Article:

Fu, W and Nijhoff, F (2022) On A Coupled Kadomtsev–Petviashvili System Associated With an Elliptic Curve. Studies in Applied Mathematics, 149 (4). pp. 1086-1122. ISSN 0022-2526

https://doi.org/10.1111/sapm.12529

ON A COUPLED KADOMTSEV–PETVIASHVILI SYSTEM ASSOCIATED WITH AN ELLIPTIC CURVE

WEI FU AND FRANK W. NIJHOFF

Abstract. The coupled Kadomtsev–Petviashvili system associated with an elliptic curve, proposed by Date, Jimbo and Miwa [J. Phys. Soc. Jpn., 52:766–771, 1983], is reinvestigated within the direct linearisation framework, which provides us with more insights into the integrability of this elliptic model from the perspective of a general linear integral equation. As a result, we successfully construct for the elliptic coupled Kadomtsev–Petviashvili system not only a Lax pair composed of differential operators in 2×2 matrix form but also multi-soliton solutions with phases parametrised by points on the elliptic curve. Dimensional reductions to the elliptic coupled Korteweg-de Vries and Boussinesq systems are also disscused based on the direct linearisation. In addition, a novel class of solutions are obtained for the D_∞-type Kadomtsev–Petviashvili equation with nonzero constant background as a byproduct.

1. Introduction

It is well-known that integrable systems often come in three different classes comprising rational, trigonometric/hyperbolic, and elliptic models. Elliptic models are described by equations where there are parameters which are essentially moduli of elliptic curves, or where the dependent variable appears in the argument of elliptic functions. The recent history of the subject has taught us the remarkable finding that the theory of integrable systems is intimately linked to that of elliptic functions and curves. One aspect of this connection is the fact that, as far as we know, the richest class of integrable systems are the ones associated with those curves. For example, the Adler equation (i.e. the discrete Krichever–Novikov (KN) equation) [1] acts as the master equation among first-order partial difference equations; the elliptic Painlevé equation is on the top in Sakai's classification [38] of the discrete Painlevé equations.

Here we focus on a three-component partial differential system given by

$$\partial_3 u = \frac{1}{4}\partial_1^3 u + \frac{3}{2}(\partial_1 u)^2 + \frac{3}{4}\partial_1^{-1}\partial_2^2 u + 3g(1 - vw), \tag{1.1a}$$

$$\partial_3 v = -\frac{1}{2}\partial_1^3 v - 3(\partial_1 u - 3e)\partial_1 v + \frac{3}{2}\partial_1\partial_2 v + 3(\partial_2 u)v, \tag{1.1b}$$

$$\partial_3 w = -\frac{1}{2}\partial_1^3 w - 3(\partial_1 u - 3e)\partial_1 w - \frac{3}{2}\partial_1\partial_2 w - 3(\partial_2 u)w, \tag{1.1c}$$

where u, v and w are potentials relying on the independent variables x_1, x_2 and x_3, the parameters e and g are moduli of an elliptic curve, given below in (2.1), and ∂_j and ∂_j^{-1} respectively denote the partial-differential operators with respect to x_j and their corresponding inverses (i.e. pseudo-differential operators, see e.g. [6]) defined by

$$\partial_j[\,\cdot\,] \doteq \frac{\partial}{\partial x_j}[\,\cdot\,] \quad \text{and} \quad \partial_j^{-1}[\,\cdot\,] \doteq \int^{x_j} [\,\cdot\,]\mathrm{d}x_j, \tag{1.2}$$

which obey the relation $\partial_j\partial_j^{-1} = \partial_j^{-1}\partial_j = \mathrm{id}$. Equation (1.1) is an alternative presentation of an elliptic Kadomtsev–Petviashvili (KP)-type system that was originally proposed by Date, Jimbo and Miwa, the form of which actually suggests a $(3+1)$-dimensional differential-difference system, see section 3 of [5]. It was also pointed out by those authors that such a system follows from a similar construction of solutions of the fully anisotropic Landau–Lifshitz (LL) equation from a bilinear perspective as in [4]. Thus, one may infer that (1.1) in a sense is a KP (higher-dimensional) analogue of the LL equation. In addition, the form (1.1) considered in the present paper implies that there is a connection (see

Key words and phrases. elliptic coupled KP, DKP, direct linearisation, dimensional reduction, Lax pair, τ-function, soliton, nonzero constant background.

section 7 below) between (1.1) and the coupled KP system

$$\partial_t \mathcal{U} = \frac{1}{4}\partial_x^3 \mathcal{U} + \frac{3}{2}\mathcal{U}\partial_x \mathcal{U} + \frac{3}{4}\partial_x^{-1}\partial_y^2 \mathcal{U} - 6\partial_x(\mathcal{V}\mathcal{W}), \tag{1.3a}$$

$$\partial_t \mathcal{V} = -\frac{1}{2}\partial_x^3 \mathcal{V} - \frac{3}{2}\mathcal{U}\partial_x \mathcal{V} + \frac{3}{2}\partial_x\partial_y \mathcal{V} + \frac{3}{2}(\partial_x^{-1}\partial_y \mathcal{U})\mathcal{V}, \tag{1.3b}$$

$$\partial_t \mathcal{W} = -\frac{1}{2}\partial_x^3 \mathcal{W} - \frac{3}{2}\mathcal{U}\partial_x \mathcal{W} - \frac{3}{2}\partial_x\partial_y \mathcal{W} - \frac{3}{2}(\partial_x^{-1}\partial_y \mathcal{U})\mathcal{W} \tag{1.3c}$$

proposed by Hirota and Ohta (cf. [18] and also formula (3.94) in [17]), where $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are the potential variables dependent on the independent variables x, y and t, and ∂_x, ∂_y, ∂_t and ∂_x^{-1} denote the partial-differential and pseudo-differential operators (similar to (1.2)) with respect to the corresponding independent variables. Since (1.3) is one of the members in the DKP hierarchy (a KP-type hierarchy associated with the infinite-dimensional Lie algebra D_∞, see [22]) which possesses a rich integrable structure in the theory of integrable systems, we believe the understanding of elliptic model (1.1) will certainly yield additional insights into the integrability of many other nonlinear systems. However, this remarkable elliptic integrable system has attracted little attention in the literature since the paper [5], as far as we are aware. For this reason, we believe that the elliptic coupled KP system (1.1) deserves reinvestigation to explore further its integrability.

In the present paper, we shall adopt the direct linearisation (DL) method to study the elliptic coupled KP equation (1.1). Originally proposed by Fokas, Ablowitz and Santini for constructing general solutions of nonlinear integrable partial differential equations, the method which is based on a formal singular linear integral equation [7,8,39], was subsequently developed into a comprehensive framework to construct (discrete and continuous) integrable systems and study their underlying algebraic structures, see e.g. [30, 36], and [35, 37] for constructions of integrable discretisation of nonlinear partial differential equations, and [28, 29, 31–33] for the treatment of three-dimensional equations of KP-type. In [11–13] the connection between the DL and integrable systems on Lie algebras was developed. A powerful tool emerging from DL, first developed in [36], was an infinite matrix structure in the space of the spectral variable. The associated infinite matrix representation of the linear integral equation allows us to turn the DL into an algebraic method which has been very effective in forging an understanding of the underlying integrability of the nonlinear integrable systems and their interconnections. In [34] and [21], the notion of elliptic infinite matrix (effectively amounting to an index-relabelling system of infinite matrices) was introduced. This allows us to study integrable equations associated with elliptic curves within the DL framework.

By reparametrising the time evolution and the Cauchy kernel for the LL equation in the fermionic construction [4] and simultaneously considering a linear integral equation with a skew-symmetric integration measure which was introduced in [11–13] for the BKP-type equations, we establish in this paper the DL scheme for the elliptic coupled KP system (1.1). This allows us to study the integrability of the elliptic coupled KP system from a unified perspective. As a result, we successfully derive the nonlinear equation (1.1) together with a suitable Lax pair from the elliptic infinite matrix structure. Meanwhile, we construct the elliptic soliton solutions of (1.1) in terms of the τ-function, which possesses a Pfaffian structure, using a Pfaffian version of the well-known Cauchy–Binet formula (see [20, 23] and also appendix B), as well as a Pfaffian analogue of the famous Frobenius formula for the determinants of elliptic Cauchy matrices (see [4] and also appendix C). These results underpin not only the integrability of the elliptic coupled KP system from the viewpoint of solvability, but also induce a new class of solutions of the DKP equation with nonzero constant background as a byproduct. In addition, we discuss dimensional reductions of (1.1), from which we obtain the elliptic coupled Korteweg–de Vries (KdV) and Boussinesq (BSQ) systems together with their respective Lax pairs.

The paper is organised as follows. In section 2, we introduce the fundamental objects that will be used in construction of elliptic integrable systems, including the notion of elliptic index-raising matrices and elliptic index labels. The DL scheme of the elliptic coupled KP system is established in section 3 in the language of infinite matrices. Section 4 is concerned with the construction of the elliptic coupled KP system (1.1) and its Lax pair. In the subsequent section 5, we discuss dimensional reductions to the elliptic coupled KdV and BSQ systems. The formulae of the elliptic soliton solutions to (1.1) are presented section 6. Finally, we explain in section 7 how the soliton solutions to (1.1) generate those to the DKP equation with nonzero constant background.

2. Infinite matrices and elliptic index labels

We present an introduction to the fundamental objects that are needed in this paper, including the elliptic curve, infinite matrices, elliptic index-raising operators, etc. These objects were introduced in [21, 34] for the DL construction of the so-called discrete and continuous elliptic KdV and KP equations.

The elliptic curve that we consider in this paper is of the form

$$k^2 = K + 3e + \frac{g}{K}, \tag{2.1}$$

in which

$$e \quad \text{and} \quad g = (e - e')(e - e'') \tag{2.2}$$

are the moduli of the curve, for e, e' and e'' being the branch points of the standard Weierstrass elliptic curve $z^2 = 4Z^3 - g_2 Z - g_3$. The elliptic curve (2.1) is parametrised by a uniformising variable κ through the coordinates

$$k = \frac{1}{2}\frac{\wp'(\kappa)}{\wp(\kappa) - e} \quad \text{and} \quad K = \wp(\kappa) - e, \tag{2.3}$$

where $\wp$ and $\wp'$ denote the standard Weierstrass elliptic function and its first-order derivative, respectively.

We consider infinite matrices taking the form of $\boldsymbol{U} = (U_{i,j})_{\infty\times\infty}$ and infinite column and row vectors $\boldsymbol{a} = (a_i)_{\infty\times 1}$ and ${}^t\boldsymbol{a} = (a_i)_{1\times\infty}$. We adopt the notations ${}^t(\cdot)$ for the transpose, $(\cdot)^{(i,j)}$ for the (i,j)-entry of an infinite matrix, and $(\cdot)^{(i)}$ for the ith-component of an infinite vector.

Definition 2.1. *The index-raising infinite matrix Λ and its transpose ${}^t\Lambda$ are defined by their respective (i,j)-entries*

$$\Lambda^{(i,j)} \doteq \delta_{i+1,j} \quad \text{and} \quad {}^t\Lambda^{(i,j)} \doteq \delta_{i,j+1}, \quad \forall i,j \in \mathbb{Z}, \tag{2.4}$$

where $\delta_{\cdot,\cdot}$ is the standard Kronecker δ-function defined as

$$\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Remark 2.2. *The infinite matrices Λ and ${}^t\Lambda$ are entitled index-raising matrices because of the identities*

$$(\Lambda \boldsymbol{U})^{(i,j)} = U_{i+1,j} \quad \text{and} \quad (\boldsymbol{U}\,{}^t\Lambda)^{(i,j)} = U_{i,j+1}, \quad \forall i,j \in \mathbb{Z};$$

in other words, the operation of Λ (resp. ${}^t\Lambda$) from the left (resp. right) raises all the row (resp. column) indices of an infinite matrix by 1. Similarly, we have the identity

$$(\Lambda \boldsymbol{a})^{(i)} = ({}^t\boldsymbol{a}\,{}^t\Lambda)^{(i)} = a_{i+1}, \quad \forall i \in \mathbb{Z},$$

for infinite column and row vectors. In addition, by following (2.4) we observe that the infinite matrices Λ and ${}^t\Lambda$ are indeed the inverse of each other. But since there are no multiplications between Λ and ${}^t\Lambda$ in concrete calculation throughout the paper, we shall treat Λ and ${}^t\Lambda$ as two separate symbols.

Definition 2.3. *The infinite projection matrix $\boldsymbol{O}$ is defined by its (i,j)-entries*

$$\boldsymbol{O}^{(i,j)} \doteq \delta_{i,0}\delta_{0,j}, \quad \forall i,j \in \mathbb{Z}.$$

Remark 2.4. *It is easily verified that the multiplication between $\boldsymbol{U}$ and $\boldsymbol{O}$ results in identities*

$$(\boldsymbol{O}\boldsymbol{U})^{(i,j)} = \delta_{i,0}\boldsymbol{U}^{(0,j)} \quad \text{and} \quad (\boldsymbol{U}\boldsymbol{O})^{(i,j)} = \boldsymbol{U}^{(i,0)}\delta_{0,j}, \quad \forall i,j \in \mathbb{Z}. \tag{2.5}$$

This implies that the projection infinite matrix $\boldsymbol{O}$ plays the role of mapping arbitrary $\boldsymbol{U}$ to an infinite matrix of rank one. Likewise, we have for arbitrary $\boldsymbol{a}$ the identities

$$(\boldsymbol{O}\boldsymbol{a})^{(i)} = \delta_{i,0}\boldsymbol{a}^{(0)} \quad \text{and} \quad ({}^t\boldsymbol{a}\boldsymbol{O})^{(i)} = {}^t\boldsymbol{a}^{(0)}\delta_{0,i}, \quad \forall i \in \mathbb{Z}. \tag{2.6}$$

Definition 2.5. *We define a special particular unit column vector $\boldsymbol{e}$ and its transpose ${}^t\boldsymbol{e}$ by their respective components*

$$\boldsymbol{e}^{(i)} = {}^t\boldsymbol{e}^{(i)} = \delta_{i,0}, \quad \forall i \in \mathbb{Z}.$$

Remark 2.6. *The infinite projection matrix* $\boldsymbol{O}$ *can be written as the multiplication of* $\boldsymbol{e}$ *and* ${}^t\boldsymbol{e}$*, namely* $\boldsymbol{O} = \boldsymbol{e}\,{}^t\boldsymbol{e}$*. It is easily verified that* $\boldsymbol{o}$ *and* ${}^t\boldsymbol{o}$ *possess the properties*

$$({}^t\boldsymbol{e}\,\boldsymbol{U})^{(i)} = \boldsymbol{U}^{(0,i)}, \quad (\boldsymbol{U}\boldsymbol{e})^{(i)} = \boldsymbol{U}^{(i,0)} \quad \text{and} \quad {}^t\boldsymbol{e}\,\boldsymbol{U}\boldsymbol{e} = \boldsymbol{U}^{(0,0)}, \quad \forall i \in \mathbb{Z} \tag{2.7}$$

for an arbitrary infinite matrix $\boldsymbol{U}$ *as well as*

$${}^t\boldsymbol{e}\,\boldsymbol{a} = \boldsymbol{a}^{(0)} \quad \text{and} \quad {}^t\boldsymbol{a}\,\boldsymbol{e} = {}^t\boldsymbol{a}^{(0)}$$

for arbitrary infinite column and row vectors $\boldsymbol{a}$ *and* ${}^t\boldsymbol{a}$*.*

To deal with the elliptic integrable systems, we introduce the notion of elliptic index-raising matrices for future convenience.

Definition 2.7. *The elliptic index-raising operators* $\boldsymbol{\Lambda}$ *and* $\boldsymbol{L}$ *are defined as*

$$\boldsymbol{\Lambda} \doteq \frac{1}{2}\frac{\wp'(\Lambda)}{\wp(\Lambda)-e} \quad \text{and} \quad \boldsymbol{L} \doteq \wp(\Lambda) - e,$$

respectively; similarly, their respective transposes are defined by

$${}^t\boldsymbol{\Lambda} \doteq \frac{1}{2}\frac{\wp'({}^t\Lambda)}{\wp({}^t\Lambda)-e} \quad \text{and} \quad {}^t\boldsymbol{L} \doteq \wp({}^t\Lambda) - e.$$

These elliptic index-raising operators should be understood as formal series expansions of Λ *and* ${}^t\Lambda$*, respectively.*

Corollary 2.8. *The elliptic index-raising operators* $\boldsymbol{\Lambda}$*,* $\boldsymbol{L}$ *and* ${}^t\boldsymbol{\Lambda}$*,* ${}^t\boldsymbol{L}$ *obey the elliptic curve relations*

$$\boldsymbol{\Lambda}^2 = \boldsymbol{L} + 3e + \frac{g}{\boldsymbol{L}} \quad \text{and} \quad {}^t\boldsymbol{\Lambda}^2 = {}^t\boldsymbol{L} + 3e + \frac{g}{{}^t\boldsymbol{L}}, \tag{2.8}$$

respectively, as consequences of formulae (2.1) *and* (2.3)*. Here* $\frac{1}{\boldsymbol{L}}$ *and* $\frac{1}{{}^t\boldsymbol{L}}$ *denote the inverses of* $\boldsymbol{L}$ *and* ${}^t\boldsymbol{L}$*, respectively.*

Remark 2.9. *The formal operators* $\boldsymbol{\Lambda}$*,* ${}^t\boldsymbol{\Lambda}$*, and* $\boldsymbol{L}$ *and* ${}^t\boldsymbol{L}$ *should be understood in the following way. Since* $\boldsymbol{\Lambda}$ *and* $\boldsymbol{L}$ *commute, and similarly,* ${}^t\boldsymbol{\Lambda}$ *and* ${}^t\boldsymbol{L}$ *commute, we can consider a joint set of formal eigenvectors* $\boldsymbol{c}$ *and* ${}^t\boldsymbol{c}$ *respectively, obeying the eigenvalue equations*

$$\boldsymbol{\Lambda}\boldsymbol{c}(\kappa) = k\boldsymbol{c}(\kappa), \quad \boldsymbol{L}\boldsymbol{c}(\kappa) = K\boldsymbol{c}(\kappa) \quad \text{and} \quad {}^t\boldsymbol{c}(\kappa')^t\boldsymbol{\Lambda} = k'\boldsymbol{c}(\kappa), \quad {}^t\boldsymbol{c}(\kappa)^t\boldsymbol{L} = K'\boldsymbol{c}(\kappa), \tag{2.9}$$

where (k, K) *and* (k', K') *are the points on the elliptic curve* (2.1)*, parametrised by their respective uniformising spectral parameters* κ *and* κ' *for* $\kappa, \kappa' \in \mathbb{C}$*, namely we have relations*

$$k = \frac{1}{2}\frac{\wp'(\kappa)}{\wp(\kappa)-e}, \quad K = \wp(\kappa) - e, \quad k' = \frac{1}{2}\frac{\wp'(\kappa')}{\wp(\kappa')-e} \quad \text{and} \quad K' = \wp(\kappa') - e.$$

Thus, we can think of the points (k, K) *and* (k', K') *on the curve as the 'symbols' of these operators, in some representation defined by the basis of infinite vectors* $\boldsymbol{c}(\kappa) = (\kappa^i)_{\infty\times 1}$ *and* ${}^t\boldsymbol{c}(\kappa') = (\kappa'^i)_{1\times\infty}$ *respectively, composed of the monomials of uniformising spectral parameters* κ *and* κ'*. Furthermore, these variables will play the role of the spectral parameters for the elliptic coupled KP system.*

In the sections below, we shall only deal with the elliptic index-raising operations. For this reason, we introduce elliptic index labels $[i, j]$ and $[i]$ for infinite matrices and infinite vectors, respectively, compared with the above non-elliptic index labels (i, j) and (i).

