Article:

Dummigan, N. (2022) Lifting congruences to half-integral weight. Research in Number Theory, 8. 59. ISSN 2363-9555

https://doi.org/10.1007/s40993-022-00356-3

LIFTING CONGRUENCES TO HALF-INTEGRAL WEIGHT

NEIL DUMMIGAN

Abstract. Given a congruence of Hecke eigenvalues between newforms f and g of odd, square free level, and weight $2\kappa - 2$, with even $\kappa \geq 6$, we show that, under weak conditions, there is a congruence of Fourier coefficients between corresponding newforms of half-integral weight.

1. Introduction

Shimura [Sh1] gave a way of associating, to a Hecke eigenform of half-integral weight, a Hecke eigenform of integral weight. This is in general many-to-one, but by imposing a condition on the half-integral weight form, Kohnen made it one-to-one (up to scaling). For precise definitions we refer the reader to his paper.

Theorem 1.1. *(Kohnen,* [Ko1]*) Suppose M is odd and squarefree, $\kappa \geq 2$ an integer. For each normalised newform $f \in S_{2\kappa-2}(\Gamma_0(M))$, there is a unique (up to scaling) $\tilde{f} \in S^{+,\mathrm{new}}_{\kappa-(1/2)}(\Gamma_0(4M))$ such that for any fundamental discriminant $(-1)^{\kappa-1}D$ with $D > 0$,*

$$L(s-(\kappa-2), \chi_{(-1)^{\kappa-1}D}) \sum_{n=1}^{\infty} a_{Dn^2}(\tilde{f}) n^{-s} = a_D(\tilde{f}) \sum_{n=1}^{\infty} a_n(f) n^{-s},$$

where $f = \sum_{n=1}^{\infty} a_n(f) q^n$ and $\tilde{f} = \sum_{n=1}^{\infty} a_n(\tilde{f}) q^n$.

The "+" means that $a_m(\tilde{f}) = 0$ unless $(-1)^{\kappa-1}m \equiv 0$ or $1 \pmod 4$, and $\chi_{(-1)^{\kappa-1}D} = \left(\frac{(-1)^{\kappa-1}D}{\cdot}\right)$ is the quadratic character associated to the extension $\mathbb{Q}(\sqrt{(-1)^{\kappa-1}D})/\mathbb{Q}$.

The Hecke eigenvalues $a_p(f)$ for $T(p)$ on f are the eigenvalues for half-integral weight Hecke operators $T(p^2)$ on $\tilde{f}$, but it is evident from the above formula that the $a_p(f)$ determine only ratios $a_{Dn^2}(\tilde{f})/a_D(\tilde{f})$ of certain Fourier coefficients for $\tilde{f}$. They say nothing about ratios $a_{D'}(\tilde{f})/a_D(\tilde{f})$ for different fundamental discriminants $(-1)^{\kappa-1}D' \neq (-1)^{\kappa-1}D$. The problem of what the full set of Fourier coefficients tells us is addressed by the following explicit version of a theorem of Waldspurger. (The case $M = 1$ was an earlier result of Kohnen and Zagier.)

Theorem 1.2. *(Kohnen,* [Ko2, Corollary 1, Remark]*) With $f, \tilde{f}$ as above, and any fundamental discriminant $(-1)^{\kappa-1}D$ with $D > 0$ such that $\chi_{(-1)^{\kappa-1}D}(q) = \epsilon_q(f)$*

Date: March 18th, 2022.
2010 *Mathematics Subject Classification.* 11F37, 11F33, 11F46.
Key words and phrases. Modular forms of half-integral weight, Saito-Kurokawa lifts, congruences of modular forms.

(the Atkin-Lehner eigenvalue for f) for all primes $q \mid M$,

$$\frac{a_D(\tilde{f})^2}{\langle \tilde{f}, \tilde{f}\rangle} = 2^{\omega(M)}\frac{(\kappa-2)!}{\pi^{\kappa-1}}D^{\kappa-(3/2)}\frac{L(\kappa-1, f, \chi_{(-1)^{\kappa-1}D})}{\langle f, f\rangle}.$$

Furthermore, $a_D(\tilde{f}) = 0$ if $\chi_{(-1)^{\kappa-1}D}(q) = -\epsilon_q(f)$ for some prime $q \mid M$.

A problem raised by Hida is whether, given a congruence between newforms of integral weight, there is a non-trivial congruence *of Fourier coefficients* between forms of half-integral weight mapping to them via the Shimura lift. Bearing in mind the above, whereas a congruence of Hecke eigenvalues in half-integral weight is a triviality (because they are the same eigenvalues as in integral weight), a congruence of Fourier coefficients is something much stronger, and not at all obvious. Maeda [Ma] proved one instance of this, where the newforms are in $S_8(\Gamma_0(26))$ and the modulus is a divisor of 433. Since here M is even, Theorem 1.1 does not apply. In [DK], under certain hypotheses (including an assumption on the linear independence mod λ of certain ternary theta series arising from quaternion algebras), we proved a fairly general result on lifting congruences from weight 2 to weight 3/2. The main result of this paper uses a completely different method, but does not apply to either of these situations, since the weight has to be at least 10.

Theorem 1.3. *Let $f, g \in S_{2\kappa-2}(\Gamma_0(M))$, with M odd and squarefree, be normalised newforms, with even $\kappa \geq 6$ (so $2\kappa - 2 \geq 10$, twice an odd number), and $\lambda \mid \ell$ a prime divisor in a number field K containing all the Hecke eigenvalues of f and g. Suppose the following.*

(1) $\overline{\rho}_{f,\lambda}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ *contains* $\mathrm{SL}_2(\mathbb{F}_\ell)$*, where $\rho_{f,\lambda}$ is the 2-dimensional λ-adic Galois representation attached to f by Deligne* [De2]*, and $\overline{\rho}_{f,\lambda}$ is a residual representation. (Thanks to the condition, it is irreducible and therefore well-defined up to isomorphism.)*
(2) $\ell \nmid (2\kappa-2)!M\prod_{\text{prime } q \mid M}(q^2-1)$.
(3) *There exists a fundamental discriminant $-D < 0$ such that $\left(\frac{-D}{p}\right) = \epsilon_p(f)$ for all primes $p \mid M$, and an even character χ of conductor $N > 1$, with $M \mid N$ and $\ell \nmid N$, such that*

$$\mathrm{ord}_\lambda\left(\frac{L^N(3-\kappa, \chi)L^N_{\mathrm{alg}}(1, f, \chi)L^N_{\mathrm{alg}}(2, f, \chi)L_{\mathrm{alg}}(\kappa-1, f, \chi_{-D})}{[\Gamma_0^{(2)}(M) : \Gamma_0^{(2)}(N)]}\right) \leq 0.$$

(See below for the definitions of these algebraic parts, and the next section for the definition of $\Gamma_0^{(2)}(M)$. The superscript N on an L-function indicates that Euler factors at $p \mid N$ are omitted.)
(4) $\mathrm{ord}_\lambda(L^M(\kappa, f)/L(\kappa, f)) = 0$.
(5)

$$a_p(f) \equiv a_p(g) \pmod{\lambda} \text{ for all primes } p,$$

and g is the only Hecke eigenform in $S_{2\kappa-2}(\Gamma_0(M))$, not a multiple of f, satisfying this congruence for all $p \nmid M$.

