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Optimal toll design problem with improved behavioural equilibrium model: the case of the Probit model

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ABSTRACT

This paper considers the optimal toll design problem that uses the Probit model to determine travellers' route-choices. Under probit, the route flow solution to the resulting stochastic user equilibrium (SUE) is unique and can be stated implicitly as a function of tolls. This reduces the toll design problem to an optimisation problem with only nonnegativity constraints. Additionally, the gradient of the objective function can be approximated using the chain rule and the first order Taylor approximation of the equilibrium condition.

To determine SUE, this paper considers two techniques. One uses Monte-Carlo simulation to estimate route choice probabilities and the method of successive averages with its prescribed step length. The other relies on the Clark approximation and computes an optimal step length. Although both are effective at solving the toll design problem, numerical experiments show that the technique with the Clark approximation is more robust on a small network.

1. INTRODUCTION

Transport can be considered as an economic market where travellers are economics agents with the aim of maximising (or minimising) their utility (or disutility). With the cross-effect of one user's strategy on another through the congestion in the network, the concept of Nash's equilibrium can be invoked to define the converged travellers' strategies (e.g. route, mode, departure time, or destination choices). The Nash equilibrium occurs when no individual (traveller) can change their strategy to decrease their own disutility. However, it is well known that under the assumption of individual utility maximization, the converged equilibrium point of the transport system may not be the optimal travel pattern for the overall system, nor for

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other aggregated objectives of the traffic system manager (e.g. total travel time, environmental impact, or social welfare).

Road pricing has been proposed as the means to direct the traffic equilibrium condition to a more desireable state (Knight 1924; Walters 1961). Early developments of the theory of road pricing have been mainly associated with the concept of deterministic user equilibrium, namely Wardrop's user equilibrium (UE) principle (Wardrop, 1952). UE is a special case of Nash's equilibrium condition and has been widely adopted as the modelling assumption for representing travellers' behaviour. The key assumption of UE is that the traveller has perfect information regarding their travel choices and the alternatives. Despite questions about the realism of the assumption, the UE model has played a major role in the analysis of road pricing, in which a number of researchers over the years have focussed on deriving optimal toll patterns under the UE condition (e.g. Yang & Huang 1998; Santos et al. 2001; Verhoef 2002; May et al. 2002; Shepherd and Sumalee 2004; Sumalee 2004).

The key element of microeconomic theory lies in understanding the consumer's behaviour. The concept of a random utility model (RUM) has been developed to better represent the individual's choice making process. RUM may be integrated with the traffic equilibrium model by representing the payoff function, or disutility, as a random utility term. This random disutility of travel is widely referred to as the *perceived* disutility/cost of travel. The equilibrium point can then be defined as the situation where no traveller can switch his/her strategy to improve his or her *perceived* cost of travel. With this setting, we obtain the concept of Stochastic User Equilibrium (SUE). Apart from the enhanced realism of the behavioural model underlying the SUE model, the algorithmic advantage of using an SUE model in optimal toll design has also been previously implied (e.g. Davis, 1994; Patriksson & Rockafellar, 2003). This issue will be discussed later on in the paper.

Many error structures have been proposed for SUE. They include the commonly used independent Weibull and multivariate normal that lead to the logit and probit models respectively (Sheffi, 1985), as well as more general cross-nested logit models (Prashker & Bekhor, 1999), mixed error component models (Nielsen *et al,* 2002) and gamma link component distributions (Cantarella & Binetti, 2002).

Among the logit and probit models, the former is more popular because of its closed form expression for the choice probabilities. Several researchers (e.g., Smith et al., 1994, Akamatsu and Kuwahara, 1989, and Yang, 1999) have used the logit model to study toll pricing under SUE. However, the underlying assumption for the logit model is rather restrictive. In particular, it assumes that travel alternatives are uncorrelated and have no overlapping structure. Generally, this is referred to as the 'independence of irrelevant alternatives' assumption or IIA. On the other hand, despite its complexity the probit model can overcome the IIA issue of the overlapping routes. Thus, the probit SUE will be adopted as the model for travellers' behaviour in this paper.

The paper is organised into five further sections. The next section presents the formulation of the optimal toll design problem with SUE and the definition of SUE. Then, section 3 explains the treatment of variable demand (elastic demand) with the probit SUE and the different computational methods adopted for solving the probit SUE. Section 4 reformulates the optimal toll design with SUE in the form of an implicit program, and the algorithm for solving this problem is presented. Section 5 provides numerical results using a test network. Finally, section 6 concludes the paper.

2. PROBLEM FORMULATION OF OPTIMAL TOLL DESIGN WITH STOCHASTIC USER EQUILIBRIUM

The problem discussed in this paper is the optimal toll design problem where the response from the users to the toll imposed is assumed to follow a random utility model. We focus on the case of an automobile network with a single mode, single user class, and single time period. The underlying network is a directed graph with *N* nodes and a set of links denoted $\mathcal{A}$. The demand matrix $\boldsymbol{q}$ has entries q_{rs}, representing the travel demand from origin *r* to destination *s*, where *r, s* = 1…*N*. The vector of link flows is $\mathbf{x}$, with link costs $\mathbf{t}(\mathbf{x})$, so that $t_a(x_a)$ is the cost (without toll) of travelling along link $a \in \mathcal{A}$ when the link flow is x_a. Let β_a denote the toll level of link $a \in \mathcal{A}$. Then the generalised travel cost on link *a* is $t_a(x_a)+\beta_a$. In addition, let K_{rs} be the set of routes connecting node *r* to node *s*. Associated with K_{rs} is the link-route incidence matrix, Δ^{rs}, whose element, $\delta_{a,k}^{rs}$, equals 1 if link a is on route k that connects node r to node s. An assignment of flows to all routes is denoted by the vector $\mathbf{f}$, with $f_k^{rs} \geq 0 \;\; \forall k, r, s$. The assignment $\mathbf{f}$ is *feasible* for demand $\mathbf{q}$ if and only if

$$\sum_{k \in \mathcal{K}_{rs}} f_k^{rs} = q_{rs} \quad \forall r, s,$$

and the (convex) set of feasible route flows is denoted F. For any $\mathbf{f} \in F, \mathbf{c}(\mathbf{f})$ denotes the associated vector of route costs where

$$c_k^{rs}(\mathbf{f}) = \sum_{a \in A} \left(t_a(\mathbf{x}(\mathbf{f})) + \beta_a \right) \, \delta_{a,k}^{rs} .$$

Travellers are allowed to respond to the toll imposed by changing their routes or deciding not to travel (the precise mechanism for achieving this is described in section 3). The responses of the travellers are assumed to follow the Stochastic User Equilibrium condition (SUE). Let Φ be a mapping from $\Re^{|\beta|} \to \Re^{|\kappa|}$ that gives the vector $\mathbf{f}$ of feasible route flows satisfying the SUE condition, given a toll vector β. Let Z($\mathbf{f}$, β) be the objective function that we wish to optimise. We can then formulate the optimisation problem for determining the optimal toll as:

$$\begin{aligned} &\max_{(\mathbf{f},\beta)} \mathrm{Z}\left(\mathbf{f}, \beta\right) \\ &s.t.\, \mathbf{f} = \Phi\left(\beta\right) \\ &\beta \geq \mathbf{0} \end{aligned}$$

Note that this problem can be considered as a mathematical program with equilibrium constraints (MPEC). As noted previously by many authors, this formulation can also be applied to the UE case, but with a mapping between the link flow vector and the toll vector, since the route flow in UE is not unique.

This paper assumes that the route choice behaviour follows a random utility model. In particular, the perceived cost of the k-th route is a random variable of the form:

$$C_k = c_k + \varepsilon_k$$

where $c_k = c_k(\mathbf{f})$ is the mean perceived route cost and the random errors $(\varepsilon_1, \varepsilon_2, ...)$ follow some joint probability density function with zero mean vector. These random error terms represent the fact that individual drivers have their own assessment of both network conditions and of the cost of taking different routes (including their personal preferences for some routes over others).

Given the route cost vector $\mathbf{c}$, $P_k^{rs}(\mathbf{c})$ denotes the proportion of drivers who perceive route k to be the cheapest route from r to s, i.e.

$$\begin{aligned} P_k^{rs} &= \Pr\left(C_k^{rs} \leq C_j^{rs} \ \ \forall j \in K_{rs}, j \neq k\right) \\ &= \Pr\left(\varepsilon_k^{rs} + c_k^{rs} \ \ \leq \varepsilon_j^{rs} + c_j^{rs} \forall j \in K_{rs}, j \neq k\right), \end{aligned}$$

where Pr(.) denotes probability. Then, the stochastic user equilibrium (SUE) can be stated as follows:

At SUE, no driver can improve their perceived travel cost by unilaterally changing route.

The SUE route flow assignment (for $\mathbf{f} \in F$) is, therefore, the solution to the following fixed-point problem:

$$f_k^{rs} = q_{rs} P_k^{rs}(\mathbf{c}(\mathbf{f})) \quad \forall k \in K_{rs}, \ \forall r, s.$$

This states that, for a given OD pair, the flow on the k-th route consists of those drivers who perceive this to be the best route. Since $\mathbf{f}$ is defined to be a feasible set of flows, the total number of drivers on all routes connecting r to s matches the total travel demand from this origin to this destination. A network route flow vector satisfying SUE will be denoted **f***. This fixed-point condition defines the mapping Φ between the SUE flows and the toll vector.

With the SUE, several properties that UE does not possess can be gained. Consider first a simplified network structure in which the only routes are non-overlapping and consist of single links, and that for given tolls the vector of link travel cost functions is continuous and strictly increasing in the vector of link flows. In this case, for given link tolls, there are unique UE link flows and route flows (see e.g. Smith, 1979).

In UE, a route will be used if and only if the travel cost on this route is the minimum O-D travel cost (compared to all other routes connecting the same O-D pair). This can be represented as a complementarity condition: $0 \leq f_k^{rs} \perp \left(C_k^{rs} - C^{rs*}\right) \geq 0$, where C^{rs*} is the minimum travel cost from origin r to destination s and $x \perp y \equiv x \cdot y = 0$. This complementarity condition is non-differentiable when $f_k^{rs} = \left(C_k^{rs} - C^{rs*}\right) = 0$. Thus, when including this condition into the optimal toll design problem, one may face a non-differentiable optimisation problem. This is an example of a wider phenomenon arising from the *complementarity condition* as constraints to optimisation problems (Patriksson & Rockafellar, 2002, Luo *et al.* 1996).

In general network structures, while the set of link flow solutions to the UE model at given tolls is a singleton under the assumption that the vector of link travel cost functions is continuous and strictly monotonic (Smith, 1979), it is well known that the UE *route* flow solutions are typically non-unique. Therefore, route-based solution strategies are commonly faced with an additional hurdle of selecting a single UE route flow solution from a convex set, for example by an arbitrary choice of extreme point (e.g. Tobin & Friesz, 1988) or by an additional model selecting the 'most likely' route flows (e.g. Larsson *et al*, 2001). Still, establishing desirable properties of a sequence of such 'unique' UE route flow solutions, as the tolls are altered, may be extremely problematic.

For problems with continuous and strictly monotone link cost functions as above, under mild conditions on the choice probability model, SUE is know to give rise to solutions (a) in which *all* routes are active, at least in theory, and (b) that are unique in the *route* flow domain (e.g. Cantarella & Cascetta, 1995). Therefore, it is natural to ask, is solving the optimal toll problem with an SUE network model actually *easier* than with a UE? At the same time, one is adopting a model that, from a behavioural perspective, is arguably superior in terms of its representation of the uncertainty and heterogeneity that surely exists in traveller decisions.

3. PROBIT EQUILIBRIUM WITH VARIABLE DEMAND: FORMULATION AND SOLUTION ALGORITHM

The probit model assumes that perceived route costs are derived from normally distributed perceived link costs:

$$C_k^{rs} = \sum_a T_a \delta_{a,k}^{rs} \quad \forall k, r, s$$

with $T_a \sim N(t_a, \sigma_a^2)$, with σ_a^2 constant. In this paper we assume that the perceived link costs, $\{T_a\}$, are independent. The distribution of perceived route costs is therefore multivariate normal, $C \sim MVN(c, \Sigma)$, centred on the deterministic route costs. This results in a variance-covariance matrix, Σ, where the perceived costs of routes that have links in common are correlated.

The SUE model in section 2 assumes that travel demands are fixed. In this section, we allow demands to vary. Maher *et al* (1999) assumes that the demand for OD pair (*r*,*s*) is a function of the expected minimum travel time between the origin and destination, i.e. q_{rs} depends on $E\left[\min\left\{C_k^{rs} : k \in K_{rs}\right\}\right]$. When the logit route-choice is used, the demand function resulting from the assumption can be mathematically expressed in a closed form (see, e.g. Ben-Akiva *et al*. 1986, Gentile & Papola 2001) but this is not the case for probit.

To make our model more manageable under probit, we add to the original network a pseudo-link (*r*,*s*) for each OD pair. The amount of flow on pseudo link (*r*,*s*) represents the number of drivers who decide not to travel from *r* to *s*. The perceived travel cost on each pseudo link (or link zero) is $c_0^{rs} + \varepsilon_0^{rs}$, where c_0^{rs} represents the deterministic disutility of not travelling and ε_0^{rs} is the associated random error in accordance with the probit model. Then, the proportion of drivers who decide not to travel is given by the following expressions:

$$\begin{aligned} P_0^{rs} &= \Pr\left(c_0^{rs} + \varepsilon_0^{rs} \le c_k^{rs} + \varepsilon_k^{rs} \;\; \forall k \in K_{rs}\right) \\ &= \Pr\left(c_0^{rs} + \varepsilon_0^{rs} \le \min_{k \in K_{rs}}\left\{c_k^{rs} + \varepsilon_k^{rs}\right\}\right), \end{aligned}$$

and the condition for SUE can be written in the same manner for those with fixed demand:

$$f_k^{rs} = q_{rs} P_k^{rs}(\mathbf{c}(\mathbf{f})) \quad \forall k \in K_{rs}^0, \; \forall r, s,$$

where $K_{rs}^0 = K_{rs} \cup \{0\}$, with f_0^{rs} the number of drivers electing to not travel. Moreover, q_{rs} now represents the number of potential drivers, some of whom choose the pseudo link, i.e. decide not to travel.

To determine a solution that satisfies the above equilibrium condition, any algorithm that solves a probit-based SUE problem with fixed demand can be used. In Section 5, we consider the following algorithms:

1. The method of successive averages (MSA) algorithm (see Sheffi, 1985) with probit choice fractions estimated by a Monte Carlo (MC) simulation.
2. A step-length algorithm recently proposed by Maher and Hughes (1997) that uses the equivalent optimisation formulation of SUE (Daganzo & Sheffi, 1977) with the Clark approximation (Clark, 1961; Horowitz *et al*, 1982) for computing probit choice probabilities.

4. IMPLICIT PROGRAMMING APPROACH TO OPTIMAL TOLL DESIGN

Assume that the travel cost of link a, $t_a(x)$ is continuous for each a and the travel cost vector, $\mathbf{t}(\mathbf{x})$, is strictly monotone. Then, the route-flow solution of the probit-based SUE problem is unique (see e.g. Cantarella and Cascetta, 1995) and the optimal toll design problem can be formulated as follows:

$$\max_{\beta}\left\{\mathrm{Z}\left(\mathbf{x}^*(\beta),\beta\right):\beta\geq\mathbf{0}\right\}$$

where $\mathbf{x}^*(\beta)$ denotes a link flow solution to the probit SUE problem at toll vector β.

As stated above, the optimal toll design problem is an optimisation problem with simple bounds. Many algorithms for such a problem typically require, at minimum, calculating the gradient of the objective function at the current solution. When Z is relatively simple, its gradient can be approximated. To illustrate, consider the revenue function, i.e. $\mathrm{Z}\left(\mathbf{x}^*(\beta),\beta\right)=\beta^T\cdot\mathbf{x}^*(\beta)$. In this case,

$$\nabla_{\beta}\mathrm{Z}\left(\mathbf{x}^*(\beta),\beta\right)=\mathbf{x}^*(\beta)+\beta^T\cdot\nabla_{\beta}\mathbf{x}^*(\beta),$$

where $\nabla_{\beta}\mathbf{x}^*(\beta)$ denotes the Jacobian of $\mathbf{x}^*$ at β.

From the relationship between link and route flow, we can define the Jacobian of $\mathbf{x}^*$ at β as:

$$\nabla_{\beta}\mathbf{x}^*(\beta)=\Delta\cdot\nabla_{\beta}\mathbf{f}^*(\beta),$$

where $\mathbf{f}^*(\beta)$ is a vector of SUE route flow solution at β, Δ is the link-route incidence matrix whose element, $\delta_{a,k}$, equals 1 if link *a* is on route *k*, and $\nabla_\beta \mathbf{f}^*(\beta)$ denotes the Jacobian of $\mathbf{f}^*$ at β. To approximate $\nabla_\beta \mathbf{f}^*(\beta)$, consider the 'gap' function:

$$\Psi(\mathbf{f},\beta) = \mathbf{f} - \mathbf{q}\cdot\mathbf{P}(\mathbf{c}(\mathbf{f},\beta)),$$

where **P** is the route-choice probability operator as defined in Section 2. Assuming all functions are differentiable, the first order Taylor approximation of $\Psi\left(\mathbf{f}^*(\beta),\beta\right)$ at $(\mathbf{f},\beta) = \left(\mathbf{f}^*(\beta_0),\beta_0\right)$ is:

$$\Psi(\mathbf{f},\beta) \approx \Psi\left(\mathbf{f}^*(\beta_0),\beta_0\right) + J_1\left(\mathbf{f} - \mathbf{f}^*(\beta_0)\right) + J_2(\beta-\beta_0),$$

where J_1 and J_2 are the Jacobians of Ψ evaluated at $\left(\mathbf{f}^*(\beta_0),\beta_0\right)$ with respect to $\mathbf{f}$ at β, respectively, i.e., $J_1 = \nabla_{\mathbf{f}}\Psi\left(\mathbf{f}^*(\beta_0),\beta_0\right)$ and $J_2 = \nabla_\beta\Psi\left(\mathbf{f}^*(\beta_0),\beta_0\right)$. (See Bell and Iida, 1997, Daganzo, 1979, and Clark and Watling, 2002 for the calculation of J_1 and J_2). Because $\Psi\left(\mathbf{f}^*(\beta),\beta\right) = 0$ for all β, the above reduces to

$$0 \approx 0 + J_1\left(\mathbf{f} - \mathbf{f}^*(\beta_0)\right) + J_2(\beta-\beta_0).$$

When J_1 is non-singular, the above implies that $-J_1^{-1}J_2$ is an approximation of the Jacobian of $\mathbf{f}^*(\beta)$ at β_0, i.e.,

$$\mathbf{f}^*(\beta) - \mathbf{f}^*(\beta_0) \approx -J_1^{-1}J_2(\beta-\beta_0) \text{ or } \lim_{\beta\to\beta_0}\frac{\mathbf{f}^*(\beta) - \mathbf{f}^*(\beta_0) + J_1^{-1}J_2(\beta-\beta_0)}{\|\beta-\beta_0\|} \approx 0.$$

For the above example, $\nabla_\beta Z\left(\mathbf{x}^*(\beta),\beta\right) = \mathbf{x}^*(\beta) - \Delta J_1^{-1}J_2\beta.$

5. NUMERICAL EXPERIMENTS

5.1 Definition of the test network

The network adopted for the test has seven nodes connected by 18 links, with six pseudo-links representing the no-travel options for each OD movement (as required in the variable demand probit SUE model). Figure 1 shows the topology of the network. There are six OD pairs: (1, 5), (1, 7), (5, 1), (5, 7), (7, 1), and (7, 5). Table A.1 in the Appendix gives the origin-destination 'potential demand' matrix. The link cost functions are based on the BPR function $t_i(x_i) = a_i + b_i\left(\frac{x_i}{\kappa_i}\right)^{n_i}$ where a_i, b_i, n_i, and κ_i are given in Table A.2 in the Appendix. Note that for the 'no travel' or 'pseudo' links, there is only a constant parameter associated with the disutility of not conducting a trip, i.e. $b_i = 0$ for such links. As in , e.g. Sheffi (1985), the probit link error terms are independent and normally distributed with zero mean and standard deviations as listed in Table A.2 in the Appendix.

Tolls are implemented by adding the tolls to the free flow costs. When an additional cost is added to the free flow parameter for a pseudo-link, this can be thought of as representing an increase in the no-travel cost (for that OD movement) representing the increase in the utility of conducting a trip rather. There are 36 routes among the six OD pairs.

The variance-covariance matrix is constructed by setting the probit variance for link j to be $\sigma_j^2 = \alpha \cdot a_j^2$ where α is a scaling factor and a_j is the free flow cost (see Sheffi 1985). The entire link cost is written into the free flow parameter for the pseudo-links, leading to huge variances on these links, relative to the rest of the network This seems unrealistic, and is problematic for the Clark approximation, so we set each pseudo-link variance to be the mean of the real link variances.

Different values of α can be used to define different levels of the perception error of the travellers on the travel time/cost resulting in different behavioural models. Several values α are adopted to investigate effect of the behavioural model on the route choice behaviour, the resulting flows, and the optimal toll levels (α = 0, 0.3, 1, 3).

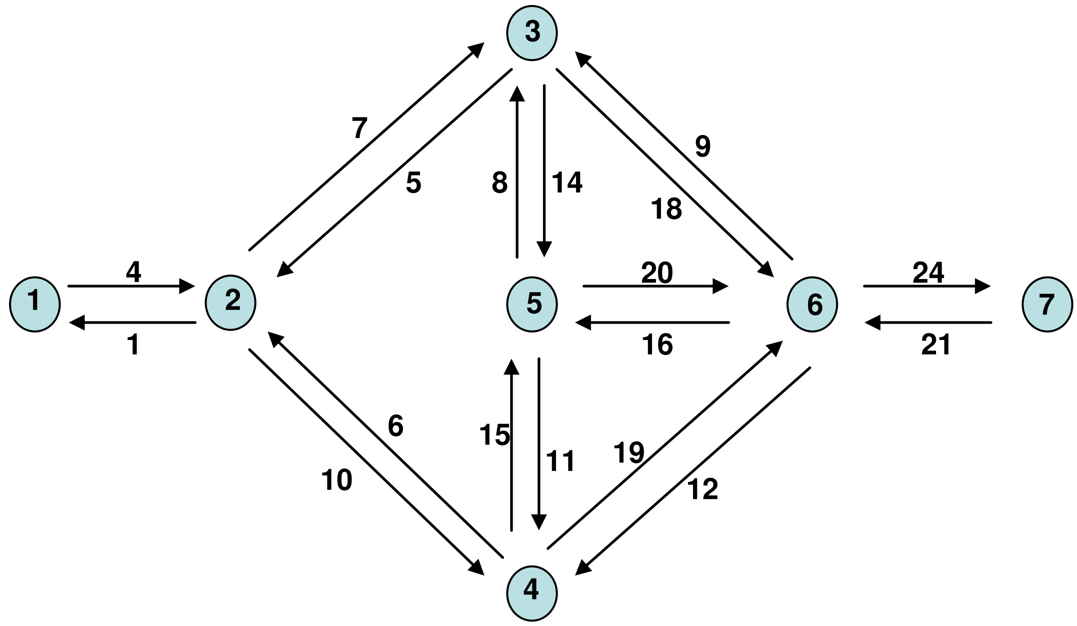

Figure 1: The topology of the test network (without the pseudo-links)

The objective function adopted in the test is a combination of the revenue and the actual total travel time. The revenue, *R*, is simply calculated by summing the tolls multiplied by the relevant link flows for the tolled links. The total travel time, *TTT*, is calculated from only the real links (since the flow on the pseudo-links does not travel); it is the sum of the link flows multiplied by the link travel times (without the toll included).

The objective function is $Z = \mu R + (1-\mu)(-TTT)$ where μ is a weighting factor with $0 \le \mu \le 1$. The gradient of the objective function with respect to tolls can be derived as follows:

$$\nabla_\beta Z \approx \mu\left(\mathbf{x}^*(\beta) - \Delta J_1^{-1} J_2 \beta\right) + (1-\mu)\Delta J_1^{-1} J_2 \left\{\mathbf{t}\left(\mathbf{x}^*(\beta)\right) - \operatorname{diag}\left[\mathbf{x}^*(\beta)\left(\nabla_{\mathbf{x}}\mathbf{t}\left(\mathbf{x}^*(\beta)\right)\right)^T\right]\right\},$$

where $\nabla_{\mathbf{x}}\mathbf{t}$ denotes the Jacobian of the travel cost with respect to flows, and J_1 and J_2 are as defined in Section 4.

To demonstrate the behaviour of the test network, the revenue generated and total travel time for different toll levels applied to each link in turn are shown in Figure 2 below. In these tests the covariance scaling factor, α, is set to 1. From the figures, the revenue levels generated are most sensitive to tolls on links 1,4, 21 and 24. The network diagram above shows that these are the links that cannot be avoided (by the relevant OD movements); the only alternative "route" is the no-travel option. Thus, it is no surprise to observe that these links can generate the highest revenues. For the other links in the network, travellers can avoid the tolled link by changing route. For the total travel time, tolling on certain links (e.g. link 8) increases the total travel time as we increase the toll. For other links (e.g. link 4) the opposite occurs. With the weighting factor $\mu = 0.5$ the objective function values as each link is tolled individually are shown in Figure 3.

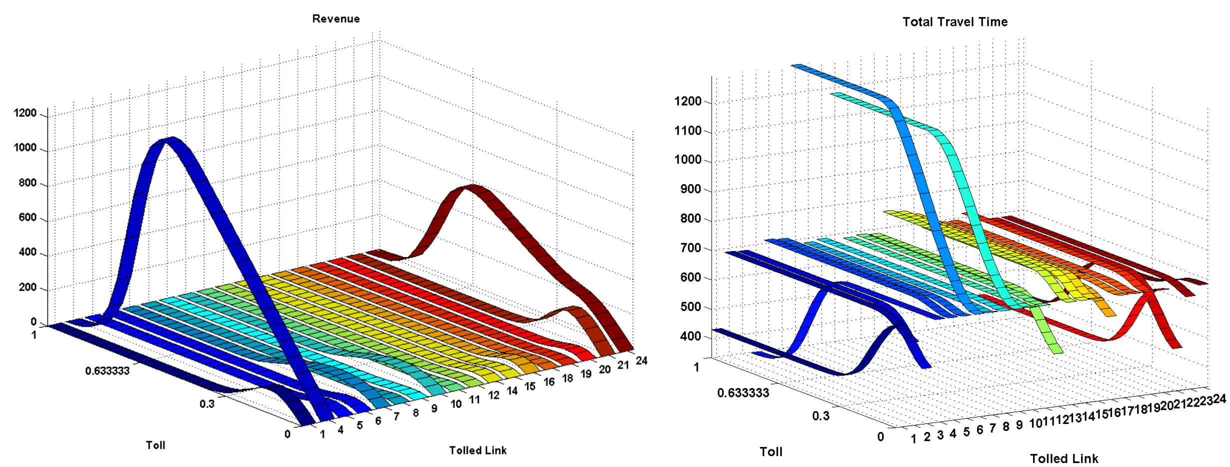

Figure 2: Revenue and total travel time for different toll levels on each link

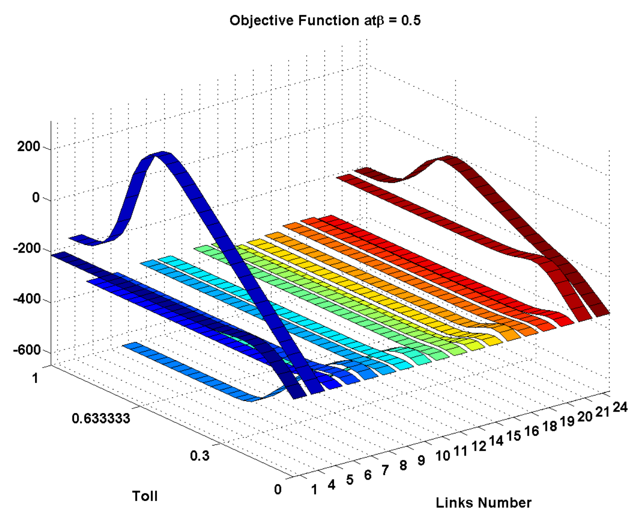

Figure 3: Objective function levels for different toll levels applied to each link in turn

5.2 Comparison of different SUE solution algorithms

In this section, the two alternative algorithms proposed for solving the SUE problem (described in section 3.2) are tested. We consider the case of tolling links 14, 15, and 16 simultaneous with a uniform toll. For this one-dimensional problem the gradient of the objective function at each toll level can be plotted as shown in Figures 4 and 5. Three different levels of α are adopted for the test (α = 0.3, 1, and 3). Six curves are plotted, three for each method with different α in each figure. Figure 4 compares the gradient of the objective as calculated by 'numerical differencing' (a finite difference approximation) and by sensitivity analysis (see Section 4) in which the SUE flows are calculated by the first method (MSA + MC-estimated choice probabilities). Figure 5 shows the same comparison but the SUE flows are calculated by the second method (Clark's approximation + optimal step length). In both figures, the curves with the bold line are the gradients calculated from numerical differencing and the broken lines are the gradients from the sensitivity analysis.

In both cases, the gradients calculated by the sensitivity analysis method are reasonably smooth. In Figure 4, the numerical differencing produces a non-smooth gradient that is caused by the non-smooth objective function as calculated from the MC simulation and pre-defined step length. Although the Clark's approximation does have disadvantages (in terms of where this approximation is valid) the resulting link flows (and hence objective function values) are much smoother than the corresponding values calculated on the basis of the MC simulation. The gradients calculated by numerical differencing of the SUE flows resulting from the Clark's approximation based approach (bold line in Figure 5) are visually as smooth as the gradients calculated via sensitivity analysis in the same figure (broken line).

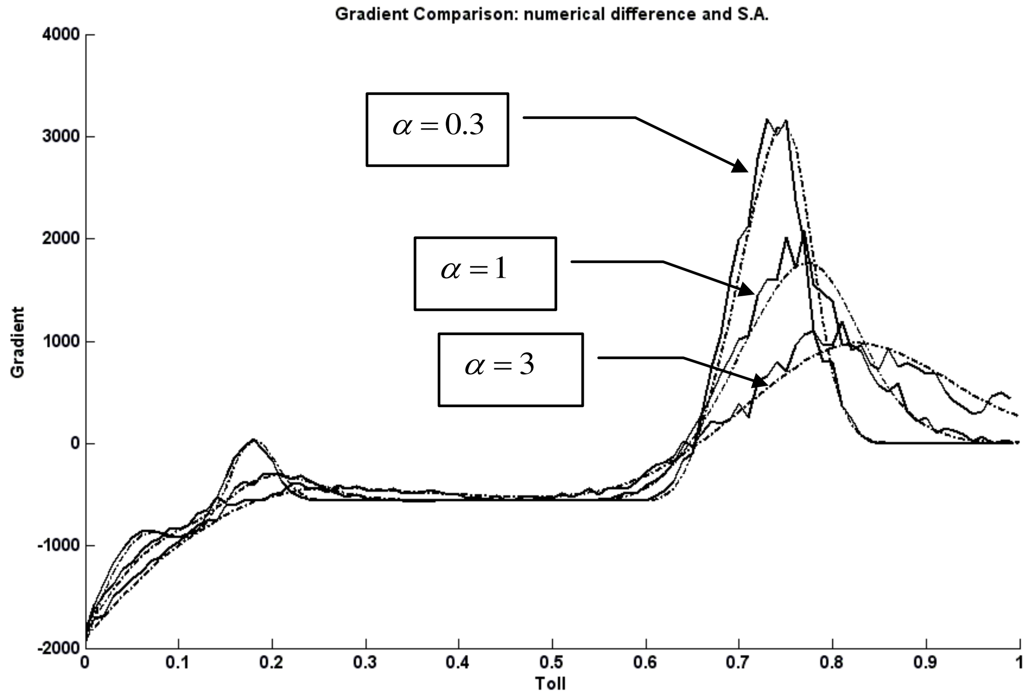

Figure 4: Gradients of the objective at different toll levels calculated from the numerical differencing (solid line) and sensitivity analysis (broken line). MSA calculations using the MC simulation and predefined step-length.

Obviously, different methods significantly influence the smoothness of the objective function. The MC based method does suffer from the unpredictability of the random trial process which may not guarantee the same SUE flows/route choice probabilities with different runs. On the other hand, the benefit of the MC based method is that with a high number of the trials the accuracy of the estimation of the route choice probability may be improved, but one can never be sure what constitutes a sufficient number of trials. The Clark approximation, despite its possible drawback on the accuracy of the approximation, does produce very good results in terms of the smoothness of the objective function. Nevertheless, in both cases the sensitivity analysis method can eventually define a smooth trend of the gradient reflecting the real property of the problem. The reason is that the sensitivity analysis method estimates the gradient based on a single point (see previous section). Thus, it does not suffer from the poor convergence of the SUE flows from one toll level to another whereas the numerical differencing, which uses two points of SUE flows, suffers from this error.

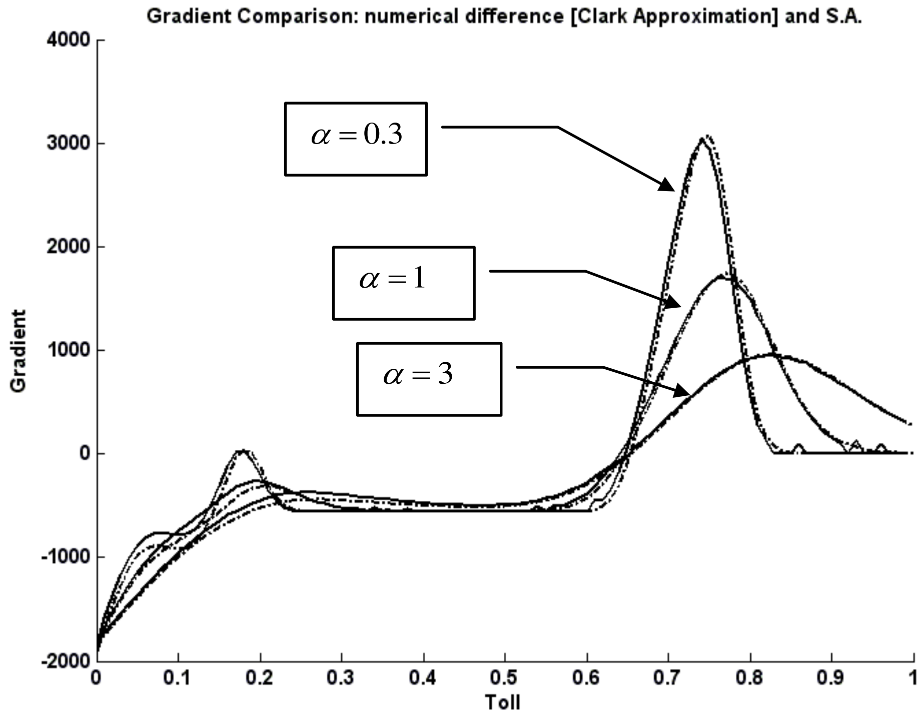

Figure 5: Gradients of the objective at different toll levels calculated from the numerical differencing (solid line) and sensitivity analysis (broken line). MSA calculations using the Clark's approximation and optimal step-length.

Based on this comparison, we decided to adopt the second approach (Clark's approximation + optimal step length) for the tests in the following sections.

5.3 Effect of probit variances on the optimal toll policy

This section presents some numerical results using the optimization approach explained in Section 4 to find the optimal tolls for different cases. The sequential quadratic programming (SQP) algorithm in MATLAB ('fmincon' solver) is adopted to solve the problem, with the Jacobian of the objective function supplied (using the approach described in Section 4). Before applying the optimization algorithm to the test, we explore the effect of the behavioural model parameters on the objective function. Three different sets of tests are conducted. In the first set of tests, we apply the uniform toll level on link 8, 11, 14, and 15 with four different values of α. Similarly, the second set of tests involves imposing the uniform tolls on link 14, 15, and 16 making a pricing cordon around node 5. The third set of tests is to put the toll on link 4 only.

For all tests, we provide the plots the corresponding objective function values (see Figure 6, 7, and 8 below). Different values of the scaling parameter α show the influence of the behavioural model on the objective function profile. The first observation is the smoothing effect of the α parameter on the objective function. When $\alpha = 0$ (UE case), non-smoothness of the objective function is apparent. This property of the MPEC with UE is well documented where the objective function can be non-differentiable at some point.

On the other hand, the objective function curves with $\alpha > 0$ appear to be smooth. As the probit variances increase (with α), so drivers become less reactive to changes due to the toll and there is non-zero probability for each route to be used. This property of the SUE model contributes to the smoothness of the objective function with respect to the toll. As mentioned earlier, although the main incentive of introducing the probit SUE in place of UE is to increase the realism of the lower level model for the optimal toll problem, the SUE model may also make the optimal toll problem become easier to deal with. The other observation is the possible change of the optimal toll solution for the different values of α. With all tests, the value of the optimal toll levels do change according to the level of α.

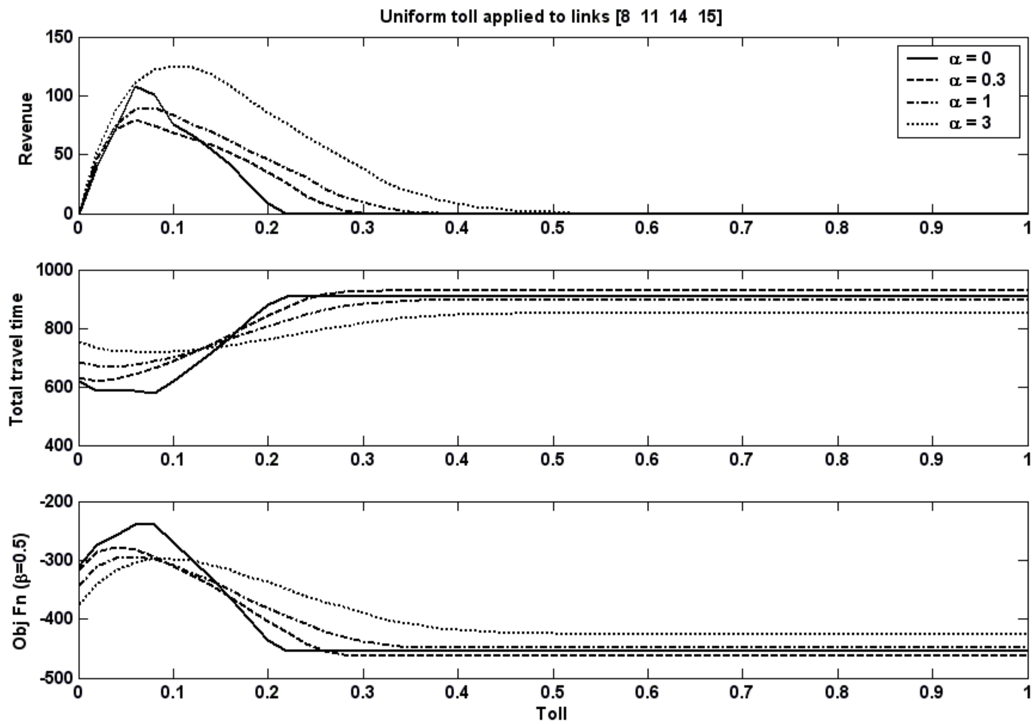

Figure 6: Revenue, total travel time, and objective function curves with different values of α and different uniform toll levels on link 8, 11, 14, and 15

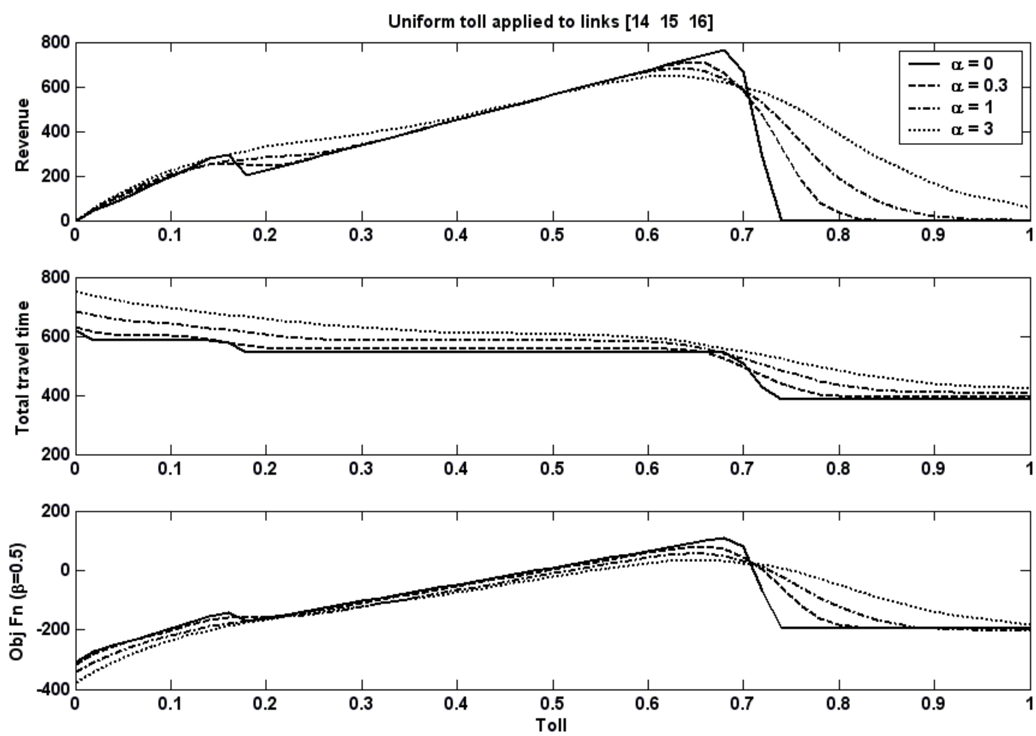

Figure 7: Revenue, total travel time, and objective function curves with different values of α and different uniform toll levels on link 14, 15, and 16

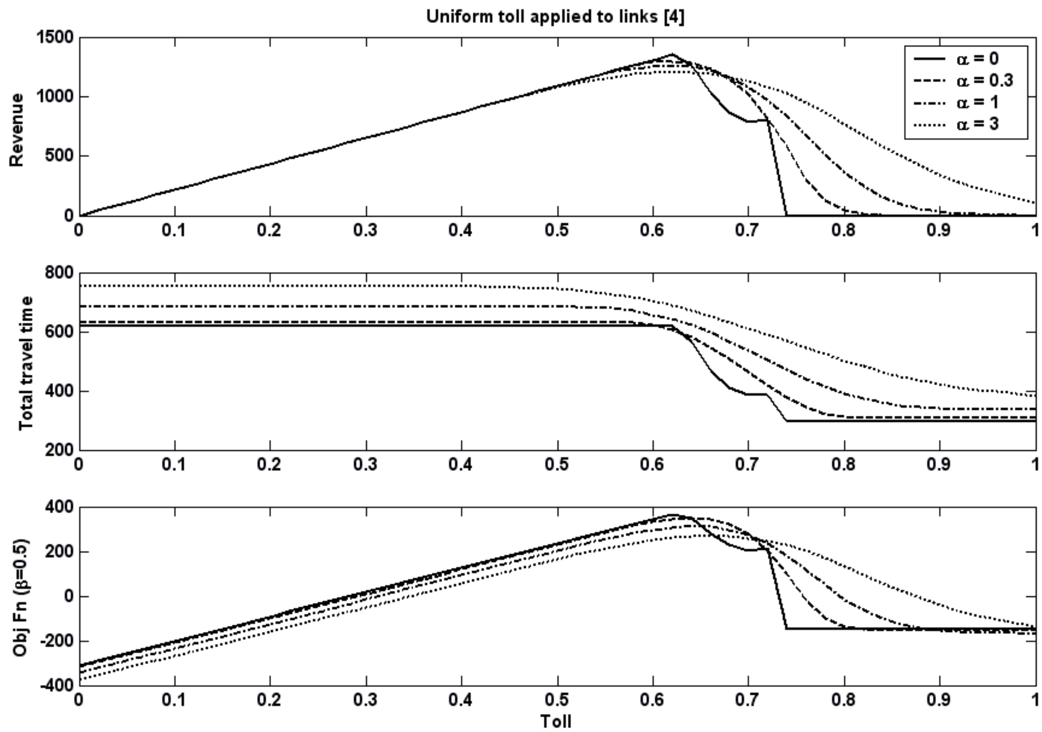

Figure 8: Revenue, total travel time, and objective function curves with different values of α and different toll levels on link 4

Table 1 shows the results from applying the optimization algorithm to find the optimal uniform toll applied to links 14, 15, and 16 with different values of α.

α	**Optimal Toll**	**Objective at optimal toll**
0.0001	0.68867	114.7479
0.3	0.65346	80.7625
1	0.64773	55.7007
3	0.6566	33.6732
10	0.7	25.6591

Table 1: Optimal toll on links 14, 15, 16 with different α found by the optimization algorithm

Figure 7 can be used to verify that the optimization algorithm can find the real optimal toll level for each case. Again, as mentioned the optimal toll levels change with the levels of α. Unfortunately, we cannot observe any clear relationship between the optimal toll and the level of α from the results.

The optimization algorithm is also applied to the find the optimal toll level on all links (except the pseudo links) simultaneously and the optimal toll level on each link in turn, again with different levels of α. Table 2 shows the result with the optimal toll on each link simultaneously and Table 3 shows the results with the toll on each link in turn.

α	Link Number	Optimal toll (for this link)	Objective function at optimal toll	Benefit
0.3	1	0.0621	570.7513	886.44
	4	0.2431		
	5	0.0688		
	6	0.0660		
	7	0.2001		
	8	0.0815		
	9	0.0688		
	10	0.1930		
	11	0.0839		
	12	0.0659		
	14	0.2149		
	15	0.2171		
	16	0.1028		
	18	0.1228		
	19	0.1242		
	20	0.0952		
	21	0.0430		
	24	0.0449		
1	1	0.0379	554.0486	896.30
	4	0.2224		
	5	0.0527		
	6	0.0551		
	7	0.2003		
	8	0.0696		
	9	0.0617		
	10	0.2053		
	11	0.0726		
	12	0.0646		
	14	0.2256		
	15	0.2231		
	16	0.1177		
	18	0.1494		
	19	0.1468		
	20	0.0935		
	21	0.0395		
	24	0.0492		
3	1	0.0572	564.7796	933.9598
	4	0.2412		
	5	0.0723		
	6	0.0749		
	7	0.2185		
	8	0.0794		
	9	0.0827		
	10	0.1902		
	11	0.0791		
	12	0.0846		
	14	0.1627		
	15	0.1913		
	16	0.1134		
	18	0.1510		
	19	0.1798		
	20	0.0577		
	21	0.0543		
	24	0.0649		

Table 2: Results from optimizing all link tolls simultaneously

α	Link Number	Optimal toll (for this link)	Objective function at optimal toll	Benefit
0.3	1	0.9550	-193.856	121.8355
	4	0.6384	352.121	667.8127
	5	0.0401	-306.689	9.0025
	6	0.0368	-309.498	6.1935
	7	0.0594	-298.020	17.6715
	8	0.0253	-309.705	5.9865
	9	0.0164	-314.148	1.5435
	10	0.0490	-303.654	12.0375
	11	0.0283	-310.685	5.0065
	12	0.0207	-314.074	1.6175
	14	0.0315	-306.840	8.8515
	15	0.0263	-308.624	7.0675
	16	0.0423	-296.542	19.1495
	18	0.0206	-313.947	1.7445
	19	0.0245	-314.716	0.9755
	20	0.0373	-301.445	14.2465
	21	0.9550	-199.047	116.6445
	24	0.6218	20.640	336.3316
1	1	0.2057	-202.250	139.9976
	4	0.6436	314.731	656.9788
	5	0.0686	-327.319	14.9282
	6	0.0596	-329.466	12.7816
	7	0.0740	-320.685	21.5622
	8	0.0445	-332.526	9.7215
	9	0.0521	-334.390	7.8575
	10	0.0618	-325.366	16.881
	11	0.0507	-333.555	8.6921
	12	0.0464	-333.641	8.6059
	14	0.0344	-331.960	10.2877
	15	0.0348	-333.117	9.1302
	16	0.0559	-319.686	22.5609
	18	0.0561	-332.115	10.1319
	19	0.0561	-333.399	8.8488
	20	0.0468	-328.318	13.9294
	21	0.7388	-215.222	127.0258
	24	0.6284	-6.2829	335.9649
3	1	0.2637	-216.123	153.0572
	4	0.6656	272.968	642.1482
	5	0.0908	-348.657	20.5232
	6	0.0999	-349.395	19.7852
	7	0.1001	-342.819	26.3612
	8	0.0728	-362.081	7.0992
	9	0.1252	-356.680	12.5002
	10	0.0995	-348.549	20.6312
	11	0.0728	-363.206	5.9742
	12	0.1001	-356.138	13.0422
	14	0.0635	-361.323	7.8572
	15	0.0586	-361.701	7.4792
	16	0.0816	-346.962	22.2182
	18	0.1000	-357.077	12.1032
	19	0.0990	-356.783	12.3972
	20	0.0786	-360.340	8.8402
	21	0.2648	-195.343	173.8372
	24	0.6630	-33.8258	335.3544

Table 3: Results from optimizing each tolled link individually

Note that the column 'objective function at optimal toll' shows the absolute value of the objective function at that toll level. The objective function adopted here, as explained, is a weighted sum of the revenue and negative total travel time. Therefore, it is possible that the objective function may become negative even at the optimal toll. This does not mean the optimal toll generate dis-benefit, since the objective at the no toll scenario is a negative figure as well. Column 'benefit' in both tables presents the relative improvement of the objective of each toll policy compared with the no-toll situation. The optimization algorithm successfully solved all the scenarios reported here.

For the case with the tolls on all links, the improvement of the overall objective function increases as α increases. When all links are tolled, the links with the highest toll levels are links 4, 14, and 15. However, when each link is tolled individually, the links with the highest optimal tolls are links 1, 4, 21, and 24. Imposing the tolls on one of these link individually is actually equivalent to imposing the toll on all of the demand for some OD movement since these links are the feeding links of the demand from different OD pairs to the network (hence there is no alternative routes that avoid the tolls). The link generating the highest objective is link 4. The result may be that link 4 imposes the toll directly to a significant level of the demand in the network (the level of the demand coming from node 1 is highest compared to the other origin nodes, see Table A1 in the Appendix).

6. CONCLUSIONS

The traditional assumption of travellers' response to a road toll is the deterministic user equilibrium model. We have argued in this paper that a better representation of travellers' responses may be achieved through an improved behavioural model following random utility theory, as achieved through the probit SUE model. Optimal toll design with the probit SUE is then formulated, with the probit SUE framework extended in a novel way to include variable demand, by adding pseudo links to the network. The optimal toll problem with probit SUE can be categorised as an MPEC. However, the uniqueness and smoothness of the route choice probabilities in probit SUE, given a toll vector, help us in developing an optimisation algorithm for tackling this problem, by reformulating the MPEC as an implicit programming problem. The key element in developing an algorithm to solve the reformulated optimal toll problem is the Jacobian of the objective function with respect to the tolls, which can be estimated in practice by applying the sensitivity analysis method.

In particular, we used the Sequential Quadratic Programming (SQP) algorithm in MATLAB to solve the optimal toll problems. The algorithm was applied to a test network (with 18 links and six OD pairs). Firstly, we tested the accuracy of two different algorithms for solving the probit SUE, one combining MSA with MC-based choice probabilities, and a second using Clark's approximation method with optimal step length computation. The results show the instability of the MC based method. This is thought to be due to the lack of consistency in the convergence properties of the MC method at 'adjacent' (very similar) tolls. Clark's approximation, on the other hand, produces a smoother objective function. However, there exists some uncertainty regarding the accuracy of the Clark approximation in estimating the probit route choice probabilities. Nevertheless, with both methods the sensitivity analysis can produce a reasonably smooth gradient due to the fact that in deriving the gradient of the objective, the sensitivity analysis method is only based on a single point of solution, hence reducing the uncertainty of the converged solution between two toll levels.

The second test concerned the influence of the behavioural parameters on the optimal toll solution. Different scaling parameters, which determine the magnitude of terms in the variance-covariance matrix of the probit model, were tested. The results showed some changes of the objective function curves with different scaling parameters, resulting in changes to the optimal toll solution. This result highlights the importance of calibrating the behavioural model in order to accurately determine the optimal toll policy. The last set of tests applied the optimisation algorithm to the test (tolls on all links simultaneously and tolls on each link individually). The optimisation algorithm successfully solved all test problems.

Despite encouraging results from these tests, further research is still required in order to make the algorithm work efficiently with a large scale application. Firstly, although the theory of the probit model suggests that all routes will always be used, in practice some routes may have a very small probability of being used, and these routes will be eliminated from the choice set due to the limitation of machine precision. In the current algorithm, we assume a fixed set of predetermined used routes, even when the toll is varied. This assumption can be relaxed easily within the iterative procedure to allow the set of used routes to be changed dynamically with the toll level, updating the route set at each iteration. The second issue is concerned with the computational burden of the calculation of the probit SUE. A more efficient algorithm exploiting other estimation techniques of the multi-dimensional integral is being investigated in order to increase the efficiency of the algorithm in solving a large scale SUE problem. Last but not least, we wish to explore the development of the optimisation algorithm itself, aiming to improve it by better exploiting the structure of the problem, or through alternative reformulations of the problem.

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Appendix

O/D	1	5	7
1	-	1125	1050
5	675	-	850
7	1050	850	-

Table A1: OD potential demand matrix for the test network

Link Number i	a_i	b_i	c_i	n_i	σ_i
1	0.0125	0.0026515	1800	4.5	0.0125
2	0.16	0	1	1	0.041498
3	0.25	0	1	1	0.041498
4	0.0125	0.0026515	1800	4.5	0.0125
5	0.03	0.03	1100	3	0.03
6	0.033333	0.033333	1100	3.1	0.033333
7	0.03	0.03	1100	3	0.03
8	0.025	0.025	1100	3.2	0.025
9	0.075	0.015909	1100	3.5	0.075
10	0.033333	0.033333	1100	3.1	0.033333
11	0.026667	0.026667	1100	3.1	0.026667
12	0.07625	0.016174	1100	3	0.07625
13	0.8	0	1	1	0.041498
14	0.025	0.025	1100	3.2	0.025
15	0.026667	0.026667	1100	3.1	0.026667
16	0.02	0.02	1100	3.1	0.02
17	0.2	0	1	1	0.041498
18	0.075	0.015909	1100	3.5	0.075
19	0.07625	0.016174	1100	3	0.07625
20	0.02	0.02	1100	3.1	0.02
21	0.0125	0.0026515	1800	4.5	0.0125
22	0.8	0	1	1	0.041498
23	0.2	0	1	1	0.041498
24	0.0125	0.0026515	1800	4.5	0.0125

*links 13 and 22 are the pseudo links for O-D 1-5 and 1-7 respectively. links 2 and 23 are the pseudo links for O-D 5-1 and 5-7 respectively. links 3 and 17 are the pseudo links for O-D 7-1 and 7-5 respectively.

Table A2: Link travel time parameters for the test network