Definition 2.10. *In the elliptic index-labelling system, the* $[i, j]$*-entries of the infinite matrix* $\boldsymbol{U}$ *are defined by the following:*

$$\boldsymbol{U}^{[2i,2j]} \doteq (\boldsymbol{L}^i\boldsymbol{U}{}^t\boldsymbol{L}^j)^{(0,0)}, \quad \boldsymbol{U}^{[2i+1,2j+1]} \doteq (\boldsymbol{\Lambda}\boldsymbol{L}^i\boldsymbol{U}{}^t\boldsymbol{L}^j{}^t\boldsymbol{\Lambda})^{(0,0)}, \tag{2.10a}$$

$$\boldsymbol{U}^{[2i+1,2j]} \doteq (\boldsymbol{\Lambda}\boldsymbol{L}^i\boldsymbol{U}{}^t\boldsymbol{L}^j)^{(0,0)}, \quad \boldsymbol{U}^{[2i,2j+1]} \doteq (\boldsymbol{L}^i\boldsymbol{U}{}^t\boldsymbol{L}^j{}^t\boldsymbol{\Lambda})^{(0,0)}, \tag{2.10b}$$

for all $i, j \in \mathbb{Z}$*. Likewise, we define the* $[i]$*th components of the infinite vectors* $\boldsymbol{a}$ *and* ${}^t\boldsymbol{a}$ *as follows:*

$$\boldsymbol{a}^{[2i]} \doteq (\boldsymbol{L}^i\boldsymbol{a})^{(0,0)}, \quad \boldsymbol{a}^{[2i+1]} \doteq (\boldsymbol{\Lambda}\boldsymbol{L}^i\boldsymbol{a})^{(0,0)}, \tag{2.11a}$$

$${}^t\boldsymbol{a}^{[2i]} \doteq ({}^t\boldsymbol{a}{}^t\boldsymbol{L}^i)^{(0)}, \quad {}^t\boldsymbol{a}^{[2i+1]} \doteq ({}^t\boldsymbol{a}{}^t\boldsymbol{L}^{i}{}^t\boldsymbol{\Lambda})^{(0)}, \tag{2.11b}$$

for arbitrary $i \in \mathbb{Z}$*.*

In the elliptic index-labelling system, $\boldsymbol{\Lambda}$ and ${}^t\boldsymbol{\Lambda}$ (resp. $\boldsymbol{L}$ and ${}^t\boldsymbol{L}$) play roles of order 1 (resp. order 2) index-raising operators. We also comment that in concrete calculation, sometimes the curve relations in (2.8) are useful to reduce the powers of $\boldsymbol{\Lambda}$ and ${}^t\boldsymbol{\Lambda}$. For example, we have

$$(\boldsymbol{\Lambda}^2\boldsymbol{U})^{(0,0)} = \boldsymbol{U}^{[2,0]} + 3e\boldsymbol{U}^{[0,0]} + g\boldsymbol{U}^{[-2,0]} \quad \text{and} \quad (\boldsymbol{\Lambda}^3\boldsymbol{U})^{(0,0)} = \boldsymbol{U}^{[3,0]} + 3e\boldsymbol{U}^{[1,0]} + g\boldsymbol{U}^{[-1,0]},$$

because of $\boldsymbol{\Lambda}^2\boldsymbol{U} = (\boldsymbol{L} + 3e + g/\boldsymbol{L})\boldsymbol{U}$ and $\boldsymbol{\Lambda}^3\boldsymbol{U} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^2\boldsymbol{U} = \boldsymbol{\Lambda}(\boldsymbol{L} + 3e + g/\boldsymbol{L})\boldsymbol{U}$.

3. Infinite matrix representation of the elliptic coupled KP system

The essential ingredients in the construction of the elliptic coupled KP system are the plane wave factor, which defines the dynamics of the system in terms of the independent variables x_j, given by

$$\rho_n(\kappa) \doteq \exp\left\{\sum_{j=0}^{\infty} k^{2j+1}x_{2j+1} + \sum_{j=1}^{\infty}\left[K^j - \left(\frac{g}{K}\right)^j\right]x_{2j}\right\}\left(\frac{K}{\sqrt{g}}\right)^n, \tag{3.1}$$

and the skew-symmetric Cauchy kernel

$$\Omega(\kappa,\kappa') \doteq \frac{K-K'}{k+k'} = \frac{k-k'}{1-\dfrac{g}{KK'}}. \tag{3.2}$$

The plane wave factor (3.1) and the kernel (3.2) are reparametrisation of those objects first presented in [4] in the different context of a fermionic construction of the LL equation.

Definition 3.1. *For the given plane wave factor* (3.1) *and the Cauchy kernel* (3.2), *the linear integral equation of the elliptic coupled KP system takes the form of*

$$\boldsymbol{u}_n(\kappa) + \iint_D \mathrm{d}\zeta(\lambda,\lambda')\rho_n(\kappa)\Omega(\kappa,\lambda')\rho_n(\lambda')\boldsymbol{u}_n(\lambda) = \rho_n(\kappa)\boldsymbol{c}(\kappa), \tag{3.3}$$

where the infinite column vector $\boldsymbol{u}_n(\kappa)$ is the wave function whose components depend on the independent variables x_j for $j \in \mathbb{Z}^+$ and the spectral parameter κ, and the integration measure $\mathrm{d}\zeta$ and the integration domain D must obey the antisymmetry property

$$\mathrm{d}\zeta(\kappa,\kappa') = -\mathrm{d}\zeta(\kappa',\kappa), \quad \forall(\kappa,\kappa') \in D. \tag{3.4}$$

Remark 3.2. *At the current stage, we do not specify the form of the integration measure. The only requirement is that the associated homogeneous integral equation of* (3.3) *has only zero solution, which guarantees the most general solution space from the perspective of the DL approach.*

For the sake of construction of integrable systems, we need the infinite matrix $\boldsymbol{C}_n$ defined as

$$\boldsymbol{C}_n \doteq \iint_D \mathrm{d}\zeta(\kappa,\kappa')\rho_n(\kappa)\boldsymbol{c}(\kappa)\,{}^t\boldsymbol{c}(\kappa')\rho_n(\kappa'), \tag{3.5}$$

in which the integration measure and domain must satisfy the same antisymmetry property (3.4), and also the infinite matrix $\boldsymbol{\Omega}$ defined by

$${}^t\boldsymbol{c}(\kappa')\boldsymbol{\Omega}\boldsymbol{c}(\kappa) \doteq \Omega(\kappa,\kappa'). \tag{3.6}$$

From the definitions, it is reasonable to think of $\boldsymbol{C}_n$ and $\boldsymbol{\Omega}$ as the infinite matrix representation of the plane wave factor (3.1) and the kernel (3.2). Due to the antisymmetry of the integration measure and the kernel, it is verified that both $\boldsymbol{C}_n$ and $\boldsymbol{\Omega}$ are skew-symmetric, i.e. ${}^t\boldsymbol{C}_n = -\boldsymbol{C}_n$ and ${}^t\boldsymbol{\Omega} = -\boldsymbol{\Omega}$. The key object towards nonlinear integrable systems in the direct linearisation is a potential matrix. We give its definition as follows.

Definition 3.3. *The potential matrix in the DL is a double integral in terms of the spectral parameters, defined as*

$$\boldsymbol{U}_n \doteq \iint_D \mathrm{d}\zeta(\kappa,\kappa')\boldsymbol{u}_n(\kappa)\,{}^t\boldsymbol{c}(\kappa')\rho_n(\kappa'), \tag{3.7}$$

in which $\boldsymbol{u}_n(\kappa)$ satisfies the linear integral equation (3.3), *and the measure is the same as the one for* (3.3), *obeying the antisymmetry property* (3.4).

By following (3.7) and (3.6), we can reformulate the linear integral equation (3.3) as

$$\boldsymbol{u}_n(\kappa) = (1 - \boldsymbol{U}_n\boldsymbol{\Omega})\rho_n(\kappa)\boldsymbol{c}(\kappa). \tag{3.8}$$

Performing the operation $\iint_D \mathrm{d}\zeta(\kappa,\kappa')(3.8)^t\boldsymbol{c}(\kappa')\rho_n(\kappa')$, we can further derive

$$\boldsymbol{U}_n = (1 - \boldsymbol{U}_n\boldsymbol{\Omega})\boldsymbol{C}_n, \quad \text{which can alternatively be written as} \quad \boldsymbol{U}_n = \boldsymbol{C}_n(1 + \boldsymbol{\Omega}\boldsymbol{C}_n)^{-1}. \tag{3.9}$$

Equation (3.9) in some sense can be considered as the infinite matrix version of (3.8).

We now introduce the τ-function associated with the elliptic coupled KP system.

Definition 3.4. *The τ-function is formally defined by*

$$\tau_n^2 \doteq \det(1 + \boldsymbol{\Omega}\boldsymbol{C}_n), \tag{3.10}$$

where $\boldsymbol{\Omega}$ and $\boldsymbol{C}_n$ are given by (3.6) and (3.5), respectively. The determinant should be understood as the formal expansion

$$\det(1 + \boldsymbol{\Omega}\boldsymbol{C}_n) = 1 + \sum_i (\boldsymbol{\Omega}\boldsymbol{C}_n)^{(i,i)} + \sum_{i<j} \begin{vmatrix} (\boldsymbol{\Omega}\boldsymbol{C}_n)^{(i,i)} & (\boldsymbol{\Omega}\boldsymbol{C}_n)^{(i,j)} \\ (\boldsymbol{\Omega}\boldsymbol{C}_n)^{(j,i)} & (\boldsymbol{\Omega}\boldsymbol{C}_n)^{(j,j)} \end{vmatrix} + \cdots .$$

We also remark that the determinant satisfies the identity $\ln[\det(1 + \boldsymbol{\Omega}\boldsymbol{C}_n)] = \mathrm{tr}[\ln(1 + \boldsymbol{\Omega}\boldsymbol{C}_n)]$.

Our aim is to derive the dynamical relations of $\boldsymbol{U}_n$ in terms of the continuous variables x_j and the discrete variable n. To realise this, we first investigate the evolutions of $\boldsymbol{C}_n$. Observing the identity (2.9) and the form of the plane wave factor (3.1), we obtain the following dynamical relations:

$$\partial_{2j+1}\boldsymbol{C}_n = \boldsymbol{\Lambda}^{2j+1}\boldsymbol{C}_n + \boldsymbol{C}_n{}^t\boldsymbol{\Lambda}^{2j+1}, \tag{3.11a}$$

$$\partial_{2j}\boldsymbol{C}_n = \left[\boldsymbol{L}^j - \left(\frac{g}{\boldsymbol{L}}\right)^j\right]\boldsymbol{C}_n + \boldsymbol{C}_n\left[{}^t\boldsymbol{L}^j - \left(\frac{g}{{}^t\boldsymbol{L}}\right)^j\right], \tag{3.11b}$$

$$\boldsymbol{C}_{n+1}\frac{\sqrt{g}}{{}^t\boldsymbol{L}} = \frac{\boldsymbol{L}}{\sqrt{g}}\boldsymbol{C}_n, \tag{3.11c}$$

by differentiating $\boldsymbol{C}_n$ with respect to x_j and shifting $\boldsymbol{C}_n$ with respect to n. Next, from equations (3.2) and (3.6) we are able to derive

$$\boldsymbol{\Omega}\boldsymbol{\Lambda} + {}^t\boldsymbol{\Lambda}\boldsymbol{\Omega} = \boldsymbol{O}\boldsymbol{L} - {}^t\boldsymbol{L}{}^t\boldsymbol{O} \quad \text{and} \quad \boldsymbol{\Omega} - g\,{}^t\boldsymbol{L}^{-1}\boldsymbol{\Omega}\boldsymbol{L}^{-1} = \boldsymbol{O}\boldsymbol{\Lambda} - {}^t\boldsymbol{\Lambda}\boldsymbol{O}. \tag{3.12}$$

In fact, multiplying the two equations in (3.12) respectively by ${}^t\boldsymbol{c}(\kappa')$ from left and $\boldsymbol{c}(\kappa)$ from the right yield

$$\Omega(\kappa,\kappa')k + k'\Omega(\kappa,\kappa') = K - K' \quad \text{and} \quad \Omega(\kappa,\kappa') - \frac{g}{KK'}\Omega(\kappa,\kappa') = k - k',$$

namely (3.2); in other words, equations in (3.12) are nothing but the infinite matrix representation of the elliptic Cauchy kernel. With the help of (3.12), we can further derive the following relations for the infinite matrix $\boldsymbol{\Omega}$ by mathematical induction:

$$\boldsymbol{\Omega}\boldsymbol{\Lambda}^{2j+1} + {}^t\boldsymbol{\Lambda}^{2j+1}\boldsymbol{\Omega} = \boldsymbol{O}_{2j+1}\boldsymbol{L} - {}^t\boldsymbol{L}\boldsymbol{O}_{2j+1}, \tag{3.13a}$$

$$\boldsymbol{\Omega}\left[\boldsymbol{L}^j - \left(\frac{g}{\boldsymbol{L}}\right)^j\right] + \left[{}^t\boldsymbol{L}^j - \left(\frac{g}{{}^t\boldsymbol{L}}\right)^j\right]\boldsymbol{\Omega} = \boldsymbol{O}'_j\boldsymbol{\Lambda} - {}^t\boldsymbol{\Lambda}\boldsymbol{O}'_j, \tag{3.13b}$$

$$\boldsymbol{\Omega}\frac{\boldsymbol{L}}{\sqrt{g}} - \frac{\sqrt{g}}{{}^t\boldsymbol{L}}\boldsymbol{\Omega} = \boldsymbol{O}_2\frac{\boldsymbol{L}}{\sqrt{g}}, \tag{3.13c}$$

in which

$$\boldsymbol{O}_j \doteq \sum_{i=0}^{j-1}\left(-{}^t\boldsymbol{\Lambda}\right)^i\boldsymbol{O}\boldsymbol{\Lambda}^{j-1-i}, \quad \boldsymbol{O}'_j \doteq \sum_{i=0}^{j-1} g^i\left({}^t\boldsymbol{L}^{-i}\boldsymbol{O}\boldsymbol{L}^{j-i} + {}^t\boldsymbol{L}^{j-i}\boldsymbol{O}\boldsymbol{L}^{-i}\right).$$

Proposition 3.5. *The infinite matrix* $\boldsymbol{U}_n$ *defined by* (3.7) *satisfies the following continuous and discrete dynamical evolutions:*

$$\partial_{2j+1}\boldsymbol{U}_n = \boldsymbol{\Lambda}^{2j+1}\boldsymbol{U}_n + \boldsymbol{U}_n {}^t\boldsymbol{\Lambda}^{2j+1} - \boldsymbol{U}_n \left(\boldsymbol{O}_{2j+1}\boldsymbol{L} - {}^t\boldsymbol{L}\boldsymbol{O}_{2j+1}\right)\boldsymbol{U}_n, \tag{3.14a}$$

$$\partial_{2j}\boldsymbol{U}_n = \left[\boldsymbol{L}^j - \left(\frac{g}{\boldsymbol{L}}\right)^j\right]\boldsymbol{U}_n + \boldsymbol{U}_n\left[{}^t\boldsymbol{L}^j - \left(\frac{g}{{}^t\boldsymbol{L}}\right)^j\right] - \boldsymbol{U}_n\left(\boldsymbol{O}'_j\boldsymbol{\Lambda} - {}^t\boldsymbol{\Lambda}\boldsymbol{O}'_j\right)\boldsymbol{U}_n, \tag{3.14b}$$

$$\boldsymbol{U}_{n+1}\frac{\sqrt{g}}{{}^t\boldsymbol{L}} = \frac{\boldsymbol{L}}{\sqrt{g}}\boldsymbol{U}_n - \boldsymbol{U}_{n+1}\boldsymbol{O}_2\frac{\boldsymbol{L}}{\sqrt{g}}\boldsymbol{U}_n. \tag{3.14c}$$

Proof. We only prove (3.14b) and (3.14c). Differentiating (3.9) with respect to x_{2j} gives rise to

$$\partial_{2j}\boldsymbol{U}_n = (1 - \boldsymbol{U}_n\boldsymbol{\Omega})(\partial_{2j}\boldsymbol{C}_n) - (\partial_{2j}\boldsymbol{U}_n)\boldsymbol{\Omega}\boldsymbol{C}, \quad \text{namely} \quad (\partial_{2j}\boldsymbol{U}_n)(1 + \boldsymbol{\Omega}\boldsymbol{C}_n) = (1 - \boldsymbol{U}_n\boldsymbol{\Omega})(\partial_{2j}\boldsymbol{C}_n).$$

Notice that the infinite matrix $\boldsymbol{C}_n$ obeys the evolution given by (3.11b). The above equation is reformulated as

$$(\partial_{2j}\boldsymbol{U}_n)(1 + \boldsymbol{\Omega}\boldsymbol{C}_n) = (1 - \boldsymbol{U}_n\boldsymbol{\Omega})\left[\boldsymbol{L}^j - \left(\frac{g}{\boldsymbol{L}}\right)^j\right]\boldsymbol{C}_n + \boldsymbol{U}_n\left[{}^t\boldsymbol{L}^j - \left(\frac{g}{{}^t\boldsymbol{L}}\right)^j\right].$$