Let $\tilde{f}, \tilde{g} \in S^+_{\kappa-(1/2)}(\Gamma_0(4M))$ be images of f and g respectively under Kohnen's correspondence (Theorem 1.1). Then $\tilde{f}, \tilde{g}$ may be scaled in such a way that

(1) *the Fourier coefficients of $\tilde{f}$ are in K, all integral at λ, but not all divisible by λ, and likewise for $\tilde{g}$.*

(2) *There is a congruence of Fourier coefficients*
$$a_n(\tilde{f}) \equiv a_n(\tilde{g}) \pmod{\lambda}$$
for all $n \geq 1$.

The important condition is of course the last one, the existence of the congruence between f and g. The rest, despite their number and complexity, are fairly weak, as the example in §3 will illustrate.

For integers $1 \leq t \leq 2\kappa - 3$ the algebraic parts are defined by
$$L_{\mathrm{alg}}(t, f) := \frac{L(t,f)}{(2\pi i)^t \omega^{(-1)^t}}, \quad L_{\mathrm{alg}}(t, f, \chi_{-D}) := \frac{L(t, f, \chi_{-D})}{i\sqrt{D}(2\pi i)^t \omega^{(-1)^{t-1}}},$$
where ω^+ and ω^- are canonically scaled Deligne periods as in [Du1, §5] (where I called them Ω^+ and Ω^-). These algebraic parts belong to K.

Here we are using integral premotivic structures $\mathcal{M}_{f,\mathrm{dR}}$ and $\mathcal{M}_{f,B}$, and a comparison map $I^\infty : \mathcal{M}_{f,\mathrm{dR}} \otimes \mathbb{C} \to \mathcal{M}_{f,B} \otimes \mathbb{C}$, as in [DFG, §1.2.4, §1.6.2]. The restriction of I^∞ to $\mathrm{Fil}^{k-1}\mathcal{M}_{f,\mathrm{dR}} \otimes \mathbb{C}$ yields isomorphisms $I^\infty : \mathrm{Fil}^{k-1}\mathcal{M}_{f,\mathrm{dR}} \otimes \mathbb{C} \simeq \mathcal{M}^{\pm}_{f,B} \otimes \mathbb{C}$, and $\omega^{\pm}$ are the determinants with respect to the integral structures $\mathrm{Fil}^{k-1}\mathcal{M}_{f,\mathrm{dR}} = \langle f \rangle$ and $\mathcal{M}^{\pm}_{f,B}$. Note that although the definition of Deligne periods in [De1] would be in terms of $(I^\infty)^{-1} : \mathcal{M}^{\pm}_{f,B} \otimes \mathbb{C} \to (\mathcal{M}_{f,\mathrm{dR}}/\mathrm{Fil}^{k-1}\mathcal{M}_{f,\mathrm{dR}}) \otimes \mathbb{C}$, using [De1, Lemma 5.1.6, (5.1.7)] one may show that they are the same.

Lemma 1.4. *The map* $I^\infty : \mathbb{C}f \to \mathcal{M}^{\pm}_{f,B} \otimes \mathbb{C}$ *coincides with Kato's map* per_f*, referred to in* [Kato, Proposition 14.21]*, defined in his §§6.3, 4.10, 4.5.*

This is justified by a close comparison of [Kato, §4.10] and [DFG, §1.2.4]. In particular, $\mathrm{coLie}(\overline{E})^{\otimes(k-2)}$ in [Kato, (4.10.2)] is $\mathcal{F}^k_{\mathrm{dR}}$ in [DFG, §1.2.4], and the resolution referred to in the last paragraph of [DFG, §1.2.4] is the result of tensoring [Kato, (4.10.3)] with (the left hand side of) [Kato, (4.10.2)]. Using this, the relationship between our $\omega^{\pm}$ and the periods Ω (which depend on r) in [Kato, Proposition 14.21] is that
$$\omega^{(-1)^r} = (2\pi i)^{1-k}\Omega.$$
(The factor $(2\pi i)^{1-k}$ is accounted for by its appearance in the trivialisation on the line preceding [DFG, (4) in §1.2.4].) Hence Kato's
$$(2\pi i)^{r-1}\frac{L(k-r,f)}{\Omega} = (2\pi i)^{k-1}\frac{L(k-r,f)}{(2\pi i)^{k-r}\Omega} = \frac{L(k-r,f)}{(2\pi i)^{k-r}\omega^{(-1)^{k-r}}}$$
is our $L_{\mathrm{alg}}(k-r, f)$. Note that for us, the "associated" $\mathcal{O}_\lambda$-lattices T and D in the λ-adic and crystalline realisations, required by [Kato, Proposition 14.21], will be those coming from the integral premotivic structures in [DFG].

