimir (A.17) coincides with the spin$(N,1)$ Casimir $C_2([F])$ in eq. (A.9).

***N* odd**. By working as in the case with N even, we find that the quadratic Casimir for the representation of spin$(N,1)$ (N odd) formed by the analytically continued STSSH's with imaginary mass parameter on dS_N is given again by eq. (A.17).

For later convenience, recall that the TT eigenmodes for the strictly massless spin-3/2 theory are given by the analytically continued STSSH's of rank $r=1$ (see Section 7) with imaginary mass parameter given by eq. (1.5) with $r = \tau = 1$. Similarly, the TT eigenmodes for the strictly (partially) massless spin-5/2 theory are given by the analytically continued STSSH's of rank $r=2$ with imaginary mass parameter given by eq. (1.5) with $r = \tau+1 = 2$ ($r = \tau = 2$). All TT eigenmodes with spin $s = r+1/2$ ($r \in \{1, 2\}$) on dS_N are constructed in terms of STSSH's of rank $\tilde{r}$ ($0 \le \tilde{r} \le r$) on S^{N-1} (see Sections 4, 5 and 7). The (strictly and partially) massless representations of spin$(N,1)$ are formed by the non-zero-norm TT eigenmodes. The latter consist only of the TT eigenmodes on dS_N for which the corresponding STSSH's of rank $\tilde{r}$ on S^{N-1} satisfy $r - \tau + 1 \le \tilde{r} \le r$ (see Subsection 7.2). In other words, the strictly massless spin-3/2 representation ($r = \tau = 1$) is formed by type-*II* modes ($\tilde{r} = 1$). The strictly massless spin-5/2 representation ($r = \tau+1 = 2$) is formed by type-*III* modes ($\tilde{r} = 2$). The partially massless spin-5/2 representation ($r = \tau = 2$) is formed by type-*II* ($\tilde{r} = 1$) and type-*III* ($\tilde{r} = 2$) modes.

A.3 Verifying the non-unitarity of the massless theories with spin $s \in \{3/2, 5/2\}$ for odd $N = 2p+1$

We will show that there are no UIR's of spin$(2p+1,1)$ that correspond to the massless fields with spin $s \in \{3/2, 5/2\}$ on dS_{2p+1} ($p \ge 1$). This will confirm that the representations of spin$(2p+1,1)$ corresponding to these fields are non-unitary.

Let us first discuss the cases with $N \geq 5$ ($p \geq 2$). Recall that the spin$(2p+1)$ content of the massless spin$(2p+1,1)$ representation with spin $s = r+1/2$ ($r \in \{1,2\}$) corresponds to the STSSH's of rank $\tilde{r}$ (with $r \geq \tilde{r} \geq r - \tau + 1$) on $S^{N-1} = S^{2p}$. Thus, the spin$(2p+1)$ content is

$$[f] = (f_1, f_2, ,..., f_p) = \left(\ell + \frac{1}{2}, \tilde{r} + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right), \qquad \text{for } p \geq 3 \quad (\ell \geq r \geq \tilde{r} \geq r - \tau + 1) \tag{A.18}$$

$$[f] = (f_1, f_2) = \left(\ell + \frac{1}{2}, \tilde{r} + \frac{1}{2}\right), \qquad \text{for } p = 2 \quad (\ell \geq r \geq \tilde{r} \geq r - \tau + 1). \tag{A.19}$$

As for the spin$(2p+1,1)$ labels, $F_1, F_2, ..., F_p$, they must all be half-odd-integers. It is clear that these values for $F_1, ..., F_p$ correspond neither to the UIR's of the Exceptional Series $D_{\text{ex}}([F])$, nor to the UIR's of the Complementary Series $D_{\text{comp}}([F])$, since these allow only integer values for $F_1, ..., F_p$. Finally, we can verify that the Principal Series $D_{\text{prin}}([F])$, where $F_0 = -p + iy$ ($y \in \mathbb{R}$), cannot describe the massless fields with spin $s \in \{3/2, 5/2\}$, since the allowed values for the representation labels $F_1, F_2, ..., F_p$ (with $F_0 = -p + iy$) do not give the correct value for the quadratic Casimir. This is readily understood by comparing the two expressions for the quadratic Casimir, i.e. comparing eq. (A.17) (with $\tilde{M}^2 = (r - \tau + \frac{N-2}{2})^2$) and eq. (A.14).

Now, let us examine the case with $p = 1$. Let us make the following observation for the strictly massless vector-spinor field on dS_3. This field has only type-I eigenmodes, i.e. all TT eigenmodes of this field are expressed in the 'pure gauge' form (7.13). This means that the spin$(3,1)$ representation corresponding to the strictly massless vector-spinor field on dS_3 and the representation corresponding to the spinors $\Lambda_{\pm}^{(\tilde{\ell})}$ in eq. (7.13) are equivalent. The spinors $\Lambda_{\pm}^{(\tilde{\ell})}$ have an imaginary mass parameter and thus they form a non-unitary representation of spin$(3,1)$. (By using the results of Section VB in Ref. [22], one can straightforwardly show that there is no de Sitter invariant scalar product for spinors with imaginary mass parameter on odd-dimensional dS_N. The argument is similar to that for the vector-spinors and rank-2 tensor-spinors presented in Subsection 8.3.) Thus, we have verified the non-unitarity of the strictly massless vector-spinor field on dS_3. Similarly, we can verify the non-unitarity of the strictly and partially massless rank-2 symmetric tensor-spinor fields on dS_3. As in the case of the strictly massless vector-spinor field, only type-I symmetric tensor-spinor eigenmodes exist on dS_3. Hence, the non-unitary spin$(3,1)$ representation corresponding to the strictly (partially) massless symmetric tensor-spinor field on dS_3 is equivalent to a non-unitary representation corresponding to a vector-spinor (spinor) field with imaginary mass parameter on dS_3 [see eq. (7.15) for the strictly massless case and eq. (7.18) for the partially massless case].

A.4 Verifying that the massless theories with spin $s \in \{3/2, 5/2\}$ for even $N = 2p$ are unitary only for $p = 2$

We will show that there are no UIR's of spin$(2p,1)$ that correspond either to the strictly massless spin-3/2 field or to the strictly and partially massless spin-5/2 fields on dS_{2p}

for $p \geq 3$. Then, for $p = 2$, we will identify the UIR's of spin$(4,1)$ that correspond to the unitary strictly massless spin-$3/2$ field on dS_4 and those that correspond to the unitary strictly and partially massless spin-$5/2$ fields on dS_4. Recall that a representation of spin$(2p,1)$ is labelled by $[F] = (F_0, F_1, ..., F_{p-1})$, while the spin$(2p)$ content is labelled by $[f] = (f_1, ..., f_p)$.

Even $N \geq 6$ ($p \geq 3$). The STSSH's of rank $\tilde{r}$ on S^{2p-1} - determining the spin$(2p)$ content for the massless theory with spin $s = r + 1/2$ ($r \in \{1, 2\}$) on dS_{2p} - are labelled by:

$$[f] = (f_1, f_2, , ..., f_p) = \left(\ell + \frac{1}{2}, \tilde{r} + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \pm\frac{1}{2}\right), \text{ for } p \geq 3 \quad (\ell \geq r \geq \tilde{r} \geq r - \tau + 1). \tag{A.20}$$

Also, the spin$(2p,1)$ labels $F_1, ..., F_{p-1}$ must be all half-odd-integers. The only Series of UIR's that allow these values for $F_1, ..., F_{p-1}$ are the Principal Series $D_{\text{prin}}([F])$ - where $F_0 = -p + 1/2 + iy$ ($y > 0$) - and the Discrete Series $D^{\pm}([F])$, where the unitarity condition for the Discrete Series means that F_0 has to be given by $F_0 = -p + 1/2 = -(N-1)/2$. It can be readily shown that for $F_0 = -p + 1/2 + iy$ and for $F_0 = -p + 1/2$ there are no allowed values for $F_1, F_2, ..., F_{p-1}$ that give the correct value for the quadratic Casimir. In other words, there are no UIR's of spin$(2p,1)$ ($p \geq 3$) that correspond to the massless fields with spin $s \in \{3/2, 5/2\}$ on dS_{2p} (because these field theories are non-unitary).

$N = 4$ ($p = 2$). The STSSH's of rank $\tilde{r}$ on S^3 - determining the spin(4) content for the massless theory with spin $s = r + 1/2$ ($r \in \{1, 2\}$) on dS_4 - are labelled by:

$$[f] = (f_1, f_2) = \left(\ell + \frac{1}{2}, \pm(\tilde{r} + \frac{1}{2})\right), \qquad (\ell \geq r \geq \tilde{r} \geq r - \tau + 1). \tag{A.21}$$

As discussed in Section 8, the (non-zero-norm) eigenmodes with negative spin projection and the ones with positive spin projection on dS_4 form a direct sum of UIR's of spin$(4,1)$. (Recall that the massless tunings for the imaginary mass parameter $M = i\tilde{M}$ are $|\tilde{M}| = r - \tau + (N-2)/2 = r - \tau + 1$ - see eq. (1.5).) By studying the rules (A.8) for the Discrete Series, we straightforwardly find that the eigenmodes with negative spin projection correspond to the following labelling:

$$\begin{aligned} [F] = (F_0, F_1) &= \left(|\tilde{M}| - \frac{3}{2}, r + \frac{1}{2}\right) = \left(r - \tau - \frac{1}{2}, r + \frac{1}{2}\right) \\ [f] = (f_1, f_2) &= \left(\ell + \frac{1}{2}, -(\tilde{r} + \frac{1}{2})\right), \qquad (\ell \geq r \geq \tilde{r} \geq r - \tau + 1). \end{aligned} \tag{A.22}$$

The corresponding UIR of spin$(4,1)$ is the Discrete Series $D^{-}(r - \tau - 1/2,\, r + 1/2)$. Similarly, we find that the eigenmodes with positive spin projection correspond to:

$$\begin{aligned} [F] = (F_0, F_1) &= \left(r - \tau - \frac{1}{2}, r + \frac{1}{2}\right) \\ [f] = (f_1, f_2) &= \left(\ell + \frac{1}{2}, \tilde{r} + \frac{1}{2}\right), \qquad (\ell \geq r \geq \tilde{r} \geq r - \tau + 1). \end{aligned} \tag{A.23}$$

The corresponding UIR of spin$(4,1)$ is the Discrete Series $D^{+}(r - \tau - 1/2,\, r + 1/2)$. Thus, for the massless fields with spin $s = r + 1/2$ ($r \in \{1, 2\}$) on dS_4, the eigenmodes with negative spin projection and the ones with positive spin projection together form the direct sum of UIR's:

$$D^{-}\left(r-\tau-\frac{1}{2},\, r+\frac{1}{2}\right)\oplus D^{+}\left(r-\tau-\frac{1}{2},\, r+\frac{1}{2}\right).$$

(Note that the eigenmodes with $M=+i|\tilde{M}|=i(r-\tau+1)$ and the ones with $M=-i(r-\tau+1)$ form equivalent representations, because if we act with γ^5 [eq. (2.13)] on one set of eigenmodes, we obtain the other set of eigenmodes and vice versa.)

A.5 Unitary representations of spin$(N,1)$ for fields with spin $s\in\{3/2,5/2\}$ and real mass parameter

All representations corresponding to fields of spin $s\in\{3/2,5/2\}$ with real mass parameter on dS_N are unitary. The TT eigenmodes with real mass parameter satisfy eq. (1.3) with $M\in\mathbb{R}$ on dS_N. For two such eigenmodes, $\Psi^{(1)}_{\mu_1\dots\mu_r}$ and $\Psi^{(2)}_{\mu_1\dots\mu_r}$, a dS invariant positive-definite inner product is

$$\langle\Psi^{(1)}|\Psi^{(2)}\rangle=\cosh^{N-1}t\int_{S^{N-1}}\sqrt{\tilde{g}}\,d\boldsymbol{\theta}_{N-1}\,\Psi^{(1)\dagger}_{\mu_1\dots\mu_r}\Psi^{(2)\mu_1\dots\mu_r}. \tag{A.24}$$

For real mass parameter, all types of TT eigenmodes have positive norms.

The TT eigenmodes with real mass parameter on dS_N are given by the analytic continuation of STSSH's on S^N. In order to obtain these eigenmodes on dS_N we just apply the analytic continuation techniques of Section 7, where instead of the replacements (7.5), we have to make the following replacements:

$$\theta_N\to x(t)\equiv\frac{\pi}{2}-it,\qquad n\to -iM-\frac{N}{2}\qquad (t\in\mathbb{R},\ M\in\mathbb{R}). \tag{A.25}$$

For N even, we can find the formulae for the spin$(N,1)$ transformation of the eigenmodes by replacing $\tilde{M}$ by $-iM$ in eqs. (8.2), (8.3) and (8.4). For N even and $M=0$, the two spin projections do not mix with each other under spin$(N,1)$ - see eqs. (8.10)-(8.12). Furthermore, for N even and $M=0$, the eigenmodes with negative spin projection are eigenfunctions of γ^{N+1} [eq. (2.13)] with eigenvalue $+1$, while the eigenmodes with positive spin projection are eigenfunctions of γ^{N+1} with eigenvalue -1. This is easily understood as follows. By analytically continuing the functions defined by eqs. (3.2), (4.10), (4.12), (5.16), (5.18) and (5.20) [by making the replacements (A.25)], it is easy to check that the analytically continued versions of all these functions vanish for $M=0$. This leads to the vanishing of the lower (upper) component of the eigenmodes with negative (positive) spin projection and thus these eigenmodes become eigenfunctions of γ^{N+1}. For N odd, we find the formulae for the spin$(N,1)$ transformation of the eigenmodes by replacing $\tilde{M}$ by $-iM$ in eqs. (8.27)-(8.29).

Let us now identify the unitary representations of spin$(N,1)$ formed by the analytically continued STSSH's of rank $r\in\{1,2\}$ with real mass parameter M on dS_N.

Odd $N=2p+1$. We have:

- For $p = 1$, the representation formed by the TT eigenmodes on dS_3 is labelled by:

$$\begin{aligned}[F] &= (F_0, F_1) = \left(-p - iM,\, r + \frac{1}{2}\right) = \left(-1 - iM,\, r + \frac{1}{2}\right)\\ [f] &= f_1 = \ell + \frac{1}{2} \qquad (\ell \geq r \geq 0).\end{aligned} \tag{A.26}$$

  The corresponding UIR of spin$(3,1)$ is $D_{\text{prin}}([F]) = D_{\text{prin}}(-1-iM,\, r+1/2)$ $(M \in \mathbb{R})$. (Recall that on dS_3 there are only type-I modes.)

- For $p = 2$, the representation formed by the TT eigenmodes on dS_5 is labelled by:

$$\begin{aligned}[F] &= (F_0, F_1, F_2) = \left(-p - iM,\, r + \frac{1}{2}, \frac{1}{2}\right) = \left(-2 - iM,\, r + \frac{1}{2},\, \frac{1}{2}\right)\\ [f] &= (f_1, f_2) = \left(\ell + \frac{1}{2},\, \tilde{r} + \frac{1}{2}\right) \qquad (\ell \geq r \geq \tilde{r} \geq 0).\end{aligned} \tag{A.27}$$

  The corresponding UIR of spin$(5,1)$ is $D_{\text{prin}}([F]) = D_{\text{prin}}(-2 - iM,\, r + 1/2, 1/2)$ $(M \in \mathbb{R})$.

- For $p \geq 3$, the representation formed by the TT eigenmodes on dS_{2p+1} is labelled by:

$$\begin{aligned}[F] &= (F_0, F_1, F_2, ..., F_p) = \left(-p - iM,\, r + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)\\ [f] &= (f_1, f_2, f_3, ..., f_p) = \left(\ell + \frac{1}{2},\, \tilde{r} + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right) \qquad (\ell \geq r \geq \tilde{r} \geq 0).\end{aligned} \tag{A.28}$$

  The corresponding UIR of spin$(2p+1,1)$ is $D_{\text{prin}}([F]) = D_{\text{prin}}(-p-iM,\, r+1/2, 1/2, ..., 1/2)$ $(M \in \mathbb{R})$.

Even $N = 2p \geq 4$. We have:

Case 1. $M \neq 0$. For M real and nonzero, the TT eigenmodes with different spin projections on dS_{2p} mix with each other under spin$(2p, 1)$ [see the transformation formulae (8.2)-(8.4)].

- For $p = 2$, the representation formed by the TT eigenmodes with $M \neq 0$ on dS_4 is labelled by:

$$\begin{aligned}[F] &= (F_0, F_1) = \left(-p + \frac{1}{2} + i|M|,\, r + \frac{1}{2}\right) = \left(-\frac{3}{2} + i|M|,\, r + \frac{1}{2}\right)\\ [f] &= (f_1, f_2) = \left(\ell + \frac{1}{2},\, \pm(\tilde{r} + \frac{1}{2})\right), \qquad (\ell \geq r \geq \tilde{r} \geq 0).\end{aligned} \tag{A.29}$$

  The corresponding UIR of spin$(4,1)$ is $D_{\text{prin}}([F]) = D_{\text{prin}}(-3/2 + i|M|,\, r + 1/2)$ (for all real $M \neq 0$). The eigenmodes with $M = +|M|$ and the ones with $M = -|M|$ form equivalent representations.

- For $p \geq 3$, the representation formed by the TT eigenmodes with $M \neq 0$ on dS_{2p} is labelled by:

$$\begin{aligned}[F] &= (F_0, F_1, ..., F_{p-1}) = \left(-p + \frac{1}{2} + i|M|,\, r + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)\\ [f] &= (f_1, f_2, ..., f_p) = \left(\ell + \frac{1}{2},\, \tilde{r} + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \pm\frac{1}{2}\right), \qquad (\ell \geq r \geq \tilde{r} \geq 0).\end{aligned} \tag{A.30}$$

The corresponding UIR of spin$(2p,1)$ is

$$D_{\text{prin}}([F]) = D_{\text{prin}}\left(-p+\frac{1}{2}+i|M|,\, r+\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$$

(for all real $M \neq 0$). The eigenmodes with $M = +|M|$ and the ones with $M = -|M|$ form equivalent representations.

Case 2. $M = 0$. Recall that, for $M = 0$, the two sets of eigenmodes with different spin projections on dS_{2p} separately form UIR's of spin$(2p,1)$.