We can now replace $\boldsymbol{\Omega}\left[\boldsymbol{L}^j - \left(\frac{g}{\boldsymbol{L}}\right)^j\right]$ by following (3.13b). This in turn implies

$$(\partial_{2j}\boldsymbol{U}_n)(1 + \boldsymbol{\Omega}\boldsymbol{C}_n) = \left[\boldsymbol{L}^j - \left(\frac{g}{\boldsymbol{L}}\right)^j\right]\boldsymbol{C}_n + \boldsymbol{U}_n\left[{}^t\boldsymbol{L}^j - \left(\frac{g}{{}^t\boldsymbol{L}}\right)^j\right](1 + \boldsymbol{\Omega}\boldsymbol{C}_n) - \boldsymbol{U}_n\left(\boldsymbol{O}'_j\boldsymbol{\Lambda} - {}^t\boldsymbol{\Lambda}\boldsymbol{O}'_j\right)\boldsymbol{C}_n,$$

which immediately results in (3.14b) by multiplying $(1 + \boldsymbol{\Omega}\boldsymbol{C}_n)^{-1}$ from the right. Equation (3.14c) is derived by a similar approach. We shift (3.9) with respect to n and obtain

$$\boldsymbol{U}_{n+1}\frac{\sqrt{g}}{{}^t\boldsymbol{L}} = (1 - \boldsymbol{U}_{n+1}\boldsymbol{\Omega})\frac{\boldsymbol{L}}{\sqrt{g}}\boldsymbol{C}_n$$

in virtue of (3.11c). By substituting $\boldsymbol{\Omega}\frac{\boldsymbol{L}}{\sqrt{g}}$ with the help of (3.13c), this equation turns out to be

$$\boldsymbol{U}_{n+1}\frac{\sqrt{g}}{{}^t\boldsymbol{L}} = \frac{\boldsymbol{L}}{\sqrt{g}}\boldsymbol{C}_n - \boldsymbol{U}_{n+1}\left(\boldsymbol{O}_2\frac{\boldsymbol{L}}{\sqrt{g}} + \frac{\sqrt{g}}{{}^t\boldsymbol{L}}\boldsymbol{\Omega}\right)\boldsymbol{C}_n,$$

i.e.

$$\boldsymbol{U}_{n+1}\frac{\sqrt{g}}{{}^t\boldsymbol{L}}(1 + \boldsymbol{\Omega}\boldsymbol{C}_n) = \frac{\boldsymbol{L}}{\sqrt{g}}\boldsymbol{C}_n - \boldsymbol{U}_{n+1}\boldsymbol{O}_2\frac{\boldsymbol{L}}{\sqrt{g}}\boldsymbol{C}_n.$$

Multiplying $(1 + \boldsymbol{\Omega}\boldsymbol{C}_n)^{-1}$ from the right, we end up with (3.14c). Equation (3.14a) is proven in a similar way. □

Proposition 3.6. *The infinite matrix* $\boldsymbol{U}_n$ *satisfies the antisymmetry condition*

$${}^t\boldsymbol{U}_n = -\boldsymbol{U}_n, \quad \text{and consequently} \quad \boldsymbol{U}_n^{[j,i]} = -\boldsymbol{U}_n^{[i,j]} \tag{3.15}$$

in terms of the elliptic index labels $[i, j]$ *for all* $i, j \in \mathbb{Z}$.

Proof. Notice that $\boldsymbol{C}_n$ and $\boldsymbol{\Omega}$ are both skew-symmetric. We from (3.9) obtain

$$\begin{aligned}{}^t\boldsymbol{U}_n &= {}^t\left[\boldsymbol{C}_n(1 + \boldsymbol{\Omega}\boldsymbol{C}_n)^{-1}\right] = {}^t\left[\left(\boldsymbol{C}_n^{-1} + \boldsymbol{\Omega}\right)^{-1}\right] \\ &= \left({}^t\boldsymbol{C}_n^{-1} + {}^t\boldsymbol{\Omega}\right)^{-1} = -\left(\boldsymbol{C}_n^{-1} + \boldsymbol{\Omega}\right)^{-1} = -\boldsymbol{C}_n(1 + \boldsymbol{\Omega}\boldsymbol{C}_n)^{-1} = -\boldsymbol{U}_n,\end{aligned}$$

and subsequently $\boldsymbol{U}_n^{[i,j]} = -\boldsymbol{U}_n^{[j,i]}$ for all $i, j \in \mathbb{Z}$ by following the elliptic index labels defined by (2.10). □

We can also follow the derivation of (3.14) and construct the dynamical relations for the wave function $\boldsymbol{u}_n(\kappa)$.

Proposition 3.7. *The wave function of the linear integral equation* (3.3) *obeys dynamical evolutions with respect to the continuous variables* x_j *and the discrete variable* n *as follows:*

$$\partial_{2j+1}\boldsymbol{u}_n(\kappa) = \boldsymbol{\Lambda}^{2j+1}\boldsymbol{u}_n(\kappa) - \boldsymbol{U}_n\left(\boldsymbol{O}_{2j+1}\boldsymbol{L} - {}^t\boldsymbol{L}\boldsymbol{O}_{2j+1}\right)\boldsymbol{u}_n(\kappa), \tag{3.16a}$$

$$\partial_{2j}\boldsymbol{u}_n(\kappa) = \left[\boldsymbol{L}^j - \left(\frac{g}{\boldsymbol{L}}\right)^j\right]\boldsymbol{u}_n(\kappa) - \boldsymbol{U}_n\left(\boldsymbol{O}'_j\boldsymbol{\Lambda} - {}^t\boldsymbol{\Lambda}\boldsymbol{O}'_j\right)\boldsymbol{u}_n(\kappa), \tag{3.16b}$$

$$\boldsymbol{u}_{n+1}(\kappa) = \frac{\boldsymbol{L}}{\sqrt{g}}\boldsymbol{u}_n(\kappa) - \boldsymbol{U}_{n+1}\boldsymbol{O}_2\frac{\boldsymbol{L}}{\sqrt{g}}\boldsymbol{u}_n(\kappa). \tag{3.16c}$$

Proof. We only present the proof of (3.16b) and (3.16c). By differentiating (3.8) with respect to x_{2j}, we obtain

$$\partial_{2j}\boldsymbol{u}_n(k) = (1 - \boldsymbol{U}_n\boldsymbol{\Omega})[\partial_{2j}\rho_n(\kappa)]\boldsymbol{c}(\kappa) - (\partial_{2j}\boldsymbol{U}_n)\boldsymbol{\Omega}\rho_n(\kappa')\boldsymbol{c}(\kappa').$$

Equations (2.9) and (3.14b) can help us to reformulate the above equation as

$$\begin{aligned}\partial_{2j}\boldsymbol{u}_n(k) = &\left[\boldsymbol{L}^j - \left(\frac{g}{\boldsymbol{L}}\right)^j\right](1 - \boldsymbol{U}_n\boldsymbol{\Omega})\rho_n(\kappa)\boldsymbol{c}(\kappa)\\ &- \boldsymbol{U}_n\left\{\boldsymbol{\Omega}\left[\boldsymbol{L}^j - \left(\frac{g}{\boldsymbol{L}}\right)^j\right] + \left[{}^t\boldsymbol{L}^j - \left(\frac{g}{{}^t\boldsymbol{L}}\right)^j\right]\boldsymbol{\Omega}\right\}\rho(\kappa)\boldsymbol{c}(\kappa)\\ &+ \boldsymbol{U}_n\left(\boldsymbol{O}'_j\boldsymbol{\Lambda} - {}^t\boldsymbol{\Lambda}\boldsymbol{O}'_j\right)\boldsymbol{U}_n\boldsymbol{\Omega}\rho_n(\kappa')\boldsymbol{c}(\kappa').\end{aligned}$$

Replacing $\boldsymbol{\Omega}\left[\boldsymbol{L}^j - \left(\frac{g}{\boldsymbol{L}}\right)^j\right] + \left[{}^t\boldsymbol{L}^j - \left(\frac{g}{{}^t\boldsymbol{L}}\right)^j\right]\boldsymbol{\Omega}$ by $\boldsymbol{O}'_j\boldsymbol{\Lambda} - {}^t\boldsymbol{\Lambda}\boldsymbol{O}'_j$ according to (3.13b), we reach to

$$\partial_{2j}\boldsymbol{u}_n(k) = \left[\boldsymbol{L}^j - \left(\frac{g}{\boldsymbol{L}}\right)^j\right](1 - \boldsymbol{U}_n\boldsymbol{\Omega})\rho_n(\kappa)\boldsymbol{c}(\kappa) - \boldsymbol{U}_n\left(\boldsymbol{O}'_j\boldsymbol{\Lambda} - {}^t\boldsymbol{\Lambda}\boldsymbol{O}'_j\right)(1 - \boldsymbol{U}_n\boldsymbol{\Omega})\rho_n(\kappa')\boldsymbol{c}(\kappa'),$$

which is nothing but equation (3.16b) according to (3.8). To derive (3.16c), we shift (3.8) with respect to n. This gives rise to

$$\boldsymbol{u}_{n+1} = (1 - \boldsymbol{U}_{n+1}\boldsymbol{\Omega})\rho_{n+1}(\kappa')\boldsymbol{c}(\kappa') = (1 - \boldsymbol{U}_{n+1}\boldsymbol{\Omega})\frac{\boldsymbol{L}}{\sqrt{g}}\rho_n(\kappa')\boldsymbol{c}(\kappa').$$

Then (3.13c) leads this equation to

$$\boldsymbol{u}_{n+1} = \frac{\boldsymbol{L}}{\sqrt{g}}\rho_n(\kappa')\boldsymbol{c}(\kappa') - \boldsymbol{U}_{n+1}\left(\frac{\sqrt{g}}{{}^t\boldsymbol{L}}\boldsymbol{\Omega} + \boldsymbol{O}_2\frac{\boldsymbol{L}}{\sqrt{g}}\right)\rho_n(\kappa')\boldsymbol{c}(\kappa').$$

Finally by substituting $\boldsymbol{U}_{n+1}\frac{\sqrt{g}}{{}^t\boldsymbol{L}}$ with the help of (3.14c), the above equation turns out to be

$$\boldsymbol{u}_{n+1} = \frac{\boldsymbol{L}}{\sqrt{g}}(1 - \boldsymbol{U}_n\boldsymbol{\Omega})\rho_n(\kappa')\boldsymbol{c}(\kappa') - \boldsymbol{U}_{n+1}\boldsymbol{O}_2\frac{\boldsymbol{L}}{\sqrt{g}}(1 - \boldsymbol{U}_n\boldsymbol{\Omega})\rho_n(\kappa')\boldsymbol{c}(\kappa'),$$

namely equation (3.16c) is proven in virtue of (3.8). Equation (3.16a) can be proven through the same procedure. □

Finally, we present the dynamics of the τ-function in terms of the indices of the infinite matrix $\boldsymbol{U}_n$.

Proposition 3.8. *The* τ*-function satisfies dynamical evolutions*

$$2\partial_{2j+1}\ln\tau_n = \sum_{i=0}^{j-1}(-1)^i\left(\boldsymbol{\Lambda}^{j-1-i}\boldsymbol{U}_n{}^t\boldsymbol{\Lambda}^i\right)^{(0,0)} \tag{3.17a}$$

and

$$2\partial_{2j}\ln\tau_n = \sum_{i=0}^{j-1}g^i\left(\boldsymbol{L}^{j-i}\boldsymbol{U}_n{}^t\boldsymbol{L}^{-i} + \boldsymbol{L}^{-i}\boldsymbol{U}_n{}^t\boldsymbol{L}^{j-i}\right)^{(0,0)} \tag{3.17b}$$

with respect to the continuous arguments x_j*, as well as*

$$\frac{\tau_{n+1}}{\tau_n} = 1 + g^{-1}\boldsymbol{U}_n^{[3,2]} \quad \text{and} \quad \frac{\tau_{n-1}}{\tau_n} = 1 - \boldsymbol{U}_n^{[1,0]} \tag{3.17c}$$

with respect to the discrete argument n*.*

Proof. We first prove (3.17a). Differentiating the logarithm of the τ-function with respective to x_{2j+1} gives rise to

$$\partial_{2j+1}\ln\tau_n^2=\partial_{2j+1}\ln[\det(1+\boldsymbol{\Omega}\boldsymbol{C}_n)]=\partial_1\operatorname{tr}[\ln(1+\boldsymbol{\Omega}\boldsymbol{C}_n)]=\operatorname{tr}[(1+\boldsymbol{\Omega}\boldsymbol{C}_n)^{-1}\boldsymbol{\Omega}(\partial_{2j+1}\boldsymbol{C}_n)].$$

Replacing $\partial_{2j+1}\boldsymbol{C}_n$ with the help of (3.11a) and (3.9), we then obtain

$$\partial_{2j+1}\ln\tau_n^2=\operatorname{tr}[(1+\boldsymbol{\Omega}\boldsymbol{C}_n)^{-1}\boldsymbol{\Omega}(\boldsymbol{\Lambda}^{2j+1}\boldsymbol{C}_n+\boldsymbol{C}_n{}^t\boldsymbol{\Lambda}^{2j+1})]=\operatorname{tr}[(\Omega\boldsymbol{\Lambda}^{2j+1}+{}^t\boldsymbol{\Lambda}^{2j+1}\boldsymbol{\Omega})\boldsymbol{U}_n].$$

Recall that $\boldsymbol{\Omega}\boldsymbol{\Lambda}+{}^t\boldsymbol{\Lambda}\boldsymbol{\Omega}=\boldsymbol{O}_{2j+1}$. We end up with

$$2\ln\tau_n=\ln\tau_n^2=\operatorname{tr}(\boldsymbol{O}_{2j+1}\boldsymbol{U}_n),$$

which is nothing but the first equation of (3.17a). Equation (3.17b) follows from a similar derivation. Next, performing the shift operation on (3.10) we obtain

$$\begin{aligned}\tau_{n+1}^2&=\det(1+\boldsymbol{\Omega}\boldsymbol{C}_{n+1})=\det\left(1+\boldsymbol{\Omega}\frac{\boldsymbol{L}}{\sqrt{g}}\boldsymbol{C}_n\frac{{}^t\boldsymbol{L}}{\sqrt{g}}\right)=\det\left[1+\left(\boldsymbol{O}_2\frac{\boldsymbol{L}}{\sqrt{g}}+\frac{\sqrt{g}}{{}^t\boldsymbol{L}}\boldsymbol{\Omega}\right)\boldsymbol{C}_n\frac{{}^t\boldsymbol{L}}{\sqrt{g}}\right]\\&=\det\left[1+\boldsymbol{\Omega}\boldsymbol{C}_n+(1+\boldsymbol{\Omega}\boldsymbol{C}_n)^{-1}\frac{{}^t\boldsymbol{L}}{\sqrt{g}}\boldsymbol{O}_2\frac{\boldsymbol{L}}{\sqrt{g}}\boldsymbol{C}_n\right]=\tau_n^2\det\left[1+(1+\boldsymbol{\Omega}\boldsymbol{C}_n)^{-1}\frac{{}^t\boldsymbol{L}}{\sqrt{g}}\boldsymbol{O}_2\frac{\boldsymbol{L}}{\sqrt{g}}\boldsymbol{C}_n\right],\end{aligned}$$

where the second and third equalities hold because of (3.11c) and (3.13c), respectively. Hence, this equation can further be rewritten as

$$\begin{aligned}\frac{\tau_{n+1}^2}{\tau_n^2}&=\det\left[1+g^{-1}(1+\boldsymbol{\Omega}\boldsymbol{C}_n)^{-1}({}^t\boldsymbol{L},-{}^t\boldsymbol{L}{}^t\boldsymbol{\Lambda})\boldsymbol{O}\begin{pmatrix}\boldsymbol{\Lambda}\boldsymbol{L}\\\boldsymbol{L}\end{pmatrix}\boldsymbol{C}_n\right]\\&=\det\left[1+g^{-1}\left((1+\boldsymbol{\Omega}\boldsymbol{C}_n)^{-1t}\boldsymbol{L}\boldsymbol{e},-(1+\boldsymbol{\Omega}\boldsymbol{C}_n)^{-1t}\boldsymbol{L}^t\boldsymbol{\Lambda}\boldsymbol{e}\right)\begin{pmatrix}{}^t\boldsymbol{e}\boldsymbol{\Lambda}\boldsymbol{L}\boldsymbol{C}_n\\{}^t\boldsymbol{e}\boldsymbol{L}\boldsymbol{C}_n\end{pmatrix}\right],\end{aligned}$$

where we have used the identity $\boldsymbol{O}=\boldsymbol{e}\,{}^t\boldsymbol{e}$. This implies that τ_{n+1}/τ_n is of the form

$$\det\left[1+(\boldsymbol{a}_1,\boldsymbol{a}_2)\begin{pmatrix}{}^t\boldsymbol{b}_1,\\{}^t\boldsymbol{b}_2\end{pmatrix}\right]$$

for infinite column vectors $\boldsymbol{a}_i$ and infinite row vectors ${}^t\boldsymbol{b}_i$ for $i=1,2$. Using the rank 2 Weinstein–Aronszajn formula

$$\det\left[1+(\boldsymbol{a}_1,\boldsymbol{a}_2)\begin{pmatrix}{}^t\boldsymbol{b}_1\\{}^t\boldsymbol{b}_2\end{pmatrix}\right]=\det\left[1+\begin{pmatrix}{}^t\boldsymbol{b}_1\boldsymbol{a}_1&{}^t\boldsymbol{b}_1\boldsymbol{a}_2\\{}^t\boldsymbol{b}_2\boldsymbol{a}_1&{}^t\boldsymbol{b}_2\boldsymbol{a}_2\end{pmatrix}\right]$$

and also equation (3.9), we obtain

$$\begin{aligned}\frac{\tau_{n+1}^2}{\tau_n^2}&=\det\begin{bmatrix}1+g^{-1t}\boldsymbol{e}\boldsymbol{\Lambda}\boldsymbol{L}\boldsymbol{U}_n{}^t\boldsymbol{\Lambda}\boldsymbol{e}&-g^{-1t}\boldsymbol{e}\boldsymbol{\Lambda}\boldsymbol{L}\boldsymbol{U}_n{}^t\boldsymbol{L}^t\boldsymbol{\Lambda}\boldsymbol{e}\\g^{-1t}\boldsymbol{e}\boldsymbol{L}\boldsymbol{U}_n{}^t\boldsymbol{\Lambda}\boldsymbol{e}&1-g^{-1t}\boldsymbol{e}\boldsymbol{L}\boldsymbol{U}_n{}^t\boldsymbol{L}^t\boldsymbol{\Lambda}\boldsymbol{e}\end{bmatrix}\\&=\det\begin{bmatrix}1+g^{-1}(\boldsymbol{\Lambda}\boldsymbol{L}\boldsymbol{U}_n{}^t\boldsymbol{\Lambda})^{(0,0)}&-g^{-1}(\boldsymbol{\Lambda}\boldsymbol{L}\boldsymbol{U}_n{}^t\boldsymbol{L}^t\boldsymbol{\Lambda})^{(0,0)}\\g^{-1}(\boldsymbol{L}\boldsymbol{U}_n{}^t\boldsymbol{\Lambda})^{(0,0)}&1-g^{-1}(\boldsymbol{L}\boldsymbol{U}_n{}^t\boldsymbol{L}^t\boldsymbol{\Lambda})^{(0,0)}\end{bmatrix}\\&=\det\begin{bmatrix}1+g^{-1}\boldsymbol{U}_n^{[3,2]}&-g^{-1}\boldsymbol{U}_n^{[3,3]}\\g^{-1}\boldsymbol{U}_n^{[2,2]}&1-g^{-1}\boldsymbol{U}_n^{[2,3]}\end{bmatrix}=\left(1+g^{-1}\boldsymbol{U}_n^{[3,2]}\right)^2,\end{aligned}$$

where the second equality holds because of the third equation in (2.7), and we have used the antisymmetry property $\boldsymbol{U}_n^{[j,i]}=-\boldsymbol{U}_n^{[i,j]}$ for the last equality. Without loss of generality, we have $\tau_{n+1}/\tau_n=1+g^{-1}\boldsymbol{U}_n^{[3,2]}$. Likewise, by performing the backward shift operation on (3.10), we derive the other relation in (3.17c). □

Equations (3.14a), (3.14b) and (3.15) together form the infinite matrix representation of the elliptic coupled KP hierarchy, describing possible dynamical evolutions and algebraic constraints of the potential matrix. Meanwhile, equations in (3.16) form the infinite vector representation of the Lax pair of the elliptic coupled KP hierarchy. In the subsequent section, we shall present a closed-form multi-component nonlinear system composed of certain entries of the infinite matrix $\boldsymbol{U}_n$ and a linear system based on particular components of the infinite vector $\boldsymbol{u}_n(\kappa)$, which respectively form the closed-form elliptic coupled KP system and its Lax pair.