To prove Theorem 1.3, we shall consider Siegel modular forms $\hat{f}$ and $\hat{g}$ (Saito-Kurokawa lifts) of genus 2 and weight κ, for a congruence subgroup $\Gamma_0^{(2)}(M)$. The Fourier coefficients of $\hat{f}$ are intimately related to those of $\tilde{f}$, and it will suffice to prove a congruence of Fourier coefficients between $\hat{f}$ and $\hat{g}$ (with appropriate scaling). To prove the congruence, we find multiples of $\hat{f}(Z)\hat{f}(W)$ and $\hat{g}(Z)\hat{g}(W)$ in a formula for the restriction of a certain genus 4 Eisenstein series from $\mathfrak{H}_4$ to $\mathfrak{H}_2 \times \mathfrak{H}_2$. We need the coefficient of $\hat{f}(Z)\hat{f}(W)$ to have λ in the denominator. For this we use a formula of Agarwal and Brown expressing $\langle \hat{f}, \hat{f} \rangle$ (which naturally appears in the denominator of an expression for the coefficient) as a multiple of

$\langle f,f\rangle L(\kappa,f)$. A theorem of Hida and Ribet, that the congruence prime λ appears in the numerator of a ratio of periods $\frac{\langle f,f\rangle}{i\omega^+\omega^-}$, then shows that the first factor $\langle f,f\rangle$ contributes a factor of λ. We also need to apply elements of the Hecke algebra to kill all but the $\hat{f}(Z)\hat{f}(W)$ and $\hat{g}(Z)\hat{g}(W)$ terms without cancelling the λ. For this we use the uniqueness of g to rule out congruences of Hecke eigenvalues between $\hat{f}$ and other Saito-Kurokawa lifts. Any congruences of Hecke eigenvalues between $\hat{f}$ and non-lifts produce elements in a certain Selmer group, by a Ribet-style construction used in [AB1]. By a theorem of Kato, its "order at λ" is bounded by that of $L^M_{\mathrm{alg}}(\kappa,f)$, and any power of λ introduced by killing the non-lift terms is soaked up by the $L(\kappa,f)$ factor in $\langle\hat{f},\hat{f}\rangle$.

We make much use of the work of Agarwal and Brown [AB1], [AB2], but they do not prove congruences of Fourier coefficients between Hecke eigenforms. Their concern is to prove congruences of Hecke eigenvalues between Saito-Kurokawa lifts and non-lifts (and hence construct elements in Selmer groups), by limiting those between different Saito-Kurokawa lifts. By contrast, ours is to limit congruences of Hecke eigenvalues between lifts and non-lifts (using Kato's theorem to bound Selmer groups) enough to allow the deduction of congruences of Fourier coefficients between different Saito-Kurokawa lifts. The way in which the congruence between f and g implies that between $\hat{f}$ and $\hat{g}$ is, as outlined above, quite subtle, starting with the Hida-Ribet theorem about congruence primes appearing in Petersson norms, which depends on congruences being cohomological.

In [Du2] we looked at congruences between newforms of different weights, in a Hida family, and showed that sometimes they can be lifted to half-integral weight. Using Theorem 1.2, it follows that when one twisted L-value vanishes, the other has algebraic part divisible by λ. We made an application to the Bloch-Kato conjecture, especially in the case when the smaller weight is 2. This theme was further developed by McGraw and Ono [MO]. The theorem in this paper is not so suitable for such applications, since heuristics from random matrix theory suggest that for f of weight $2\kappa-2\geq 6$, at most finitely many of the twisted L-values will vanish [CKRS].

Questions by Tobias Berger and Narasimha Kumar during a seminar, and comments by Masataka Chida, Hidenori Katsurada and an anonymous referee, led to improvements to earlier versions of this paper. All data generated or analysed during this study are included in this published article .

2. Congruences between Saito-Kurokawa lifts

Let $\mathrm{Sp}_2(\mathbb{Z}) := \left\{ g\in M_4(\mathbb{Z}) :\ g^t \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \right\}$, and

$$\Gamma_0^{(2)}(M) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z}) :\ \ C\in M\, M_2(\mathbb{Z}) \right\}.$$

Consider any Siegel cusp form $F\in S_\kappa(\Gamma_0^{(2)}(M))$, so F is holomorphic,

$$F((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^\kappa F(Z)$$

for all $Z \in \mathfrak{H}_2 := \{Z \in M_2(\mathbb{C}) :\ {}^tZ = Z, \mathrm{Im}(Z) > 0\}$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(2)}(M)$, with a vanishing condition at cusps. There is a Fourier expansion

$$F(Z) = \sum_S a(F, S) e^{2\pi i \mathrm{tr}(SZ)},$$

where $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, with $a, b, c \in \mathbb{Z}$, $a > 0$, $\mathrm{disc}(S) := b^2 - 4ac < 0$.

Following Agarwal and Brown [AB2, §3] we summarise how one obtains (along the lines of Manickam, Ramakrishnan and Vasudevan [MRV]) a Saito-Kurokawa lift $\hat{f} \in S_\kappa(\Gamma_0^{(2)}(M))$ of the normalised newform $f \in S_{2\kappa-2}(\Gamma_0(M)$. (Note that at the bottom of [AB2, p. 646], "$(2n - j^2)$" should be "$(2n - j)^2$".) First one takes an $\tilde{f} \in S^+_{\kappa-(1/2)}(\Gamma_0(4M))$, determined only up to scaling, using Kohnen's correspondence (Theorem 1.1). Next one applies an isomorphism

$$\mathcal{J} : S^+_{\kappa-(1/2)}(\Gamma_0(4M)) \to J^c_{\kappa,1}(\Gamma_0(M)^J)$$

to a space of Jacobi cusp forms of weight κ and index 1. Then

$$\hat{f}(Z) = \hat{f}\left(\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}\right) := \sum_{m \geq 1} V_m(\mathcal{J}(\tilde{f})) e^{2\pi i m \tau'},$$

where $V_m : J^c_{\kappa,1}(\Gamma_0(M)^J) \to J^c_{\kappa,m}(\Gamma_0(M)^J)$ are certain index-shifting operators.

As in [AB2, Theorem 3.2, Corollary 3.4], if the Fourier coefficients of $\tilde{f}$ are in K, all integral at λ, then it is immediate from the explicit formulas defining $\mathcal{J}$ and the V_m that the same is true of $\mathcal{J}(\tilde{f})$ and $\hat{f}$. We actually need to go in the opposite direction. Choosing a scaling of $\tilde{f}$ and choosing a scaling of $\hat{f}$ are equivalent. By a theorem of Shimura [Sh2], $S_\kappa(\Gamma_0^{(2)}(M))$ has a basis comprising forms with rational Fourier coefficients. For each prime $p \nmid M$ the Hecke operator $T(p)$ on $S_\kappa(\Gamma_0^{(2)}(M))$ preserves rationality of Fourier coefficients. Its eigenvalue on $\hat{f}$ is $a_p(f) + p^{\kappa-2} + p^{\kappa-1}$. Taking the intersection of the kernels of $T(p) - (a_p(f) + p^{\kappa-2} + p^{\kappa-1})$ for sufficiently many primes p, we arrive at a 1-dimensional space spanned by an $\hat{f}$ with coefficients in K. If $-D < 0$ is a fundamental discriminant then the formulas show that for any $r, a \in \mathbb{Z}$ with $a \geq 1$ and $r^2 - 4a = -D$ (one can always choose either $r = 0$ or $r = 1$), the coefficient of $e^{2\pi i(a\tau + \tau' + rz)}$ in $\hat{f}$ is $c(D)$, i.e. $a\left(\hat{f}, \begin{pmatrix} a & r/2 \\ r/2 & 1 \end{pmatrix}\right) = c(D)$, where $\tilde{f} = \sum_{n \geq 1} c(n) q^n$. Hence $c(D) \in K$ for all such D.