- For $p = 2$, the representation formed by the TT eigenmodes with negative spin projection on dS_4 is labelled by:

$$\begin{aligned}[F] &= (F_0, F_1) = \left(-p+\frac{1}{2},\, r+\frac{1}{2}\right) = \left(-\frac{3}{2},\, r+\frac{1}{2}\right)\\ [f] &= (f_1, f_2) = \left(\ell+\frac{1}{2},\, -(\tilde{r}+\frac{1}{2})\right), \qquad (\ell \geq r \geq \tilde{r} \geq 0).\end{aligned} \tag{A.31}$$

  The corresponding UIR of spin$(4,1)$ is the Discrete Series UIR

$$D^{-}\left(-\frac{3}{2},\, r+\frac{1}{2}\right).$$

  The representation formed by the TT eigenmodes with positive spin projection on dS_4 is labelled by:

$$\begin{aligned}[F] &= (F_0, F_1) = \left(-p+\frac{1}{2},\, r+\frac{1}{2}\right) = \left(-\frac{3}{2},\, r+\frac{1}{2}\right)\\ [f] &= (f_1, f_2) = \left(\ell+\frac{1}{2},\, \tilde{r}+\frac{1}{2}\right), \qquad (\ell \geq r \geq \tilde{r} \geq 0).\end{aligned} \tag{A.32}$$

  The corresponding UIR of spin$(4,1)$ is

$$D^{+}\left(-\frac{3}{2},\, r+\frac{1}{2}\right).$$

- For $p \geq 3$, the representation formed by the TT eigenmodes with negative spin projection on dS_{2p} is labelled by:

$$\begin{aligned}[F] &= (F_0, F_1, ..., F_{p-1}) = \left(-p+\frac{1}{2},\, r+\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)\\ [f] &= (f_1, f_2, ..., f_p) = \left(\ell+\frac{1}{2},\, \tilde{r}+\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2}\right), \qquad (\ell \geq r \geq \tilde{r} \geq 0).\end{aligned} \tag{A.33}$$

  The corresponding UIR of spin$(2p,1)$ is the Discrete Series UIR

$$D^{-}\left(-p+\frac{1}{2},\, r+\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right).$$

The representation formed by the TT eigenmodes with positive spin projection on dS_{2p} is labelled by:

$$[F] = (F_0, F_1, ..., F_{p-1}) = \left(-p + \frac{1}{2}, r + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$$
$$[f] = (f_1, f_2, ..., f_p) = \left(\ell + \frac{1}{2}, \tilde{r} + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right), \qquad (\ell \geq r \geq \tilde{r} \geq 0). \tag{A.34}$$

The corresponding UIR of spin$(2p,1)$ is the Discrete Series UIR

$$D^+\left(-p + \frac{1}{2}, r + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right).$$

B Raising and lowering operators for the Gauss hypergeometric function and other useful formulae

The Gauss hypergeometric function $F(a,b;c;z)$ satisfies [38]

$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z), \tag{B.1}$$
$$(z\frac{d}{dz} + c - 1)F(a,b;c;z) = (c-1)F(a,b;c-1;z), \tag{B.2}$$
$$(z\frac{d}{dz} + a)F(a,b;c;z) = aF(a+1,b;c;z). \tag{B.3}$$

By combining eq. (B.3) with the following relation [39]:

$$(c-b)F(a+1,b-1;c;z) + (b-a-1)(1-z)F(a+1,b;c;z) = (c-a-1)F(a,b;c;z), \tag{B.4}$$

we find

$$\left(a(b-c) + a(-b+a+1)z - (-b+a+1)z(1-z)\frac{d}{dz}\right)F(a,b;c;z)$$
$$= a(b-c)F(a+1,b-1;c;z). \tag{B.5}$$

Using eqs. (B.1) and (B.2) we can show the ladder relations (F.21) and (F.22), while using eq. (B.5) we can show the ladder relations (F.23) and (F.24).

The behaviour of the functions (3.1) and (3.2) in the limit $\theta_N \to \pi$ is studied by using the transformation formula [24]

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\,F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z)$$
$$+ (1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(-\gamma+\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z). \tag{B.6}$$

Equation (8.26) is proved using [40]

$$F\left(a,b,\frac{a+b}{2};\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma\left(\frac{a+b}{2}\right)\left[\frac{1}{\Gamma((a+1)/2)\Gamma(b/2)} + \frac{1}{\Gamma((b+1)/2)\Gamma(a/2)}\right] \tag{B.7}$$

and [41]

$$F\left(a,b,\frac{a+b}{2}+1;\frac{1}{2}\right)=\frac{2\sqrt{\pi}}{a-b}\Gamma\left(\frac{a+b}{2}+1\right)\times\left[\frac{1}{\Gamma((b+1)/2)}\frac{1}{\Gamma(a/2)}-\frac{1}{\Gamma((a+1)/2)\Gamma(b/2)}\right]. \tag{B.8}$$

C Spinor eigenmodes of the Dirac operator on the $(N-1)$-sphere

The spinor eigenmodes of the Dirac operator (i.e. the STSSH's of rank 0) on spheres of arbitrary dimension have been computed in Ref. [21]. Here we write down explicitly the eigenspinors on S^{N-1} that satisfy eq. (2.22). These eigenspinors play an important role in the derivation of the formulae for the spin$(N+1)$ transformation of the STSSH's in Appendix F.

Case 1: $N-1$ odd. We denote the eigenspinors on S^{N-1} as $\chi_{\pm\ell m\rho}(\theta_{N-1},\boldsymbol{\theta}_{N-2})$, where ρ stands for labels other than ℓ and m. These eigenspinors are given by

$$\chi_{\pm\ell m\rho}(\theta_{N-1},\boldsymbol{\theta}_{N-2})=\frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\left\{\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})\pm i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right\}, \tag{C.1}$$

where $\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})$ and $\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})$ are given by eqs. (F.8) and (F.9), respectively, and

$$\hat{\tilde{\chi}}_{\pm m\rho}(\boldsymbol{\theta}_{N-2})=\frac{1}{\sqrt{2}}(1+i\tilde{\gamma}^{N-1})\tilde{\chi}_{\pm m\rho}(\boldsymbol{\theta}_{N-2}), \tag{C.2}$$

$$\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})=\tilde{\gamma}^{N-1}\tilde{\chi}_{-m\rho}(\boldsymbol{\theta}_{N-2}), \tag{C.3}$$

where the spinors $\tilde{\chi}_{\pm m\rho}(\boldsymbol{\theta}_{N-2})$ are the eigenspinors of the Dirac operator on S^{N-2}. [The gamma matrices on S^{N-1} are denoted as $\tilde{\gamma}^a$ - see eq. (2.11).] In order for the eigenspinors (C.1) to be non-singular we require $\ell\geq m$ and $\ell=0,1,...$ [21]. The eigenspinors (C.1) satisfy the normalisation condition (2.23), while the normalisation factor is given by [21]

$$\left|\frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\right|^2=\frac{\Gamma(\ell-m+1)\Gamma(\ell+N-1+m)}{2^{N-2}|\Gamma(\frac{N-1}{2}+\ell)|^2}. \tag{C.4}$$

Case 2: $N-1$ even. We denote the eigenspinors on S^{N-1} as $\chi^{(\sigma_{N-1})}_{\pm\ell m\rho}(\theta_{N-1},\boldsymbol{\theta}_{N-2})$, where $\sigma_{N-1}=\pm$ is the spin projection index on S^{N-1} and ρ stands for labels other than σ_{N-1},ℓ and m. The eigenspinors with negative spin projection are given by

$$\chi^{(-)}_{\pm\ell m\rho}(\theta_{N-1},\boldsymbol{\theta}_{N-2})=\frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\begin{pmatrix}\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\,\tilde{\chi}_{-m\rho}(\boldsymbol{\theta}_{N-2})\\ \pm i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\,\tilde{\chi}_{-m\rho}(\boldsymbol{\theta}_{N-2})\end{pmatrix} \tag{C.5}$$

and those with positive spin projection are given by

$$\chi^{(+)}_{\pm\ell m\rho}(\theta_{N-1},\boldsymbol{\theta}_{N-2})=\frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\begin{pmatrix}i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\,\tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2})\\ \pm\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\,\tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2})\end{pmatrix} \tag{C.6}$$

and they both satisfy eq. (2.22). The normalisation factors $\tilde{c}_{N-1}(\ell,m)$, as well as the functions $\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})$ and $\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})$, have the same expressions as in the case with $N-1$ odd.

D Some useful formulae on S^{N-1}

Let $\tilde{g}_{\mu\nu}$ be the metric tensor on S^{N-1}. The Riemann tensor on S^{N-1} is

$$\tilde{R}_{\mu\nu\kappa\lambda} = \tilde{g}_{\mu\kappa}\tilde{g}_{\nu\lambda} - \tilde{g}_{\nu\kappa}\tilde{g}_{\mu\lambda}. \tag{D.1}$$

Let $\tilde{\psi}, \tilde{\psi}_\mu$ and $\tilde{\psi}_{\mu\nu}$ be any spinor, vector-spinor and rank-2 tensor-spinor field, respectively, on S^{N-1}. The commutator of covariant derivatives acting on these fields is given by

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\tilde{\psi} = \frac{1}{4}\tilde{R}_{\mu\nu\kappa\lambda}\tilde{\gamma}^\kappa\tilde{\gamma}^\lambda\tilde{\psi} \tag{D.2}$$

$$= \frac{1}{2}(\tilde{\gamma}_\mu\tilde{\gamma}_\nu - \tilde{g}_{\mu\nu})\tilde{\psi}, \tag{D.3}$$

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\tilde{\psi}_\alpha = \frac{1}{4}\tilde{R}_{\mu\nu\kappa\lambda}\tilde{\gamma}^\kappa\tilde{\gamma}^\lambda\tilde{\psi}_\alpha + \tilde{R}^\lambda{}_{\alpha\nu\mu}\tilde{\psi}_\lambda \tag{D.4}$$

$$= \frac{1}{2}(\tilde{\gamma}_\mu\tilde{\gamma}_\nu - \tilde{g}_{\mu\nu})\tilde{\psi}_\alpha + 2\tilde{g}_{\alpha[\mu}\tilde{\psi}_{\nu]}, \tag{D.5}$$

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\tilde{\psi}_{\alpha\beta} = \frac{1}{2}(\tilde{\gamma}_\mu\tilde{\gamma}_\nu - \tilde{g}_{\mu\nu})\tilde{\psi}_{\alpha\beta} + 2\tilde{g}_{\alpha[\mu}\tilde{\psi}_{\nu]\beta} + 2\tilde{\psi}_{\alpha[\nu}\tilde{g}_{\mu]\beta}. \tag{D.6}$$

The Laplace-Beltrami operator on S^{N-1} is defined as $\tilde{\Box} \equiv \tilde{g}^{\kappa\lambda}\tilde{\nabla}_\kappa\tilde{\nabla}_\lambda$. The eigenspinors on S^{N-1} [see eq. (2.22)] satisfy [21]

$$\tilde{\Box}\chi_{\pm\ell\tilde{\rho}} = \left[\tilde{\nabla\!\!\!/}^2 + \frac{(N-1)(N-2)}{4}\right]\chi_{\pm\ell\tilde{\rho}}$$
$$= \left[-\left(\ell + \frac{N-1}{2}\right)^2 + \frac{(N-1)(N-2)}{4}\right]\chi_{\pm\ell\tilde{\rho}}. \tag{D.7}$$

Note also the following relations:

$$\tilde{\gamma}^{\theta_i}\tilde{\nabla}_{(\theta_i}\tilde{\nabla}_{\theta_j)}\chi_{\pm\ell\tilde{\rho}} = \pm i\left(\ell + \frac{N-1}{2}\right)\tilde{\nabla}_{\theta_j}\chi_{\pm\ell\rho} + \frac{N-2}{4}\tilde{\gamma}_{\theta_j}\chi_{\pm\ell\tilde{\rho}}, \tag{D.8}$$

$$\tilde{\gamma}^{\theta_i}\tilde{\gamma}_{(\theta_i}\tilde{\nabla}_{\theta_j)}\chi_{\pm\ell\tilde{\rho}} = \frac{N+1}{2}\tilde{\nabla}_{\theta_j}\chi_{\pm\ell\rho} \mp i\frac{\ell + \frac{N-1}{2}}{2}\tilde{\gamma}_{\theta_j}\chi_{\pm\ell\tilde{\rho}}, \tag{D.9}$$

$$\tilde{\nabla}^{\theta_i}\tilde{\nabla}_{(\theta_i}\tilde{\nabla}_{\theta_j)}\chi_{\pm\ell\tilde{\rho}} = \tilde{\nabla}_{\theta_j}\left(\tilde{\Box} + N - \frac{5}{4}\right)\chi_{\pm\ell\tilde{\rho}} \mp \frac{3}{4}i\left(\ell + \frac{N-1}{2}\right)\tilde{\gamma}_{\theta_j}\chi_{\pm\ell\tilde{\rho}}, \tag{D.10}$$

$$\tilde{\nabla}^{\theta_i}\tilde{\gamma}_{(\theta_i}\tilde{\nabla}_{\theta_j)}\chi_{\pm\ell\tilde{\rho}} = \pm i\frac{\ell + \frac{N-1}{2}}{2}\tilde{\nabla}_{\theta_j}\chi_{\pm\ell\tilde{\rho}} + \frac{1}{2}\tilde{\gamma}_{\theta_j}\left(\tilde{\Box} + \frac{N-2}{2}\right)\chi_{\pm\ell\tilde{\rho}}, \tag{D.11}$$

where in order to prove eqs. (D.8) and (D.11) we have to use eq. (D.3), while in order to prove eq. (D.10) we have to use eqs. (D.3) and (D.5).

The TT vector-spinor eigenmodes [see eqs. (4.14)-(4.15)] satisfy

$$\tilde{\Box}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j} = \left[-\left(\ell + \frac{N-1}{2}\right)^2 + \frac{(N-1)(N-2)}{4} + 1\right]\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j} \tag{D.12}$$

($j=1,...,N-1$). By combining this equation with eq. (D.5) we can prove the following relation:

$$\tilde{\nabla}^{\theta_i}\tilde{\nabla}_{(\theta_i}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_k)} = \frac{1}{2}\left(\tilde{\Box}+N-\frac{3}{2}\right)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_k}$$
$$= \frac{1}{2}\left(\tilde{\nabla}^2+\frac{N(N+1)}{4}\right)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_k}. \tag{D.13}$$

The rank-2 STSSH's on S^{N-1} [see eqs. (5.6)- (5.8)] satisfy

$$\tilde{\Box}\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_k} = \left[-\left(\ell+\frac{N-1}{2}\right)^2+\frac{(N-1)(N-2)}{4}+2\right]\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_k} \tag{D.14}$$

($j,k=1,...,N-1$).

E Constructing the STSSH's of rank 2 on the N-sphere

In this Appendix, we construct the STSSH's of rank 2 on S^N. These STSSH's satisfy eqs. (5.1)-(5.3) and we construct them explicitly by using the method of separation of variables in geodesic polar coordinates (2.2), as in Refs. [18, 21]. In the method of separation of variables, the STSSH's of rank 2 on S^N are expressed in terms of STSSH's of rank $\tilde{r}$ (with $\tilde{r}=0,1,2$) on S^{N-1}.

For later convenience, note that the functions $\phi^{(a)}_{n\ell}(\theta_N)$ [eq. (3.1)] satisfy the following differential equation:

$$D_{(a)}\phi^{(a)}_{n\ell}(\theta_N) = -\zeta^2_{n,N}\phi^{(a)}_{n\ell}(\theta_N), \tag{E.1}$$

where $\zeta^2_{n,N}\equiv\zeta^2=(n+\frac{N}{2})^2$ is the eigenvalue of the STSSH in eq. (1.6), while the differential operator is given by

$$D_{(a)} = \frac{\partial^2}{\partial\theta_N^2}+(N+2a-1)\cot\theta_N\frac{\partial}{\partial\theta_N}+\left(\ell+\frac{N-1}{2}\right)\frac{\cos\theta_N}{\sin^2\theta_N}$$
$$-\frac{(\ell+\frac{N-1}{2})^2-\frac{1}{4}(N+2a-1)(N+2a-3)}{\sin^2\theta_N}-\frac{(N+2a-1)^2}{4}. \tag{E.2}$$

One can readily verify that the functions $\phi^{(a)}_{n\ell}(\theta_N)$ [eq. (3.1)] are the unique regular solutions (up to a normalisation constant) of the differential equation (E.1) by using the results of Ref. [21], as follows. By expressing $\phi^{(a)}_{n\ell}$ as

$$\phi^{(a)}_{n\ell}(\theta_N) = \left(\sin\frac{\theta_N}{2}\cos\frac{\theta_N}{2}\right)^{-a}\phi^{(0)}_{n\ell}(\theta_N) \tag{E.3}$$

[see eq. (3.1)] we rewrite eq. (E.1) as $D_{(0)}\phi^{(0)}_{n\ell}=-\zeta^2_{n,N}\phi^{(0)}_{n\ell}$. The latter has been solved in Ref. [21] and it was found that the unique regular solutions $\phi^{(0)}_{n\ell}$ are the ones given by eq. (3.1) (with $a=0$). For the rank-1 STSSH's on S^N the integer a takes the values $a=-1,1$ (see Section 4), while for rank-2 STSSH's a takes the values $a=-2,0,2$ (see

Section 5). The functions $\phi^{(a)}_{n\ell}(\theta_N)$ are regular for $a = 1$ and $a = 2$ despite the factor $\left(\sin\frac{\theta_N}{2}\cos\frac{\theta_N}{2}\right)^{-a}$ in eq. (E.3) because of the restriction $\ell \geq r$ (this restriction on ℓ is proved in Section 4 for $r = 1$ and in Section 5 for $r = 2$).

The differential equation satisfied by the functions $\psi^{(a)}_{n\ell}(\theta_N)$ [eq. (3.2)] is obtained from eq. (E.1) by making the replacement $\theta_N \to \pi - \theta_N$ in the expression (E.2) for the differential operator $D_{(a)}$.

Let us also briefly explain how to obtain the condition $n \geq \ell$ [eq. (3.4)]. By taking the limit $\theta_N \to \pi$ for $\phi^{(a)}_{n\ell}(\theta_N)$ and using the transformation formula (B.6) for the Gauss hypergeometric function, we readily find that the requirement for absence of singularity in $\phi^{(a)}_{n\ell}(\theta_N)$ gives rise to the condition $n \geq \ell$, as well as to the quantisation condition

$$|\zeta_{n,N}| = n + \frac{N}{2}, \quad n \in \mathbb{N}_0. \tag{E.4}$$

E.1 Constructing the STSSH's of rank 2 for N even

Our aim is to obtain the STSSH's $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\mu\nu}$ that satisfy eqs. (5.1)-(5.3), where the gamma matrices for N even are given by eq. (2.11). As in Ref. [21], we write $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\mu\nu}$ in terms of upper and lower $2^{N/2-1}$-dimensional spinor components

$$\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu\nu}(\theta_N, \boldsymbol{\theta}_{N-1}) = \begin{pmatrix} {}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu\nu}(\theta_N, \boldsymbol{\theta}_{N-1}) \\ \\ {}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu\nu}(\theta_N, \boldsymbol{\theta}_{N-1}) \end{pmatrix}. \tag{E.5}$$

It is clear that eqs. (5.1)-(5.3) - which determine the form of our STSSH's - reduce to a system of equations for the upper and lower components. We will now obtain the system of equations for the upper and lower components. By using eqs. (2.4), (2.9), (2.11), (2.18), (2.19), (5.1) and (5.2) and by expressing $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu\nu}$ in terms of the upper and lower components as in (E.5), we find that the eigenvalue equation $\nabla\!\!\!/\,\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} = \pm i|\zeta_{n,N}|\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}$ is written as

$$\left(\frac{\partial}{\partial\theta_N} + \frac{N+3}{2}\cot\theta_N + \frac{i}{\sin\theta_N}\tilde{\nabla\!\!\!/}\right){}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} = \pm i|\zeta_{n,N}|\,{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}, \tag{E.6a}$$

$$\left(\frac{\partial}{\partial\theta_N} + \frac{N+3}{2}\cot\theta_N - \frac{i}{\sin\theta_N}\tilde{\nabla\!\!\!/}\right){}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} = \pm i|\zeta_{n,N}|\,{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}. \tag{E.6b}$$