4. Closed-form nonlinear system and associated linear problem

We now construct the closed-form elliptic coupled KP system and its Lax pair. Our attention is only paid to the first nontrivial flow in the hierarchy, namely the equation evolving with respect to the flow variables x_1, x_2 and x_3. Our construction of the elliptic coupled KP system is based on new variables (i.e. potentials) as follows:

$$u_n \doteq \boldsymbol{U}_n^{[2,0]} \equiv -\boldsymbol{U}_n^{[0,2]}, \tag{4.1a}$$

$$v_n \doteq 1 + g^{-1}\boldsymbol{U}_n^{[3,2]} \equiv 1 - g^{-1}\boldsymbol{U}_n^{[2,3]}, \tag{4.1b}$$

$$w_n \doteq 1 - \boldsymbol{U}_n^{[1,0]} \equiv 1 + \boldsymbol{U}_n^{[0,1]}. \tag{4.1c}$$

From the dynamical equations in (3.14), we are able to find a closed-form three-component system composed of the variables u_n, v_n and w_n. We present the result as the following theorem.

Theorem 4.1. *Suppose that $\boldsymbol{u}_n(\kappa)$ is a solution to the linear integral equation* (3.3) *subject to* (3.1), (3.2) *and* (3.4). *The potentials u_n, v_n and w_n defined by* (4.1) *provide solutions to the elliptic coupled KP system*

$$\partial_3 u_n = \frac{1}{4}\partial_1^3 u_n + \frac{3}{2}(\partial_1 u_n)^2 + \frac{3}{4}\partial_1^{-1}\partial_2^2 u_n + 3g(1 - v_n w_n), \tag{4.2a}$$

$$\partial_3 v_n = -\frac{1}{2}\partial_1^3 v_n - 3(\partial_1 u_n - 3e)\partial_1 v_n + \frac{3}{2}\partial_1\partial_2 v_n + 3(\partial_2 u_n)v_n, \tag{4.2b}$$

$$\partial_3 w_n = -\frac{1}{2}\partial_1^3 w_n - 3(\partial_1 u_n - 3e)\partial_1 w_n - \frac{3}{2}\partial_1\partial_2 w_n - 3(\partial_2 u_n)w_n, \tag{4.2c}$$

in which e and g are the moduli of the elliptic curve (2.1). *Equation* (4.2) *is exactly the same as* (1.1) *by ignoring the discrete variable n.*

Proof. Equations in (4.2) are verified by direct computation. To realise this, we first of all need the fundamental formulae for the first-order derivatives $\partial_{\mathrm{j}}\boldsymbol{U}_n^{[i,j]}$ for $\mathrm{j} \in \mathbb{Z}^+$ and $i, j \in \mathbb{Z}$. Notice that

$$\begin{aligned}
\partial_{\mathrm{j}}\boldsymbol{U}_n^{[2i,2j]} &= \partial_{\mathrm{j}}(\boldsymbol{L}^i\boldsymbol{U}_n{}^t\boldsymbol{L}^j)^{(0,0)} = (\boldsymbol{L}^i\partial_{\mathrm{j}}\boldsymbol{U}_n{}^t\boldsymbol{L}^j)^{(0,0)},\\
\partial_{\mathrm{j}}\boldsymbol{U}_n^{[2i+1,2j]} &= \partial_{\mathrm{j}}(\boldsymbol{\Lambda}\boldsymbol{L}^i\boldsymbol{U}_n{}^t\boldsymbol{L}^j)^{(0,0)} = (\boldsymbol{\Lambda}\boldsymbol{L}^i\partial_{\mathrm{j}}\boldsymbol{U}_n{}^t\boldsymbol{L}^j)^{(0,0)},\\
\partial_{\mathrm{j}}\boldsymbol{U}_n^{[2i,2j+1]} &= \partial_{\mathrm{j}}(\boldsymbol{L}^i\boldsymbol{U}_n{}^t\boldsymbol{L}^j{}^t\boldsymbol{\Lambda})^{(0,0)} = (\boldsymbol{L}^i\partial_{\mathrm{j}}\boldsymbol{U}_n{}^t\boldsymbol{L}^j{}^t\boldsymbol{\Lambda})^{(0,0)},\\
\partial_{\mathrm{j}}\boldsymbol{U}_n^{[2i+1,2j+1]} &= \partial_{\mathrm{j}}(\boldsymbol{\Lambda}\boldsymbol{L}^i\boldsymbol{U}_n{}^t\boldsymbol{L}^j{}^t\boldsymbol{\Lambda})^{(0,0)} = (\boldsymbol{\Lambda}\boldsymbol{L}^i\partial_{\mathrm{j}}\boldsymbol{U}_n{}^t\boldsymbol{L}^j{}^t\boldsymbol{\Lambda})^{(0,0)}
\end{aligned}$$

due to (2.10). Replacing all the $\partial_{\mathrm{j}}\boldsymbol{U}_n$ with the help of (3.14a) and (3.14b) for odd and even j, respectively, we end up with the formulae for all the $\partial_{\mathrm{j}}\boldsymbol{U}_n^{[i,j]}$ expressed by various algebraic combinations of $\boldsymbol{U}_n^{[i,j]}$. For instance, when $i = 2$ and $j = 1$ the simplest formulae are given by

$$\partial_1\boldsymbol{U}_n^{[2,0]} = \boldsymbol{U}_n^{[3,0]} + \boldsymbol{U}_n^{[2,1]} - \boldsymbol{U}_n^{[2,0]}\boldsymbol{U}_n^{[2,0]} + \boldsymbol{U}_n^{[2,2]}\boldsymbol{U}_n^{[0,0]}$$

and

$$\begin{aligned}
\partial_2\boldsymbol{U}_n^{[2,0]} = {}& \boldsymbol{U}_n^{[4,0]} - g\boldsymbol{U}_n^{[0,0]} + \boldsymbol{U}_n^{[2,2]} - g\boldsymbol{U}_n^{[2,-2]}\\
& - \boldsymbol{U}_n^{[2,0]}\boldsymbol{U}_n^{[3,0]} - \boldsymbol{U}_n^{[2,2]}\boldsymbol{U}_n^{[1,0]} + \boldsymbol{U}_n^{[2,1]}\boldsymbol{U}_n^{[2,0]} + \boldsymbol{U}_n^{[2,3]}\boldsymbol{U}_n^{[0,0]}.
\end{aligned}$$

Then by iteration, we can further derive the general formulae for the higher-order derivatives of $\boldsymbol{U}_n^{[i,j]}$ such as $\partial_{\mathrm{j}}^2\boldsymbol{U}_n^{[i,j]}$, $\partial_{\mathrm{j}}^3\boldsymbol{U}_n^{[i,j]}$, $\partial_{\mathrm{j}}^4\boldsymbol{U}_n^{[i,j]}$, $\partial_{\mathrm{i}}\partial_{\mathrm{j}}\boldsymbol{U}_n^{[i,j]}$, etc. for $\mathrm{i}, \mathrm{j} \in \mathbb{Z}^+$ and $i, j \in \mathbb{Z}$. These formulae together with the antisymmetry property (3.15) allow us to verify equations given by (4.2) in a purely algebraic way. For example, in order to verify (4.2a) we need the corresponding formulae for $\partial_1\boldsymbol{U}_n^{[2,0]}$, $\partial_1^2\boldsymbol{U}_n^{[2,0]}$, $\partial_1^4\boldsymbol{U}_n^{[2,0]}$, $\partial_2^2\boldsymbol{U}_n^{[2,0]}$ and $\partial_1\partial_3\boldsymbol{U}_n^{[2,0]}$. Then it is verified that

$$-\partial_1\partial_3\boldsymbol{U}_n^{[2,0]} + \frac{1}{4}\partial_1^4\boldsymbol{U}_n^{[2,0]} + 3\left(\partial_1\boldsymbol{U}_n^{[2,0]}\right)\left(\partial_1^2\boldsymbol{U}_n^{[2,0]}\right) + \frac{3}{4}\partial_2^2\boldsymbol{U}_n^{[2,0]} - 3g\partial_1\left[\left(1 + g^{-1}\boldsymbol{U}_n^{[3,2]}\right)\left(1 - \boldsymbol{U}_n^{[1,0]}\right)\right]$$

vanishes in virtue of $\boldsymbol{U}_n^{[j,i]} = -\boldsymbol{U}_n^{[i,j]}$. This implies that

$$\partial_1\partial_3 u_n = \partial_1\left(\frac{1}{4}\partial_1^3 u_n + \frac{3}{2}(\partial_1 u_n)^2\right) + \frac{3}{4}\partial_2^2 u_n - 3g\partial_1(v_n w_n).$$

Observing the asymptotic conditions $u_n = 0$, $v_n = 1$ and $w_n = 1$ (which are observed from (4.1)), we immediately derive (4.2a) by integration with respect to x_1. Equations (4.2b) and (4.2c) are verified in the same manner. □

Next, we derive the bilinear form of the elliptic coupled KP system. Note that the simplest cases of (3.17) give us the bilinear transform[1]

$$u_n = \partial_1 \ln \tau_n, \quad v_n = \frac{\tau_{n+1}}{\tau_n} \quad \text{and} \quad w_n = \frac{\tau_{n-1}}{\tau_n}. \tag{4.3}$$

The transformation (4.3) allows us to find a closed-form bilinear system in terms of the τ-function. The result is presented as the theorem below.

Theorem 4.2. *The τ-function defined by* (3.10) *satisfies the following system of bilinear equations:*

$$\left(\mathrm{D}_1^4 - 4\mathrm{D}_1\mathrm{D}_3 + 3\mathrm{D}_2^2\right) \tau_n \cdot \tau_n = 24g\left(\tau_{n+1}\tau_{n-1} - \tau_n^2\right), \tag{4.4a}$$

$$\left(\mathrm{D}_1^3 + 2\mathrm{D}_3 - 3\mathrm{D}_1\mathrm{D}_2 - 18e\mathrm{D}_1\right) \tau_{n+1} \cdot \tau_n = 0, \tag{4.4b}$$

$$\left(\mathrm{D}_1^3 + 2\mathrm{D}_3 + 3\mathrm{D}_1\mathrm{D}_2 - 18e\mathrm{D}_1\right) \tau_{n-1} \cdot \tau_n = 0, \tag{4.4c}$$

where D_j stand for the bilinear derivatives (see appendix A for the definition) with respect to x_j.

Proof. These equations are obtained from (4.2) by using the transformations in (4.3) and identities (A.3) and (A.4). For instance, the bilinear transformation (4.3) reformulates (4.2a) as

$$\partial_1\partial_3 \ln \tau_n = \frac{1}{4}\partial_1^4 \ln \tau_n + \frac{3}{2}(\partial_1^2 \ln \tau_n)^2 + \frac{3}{4}\partial_2^2 \ln \tau_n + 3g\left(1 - \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}\right)$$

in terms of the τ-function, which is exactly the same as (4.4a), due to the logarithmic transformations give by (A.3). Equations (4.4b) and (4.4c) are derived similarly from (4.2b) and (4.2c), respectively, where we need to make use of the bi-logarithmic transformations listed in (A.4). We also comment that these bilinear equations can alternatively be verified directly by following the same procedure of deriving (4.2). This is because equation (3.17) establishes the connection between τ_n and $\boldsymbol{U}_n^{[i,j]}$. □

Remark 4.3. *In fact, the last two equations in the bilinear equations* (4.4) *are equivalent to each other, which means either of them can be omitted. However, we would still like to reserve both equations because here the main idea is to present the bilinear elliptic coupled KP systems as a* (2+1)*-dimensional continuous closed-form system for τ_n, τ_{n+1} and τ_{n-1} in terms of the independent variables x_1, x_2 and x_3, rather than a* $(3+1)$*-dimensional differential-difference system for a single τ-function in terms of the independent variables x_1, x_2, x_3 and n. We also note that equations in* (4.4) *are reparametrisation of those bilinear equations in [5], which, up to the moduli e and g, have also appeared in [40] (see also references therein) as the first few members in the bilinear DKP hierarchy.*

In addition, by introducing the scalar wave function $\phi_n \doteq \boldsymbol{u}_n^{[0]}(\kappa)$, we can also construct from (3.16) the associated linear problem for the nonlinear system (4.2). We conclude the result as the following theorem.

Theorem 4.4. *For an arbitrary solution $\boldsymbol{u}_n(\kappa)$ to the linear integral equation* (3.3)*, the scalar wave function ϕ_n and the potentials u_n, v_n and w_n defined by* (4.1) *satisfy the linear system for ϕ_n as follows*[2]*:*

$$\partial_2\phi_n = -\left[\partial_1^2 + 2\partial_1 u_n - 3e\right]\phi_n + 2\sqrt{g}v_n w_n \phi_{n+1}, \tag{4.5a}$$

$$\partial_3\phi_n = \left[\partial_1^3 + 3(\partial_1 u_n)\partial_1 + \frac{3}{2}\left(\partial_1^2 u_n - \partial_2 u_n\right)\right]\phi_n - 3\sqrt{g}v_n(\partial_1 w_n)\phi_{n+1}, \tag{4.5b}$$

$$\begin{aligned}&\sqrt{g}v_n w_n \phi_{n+1} + \sqrt{g}\phi_{n-1}\\ &\quad= \left[\partial_1^2 - (\partial_1 \ln w_n)\partial_1 + 2\partial_1 u_n - 3e + \frac{1}{2}\left(\partial_1^2 \ln w_n + (\partial_1 \ln w_n)^2 + \partial_2 \ln w_n\right)\right]\phi_n.\end{aligned} \tag{4.5c}$$

[1]The last two equations imply that v_n and w_n are connected with each other through $v_n w_{n+1} = 1$. However, our aim is to present the elliptic coupled KP system as a $(2+1)$-dimensional continuous integrable model, where there is no dynamical evolution with respect to the discrete independent variable n. Hence, v_n and w_n are treated as two separate variables, as we have done in (4.2).

[2]These equations are effectively reparametrisation of the linear equations listed in [5]. We also remark that a typo in the third equation in [5] has been fixed here.