By the formula in Theorem 1.1, all the $c(n)$ are determined by the $c(D)$ for fundamental $-D$, in such a way that if we scale $\tilde{f}$ so that the minimum of $\mathrm{ord}_\lambda(c(D))$ (with $-D$ fundamental) is 0, then all the $c(n)$ belong to K, all integral at λ, not all divisible by λ. Clearly all the Fourier coefficients $a(\hat{f}, S)$ are integral at λ, not all divisible by λ. Furthermore, given how the $c(n)$ can be recovered from the $a(\hat{f}, S)$ and the $a_m(f)$, to prove Theorem 1.3 it now suffices to prove the following.

Proposition 2.1. *Let $f, g \in S_{2\kappa-2}(\Gamma_0(M))$, with M odd and squarefree, be normalised newforms, with even $\kappa \geq 6$, and $\lambda \mid \ell$ a prime divisor in a number field K containing all the Hecke eigenvalues of f and g. Suppose the following.*

(1) $\overline{\rho}_{f,\lambda}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ *contains* $\mathrm{SL}_2(\mathbb{F}_\ell)$.
(2) $\ell \nmid (2\kappa - 2)! M \prod_{\text{prime } q \mid M} (q^2 - 1)$.

(3) *There exists a fundamental discriminant $-D<0$ such that $\left(\frac{-D}{p}\right)=\epsilon_p(f)$ for all primes $p\mid M$, and an even character χ of conductor $N>1$, with $M\mid N$ and $\ell\nmid N$, such that*

$$\mathrm{ord}_\lambda\left(\frac{L^N(3-\kappa,\chi)L^N_{\mathrm{alg}}(1,f,\chi)L^N_{\mathrm{alg}}(2,f,\chi)L_{\mathrm{alg}}(\kappa-1,f,\chi_{-D})}{[\Gamma_0^{(2)}(M):\Gamma_0^{(2)}(N)]}\right)\leq 0.$$

(4) $\mathrm{ord}_\lambda(L^M(\kappa,f)/L(\kappa,f))=0$.

(5)
$$a_p(f)\equiv a_p(g)\pmod{\lambda}\ \textit{for all primes } p,$$
and g is the only Hecke eigenform in $S_{2\kappa-2}(\Gamma_0(M))$, not a multiple of f, satisfying this congruence for all $p\nmid M$.

Let $\hat{f},\hat{g}\in S_\kappa(\Gamma_0^{(2)}(M))$ be Saito-Kurokawa lifts of f and g respectively. Then $\hat{f},\hat{g}$ may be scaled in such a way that

(1) *the Fourier coefficients of $\hat{f}$ are in K, all integral at λ, but not all divisible by λ, and likewise for $\hat{g}$.*

(2) *There is a congruence of Fourier coefficients*
$$a(\hat{f},S)\equiv a(\hat{g},S)\pmod{\lambda}$$
for all S.

Proof. By [AB1, Lemma 6.3],
$$\mathcal{E}_M(Z,W)=\sum_{i=1}^{m+r}c_iF_i(Z)F_i^c(W),$$
for $Z,W\in\mathfrak{H}_2$. Here $\mathcal{E}_M(Z,W)$, the restriction to $\mathfrak{H}_2\times\mathfrak{H}_2$ of some Eisenstein series of weight κ on $\mathfrak{H}_4$ (for which we need $\kappa\geq 6$) has rational Fourier coefficients, integral at ℓ (using $\ell\geq 5$ and $\ell\nmid M$), $\{F_1,\ldots,F_{m+r}\}$ is an orthogonal basis for $S_\kappa(\Gamma_0^{(2)}(M))$, all Hecke eigenforms (for all $T(p)$ with $p\nmid M$), and $F^c(W):=\overline{F(-\overline{W})}$. For $1\leq i\leq m$ (and only for those i), F_i belongs to the Saito-Kurokawa subspace, meaning that
$$T(p)(F_i)=(a_p(h_i)+p^{\kappa-2}+p^{\kappa-1})F_i,\ \text{ for all primes } p\nmid M,$$
for a newform h_i of weight $2\kappa-2$ and level $\Gamma_0(M')$ for some $M'\mid M$, cf. [AB1, Definition 5.3]. Note that $F_i^c=F_i$ for all $1\leq i\leq m$. We choose $h_1=f$ and $h_2=g$.

First we have to eliminate the possibility that $h_i=f$ or g for some $3\leq i\leq m$, by checking the proof of [AB1, Theorem 5.4]. As in [DPSS, §3.1], we may assume that the adelization of F_i generates an irreducible automorphic representation of $\mathrm{GSp}_2(\mathbb{A})$, type IIb at primes $p\nmid N$ and type VIb at $p\mid N$. (That it is necessarily non-spherical, hence VIb rather than IIb, at $p\mid N$, follows from [P-S, Theorem 2.4(2)].) Now that the local components of the automorphic representation are uniquely determined, it then follows from [Sc, Theorem 5.2(ii)] that F_i is a scalar multiple of F_1 or F_2, which is a contradiction.