Similarly, we find that the eigenvalue equation $\nabla\!\!\!/\,\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j} = \pm i|\zeta_{n,N}|\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}$ ($j = 1, ..., N-1$) is written as

$$\begin{aligned}\left(\frac{\partial}{\partial\theta_N} + \frac{N-1}{2}\cot\theta_N + \frac{i}{\sin\theta_N}\tilde{\nabla\!\!\!/}\right){}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j} &+ i\cos\theta_N\tilde{\gamma}_{\theta_j}{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} \\ = \pm i|\zeta_{n,N}|\,{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}&,\end{aligned} \tag{E.7a}$$

$$\begin{aligned}\left(\frac{\partial}{\partial\theta_N} + \frac{N-1}{2}\cot\theta_N - \frac{i}{\sin\theta_N}\tilde{\nabla\!\!\!/}\right){}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j} &- i\cos\theta_N\tilde{\gamma}_{\theta_j}{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} \\ = \pm i|\zeta_{n,N}|\,{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}&,\end{aligned} \tag{E.7b}$$

while $\tilde{\nabla\!\!\!/}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k} = \pm i|\zeta_{n,N}|\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ $(j,k=1,...,N-1)$ is written as

$$\begin{aligned}
&\Big(\frac{\partial}{\partial\theta_N}+\frac{N-5}{2}\cot\theta_N+\frac{i}{\sin\theta_N}\tilde{\nabla\!\!\!/}\Big)^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}+2i\cos\theta_N\tilde{\gamma}_{(\theta_j}{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_k)\theta_N}\\
&\quad=\pm i|\zeta_{n,N}|\,{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}, \qquad\qquad \text{(E.8a)}\\
&\Big(\frac{\partial}{\partial\theta_N}+\frac{N-5}{2}\cot\theta_N-\frac{i}{\sin\theta_N}\tilde{\nabla\!\!\!/}\Big)^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}-2i\cos\theta_N\tilde{\gamma}_{(\theta_j}{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_k)\theta_N}\\
&\quad=\pm i|\zeta_{n,N}|\,{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}. \qquad\qquad \text{(E.8b)}
\end{aligned}$$

By making use of eq. (E.5), we express the gamma-tracelessness condition (5.2) as

$$\begin{cases}
{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\mu}+\dfrac{i}{\sin\theta_N}\tilde{\gamma}^{\theta_i}\;{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\mu}=0,\\[2ex]
{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\mu}-\dfrac{i}{\sin\theta_N}\tilde{\gamma}^{\theta_i}\;{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\mu}=0, \qquad (\mu=\theta_1,...,\theta_N \;\;\text{and}\;\; \theta_i=\theta_1,...,\theta_{N-1})
\end{cases} \tag{E.9}$$

and the tracelessness condition (5.3) as

$$\begin{cases}
{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}+\dfrac{1}{\sin^2\theta_N}\tilde{g}^{\theta_i\theta_j}\;{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\theta_j}=0,\\[2ex]
{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}+\dfrac{1}{\sin^2\theta_N}\tilde{g}^{\theta_i\theta_j}\;{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\theta_j}=0.
\end{cases} \tag{E.10}$$

Similarly, by substituting eq. (E.5) into the divergence-free condition (5.2), we may express the condition $\nabla^\alpha\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha\theta_N}=0$ as

$$\begin{cases}
\Big[\frac{\partial}{\partial\theta_N}+(N+\frac{1}{2})\cot\theta_N\Big]{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}+\dfrac{1}{\sin^2\theta_N}\tilde{\nabla}^{\theta_i}\;{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\theta_N}=0,\\[2ex]
\Big[\frac{\partial}{\partial\theta_N}+(N+\frac{1}{2})\cot\theta_N\Big]{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}+\dfrac{1}{\sin^2\theta_N}\tilde{\nabla}^{\theta_i}\;{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\theta_N}=0,
\end{cases} \tag{E.11}$$

while the condition $\nabla^\alpha\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha\theta_j}=0$ $(j=1,...,N-1)$ is expressed as

$$\begin{cases}
\Big[\frac{\partial}{\partial\theta_N}+(N-\frac{1}{2})\cot\theta_N\Big]{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}+\dfrac{1}{\sin^2\theta_N}\tilde{\nabla}^{\theta_i}\;{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\theta_j}=0,\\[2ex]
\Big[\frac{\partial}{\partial\theta_N}+(N-\frac{1}{2})\cot\theta_N\Big]{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}+\dfrac{1}{\sin^2\theta_N}\tilde{\nabla}^{\theta_i}\;{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\theta_j}=0.
\end{cases} \tag{E.12}$$

Type-I STSSH's of rank 2 for N even. Let us start by describing how to obtain the type-I modes, given by eqs. (5.10)-(5.12). The component $\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}$ is a spinor on S^{N-1}. Thus, in order to solve the system of equations (E.6) we separate variables as in the case of spinor eigenmodes in Ref. [21], i.e.

$$
\begin{aligned}
{}^{(\uparrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \phi^{(2)}_{n\ell}(\theta_N)\,\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm i\psi^{(2)}_{n\ell}(\theta_N)\,\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})
\end{aligned}
\tag{E.13}
$$

$$
\begin{aligned}
{}^{(\uparrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) &= i\psi^{(2)}_{n\ell}(\theta_N)\,\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm\phi^{(2)}_{n\ell}(\theta_N)\,\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),
\end{aligned}
\tag{E.14}
$$

where $\chi_{\pm\ell\tilde{\rho}}$ are the eigenspinors on S^{N-1} (see eq. (2.22)). By substituting eq. (E.13) [or eq. (E.14)] into the system of equations (E.6) and eliminating ${}^{(\downarrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}$ (or ${}^{(\downarrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}$) we find that $\phi^{(2)}_{n\ell}$ has to satisfy the differential equation (E.1) (with $a=2$), while $\psi^{(2)}_{n\ell}$ has to satisfy the differential equation (E.1) ($a=2$) with θ_N replaced by $\pi-\theta_N$ in the differential operator $D_{(2)}$ [eq. (E.2)]. Thus, we find that $\phi^{(2)}_{n\ell}$ and $\psi^{(2)}_{n\ell}$ are given by eqs. (3.1) and (3.2), respectively. As a check, one readily finds that the components defined by eqs. (E.13) and (E.14) satisfy the system of equations (E.6) by making use of the formulae (3.5) and (3.6).

The components $\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}$ ($j=1,...,N-1$) are vector-spinors on S^{N-1} and thus we may separate variables analogously to eqs. (4.6) and (4.8). Thus, for STSSH's with negative spin projection ($\sigma=-$) we separate variables as

$$
\begin{aligned}
{}^{(\uparrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= C^{(\uparrow)(2)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) + D^{(\uparrow)(2)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm\, iC^{(\downarrow)(2)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \pm iD^{(\downarrow)(2)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),
\end{aligned}
\tag{E.15}
$$

while for STSSH's with positive spin projection ($\sigma=+$) we separate variables as

$$
\begin{aligned}
{}^{(\uparrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= iC^{(\downarrow)(2)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) - iD^{(\downarrow)(2)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm\, C^{(\uparrow)(2)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \mp D^{(\uparrow)(2)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}).
\end{aligned}
\tag{E.16}
$$

By using the gamma-tracelessness condition (E.9) we readily find that the functions $D^{(b)(2)}_{n\ell}$ and $C^{(b)(2)}_{n\ell}$ ($b=\uparrow,\downarrow$) are related to each other by eqs. (4.11) and (4.12). Then, using the divergence-free condition (E.11), we find that $C^{(\uparrow)(2)}_{n\ell}$ is given by eq. (4.9) and $C^{(\downarrow)(2)}_{n\ell}$ is given by eq. (4.10), where we also have used eqs. (3.5), (3.6) and eq. (D.7). One can straightforwardly verify that the components defined by eqs. (E.15) and (E.16) are solutions of the system of equations (E.7), where the calculations are significantly simplified by using the following formulae:

$$\left(\frac{\partial}{\partial\theta_N}+\frac{N-1}{2}\cot\theta_N-\frac{\ell+\frac{N-1}{2}}{\sin\theta_N}\right)C^{(\uparrow)(2)}_{n\ell}(\theta_N)-\frac{2i}{\sin\theta_N}D^{(\uparrow)(2)}_{n\ell}(\theta_N)$$
$$=-\left(n+\frac{N}{2}\right)\ C^{(\downarrow)(2)}_{n\ell}(\theta_N), \tag{E.17}$$
$$\left(\frac{\partial}{\partial\theta_N}+\frac{N-1}{2}\cot\theta_N+\frac{\ell+\frac{N-1}{2}}{\sin\theta_N}\right)C^{(\downarrow)(2)}_{n\ell}(\theta_N)+\frac{2i}{\sin\theta_N}D^{(\downarrow)(2)}_{n\ell}(\theta_N)$$
$$=\left(n+\frac{N}{2}\right)\ C^{(\uparrow)(2)}_{n\ell}(\theta_N), \tag{E.18}$$

which can be proved by using the formulae (3.5) and (3.6).

The components $\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ $(j,k=1,...,N-1)$ are rank-2 symmetric tensor-spinors on S^{N-1}. Let us first discuss the case with negative spin projection ($\sigma=-$). We choose to separate variables for $\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ as follows:

$$\begin{aligned}
{}^{(\uparrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =&K^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{g}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\
&+W^{(\uparrow)}_{n\ell}(\theta_N)\left(\tilde{\nabla}_{(\theta_j}\tilde{\nabla}_{\theta_k)}-\tilde{g}_{\theta_j\theta_k}\frac{\tilde{\Box}}{N-1}\right)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\
&+T^{(\uparrow)}_{n\ell}(\theta_N)\left(\tilde{\gamma}_{(\theta_j}\tilde{\nabla}_{\theta_k)}-\tilde{g}_{\theta_j\theta_k}\frac{\tilde{\nabla\!\!\!/}}{N-1}\right)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =&\pm iK^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{g}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\
&\pm iW^{(\downarrow)}_{n\ell}(\theta_N)\left(\tilde{\nabla}_{(\theta_j}\tilde{\nabla}_{\theta_k)}-\tilde{g}_{\theta_j\theta_k}\frac{\tilde{\Box}}{N-1}\right)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\
&\pm iT^{(\downarrow)}_{n\ell}(\theta_N)\left(\tilde{\gamma}_{(\theta_j}\tilde{\nabla}_{\theta_k)}-\tilde{g}_{\theta_j\theta_k}\frac{\tilde{\nabla\!\!\!/}}{N-1}\right)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),
\end{aligned} \tag{E.19}$$

where $\tilde{\nabla\!\!\!/}\chi_{-\ell\tilde{\rho}}=-i\left(\ell+\frac{N-1}{2}\right)\chi_{-\ell\tilde{\rho}}$ (see eq. (2.22)) and $\tilde{\Box}\chi_{-\ell\tilde{\rho}}\equiv\nabla^{\theta_k}\nabla_{\theta_k}\chi_{-\ell\tilde{\rho}}$ is given by eq. (D.7). By using the tracelessness condition (E.10), we find that $K^{(\uparrow)}_{n\ell}$ and $K^{(\downarrow)}_{n\ell}$ are given by eqs. (5.15) and (5.16), respectively. Then, by using the gamma-tracelessness condition (E.9) (and by making use of eqs. (D.8) and (D.9)) we find that the function $T^{(\uparrow)}_{n\ell}$ ($T^{(\downarrow)}_{n\ell}$) is expressed in terms of $W^{(\uparrow)}_{n\ell}$ ($W^{(\downarrow)}_{n\ell}$) as in eq. (5.17) (eq. (5.18)). Then, by making use of the divergence-free condition (E.12) (and using eqs. (D.10) and (D.11)) we find

$$\begin{aligned}
&\left(\frac{\partial}{\partial\theta_N}+(N-\frac{1}{2})\cot\theta_N\right)C^{(b)(2)}_{n\ell}(\theta_N)+\frac{1}{\sin^2\theta_N}K^{(b)}_{n\ell}(\theta_N)\\
&\quad+\frac{1}{\sin^2\theta_N}W^{(b)}_{n\ell}(\theta_N)\left\{-\frac{\left(\ell+\frac{N-1}{2}\right)^2(N-2)}{N-1}+\frac{N^2-1}{4}\right\}\\
&\quad-i\frac{1}{2\sin^2\theta_N}\frac{\left(\ell+\frac{N-1}{2}\right)(N-3)}{N-1}T^{(b)}_{n\ell}(\theta_N)=0,\qquad b=\uparrow,\downarrow\,.
\end{aligned} \tag{E.20}$$

Finally, by solving the system of equations consisting of eqs. (5.17), (5.18) and (E.20) (and using eqs. (E.17) and (E.18)) we find that $W^{(\uparrow)}_{n\ell}$ is given by eq. (5.19), while $W^{(\downarrow)}_{n\ell}$ is given by eq. (5.20).

By working as in the case with negative spin projection, we find that the components $\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ with positive spin projection are expressed in terms of upper and lower spinorial components as follows:

$$\begin{aligned}
{}^{(\uparrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =& iK^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{g}_{\theta_j\theta_k}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\
&+ iW^{(\downarrow)}_{n\ell}(\theta_N)\left(\tilde{\nabla}_{(\theta_j}\tilde{\nabla}_{\theta_k)} - \tilde{g}_{\theta_j\theta_k}\frac{\tilde{\Box}}{N-1}\right)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\
&- iT^{(\downarrow)}_{n\ell}(\theta_N)\left(\tilde{\gamma}_{(\theta_j}\tilde{\nabla}_{\theta_k)} - \tilde{g}_{\theta_j\theta_k}\frac{\tilde{\not{\nabla}}}{N-1}\right)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}), \\
{}^{(\downarrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =& \pm K^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{g}_{\theta_j\theta_k}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\
&\pm W^{(\uparrow)}_{n\ell}(\theta_N)\left(\tilde{\nabla}_{(\theta_j}\tilde{\nabla}_{\theta_k)} - \tilde{g}_{\theta_j\theta_k}\frac{\tilde{\Box}}{N-1}\right)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\
&\mp T^{(\uparrow)}_{n\ell}(\theta_N)\left(\tilde{\gamma}_{(\theta_j}\tilde{\nabla}_{\theta_k)} - \tilde{g}_{\theta_j\theta_k}\frac{\tilde{\not{\nabla}}}{N-1}\right)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}).
\end{aligned} \tag{E.21}$$

We have verified using Mathematica 11.2 that the components defined by eqs. (E.19) and (E.21) satisfy the system of equations (E.8).

Type-*II* STSSH's of rank 2 for *N* even. Now let us describe how to obtain the type-*II* modes given by eqs. (5.22) and (5.23). The type-*II* modes satisfy $\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} = 0$ by definition. The components $\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}$ $(j = 1,...,N-1)$ may be expressed as

$$\begin{aligned}
{}^{(\uparrow)}\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \phi^{(0)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}), \\
{}^{(\downarrow)}\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm i\psi^{(0)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}),
\end{aligned} \tag{E.22}$$

$$\begin{aligned}
{}^{(\uparrow)}\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= i\psi^{(0)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}(\boldsymbol{\theta}_{N-1}) \\
{}^{(\downarrow)}\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm\phi^{(0)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}(\boldsymbol{\theta}_{N-1}).
\end{aligned} \tag{E.23}$$

The TT eigenvector-spinors $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ $(j = 1,...,N-1)$ on S^{N-1} satisfy eqs. (4.14) and (4.15). By working as in the case of type-*I* modes presented above, we find that $\phi^{(0)}_{n\ell}$ has to satisfy the differential equation (E.1) with $a = 0$, while $\psi^{(0)}_{n\ell}$ has to satisfy the differential equation (E.1) $(a = 0)$ with θ_N replaced by $\pi-\theta_N$ in the differential operator $D_{(0)}$ [eq. (E.2)]. Thus, we find that $\phi^{(0)}_{n\ell}$ and $\psi^{(0)}_{n\ell}$ are given by eqs. (3.1) and (3.2), respectively. By making use of the formulae (3.5) and (3.6), one can readily verify that the components defined by eqs. (E.22) and (E.23) are solutions of the system of equations (E.7).

The components $\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ $(j,k = 1,...,N-1)$ are symmetric rank-2 tensor-spinors on S^{N-1}. Let us first discuss the case with negative spin projection $(\sigma = -)$. We separate variables as

$$
\begin{aligned}
&{}^{(\uparrow)}\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = \Gamma^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) + \Delta^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}),\\
&{}^{(\downarrow)}\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = \pm\, i\Gamma^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) \pm i\Delta^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}),
\end{aligned}
\tag{E.24}
$$

where we have to determine the functions $\Gamma^{(b)}_{n\ell}$ and $\Delta^{(b)}_{n\ell}$ (with $b =\uparrow,\downarrow$). By using the TT conditions as in the case of type-I modes, we find that $\Delta^{(\uparrow)}_{n\ell}$ and $\Delta^{(\downarrow)}_{n\ell}$ are given by eqs. (5.24) and (5.25), respectively, while $\Gamma^{(\uparrow)}_{n\ell}$ and $\Gamma^{(\downarrow)}_{n\ell}$ are given by eqs. (5.26) and (5.27), respectively, where we also have used eqs. (3.5), (3.6) and (D.13). By using the formulae (3.5) and (3.6), we can also prove the following formulae:

$$
\left(\frac{\partial}{\partial\theta_N} + \frac{N-5}{2}\cot\theta_N - \frac{\ell+\frac{N-1}{2}}{\sin\theta_N}\right)\Gamma^{(\uparrow)}_{n\ell}(\theta_N) - \frac{2i}{\sin\theta_N}\Delta^{(\uparrow)}_{n\ell}(\theta_N) = -(n+\frac{N}{2})\Gamma^{(\downarrow)}_{n\ell}(\theta_N),
\tag{E.25}
$$

$$
\left(\frac{\partial}{\partial\theta_N} + \frac{N-5}{2}\cot\theta_N + \frac{\ell+\frac{N-1}{2}}{\sin\theta_N}\right)\Gamma^{(\downarrow)}_{n\ell}(\theta_N) + \frac{2i}{\sin\theta_N}\Delta^{(\downarrow)}_{n\ell}(\theta_N) = (n+\frac{N}{2})\Gamma^{(\uparrow)}_{n\ell}(\theta_N).
\tag{E.26}
$$

Similarly, we find that the upper and lower components of $\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ $(j,k=1,...,N-1)$ are given by

$$
\begin{aligned}
&{}^{(\uparrow)}\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = i\Gamma^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}(\boldsymbol{\theta}_{N-1}) - i\Delta^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}(\boldsymbol{\theta}_{N-1}),\\
&{}^{(\downarrow)}\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = \pm\,\Gamma^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}(\boldsymbol{\theta}_{N-1}) \mp \Delta^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}(\boldsymbol{\theta}_{N-1}).
\end{aligned}
\tag{E.27}
$$

By making use of the formulae (E.25) and (E.26), as well as eq. (D.5), one can readily verify that the system of equations (E.8) is satisfied by the type-II modes in eqs. (E.24) and (E.27).

Type-*III* STSSH's of rank 2 for *N* even. Finally, let us construct the type-III mode, given by eqs. (5.30) and (5.33). The type-III modes satisfy $\psi^{(III\text{-}\tilde{B};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} = 0$ and $\psi^{(III\text{-}\tilde{B};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_i} = 0$ $(i = 1,...,N-1)$ by definition. The components $\psi^{(III\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ $(j,k = 1,...,N-1)$ are rank-2 symmetric tensor-spinors on S^{N-1}. Since type-III modes are divergence-free and gamma-traceless, we separate variables in the following way:

$$
\begin{aligned}
&{}^{(\uparrow)}\psi^{(III\text{-}\tilde{B};-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = \phi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}),\\
&{}^{(\downarrow)}\psi^{(III\text{-}\tilde{B};-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = \pm i\psi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}),
\end{aligned}
\tag{E.28}
$$

$$
\begin{aligned}
&{}^{(\uparrow)}\psi^{(III\text{-}\tilde{B};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = i\psi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}),\\
&{}^{(\downarrow)}\psi^{(III-\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = \pm\phi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}),
\end{aligned}
\tag{E.29}
$$

where eq. (E.28) describes the type-*III* STSSH with negative spin projection, while eq. (E.29) describes the type-*III* STSSH with positive spin projection. The functions $\phi^{(-2)}_{n\ell}$ and $\psi^{(-2)}_{n\ell}$ are given by eqs. (3.1) and (3.2), respectively. It is straightforward to verify that the type-*III* modes in eqs. (E.28) and (E.29) are solutions of the system of equations (E.8) (with the use of eqs. (3.5) and (3.6)).

E.2 Constructing the STSSH's of rank 2 for N odd

Now the gamma matrices are given by eq. (2.14). By combining eqs. (2.4), (2.9), (2.14), (2.18) and eq. (2.19) we find

$$\nabla\!\!\!\!/\,\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} = \left[\left(\frac{\partial}{\partial\theta_N} + \frac{N+3}{2}\cot\theta_N\right)\gamma^N + \frac{1}{\sin\theta_N}\tilde{\nabla\!\!\!\!/}\right]\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} = \pm i|\zeta_{n,N}|\,\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}, \tag{E.30}$$

where we have used the gamma-tracelessness condition

$$\gamma^N\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} = -\gamma^{\theta_j}\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_N}$$

(see eq. (5.2)). Similarly, we find

$$\begin{aligned}\nabla\!\!\!\!/\,\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j} &= \left[\left(\frac{\partial}{\partial\theta_N} + \frac{N-1}{2}\cot\theta_N\right)\gamma^N + \frac{1}{\sin\theta_N}\tilde{\nabla\!\!\!\!/}\right]\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j} + \cot\theta_N\gamma_{\theta_j}\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} \\ &= \pm i|\zeta_{n,N}|\,\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}\end{aligned} \tag{E.31}$$

($j = 1, ..., N-1$) and

$$\begin{aligned}\nabla\!\!\!\!/\,\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k} &= \left[\left(\frac{\partial}{\partial\theta_N} + \frac{N-5}{2}\cot\theta_N\right)\gamma^N + \frac{1}{\sin\theta_N}\tilde{\nabla\!\!\!\!/}\right]\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k} + 2\cot\theta_N\gamma_{(\theta_j}\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_k)\theta_N} \\ &= \pm i|\zeta_{n,N}|\,\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}\end{aligned} \tag{E.32}$$

($j, k = 1, ..., N-1$). Note that for N odd we have

$$\gamma^N\tilde{\nabla\!\!\!\!/} + \tilde{\nabla\!\!\!\!/}\gamma^N = 0, \tag{E.33}$$

since $\{\gamma^N, \tilde{\gamma}^j\} = 0$ ($j = 1, ..., N-1$) - see eq. (2.14). Now let us separate variables in eqs. (E.30)-(E.32).

Type-*I* STSSH's of rank 2 for *N* odd. As in Ref. [21], since N is odd we choose to express the type-I modes in terms of the following spinors on S^{N-1}:

$$\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \equiv \frac{1}{\sqrt{2}}(\mathbf{1} + i\gamma^N)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \tag{E.34}$$

$$\hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \equiv \gamma^N\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) = \frac{1}{\sqrt{2}}(\mathbf{1} + i\gamma^N)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}), \tag{E.35}$$

where $\chi_{\pm\ell\tilde{\rho}}$ are the eigenspinors on S^{N-1} (satisfying eq. (2.22)). Since N is odd, $\chi_{+\ell\tilde{\rho}}$ and $\chi_{-\ell\tilde{\rho}}$ are related to each other as follows [21]:

$$\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) = \gamma^N \chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}). \tag{E.36}$$

The spinors $\hat{\chi}_{\pm\ell\tilde{\rho}}$ are eigenfunctions of the operator $\gamma^N \tilde{\nabla\!\!\!/}$ (that commutes with $\nabla\!\!\!/^2$) and they satisfy [21]

$$\gamma^N \tilde{\nabla\!\!\!/} \hat{\chi}_{\pm\ell\tilde{\rho}} = \pm \left(\ell + \frac{N-1}{2}\right) \hat{\chi}_{\pm\ell\tilde{\rho}}. \tag{E.37}$$

In order to construct the rank-2 type-I modes on S^N, we separate variables as follows:

$$\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) = \phi^{(2)}_{n\ell}(\theta_N)\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \pm i\psi^{(2)}_{n\ell}(\theta_N)\hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \tag{E.38}$$

$$\begin{aligned}\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) =& C^{(\uparrow)(2)}_{n\ell}(\theta_N)\tilde{\nabla}_{\theta_j}\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \pm iC^{(\downarrow)(2)}_{n\ell}(\theta_N)\tilde{\nabla}_{\theta_j}\hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ & - iD^{(\uparrow)(2)}_{n\ell}(\theta_N)\tilde{\gamma}_{\theta_j}\hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \mp D^{(\downarrow)(2)}_{n\ell}(\theta_N)\tilde{\gamma}_{\theta_j}\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{aligned} \tag{E.39}$$

$$\begin{aligned}\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =& \tilde{g}_{\theta_j\theta_k}\left(\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})K^{(\uparrow)}_{n\ell}(\theta_N) \pm \hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\, iK^{(\downarrow)}_{n\ell}(\theta_N)\right)\\ & + \left[\tilde{\nabla}_{(\theta_j}\tilde{\nabla}_{\theta_k)} - \frac{\tilde{g}_{\theta_j\theta_k}}{N-1}\tilde{\Box}\right]\\ & \times \left(\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})W^{(\uparrow)}_{n\ell}(\theta_N) \pm \hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\, iW^{(\downarrow)}_{n\ell}(\theta_N)\right)\\ & + \left[\tilde{\gamma}_{(\theta_j}\tilde{\nabla}_{\theta_k)} - \frac{\tilde{g}_{\theta_j\theta_k}}{N-1}\tilde{\nabla\!\!\!/}\right]\\ & \times \left(-\hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\, iT^{(\uparrow)}_{n\ell}(\theta_N) \mp \hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\, T^{(\downarrow)}_{n\ell}(\theta_N)\right),\end{aligned} \tag{E.40}$$

$(j,k=1,...,N-1)$. By working as in the case with N even, we find that the functions $\phi^{(2)}_{n\ell}, \psi^{(2)}_{n\ell}, C^{(b)(2)}_{n\ell}, D^{(b)(2)}_{n\ell}, K^{(b)}_{n\ell}, W^{(b)}_{n\ell}$ and $T^{(b)}_{n\ell}$ (where $b=\uparrow,\downarrow$), describing the dependence on θ_N, are the same functions as the ones used in the even-dimensional case (see eqs. (5.10)-(5.12)). By expressing $\hat{\chi}_{\pm\ell\tilde{\rho}}$ in terms of $\chi_{\pm\ell\tilde{\rho}}$ (by making use of eqs. (E.34) and (E.35)), it is straightforward to show that eqs. (E.38), (E.39) and (E.40) are equal to eqs. (5.35), (5.36) and (5.37), respectively, as presented in Subsection 5.2.

Type-*II* STSSH's of rank 2 for *N* odd. In order to construct the type-II STSSH's of rank 2 on S^N, we use the following vector-spinors on S^{N-1}:

$$\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \equiv \frac{1}{\sqrt{2}}(\mathbf{1} + i\gamma^N)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \tag{E.41}$$

$$\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}(\boldsymbol{\theta}_{N-1}) \equiv \gamma^N \hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}), \tag{E.42}$$

where $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ $(j=1,...,N-1)$ are the TT eigevector-spinors on S^{N-1} (satisfying eqs. (4.14) and (4.15)) and $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j} = \gamma^N \tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}$. The vector-spinors $\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ satisfy

$$\gamma^N \tilde{\nabla\!\!\!/} \hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j} = \pm\left(\ell + \frac{N-1}{2}\right)\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j} \tag{E.43}$$

$$\tilde{\gamma}^{\theta_i}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_i} = \tilde{\nabla}^{\theta_i}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_i} = 0. \tag{E.44}$$

By making use of the vector-spinors $\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$, we separate variables for the type-*II* STSSH's $\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\mu\nu}$ on S^N as follows:

$$\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) = \phi^{(0)}_{n\ell}(\theta_N)\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \pm i\psi^{(0)}_{n\ell}(\theta_N)\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}(\boldsymbol{\theta}_{N-1}) \tag{E.45}$$

$$\begin{aligned}\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =& \Gamma^{(\uparrow)}_{n\ell}(\theta_N)\tilde{\nabla}_{(\theta_j}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) \pm i\Gamma^{(\downarrow)}_{n\ell}(\theta_N)\tilde{\nabla}_{(\theta_j}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}(\boldsymbol{\theta}_{N-1})\\ &- i\Delta^{(\uparrow)}_{n\ell}(\theta_N)\tilde{\gamma}_{(\theta_j}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}(\boldsymbol{\theta}_{N-1}) \mp \Delta^{(\downarrow)}_{n\ell}(\theta_N)\tilde{\gamma}_{(\theta_j}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1})\end{aligned} \tag{E.46}$$

$(j,k=1,...,N-1)$, while $\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}=0$ by definition. By working as in the case with N even, we find that the functions $\phi^{(0)}_{n\ell}, \psi^{(0)}_{n\ell}, \Delta^{(b)}_{n\ell}$ and $\Gamma^{(b)}_{n\ell}$ (where $b=\uparrow,\downarrow$) are given by the same expressions as in the even-dimensional case (see eqs. (5.39) and (5.40)). By expressing $\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ in terms of $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ (with the use of eqs. (E.41) and (E.42)), we straightforwardly find that eqs. (E.45) and (E.46) are equal to eqs. (5.39) and (5.40), respectively.

Type-*III* STSSH's of rank 2 for *N* odd. In order to construct the type-*III* STSSH's of rank 2 on S^N, we use the following rank-2 symmetric tensor-spinors on S^{N-1}:

$$\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \equiv \frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \tag{E.47}$$

$$\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \equiv \gamma^N\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}), \tag{E.48}$$

where $\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_K}$ $(j,k=1,...,N-1)$ are the STSSH's of rank 2 on S^{N-1} (satisfying eqs. (5.6)-(5.8)). Also, note that $\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}=\gamma^N\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}$. The tensor-spinors $\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_k}$ satisfy

$$\gamma^N\tilde{\nabla\!\!\!/}\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_k} = \pm\left(\ell+\frac{N-1}{2}\right)\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_k} \tag{E.49}$$

$$\tilde{\gamma}^{\theta_i}\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_k} = \tilde{\nabla}^{\theta_i}\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_k}=0 \tag{E.50}$$

$$\tilde{g}^{\theta_i\theta_j}\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}=0 \tag{E.51}$$

$(i,j,k=1,...,N-1)$.

By making use of the tensor-spinors $\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_k}$, we separate variables for the type-*III* STSSH's $\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\mu\nu}$ on S^N as follows:

$$\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = \phi^{(-2)}_{n\ell}(\theta_N)\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \pm i\psi^{(-2)}_{n\ell}(\theta_N)\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \tag{E.52}$$

$(j,k=1,...,N-1)$, while $\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}=0$ and $\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}=0$ (by definition). By working as in the case with N even, we find that the functions $\phi^{(-2)}_{n\ell}$ and $\psi^{(-2)}_{n\ell}$ are given by eqs. (3.1) and (3.2), respectively [and, thus, eq. (E.52) is equal to eq. (5.43)].

F Deriving the formulae for the spin$(N+1)$ transformation of the STSSH's of ranks 1 and 2 on S^N and determining their normalisation factors

In Subsections F.1-F.3 of this Appendix we derive the transformation formulae (6.10), (6.15), (6.29) and (6.30) for STSSH's of rank 1 on S^N and we calculate the normalisation factors $c_N^{(I;r=1)}(n,\ell)$ and $c_N^{(II;r=1)}(n,\ell)$ [eq. (6.8)]. The derivation of the transformation formulae and the calculation of the normalisation factors for the STSSH's of rank 2 have many similarities with the case of rank-1 STSSH's and, thus, we discuss them in less detail in Subsection F.4.

F.1 Calculating $c_N^{(II;r=1)}(n,\ell)$ and making the first step towards the calculation of $c_N^{(I;r=1)}(n,\ell)$

Since it is a quite simple task, let us start by calculating directly the normalisation factor for type-II STSSH's of rank 1 for arbitrary N. For N even, we substitute the unnormalised type-II modes (4.17) (or (4.18)) into the inner product (6.7). Then, by performing the integration over S^{N-1} using eq. (4.16), we find

$$\begin{aligned}\left|\frac{c_N^{(II;r=1)}(n,\ell)}{\sqrt{2}}\right|^{-2} &= \int_0^\pi d\theta_N \sin^{N-3}\theta_N \left[\left(\phi_{n\ell}^{(-1)}(\theta_N)\right)^2 + \left(\psi_{n\ell}^{(-1)}(\theta_N)\right)^2\right] \\ &= \frac{1}{4}\int_0^\pi d\theta_N \sin^{N-1}\theta_N \left[\left(\phi_{n\ell}^{(0)}(\theta_N)\right)^2 + \left(\psi_{n\ell}^{(0)}(\theta_N)\right)^2\right], \end{aligned} \tag{F.1}$$

where the functions $\phi_{n\ell}^{(0)}$ and $\psi_{n\ell}^{(0)}$ are given by eqs. (3.1) and (3.2), respectively. The integral in the last line is the same integral that appears in the normalisation of spinor eigenfunctions on S^N in Ref. [21]. Thus, using the result of Ref. [21] we readily find

$$\left|\frac{c_N^{(II;r=1)}(n,\ell)}{\sqrt{2}}\right|^{2} = \frac{1}{2^{N-3}}\frac{\Gamma(n-\ell+1)\Gamma(n+\ell+N)}{|\Gamma(n+\frac{N}{2})|^2}, \tag{F.2}$$

which is a special case of eq. (6.8). For N odd, the calculation is similar and we find again that the normalisation factor is given by eq. (6.8).

The normalisation factor of the type-I modes can be found by calculating the following integral:

$$\begin{aligned}\left|\frac{c_N^{(I;r=1)}(n,\ell)}{\sqrt{2}}\right|^{-2} &= \int_0^\pi d\theta_N \sin^{N-1}\theta_N \left[\left(\phi_{n\ell}^{(1)}(\theta_N)\right)^2 + \left(\psi_{n\ell}^{(1)}(\theta_N)\right)^2\right] \\ &+ \left[\left(\ell+\frac{N-1}{2}\right)^2 - \frac{(N-1)(N-2)}{4}\right] \\ &\times \int_0^\pi d\theta_N \sin^{N-3}\theta_N \left[\left(C_{n\ell}^{(\uparrow)(1)}(\theta_N)\right)^2 + \left(C_{n\ell}^{(\downarrow)(1)}(\theta_N)\right)^2\right] \\ &+ (N-1)\int_0^\pi d\theta_N \sin^{N-3}\theta_N \left[\left|D_{n\ell}^{(\uparrow)(1)}(\theta_N)\right|^2 + \left|D_{n\ell}^{(\downarrow)(1)}(\theta_N)\right|^2\right] \\ &+ 2i\left(\ell+\frac{N-1}{2}\right) \\ &\times \int_0^\pi d\theta_N \sin^{N-3}\theta_N \\ &\times \left[C_{n\ell}^{(\uparrow)(1)}(\theta_N)D_{n\ell}^{(\uparrow)(1)}(\theta_N) + C_{n\ell}^{(\downarrow)(1)}(\theta_N)D_{n\ell}^{(\downarrow)(1)}(\theta_N)\right], \end{aligned} \tag{F.3}$$

where $C_{n\ell}^{(\uparrow)(1)}, C_{n\ell}^{(\downarrow)(1)}, D_{n\ell}^{(\uparrow)(1)}$ and $D_{n\ell}^{(\downarrow)(1)}$ are given by eqs. (4.9), (4.10), (4.11) and (4.12), respectively. For N even, eq. (F.3) is derived by substituting the expressions (4.5) and (4.6) for type-I modes into the inner product (6.7) and then performing the integration over S^{N-1} (with the use of eqs. (2.23) and (D.7)). For N odd, by working similarly we find again eq. (F.3). Since the integrals in eq. (F.3) are not as simple as in the case of type-II modes, we are going to take an indirect route. To be specific, we first obtain by direct calculation the normalisation factor of the type-I modes with the highest allowed value for ℓ, i.e. $c_N^{(I;r=1)}(n, \ell = n)$. Then, once we have obtained the transformation formulae of the type-I modes under spin$(N+1)$, the normalisation factor $c_N^{(I;r=1)}(n, \ell)$ (for $\ell = 1, 2, \ldots .n-1$) will be constructed in terms of $c_N^{(I;r=1)}(n, n)$ by exploiting the spin$(N+1)$ invariance of the inner product (6.4). To calculate $c_N^{(I;r=1)}(n, n)$ we let $\ell = n$ in eq. (F.3) and by calculating the integrals using Mathematica 11.2 we find

$$\left| \frac{c_N^{(I;r=1)}(n, n)}{\sqrt{2}} \right|^2 = \frac{n(N-2)\Gamma(n + \frac{N}{2} + \frac{1}{2})}{4^{1-n}(1+n)(N-1)\sqrt{\pi}\Gamma(n + \frac{N}{2})}. \tag{F.4}$$

F.2 Derivation of the transformation formulae of type-I and type-II-I STSSH's of rank 1 and calculation of the normalisation factor $c_N^{(I;r=1)}(n, \ell)$ for N even

Below we give details for the derivation of the transformation formulae (6.10) and (6.15) for rank-1 ($r = 1$) modes with positive spin projection [these modes are given by eqs. (4.7), (4.8) and (4.18)]. The calculations for the rank-1 modes with negative spin projection are not presented here, as they can be performed in the same way.

In order to derive the desired transformation formulae (6.10) and (6.15), it is sufficient to study the following two components of the Lie-Lorentz derivative (6.1): $\mathbb{L}_{\mathscr{S}}\ \psi_{\theta_N}$ and $\mathbb{L}_{\mathscr{S}}\ \psi_{\theta_{N-1}}$. After a straightforward calculation we find

$$\mathbb{L}_{\mathscr{S}}\ \psi_{\theta_N} = \left(\mathscr{S}^\mu \partial_\mu + \frac{\sin\theta_{N-1}}{2\sin\theta_N} \gamma^N \gamma^{N-1} \right) \psi_{\theta_N} + \frac{\sin\theta_{N-1}}{\sin^2\theta_N} \psi_{\theta_{N-1}} \tag{F.5}$$

and

$$\mathbb{L}_{\mathscr{S}}\ \psi_{\theta_{N-1}} = \left(\mathscr{S}^\mu \partial_\mu - \cot\theta_N\, \cos\theta_{N-1} + \frac{\sin\theta_{N-1}}{2\sin\theta_N} \gamma^N \gamma^{N-1} \right) \psi_{\theta_{N-1}} - \sin\theta_{N-1}\ \psi_{\theta_N}, \tag{F.6}$$

where we have substituted eqs. (2.4), (2.9), (2.17) and (6.6) into eq. (6.1). Since N is even, we express $\gamma^N\gamma^{N-1}$ in eqs. (F.5) and (F.6) as

$$\gamma^N \gamma^{N-1} = \begin{pmatrix} -i\widetilde{\gamma}^{N-1} & 0 \\ 0 & i\widetilde{\gamma}^{N-1} \end{pmatrix}, \tag{F.7}$$

where we have used eq. (2.11).

The partial derivatives in eqs. (F.5) and (F.6) act only on the coordinates $\{\theta_N, \theta_{N-1}\}$. Thus, for later convenience let us introduce the functions $\tilde{\phi}_{\ell m}^{(\tilde{a})}(\theta_{N-1})$ and $\tilde{\psi}_{\ell m}^{(\tilde{a})}(\theta_{N-1})$ describing the θ_{N-1}-dependence of the STSSH's on S^{N-1} . In analogy to eqs. (3.1) and (3.2), these functions are given by

$$\tilde{\phi}^{(\tilde{a})}_{\ell m}(\theta_{N-1}) = \tilde{\kappa}_{\tilde{\phi}}(\ell, m) \left(\cos\frac{\theta_{N-1}}{2}\right)^{m+1-\tilde{a}} \left(\sin\frac{\theta_{N-1}}{2}\right)^{m-\tilde{a}} \times F\left(-\ell+m, \ell+m+N-1; m+\frac{N-1}{2}; \sin^2\frac{\theta_{N-1}}{2}\right), \tag{F.8}$$

and

$$\tilde{\psi}^{(\tilde{a})}_{\ell m}(\theta_{N-1}) = \tilde{\kappa}_{\tilde{\phi}}(\ell, m) \frac{\ell+\frac{N-1}{2}}{m+\frac{N-1}{2}} \left(\cos\frac{\theta_{N-1}}{2}\right)^{m-\tilde{a}} \left(\sin\frac{\theta_{N-1}}{2}\right)^{m+1-\tilde{a}} \times F\left(-\ell+m, \ell+m+N-1; m+\frac{N+1}{2}; \sin^2\frac{\theta_{N-1}}{2}\right), \tag{F.9}$$

where the normalisation factor is given by

$$\tilde{\kappa}_{\tilde{\phi}}(\ell, m) = \frac{\Gamma(\ell+\frac{N-1}{2})}{\Gamma(\ell-m+1)\,\Gamma(m+\frac{N-1}{2})}. \tag{F.10}$$

The number $\tilde{a}$ in eqs. (F.8) and (F.9) is an integer and m is the angular momentum quantum number on S^{N-2} [with $\ell \geq m$, in analogy with eq. (3.4)]. The formulae analogous to eqs. (3.5) and (3.6) are given by

$$\left(\frac{d}{d\theta_{N-1}} + \frac{N+2\tilde{a}-2}{2}\cot\theta_{N-1} + \frac{m+\frac{N-2}{2}}{\sin\theta_{N-1}}\right)\tilde{\psi}^{(\tilde{a})}_{\ell m} = \left(\ell+\frac{N-1}{2}\right)\tilde{\phi}^{(\tilde{a})}_{\ell m} \tag{F.11}$$

and

$$\left(\frac{d}{d\theta_{N-1}} + \frac{N+2\tilde{a}-2}{2}\cot\theta_{N-1} - \frac{m+\frac{N-2}{2}}{\sin\theta_{N-1}}\right)\tilde{\phi}^{(\tilde{a})}_{\ell m} = -\left(\ell+\frac{N-1}{2}\right)\tilde{\psi}^{(\tilde{a})}_{\ell m}, \tag{F.12}$$

respectively.

Motivated by the techniques used in Refs. [13] and [22], in order to derive the transformation formulae of our STSSH's we introduce the ladder operators for ℓ, sending ℓ to $\ell \pm 1$ when acting on the functions $\phi^{(a)}_{n\ell}(\theta_N), \psi^{(a)}_{n\ell}(\theta_N), \tilde{\phi}^{(\tilde{a})}_{\ell m}(\theta_{N-1})$ and $\tilde{\psi}^{(\tilde{a})}_{\ell m}(\theta_{N-1})$. The ladder operators are given by the following expressions:

$$T^{(+;a)}_{\phi} = \frac{d}{d\theta_N} + \left(-\ell+a-\frac{1}{2}\right)\cot\theta_N + \frac{1}{2\sin\theta_N}, \tag{F.13}$$

$$T^{(+;a)}_{\psi} = \frac{d}{d\theta_N} + \left(-\ell+a-\frac{1}{2}\right)\cot\theta_N - \frac{1}{2\sin\theta_N}, \tag{F.14}$$

$$T^{(-;a)}_{\phi} = \frac{d}{d\theta_N} + \left(\ell+N+a-\frac{3}{2}\right)\cot\theta_N - \frac{1}{2\sin\theta_N}, \tag{F.15}$$

$$T^{(-;a)}_{\psi} = \frac{d}{d\theta_N} + \left(\ell+N+a-\frac{3}{2}\right)\cot\theta_N + \frac{1}{2\sin\theta_N}, \tag{F.16}$$

$$\tilde{\Pi}^{(+;\tilde{a})}_{\tilde{\phi}} = \sin\theta_{N-1}\frac{d}{d\theta_{N-1}} + \left(\ell + \tilde{a} + N - \frac{3}{2}\right)\cos\theta_{N-1} - \frac{m + \frac{N-2}{2}}{2(\ell + \frac{N}{2})}, \tag{F.17}$$

$$\tilde{\Pi}^{(+;\tilde{a})}_{\tilde{\psi}} = \sin\theta_{N-1}\frac{d}{d\theta_{N-1}} + \left(\ell + \tilde{a} + N - \frac{3}{2}\right)\cos\theta_{N-1} + \frac{m + \frac{N-2}{2}}{2(\ell + \frac{N}{2})}, \tag{F.18}$$

$$\tilde{\Pi}^{(-;\tilde{a})}_{\tilde{\phi}} = \sin\theta_{N-1}\frac{d}{d\theta_{N-1}} + \left(-\ell + \tilde{a} - \frac{1}{2}\right)\cos\theta_{N-1} + \frac{m + \frac{N-2}{2}}{2(\ell + \frac{N-2}{2})}, \tag{F.19}$$

$$\tilde{\Pi}^{(-;\tilde{a})}_{\tilde{\psi}} = \sin\theta_{N-1}\frac{d}{d\theta_{N-1}} + \left(-\ell + \tilde{a} - \frac{1}{2}\right)\cos\theta_{N-1} - \frac{m + \frac{N-2}{2}}{2(\ell + \frac{N-2}{2})}. \tag{F.20}$$

These operators act as follows:

$$T^{(+;a)}_{f} f^{(a)}_{n\ell}(\theta_N) = k^{(+)} f^{(a)}_{n\,\ell+1}(\theta_N), \tag{F.21}$$

$$T^{(-;a)}_{f} f^{(a)}_{n\ell}(\theta_N) = k^{(-)} f^{(a)}_{n\,\ell-1}(\theta_N), \tag{F.22}$$

$$\tilde{\Pi}^{(+;\tilde{a})}_{\tilde{f}} \tilde{f}^{(\tilde{a})}_{\ell m}(\theta_{N-1}) = \tilde{k}^{(+)} \tilde{f}^{(\tilde{a})}_{\ell+1\,m}(\theta_{N-1}), \tag{F.23}$$

$$\tilde{\Pi}^{(-;\tilde{a})}_{\tilde{f}} \tilde{f}^{(\tilde{a})}_{\ell m}(\theta_{N-1}) = \tilde{k}^{(-)} \tilde{f}^{(\tilde{a})}_{\ell-1\,m}(\theta_{N-1}), \tag{F.24}$$

where $f^{(a)}_{n\ell}(\theta_N) \in \{\,\phi^{(a)}_{n\ell}(\theta_N), \psi^{(a)}_{n\ell}(\theta_N)\,\}$, $\tilde{f}^{(\tilde{a})}_{\ell m}(\theta_{N-1}) \in \{\,\tilde{\phi}^{(\tilde{a})}_{\ell m}(\theta_{N-1}), \tilde{\psi}^{(\tilde{a})}_{\ell m}(\theta_{N-1})\,\}$ and

$$k^{(+)} = -(n + \ell + N), \tag{F.25}$$

$$k^{(-)} = n - \ell + 1, \tag{F.26}$$

$$\tilde{k}^{(+)} = \frac{(\ell + N - 1 + m)(\ell - m + 1)}{\ell + N/2}, \tag{F.27}$$

$$\tilde{k}^{(-)} = -\frac{(\ell + \frac{N-1}{2} - 1)(\ell + \frac{N-1}{2})}{\ell + (N-2)/2}. \tag{F.28}$$

One can straightforwardly prove the ladder relations (F.21)-(F.24) using the raising and lowering operators for the parameters of the Gauss hypergeometric function given in Appendix B. (Similar ladder relations have been obtained by the author in Ref. [22] while studying the Dirac field on dS_N.)

Let us now proceed to the derivation of the transformation formulae of the type-I and type-II-I modes. It is clear from the expressions (F.5) and (F.6) for the Lie-Lorentz derivative that we need to express the type-I and type-II-I modes in a form where the dependence on both θ_N and θ_{N-1} is written out explicitly. By substituting eq. (C.1) into eqs. (4.7) and (4.8), we express the type-I modes with positive spin projection as

$$\begin{aligned}
&\psi^{(I;+;n\ell m;\rho)}_{\pm\theta_N}(\theta_N, \theta_{N-1}, \boldsymbol{\theta}_{N-2}) \\
&= \frac{\tilde{c}_{N-1}(\ell, m)}{\sqrt{2}} \begin{pmatrix} i\psi^{(1)}_{n\ell}(\theta_N)\left[\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) + i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \\ \pm\phi^{(1)}_{n\ell}(\theta_N)\left[\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) + i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \end{pmatrix}
\end{aligned} \tag{F.29}$$

$$\psi^{(I;+;n\ell m;\rho)}_{\pm\theta_{N-1}}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2})$$
$$= \frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\begin{pmatrix} i\left[E^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})+i\Sigma^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \\ \pm\left[H^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})+iO^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right]\end{pmatrix}, \tag{F.30}$$

where $\tilde{c}_{N-1}(\ell,m)$ is the normalisation factor (C.4) for the eigenspinors on S^{N-1}, while the spinors $\hat{\tilde{\chi}}_{\pm m\rho}(\boldsymbol{\theta}_{N-2})$ on S^{N-2} are defined by eq. (C.2). Also, we have defined

$$O^{(a)}_{n\ell m}(\theta_N,\theta_{N-1}) = C^{(\uparrow)(a)}_{n\ell}(\theta_N)\,\frac{\partial}{\partial\theta_{N-1}}\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})+iD^{(\uparrow)(a)}_{n\ell}(\theta_N)\,\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1}) \tag{F.31}$$
$$H^{(a)}_{n\ell m}(\theta_N,\theta_{N-1}) = C^{(\uparrow)(a)}_{n\ell}(\theta_N)\,\frac{\partial}{\partial\theta_{N-1}}\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})-iD^{(\uparrow)(a)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1}) \tag{F.32}$$
$$E^{(a)}_{n\ell m}(\theta_N,\theta_{N-1}) = C^{(\downarrow)(a)}_{n\ell}(\theta_N)\,\frac{\partial}{\partial\theta_{N-1}}\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})-iD^{(\downarrow)(a)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1}) \tag{F.33}$$
$$\Sigma^{(a)}_{n\ell m}(\theta_N,\theta_{N-1}) = C^{(\downarrow)(a)}_{n\ell}(\theta_N)\,\frac{\partial}{\partial\theta_{N-1}}\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})+iD^{(\downarrow)(a)}_{n\ell}(\theta_N)\,\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1}). \tag{F.34}$$

(Recall that $C^{(\uparrow)(a)}_{n\ell}, C^{(\downarrow)(a)}_{n\ell}, D^{(\uparrow)(a)}_{n\ell}$ and $D^{(\downarrow)(a)}_{n\ell}$ are given by eqs. (4.9), (4.10), (4.11) and (4.12), respectively.) Similarly, the type-I modes with negative spin projection are expressed as

$$\psi^{(I;-;n\ell m;\rho)}_{\pm\theta_{N}}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2})$$
$$= \frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\begin{pmatrix} \phi^{(1)}_{n\ell}(\theta_N)\left[\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})-i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \\ \pm i\psi^{(1)}_{n\ell}(\theta_N)\,\left[\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})-i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right]\end{pmatrix} \tag{F.35}$$

$$\psi^{(I;-;n\ell m;\rho)}_{\pm\theta_{N-1}}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2})$$
$$= \frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\begin{pmatrix} H^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})-iO^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2}) \\ \pm i\left[E^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})-i\Sigma^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right]\end{pmatrix}. \tag{F.36}$$

Similarly, it is straightforward to express the type-II-I modes with positive spin projection (4.18) as follows:

$$\psi^{(II\text{-}I;+;n\ell m;\rho)}_{\pm\theta_N}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2}) = 0, \tag{F.37}$$

$$\psi_{\pm\theta_{N-1}}^{(\textit{II-I};+;n\ell m;\rho)}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2})$$
$$= \frac{\tilde{c}_{N-1}^{(I;\tilde{r}=1)}(\ell,m)}{\sqrt{2}} \begin{pmatrix} i\psi_{n\ell}^{(-1)}(\theta_N)\left[\tilde{\phi}_{\ell m}^{(1)}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) + i\tilde{\psi}_{\ell m}^{(1)}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \\ \pm\phi_{n\ell}^{(-1)}(\theta_N)\left[\tilde{\phi}_{\ell m}^{(1)}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) + i\tilde{\psi}_{\ell m}^{(1)}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \end{pmatrix}, \tag{F.38}$$

where $\tilde{c}_{N-1}^{(I;r=1)}(\ell,m)$ is the normalisation factor of the STSSH's of rank 1 on S^{N-1} and it will be determined later. The type-*II-I* modes with negative spin projection (4.17) are expressed as

$$\psi_{\pm\theta_N}^{(\textit{II-I};-;n\ell m;\rho)}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2}) = 0, \tag{F.39}$$

$$\psi_{\pm\theta_{N-1}}^{(\textit{II-I};-;n\ell m;\rho)}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2})$$
$$= \frac{\tilde{c}_{N-1}^{(I;\tilde{r}=1)}(\ell,m)}{\sqrt{2}} \begin{pmatrix} \phi_{n\ell}^{(-1)}(\theta_N)\left[\tilde{\phi}_{\ell m}^{(1)}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) - i\tilde{\psi}_{\ell m}^{(1)}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \\ \pm i\psi_{n\ell}^{(-1)}(\theta_N)\left[\tilde{\phi}_{\ell m}^{(1)}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) - i\tilde{\psi}_{\ell m}^{(1)}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \end{pmatrix}, \tag{F.40}$$

F.2.1 Derivation of the transformation formula (6.10) for type-I modes of rank 1 and calculation of the normalisation factor $c_N^{(I;r=1)}(n,\ell)$

By using the expressions (F.29) and (F.30) for the type-I modes, we express the Lie-Lorentz derivative (F.5) as

$$\mathbb{L}_{\mathscr{S}}\psi_{\pm\theta_N}^{(I;+;n\ell m;\rho)}$$
$$= \frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}} \begin{pmatrix} i\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}_3^{(I)}(\theta_N,\theta_{N-1}) - \hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}_4^{(I)}(\theta_N,\theta_{N-1}) \\ \pm\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}_1^{(I)}(\theta_N,\theta_{N-1}) \pm i\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}_2^{(I)}(\theta_N,\theta_{N-1}) \end{pmatrix}, \tag{F.41}$$

where

$$\mathbb{T}_1^{(I)} = \mathscr{S}^\mu\partial_\mu\left[\phi_{n\ell}^{(1)}\tilde{\phi}_{\ell m}^{(0)}\right] - \frac{\sin\theta_{N-1}}{2\sin\theta_N}\phi_{n\ell}^{(1)}\tilde{\psi}_{\ell m}^{(0)} + \frac{\sin\theta_{N-1}}{\sin^2\theta_N}H_{n\ell m}^{(1)}, \tag{F.42}$$
$$\mathbb{T}_2^{(I)} = \mathscr{S}^\mu\partial_\mu\left[\phi_{n\ell}^{(1)}\tilde{\psi}_{\ell m}^{(0)}\right] + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\phi_{n\ell}^{(1)}\tilde{\phi}_{\ell m}^{(0)} + \frac{\sin\theta_{N-1}}{\sin^2\theta_N}O_{n\ell m}^{(1)}, \tag{F.43}$$
$$\mathbb{T}_3^{(I)} = \mathscr{S}^\mu\partial_\mu\left[\psi_{n\ell}^{(1)}\tilde{\phi}_{\ell m}^{(0)}\right] + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\psi_{n\ell}^{(1)}\tilde{\psi}_{\ell m}^{(0)} + \frac{\sin\theta_{N-1}}{\sin^2\theta_N}E_{n\ell m}^{(1)}, \tag{F.44}$$
$$\mathbb{T}_4^{(I)} = \mathscr{S}^\mu\partial_\mu\left[\psi_{n\ell}^{(1)}\tilde{\psi}_{\ell m}^{(0)}\right] - \frac{\sin\theta_{N-1}}{2\sin\theta_N}\psi_{n\ell}^{(1)}\tilde{\phi}_{\ell m}^{(0)} + \frac{\sin\theta_{N-1}}{\sin^2\theta_N}\Sigma_{n\ell m}^{(1)}. \tag{F.45}$$

(Recall that $O^{(1)}_{n\ell m}$, $H^{(1)}_{n\ell m}$, $E^{(1)}_{n\ell m}$ and $\Sigma^{(1)}_{n\ell m}$ are given by eqs. (F.31), (F.32), (F.33) and (F.34), respectively.) In order to proceed we need to make use of the following relations:

$$\mathbb{T}^{(I)}_1 = \mathscr{R}^{(I)} k^{(+)} \tilde{k}^{(+)} \phi^{(1)}_{n\,\ell+1} \tilde{\phi}^{(0)}_{\ell+1\,m} + \mathscr{L}^{(I)} k^{(-)} \tilde{k}^{(-)} \phi^{(1)}_{n\,\ell-1} \tilde{\phi}^{(0)}_{\ell-1\,m} + \varkappa^{(I)} \psi^{(1)}_{n\ell} \tilde{\phi}^{(0)}_{\ell m}, \tag{F.46}$$

$$\mathbb{T}^{(I)}_2 = \mathscr{R}^{(I)} k^{(+)} \tilde{k}^{(+)} \phi^{(1)}_{n\,\ell+1} \tilde{\psi}^{(0)}_{\ell+1\,m} + \mathscr{L}^{(I)} k^{(-)} \tilde{k}^{(-)} \phi^{(1)}_{n\,\ell-1} \tilde{\psi}^{(0)}_{\ell-1\,m} - \varkappa^{(I)} \psi^{(1)}_{n\ell} \tilde{\psi}^{(0)}_{\ell m}, \tag{F.47}$$

$$\mathbb{T}^{(I)}_3 = \mathscr{R}^{(I)} k^{(+)} \tilde{k}^{(+)} \psi^{(1)}_{n\,\ell+1} \tilde{\phi}^{(0)}_{\ell+1\,m} + \mathscr{L}^{(I)} k^{(-)} \tilde{k}^{(-)} \psi^{(1)}_{n\,\ell-1} \tilde{\phi}^{(0)}_{\ell-1\,m} - \varkappa^{(I)} \phi^{(1)}_{n\ell} \tilde{\phi}^{(0)}_{\ell m}, \tag{F.48}$$

$$\mathbb{T}^{(I)}_4 = \mathscr{R}^{(I)} k^{(+)} \tilde{k}^{(+)} \psi^{(1)}_{n\,\ell+1} \tilde{\psi}^{(0)}_{\ell+1\,m} + \mathscr{L}^{(I)} k^{(-)} \tilde{k}^{(-)} \psi^{(1)}_{n\,\ell-1} \tilde{\psi}^{(0)}_{\ell-1\,m} + \varkappa^{(I)} \phi^{(1)}_{n\ell} \tilde{\psi}^{(0)}_{\ell m}, \tag{F.49}$$

where $k^{(+)}, k^{(-)}, \tilde{k}^{(+)}$ and $\tilde{k}^{(-)}$ are given by eqs. (F.25), (F.26), (F.27) and (F.28), respectively, while $\varkappa^{(I)}$ is the coefficient defined in eq. (6.13) (with $r = 1$) and

$$\mathscr{R}^{(I)} = \frac{\ell + N}{2(\ell + \frac{N-1}{2})(\ell + N - 1)}, \qquad \mathscr{L}^{(I)} = \frac{1-\ell}{2\ell(\ell + \frac{N-1}{2})}. \tag{F.50}$$

Let us outline the steps required for proving eq. (F.46). (Equations (F.47)-(F.49) are proved similarly.) First, we express $\mathbb{T}^{(I)}_1$ on the left-hand side of eq. (F.46) in terms of $\phi^{(1)}_{n\ell}$, $d\phi^{(1)}_{n\ell}/d\theta_N$, $\tilde{\phi}^{(0)}_{\ell m}$ and $d\tilde{\phi}^{(0)}_{\ell m}/d\theta_{N-1}$ by making use of eqs. (F.42), (F.32), (F.12), (4.9), (4.11) and (3.6). As for the right-hand side, we express $\phi^{(1)}_{n\,\ell\pm1}$ and $\tilde{\phi}^{(0)}_{\ell\pm1\,m}$ in terms of $\phi^{(1)}_{n\ell}$, $d\phi^{(1)}_{n\ell}/d\theta_N$ and $\tilde{\phi}^{(0)}_{\ell m}$, $d\tilde{\phi}^{(0)}_{\ell m}/d\theta_{N-1}$, respectively, by making use of the ladder relations (F.21)-(F.24) and we also express $\psi^{(1)}_{n\ell}$ in terms of $\phi^{(1)}_{n\ell}$ and $d\phi^{(1)}_{n\ell}/d\theta_N$ by making use of eq. (3.6). Then, it is straightforward to show that the two sides of eq. (F.46) are equal. We have verified the calculations using Mathematica 11.2.

Then, by substituting eqs. (F.46)-(F.49) into eq. (F.41), we express the latter as

$$\begin{aligned}
\mathbb{L}_{\mathscr{S}} \psi^{(I;+;n\ell m;\rho)}_{\pm\theta_N} = \frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}} \Bigg\{ & \mathscr{R}^{(I)} k^{(+)} \tilde{k}^{(+)} \begin{pmatrix} i\psi^{(1)}_{n\,\ell+1} \left[\tilde{\phi}^{(0)}_{\ell+1\,m}\, \hat{\tilde{\chi}}_{-m\rho} + i\tilde{\psi}^{(0)}_{\ell+1\,m}\, \hat{\tilde{\chi}}_{+m\rho} \right] \\ \\ \pm\phi^{(1)}_{n\,\ell+1} \left[\tilde{\phi}^{(0)}_{\ell+1\,m}\, \hat{\tilde{\chi}}_{-m\rho} + i\tilde{\psi}^{(0)}_{\ell+1\,m}\, \hat{\tilde{\chi}}_{+m\rho} \right] \end{pmatrix} \\
& + \mathscr{L}^{(I)} k^{(-)} \tilde{k}^{(-)} \begin{pmatrix} i\psi^{(1)}_{n\,\ell-1} \left[\tilde{\phi}^{(0)}_{\ell-1\,m}\, \hat{\tilde{\chi}}_{-m\rho} + i\tilde{\psi}^{(0)}_{\ell-1\,m}\, \hat{\tilde{\chi}}_{+m\rho} \right] \\ \\ \pm\phi^{(1)}_{n\,\ell-1} \left[\tilde{\phi}^{(0)}_{\ell-1\,m}\, \hat{\tilde{\chi}}_{-m\rho} + i\tilde{\psi}^{(0)}_{\ell-1\,m}\, \hat{\tilde{\chi}}_{+m\rho} \right] \end{pmatrix} \\
& - i\varkappa^{(I)} \begin{pmatrix} \phi^{(1)}_{n\ell} \left[\tilde{\phi}^{(0)}_{\ell m}\, \hat{\tilde{\chi}}_{-m\rho} - i\tilde{\psi}^{(0)}_{\ell m}\, \hat{\tilde{\chi}}_{+m\rho} \right] \\ \\ \pm i\psi^{(1)}_{n\ell} \left[\tilde{\phi}^{(0)}_{\ell m}\, \hat{\tilde{\chi}}_{-m\rho} - i\tilde{\psi}^{(0)}_{\ell m}\, \hat{\tilde{\chi}}_{+m\rho} \right] \end{pmatrix} \Bigg\}
\end{aligned} \tag{F.51}$$

and we straightforwardly rewrite this as

$$\mathbb{L}_{\mathscr{S}} \psi^{(I;+;n\ell m;\rho)}_{\pm\theta_N} = \mathscr{A}^{(I)} \psi^{(I;+;n\,(\ell+1)\,m;\rho)}_{\pm\theta_N} + \mathscr{B}^{(I)} \psi^{(I;+;n\,(\ell-1)\,m;\rho)}_{\pm\theta_N} - i\varkappa^{(I)} \psi^{(I;-;n\ell m;\rho)}_{\pm\theta_N}, \tag{F.52}$$

as in eq. (6.10), where we have defined

$$\mathscr{A}^{(I)} \equiv \mathscr{R}^{(I)} k^{(+)} \tilde{k}^{(+)} \frac{\tilde{c}_{N-1}(\ell,m)}{\tilde{c}_{N-1}(\ell+1,m)}, \tag{F.53}$$

$$\mathscr{B}^{(I)} \equiv \mathscr{L}^{(I)} k^{(-)} \tilde{k}^{(-)} \frac{\tilde{c}_{N-1}(\ell,m)}{\tilde{c}_{N-1}(\ell-1,m)}. \tag{F.54}$$

It easy to verify that these expressions for $\mathscr{A}^{(I)}$ and $\mathscr{B}^{(I)}$ agree with the expressions given by eqs. (6.11) (with $r=1$) and (6.12) (with $r=1$), respectively.

Now, we can determine the normalisation factor $c_N^{(I;r=1)}(n,\ell)$ for the type-I modes. By using the spin$(N+1)$ invariance of the inner product (6.5) between $\psi_{\pm\mu}^{(I;\sigma;n\ell m;\rho)}$ and $\psi_{\pm\mu}^{(I;\sigma;n(\ell+1)m;\rho)}$ and using the transformation formula (6.10) we find

$$\left|\frac{c_N^{(I;r=1)}(n,\ell)}{c_N^{(I;r=1)}(n,\ell+1)}\right|^2=\frac{(n-\ell)\,\ell\,(\ell+N-1)}{(\ell+1)(\ell+N)(n+\ell+N)}. \tag{F.55}$$

By iterating this equation and using eq. (F.4), one can straightforwardly find

$$\begin{aligned}\left|\frac{c_N^{(I;r=1)}(n,\ell)}{\sqrt{2}}\right|^2=&\frac{1}{2^{N+1}}\frac{\Gamma(n-\ell+1)\Gamma(n+\ell+N)}{|\Gamma(n+\frac{N}{2})|^2}\\&\times\frac{(N-2)\ell(\ell+N-1)}{(N-1)\left([n+N/2]^2-[N-2]^2/4\right)},\end{aligned} \tag{F.56}$$

which is eq. (6.8) with $r=1$ and $\tilde{r}_{(B)}=\tilde{r}_{(I)}=0$. For later convenience, note that we can easily deduce the form of the normalisation factor for the type-I STSSH's of rank 1 on S^{N-1} by making the replacements $N\to N-1$, $n\to\ell$ and $\ell\to m$ in eq. (F.56), as

$$\begin{aligned}\left|\frac{\tilde{c}_{N-1}^{(I;\tilde{r}=1)}(\ell,m)}{\sqrt{2}}\right|^2=&\frac{1}{2^{N}}\frac{\Gamma(\ell-m+1)\Gamma(\ell+m+N-1)}{|\Gamma(\ell+\frac{N-1}{2})|^2}\\&\times\frac{(N-3)m(m+N-2)}{(N-2)(\ell+1)(\ell+N-2)}.\end{aligned} \tag{F.57}$$

Let us now discuss the mixing between type-I and type-II-I modes under the spin$(N+1)$ transformation. By using the equation $\psi_{\pm\theta_N}^{(II\text{-}I;\sigma;n\ell m;\rho)}=0$ and eqs. (F.29) and (F.38) (or eqs. (F.35) and (F.40)), one readily finds that the component given by (F.5) of the infinitesimal transformation of a type-II-I mode is proportional to a type-I mode, as

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi_{\pm\theta_N}^{(II\text{-}I;\sigma;n\ell m;\rho)}=&\frac{\sin\theta_{N-1}}{\sin^2\theta_N}\psi_{\pm\theta_{N-1}}^{(II\text{-}I;\sigma;n\ell m;\rho)}\\=&\mathscr{K}^{(II\to I)}\psi_{\pm\theta_N}^{(I;\sigma;n\ell m;\rho)},\end{aligned} \tag{F.58}$$

in agreement with eq. (6.15), where we have defined

$$\mathscr{K}^{(II\to I)}\equiv\frac{1}{2}\frac{\tilde{c}_{N-1}^{(I;\tilde{r}=1)}(\ell,m)}{\tilde{c}_{N-1}(\ell,m)}. \tag{F.59}$$

It is easy to show that this expression for $\mathscr{K}^{(II\to I)}$ is equal to the expression given by eq. (6.20) (with $r=1$). Then, since type-II-I modes transform into type-I modes under the spin$(N+1)$ transformation, the spin$(N+1)$ invariance of the inner product (6.5) (between $\psi_{\pm\mu}^{(I;\sigma;n\ell m;\rho)}$ and $\psi_{\pm\mu}^{(II\text{-}I;\sigma;n\ell m;\rho)}$) implies that

$$\mathbb{L}_{\mathscr{S}}\psi_{\pm\mu}^{(I;\sigma;n\ell m;\rho)}=...+\mathscr{K}^{(I\to II)}\psi_{\pm\mu}^{(II\text{-}I;\sigma;n\ell m;\rho)}, \tag{F.60}$$

where all the STSSH's in '...' are type-I modes, while $\mathscr{K}^{(I\to II)}$ is given by

$$\mathscr{K}^{(I\to II)} = -\mathscr{K}^{(II\to I)*}\left|\frac{c_N^{(II;r=1)}(n,\ell)}{c_N^{(I;r=1)}(n,\ell)}\right|^2, \tag{F.61}$$

where the asterisk denotes complex conjugation. Then, by using the expression for $\mathscr{K}^{(II\to I)}$ [eq. (6.20)] and the expressions for the normalisation factors [eq. (6.8)] we find that $\mathscr{K}^{(I\to II)}$ in eq. (F.61) is equal to the expression given by eq. (6.14) (with $r=1$).

F.2.2 Derivation of the transformation formula (6.15) for type-II-I modes of rank 1

By substituting the type-II-I mode (F.38) into the Lie-Lorentz derivative (F.6) we find

$$\begin{aligned}&\mathbb{L}_{\mathscr{S}}\psi^{(II\text{-}I;+;n\ell m;\rho)}_{\pm\theta_{N-1}}\\&=\frac{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell,m)}{\sqrt{2}}\begin{pmatrix} i\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}^{(II)}_3(\theta_N,\theta_{N-1}) - \hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}^{(II)}_4(\theta_N,\theta_{N-1}) \\ \pm\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}^{(II)}_1(\theta_N,\theta_{N-1}) \pm i\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}^{(II)}_2(\theta_N,\theta_{N-1})\end{pmatrix},\end{aligned} \tag{F.62}$$

where

$$\mathbb{T}^{(II)}_1 = (\mathscr{S}^\mu\partial_\mu - \cot\theta_N\cos\theta_{N-1})\left[\phi^{(-1)}_{n\ell}\tilde{\phi}^{(1)}_{\ell m}\right] - \frac{\sin\theta_{N-1}}{2\sin\theta_N}\phi^{(-1)}_{n\ell}\tilde{\psi}^{(1)}_{\ell m}, \tag{F.63}$$

$$\mathbb{T}^{(II)}_2 = (\mathscr{S}^\mu\partial_\mu - \cot\theta_N\cos\theta_{N-1})\left[\phi^{(-1)}_{n\ell}\tilde{\psi}^{(1)}_{\ell m}\right] + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\phi^{(-1)}_{n\ell}\tilde{\phi}^{(1)}_{\ell m}, \tag{F.64}$$

$$\mathbb{T}^{(II)}_3 = (\mathscr{S}^\mu\partial_\mu - \cot\theta_N\cos\theta_{N-1})\left[\psi^{(-1)}_{n\ell}\tilde{\phi}^{(1)}_{\ell m}\right] + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\psi^{(-1)}_{n\ell}\tilde{\psi}^{(1)}_{\ell m}, \tag{F.65}$$

$$\mathbb{T}^{(II)}_4 = (\mathscr{S}^\mu\partial_\mu - \cot\theta_N\cos\theta_{N-1})\left[\psi^{(-1)}_{n\ell}\tilde{\psi}^{(1)}_{\ell m}\right] - \frac{\sin\theta_{N-1}}{2\sin\theta_N}\psi^{(-1)}_{n\ell}\tilde{\phi}^{(1)}_{\ell m}. \tag{F.66}$$

Then, as in the case of the type-I modes, we prove the following relations:

$$\mathbb{T}^{(II)}_1 = \mathscr{R}^{(II)}k^{(+)}\tilde{k}^{(+)}\phi^{(-1)}_{n\,\ell+1}\tilde{\phi}^{(1)}_{\ell+1\,m} + \mathscr{L}^{(II)}k^{(-)}\tilde{k}^{(-)}\phi^{(-1)}_{n\,\ell-1}\tilde{\phi}^{(1)}_{\ell-1\,m} + \varkappa^{(II)}\psi^{(-1)}_{n\ell}\tilde{\phi}^{(1)}_{\ell m} + \frac{H^{(1)}_{n\ell m}}{2}, \tag{F.67}$$

$$\mathbb{T}^{(II)}_2 = \mathscr{R}^{(II)}k^{(+)}\tilde{k}^{(+)}\phi^{(-1)}_{n\,\ell+1}\tilde{\psi}^{(1)}_{\ell+1\,m} + \mathscr{L}^{(II)}k^{(-)}\tilde{k}^{(-)}\phi^{(-1)}_{n\,\ell-1}\tilde{\psi}^{(1)}_{\ell-1\,m} - \varkappa^{(II)}\psi^{(-1)}_{n\ell}\tilde{\psi}^{(1)}_{\ell m} + \frac{O^{(1)}_{n\ell m}}{2}, \tag{F.68}$$

$$\mathbb{T}^{(II)}_3 = \mathscr{R}^{(II)}k^{(+)}\tilde{k}^{(+)}\psi^{(-1)}_{n\,\ell+1}\tilde{\phi}^{(1)}_{\ell+1\,m} + \mathscr{L}^{(II)}k^{(-)}\tilde{k}^{(-)}\psi^{(-1)}_{n\,\ell-1}\tilde{\phi}^{(1)}_{\ell-1\,m} - \varkappa^{(II)}\phi^{(-1)}_{n\ell}\tilde{\phi}^{(1)}_{\ell m} + \frac{E^{(1)}_{n\ell m}}{2}, \tag{F.69}$$

$$\mathbb{T}^{(II)}_4 = \mathscr{R}^{(II)}k^{(+)}\tilde{k}^{(+)}\psi^{(-1)}_{n\,\ell+1}\tilde{\psi}^{(1)}_{\ell+1\,m} + \mathscr{L}^{(II)}k^{(-)}\tilde{k}^{(-)}\psi^{(-1)}_{n\,\ell-1}\tilde{\psi}^{(1)}_{\ell-1\,m} + \varkappa^{(II)}\phi^{(-1)}_{n\ell}\tilde{\psi}^{(1)}_{\ell m} + \frac{\Sigma^{(1)}_{n\ell m}}{2}, \tag{F.70}$$

where $\varkappa^{(II)}$ is given by eq. (6.18) (with $r = 1$) and

$$\mathscr{R}^{(II)} = \frac{\ell + N - 2}{2(\ell + \frac{N-1}{2})(\ell + N - 1)}, \qquad \mathscr{L}^{(II)} = \frac{-(1+\ell)}{2\ell(\ell + \frac{N-1}{2})}. \tag{F.71}$$

By substituting eqs. (F.67)-(F.70) into eq. (F.62) we find

$$\begin{aligned} \mathbb{L}_{\mathscr{S}} \psi^{(II\text{-}I;+;n\ell m;\rho)}_{\pm\theta_{N-1}} = \; & \mathscr{A}^{(II)} \psi^{(II\text{-}I;+;n\,(\ell+1)\,m;\rho)}_{\pm\theta_{N-1}} + \mathscr{B}^{(II)} \psi^{(II\text{-}I;+;n\,(\ell-1)\,m;\rho)}_{\pm\theta_{N-1}} \\ & - i\varkappa^{(II)} \psi^{(II\text{-}I;-;n\ell m;\rho)}_{\pm\theta_{N-1}} + \mathscr{K}^{(II\to I)} \psi^{(I;+;n\ell m;\rho)}_{\pm\theta_{N-1}}, \end{aligned} \tag{F.72}$$

in precise agreement with the transformation formula (6.15), where we have defined

$$\mathscr{A}^{(II)} \equiv \mathscr{R}^{(II)} k^{(+)} \tilde{k}^{(+)} \frac{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell, m)}{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell+1, m)}, \tag{F.73}$$

$$\mathscr{B}^{(II)} \equiv \mathscr{L}^{(II)} k^{(-)} \tilde{k}^{(-)} \frac{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell, m)}{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell-1, m)}. \tag{F.74}$$

It easy to verify that these expressions for $\mathscr{A}^{(II)}$ and $\mathscr{B}^{(II)}$ agree with the expressions given by eqs. (6.16) (with $r = 1$) and (6.17) (with $r = 1$), respectively.

F.3 Derivation of the transformation formulae of type-I and type-II-I STSSH's of rank 1 and calculation of the normalisation factor $c^{(I;r=1)}_N(n, \ell)$ for N odd

The Lie-Lorentz derivative is given by eqs. (F.5) and (F.6), where $\gamma^N \gamma^{N-1}$ is given by

$$\gamma^N \gamma^{N-1} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \tag{F.75}$$

where $\mathbf{1}$ is the identity spinorial matrix of dimension $2^{\frac{N-1}{2}}/2$.

The type-I modes on S^N with positive spin projection index on S^{N-1} ($\sigma_{N-1} = +$) are found by substituting eq. (C.6) into eqs. (4.20) and (4.21), as

$$\begin{aligned} & \psi^{(I;n\ell;+;m;\rho)}_{\pm\theta_N}(\theta_N, \theta_{N-1}, \boldsymbol{\theta}_{N-2}) \\ & = \frac{\tilde{c}_{N-1}(\ell, m)}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} (1+i)\, i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1}) \left[\phi^{(1)}_{n\ell}(\theta_N) \pm i\psi^{(1)}_{n\ell}(\theta_N)\right] \tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2}) \\ (1-i)\, \tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1}) \left[-\phi^{(1)}_{n\ell}(\theta_N) \pm i\psi^{(1)}_{n\ell}(\theta_N)\right] \tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2}) \end{pmatrix}, \end{aligned} \tag{F.76}$$

$$\begin{aligned} & \psi^{(I;n\ell;+;m;\rho)}_{\pm\theta_{N-1}}(\theta_N, \theta_{N-1}, \boldsymbol{\theta}_{N-2}) \\ & = \frac{\tilde{c}_{N-1}(\ell, m)}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} (1+i) \left[iO^{(1)}_{n\ell m}(\theta_N, \theta_{N-1}) \mp \Sigma^{(1)}_{n\ell m}(\theta_N, \theta_{N-1})\right] \tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2}) \\ (1-i) \left[-H^{(1)}_{n\ell m}(\theta_N, \theta_{N-1}) \pm iE^{(1)}_{n\ell m}(\theta_N, \theta_{N-1})\right] \tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2}) \end{pmatrix} \end{aligned} \tag{F.77}$$

(the functions describing the dependence on θ_N and θ_{N-1} in eq. (F.77) are given by eqs. (F.31)-(F.34)). The component $\psi^{(I;n\ell;-;m;\rho)}_{\pm\theta_N}$ is obtained from eq. (F.76) by making the replacement $\tilde{\chi}_{+m\rho} \rightarrow \tilde{\chi}_{-m\rho}$ and exchanging $i\tilde{\psi}^{(0)}_{\ell m}$ and $\tilde{\phi}^{(0)}_{\ell m}$. The component $\psi^{(I;n\ell;-;m;\rho)}_{\pm\theta_{N-1}}$ is obtained from eq. (F.77) by making the replacement $\tilde{\chi}_{+m\rho} \rightarrow \tilde{\chi}_{-m\rho}$ and exchanging $iO^{(1)}_{n\ell m}$ and $H^{(1)}_{n\ell m}$, as well as exchanging $\mp\Sigma^{(1)}_{n\ell m}$ and $\pm iE^{(1)}_{n\ell m}$. The ladder relations for the functions $\phi^{(a)}_{n\ell}(\theta_N), \psi^{(a)}_{n\ell}(\theta_N), \tilde{\phi}^{(\tilde{a})}_{\ell m}(\theta_{N-1}), \tilde{\psi}^{(\tilde{a})}_{\ell m}(\theta_{N-1})$ are given again by eqs. (F.21)-(F.24). Equations (F.46)-(F.49) hold as in the even-dimensional case.

The type-*II-I* modes on S^N with positive spin projection index on S^{N-1} ($\sigma_{N-1} = +$) are expressed as

$$\psi^{(II\text{-}I;n\ell;+;m;\rho)}_{\pm\theta_{N-1}}(\theta_N, \theta_{N-1}, \boldsymbol{\theta}_{N-2}) = \frac{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell,m)}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} (1+i)\, i\tilde{\psi}^{(1)}_{\ell m}(\theta_{N-1}) \left[\phi^{(-1)}_{n\ell}(\theta_N) \pm i\psi^{(-1)}_{n\ell}(\theta_N)\right] \tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2}) \\ (1-i)\, \tilde{\phi}^{(1)}_{\ell m}(\theta_{N-1}) \left[-\phi^{(-1)}_{n\ell}(\theta_N) \pm i\psi^{(-1)}_{n\ell}(\theta_N)\right] \tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2}) \end{pmatrix}, \tag{F.78}$$

while $\psi^{(II\text{-}I;n\ell;-;m;\rho)}_{\pm\theta_{N-1}}$ is obtained from eq. (F.78) by making the replacement $\tilde{\chi}_{+m\rho} \rightarrow \tilde{\chi}_{-m\rho}$ and exchanging $i\tilde{\psi}^{(1)}_{\ell m}$ and $\tilde{\phi}^{(1)}_{\ell m}$. Equations (F.67)-(F.70) hold as in the even-dimensional case.

The rest of the derivation of the transformation formulae is similar to that for the even-dimensional case. We find that the transformation formulae for the type-*I* and type-*II-I* modes are given by eqs. (6.29) and (6.30), respectively, while the normalisation factor $c^{(I;r=1)}_N(n,\ell)$ is given by eq. (F.56).

F.4 Transformation properties under spin$(N+1)$ and normalisation factors for STSSH's of rank 2 on S^N

As mentioned in the beginning of this Appendix, the calculations needed in order to derive the transformation formulae and determine the normalisation factors for STSSH's of rank 2 on S^N have many similarities with the case of rank-1 STSSH's, which was presented above. Therefore, below we just provide a brief description of the basic steps.

Let us begin by determining the normalisation factor for type-*III* STSSH's of rank 2, $c^{(III;r=2)}_N(n,\ell)$. In the case with N even, we substitute the rank-2 type-*III* modes (5.28)-(5.30) into the inner product (6.7), while in the case with N odd we substitute the type-*III* modes (5.41)-(5.43) into the inner product (6.27). By working as in the case of rank-1 type-*II* modes, we readily find (with the use of eq. (5.9))

$$\left|\frac{c^{(III;r=2)}_N(n,\ell)}{\sqrt{2}}\right|^2 = \frac{1}{2^{N-5}} \frac{\Gamma(n-\ell+1)\Gamma(n+\ell+N)}{|\Gamma(n+\frac{N}{2})|^2}, \tag{F.79}$$

for both N even and N odd, which is eq. (6.8) with $r = \tilde{r}_{(III)} = 2$.

Now we will determine the normalisation factor for type-*II* STSSH's of rank 2, $c^{(II;r=2)}_N(n,\ell)$. For N even we substitute eqs. (5.21)-(5.23) into the inner product (6.7), while for N odd we substitute eqs. (5.38)-(5.40) into the inner product (6.27). By performing the integration over S^{N-1} (using eqs. (D.13) and (4.16)), we straightforwardly find

$$\begin{aligned}
\left|\frac{c_N^{(II;r=2)}(n,\ell)}{\sqrt{2}}\right|^{-2} &= 2\int_0^\pi d\theta_N \sin^{N-3}\theta_N \left[\left(\phi_{n\ell}^{(0)}(\theta_N)\right)^2 + \left(\psi_{n\ell}^{(0)}(\theta_N)\right)^2\right] \\
&+ 2\left[\left(\ell+\frac{N-1}{2}\right)^2 - \frac{N(N+1)}{4}\right] \\
&\times \int_0^\pi d\theta_N \sin^{N-5}\theta_N \left[\left(\frac{\Gamma_{n\ell}^{(\uparrow)}(\theta_N)}{2}\right)^2 + \left(\frac{\Gamma_{n\ell}^{(\downarrow)}(\theta_N)}{2}\right)^2\right] \\
&+ 2(N+1)\int_0^\pi d\theta_N \sin^{N-5}\theta_N \left[\left|\frac{\Delta_{n\ell}^{(\uparrow)}(\theta_N)}{2}\right|^2 + \left|\frac{\Delta_{n\ell}^{(\downarrow)}(\theta_N)}{2}\right|^2\right] \\
&+ i\left(\ell+\frac{N-1}{2}\right) \\
&\times \int_0^\pi d\theta_N \sin^{N-5}\theta_N \left[\Gamma_{n\ell}^{(\uparrow)}(\theta_N)\Delta_{n\ell}^{(\uparrow)}(\theta_N) + \Gamma_{n\ell}^{(\downarrow)}(\theta_N)\Delta_{n\ell}^{(\downarrow)}(\theta_N)\right], \quad \text{(F.80)}
\end{aligned}$$

where $\Gamma_{n\ell}^{(\uparrow)}, \Gamma_{n\ell}^{(\downarrow)}, \Delta_{n\ell}^{(\uparrow)}$ and $\Delta_{n\ell}^{(\downarrow)}$ are given by eqs. (5.26), (5.27), (5.24) and (5.25), respectively. The calculations can be significantly simplified by making use of the following relations:

$$\begin{aligned}
&\frac{4}{\sin^2\theta}\phi_{n\ell}^{(0)}(\theta) = \phi_{n'\ell'}^{(1)}(\theta)\big|_{N\to N+2}, \\
&\frac{4}{\sin^2\theta}\psi_{n\ell}^{(0)}(\theta) = \psi_{n'\ell'}^{(1)}(\theta)\big|_{N\to N+2}, \\
&\frac{2}{\sin^2\theta}\Gamma_{n\ell}^{(b)}(\theta) = C_{n'\ell'}^{(b)(1)}(\theta)\big|_{N\to N+2}, \\
&\frac{2}{\sin^2\theta}\Delta_{n\ell}^{(b)}(\theta) = D_{n'\ell'}^{(b)(1)}(\theta)\big|_{N\to N+2}, \qquad (b=\uparrow,\downarrow) \qquad \text{(F.81)}
\end{aligned}$$

where $\theta \in [0,\pi]$, $n' = n-1$ and $\ell' = \ell - 1$, while on the right-hand sides of the relations in eq. (F.81) we have denoted the replacement of N by $N+2$ as $N \to N+2$. The relations in eq. (F.81) can be readily proved by using eqs. (3.1), (3.2), (4.9), (4.10), (4.11), (4.12), (5.24), (5.25) (5.26) and (5.27). By comparing eqs. (F.80) and (F.3) and by using eq. (F.81), we straightforwardly find

$$\left|\frac{c_N^{(II;r=2)}(n,\ell)}{\sqrt{2}}\right|^2 = 2^3\left|\frac{c_{N+2}^{(I;r=1)}(n-1,\ell-1)}{\sqrt{2}}\right|^2 \qquad \text{(F.82)}$$

$$\begin{aligned}
&= \frac{1}{2^N}\frac{\Gamma(n-\ell+1)\Gamma(n+\ell+N)}{|\Gamma(n+\frac{N}{2})|^2} \\
&\times \frac{N(\ell-1)(\ell+N)}{(N+1)\left([n+N/2]^2 - N^2/4\right)}, \qquad \text{(F.83)}
\end{aligned}$$

which is eq. (6.8) with $r=2$ and $\tilde{r}_{(B)} = \tilde{r}_{(II)} = 1$.

As for the normalisation of rank-2 type-I modes, by working as in the case of rank-1 type-I modes, we calculate the normalisation factor for $\ell = n$ using Mathematica 11.2

$$\left|\frac{c_N^{(I;r=2)}(n,n)}{\sqrt{2}}\right|^2 = \frac{(n-1)(N-2)\Gamma(n+\frac{N}{2}+\frac{1}{2})}{4^{2-n}(n+1)(N+1)\sqrt{\pi}\Gamma(n+\frac{N}{2})}, \tag{F.84}$$

while the normalisation factor $c_N^{(I;r=2)}(n,\ell)$ (for $\ell = 2,3,...,n-1$) will be determined using the spin$(N+1)$ invariance of the inner product (6.5).

In order to derive the transformation formulae (6.10), (6.15), (6.22), (6.29), (6.30) and (6.31) for the STSSH's of rank 2 it is sufficient to study the following components of the Lie-Lorentz derivative (6.1):

$$\mathbb{L}_{\mathscr{S}}\psi_{\theta_N\theta_N} = \Big(\mathscr{S}^\mu\partial_\mu + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\gamma^N\gamma^{N-1}\Big)\psi_{\theta_N\theta_N} + \frac{2\,\sin\theta_{N-1}}{\sin^2\theta_N}\psi_{\theta_N\theta_{N-1}}, \tag{F.85}$$

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi_{\theta_N\theta_{N-1}} = {}& \Big(\mathscr{S}^\mu\partial_\mu - \cos\theta_{N-1}\,\cot\theta_N + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\gamma^N\gamma^{N-1}\Big)\psi_{\theta_N\theta_{N-1}} \\ &+ \frac{\sin\theta_{N-1}}{\sin^2\theta_N}\psi_{\theta_{N-1}\theta_{N-1}} - \sin\theta_{N-1}\psi_{\theta_N\theta_N}\end{aligned} \tag{F.86}$$

and

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi_{\theta_{N-1}\theta_{N-1}} = {}& \Big(\mathscr{S}^\mu\partial_\mu - 2\cos\theta_{N-1}\,\cot\theta_N + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\gamma^N\gamma^{N-1}\Big)\psi_{\theta_{N-1}\theta_{N-1}} \\ &- 2\sin\theta_{N-1}\,\psi_{\theta_N\theta_{N-1}}.\end{aligned} \tag{F.87}$$

By working as in the case of rank-1 STSSH's, we make use of the ladder operators (F.21)-(F.24) and (after a long calculation) we find the transformation formulae (6.10), (6.15) and (6.22) for N even, and the transformation formulae (6.29)-(6.31) for N odd. Then, as in the case of rank-1 type-I modes, the normalisation factor of rank-2 type-I modes is found by combining the spin$(N+1)$ invariance of the inner product between $\psi^{(I;\sigma;n\ell m;\rho)}_{\pm\mu_1\mu_2}$ and $\psi^{(I;\sigma;n(\ell+1)m;\rho)}_{\pm\mu_1\mu_2}$ with eq. (F.84), as

$$\begin{aligned}\left|\frac{c_N^{(I;r=2)}(n,\ell)}{\sqrt{2}}\right|^2 = {}& \frac{1}{2^{N+3}}\frac{\Gamma(n-\ell+1)\Gamma(n+\ell+N)}{|\Gamma(n+\frac{N}{2})|^2} \\ &\times \frac{N-2}{N+1}\frac{\ell(\ell+N-1)(\ell-1)(\ell+N)}{\Big([n+N/2]^2-[N-2]^2/4\Big)\Big([n+N/2]^2-N^2/4\Big)},\end{aligned} \tag{F.88}$$

(for both N even and N odd) which is eq. (6.8) with $r=2$ and $\tilde{r}_{(B)} = \tilde{r}_{(I)} = 0$.

G Pure gauge modes

In this Appendix, we present details for the derivation of the pure gauge expressions (7.13), (7.15) and (7.18) for N even. The calculations for N odd are similar and, thus, we do not present them here.

For later convenience, note that by making the replacements $\theta_N \to x(t) = \pi/2 - it$, $n \to \tilde{M} - N/2$ [eq. (7.5)] in the formulae (3.5) and (3.6) we find

$$\left(\frac{d}{dx} + \frac{N+2a-1}{2}\cot x + \frac{\ell + (N-1)/2}{\sin x}\right)\hat{\psi}^{(a)}_{\tilde{M}\ell}(t) = \tilde{M}\,\hat{\phi}^{(a)}_{\tilde{M}\ell}(t) \tag{G.1}$$

and

$$\left(\frac{d}{dx} + \frac{N+2a-1}{2}\cot x - \frac{\ell + (N-1)/2}{\sin x}\right)\hat{\phi}^{(a)}_{\tilde{M}\ell}(t) = -\tilde{M}\hat{\psi}^{(a)}_{\tilde{M}\ell}(t), \tag{G.2}$$

respectively, where $\cot x = i\tanh t$ and $\sin x = \cosh t$. Also, let us obtain lowering operators for $\tilde{M}$ as follows. By making the replacements $N \to N+1$, $\theta_{N-1} \to x(t) = \pi/2 - it$, $\ell \to \tilde{M} - N/2$, $\tilde{a} \to a$ and $m \to \ell$ in the lowering operator (F.19) we find

$$\begin{aligned}\hat{L}^{(\tilde{M};a)}_{\phi}\hat{\phi}^{(a)}_{\tilde{M}\ell}(t) &\equiv \left[\sin x\frac{\partial}{\partial x} + \left(-\tilde{M} + \frac{N-1}{2} + a\right)\cos x + \frac{\ell + \frac{N-1}{2}}{2(\tilde{M}-1/2)}\right]\hat{\phi}^{(a)}_{\tilde{M}\ell}(t)\\ &= -\frac{\tilde{M}\left(\tilde{M} - \ell - N/2\right)}{\tilde{M}-1/2}\hat{\phi}^{(a)}_{(\tilde{M}-1)\ell}(t),\end{aligned} \tag{G.3}$$

while by making the same replacements in the lowering operator (F.20) we find

$$\begin{aligned}\hat{L}^{(\tilde{M};a)}_{\psi}\hat{\psi}^{(a)}_{\tilde{M}\ell}(t) &\equiv \left[\sin x\frac{\partial}{\partial x} + \left(-\tilde{M} + \frac{N-1}{2} + a\right)\cos x - \frac{\ell + \frac{N-1}{2}}{2(\tilde{M}-1/2)}\right]\hat{\psi}^{(a)}_{\tilde{M}\ell}(t)\\ &= -\frac{\tilde{M}\left(\tilde{M} - \ell - N/2\right)}{\tilde{M}-1/2}\hat{\psi}^{(a)}_{(\tilde{M}-1)\ell}(t).\end{aligned} \tag{G.4}$$

G.1 Pure gauge modes for the strictly massless spin-3/2 field, N even

The type-I modes (7.13) for the strictly massless spin-3/2 field (with $\tilde{M} = \pm(N-2)/2$) are 'pure gauge' modes. In this Subsection, we prove explicitly the t-component of eq. (7.13) and we describe the calculations needed in order to prove the rest of the components. Let us denote the spinors $\Lambda^{(\tilde{\ell})}_{\pm}$ in eq. (7.13) as $\Lambda^{(\sigma;\ell;\tilde{\rho})}_{\pm}$, where we have written out explicitly the dependence on the spin projection index $\sigma = \pm$ and the angular momentum quantum number $\ell = 1, 2, \ldots$. Since these spinors satisfy the Dirac equation $(\nabla\!\!\!/ \pm iN/2)\Lambda^{(\sigma;\ell;\tilde{\rho})}_{\pm} = 0$, they are given by [22]

$$\Lambda^{(-;\ell;\tilde{\rho})}_{\pm}(t,\boldsymbol{\theta}_{N-1}) = \frac{2}{\ell}\begin{pmatrix}\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t)\,\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \mp i\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t)\,\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix}, \tag{G.5}$$

$$\Lambda^{(+;\ell;\tilde{\rho})}_{\pm}(t,\boldsymbol{\theta}_{N-1}) = \frac{2}{\ell}\begin{pmatrix}i\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t)\,\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \mp\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t)\,\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix}, \tag{G.6}$$

where $\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t)$ and $\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t)$ are found by letting $\tilde{M} = N/2$ in eqs. (7.7) and (7.9), respectively, while $\chi_{\pm\ell\tilde{\rho}}$ are the eigenspinors (2.22) of the Dirac operator on S^{N-1}. The factor of $2/\ell$

will be motivated naturally below [it arises from the use of the lowering operators (G.3) and (G.4)]. Below we prove the t-component of eq. (7.13) only for negative spin projection $\sigma = -$. The case with $\sigma = +$ can be proved in the same way.

The type-I modes $\Psi_\mu^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})}$ for the strictly massless spin-3/2 field ($\tilde{M} = \pm(N-2)/2$) are found by combining eq. (7.11) with eqs. (4.5) and (4.6), as

$$\Psi_t^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})}(t,\boldsymbol{\theta}_{N-1}) = -i \begin{pmatrix} \hat{\phi}^{(1)}_{(\frac{N-2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \mp i\hat{\psi}^{(1)}_{(\frac{N-2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \tag{G.7}$$

$$\begin{aligned}\Psi_{\theta_j}^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})}&(t,\boldsymbol{\theta}_{N-1}) \\ &= \begin{pmatrix} \hat{C}^{(\uparrow)(1)}_{(\frac{N-2}{2})\ell}(t)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) + \hat{D}^{(\uparrow)(1)}_{(\frac{N-2}{2})\ell}(t)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \mp i\hat{C}^{(\downarrow)(1)}_{(\frac{N-2}{2})\ell}(t)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \mp i\hat{D}^{(\downarrow)(1)}_{(\frac{N-2}{2})\ell}(t)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix},\end{aligned} \tag{G.8}$$

where the functions $\hat{C}^{(b)(1)}_{\tilde{M}\ell}(t)$ and $\hat{D}^{(b)(1)}_{\tilde{M}\ell}(t)$ ($b =\uparrow,\downarrow$) are obtained by making the replacements $\theta_N \to \pi/2 - it$, $n \to \tilde{M} - N/2$, $\phi^{(1)}_{n\ell}(\theta_N) \to \hat{\phi}^{(1)}_{\tilde{M}\ell}(t)$, $\psi^{(1)}_{n\ell}(\theta_N) \to \hat{\psi}^{(1)}_{\tilde{M}\ell}(t)$ in the functions $C^{(b)(1)}_{n\ell}(\theta_N)$ and $D^{(b)(1)}_{n\ell}(\theta_N)$ ($b =\uparrow,\downarrow$), respectively, in eq. (4.6).

Now, let us prove eq. (7.13) for the t-component of $\Psi_\mu^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})}$. We will show that the two sides of eq. (7.13) are equal by making use of the lowering operators (G.3) and (G.4). We want to show

$$\Psi_t^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})} = \left(\frac{\partial}{\partial t} \pm \frac{i}{2}\gamma_t\right)\Lambda_{\pm}^{(-;\ell;\tilde{\rho})} \tag{G.9}$$

which is expressed in terms of upper and lower components as

$$-i \begin{pmatrix} \hat{\phi}^{(1)}_{(\frac{N-2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \mp i\hat{\psi}^{(1)}_{(\frac{N-2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} = \frac{2}{\ell}\begin{pmatrix} \left[\frac{\partial}{\partial t}\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t) - \frac{i}{2}\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t)\right]\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \mp\left[i\frac{\partial}{\partial t}\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t) - \frac{1}{2}\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t)\right]\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}, \tag{G.10}$$

[where we have used eq. (2.11) and $\gamma^t = i\gamma^N$] or equivalently

$$\frac{\ell}{2}\,\hat{\phi}^{(1)}_{(\frac{N-2}{2})\ell}(t) = \frac{\ell}{\sin x}\,\hat{\phi}^{(0)}_{(\frac{N-2}{2})\ell}(t) = \frac{\partial}{\partial x}\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t) + \frac{1}{2}\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t) \tag{G.11}$$

$$\frac{\ell}{2}\,\hat{\psi}^{(1)}_{(\frac{N-2}{2})\ell}(t) = \frac{\ell}{\sin x}\,\hat{\psi}^{(0)}_{(\frac{N-2}{2})\ell}(t) = \frac{\partial}{\partial x}\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t) - \frac{1}{2}\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t), \tag{G.12}$$

where we have used eqs. (7.7) and (7.9). Then, by using the formulae (G.1) and (G.2) we rewrite eqs. (G.11) and (G.12) as

$$\left(\sin x\frac{d}{dx} - \frac{1}{2}\cot x + \frac{\ell + (N-1)/2}{N-1}\right)\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t) = \frac{N\,\ell}{N-1}\hat{\phi}^{(0)}_{(\frac{N-2}{2})\ell}(t) \tag{G.13}$$

and

$$\left(\sin x \frac{d}{dx} - \frac{1}{2}\cot x - \frac{\ell + (N-1)/2}{N-1}\right) \hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t) = \frac{N\,\ell}{N-1} \hat{\psi}^{(0)}_{(\frac{N-2}{2})\ell}(t), \tag{G.14}$$

respectively. It is easy to verify that eq. (G.13) is equal to the lowering operator (G.3) acting on $\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t)$, while eq. (G.14) is equal to the lowering operator (G.4) acting on $\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t)$. Hence, the two sides of the time component of eq. (7.13) are equal. The rest of the components of eq. (7.13), i.e. $\Psi^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})}_{\theta_j} = \left(\nabla_{\theta_j} \pm \frac{i}{2}\gamma_{\theta_j}\right)\Lambda^{(-;\ell;\tilde{\rho})}_{\pm}$ $(j = 1, ..., N-1)$, can be proved straightforwardly just by using eqs. (G.13) and (G.14), as well as formulae (G.1) and (G.2).

G.2 Pure gauge modes for the strictly massless spin-5/2 field, N even

The type-*I* and type-*II* modes for the strictly massless spin-5/2 field (with $\tilde{M} = \pm N/2$ - see eq. (7.15)) are 'pure gauge' modes. In this Subsection, we briefly describe how to obtain the 'pure gauge' expression in eq. (7.15). We denote the vector-spinors $\lambda^{(B;\tilde{\ell})}_{\pm\nu}(t, \boldsymbol{\theta}_{N-1})$ in eq. (7.15) as $\lambda^{(B;\sigma;\ell;\tilde{\rho})}_{\pm\nu}(t, \boldsymbol{\theta}_{N-1})$ ($\sigma = \pm$, $B = I, II$ and $\ell = 2, 3, ...$). Since the calculations for $\sigma = -$ and $\sigma = +$ are similar, below we discuss only the case with $\sigma = -$.

Pure gauge modes of type-*I*. The type-*I* modes $\Psi^{(I;-;(\pm\frac{N}{2})\ell;\tilde{\rho})}_{\mu\nu}$ for the strictly massless spin-5/2 field ($\tilde{M} = \pm N/2$) are found by combining eq. (7.11) with eqs. (5.10). The 'time-time component' is

$$\Psi^{(I;-;(\pm\frac{N}{2})\ell;\tilde{\rho})}_{tt}(t, \boldsymbol{\theta}_{N-1}) = (-1) \times \begin{pmatrix} \hat{\phi}^{(2)}_{\frac{N}{2},\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \mp i\hat{\psi}^{(2)}_{\frac{N}{2},\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}. \tag{G.15}$$

Similarly, since the TT vector-spinors $\lambda^{(B;-;\ell;\tilde{\rho})}_{\pm\mu}(t, \boldsymbol{\theta}_{N-1})$ in eq. (7.15) satisfy

$$\left(\nabla\!\!\!/ \pm i\frac{N+2}{2}\right)\lambda^{(B;-;\ell;\tilde{\rho})}_{\pm\mu} = 0,$$

they are given by the analytic continuation of the type-*I* STSSH's of rank 1 in eqs. (4.5) and (4.6). The 'time component' is given by

$$\lambda^{(I;-;\ell;\tilde{\rho})}_{\pm t}(t, \boldsymbol{\theta}_{N-1}) = -\frac{2i}{\ell-1}\begin{pmatrix} \hat{\phi}^{(1)}_{(\frac{N+2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \mp i\hat{\psi}^{(1)}_{(\frac{N+2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}. \tag{G.16}$$

(The factor of $2/(\ell-1)$ is inserted for the same reason as the factor of $2/\ell$ in eqs. (G.5) and (G.6).) Then, by using eqs. (G.15) and (G.16), we expand the two sides of $\Psi^{(I;-;(\pm\frac{N}{2})\ell;\tilde{\rho})}_{tt} = \left(\nabla_t \pm \frac{i}{2}\gamma_t\right)\lambda^{(I;-;\ell;\tilde{\rho})}_{\pm t}$ [see eq. (7.15)] and find

$$\frac{\ell-1}{\sin x}\hat{\phi}^{(1)}_{\frac{N}{2},\ell}(t) = \frac{\partial}{\partial x}\hat{\phi}^{(1)}_{(\frac{N+2}{2})\ell}(t) + \frac{1}{2}\hat{\psi}^{(1)}_{(\frac{N+2}{2})\ell}(t) \tag{G.17}$$

$$\frac{\ell-1}{\sin x}\hat{\psi}^{(1)}_{\frac{N}{2},\ell}(t) = \frac{\partial}{\partial x}\hat{\psi}^{(1)}_{(\frac{N+2}{2})\ell}(t) - \frac{1}{2}\hat{\phi}^{(1)}_{(\frac{N+2}{2})\ell}(t). \tag{G.18}$$

These equations are proved in the same way as eqs. (G.11) and (G.12). Thus, we have verified the 'time-time component' of the 'pure gauge' expression (7.15). The rest of the components of eq. (7.15), i.e. $\Psi^{\left(I;-;(\pm\frac{N}{2})\ell;\tilde{\rho}\right)}_{t\theta_j} = \left(\nabla_{(t} \pm \frac{i}{2}\gamma_{(t}\right)\lambda^{(I;-;\ell;\tilde{\rho})}_{\pm\theta_j)}$ and $\Psi^{\left(I;-;(\pm\frac{N}{2})\ell;\tilde{\rho}\right)}_{\theta_k\theta_j} = \left(\nabla_{(\theta_k} \pm \frac{i}{2}\gamma_{(\theta_k}\right)\lambda^{(I;-;\ell;\tilde{\rho})}_{\pm\theta_j)}$, can be proved using eqs. (G.17) and (G.18).

Pure gauge modes of type-*II*. By working as in the case of type-I modes presented above, we find

$$\Psi^{\left(II\text{-}\tilde{A};-;(\pm\frac{N}{2})\ell;\tilde{\rho}\right)}_{t\theta_j}(t,\boldsymbol{\theta}_{N-1}) = \ (-i)\times\begin{pmatrix}\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1})\\ \mp i\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1})\end{pmatrix} \tag{G.19}$$

and

$$\lambda^{(II\text{-}\tilde{A};-;\ell;\tilde{\rho})}_{\pm\theta_j}(t,\boldsymbol{\theta}_{N-1}) = \ \frac{4}{\ell-1}\begin{pmatrix}\hat{\phi}^{(-1)}_{(\frac{N+2}{2})\ell}(t)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1})\\ \mp i\hat{\psi}^{(-1)}_{(\frac{N+2}{2})\ell}(t)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1})\end{pmatrix}. \tag{G.20}$$

(Recall that for type-II modes we have $\Psi^{\left(II\text{-}\tilde{A};\sigma;(\pm\frac{N}{2})\ell;\tilde{\rho}\right)}_{tt} = 0$ and $\lambda^{\left(II\text{-}\tilde{A};\sigma;\ell;\tilde{\rho}\right)}_{\pm t} = 0$.) Then, we can verify the 'pure gauge' expression (7.15) by working as in the case of type-I modes presented above.

G.3 Pure gauge modes for the partially massless spin-5/2 field, N even

The type-I modes [eq. (7.18)] for the partially massless spin-5/2 field (with $\tilde{M} = \pm(N-2)/2$) are 'pure gauge' modes. Below we describe briefly how to obtain the 'pure gauge' expression in eq. (7.18) for N even. (We present the proof only for the tt-component of eq. (7.18).) We denote the Dirac spinors $\varphi^{(\tilde{\ell})}_{\pm}(t,\boldsymbol{\theta}_{N-1})$ in eq. (7.18) as $\varphi^{(\sigma;\ell;\tilde{\rho})}_{\pm}(t,\boldsymbol{\theta}_{N-1})$ ($\sigma = \pm$ and $\ell = 2,3,...$). Again, the calculations for $\sigma = -$ and $\sigma = +$ are similar and, thus, we discuss only the case with $\sigma = -$.

For later convenience let us write down explicit expressions for lowering operators that lower the parameter $\tilde{M}$ to $\tilde{M}-2$ of the functions $\hat{f}^{(a)}_{\tilde{M}\ell}(t) \in \{\hat{\phi}^{(a)}_{\tilde{M}\ell}(t), \hat{\psi}^{(a)}_{\tilde{M}\ell}(t)\}$. By applying each of the lowering operators (G.3), (G.4) twice, we find

$$\begin{aligned}\hat{L}^{(\tilde{M}-1;a)}_{f}\hat{L}^{(\tilde{M};a)}_{f}\hat{f}^{(a)}_{\tilde{M}\ell}(t) =& \left[\sin^2 x\,\frac{\partial^2}{\partial x^2} + b_f(x)\,\frac{\partial}{\partial x} + c_f(x)\right]\hat{f}^{(a)}_{\tilde{M}\ell}(t)\\ =& \frac{\tilde{M}(\tilde{M}-1)(\tilde{M}-\ell-\frac{N}{2})(\tilde{M}-1-\ell-\frac{N}{2})}{(\tilde{M}-\frac{1}{2})(\tilde{M}-\frac{3}{2})}\hat{f}^{(a)}_{(\tilde{M}-2)\ell}(t),\end{aligned} \tag{G.21}$$

(recall $x = \pi/2 - it$) where

$$b_f(x) = \sin x\cos x\left(-2\tilde{M}+1+2a+N\right) + s_f\,\frac{(\ell+\frac{N-1}{2})(\tilde{M}-1)\sin x}{(\tilde{M}-1/2)(\tilde{M}-3/2)} \tag{G.22}$$

and

$$\begin{aligned}
c_f(x) =& \frac{(\ell+\frac{N-1}{2})^2}{4(\tilde{M}-1/2)(\tilde{M}-3/2)} \\
&+ s_f \frac{(\ell+\frac{N-1}{2})\cos x}{2}\left(\frac{-\tilde{M}+a+\frac{N-1}{2}}{\tilde{M}-3/2}+\frac{1-\tilde{M}+a+\frac{N-1}{2}}{\tilde{M}-1/2}\right) \\
&+ \left(-\tilde{M}+a+\frac{N-1}{2}\right)\left(1-\tilde{M}+a+\frac{N-1}{2}\right) \\
&- \sin^2 x\left(-\tilde{M}+a+\frac{N-1}{2}\right)\left(2-\tilde{M}+a+\frac{N-1}{2}\right),
\end{aligned} \tag{G.23}$$

with $s_f = 1$ if $\hat{f}^{(a)}_{\tilde{M}\ell}(t) = \hat{\phi}^{(a)}_{\tilde{M}\ell}(t)$ and $s_f = -1$ if $\hat{f}^{(a)}_{\tilde{M}\ell}(t) = \hat{\psi}^{(a)}_{\tilde{M}\ell}(t)$.

Now we will verify the 'time-time component' of eq. (7.18) with negative spin projection ($\sigma = \; -$), i.e.

$$\Psi_{tt}^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})}(t,\boldsymbol{\theta}_{N-1}) = \left(\nabla_t\nabla_t \pm i\gamma_t\nabla_t + \frac{3}{4}g_{tt}\right)\varphi_\pm^{(-;\ell;\tilde{\rho})}(t,\boldsymbol{\theta}_{N-1}). \tag{G.24}$$

Since the spinors $\varphi_\pm^{(\sigma;\ell;\tilde{\rho})}(t,\boldsymbol{\theta}_{N-1})$ satisfy the Dirac equation $\left[\nabla\!\!\!/ \pm i(N+2)/2\right]\varphi_\pm^{(\sigma;\ell;\tilde{\rho})} = 0$, they are given by [22]

$$\varphi_\pm^{(-;\ell;\tilde{\rho})}(t,\boldsymbol{\theta}_{N-1}) = \frac{4}{\ell(\ell-1)}\begin{pmatrix} \hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}(t)\,\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \mp i\hat{\psi}^{(0)}_{(\frac{N+2}{2})\ell}(t)\,\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}, \tag{G.25}$$

$$\varphi_\pm^{(+;\ell;\tilde{\rho})}(t,\boldsymbol{\theta}_{N-1}) = \frac{4}{\ell(\ell-1)}\begin{pmatrix} i\hat{\psi}^{(0)}_{(\frac{N+2}{2})\ell}(t)\,\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \mp\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}(t)\,\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}, \tag{G.26}$$

where the factor $4/\left(\ell\left[\ell-1\right]\right)$ is motivated naturally below. On the other hand, the tt-component of the type-I mode of the partially massless spin-5/2 field is given by

$$\Psi_{tt}^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})}(t,\boldsymbol{\theta}_{N-1}) = \;(-1)\times\begin{pmatrix} \hat{\phi}^{(2)}_{(\frac{N-2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \mp i\hat{\psi}^{(2)}_{(\frac{N-2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}. \tag{G.27}$$

By substituting eqs. (G.25) and (G.27) into eq. (G.24) we find

$$\begin{aligned}
\frac{\ell(\ell-1)}{4}\hat{\phi}^{(2)}_{(\frac{N-2}{2})\ell}(t) &= \frac{\ell(\ell-1)}{\sin^2 x}\hat{\phi}^{(0)}_{(\frac{N-2}{2})\ell}(t) \\
&= \left(\frac{\partial^2}{\partial x^2}+\frac{3}{4}\right)\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}(t) + \frac{\partial}{\partial x}\hat{\psi}^{(0)}_{(\frac{N+2}{2})\ell}(t)
\end{aligned} \tag{G.28}$$

$$\begin{aligned}
\frac{\ell(\ell-1)}{4}\hat{\psi}^{(2)}_{(\frac{N-2}{2})\ell}(t) &= \frac{\ell(\ell-1)}{\sin^2 x}\hat{\psi}^{(0)}_{(\frac{N-2}{2})\ell}(t) \\
&= \left(\frac{\partial^2}{\partial x^2}+\frac{3}{4}\right)\hat{\psi}^{(0)}_{(\frac{N+2}{2})\ell}(t) - \frac{\partial}{\partial x}\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}(t).
\end{aligned} \tag{G.29}$$

Equation (G.28) is proved using the lowering operator (G.21) as follows. First, we express $\partial\hat{\psi}^{(0)}_{(\frac{N+2}{2})\ell}/\partial x$ in eq. (G.28) in terms of $\partial\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}/\partial x$ and $\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}$ by making use of the formulae (G.1) and (G.2). Then, after a long calculation, we rewrite eq. (G.28) as

$$\frac{\ell(\ell-1)}{\sin^2 x}\hat{\phi}^{(0)}_{(\frac{N-2}{2})\ell} = \frac{(N-1)(N+1)}{N(N+2)\sin^2 x}\left(\hat{L}_f^{(\frac{N}{2};a=0)}\hat{L}_f^{(\frac{N+2}{2};a=0)}\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}\right), \tag{G.30}$$

which is readily verified using the lowering relation (G.21). Equation (G.29) is proved in the same way. Thus, we have verified the tt-component of the 'pure gauge' expression (7.18).

Let us now show that our 'pure gauge' expression for the type-I modes $\Psi_{\mu\nu}^{(I;\sigma;(\tilde{M}=+1)\ell;\tilde{\rho})}$ on dS_4 in eq. (7.18) is equal to the gamma-traceless part of the gauge transformation that is proposed in Ref. [8] (for a specific choice of the spinor gauge function in the gauge transformation of Ref. [8]). In order to compare our results with the results of Ref. [8] we let $N=4$ and $\tilde{M}=+(N-2)/2=+1$ in eq. (7.18). [Now, the spinors $\varphi_+^{(\sigma;\ell;\tilde{\rho})}$ in eq. (7.18) satisfy $\nabla\!\!\!\!/\,\varphi_+^{(\sigma;\ell;\tilde{\rho})}=-3i\varphi_+^{(\sigma;\ell;\tilde{\rho})}$.] By using units in which the cosmological constant is $\Lambda=3$, the gauge transformation for the partially massless spin-5/2 field $\psi_{\mu\nu}$ in Ref. [8] is

$$\delta\psi_{\mu\nu} = \left(\nabla_{(\mu}\nabla_{\nu)} - \frac{1}{4}\gamma_{(\mu}\nabla_{\nu)}\nabla\!\!\!\!/ + \frac{15}{16}g_{\mu\nu}\right)\epsilon \tag{G.31}$$

$$= \left(\nabla_{(\mu}\nabla_{\nu)} + \frac{3i}{4}\gamma_{(\mu}\nabla_{\nu)} + \frac{15}{16}g_{\mu\nu}\right)\epsilon, \tag{G.32}$$

where we have chosen ϵ to be a solution of the equation $\nabla\!\!\!\!/\,\epsilon=-3i\,\epsilon$. (For this choice it is clear that our spinors $\varphi_+^{(\sigma;\ell;\tilde{\rho})}$ are the mode functions corresponding to the field ϵ.) Note that for this choice of ϵ the gauge transformation of the auxiliary field is zero - see Ref. [8]. Also, for this choice of ϵ it can be readily verified that $g^{\mu\nu}\delta\psi_{\mu\nu}=0$, but $\gamma^\mu\delta\psi_{\mu\nu}\neq 0$. Let $\delta\psi'_{\mu\nu}$ be the gamma-traceless part of $\delta\psi_{\mu\nu}$, i.e.

$$\delta\psi'_{\mu\nu} = \delta\psi_{\mu\nu} - \frac{\gamma_\mu}{6}\gamma^\alpha\delta\psi_{\alpha\nu} - \frac{\gamma_\nu}{6}\gamma^\alpha\delta\psi_{\alpha\mu}, \tag{G.33}$$

where $\gamma^\alpha\delta\psi'_{\alpha\nu}=0$ and $g^{\mu\nu}\delta\psi'_{\mu\nu}=0$. Then, we can straightforwardly show that

$$\delta\psi'_{\mu\nu} = \left(\nabla_{(\mu}\nabla_{\nu)} + i\gamma_{(\mu}\nabla_{\nu)} + \frac{3}{4}g_{\mu\nu}\right)\epsilon, \tag{G.34}$$

which is in precise agreement with the expression for our type-I modes in eq. (7.18).

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