Proof. The idea of the proof is very similar to that of verifying (4.2). We start with deriving from (3.16) the algebraic expressions of the derivatives $\partial_{\mathrm{j}}\boldsymbol{u}_n^{[i]}(\kappa)$. Following the elliptic index labels introduced by (2.11), we have

$$\partial_{\mathrm{j}}\boldsymbol{u}_n^{[2i]}(\kappa) = \partial_{\mathrm{j}}[\boldsymbol{L}^i\boldsymbol{u}_n(\kappa)]^{(0)} = [\boldsymbol{L}^i\partial_{\mathrm{j}}\boldsymbol{u}_n(\kappa)]^{(0)}$$

and

$$\partial_{\mathrm{j}}\boldsymbol{u}_n^{[2i+1]}(\kappa) = \partial_{\mathrm{j}}[\boldsymbol{\Lambda}\boldsymbol{L}^i\boldsymbol{u}_n(\kappa)]^{(0)} = [\boldsymbol{\Lambda}\boldsymbol{L}^i\partial_{\mathrm{j}}\boldsymbol{u}_n(\kappa)]^{(0)},$$

which can be further expressed by algebraic expressions of $\boldsymbol{U}_n^{[i,j]}$ and $\boldsymbol{u}_n^{[i]}$ by replacing $\partial_{\mathrm{j}}\boldsymbol{u}_n(\kappa)$ with the help of (3.16a) and (3.16b). For instance, we have for $i = 0$ the simplest relations as follows:

$$\begin{aligned}
\partial_1\boldsymbol{u}_n^{[0]}(\kappa) &= \boldsymbol{u}_n^{[1]}(\kappa) + \boldsymbol{U}_n^{[0,2]}\boldsymbol{u}_n^{[0]}(\kappa),\\
\partial_2\boldsymbol{u}_n^{[0]}(\kappa) &= \left(1 + \boldsymbol{U}_n^{[0,1]}\right)\boldsymbol{u}_n^{[2]}(\kappa) - \boldsymbol{U}_n^{[0,2]}\boldsymbol{u}_n^{[1]}(\kappa) + \boldsymbol{U}_n^{[0,3]}\boldsymbol{u}_n^{[0]}(\kappa) - g\boldsymbol{u}_n^{[-2]}(\kappa),\\
\boldsymbol{u}_{n+1}^{[0]}(\kappa) &= \frac{1}{\sqrt{g}}\left(1 + \boldsymbol{U}_{n+1}^{[0,1]}\right)\boldsymbol{u}_n^{[2]}(\kappa),\\
\boldsymbol{u}_{n-1}^{[0]}(\kappa) &= \frac{1}{\sqrt{g}}\boldsymbol{U}_{n-1}^{[0,2]}\boldsymbol{u}_n^{[1]}(\kappa) - \frac{1}{\sqrt{g}}\boldsymbol{U}_{n-1}^{[0,3]}\boldsymbol{u}_n^{[0]}(\kappa) + \sqrt{g}\boldsymbol{u}_n^{[-2]}(\kappa).
\end{aligned}$$

The formulae for the higher-order derivatives of $\boldsymbol{u}_n^{[i]}$, such as $\partial_{\mathrm{j}}^2\boldsymbol{u}_n^{(i)}$, $\partial_{\mathrm{j}}^3\boldsymbol{u}_n^{(i)}$, etc. for $\mathrm{j} \in \mathbb{Z}^+$ and $i, j \in \mathbb{Z}$ are obtained by iteration. Following such an algorithm, we are able to verify the linear equations in (4.5). For example, by replacing all the derivatives by algebraic expressions of $\boldsymbol{U}_n^{[i,j]}$ and $\boldsymbol{u}_n^{[i]}(\kappa)$, direct calculation shows that

$$\partial_2\boldsymbol{u}_n^{[0]} + \partial_1^2\boldsymbol{u}_n^{[0]} + 2\left(\boldsymbol{U}_n^{[3,0]} + \boldsymbol{U}_n^{[2,1]} - \boldsymbol{U}_n^{[2,0]}\boldsymbol{U}_n^{[2,0]} + \boldsymbol{U}_n^{[2,2]}\boldsymbol{U}_n^{[0,0]}\right)\boldsymbol{u}_n^{[0]} - 3e\boldsymbol{u}_n^{[0]} - 2\sqrt{g}v_nw_n\boldsymbol{u}_{n+1}^{[0]}$$

is identically zero in virtue of the antisymmetry condition $\boldsymbol{U}_n^{[i,j]} = -\boldsymbol{U}_n^{[j,i]}$ as well as the algebraic relation (which helps us to eliminate those shifted variables $U_{n+1}^{[i,j]}$ in terms of n) that follows from taking $[i, j]$-labels of (3.14c). Equations (4.5b) and (4.5c) are proven similarly. □

Remark 4.5. *The third equation in* (4.5) *allows us to rewrite the other two equations as a two-component linear system composed of*

$$\partial_2\begin{pmatrix}\phi_n\\ \phi_{n+1}\end{pmatrix} = \boldsymbol{P}\begin{pmatrix}\phi_n\\ \phi_{n+1}\end{pmatrix} \quad \textit{and} \quad \partial_3\begin{pmatrix}\phi_n\\ \phi_{n+1}\end{pmatrix} = \boldsymbol{Q}\begin{pmatrix}\phi_n\\ \phi_{n+1}\end{pmatrix}, \tag{4.6}$$

where $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *are* 2×2 *matrix operators given by*

$$\boldsymbol{P} = \begin{pmatrix} -\partial_1^2 - 2\partial_1 u_n + 3e & 2\sqrt{g}v_nw_n \\ -2\sqrt{g} & \partial_1^2 + 2(\partial_1\ln v_n)\partial_1 + 2\partial_1 u_n - 3e + \partial_1^2\ln v_n + (\partial_1\ln v_n)^2 - \partial_2\ln v_n \end{pmatrix} \tag{4.7a}$$

and

$$\boldsymbol{Q} = \begin{pmatrix} \partial_1^3 + 3(\partial_1 u_n)\partial_1 + \frac{3}{2}\left(\partial_1^2 u_n - \partial_2 u_n\right) & -3\sqrt{g}v_n(\partial_1 w_n) \\ -3\sqrt{g}\partial_1\ln v_n & * \end{pmatrix}, \tag{4.7b}$$

respectively, in which

$$\begin{aligned}
* = {}& \partial_1^3 + 3(\partial_1\ln v_n)\partial_1^2 + 3\left(\partial_1 u_n + \partial_1^2\ln v_n + (\partial_1\ln v_n)^2\right)\partial_1\\
&+ \frac{3}{2}\left(\partial_1^2 u_n - \partial_2 u_n\right) + 3(\partial_1\ln v_n)(2\partial_1 u_n - 3e)\\
&+ \frac{3}{2}(\partial_1 + \partial_1\ln v_n)\left(\partial_1^2\ln v_n + (\partial_1\ln v_n)^2 - \partial_2\ln v_n\right).
\end{aligned}$$

The linear equations in (4.6) *form the Lax pair for the elliptic coupled KP system, namely the zero curvature equation*

$$\partial_3\boldsymbol{P} - \partial_2\boldsymbol{Q} + [\boldsymbol{P}, \boldsymbol{Q}] = 0$$

with $[\boldsymbol{P}, \boldsymbol{Q}] \doteq \boldsymbol{PQ} - \boldsymbol{QP}$ *yields the nonlinear system* (4.2).

5. Dimensional reductions

In this section, we further discuss the relevant $(1+1)$-dimensional elliptic integrable models by performing dimensional reductions on the elliptic coupled KP system (4.2). This is realised by imposing a restriction of the form $\mathcal{F}(\kappa,\kappa')=0$ on the spectral variables in the linear integral equation (3.3), such that the effective plane wave factor $\rho_n(\kappa)\rho_n(\kappa')$ turns out to be independent of a certain argument x_j. Recall that the dynamics of the elliptic coupled KP system completely relies on the effective plane wave factor

$$\rho_n(\kappa)\rho_n(\kappa')=\exp\left\{\sum_{j=0}^{\infty}\left(k^{2j+1}+k'^{2j+1}\right)x_{2j+1}+\sum_{j=1}^{\infty}\left[K^j-\left(\frac{g}{K}\right)^j+K'^j-\left(\frac{g}{K'}\right)^j\right]x_{2j}\right\}\left(\frac{KK'}{g}\right)^n. \tag{5.1}$$

Thus, we can impose

$$k^{2j_0+1}+k'^{2j_0+1}=0 \quad\Rightarrow\quad k+k'=0 \quad\text{or}\quad \sum_{i=0}^{2j_0}k^{2j_0-i}(-k')^i=0$$

and

$$K^{j_0}-\left(\frac{g}{K}\right)^{j_0}+K'^{j_0}-\left(\frac{g}{K'}\right)^{j_0}=0 \quad\Rightarrow\quad \left(\frac{KK'}{g}\right)^{j_0}=1 \quad\text{or}\quad K^{j_0}+K'^{j_0}=0,$$

in order to realise x_{2j_0+1}- and x_{2j_0}-independence, respectively. But in practice, we are only allowed to set

$$\mathcal{F}_{2j_0+1}(\kappa,\kappa')\doteq\sum_{i=0}^{2j_0}k^{2j_0-i}(-k')^i=0 \tag{5.2}$$

and

$$\mathcal{F}_{2j_0}(\kappa,\kappa')\doteq K^{j_0}+K'^{j_0}=0 \tag{5.3}$$

to respectively perform x_{2j_0+1}- and x_{2j_0}-reductions. This is because either $k+k'=0$ or $(KK'/g)^{j_0}=1$ will lead to trivialities; to be more precise, useful independent variables are also eliminated in these two cases. Below we shall give the simplest examples including the elliptic coupled KdV and BSQ systems arising from the x_2- and x_3-reductions of (4.2).

To construct the elliptic coupled KdV system, we set

$$K+K'=0,$$

namely the $j_0=1$ case of (5.3) leading to the x_2-independence. This in turn implies

$$k^2+k'^2=6e$$

due to the elliptic curve relation (2.1). The constraint on the spectral points (k,K) and (k',K') results in $\partial_{4j+2}\rho_n(\kappa)\rho_n(\kappa')=0$. From the definition of $\boldsymbol{C}_n$, i.e. (3.5), we can easily prove $\partial_{4j+2}\boldsymbol{C}_n=0$ and subsequently $\partial_{4j+2}\boldsymbol{U}_n=0$ by (3.9). Recall that the τ-function and the potentials are defined as (3.10) and (4.1). We obtain reduction conditions

$$\partial_{4j+2}\tau_n=\partial_{4j+2}u_n=\partial_{4j+2}v_n=\partial_{4j+2}w_n=0$$

for $j=0,1,2,\cdots$. Performing such reductions on the elliptic coupled KP system (4.2), we obtain a coupled system

$$\partial_3u_n=\frac{1}{4}\partial_1^3u_n+\frac{3}{2}(\partial_1u_n)^2+3g(1-v_nw_n), \tag{5.4a}$$

$$\partial_3v_n=-\frac{1}{2}\partial_1^3v_n-3(\partial_1u_n-3e)\partial_1v_n, \tag{5.4b}$$

$$\partial_3w_n=-\frac{1}{2}\partial_1^3w_n-3(\partial_1u_n-3e)\partial_1w_n, \tag{5.4c}$$

which we shall refer to as the elliptic coupled KdV system. The bilinear form of equation (5.4) is obtained from (4.4), which is a multi-component system given by

$$\left(\mathrm{D}_1^4-4\mathrm{D}_1\mathrm{D}_3\right)\tau_n\cdot\tau_n=24g\left(\tau_{n+1}\tau_{n-1}-\tau_n^2\right), \tag{5.5a}$$

$$\left(\mathrm{D}_1^3+2\mathrm{D}_3-18e\mathrm{D}_1\right)\tau_{n+1}\cdot\tau_n=0, \tag{5.5b}$$

$$\left(\mathrm{D}_1^3+2\mathrm{D}_3-18e\mathrm{D}_1\right)\tau_{n-1}\cdot\tau_n=0. \tag{5.5c}$$

Notice that $\partial_{4j+2}\boldsymbol{U}_n=0$ and $\boldsymbol{u}(\kappa)$ satisfies (3.8). We can further derive

$$\partial_{4j+2}\phi_n=\left(K^{2j+1}-(g/K)^{2j+1}\right)\phi_n.$$

This provides us with a reduction on the Lax pair (4.6). Therefore, the Lax pair of the elliptic coupled KdV system (5.4) is composed of

$$\boldsymbol{P}\begin{pmatrix}\phi_n\\ \phi_{n+1}\end{pmatrix}=(K-g/K)\begin{pmatrix}\phi_n\\ \phi_{n+1}\end{pmatrix}\quad\text{and}\quad\partial_3\begin{pmatrix}\phi_n\\ \phi_{n+1}\end{pmatrix}=\boldsymbol{Q}\begin{pmatrix}\phi_n\\ \phi_{n+1}\end{pmatrix}. \tag{5.6}$$

Here the Lax matrices are given by

$$\boldsymbol{P}=\begin{pmatrix}-\partial_1^2-2\partial_1u_n+3e & 2\sqrt{g}v_nw_n\\ -2\sqrt{g} & \partial_1^2+2(\partial_1\ln v_n)\partial_1+2\partial_1u_n-3e+\partial_1^2\ln v_n+(\partial_1\ln v_n)^2\end{pmatrix} \tag{5.7a}$$

and

$$\boldsymbol{Q}=\begin{pmatrix}\partial_1^3+3(\partial_1u_n)\partial_1+\frac{3}{2}\left(\partial_1^2u_n-\partial_2u_n\right) & -3\sqrt{g}v_n(\partial_1w_n)\\ -3\sqrt{g}\partial_1\ln v_n & *\end{pmatrix}, \tag{5.7b}$$

respectively, in which

$$\begin{aligned}*={}&\partial_1^3+3(\partial_1\ln v_n)\partial_1^2+3\left(\partial_1u_n+\partial_1^2\ln v_n+(\partial_1\ln v_n)^2\right)\partial_1\\&+\frac{3}{2}\partial_1^2u_n+3(\partial_1\ln v_n)(2\partial_1u_n-3e)+\frac{3}{2}(\partial_1+\partial_1\ln v_n)\left(\partial_1^2\ln v_n+(\partial_1\ln v_n)^2\right).\end{aligned}$$

For the elliptic coupled BSQ system, we set $j_0=1$ in (5.2), namely

$$k^2-kk'+k'^2=0,$$

in order to induce the x_3-independence. This simultaneously leads to the fact that K and K' also obey a restriction in the form of

$$\left(K+3e+\frac{g}{K}\right)^2+\left(K+3e+\frac{g}{K}\right)\left(K'+3e+\frac{g}{K'}\right)+\left(K'+3e+\frac{g}{K'}\right)^2=0,$$

which follows form the curve relation (2.1). In this case, the reduction conditions are given by

$$\partial_{3j}\tau_n=\partial_{3j}u_n=\partial_{3j}v_n=\partial_{3j}w_n=0\quad\text{as well as}\quad\partial_{3j}\phi_n=k^{3j}\phi_n$$

for $j\in\mathbb{Z}^+$. Therefore, we obtain from (4.2) the elliptic coupled BSQ system

$$\partial_2^2u_n=-\frac{1}{3}\partial_1^4u_n-4(\partial_1u_n)\partial_1^2u_n+4g\partial_1(v_nw_n), \tag{5.8a}$$

$$\partial_1\partial_2v_n=\frac{1}{3}\partial_1^3v_n+2(\partial_1u_n-3e)\partial_1v_n-2(\partial_2u_n)v_n, \tag{5.8b}$$

$$\partial_1\partial_2w_n=-\frac{1}{3}\partial_1^3w_n-2(\partial_1u_n-3e)\partial_1w_n-2(\partial_2u_n)w_n. \tag{5.8c}$$

The corresponding bilinear form is a consequence of the (4.4), taking the form of the following:

$$\left(\mathrm{D}_1^4+3\mathrm{D}_2^2\right)\tau_n\cdot\tau_n=24g\left(\tau_{n+1}\tau_{n-1}-\tau_n^2\right), \tag{5.9a}$$

$$\left(\mathrm{D}_1^3-3\mathrm{D}_1\mathrm{D}_2-18e\mathrm{D}_1\right)\tau_{n+1}\cdot\tau_n=0, \tag{5.9b}$$

$$\left(\mathrm{D}_1^3+3\mathrm{D}_1\mathrm{D}_2-18e\mathrm{D}_1\right)\tau_{n-1}\cdot\tau_n=0. \tag{5.9c}$$

The Lax pair of the elliptic coupled BSQ equation (5.8) is composed of

$$\boldsymbol{Q}\begin{pmatrix}\phi_n\\ \phi_{n+1}\end{pmatrix}=k^3\begin{pmatrix}\phi_n\\ \phi_{n+1}\end{pmatrix}\quad\text{and}\quad\partial_2\begin{pmatrix}\phi_n\\ \phi_{n+1}\end{pmatrix}=\boldsymbol{P}\begin{pmatrix}\phi_n\\ \phi_{n+1}\end{pmatrix}, \tag{5.10}$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are the same as the ones given by (4.7).

We note that the third equations in (5.5) and (5.9) can be omitted, since they are equivalent to their respective second equations. But here we still reserve these equations in order to treat both (5.5) and (5.9) as $(1+1)$-dimensional three-component partial-differential systems of τ_n, τ_{n+1} and τ_{n-1}, instead of $(2+1)$-dimensional differential-difference systems.