By assumption (the uniqueness in (5)), for each $3\leq i\leq m$ there exists a prime $q_i\nmid M$ such that $a_{q_i}(h_i)\not\equiv a_{q_i}(f)\pmod{\lambda}$. (We temporarily extend K to contain all the Hecke eigenvalues for $F_1,\ldots,F_{m+r}$.) It follows that if $\mu_p(F_i)$ denotes the eigenvalue of $T(p)$ acting on F_i then
$$\mu_{q_i}(F_i)\not\equiv\mu_{q_i}(F_1)\pmod{\lambda},\quad\text{for } 3\leq i\leq m.$$

Let $\mathbb{T}$ be an algebra of Hecke operators, with coefficients in the localisation $\mathcal{O}_{K,(\lambda)}$, acting on $S_\kappa(\Gamma_0^{(2)}(M))$, cf. [AB1, §4.2, §7.3]. (Though they use more, $T(p)$ for $p \nmid M$ would suffice.) Let $\mathbb{T}^X$ and $\mathbb{T}^Y$ be the quotients through which $\mathbb{T}$ acts on the subspaces $X := \mathbb{C}\hat{f}$ and $Y := \langle F_{m+1}, \ldots, F_{m+r}\rangle_{\mathbb{C}}$, with surjective restriction homomorphisms $\pi_X : \mathbb{T} \to \mathbb{T}^X$ and $\pi_Y : \mathbb{T} \to \mathbb{T}^Y$, kernels I_X and I_Y respectively. Using the elementary isomorphisms

$$\frac{\mathbb{T}^X}{\pi_X(I_Y)} \simeq \frac{\mathbb{T}}{I_Y + I_X} \simeq \frac{\mathbb{T}^Y}{\pi_Y(I_X)},$$

there exists an element $t \in \mathbb{T}$ such that $t(F_i) = 0$ for all $m+1 \leq i \leq m+r$, and $t(\hat{f}) = \alpha\hat{f}$, where $\mathrm{ord}_\lambda(\alpha) = \mathrm{ord}_\lambda\left(\mathrm{Fitt}\left(\frac{\mathbb{T}^Y}{\pi_Y(I_X)}\right)\right)$.

For every $3 \leq i \leq m$, there exists a prime $q_i \nmid M$ such that

$$\mu_{q_i}(F_i) \not\equiv \mu_{q_i}(F_1) \pmod{\lambda}.$$

Recall that

$$\mathcal{E}_M(Z, W) = \sum_{i=1}^{m+r} c_i F_i(Z) F_i^c(W).$$

Now apply $t\prod_{i=3}^m (T(q_i) - \mu_{q_i}(F_i))$ to both sides (in the variable Z). This kills all the terms for $i \geq 3$ on the right hand side. If we further take a partial Fourier coefficient of $e^{2\pi i \mathrm{Tr}(SW)}$, with $\mathrm{disc}(S) = -D$ a fundamental discriminant, we get

$$\mathcal{F}(Z) = b_1\hat{f}(Z) + b_2\hat{g}(Z),$$

where the Fourier coefficients of $\mathcal{F}$ are integral at λ and

$$b_1 = c_1 c(D)\alpha \prod_{i=3}^m (\mu_{q_i}(F_1) - \mu_{q_i}(F_i)),$$

so that if we choose D with $\mathrm{ord}_\lambda(c(D)) = 0$ then

$$\mathrm{ord}_\lambda(b_1) = \mathrm{ord}_\lambda(c_1) + \mathrm{ord}_\lambda\left(\mathrm{Fitt}\left(\frac{\mathbb{T}^Y}{\pi_Y(I_X)}\right)\right).$$

We aim now to show that $\mathrm{ord}_\lambda(b_1) < 0$. According to [AB1, Theorem 6.2] (and with a less peculiar normalisation of the standard L-function),

$$c_1 = \mathcal{B}_{\kappa,M}\frac{L^M(3-\kappa, \hat{f}, \mathrm{st}, \chi)}{\pi^3\langle\hat{f}, \hat{f}\rangle},$$

with $\mathcal{B}_{\kappa,M} = \frac{\pm 2^{2\kappa-3}}{3[\mathrm{Sp}_2(\mathbb{Z}):\Gamma_0^{(2)}(N)]}$. By [AB1, Theorem 5.8], which is also [AB2, Corollary 4.7],

$$\langle\hat{f}, \hat{f}\rangle = \mathcal{A}_{\kappa,M}\frac{c(D)^2}{D^{\kappa-3/2}}\frac{L(\kappa, f)}{\pi L(\kappa-1, f, \chi_{-D})}\langle f, f\rangle,$$

with $\mathcal{A}_{\kappa,M} = \frac{M^\kappa \zeta_M(4)\zeta_M(1)^2(\kappa-1)\prod_{p\mid M}(1+p^2)(1+p^{-1})}{2^{\omega(M)+3}[\Gamma_0(M):\Gamma_0(4M)][\mathrm{Sp}_2(\mathbb{Z}):\Gamma_0^{(2)}(M)]}$. Here, $\zeta_M(s)$ is a product of Euler factors just for primes $p \mid M$. Also, looking at [AB2, Theorem 4.1], $-D < 0$ is a fundamental discriminant such that $\left(\frac{-D}{p}\right) = \epsilon_p(f)$ for all primes $p \mid M$ (which therefore ought to be a condition in [AB1, Theorems 5.8, 6.5]). Note that in their citation of [Ko2, Corollary 1], it is not necessary to view $\tilde{f}$ as a Shintani lift.

Using the conditions on ℓ, the fact that $[\Gamma_0(M):\Gamma_0(4M)]=6$, and that $L(s,\hat{f},\mathrm{st})=\zeta(s)L(s+(\kappa-2),f)L(s+(\kappa-1),f)$, we need to show that

$$\mathrm{ord}_\lambda\left(\frac{\alpha D^{\kappa-3/2}}{c(D)^2}\frac{L^N(3-\kappa,\chi)L^N(1,f,\chi)L^N(2,f,\chi)L(\kappa-1,f,\chi_{-D})}{\pi^2 L(\kappa,f)\langle f,f\rangle[\Gamma_0^{(2)}(M):\Gamma_0^{(2)}(N)]}\right)<0.$$

Multiplying both the numerator and the denominator by $(2\pi i)^{\kappa+2}\sqrt{-D}\omega^-(\omega^+)^2$, and using the hypothesis (3) of the proposition, it is good enough to show that

$$\mathrm{ord}_\lambda\left(\frac{\langle f,f\rangle}{i\omega^+\omega^-}\right)+\mathrm{ord}_\lambda\left(\frac{L_{\mathrm{alg}}(\kappa,f)}{\alpha}\right)>0.$$

As in [Du1, (4)], using work of Hida [Hi1, §6], the ratio $\frac{\langle f,f\rangle}{i\omega^+\omega^-}$ is, up to S-units (where S is the set of primes dividing $(2\kappa-2)!M$), an integral cohomological congruence ideal η_f. A good additional reference is [Hi2, (5.18)]. The $\langle\zeta_+,\zeta_-\rangle$ in [Hi2, Theorem 5.16] is our η_f. A useful alternative reference is [Be, Theorem 5.4.13(a)], though note that in the proof, his $(\Omega_f^+\Omega_f^-)^2$ should be $(\Omega_f^+\Omega_f^-)$. Here, as in [Hi1], a map defined using integration takes the place of our I^∞ (cf. Lemma 1.4), in the definition of the periods. They are presumably the same, but in any case the computation is formally identical, to prove the same relation for our periods. The necessary compatibility between cup-product and Petersson norm, expressed in [Be, Lemma 5.3.26], may be replaced by the version in [DFG, §1.5.1].

It follows from a theorem of Ribet [Ri2, Theorems 1.3, 1.4] (which removes an ordinarity assumption from an earlier theorem of Hida) that, since λ is a "congruence prime" for f (and $\ell\nmid k!N$), λ divides η_f, which helps. (Although Hida and Ribet worked with rational coefficients, combining Galois orbits of newforms, this is not necessary.)

To obtain $\mathrm{ord}_\lambda(b_1)<0$, it remains to show that

$$\mathrm{ord}_\lambda\left(\mathrm{Fitt}\left(\frac{\mathbb{T}^Y}{\pi_Y(I_X)}\right)\right)\leq\mathrm{ord}_\lambda(L_{\mathrm{alg}}(\kappa,f)).$$

The left hand side measures mod λ congruences of Hecke eigenvalues between $\hat{f}$ and the non-lifts $F_{m+1},\dots,F_{m+r}$. Suppose that $m+1\leq j\leq m+r$ and that

$$\mu_q(F_j)\equiv\mu_q(F_1)\pmod{\lambda},\quad\text{for all primes } q\nmid M.$$

By [AB1, Theorem 7.3, Theorem 7.4, Corollary 7.5], using that

$$\ell\nmid(2\kappa-2)!M\prod_{\text{prime } q\mid M}(q^2-1),$$

F_j is not a weak endoscopic lift, and since also it does not belong to the Saito-Kurokawa subspace, the 4-dimensional λ-adic representation $\rho_{F_j,\lambda}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ associated with F_j by Weissauer [We] must be irreducible, cf. [AB1, beginning of §7].

The congruence of Hecke eigenvalues (viewed as traces of Frobenius elements) implies that a residual representation $\overline{\rho}_{F_j,\lambda}$ has composition factors $\overline{\rho}_{f,\lambda}$ and the Tate twists $\mathbb{F}_\lambda(1-\kappa),\mathbb{F}_\lambda(2-\kappa)$ of the trivial representation. Using the irreducibility of $\rho_{F_j,\lambda}$, and adapting an argument used by Ribet [Ri1] it is possible to choose a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant lattice for $\rho_{F_j,\lambda}$ whose reduction provides a non-split extension of $\mathbb{F}_\lambda(2-\kappa)$ by $\overline{\rho}_{f,\lambda}$, hence of $\mathbb{F}_\lambda$ by $\overline{\rho}_{f,\lambda}(\kappa-2)$. As in the proof of [AB1, Theorem 8.8], one can show that this gives a non-zero class in $H^1(\mathbb{Q},W_{f,\lambda}(\kappa-2))$, where $\rho_{f,\lambda}$

is on a space $V_{f,\lambda}$, with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant lattice $T_{f,\lambda}$, and $W_{f,\lambda} := V_{f,\lambda}/T_{f,\lambda}$. Furthermore, this class satisfies the Bloch-Kato local conditions at all primes $p \nmid M$, including $p = \ell$. In the notation of [AB1], it gives us a non-zero element of the Selmer group $\mathrm{Sel}_{\{p|\ell M\}}(\{p \mid M\}, W_{f,\lambda}(\kappa - 2))$.

In fact, Theorem 8.8 in [AB1] gives us something stronger, that

$$\mathrm{ord}_\lambda \left(\mathrm{Fitt} \left(\frac{\mathbb{T}^Y}{\pi_Y(I_X)} \right) \right) \leq \mathrm{ord}_\lambda \left(\mathrm{Fitt} \left(\mathrm{Sel}_{\{p|\ell M\}}(\{p \mid M\}, W_{f,\lambda}(\kappa - 2)) \right) \right).$$

Letting $k = 2\kappa - 2, r = \kappa - 2, k - r = \kappa, T = T_{f,\lambda}(\kappa - 2)$ in [Kato, Proposition 14.21(2)], (where $\mathcal{S}(T(r))$ should be $\mathcal{S}(T)$ on the left hand side), it would say that $\mathrm{ord}_\lambda (L_{\mathrm{alg}}(\kappa, f))$ is what the Bloch-Kato conjecture [BK] predicts it should be, as long as (in his notation) $\mu = 1$, cf. the end of [Kato, §14.5]. (Recall that our Deligne period ω^+ is $(2\pi i)^{1-k}\Omega$ in [Kato, Proposition 14.21].) Using the condition that $\overline{\rho}_{f,\lambda}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ contains $\mathrm{SL}_2(\mathbb{F}_\ell)$ (which implies the condition [Kato, (12.5.2)] by [Se, §3.4, Lemma 3]), it follows from [Kato, Theorem 14.5(3)] and its proof that $\mathrm{ord}_\lambda(\mu) \geq 0$. Hence [Kato, Proposition 14.21(2)] says that $\mathrm{ord}_\lambda (L_{\mathrm{alg}}(\kappa, f))$ is at least what the Bloch-Kato conjecture predicts it should be. Since the truth of the Bloch-Kato conjecture is invariant under relaxing local conditions and dropping Euler factors at a finite set of primes, this implies that

$$\mathrm{ord}_\lambda \left(\mathrm{Fitt} \left(\mathrm{Sel}_{\{p|\ell M\}}(\{p \mid M\}, W_{f,\lambda}(\kappa - 2)) \right) \right) \leq \mathrm{ord}_\lambda \left(L^M_{\mathrm{alg}}(\kappa, f) \right).$$

Using hypothesis (4), this gives us the desired

$$\mathrm{ord}_\lambda \left(\mathrm{Fitt} \left(\frac{\mathbb{T}^Y}{\pi_Y(I_X)} \right) \right) \leq \mathrm{ord}_\lambda(L_{\mathrm{alg}}(\kappa, f)).$$

To complete the proof, recall the equation

$$\mathcal{F}(Z) = b_1 \hat{f}(Z) + b_2 \hat{g}(Z),$$

where $\mathcal{F}(Z)$ has integral Fourier coefficients, and we now know that $\mathrm{ord}_\lambda(b_1) < 0$. Dividing both sides of the equation by b_1, we see that there is a congruence of Fourier coefficients of $\hat{f}$ and the re-scaled $(b_2/b_1)\hat{g}$. Note that $\hat{f}$ is scaled as in the statement of the proposition, and the congruence forces $(b_2/b_1)\hat{g}$ to be likewise. $\square$

Remark 2.2. More generally, if f and g are congruent modulo λ^s with $s > 0$, one may prove similarly a congruence mod λ^s between $\tilde{f}$ and $\tilde{g}$.

3. An example

The 34-dimensional space $S_{10}(\Gamma_0(35))$ contains normalised newforms

$$f = q + 28q^2 - 116q^3 + 272q^4 + 625q^5 + \ldots$$

and

$$g = q + (-12 + \sqrt{2})q^2 + (-87 + 108\sqrt{2})q^3 + (-360 - 48\sqrt{2})q^4 + 625q^5 + \ldots,$$

among other newforms with coefficient fields of degrees 4, 5 and 6. According to the computer algebra package Magma [BCP], the q-expansions of f and g are congruent modulo $\lambda = (199, \sqrt{2} - 20)$, with $K = \mathbb{Q}(\sqrt{2})$ and $\ell = 199$, at least as far as the coefficients of q^{100}. We check that all the conditions of Theorem 1.3 are satisfied by this example.

(1) We apply a theorem of Billerey and Dieulefait [BD, Introduction, Square-free level case]. Since $199 \nmid 35$, $199 > 4(10) - 3$, and none of $5^8, 5^{10}$, 7^8 or 7^{10} is congruent to 1 modulo 199, $\overline{\rho}_{f,199}$ is irreducible and has image of order divisible by 199. By a theorem of Dickson[Di], the image of $\overline{\rho}_{f,199}$ in $\mathrm{PGL}_2(\mathbb{F}_{199})$ contains $\mathrm{PSL}_2(\mathbb{F}_{199})$. If the image of $\overline{\rho}_{f,199}$ (in $\mathrm{GL}_2(\mathbb{F}_{199})$) does not contain $\mathrm{diag}(-1,-1)$ then every element of $\mathrm{SL}_2(\mathbb{F}_{199})$ is uniquely ± 1 times something in this image, giving a well-defined character from $\mathrm{SL}_2(\mathbb{F}_{199})$ to $\{\pm 1\}$. Since $\mathrm{SL}_2(\mathbb{F}_{199})$ has no non-trivial abelian character (the smallest degree of an irreducible character being $\frac{199-1}{2}$), it follows that the image of $\overline{\rho}_{f,199}$ contains $\mathrm{SL}_2(\mathbb{F}_{199})$. (This argument was inspired by the proof of (3.1) in [Ri3].)

(2) $199 \nmid 10!(35)(5^2-1)(7^2-1)$, whose prime divisors are $2, 3, 5$ and 7.

(3) We take $N = M$, and χ quadratic of conductor 5. It is easy to check that $\mathrm{ord}_{199}(L^N(-3,\chi)) = 0$, using Bernoulli polynomials. Using the Magma command $\mathrm{LRatio}(f, 6)$, where

$$f := \mathrm{NewformDecomposition}(\mathrm{CuspidalSubspace}(\mathrm{ModularSymbols}(35, 10)))[1],$$

we find $\mathrm{LRatio}(6, f) = 24843/2$, which factorises as $3 \cdot 7^2 \cdot 13^2/2$, implying that $\mathrm{ord}_{199}(L_{\mathrm{alg}}(6, f)) = 0$. We don't really need that, but it shows that in this example there are no congruences of Hecke eigenvalues between $\hat{f}$ and non-lifts.

Since $\left(\frac{-8}{5}\right) = \left(\frac{-8}{7}\right) = \epsilon_5(f) = \epsilon_7(f) = -1$, we let $-D = -8$, and aim to show that $\mathrm{ord}_{199}(L_{\mathrm{alg}}(5, f, \chi_{-8})) = 0$. Letting f be a 2-dimensional space of modular symbols created in Magma as above, and $\phi := \mathrm{IntegralMapping}(f)$ a projection into this space, if we apply ϕ to the winding element $X^5Y^3\{0,\infty\}$ then we get $(24843, 0)$. The 24843 recovers what we obtained earlier using $\mathrm{LRatio}(6, f)$. Applying ϕ instead to a twisted winding element

$$(8X+Y)^4Y^4\{-1/8,\infty\} + (8X+3Y)^4Y^4\{-3/8,\infty\}$$
$$-(8X-3Y)^4Y^4\{3/8,\infty\} - (8X-Y)^4Y^4\{1/8,\infty\},$$

we get $(-13829760, 0)$. The 0 is a check on the correctness of the computation, and up to small prime factors, the $-13829760 = -2^7 \cdot 3^2 \cdot 5 \cdot 7^4$ gives us $L_{\mathrm{alg}}(5, f, \chi_{-8})$, by [MTT, (8.6), §3(i)].

If we plug in $X^aY^{8-a}\{0,\infty\}$ with $0 \le a \le 8$ and a even, we always get a multiple of $(-781, 1)$. So the $\pm$-parts under the natural complex conjugation action must be spanned by $v^+ := (1, 0)$ and $v^- := (-781, 1)$. Applying ϕ to the twisted winding elements

$$Y^8\{-1/5,\infty\} + Y^8\{-4/5,\infty\} - Y^8\{-2/5,\infty\} - Y^8\{-3/5,\infty\}$$

and

$$(5X+Y)Y^7\{-1/5,\infty\} + (5X+4Y)Y^7\{-4/5,\infty\}$$
$$-(5X+2Y)Y^7\{-2/5,\infty\} - (5X+3Y)Y^7\{-3/5,\infty\},$$

we obtain $2^{11} \cdot 3^2 \cdot 5^4 \cdot 7^4 \cdot 11 v^+$ and $2^3 \cdot 5^5 \cdot 7^4 \cdot 13 \cdot 1511 v^-$, respectively. This shows that $\mathrm{ord}_{199}(L_{\mathrm{alg}}(1, f, \chi)) = \mathrm{ord}_{199}(L_{\mathrm{alg}}(2, f, \chi)) = 0$. The factors by which these are multiplied to get $L^N_{\mathrm{alg}}(1, f, \chi)$ and $L^N_{\mathrm{alg}}(2, f, \chi)$ are $1 + 7^4 7^{-1} = 2^3 \cdot 43$ and $1 + 7^4 7^{-2} = 2 \cdot 5^2$.

(4) The ratio $L^M_{\mathrm{alg}}(6, f)/L_{\mathrm{alg}}(6, f)$ is a product of factors $(1 - 5^{-2}) = -\frac{2^3 \cdot 3}{5^2}$ and $(1 - 7^{-2}) = -\frac{2^4 \cdot 3}{7^2}$.

(5) Since the Sturm bound [St] is $\frac{10}{12} \cdot 35 \cdot \left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right) = 40$, the congruence already observed experimentally between f and g actually holds for all coefficients. The uniqueness of g is easily verified using the command Reductions$(g, 199)$ in Magma.

References

[AB1] M. Agarwal, J. Brown, On the Bloch-Kato conjecture for elliptic modular forms of square-free level, *Math. Z.* **276** (2014), 889–924.

[AB2] M. Agarwal, J. Brown, Saito-Kurokawa lifts of square-free level, *Kyoto J. Math.* **55** (2015), 641–662.

[Be] J. Bellaïche, *The Eigenbook. Eigenvarieties, families of Galois representations, p-adic L-functions*, Pathways in Mathematics, Birkhäuser, 2021.

[BD] N. Billerey, L. V. Dieulefait, Explicit large image theorems for modular forms, *J. London Math. Soc.* **89** (2014), 499–523.

[BK] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, *The Grothendieck Festschrift Volume I*, Progress in Mathematics, 86, Birkhäuser, Boston, 1990, 333-400.

[BCP] W. Bosma, J. J. Cannon, C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* **24** (1997), 235–265.

[CKRS] J. B. Conrey, J. P. Keating, M. O. Rubinstein, N. C. Snaith, On the frequency of vanishing of quadratic twists of modular L-functions, *Number theory for the millennium, I* (Urbana, IL, 2000), 303–315, A K Peters, Natick, MA, 2002.

[De1] P. Deligne, Valeurs de Fonctions L et Périodes d'Intégrales, *AMS Proc. Symp. Pure Math.*, Vol. 33 part 2, 1979, 313-346.

[De2] P. Deligne, Formes modulaires et représentations l-adiques, *Sém. Bourbaki*, éxp. 355, Lect. Notes Math., Vol. 179, Springer, Berlin, 1969, 139-172.

[DFG] F. Diamond, M. Flach, L. Guo, The Tamagawa number conjecture of adjoint motives of modular forms, *Ann. Sci. École Norm. Sup. (4)* **37** (2004), 663–727.

[Di] L. E. Dickson, *Linear groups with an exposition of the Galois field theory.* Teubner, Leipzig, 1901.

[DPSS] M. Dickson, A. Pitale, A. Saha, R. Schmidt, Explicit refinements of Böcherer's conjecture for Siegel modular forms of squarefree level, *J. Math. Soc. Japan* **72** (2020), 251–301.

[Du1] N. Dummigan, Symmetric square L-functions and Shafarevich-Tate groups, II, *Int. J. Number Theory* **5** (2009), 1321–1345.

[Du2] N. Dummigan, Congruences of modular forms and Selmer groups, *Math. Res. Lett.* **8** (2001), 479–494.

[DK] N. Dummigan, S. Krishnamoorthy, Lifting congruences to weight 3/2, *J. Ramanujan Math. Soc.* **32** (2017), 431–440.

[Hi1] H. Hida, Congruences for cusp forms and special values of their zeta functions, *Invent. math.* **63** (1981), 225–261.

[Hi2] H. Hida, *Modular forms and Galois cohomology.* Cambridge University Press, 2000.

[Kato] K. Kato, p-adic Hodge theory and values of zeta functions of modular forms, *Astérisque,* tome 295 (2004), 117–290.

[Ko1] W. Kohnen, Newforms of half-integral weight, *J. Reine Angew. Math.* **333** (1982), 32–72.

[Ko2] W. Kohnen, Fourier coefficients of modular forms of half-integral weight, *Math. Ann.* **271** (1985), 237–268.

[Ma] Y. Maeda, A congruence between modular forms of half-integral weight, *Hokkaido Math. J.* **12** (1983), 64–73.

[MRV] M. Manickam, B. Ramakrishnan, T. C. Vasudevan, On Saito-Kurokawa liftings for congruence subgroups, *Manuscripta Math.* **81** (1993), 161–182.

[MTT] B. Mazur, J. Tate, J. Teitelbaum, On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer, *Invent. math.* **84** (1986), 1–48.

[MO] W. J. McGraw, K. Ono, Modular form congruences and Selmer groups, *J. London Math. Soc.* **67** (2003), 302–318.

[P-S] I. I. Piatetski-Shapiro, On the Saito-Kurokawa Lifting, *Invent. math.* **71** (1983), 309–338.

[Ri1] K. Ribet, A modular construction of unramified p-extensions of $\mathbb{Q}(\mu_p)$, *Invent. math.* **34** (1976), 151–162.

[Ri2] K. Ribet, Mod p Hecke operators and congruences of modular forms, *Invent. math.* **71** (1983), 193–205.

[Ri3] K. Ribet, On ℓ-adic representations attached to modular forms, *Invent. math.* **28** (1975), 245–275.

[Sc] R. Schmidt, On classical Saito-Kurokawa liftings, *J, reine angew. Math.* **604** (2007), 211–236.

[Se] J.-P. Serre, *Abelian ℓ-adic representations and elliptic curves,* McGill University lecture notes written with the collaboration of Willem Kuyk and John Labute, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[Sh1] G. Shimura, On modular forms of half-integral weight, *Ann. of Math.* **97** (1973), 440–481.

[Sh2] G. Shimura, On the Fourier coefficients of modular forms in several variables, *Nachr. Akad. Wiss. Göttingen Math. -Phys. Kl. II* (1975), 261–268.

[St] J. Sturm, On the congruence of modular forms, *Number Theory (New York, 1984–1985),* Lect. Notes Math., Vol. 1240, Springer, Berlin, 1984, 275–280.

[We] R. Weissauer, Four dimensional Galois representations, *Astérisque* **302** (2005), 67–150.

University of Sheffield, School of Mathematics and Statistics, Hicks Building, Hounsfield Road, Sheffield, S3 7RH, U.K.

Email address: `n.p.dummigan@shef.ac.uk`