6. Multi-soliton solutions to the elliptic coupled KP system

To further explore the integrability of the elliptic coupled KP system, we now specify a class of concrete solutions (i.e. elliptic multi-soliton solutions) to (4.2). They are constructed directly from the DL framework by specifying a discrete measure associated with distinct simple poles at values κ_ν for $\nu = 1, \ldots, N$ of the uniformising spectral parameter, associated with distinct points (k_ν, K_ν) on the elliptic curve (2.1). This will lead the linear integral equation (3.3) to

$$\boldsymbol{u}_n(\kappa) + \sum_{\nu,\nu'=1}^{N} c_{\nu,\nu'} \rho_n(\kappa)\Omega(\kappa,\kappa_{\nu'})\rho_n(\kappa_{\nu'})\boldsymbol{u}_n(\kappa_\nu) = \rho_n(\kappa)\boldsymbol{c}(\kappa), \tag{6.1}$$

where the coefficients $c_{\nu,\nu'}$ are skew-symmetric in terms of the indices, i.e. $c_{\nu,\nu'} = -c_{\nu',\nu}$. Let us introduce the finite matrices $\boldsymbol{M} = (M_{\mu,\nu})_{N\times N}$ and $\boldsymbol{R} = (R_{\mu,\nu})_{N\times N}$ whose respective entries are defined as

$$M_{\mu,\nu} \doteq \Omega(\kappa_\mu,\kappa_\nu) = \frac{K_\mu - K_\nu}{k_\mu + k_\nu} = \frac{k_\mu - k_\nu}{1 - \dfrac{g}{K_\mu K_\nu}}, \tag{6.2}$$

and

$$\begin{aligned} R_{\mu,\nu} &\doteq c_{\nu,\mu}\rho_n(\kappa_\nu)\rho_n(\kappa_\mu) \\ &= c_{\nu,\mu}\exp\left\{\sum_{j=0}^{\infty}\left(k_\mu^{2j+1} + k_\nu^{2j+1}\right)x_{2j+1} + \sum_{j=1}^{\infty}\left[K_\mu^j - \left(\frac{g}{K_\mu}\right)^j + K_\nu^j - \left(\frac{g}{K_\nu}\right)^j\right]x_{2j}\right\}\left(\frac{K_\mu K_\nu}{g}\right)^n. \end{aligned} \tag{6.3}$$

Equations (6.2) and (6.3) show that both $\boldsymbol{M}$ and $\boldsymbol{R}$ are skew-symmetric. By setting $\kappa = \kappa_\mu$ for $\mu = 1, 2, \cdots, N$, we can rewrite (6.1) as

$$\boldsymbol{\phi}_\mu + \sum_{\nu,\nu'=1}^{N} M_{\mu,\nu'}R_{\nu',\nu}\boldsymbol{\phi}_\nu = \boldsymbol{c}(\kappa_\mu),$$

for the infinite vector $\boldsymbol{\phi}_\mu \doteq \boldsymbol{u}_n(\kappa_\mu)/\rho_n(\kappa_\mu)$, which can, in turn, be reformulated as a linear problem for an N-component block vector $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_N)$ given by

$$(\boldsymbol{I} + \boldsymbol{M}\boldsymbol{R})\begin{pmatrix}\boldsymbol{\phi}_1\\ \boldsymbol{\phi}_2\\ \vdots\\ \boldsymbol{\phi}_N\end{pmatrix} = \begin{pmatrix}\boldsymbol{c}(\kappa_1)\\ \boldsymbol{c}(\kappa_2)\\ \vdots\\ \boldsymbol{c}(\kappa_N)\end{pmatrix},$$

namely

$$(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_N) = (\boldsymbol{c}(\kappa_1), \boldsymbol{c}(\kappa_2), \cdots, \boldsymbol{c}(\kappa_N))(\boldsymbol{I} + \boldsymbol{R}\boldsymbol{M})^{-1}, \tag{6.4}$$

where $\boldsymbol{I} = \boldsymbol{I}_{N\times N}$ denotes the $N\times N$ identity matrix. Meanwhile, the discrete measure also reduces (3.7) to

$$\boldsymbol{U}_n = \sum_{\mu,\mu'=1}^{N} c_{\mu,\mu'}\boldsymbol{u}_n(\kappa_\mu)\,{}^t\boldsymbol{c}(\kappa_{\mu'})\rho_n(\kappa_{\mu'}) = \sum_{\mu,\mu'=1}^{N} \boldsymbol{\phi}_\mu R_{\mu',\mu}\,{}^t\boldsymbol{c}(\kappa_{\mu'}) \tag{6.5}$$

$$= -(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_N)\boldsymbol{R}\begin{pmatrix}{}^t\boldsymbol{c}(\kappa_1)\\ {}^t\boldsymbol{c}(\kappa_2)\\ \vdots\\ {}^t\boldsymbol{c}(\kappa_N)\end{pmatrix}. \tag{6.6}$$

Equation (6.5) together with (6.4) allows us to write down the final formula of the infinite matrix $\boldsymbol{U}_n$ as follows:

$$\boldsymbol{U}_n = -(\boldsymbol{c}(\kappa_1), \boldsymbol{c}(\kappa_2), \cdots, \boldsymbol{c}(\kappa_N))(\boldsymbol{I} + \boldsymbol{R}\boldsymbol{M})^{-1}\boldsymbol{R} \begin{pmatrix} {}^t\boldsymbol{c}(\kappa_1) \\ {}^t\boldsymbol{c}(\kappa_2) \\ \vdots \\ {}^t\boldsymbol{c}(\kappa_N) \end{pmatrix}. \tag{6.7}$$

Notice that the potentials u_n, v_n and w_n of the elliptic coupled KP system purely rely on the entries of $\boldsymbol{U}_n$, see (4.1). We can therefore derive the general formulae for the Cauchy matrix solutions to (4.2).

Theorem 6.1. *The elliptic coupled KP system* (4.2) *possesses Cauchy matrix solutions*

$$u_n = {}^t\boldsymbol{k}_2(\boldsymbol{I} + \boldsymbol{R}\boldsymbol{M})^{-1}\boldsymbol{R}\,\boldsymbol{k}_0, \tag{6.8a}$$

$$v_n = 1 - g^{-1}\,{}^t\boldsymbol{k}_3(\boldsymbol{I} + \boldsymbol{R}\boldsymbol{M})^{-1}\boldsymbol{R}\,\boldsymbol{k}_2, \tag{6.8b}$$

$$w_n = 1 + {}^t\boldsymbol{k}_1(\boldsymbol{I} + \boldsymbol{R}\boldsymbol{M})^{-1}\boldsymbol{k}_0, \tag{6.8c}$$

where $\boldsymbol{M}$ and $\boldsymbol{R}$ are $N \times N$ matrices introduced in (6.2) *and* (6.3), *and $\boldsymbol{k}_i$ for $i = 0, 1, 2, 3$ are N-component column vectors defined as*

$$\boldsymbol{k}_0 = {}^t(1, 1, \cdots, 1), \quad \boldsymbol{k}_1 = {}^t(k_1, k_2, \cdots, k_N),$$
$$\boldsymbol{k}_2 = {}^t(K_1, K_2, \cdots, K_N) \quad \textit{and} \quad \boldsymbol{k}_3 = {}^t(k_1K_1, k_2K_2, \cdots, k_NK_N).$$

For these solutions, the square of the τ-function (3.10) is of the form

$$\tau_n^2 = \det(\boldsymbol{I} + \boldsymbol{M}\boldsymbol{R}) = \det(\boldsymbol{I} + \boldsymbol{R}\boldsymbol{M}) = \det\left(\begin{array}{c:c} \boldsymbol{M} & \boldsymbol{I} \\ \hdashline -\boldsymbol{I} & \bar{\boldsymbol{R}} \end{array}\right). \tag{6.9}$$

Since both $\boldsymbol{M}$ and $\boldsymbol{R}$ are skew-symmetric matrices the latter $N \times N$ determinant is a square of a $(2N-1)\times(2N-1)$ Pfaffian, and hence the τ-function itself can be written as a Pfaffian[3] (see appendix B for the definition) that

$$\tau_n = \mathrm{pf}(\boldsymbol{M}|\boldsymbol{R})$$
$$\doteq \left|\begin{array}{ccccccccc} M_{1,2} & M_{1,3} & \cdots & M_{1,N} & 1 & 0 & \cdots & 0 & 0 \\ & M_{2,3} & \cdots & M_{2,N} & 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & M_{N-1,N} & 0 & 0 & \cdots & 1 & 0 \\ & & & & 0 & 0 & \cdots & 0 & 1 \\ & & & & & R_{1,2} & \cdots & R_{1,N-1} & R_{1,N} \\ & & & & & & \ddots & \vdots & \vdots \\ & & & & & & & R_{N-2,N-1} & R_{N-2,N} \\ & & & & & & & & R_{N-1,N} \end{array}\right| (-1)^{N(N-1)/2}. \tag{6.10}$$

In order to give an explicit expression for this Pfaffian (a definition is given in appendix B), we need an expansion formula for the Pfaffian, similar to the expansion for a determinant of the form $\det(\boldsymbol{1} + \boldsymbol{M}\boldsymbol{R})$ in terms of the matrix invariants of the matrix $\boldsymbol{M}\boldsymbol{R}$, but now in terms of Pfaffians of $\boldsymbol{M}$ and $\boldsymbol{R}$ separately. Such an expansion formula is given by (B.3) in appendix B. Furthermore, we can make use of the following lemma which amounts to a Pfaffian analogue of the Frobenius (i.e. elliptic Cauchy, cf. [9]) determinant formula:

Lemma 6.2. *The Pfaffian of an elliptic Cauchy matrix $\boldsymbol{M}$ with entries $M_{i,j} = \frac{K_i - K_j}{k_i + k_j}$ for $i, j = 1, 2, \ldots, m$, where (k_i, K_i) are distinct points on the elliptic curve* (2.1), *is given by*

$$\mathrm{pf}(\boldsymbol{M}) = \frac{g^{m(m-2)/8}}{\left(\prod_{i=1}^{m} K_i\right)^{(m-2)/2}} \prod_{1 \le i < j \le m} \frac{K_i - K_j}{k_i + k_j}, \tag{6.11}$$

for m is even, while the Pfaffian vanishes (by definition) for m is odd.

[3]We refer the reader to [27] for such a notation (i.e. the triangular array) of Pfaffian. Here the postfactor is chosen to comply with a normalisation of the τ-function such that it has the form $\tau_n = 1 +$ perturbation.

Essentially formula (6.11) in the lemma has appeared in [4] without a proof. A proof based on the Frobenius determinant formula for the elliptic Cauchy matrix is given in appendix C.

Together with the expansion formula (B.3) this allows us to write an explicit 'Hirota-type' formula[4] for the elliptic N-soliton solution of the elliptic coupled KP system. Combining the results of appendices B and C, we thus obtain the following explicit expression for the τ-function of the N-soliton solution to the elliptic coupled KP system. In fact, the expansion formula (B.3) gives a finite sum of terms, each of which is a product of individual sub-Pfaffians of the matrices $\boldsymbol{M}$ and $\boldsymbol{R}$. The result is given in the following theorem.

Theorem 6.3. *The τ-function for the N-soliton solution to the elliptic coupled KP system* (1.1) *is given by*

$$\tau_n = 1 + \sum_{m\in J} \sum_{1\leq i_1<i_2<\cdots<i_m\leq N} \frac{g^{m(m-2)/8}}{\left(\prod_{\nu=1}^{m} K_{i_\nu}\right)^{(m-2)/2}} \left(\prod_{1\leq\nu<\nu'\leq m} \frac{K_{i_\nu}-K_{i_{\nu'}}}{k_{i_\nu}+k_{i_{\nu'}}}\right) \mathrm{pf}(\mathfrak{C}_{i_1,\cdots,i_m}) \left(\prod_{\nu=1}^{m} \rho_n(\kappa_{i_\nu})\right) \tag{6.12}$$

with $J=\{j=2i|i=1,2,\cdots,[N/2]\}$, where $\mathfrak{C}_{i_1,\cdots,i_m}$ denotes the sub-matrix of the coefficient matrix $\mathfrak{C}=(c_{\mu,\nu})_{N\times N}$ obtained from selecting from it the rows and columns labelled by $i_1,\cdots,i_m$, and the exponential factors ρ_n are given in (3.1). *The solutions u_n, v_n and w_n of the elliptic coupled KP system* (4.2) *are subsequently inferred from the expression* (6.12) *into* (4.3).

Remark 6.4. *Above we present the formula for the elliptic N-soliton solution parametrised by the curve* (2.1). *The degenerate solutions are obtained when any two of the branch points coincide with each other. The situation when $\mathrm{e}=\mathrm{e}'$ or $\mathrm{e}=\mathrm{e}''$ corresponds to $g=0$ due to* (2.2). *In these two cases, equation* (1.1) *reduces to the scalar KP equation, but simultaneously the elliptic N-soliton solution turns out to be a trivial one when $g=0$. While the third possibility $\mathrm{e}'=\mathrm{e}''$ leads to $9\mathrm{e}^2=4g$, in which case we obtain a degenerate curve*

$$k^2 = K + 3e + \frac{9}{4}\frac{e^2}{K} = \left(\sqrt{K} + \frac{3}{2}\frac{e}{\sqrt{K}}\right)^2 \tag{6.13}$$

as well as the corresponding degeneration of (1.1) *and its solution.*

Below we give examples for $N=2,3,4,5$ ($N=1$ results in the seed solution $\tau_n=1$), namely

$N=2$:

$$\tau_n = 1 + \frac{K_1-K_2}{k_1+k_2} c_{1,2}\rho_n(\kappa_1)\rho_n(\kappa_2);$$

$N=3$:

$$\tau_n = 1 + \sum_{1\leq i_1<i_2\leq 3} \frac{K_{i_1}-K_{i_2}}{k_{i_1}+k_{i_2}} c_{i_1,i_2}\rho_n(\kappa_{i_1})\rho_n(\kappa_{i_2});$$

$N=4$:

$$\begin{aligned}\tau_n = 1 &+ \sum_{1\leq i_1<i_2\leq 4} \frac{K_{i_1}-K_{i_2}}{k_{i_1}+k_{i_2}} c_{i_1,i_2}\rho_n(\kappa_{i_1})\rho_n(\kappa_{i_2}) \\ &+ \frac{g}{\prod_{i=1}^{4} K_i}\left(\prod_{1\leq i_1<i_2\leq 4} \frac{K_{i_1}-K_{i_2}}{k_{i_1}+k_{i_2}}\right)(c_{1,2}c_{3,4}-c_{1,3}c_{2,4}+c_{1,4}c_{2,3})\left(\prod_{i=1}^{4}\rho_n(\kappa_i)\right);\end{aligned}$$

[4]Note that taking the square root of the corresponding expansion of the determinant expansion does not necessarily lead to a finite expansion formula, while the Pfaffian analogue (B.3) does provide one.

$N=5$:

$$\tau_n = 1 + \sum_{1\leq i_1<i_2\leq 5} \frac{K_{i_1}-K_{i_2}}{k_{i_1}+k_{i_2}} c_{i_1,i_2}\rho_n(\kappa_{i_1})\rho_n(\kappa_{i_2})$$
$$+ \sum_{1\leq i_1<i_2<i_3<i_4\leq 5} \frac{g}{\prod_{\nu=1}^{4} K_{i_\nu}} \left(\prod_{1\leq\nu<\nu'\leq 4} \frac{K_{i_\nu}-K_{i_{\nu'}}}{k_{i_\nu}+k_{i_{\nu'}}} \right)$$
$$\times \left(c_{i_1,i_2}c_{i_3,i_4} - c_{i_1,i_3}c_{i_2,i_4} + c_{i_1,i_4}c_{i_2,i_3}\right) \left(\prod_{\nu=1}^{4} \rho_n(\kappa_{i_\nu}) \right).$$

We observe from the τ-function (6.12) as well as the examples that the solutions for even N resemble those of the BKP hierarchy apart from extra terms as well as the elliptic kernel and plane wave factors. For instance, setting $c_{1,3}=c_{2,4}=c_{1,4}=c_{2,3}=0$ and introducing $d_{i,j}$ (which are effectively arbitrary constrants) determined by

$$\exp(d_{i,j}) \doteq c_{i,j}\frac{K_i-K_j}{k_i+k_j}.$$

in the formula for $N=4$ yields the a particular solution to the bilinear system (4.4) as follows:

$$\tau_n = 1 + \exp(\xi_1+\xi_2+d_{1,2}) + \exp(\xi_3+\xi_4+d_{3,4})$$
$$+ \frac{g}{K_1K_2K_3K_4}\frac{(K_1-K_3)(K_1-K_4)(K_2-K_3)(K_2-K_4)}{(k_1+k_3)(k_1+k_4)(k_2+k_3)(k_2+k_4)}\exp(\xi_1+\xi_2+\xi_3+\xi_4+d_{1,2}+d_{3,4})$$

in which the plane wave factor is given by

$$\exp(\xi_i) \doteq \exp\left\{ \sum_{j=0}^{\infty} k_i^{2j+1}x_{2j+1} + \sum_{j=1}^{\infty}\left[K_i^j - \left(\frac{g}{K_i}\right)^j\right]x_{2j} \right\} \left(\frac{K_i}{\sqrt{g}}\right)^n.$$

While the 'two-soliton' solution to the bilinear BKP hierarchy, including the first member

$$(\mathrm{D}_1^6 - 5\mathrm{D}_1^3\mathrm{D}_3 - 5\mathrm{D}_3^2 + 9\mathrm{D}_1\mathrm{D}_5)\tau\cdot\tau = 0,$$

takes the form of (cf. [3] and also [16])

$$\tau = 1 + \exp(\xi_1+\xi_2+d_{1,2}) + \exp(\xi_3+\xi_4+d_{3,4})$$
$$+ \frac{(k_1-k_3)(k_1-k_4)(k_2-k_3)(k_2-k_4)}{(k_1+k_3)(k_1+k_4)(k_2+k_3)(k_2+k_4)}\exp(\xi_1+\xi_2+\xi_3+\xi_4+d_{1,2}+d_{3,4})$$

with plane wave factors

$$\exp(\xi_i) \doteq \exp\left\{ \sum_{j=0}^{\infty} k_i^{2j+1}x_{2j+1} \right\}$$

and arbitrary constants $d_{i,j}$. The main difference here is that the linear dispersion is parametrised by the elliptic curve (2.1), and also, an elliptic phase shift term (as a consequence of the significant formula (6.11)) is involved. They together describe an elliptic-type (which is remarkable from our viewpoint) soliton interaction. Whilst the formulae for odd N are in a sense the respective parameter extensions of those for even numbers $N-1$.

7. Multi-soliton solutions to the DKP equation with nonzero constant background

We now discuss the connection between the elliptic coupled KP system (1.1) and the DKP equation (1.3), and show how soliton solutions to the DKP equation (1.3) with nonzero constant background are constructed as a byproduct of the result in section 6.

By introducing new variables

$$\mathcal{U} \doteq 2\partial_1 u + c_1, \quad \mathcal{V} \doteq c_2 v \quad \text{and} \quad \mathcal{W} \doteq c_3 w, \tag{7.1}$$

where c_1, c_2 and c_3 are constants obeying $c_1 = -6e$ and $c_2c_3 = g$, we are able to reformulate (1.1) as

$$\partial_3\mathcal{U} = \frac{1}{4}\partial_1^3\mathcal{U} + \frac{3}{2}(\mathcal{U}+6e)\partial_1\mathcal{U} + \frac{3}{4}\partial_1^{-1}\partial_2^2\mathcal{U} - 6\partial_1(\mathcal{V}\mathcal{W}), \tag{7.2a}$$

$$\partial_3\mathcal{V} = -\frac{1}{2}\partial_1^3\mathcal{V} - \frac{3}{2}\mathcal{U}\partial_1\mathcal{V} + \frac{3}{2}\partial_1\partial_2\mathcal{V} + \frac{3}{2}(\partial_1^{-1}\partial_2\mathcal{U})\mathcal{V}, \tag{7.2b}$$

$$\partial_3\mathcal{W} = -\frac{1}{2}\partial_1^3\mathcal{W} - \frac{3}{2}\mathcal{U}\partial_1\mathcal{W} - \frac{3}{2}\partial_1\partial_2\mathcal{W} - \frac{3}{2}(\partial_1^{-1}\partial_2\mathcal{U})\mathcal{W}. \tag{7.2c}$$

Then a transformation between differential operators (which effectively follows from a Galilean transformation between the old and new coordinates) determined by

$$\partial_1 = \partial_x, \quad \partial_2 = \partial_y \quad \text{and} \quad \partial_3 = \partial_t + 9\partial_x, \tag{7.3}$$

leads (7.2) to the DKP equation (1.3). Notice that the solutions we have obtained for the elliptic KP system in section 6 are the ones with background $u = 0$, $v = 1$ and $w = 1$. This implies that here we are able to construct soliton solutions to the DKP equation with nonzero constant background with the help of (7.1) and (7.3). We conclude this as the theorem below.

Theorem 7.1. *The N-soliton solution to the DKP equation* (1.3) *with nonzero constrant background*[5]

$$\mathcal{U} = c_1, \quad \mathcal{V} = c_2, \quad \mathcal{W} = c_3 \quad \textit{for} \quad c_2, c_3 \neq 0 \tag{7.4}$$

reads

$$\mathcal{U} = c_1 + 2\partial_x \ln\tau_n, \quad \mathcal{V} = c_2\frac{\tau_{n+1}}{\tau_n}, \quad \mathcal{W} = c_3\frac{\tau_{n-1}}{\tau_n}, \tag{7.5}$$

where τ_n is given by (6.12) *with the moduli given by $e = -c_1/6$ and $g = c_2c_3$, up to a change of coordinates determined by*

$$x \doteq x_1 + 9x_3, \quad y \doteq x_2 \quad \textit{and} \quad t \doteq x_3.$$

Remark 7.2. *The special case when $c_1^2 = 4c_2c_3$ corresponds to $9e^2 = 4g$. This will lead to the fact that the singularities in the soliton solution are parametrised by the degenerate curve* (6.13)*. We also comment that the multi-soliton solutions to DKP discussed here differ from those with zero background $\mathcal{U} = \mathcal{V} = \mathcal{W} = 0$ in the literature, cf. [14, 18, 23]. In this paper, the values c_1, c_2 and c_3 for the nonzero 'seed solution' will eventually play a role of the moduli in the elliptic curve* (2.1) *that parameterises the spectral points in the multi-solitons.*

For example, by taking $N = 2$ we obtain the simplest nontrivial τ-function for soliton solutions to the DKP equation (1.3) with the nonzero constant background (7.4) as follows:

$$\tau_n = 1 + \exp\left\{(k_1+k_2)x + \left(K_1 - \frac{c_2c_3}{K_1} + K_2 - \frac{c_2c_3}{K_2}\right)y + (k_1+k_2)(k_1^2 - k_1k_2 + k_2^2 - 9)t + d_{1,2}\right\}\left(\frac{K_1K_2}{c_2c_3}\right)^n,$$

where (k_1, K_1) and (k_2, K_2) are points on the elliptic curve

$$k^2 = K - \frac{c_1}{2} + \frac{c_2c_3}{K},$$

and $d_{1,2}$ is an arbitrary constant. In this case, the corresponding components for the 'one-soliton' solution to the DKP equation (1.3) are given by

$$\mathcal{U} = c_1 + 2\partial_x\ln\tau_0, \quad \mathcal{V} = c_2\frac{\tau_1}{\tau_0} \quad \text{and} \quad \mathcal{W} = c_3\frac{\tau_{-1}}{\tau_0},$$

where we have set $n = 0$ in (7.5) for simplicity.

[5]The constant c_1 is not necessarily nonzero, this is because the case $c_1 = 0$ only implies $e = 0$ in the elliptic curve (2.1).

We can similarly construct solutions with nonzero constant background $\mathcal{U} = c_1$, $\mathcal{V} = c_2$, $\mathcal{W} = c_3$ for the coupled KdV system (i.e. the Hirota–Satsuma equation [19])

$$\partial_t \mathcal{U} = \frac{1}{4}\partial_x^3 \mathcal{U} + \frac{3}{2}\mathcal{U}\partial_x \mathcal{U} - 6\partial_x(\mathcal{V}\mathcal{W}), \tag{7.6a}$$

$$\partial_t \mathcal{V} = -\frac{1}{2}\partial_x^3 \mathcal{V} - \frac{3}{2}\mathcal{U}\partial_x \mathcal{V}, \tag{7.6b}$$

$$\partial_t \mathcal{W} = -\frac{1}{2}\partial_x^3 \mathcal{W} - \frac{3}{2}\mathcal{U}\partial_x \mathcal{W} \tag{7.6c}$$

and the coupled BSQ system

$$\partial_y^2 \mathcal{U} = -\frac{1}{3}\partial_1^4 \mathcal{U} - 2\mathcal{U}\partial_1^2 \mathcal{U} - 2(\partial_1 \mathcal{U})^2 + 8\partial_1^2(\mathcal{V}\mathcal{W}), \tag{7.7a}$$

$$\partial_x \partial_y \mathcal{V} = \frac{1}{3}\partial_x^3 \mathcal{V} + \mathcal{U}\partial_x \mathcal{V} - (\partial_x^{-1}\partial_y \mathcal{U})\mathcal{V}, \tag{7.7b}$$

$$\partial_x \partial_y \mathcal{W} = -\frac{1}{3}\partial_x^3 \mathcal{W} - \mathcal{U}\partial_x \mathcal{W} - (\partial_x^{-1}\partial_y \mathcal{U})\mathcal{W}. \tag{7.7c}$$

based on (5.4) and (5.8), respectively. Since the procedure is the same, we omit the detail here.

8. Concluding remarks

The elliptic coupled KP system (1.1) was studied within the DL framework, which provided us with a unified perspective to understand the integrability of the elliptic model. As a consequence, we have constructed its Lax pair and elliptic soliton solutions. The elliptic coupled KdV and BSQ systems were obtained from dimensional reductions of the elliptic coupled KP system, together with their respective Lax pairs. An interesting observation is that from the elliptic coupled KP system we are able to construct a new class of solutions (i.e. solitons with nonzero constant background) to the DKP equation through a Galilean transformation.

Since the DL approach is akin to the Riemann–Hilbert method appearing in the inverse scattering, based on similar singular integral equations, we expect that the present results open the way to a comprehensive study of the initial value problems associated with these elliptic models. Also, we expect that algebro-geometric solutions of higher-genus can be treated, as well as other reductions (e.g. pole reductions) leading to possibly novel finite-dimensional integrable systems. Furthermore, the structures emerging from the DL approach in terms of the infinite matrix representation (3.14) are evidence for the possibility that these nonlinear systems are also integrable in the sense of possessing higher-order symmetries. However, it remains an open problem to explicitly construct these higher-order symmetries. In particular, we may want to establish a Sato-type scheme for the coupled elliptic KP system through a matrix pseudo-differential operator algebra, and construct recursion operators for the elliptic coupled KdV and BSQ systems.

Further aspects of the elliptic KP family of systems remain to be investigated. So far, we have only constructed the single-component KP-type equation and its dimensional reductions in the present paper. It would be interesting to find the multi-component (or matrix) KP-type hierarchy in this family, and simultaneously to see how they are related to the LL equation. In addition to the LL equations, there also exist the KN equation [25] and the elliptic analogue of the Toda equation [24]. Both are elliptic integrable systems that play roles of the master equations in their respective classes. How these particular elliptic systems and their discrete analogues are related to this elliptic family remains a problem for future work. The first step towards this goal might be searching for a discrete analogue of the elliptic coupled KP system (1.1), since the integrability of the discrete (non-elliptic) coupled KP system has been studied in [15] which we believe would bring us insights into the elliptic case.

Acknowledgments

This project was supported by the National Natural Science Foundation of China (grant no. 11901198). WF was also partially sponsored by the Science and Technology Commission of Shanghai Municipality (grant no. 18dz2271000).

Appendix A. Bilinear derivative and logarithmic transformations

In this appendix we briefly recapitulate the notion of Hirota's bilinear derivative and the relevant logarithmic transformations. We also refer the reader to the monographs [17] and [26] for more details regarding bilinear derivatives.

Definition A.1. *Suppose that F and G are differentiable functions of the independent variables x_j for $j \in \mathbb{Z}^+$. The mth-order bilinear derivative of F and G with respect to the argument x_j is defined as*

$$\mathrm{D}_j^m F \cdot G \doteq \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x'_j} \right)^m F(\boldsymbol{x}) G(\boldsymbol{x}') \bigg|_{\boldsymbol{x}'=\boldsymbol{x}} \tag{A.1}$$

for $\boldsymbol{x} = (x_1, x_2, \cdots)$ and $\boldsymbol{x}' = (x'_1, x'_2, \cdots)$.

Remark A.2. *We can alternatively define the bilinear derivative by*

$$\mathrm{e}^{\varepsilon \mathrm{D}_j} F \cdot G = F(\cdots, x_{j-1}, x_j + \varepsilon, x_{j+1}, \cdots) G(\cdots, x_{j-1}, x_j - \varepsilon, x_{j+1}, \cdots), \tag{A.2}$$

in which ε is a parameter. Then we obtain the explicit formulae for $\mathrm{D}_j^m F \cdot G$ from the coefficients of ε^m for $m = 1, 2, \cdots$ in the series expansion.

Bilinear derivatives are closely related to derivatives of logarithmic functions through the so-called logarithmic and bi-logarithmic transformations. These transformations are derived from some fundamental identities in terms of exponents of D- and ∂-operators, see e.g. [17]. Below we only list transformations that are needed in this paper (which can even be proven by definition of bilinear derivative). The logarithmic transformations include the following:

$$\frac{\mathrm{D}_j^2 F \cdot F}{F^2} = 2\partial_j^2 \ln F, \tag{A.3a}$$

$$\frac{\mathrm{D}_j^4 F \cdot F}{F^2} = 2\partial_j^4 \ln F + 12 \left(\partial_j^2 \ln F \right)^2, \tag{A.3b}$$

$$\frac{\mathrm{D}_j^6 F \cdot F}{F^2} = 2\partial_j^6 \ln F + 60 \left(\partial_j^2 \ln F \right) \left(\partial_j^4 \ln F \right) + 120 \left(\partial_j^2 \ln F \right)^3, \tag{A.3c}$$

$$\frac{\mathrm{D}_i \mathrm{D}_j F \cdot F}{F^2} = 2\partial_i \partial_j \ln F. \tag{A.3d}$$

These transformations only involve a single function F. Instead, bi-logarithmic transformations involve two functions F and G. The first few of such transformations are as follows:

$$\frac{\mathrm{D}_j F \cdot G}{FG} = \partial_j \ln \frac{F}{G}, \tag{A.4a}$$

$$\frac{\mathrm{D}_j^2 F \cdot G}{FG} = \left(\partial_j \ln \frac{F}{G} \right)^2 + \partial_j^2 \ln \frac{F}{G} + 2\partial_j^2 \ln G, \tag{A.4b}$$

$$\frac{\mathrm{D}_j^3 F \cdot G}{FG} = \left(\partial_j \ln \frac{F}{G} \right)^3 + \partial_j^3 \ln \frac{F}{G} + 3 \left(\partial_j \ln \frac{F}{G} \right) \left(\partial_j^2 \ln \frac{F}{G} + 2\partial_j^2 \ln G \right), \tag{A.4c}$$

$$\frac{\mathrm{D}_i D_j F \cdot G}{FG} = \left(\partial_i \ln \frac{F}{G} \right) \left(\partial_j \ln \frac{F}{G} \right) + \partial_i \partial_j \ln \frac{F}{G} + 2\partial_i \partial_j \ln G. \tag{A.4d}$$

Equations (A.3) and (A.4) together allow us to transfer the bilinear equations in (4.4) to nonlinear equations in (4.2). We can also use these formulae reversely, in order to reformulate the nonlinear system as its corresponding bilinear form.

Appendix B. Pfaffian and an expansion formula

Here we remind the reader of a few facts about Pfaffians (see [17, 27]), and present an expansion formula (see [20, 23]) that is effective in expressing the soliton solutions of the elliptic coupled KP system in a concise (Hirota-type) form.

Let $\boldsymbol{A}$ be a $N \times N$ skew-symmetric matrix with entries $a_{i,j}$, hence $\boldsymbol{A}$ is of the form

$$\boldsymbol{A} = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,N} \\ -a_{1,2} & 0 & a_{2,3} & \cdots & a_{2,N} \\ -a_{1,3} & -a_{2,3} & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & a_{N-1,N} \\ -a_{1,N} & -a_{2,N} & \cdots & -a_{N-1,N} & 0 \end{pmatrix}.$$

The Pfaffian $\mathrm{pf}(\boldsymbol{A})$ associated with the matrix $\boldsymbol{A}$ can be defined as follows:

Definition B.1. *The Nth-order Pfaffian associated with $\boldsymbol{A}$ is the triangular array (see e.g. [27] for such a notation)*

$$\mathrm{pf}(\boldsymbol{A}) = \left| \begin{matrix} a_{1,2} & a_{1,3} & \cdots & a_{1,N} \\ & a_{2,3} & \cdots & a_{2,N} \\ & & \ddots & \vdots \\ & & & a_{N-1,N} \end{matrix} \right|, \tag{B.1a}$$

which is uniquely defined by the recursion relation

$$\mathrm{pf}(\boldsymbol{A}) = \sum_{i=2}^{N} (-1)^i a_{1,i} \left| \begin{matrix} a_{2,3} & \cdots & a_{2,i-1} & a_{2,i+1} & a_{2,i+2} & \cdots & a_{1,N} \\ & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & a_{i-2,i-1} & a_{i-2,i+1} & a_{i-2,i+2} & \cdots & a_{i-2,N} \\ & & & a_{i-1,i+1} & a_{i-1,i+2} & \cdots & a_{i-1,N} \\ & & & & a_{i+1,i+2} & \cdots & a_{i+1,N} \\ & & & & & \ddots & \vdots \\ & & & & & & a_{N-1,N} \end{matrix} \right|, \tag{B.1b}$$

together with the initial values defined as $|\cdot| \doteq 0$ and $|a_{1,2}| \doteq a_{1,2}$ for $N = 1$ and $N = 2$, respectively.

Remark B.2. *The definition implies that a Pfaffian is only non-zero when N is even. For example, we have*

$$\left| \begin{matrix} a_{1,2} & a_{1,3} \\ & a_{2,3} \end{matrix} \right| = 0 \quad \textit{and} \quad \left| \begin{matrix} a_{1,2} & a_{1,3} & a_{1,4} \\ & a_{2,3} & a_{2,4} \\ & & a_{3,4} \end{matrix} \right| = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3},$$

for $N = 3$ and $N = 4$, respectively. We note the following important relation between the Pfaffians and determinants of skew-symmetric matrices of the form given above:

$$\det(\boldsymbol{A}) = (\mathrm{pf}(\boldsymbol{A}))^2 . \tag{B.2}$$

Hence, the Pfaffian of $\boldsymbol{A}$ can be thought of as the square root of a determinant of a skew-symmetric matrix. However, the latter relation does not define the Pfaffian uniquely, while the recursion relation does.

What we need, in order to obtain our explicit form of the soliton solutions, is a Pfaffian analogue of the expansion formula for a determinant of the type $\det(\boldsymbol{I} + \boldsymbol{AB})$ in terms of the matrix invariants of $\boldsymbol{AB}$ (which are the sums of its principal minors). This is given in the lemma below, which curiously seems to be a new result. To express the formula in a compact manner, let us introduce the notation

$$\boldsymbol{A}_{i_1,i_2,\cdots,i_m}$$

denoting the sub-matrix of $\boldsymbol{A}$ by selecting from it the rows and columns labelled by $i_1, i_2, \cdots, i_m$ for $1 \le i_1 < i_2 < \cdots < i_m < N$. For examples, we have

$$\boldsymbol{A}_{i_1} = a_{i_1,i_1}, \quad \boldsymbol{A}_{i_1,i_2} = \begin{pmatrix} a_{i_1,i_1} & a_{i_1,i_2} \\ a_{i_2,i_1} & a_{i_2,i_2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{A}_{1,2,\cdots,N} = \boldsymbol{A}.$$

In this notation, the expansion formula (B.1b) can be rewritten as

$$\mathrm{pf}(\boldsymbol{A}) = \sum_{i=2}^{N} (-1)^i a_{1,i}\, \mathrm{pf}(\boldsymbol{A}_{2,\cdots,i-1,i+1,\cdots,N}).$$

We need the following expansion formula for the solution (6.10), akin to the expansion in terms of matrix invariants for the determinant. This is provided by the following lemma, see [23] which is based on a Cauchy–Binet type of expansion for Pfaffians, cf. [20], namely

Lemma B.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $N \times N$ skew-symmetric matrices of the form given above for $\boldsymbol{A}$ and a similar form for $\boldsymbol{B}$ (with entries $b_{i,j}$). The following expansion formula holds for the special Pfaffian of the format*

$$
\begin{aligned}
\mathrm{pf}(\boldsymbol{A}|\boldsymbol{B}) &\doteq \left|\begin{array}{ccccccccc}
a_{1,2} & a_{1,3} & \cdots & a_{1,N} & 1 & 0 & \cdots & 0 & 0 \\
 & a_{2,3} & \cdots & a_{2,N} & 0 & 1 & \cdots & 0 & 0 \\
 & & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & a_{N-1,N} & 0 & 0 & \cdots & 1 & 0 \\
 & & & & 0 & 0 & \cdots & 0 & 1 \\
 & & & & & b_{1,2} & \cdots & b_{1,N-1} & b_{1,N} \\
 & & & & & & \ddots & \vdots & \vdots \\
 & & & & & & & b_{N-2,N-1} & b_{N-2,N} \\
 & & & & & & & & b_{N-1,N}
\end{array}\right| \\
&= (-1)^{N(N-1)/2}\left[1 + \sum_{m\in J}(-1)^{m(m-1)/2} \sum_{1\le i_1<i_2<\cdots<i_m\le N} \mathrm{pf}(\boldsymbol{A}_{i_1,i_2,\cdots,i_m})\,\mathrm{pf}(\boldsymbol{B}_{i_1,i_2,\cdots,i_m})\right], \qquad \text{(B.3)}
\end{aligned}
$$

for $J = \{j = 2i | i = 1, 2, \cdots, [N/2]\}$.

The crucial upshot of the lemma is that the terms in the expansion (B.3) are products of separate sub-Pfaffians of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. To give an idea how this expansion looks like, let us write them down explicitly for the values of $N = 2, 3, 4, 5$, namely

$N = 2$**:**

$$\mathrm{pf}(\boldsymbol{A}\,|\,\boldsymbol{B}) = -1 + a_{1,2}b_{1,2};$$

$N = 3$**:**

$$\mathrm{pf}(\boldsymbol{A}\,|\,\boldsymbol{B}) = -1 + a_{1,2}b_{1,2} + a_{1,3}b_{1,3} + a_{2,3}b_{2,3};$$

$N = 4$**:**

$$\mathrm{pf}(\boldsymbol{A}\,|\,\boldsymbol{B}) = 1 - \sum_{1\le i_1<i_2\le 4} a_{i_1,i_2}b_{i_1,i_2} + \left|\begin{array}{ccc} a_{1,2} & a_{1,3} & a_{1,4} \\ & a_{2,3} & a_{2,4} \\ & & a_{3,4}\end{array}\right| \left|\begin{array}{ccc} b_{1,2} & b_{1,3} & b_{1,4} \\ & b_{2,3} & b_{2,4} \\ & & b_{3,4}\end{array}\right|;$$

$N = 5$**:**

$$
\begin{aligned}
\mathrm{pf}(\boldsymbol{A}\,|\,\boldsymbol{B}) = 1 - &\sum_{1\le i_1<i_2\le 5} a_{i_1,i_2}b_{i_1,i_2} \\
+ &\sum_{1\le i_1<i_2<i_3<i_4\le 5} \left|\begin{array}{ccc} a_{i_1,i_2} & a_{i_1,i_3} & a_{i_1,i_4} \\ & a_{i_2,i_3} & a_{i_2,i_4} \\ & & a_{i_3,i_4}\end{array}\right| \left|\begin{array}{ccc} b_{i_1,i_2} & b_{i_1,i_3} & b_{i_1,i_4} \\ & b_{i_2,i_3} & b_{i_2,i_4} \\ & & b_{i_3,i_4}\end{array}\right|.
\end{aligned}
$$

In the case of the soliton solution (6.10) the fact that in this expansion we get sums of products of separate Pfaffians is crucial, as we can compute the Pfaffians of the elliptic Cauchy matrix in explicit form, with the formulae given in the next appendix.

Appendix C. Applying Frobenius formula for elliptic Cauchy matrix

We introduce the elliptic functions[6]

$$W(x) \equiv \Phi_\omega(x)e^{-\eta x}, \quad W'(x) \equiv \Phi_{\omega'}(x)e^{-\eta' x} \quad \text{and} \quad W''(x) \equiv \Phi_{\omega''}(x)e^{-\eta'' x}, \qquad \text{(C.1)}$$

[6]Note that the prime and double prime signs here are only symbols to distinguish various W-functions. They do not denote the derivatives of W with respect to the argument x.

where $\Phi_\kappa(x)$ is the Lamé function, given in terms of the Weierstrass σ-function $\sigma(x) = \sigma(x|2\omega, 2\omega')$ with half-periods ω, ω' (2ω and $2\omega'$ being the elementary periods of the period lattice):

$$\Phi_\alpha(x) \equiv \frac{\sigma(x+\alpha)}{\sigma(\alpha)\sigma(x)}, \tag{C.2}$$

where α is an arbitrary complex variable not coinciding with any zero of the σ-function, and $\eta = \zeta(\omega)$, $\eta' = \zeta(\omega')$, $\eta'' = \zeta(\omega'')$, where $\omega'' = -\omega - \omega'$ (see e.g. [2] for the standard notation of Weierstrass elliptic functions).

From the standard addition formulae for the Weierstrass functions $\sigma(x)$, $\zeta(x)$ and $\wp(x)$ we obtain the relations for the functions W, W' and W'' including the Yang–Baxter-type relation

$$W(x)W'(z) + W'(y)W''(x) + W''(z)W(y) = 0, \quad x + y + z = 0. \tag{C.3}$$

Equation (C.3) follows from the well-known 3-term addition formula for the σ-function, which can be written in the compact form of an elliptic partial fraction expansion in terms of Φ, namely

$$\Phi_\alpha(x)\Phi_\beta(y) = \Phi_{\alpha+\beta}(x)\Phi_\beta(y-x) + \Phi_\alpha(x-y)\Phi_{\alpha+\beta}(y). \tag{C.4}$$

In addition, as a consequence of the quasi-periodicity of the σ-function, namely

$$\sigma(x+2\omega) = -\sigma(x)e^{2\eta(x+\omega)} \quad \text{and} \quad \sigma(x+2\omega') = -\sigma(x)e^{2\eta'(x+\omega')},$$

we have

$$W^2(x) + e = W'^2(x) + e' = W''^2(x) + e'' = \wp(x), \tag{C.5a}$$

$$W(x)W'(x)W''(x) = -\frac{1}{2}\wp'(x), \tag{C.5b}$$

where $e = \wp(\omega)$, $e' = \wp(\omega')$, $e'' = \wp(\omega'')$.

Let us now single out one of the half-periods, say ω, and the corresponding function $W(x)$, for which we have the relation

$$W(x)W(x+\omega) = -\frac{e^{\eta\omega}}{\sigma^2(\omega)} \doteq -\sqrt{g}, \tag{C.6a}$$

and subsequently

$$(\wp(x) - e)(\wp(x+\omega) - e) = \frac{e^{2\eta\omega}}{\sigma^4(\omega)} = (e'-e)(e''-e) = \frac{1}{2}\wp''(\omega) = g, \tag{C.6b}$$

where it is the x-independent quantity on the right hand side of (C.6) that we call g, i.e. the parameter of the elliptic curve in the main text. We also introduce the corresponding parameters

$$k = \zeta(\kappa+\omega) - \zeta(\kappa) - \eta = -\frac{W'(\kappa)W''(\kappa)}{W(\kappa)} \quad \text{and} \quad K = \wp(\kappa) - e, \tag{C.7}$$

for which we note that the relations (C.6b), together with (C.5b) and (C.7) lead to the elliptic curve in the rational form (2.1), i.e.

$$k^2 = K + 3e + \frac{g}{K}.$$

The aim is to express the elliptic Cauchy matrix $\Omega(\kappa_i, \kappa_j)$ in terms of the W function which, in turn, allows us to apply the famous Frobenius determinant formula [9] for elliptic Cauchy matrices. In terms of the function W we have the following key relation:

$$\frac{W(\kappa+\kappa')}{W(\kappa)W(\kappa')} = -\frac{k-k'}{K-K'} \quad \text{or equivalently} \quad \frac{W(\kappa-\kappa')}{W(\kappa)W(\kappa')} = \frac{k+k'}{K-K'}, \tag{C.8}$$

and furthermore we can identify the Cauchy kernel of the elliptic KP system as follows:

$$\Omega(\kappa, \kappa') = \frac{K-K'}{k+k'} = -\frac{W(\kappa)W(\kappa')}{\sqrt{g}}W(\kappa-\kappa'+\omega). \tag{C.9}$$

To compute the determinants and Pfaffians, we note that since the function W is essentially a Lamé function Φ_ω, we can apply the Frobenius determinant formula for elliptic Cauchy matrices:

$$\det\left(\Phi_\alpha(\kappa_i-\kappa'_j)\right)_{i,j=1,\cdots,N} = \frac{\sigma\left(\alpha+\sum_i(\kappa_i-\kappa'_i)\right)}{\sigma(\alpha)}\frac{\prod_{1\le i<j\le N}\sigma(\kappa_i-\kappa_j)\sigma(\kappa'_j-\kappa'_i)}{\prod_{1\le i,j\le N}\sigma(\kappa_i-\kappa'_j)}, \tag{C.10}$$

cf. [9]. Setting $\alpha=\omega$ in (C.10) we get the determinant formula that is relevant for our case, which reads

$$\det\left(W\left(\kappa_{i_\nu}-\kappa_{i_{\nu'}}+\omega\right)\right)_{\nu,\nu'=1,\cdots,m} = \frac{\sigma((m+1)\omega)}{\sigma^{m+1}(\omega)}\,e^{-m\eta\omega}\,\frac{\prod_{1\le\nu<\nu'\le m}\sigma(\kappa_{i_\nu}-\kappa_{i_{\nu'}})\sigma(\kappa_{i_{\nu'}}-\kappa_{i_\nu})}{\prod_{\substack{1\le\nu,\nu'\le m\\ \nu\neq\nu'}}\sigma(\kappa_{i_\nu}-\kappa_{i_{\nu'}}+\omega)},$$

which vanishes if m is odd in accordance with the Pfaffian structure. Using (C.9) we then get

$$\det\left(\Omega(\kappa_{i_\nu},\kappa_{i_{\nu'}})\right)_{\nu,\nu'=1,\cdots,m} = (-1)^{m(m+1)/2}\frac{\sigma((m+1)\omega)}{\sigma^{(m+1)^2}(\omega)}\left(\prod_{\nu=1}^m\frac{K_{i_\nu}}{g}\right)\prod_{1\le\nu<\nu'\le m}\frac{1}{W^2(\kappa_{i_\nu}-\kappa_{i_{\nu'}})}. \tag{C.11}$$

It is not hard to prove that the prefactor

$$\frac{\sigma((m+1)\omega)}{\sigma^{(m+1)^2}(\omega)} = \begin{cases} 0, & \text{if } m \text{ odd},\\ (-1)^{m/2}\gamma_m^2, & \text{if } m \text{ even},\end{cases}$$

where $\gamma_m=g^{m(m+2)/8}$. This can be proven by induction using the quasi-periodicity of the σ-function and the relations (C.6), leading to $\gamma_{2n+2}/\gamma_{2n}=g^{n+1}$. Thus, the Pfaffian of the skew-symmetric elliptic Cauchy kernel takes the form

$$\mathrm{pf}\left(\Omega(\kappa_{i_\nu},\kappa_{i_{\nu'}})\right)_{\nu,\nu'=1,\cdots,m} = \frac{g^{m(m-2)/8}}{\left(\prod_{\nu=1}^m K_{i_\nu}\right)^{(m-2)/2}}\prod_{1\le\nu<\nu'\le m}\frac{K_{i_\nu}-K_{i_{\nu'}}}{k_{i_\nu}+k_{i_{\nu'}}}, \tag{C.12}$$

for m even, while for m odd the Pfaffian vanishes. In (C.12) we have used (C.8) and (C.5a) to express the fully skew-symmetric product in rational form.

Remark C.1. *As a curiosity we note that, as a consequence of the Frobenius–Stickelberger determinant formula, i.e. the elliptic van der Monde-determinant (see [10]), the prefactor can be written as a Hankel determinant in the following form*

$$\frac{\sigma((m+1)x)}{\sigma^{(m+1)^2}(x)} = \frac{(-1)^{m^2}}{(1!2!\cdots m!)^2}\begin{vmatrix} \wp'(x) & \wp''(x) & \cdots & \wp^{(m)}(x)\\ \wp''(x) & \wp'''(x) & \cdots & \wp^{(m+1)}(x)\\ \vdots & \vdots & & \vdots\\ \wp^{(m)}(x) & \wp^{(m+1)}(x) & \cdots & \wp^{(2m-1)}(x)\end{vmatrix},$$

see Example 20.21 in [41] and references therein, which vanishes at $x=\omega$ when m is odd while for $m=2n$ even yields (up to a sign) a perfect square, namely

$$\frac{\sigma((2n+1)\omega)}{\sigma^{(2n+1)^2}(\omega)} = (-1)^n\gamma_m^2, \tag{C.13}$$

with

$$\gamma_m \doteq \frac{1}{1!2!\cdots(2n)!}\begin{vmatrix} \wp^{(2)}(\omega) & \wp^{(4)}(\omega) & \cdots & \wp^{(2n)}(\omega) \\ \wp^{(4)}(\omega) & \wp^{(6)}(\omega) & \cdots & \wp^{(2n+2)}(\omega) \\ \vdots & \vdots & & \vdots \\ \wp^{(2n)}(\omega) & \wp^{(m+1)}(\omega) & \cdots & \wp^{(4n-2)}(\omega) \end{vmatrix},$$

which miraculously turns out to be a pure power of the modulus g alone, even though the individual entries depend on g and e.

References

[1] V.E. Adler. Bäcklund transformation for the Krichever–Novikov equation. *Int. Math. Res. Notices*, 1998:1–4, 1998.
[2] N.I. Akhiezer. *Elements of the Theory of Elliptic Functions.* American Mathematical Society, Providence, 1990.
[3] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa. KP hierarchies of orthogonal and symplectic type –Transformation groups for soliton equations VI–. *J. Phys. Soc. Jpn.*, 50:3813–3818, 1981.
[4] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa. Landau–Lifshitz equation: solitons, quasi-periodic solutions and infinite-dimensional Lie algebras. *J. Phys. A: Math. Gen.*, 16:221–236, 1983.
[5] E. Date, M. Jimbo, and T. Miwa. Method for generating discrete soliton equations. V. *J. Phys. Soc. Jpn.*, 52:766–771, 1983.
[6] L.A. Dickey. *Soliton Equations and Hamiltonian Systems.* World Scientific, Singapore, 2nd edition, 2003.
[7] A.S. Fokas and M.J. Ablowitz. Linearization of the Korteweg–de Vries and Painlevé II equations. *Phys. Rev. Lett.*, 47:1096–1110, 1981.
[8] A.S. Fokas and M.J. Ablowitz. On the inverse scattering and direct linearizing transforms for the Kadomtsev–Petviashvili equation. *Phys. Lett. A*, 94:67–70, 1983.
[9] F.G. Frobenius. Über die elliptischen functionen zweiter art. *J. reine angew. Math.*, 93:53–68, 1882.
[10] F.G. Frobenius and L. Stickelberger. Zur theorie der elliptischen functionen. *J. reine angew. Math.*, 83:175–179, 1877.
[11] W. Fu and F.W. Nijhoff. Direct linearizing transform for three-dimensional discrete integrable systems: the lattice AKP, BKP and CKP equations. *Proc. R. Soc. A*, 473:20160915, 2017.
[12] W. Fu and F.W. Nijhoff. Linear integral equations, infinite matrices, and soliton hierarchies. *J. Math. Phys.*, 59:071101, 2018.
[13] W. Fu and F.W. Nijhoff. On nonautonomous differential-difference AKP, BKP and CKP equations. *Proc. R. Soc. A*, 477:20200717, 2021.
[14] C.R. Gilson. Generalizing the KP hierarchies: Pfaffian hierarchies. *Theor. Math. Phys.*, 133:1663–1674, 2002.
[15] C.R. Gilson, J.J.C. Nimmo, and S. Tsujimoto. Pfaffianization of the discrete KP equation. *J. Phys. A: Math. Gen.*, 34:10569–10575, 2001.
[16] R. Hirota. Soliton solutions to the BKP equations. I. The Pfaffian technique. *J. Phys. Soc. Jpn.*, 58:2285–2296, 1989.
[17] R. Hirota. *The Direct Method in Soliton Theory.* Cambridge University Press, Cambridge, 2004.
[18] R. Hirota and Y. Ohta. Hierarchies of coupled soliton equations. I. *J. Phys. Soc. Jpn.*, 60:198–809, 1991.
[19] R. Hirota and J. Satsuma. Soliton solutions of a coupled Korteweg–de Vries equation. *Phys. Lett. A*, 85:407–408, 1981.
[20] M. Ishikawa and M. Wakayama. Minor summation formula of Pfaffians. *Linear Multilinear Algebra*, 39:285–305, 1995.
[21] P. Jennings and F.W. Nijhoff. On an elliptic extension of the Kadomtsev–Petviashvili equation. *J. Phys. A: Math. Theor.*, 47:055205, 2014.
[22] M. Jimbo and T. Miwa. Solitons and infinite dimensional Lie algebras. *Publ. RIMS*, 19:943–1001, 1983.
[23] Y. Kodama and L.K. Williams. N-soliton solutions to the DKP equation and Weyl group actions. *J. Phys. A: Math. Gen.*, 39:4063–4086, 2006.
[24] I.M. Krichever. Elliptic analog of the Toda lattice. *Int. Math. Res. Notices*, 2000:383–412, 2000.
[25] I.M. Krichever and S.P. Novikov. Holomorphic bundles over algebraic curves, and nonlinear equations. *Russ. Math. Surv.*, 33:53–80, 1980.
[26] T. Miwa, M. Jimbo, and E. Date. *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras.* Cambridge University Press, Cambridge, 2000.
[27] T. Muir. Developments of a Pfaffian. *Trans. S. African Philos. Soc.*, 15:35–41, 1904.
[28] F.W. Nijhoff. The direct linearizing transform for the τ function in three-dimensional lattice equations. *Phys. Lett. A*, 110:10–14, 1985.
[29] F.W. Nijhoff. Theory of integrable three-dimensional nonlinear lattice equations. *Lett. Math. Phys.*, 9:235–241, 1985.
[30] F.W. Nijhoff. Linear integral transformations and hierarchies of integrable nonlinear evolution equations. *Physica D*, 31:339–388, 1988.
[31] F.W. Nijhoff and H.W. Capel. The direct linearisation approach to hierarchies of integrable PDEs in 2+1 dimensions: I. Lattice equations and the differential-difference hierarchies. *Inverse Probl.*, 6:567–590, 1990.
[32] F.W. Nijhoff, H.W. Capel, and G.L. Wiersma. Integrable lattice systems in two and three dimensions. In R. Martini, editor, *Geometric Aspects of the Einstein Equations and Integrable Systems*, volume 239 of *Lecture Notes in Physics*, pages 263–302. Springer-Verlag, Berlin, 1985.

[33] F.W. Nijhoff, H.W. Capel, G.L. Wiersma, and G.R.W. Quispel. Bäcklund transformations and three-dimensional lattice equations. *Phys. Lett. A*, 105:267–272, 1984.

[34] F.W. Nijhoff and S.E. Puttock. On a two-parameter extension of the lattice KdV system associated with an elliptic curve. *J. Nonlinear Math. Phys.*, 10 (suppl. 1):107–123, 2003.

[35] F.W. Nijhoff, G.R.W. Quispel, and H.W. Capel. Direct linearisation of difference-difference equations. *Phys. Lett. A*, 97:125–128, 1983.

[36] F.W. Nijhoff, G.R.W. Quispel, J. van der Linden, and H.W. Capel. On some linear integral equations generating solutions of nonlinear partial differential equations. *Physica A*, 119:101–142, 1983.

[37] G.R.W. Quispel, F.W. Nijhoff, H.W. Capel, and J. van der Linden. Linear integral equations and nonlinear difference-difference equations. *Physica A*, 125:344–380, 1984.

[38] H. Sakai. Rational surfaces associated with affine root systems and geometry of the Painlevé equations. *Commun. Math. Phys.*, 220:165–229, 2001.

[39] P.M. Santini, M.J. Ablowitz, and A.S. Fokas. The direct linearization of a class of nonlinear evolution equations. *J. Math. Phys.*, 25:2614–2619, 1984.

[40] J. van de Leur. Bäcklund–Darboux transformations for the coupled KP hierarchy. *J. Phys. A: Math. Gen.*, 37:4395–4405, 2004.

[41] E.T. Whittaker and G.N. Watson. *A Course of Modern Analysis.* Cambridge University Press, Cambridge, 5th edition, 2021.

(WF) School of Mathematical Sciences and Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai 200241, People's Republic of China

(FWN) School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom