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Internal solitary waves with a weakly stratified critical layer

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Motivated by observations of solitary waves in the ocean and atmosphere, this paper considers the evolution of long weakly nonlinear internal waves in an incompressible Boussinesq fluid. The motion is restricted to the vertical plane. The basic state consists of stable horizontal shear flow and density stratification. On a long time scale, the waves evolve and reach a quasi-steady régime where weak nonlinearity and weak dispersion are in balance. In many circumstances, this régime is described by a Korteweg-de-Vries equation. However, when the linear long-wave speed equals the basic flow velocity at a certain height, the critical level, the traditional assumption of weak nonlinearity breaks down due to the appearance of a singularity in the leading-order modal equation, implying a strong modification of the flow in the so-called critical layer. Since the relevant geophysical flows have high Reynolds and Péclet numbers, we invoke nonlinear effects to resolve this singularity. Viscosity and thermal conductivity are considered small but finite. Their presence renders the nonlinear-critical-layer solution unique. Crucially, the density stratification degree is assumed small at the critical level; this has the consequence that the leading-order singularity is then identical to that in an unstratified flow. Thus the asymptotic methodology employed previously for that case can be adapted to this present study. In this critical layer, the flow is fully nonlinear but laminar and quasi-steady, with a strong rearrangement of the buoyancy and vorticity contours. This inner flow is matched at the edges of the critical layer with the outer flow. The final outcome for spatially localized solutions is an integro-differential evolution equation, whose form depends on the critical-layer shape, and especially on the wave polarity, that is, depression or elevation. For a steady travelling wave, this evolution equation when expressed in terms of the streamfunction amplitude is not a Korteweg-de Vries equation, as it contains additional nonlinear terms necessary at a certain order of the asymptotic expansion when matching with the inner flow. However, this steady evolution equation can be transformed with an appropriate change of variables into a Korteweg-de-Vries equation. An analysis of the wave mean flow interaction is given. The horizontal basic stable flow is altered at the critical level at a slow viscous time scale by the nonlinear D-wave in the quasi-steady state régime. In most cases, the mean kinetic energy is likely to decay at the same time scale. The D-wave critical-layer thickness is found inversely proportional to the amplitude of the leading-order viscous outer flow. The D-wave critical-layer symmetry vis-à-vis the critical level is broken; the departure is proportional to the local Richardson number, and so is greater than the analogous asymmetry encountered in an unstratified shear flow. The mean flow horizontal momentum and kinetic energy are unchanged by the nonlinear E-wave in the quasi-steady régime, the exchanges taking place only during the formation of the critical layer and in the transition régime when the nonlinear E-wave is unsteady.
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I. INTRODUCTION

Internal gravity waves are frequently observed in the atmosphere and ocean, where they play an important dynamical role, transporting energy over large distances, can produce strong wind or current shear, and may generate turbulence. A class of internal gravity waves that is of particular interest to mesoscale dynamics is the internal solitary wave. This is a nonlinear long localized wave that can maintain its structure due a balance between the steepening effect of nonlinearity and linear wave dispersion. This balance allows the solitary wave to emerge as an asymptotic solution for a large class of initial-value problems. They are a robust feature under stable background conditions and are then readily observable.[1–4]

In the atmosphere, solitary waves can be generated by thunderstorm outflows, mid-latitude cold fronts or tropical sea-breeze fronts.[5,6] Atmospheric internal solitary waves can be divided into two classes; those for which the waves are confined to the lower few kilometers of the troposphere and those that occupy the entire troposphere. The Earth's surface acts as the lower boundary, while the upper boundary is a low-level inversion in the first class and the tropopause in the second class. In the ocean, internal solitary waves are often generated by the interaction between tidal currents and topography.[4,7] Of topical interest, we note that oceanic internal solitary waves can also be induced by the deformation of a long surface water waves such as a tsunami. When a tsunami approaches the continental slope, oceanic internal waves may be generated when the pycnocline is displaced by the tsunami as it propagates up the slope. Moreover tsunami wavelengths are sufficiently long that a possibility exists that they might generate atmospheric internal gravity waves that propagate up to the ionosphere, and then create a disturbance travelling coherently with the ocean wave below. Such ionospheric disturbances can be observed using measurements of ionospheric electron density by Global Positioning System (GPS) networks. Observations of long ionospheric internal waves following the 2010 Chilean tsunami have been reported for instance by Hawaii based GPS network.[8]

Weakly nonlinear, long-wave theories provide good models for internal solitary waves.[9] These theories assume that internal solitary waves propagate in a waveguide, within which the background stratification and flow vary as a function of height, and which has upper and lower boundaries which effectively trap the waves, and ensure that only horizontal propagation is allowed. The Korteweg-de Vries (KdV) equation was derived for internal solitary waves in this weakly nonlinear, long-wave limit.[10,11] The coefficients of the KdV equation depend on the background properties of the waveguide. This model and its various extensions are appropriate for the coastal ocean, and possibly also for the atmosphere when the waveguide consists either of the low-level inversion where the wave energy is effectively trapped, or of the whole troposphere, above which lies a strongly stable stratosphere. However, in the deep ocean when the near-surface pynocline lies above a deep weakly stratified layer, or in the lower atmosphere when a low-level inversion is capped by a deep weakly stratified atmosphere, it may be more appropriate to use a model equation of the Benjamin-Ono type.[2] Within the framework of these model equations, internal solitary waves are very robust, and long-lived. Indeed, observations in the ocean and atmosphere confirm these predictions, see the review articles.[12,13] However, for large amplitudes, these waves may become unstable due to shear instability, leading to turbulent patches or complete convective overturning.

Another fundamental mechanism, which can lead to the breaking of an internal solitary wave, is the situation when the solitary wave rides on a background mean flow containing a critical level. In this case, there is a strong interaction between the wave and the mean flow in the vicinity of the critical level, and this aspect is the focus of this paper. The "critical layer interaction" is amenable to a self-consistent analytical theory for that subclass of cases when the breaking zone is very narrow and occurs around the critical level where the linear-long-wave phase speed equals the velocity of the background mean flow.[14] Here, linear inviscid theory fails, and needs to be replaced by a theory in which nonlinearity and/or diffusion terms are invoked. In the oceanic and atmospheric context, nonlinearity is the dominant effect, and so here, we shall present a nonlinear critical-layer theory. In the asymptotic theory we shall describe, the flow field is wave-like in the adjoining regions outside a thin critical layer, within which it is essentially nonlinear.[15] In the absence of critical layers, it can be shown that a weakly nonlinear wave does not exchange energy and momentum with the background flow, and the effect of the background shear is purely kinematic. But when a critical level is present, the effect of this shear on the wave is essential to the wave dynamics, which is strongly nonlinear. However, it transpires diffusion cannot be neglected completely. Although the diffusion terms in the momentum and buoyancy equations, whether laminar or "turbulent," are not used to remove the singularity in the critical layer, as this is achieved through the nonlinear terms, they are needed as perturbations in order to obtain a viscous secular condition, valid when the viscosity and thermal conduction go to zero, which selects a unique solution for our vorticity and buoyancy equations in the critical layer. This is especially the case within the closed region bounded by certain separatrices (often called the cat's eye) where we use an extension of the Prandtl-Batchelor theorem. A further consequence is that the small diffusive terms induce the generations of mean flow and buoyancy on either side of the critical layer, due to wave-mean flow interactions in the critical layer. This is a very slow process, and the relevant time scales are proportional to the rescaled viscosity and thermal conductivity coefficients.

When a finite-period linear internal wave is considered[16] propagating vertically and incident on a critical level, the direction of the energy flux is determined by the local Richardson number J_c at the critical level for the case when viscosity dominates. When $J_c > 1/4$, they showed that the wave energy is absorbed by the background flow near the critical level. For $J_c < 1/4$, the energy flux can go in either direction, depending on J_c and the wavelength of the wave. But when nonlinearity is the dominant mechanism in the critical layer, these results are drastically changed.[17–20] In particular, the energy exchange is strongly influenced by the critical layer-induced mean fields of current and buoyancy. For the analogous problem of a barotropic Rossby wave, in the long wavelength limit, the energy flux directions and amplitudes crucially depend on the basic-flow profile.[20] However, the energy and momentum exchanges are reduced in the asymptotic quasi-steady régime following the formation of the critical layer to the mean shear flow evolution, since the nonlinear Rossby wave conserves its shape in the frame moving with the constant wave speed.

These same conclusions seem to hold for slightly stratified fluids ($J_c \ll 1$). Generally, for stratified flows, the energy exchange during the critical-layer formation depends on the local Richardson number J_c and the mean flow and buoyancy profiles.[17] A steady stratified critical layer partially reflects and transmits an incident internal wave according to the values of a parameter λ, which measures the ratio of viscous diffusion to nonlinear advection, and the local Richardson number J_c. The reflection coefficient $|R|$ decreases as λ and J_c increase and $|R|$ is usually small. The transmission coefficient $|Tr|$ also decreases as J_c increases and $|Tr|$ tends to $\exp(-\sqrt{J_c - 1/4}\,\pi)$ as $\lambda \to \infty$ conforming with the viscous critical-layer theory. The phase jump ϕ_{jump} asymptotes to $-\pi$ as J_c increases. The vorticity jump is of order one when $J_c = O(1)$, and decreases as λ and J_c increase. The critical-layer pattern is not symmetric *vis-à-vis* the critical level owing to assumption of a small but finite viscosity, that is, when $\lambda = O(1)$. In the quasi-steady régime, the wave is also partially reflected and transmitted by diffusion boundary layers. The latter are located on either side of the critical layer and are formed by the critical-layer induced mean flow. They are therefore created during the formation of the critical layer, but they are still evolving in the asymptotic quasi-steady state by the slow diffusion of momentum from the critical layer. The wave after passing over the first diffusion boundary layer, is then partially reflected, absorbed, and transmitted by the critical layer. The transmitted wave is then scattered from the second diffusion boundary layer through the critical layer. These diffusion boundary layers evolve in time on the classic diffusion length scale $\sqrt{\lambda T}$ (T is a slow time variable defined later), and consequently, the reflected and transmitted coefficients also evolve on this slow length scale.[18] Similarly, the wave momentum flux and the vorticity jump also depend on it. If J_c is not close to 1/4, then the final transmission coefficient will be very small. In the quasi-steady régime, in the inviscid assumption, the internal wave is completely reflected and no transmission occurs: ϕ_{jump}, J_c and R approach 0^-, $1/4^+$ and -1^+, respectively, as $\lambda \to 0$. When neither nonlinearity nor viscosity is prevailing, the wave is partially reflected and transmitted.

However, the case of an internal wave packet propagating vertically and incident on a critical layer is quite different, as absorption then dominates over reflection and transmission.[21] The horizontal extent of the packet increases with time in the critical layer. The horizontal-momentum vertical flux jump does not attain a steady state, but instead decreases and remains positive highlighting the dominance of absorption over reflection, while the periodic-wave flux oscillates with respect to zero. A positive jump implies a mean-horizontal-velocity acceleration induced by the wave packet, whereas for a periodic wave, the mean flow is alternatively accelerated and decelerated. If the forcing length is increased, the momentum flux jump decreases and tends to a constant value but never reaches zero even when the wave packet tends to a monochromatic wave. The absorption rate decreases as the forcing amplitude diminishes. The local Richardson number never becomes negative in the wave packet critical layer whereas for the case of an incident periodic wave, the critical layer possesses substantial regions with $0 < J_c < 1/4$ and $J_c < 0$ at large times, and overturning is evident in the density contours. The degree of transmission is near zero, as for the averaged horizontal-momentum vertical flux. Nonlinear-critical-layer steady states can be found as the long-time outcome of a spatially periodic wave propagating and interacting with a nonlinear critical layer.[22] These states were possible when the overall Richardson number was less than a critical value always smaller than one. The local Richardson number was observed to be significantly decreased and areas of $J_c < 1/4$ existed in a thin layer along the lower side of the critical layer. Of course, these solutions may be unstable to shear instabilities. They also showed that weak stratification allowed a weak transmitted wave but as J_c increases, there is complete reflection of the primary wave. Similar steady zones have also been found.[23] In numerical simulations, large-amplitude internal waves undergo convective overturning in the critical layer and three-dimensional convective instabilities emerge leading to break-down.[24] The wave amplitude is assumed here, to be sufficiently large to allow for the creation of the critical layer but nevertheless not so large in order to avoid three-dimensional overturning outside the critical layer. In the same way, the spanwise wavenumber of our nonlinear neutral internal mode is assumed very small, smaller than the streamwise wavenumber, which precludes the transverse instability from occurring inside the critical layer. As three-dimensional convection is triggered by the transverse instability,[24,25] we will henceforth assume that the flow remains two-dimensional everywhere.

In a stratified two-dimensional flow, the nonlinear critical layer is far more complex than for its homogeneous counterpart. Instead of a single equation, it is governed by a coupled set of nonlinear partial differential equations, since now the buoyancy equation is added to the vorticity equation. Also, the singularity in the resulting modal equation is stronger. A theoretical study of the stratified nonlinear critical layer is desirable due to the difficulty in achieving high Reynolds numbers in stratified-flow experimental facilities and in numerical simulations. Generally, many modes are required in a nonlinear analysis, as it emerges that all harmonics are of the same order of magnitude as the fundamental neutral mode.

In this paper, we will assume that at the critical level, the flow is only weakly stratified, that is $J_c \ll 1$. This assumption is made essentially to keep the analysis tractable. But it is pertinent to note that, in practice, in the critical layer, the strong distortion of the buoyancy field could lead to homogenization of the latter, and hence our assumption has some physical validity. Assuming then that the neighbourhood of the critical level is weakly stratified leads us to consider two different wave structures, respectively, outside and inside the critical layer, that are coupled through matching. The outer flow is the place where a long-wavelength internal mode propagates in essentially the same way as in the study,[26] while the critical layer acts as a vortex wave guide in a similar manner as in the study[27] of a weakly stratified upper ocean. This second wave is indeed due to the vertical gradient of the background vorticity, and buoyancy affects only the kinematics of the wave. The internal solitary waves obtained here are thus fundamentally distinct from these cited studies and instead can be considered as wave-vortex structures. Importantly, the background shear flow considered here is assumed to be linearly stable, and hence the internal solitary waves we shall describe differ fundamentally from the long-wave localized structures which might emerge from nonlinear self-organization from infinitesimally small, but growing, disturbances due to shear instability. For instance, the development and saturation of the long-wave nonlinear disturbances are examined in Ref. 28, interacting with the critical layer of a weakly supercritical barotropically unstable zonal flow.

Previous studies of stratified critical layers have usually considered finite wavenumbers and finite Richardson numbers.[17,22,23] As a result, the coupled equations were too complicated for detailed theoretical analysis even when a steady travelling wave hypothesis was made. Consequently, the more tractable slightly stratified problem was examined,[29] when the local Richardson number is small of order $\epsilon^{1/2}$, where ϵ is the disturbance amplitude outside the critical layer. As in the homogeneous case, the logarithmic-phase shift of the disturbance then vanishes. It is also possible to obtain some analytical results in the study of the spatio-temporal evolution of a critical layer by considering weakly supercritical modes instead of neutral singular modes. The problem thus becomes tractable as the leading-order inner equation is linear.[28,30–32]

In this paper, we will develop a weakly nonlinear theory based upon long-wave and long-time hypotheses, with a weakly stratified critical layer, which will enable us to find analytical solutions for the critical-layer flow to a non-trivial order in our asymptotic expansion. Technically, the analysis is similar to our previous study[19] for a barotropic Rossby wave. Here, we will consider a free singular internal wave mode embedded in a vertically stratified shear flow. As already noted, the assumption of a weakly stratified critical layer is not necessarily restrictive, as the cat's eye pattern inside the critical layer, where the streamlines form closed loops, is the location where strong mixing occurs. As a result, the buoyancy and vorticity gradients may be homogenized there.[33] While our study is motivated by atmospheric and oceanic flows, it is set up as a classical fluid dynamics problem. Our assumption that, at the critical layer the stratification is weak, is motivated by analytical considerations, but we believe that this is a realistic scenario when a strongly nonlinear critical layer has formed, as then the consequent "mixing" would lead to a weak or constant stratification there. Thus, in possible applications to the atmosphere or ocean, we emphasize that our obtained solutions are the outcome of a long-time process. Hence, our results may not be directly applicable to observed atmospheric or oceanic density stratified shear flows, unless these have been present for sufficiently long times. For instance, the oceanic thermocline is often characterized by a strong density gradient and strong shear. However, if this is disturbed, then buoyancy mixing may occur, leading to a weaker stratification, but still with strong shear.

The effect of allowing the wave to coexist with a nonlinear critical layer was studied in a pioneering study[26] of internal solitary waves in stratified shear flows. In the absence of any critical layers, internal solitary waves satisfy a Korteweg-de Vries equation, whose coefficients are determined by certain integrals of the relevant modal function over the waveguide. Essentially, they assumed that this equation is not modified by the presence of a critical layer, except that the coefficients formally do not exist due to the integrals being singular, and instead need to be obtained by a different procedure. They illustrated their procedure for a three-layer flow, where only the middle layer is stratified with a constant buoyancy frequency and constant shear, with $J_c > 1/4$. Their nonlinear-critical-layer analysis assumed no phase jumps to the two first orders of an asymptotic expansion.

In this present work, the background shear velocity and buoyancy profiles will be specified arbitrarily. We suppose that the mean flow velocity equals the wave speed at a certain height, namely the critical level. Then, in the linearized theory, a critical-layer singularity occurs. This has the consequence that the nonlinearity needs to be invoked, and *inter alia* the mean flow is subsequently modified. We examine this modified mean flow at a long time after the formation of the critical layer, when we also assume that the buoyancy stratification has become small locally, due to homogenization. The thin region near this level, the so-called critical layer, is studied with an inner scaling, and for quasi-steady flows, is fully nonlinear.

The inner critical-layer expansion is first analytically determined at each order. Using the method of matched asymptotic expansions, this inner flow is then matched at the edges of the critical layer with the outer flow. Certain integration constants of the inviscid part of the flow are determined by using Redekopp's averaging technique[34] on the diffusive components using the nonlinear critical layer theory.[35] In this approach, the viscosity and thermal conductivity render the inviscid solution unique. Our main result is that the evolution equation for the modal streamfunction amplitude A has the general form

$$\partial_X \mathcal{C}[\partial_T A, A] = -\partial_X \mathcal{N}(A) - \partial_X^3 A - \mu \partial_X A\,, \tag{1}$$

where $\mathcal{C}[\partial_T A, A]$ is a nonlinear functional of $\partial_T A$ and A, and $\mathcal{N}(A)$ is a nonlinear function of A. When $\mathcal{N}(A)$ is a quadratic polynomial in A, the right-hand side has the usual KdV form. Here the left-hand side does not adopt the simple form $\partial_T A$ which would then reduce (1) to the usual KdV equation. Instead, the left-hand side contains integro-differential terms, required at a certain order of the asymptotic expansion when matching the inner flow on the dividing streamlines.

The resulting evolution equation bears some similarity to another evolution equation found[36] for a certain class of finite-amplitude internal solitary waves. Remarkably, for steady travelling waves, with an appropriate change of variables, this evolution equation can be transformed back into a KdV-type equation. The critical-layer flow is found to be asymmetric with respect to the critical level.[17,22] However, the asymmetry there was due to the choice of a dissipative ($\lambda = O(1)$) and stable stratified shear flow ($J_c > 1/4$). Here, the asymmetry is not only caused by the presence of a density stratification in spite of the fact that $J_c < 1/4$ but also to some degree by nonlinearity.[19,20] The evolution of the critical layer leads to a strong rearrangement of the streamlines and buoyancy contours, as breaking theories predict.[37] Our asymptotic procedure involves an adaption of the Prandtl-Batchelor theorem for a slowly evolving, weakly stratified and weakly forced flow within the cat's eye, which is exact after an asymptotically long time when the critical layer has settled to a steady and laminar state.

In forming recirculation regions, the isopycnal curves advected by the shear flow are nearly closed. Hence, these regions may possess convectively unstable zones. However, the buoyancy and vorticity gradients are eventually homogenized there due to diffusion and mixing. The resulting asymptotic quasi-steady state is then a stable configuration,[38] where vorticity and density are constant in space and time. Here we assume the existence of such a quasi-steady state, but due to the necessary presence of weak horizontal-momentum and buoyancy body forces, the extended Prandtl-Batchelor theorem yields a cat's eye characterized by a weak non-constant density field coupled with a weak non-constant vorticity field.

The plan of the paper is as follows. In Sec. II, we formulate the problem and give the main assumptions used in our study. Section III displays the equations of the outer flow and gives the local solutions around the singularity. The critical-layer flow is examined in Sec. IV, where in particular, the description of this flow is refined by a better parameterization of the streamlines. The flow inside the defining separatrices within the critical layer (the so-called cat's eye flow) is analyzed using an extension of the Prandtl-Batchelor theorem for quasi-steady solitary-wave motions, described in Appendix A. The outcome is an integro-differential evolution equation. Searching for travelling-wave solutions in Sec. V, the evolution equation is simplified into a KdV-type equation, whose form depends on the wave polarity. Solitary-wave solutions are found whose characteristics depend on the outer flow, the phase jumps across the critical layer and the distortions of the induced mean flow. We specify some classical mean buoyancy and velocity profiles such as a constant-shear flow. Section VI focuses on the critical-layer induced mean flow and buoyancy and gives the leading-order time scales and growth rates of their evolution. It also yields the energy budget in the critical layer and can predict the heating or cooling of the critical layer according to the various parameters of the problem. Finally, Sec. VII offers some concluding remarks.

II. FORMULATION

We consider the two-dimensional flow of a stratified, incompressible, viscous, and thermal conducting fluid. The fluid is located in the (x, y)-plane, where y is the vertical coordinate, and is confined between two rigid plane boundaries, $y = y^{\pm}$, either or both of which may be at infinity. The basic dimensionless equations of motion are the Navier-Stokes equation, and the density equation in the Boussinesq approximation,

$$\partial_t \mathbf{u} + \mathbf{u}\nabla\mathbf{u} + \nabla p = \Theta + \frac{1}{R}\Delta\mathbf{u} + F_u, \tag{2}$$

$$\partial_t \Theta + \mathbf{u}\nabla\Theta = \frac{1}{Pr\,R}\Delta\Theta + F_b, \tag{3}$$

where $\mathbf{u} = (u, w)$ is the velocity field, p the pressure, and Θ is the buoyancy. The buoyancy is defined as the variation of the density ρ relative to a reference density ρ_c, that is as $-g(\rho - \rho_c)/\rho_c$, g being the constant gravity acceleration. The subscript c characterizes quantities at the critical level. We use a linear equation of state, $\rho = \rho_c\,[1 - \beta(Te - T_c)]$ connecting the density and temperature fields. Hence, ρ_c, T_c denote the basic density, temperature, respectively, at the critical level and β is the thermal-expansion coefficient. All variables are made dimensionless using a length scale L (the shear layer thickness), $\mathcal{U}$ (the velocity jump across the shear layer), and ρ_c, T_c. The Reynolds number

R is assumed large, while the Prandtl number Pr is of order unity. The overall Richardson number is then $Ri = \beta g T_c L/\mathcal{U}^2$. The body forces $F_{q,\,b}$ are applied in order to prevent viscous and thermal diffusions of the basic mean fields and are specified later in Sec. III. As a result, the evolution of the mean fields is driven by the nonlinear critical layer and not by a laminar diffusion. Relative to the basic-state shear flow and stratification, we impose a neutral-wave disturbance with a real horizontal phase speed c. Then, in a reference frame moving with the wave speed c, the total streamfunction Ψ and buoyancy Θ are given by

$$\Psi = \int_{y_c}^{y} [U(\bar{y}, \tau_1, \tau_2, \tau_{DB}, \cdots) - c]\, d\bar{y} + \epsilon\, \psi, \qquad \Theta = T(y, \tau_1, \tau_2, \tau_{DB}, \cdots) + \epsilon\, \theta\,. \tag{4}$$

Here $U(y, \tau_1, \tau_2, \tau_{DB}, \cdots)$ and $T(y, \tau_1, \tau_2, \tau_{DB}, \cdots)$ are the dimensionless basic shear flow and buoyancy fields, where τ_1 and τ_2 are critical-layer leading-order slow times and τ_{DB} is the diffusion boundary layer slow time whose scalings are to be determined. The disturbance stream function and buoyancy are $\epsilon\, \psi$ and $\epsilon\, \theta$, respectively, where ϵ is a small parameter. The local Richardson number is $Ri\, T^{'}(y)/U^{'}(y)^2$. The relevant dimensionless equations of motion are the vorticity and buoyancy equations

$$\{\partial_t + (U - c)\partial_{\bar{x}}\}\Delta\psi + \epsilon J(\Delta\psi, \psi) = -\partial_{\bar{x}}\theta + U^{''}(y)\partial_{\bar{x}}\psi + \frac{\Delta^2\psi}{R} + \frac{U^{'''}(y)}{\epsilon R} + \frac{F_{q'}}{\epsilon}\,, \tag{5}$$

$$\{\partial_t + (U - c)\partial_{\bar{x}}\}\theta + \epsilon J(\theta, \psi) = T^{'}(y)\partial_{\bar{x}}\psi + \frac{\Delta\theta}{PrR} + \frac{T^{''}(y)}{\epsilon PrR} + \frac{F_b}{\epsilon}\,. \tag{6}$$

J is the operator Jacobian. The prime denotes a derivate with respect to y. The formulation is completed by specifying boundary conditions at the remote edges of the waveguide at $y = y_\pm$. The Reynolds number being large, the viscous boundary layers at $y = y_\pm$ are very thin and do not interact with the critical-layer induced flow. Only the inviscid boundary condition will be thus needed here, which is that $\psi = 0$ at $y = y_\pm$. As anticipated above, the mean fields are expanded in the form

$$U(y, \tau_1, \cdots) = U_0(y) + \epsilon^{\frac{1}{2}}[U_1(y, \tau_1, \cdots) - U_{u,1}(y)] + \epsilon\, U_2(y, \tau_1, \cdots) + \cdots, \tag{7}$$

$$T(y, \tau_1, \cdots) = T_0(y) + \epsilon^{\frac{1}{2}} T_1(y, \tau_1, \cdots) + \epsilon\, T_2(y, \tau_1, \cdots) + T_{ind}(y, \tau_1, \cdots) + \cdots\,, \tag{8}$$

$$c = c_0 + \epsilon^{\frac{1}{2}} c_1 + \epsilon\, c_2 + \cdots\,, \tag{9}$$

where henceforth, we let $\hat{U}_1(y, \tau_1, \cdots) = U_1(y, \tau_1, \cdots) - U_{u,1}(y)$. Here $U_0(y)$, $T_0(y)$ are the basic mean fields, $T_{ind}(y)$ is an $O(\epsilon)$ mean buoyancy induced at the critical layer by the leading-order outer viscous flow terms, and the remaining $O(\epsilon^{1/2}, \epsilon, \cdots)$ terms are to ensure continuity of the leading-order vorticity and buoyancy fields inside the critical layer, needed here as we do not impose thin viscous and thermal boundary layers along the separatrices. This additional flow then appears in the outer flow through small $O(\epsilon^{1/2})$ jumps in mean vorticity and buoyancy at either side of the critical level.[39] Our concern is with the long-time asymptotic régime after the critical-layer formation stage characterized by the diffusive spreading of the vorticity and buoyancy fields with an amplitude $O(\epsilon^{1/2})$ throughout the domain. The critical-layer theory will lead eventually to a diffusion scaling such as the inverse Reynolds number be $1/R = \lambda\, \epsilon^{7/4}$. This small diffusion outwards from the critical layer generates distorted mean fields of current and buoyancy, first noted in Ref. 40. As shown in Ref. 19, this $O(\epsilon^{1/2})$ diffusion is only possible by defining the body force so that $F_q = -[U_0^{'''}(y) - \epsilon^{1/2} U_{u,1}^{'''}(y)]/R$ in the vorticity equation. The basic flow $U_0(y) - \epsilon^{1/2} U_{u,\,1}(y)$ will remain steady through the evolution of the critical layer, $\epsilon^{1/2} U_1(y, \tau_1, \cdots) + \epsilon\, U_2(y, \tau_1, \cdots) + \cdots$, while additional momentum is created by the critical layer. Outside these diffusion boundary layers, far away from the critical layer, this new shear distribution will match with $\epsilon^{1/2} U_{u,\,1}(y)$, so that the unperturbed flow far away from the critical layer will be the basic profile $U_0(y)$, see Fig. 1. An

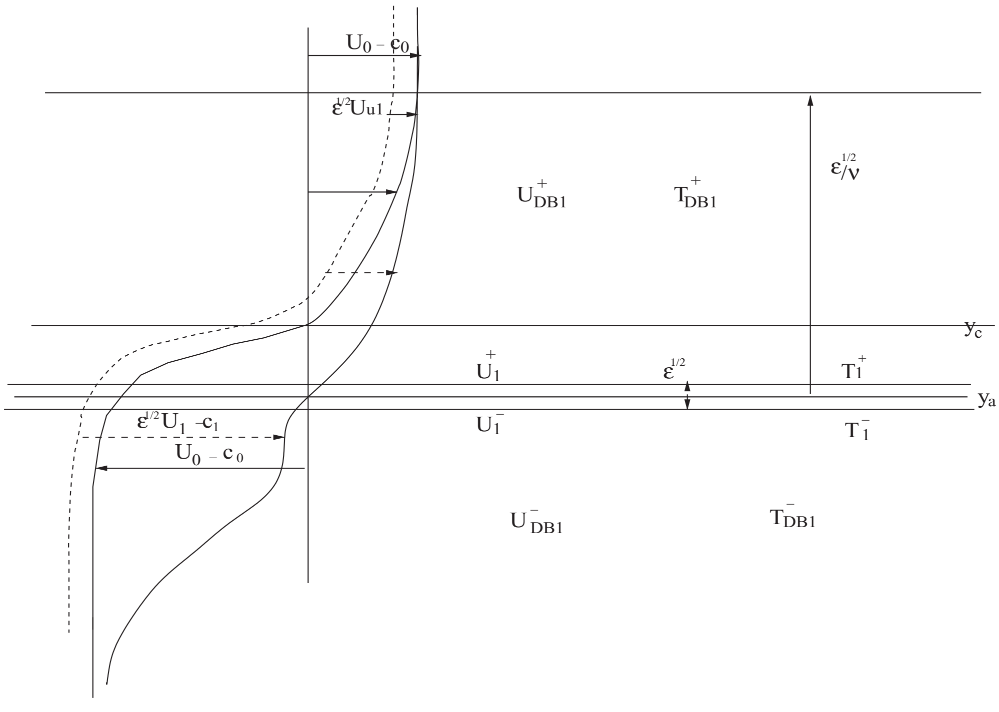


FIG. 1. Diagram of the nonlinear critical layer embedded by two diffusion boundary layers, the diffusion wave number is $\nu = (\lambda T)^{1/2}/2$. The leading-order horizontal mean velocity is $U(y) = U_0 + [U_1(y) - U_{u,1}(y)]\epsilon^{1/2}$. The critical level y_c is located inside the critical layer and does not coincide with the symmetrix axis y_a. The critical-layer induced velocity in the diffusion boundary layers $U_{DB,1}(y)$ must be matched with the mean flow $U_1(y)$ on the edges of the critical layer. The same matching must be performed for the buoyancy $T_{DB,1}(y)$ with $T_1(y)$.

analogous correction is also needed in the buoyancy equation but for simplicity's sake, it will not be formulated explicitly in the equations.

Linearizing and combining (5) and (6), we obtain the steady inviscid modal equation, in the incompressible Boussinesq limit,

$$\Delta\psi + \frac{T_0^{'}(y)}{[U_0(y) - c_0]^2}\psi - \frac{U_0^{''}(y)}{[U_0(y) - c_0]}\psi = 0\,. \tag{10}$$

This is separable, and when the x-derivative term is extracted by a Fourier expansion, it reduces to the well-known Taylor-Goldstein equation. The location y_c where the phase speed of the perturbation equals the given flow velocity, that is $U_0(y_c) = c_0$, is potentially a singularity, which is of higher order than that for a homogeneous flow. In the linearized theory, a neutral mode is only possible in the case when $T_0^{'}(y_c) = 0$ and $T_0^{''}(y_c) = U_0^{'}(y_c)U_0^{''}(y_c)$, that is, when the singularity vanishes. Here, we assume only that $T_0^{'}(y_c) = 0$ so that the leading-order singularity is merely $[U_0(y) - c_0]^{-1}$ as in an unstratified flow. Note that with $T_0^{'}(y_c) = 0\,,\, T_0^{''}(y_c) \neq 0$, the basic buoyancy profile is weakly convectively unstable in the vicinity of the critical layer, an issue which we shall return to in the conclusion. Proceeding, the mode is now singular with a critical-layer singularity, so transience, nonlinearity, viscosity, and conductivity must be reintroduced into the critical-layer leading-order solution. Here, we select nonlinearity as the principal factor, since this is the case for geophysical flows with high Reynolds numbers. Nevertheless, viscosity and conductivity are present and render this inviscid solution unique. The role of long-time transience will be discussed later. The asymptotic expansion based on the small parameter ϵ will consist of an outer expansion outside the critical layer, matched at the edges of the critical layer to an inner asymptotic expansion with different scalings valid in the critical layer. It will transpire that a balance between nonlinearity with linear long-wave dispersion in the inner flow is obtained by putting

$$X = \epsilon^{\frac{1}{4}} \bar{x}\,, \qquad T = \epsilon^{\frac{5}{4}} t\,, \quad \bar{x} = x - ct\,. \tag{11}$$

Note that the usual scaling for a Korteweg-de Vries equation derivation is, see Ref. 26 for instance,

$$X = \epsilon^{\frac{1}{2}} \bar{x}\,, \qquad T = \epsilon^{\frac{3}{2}} t\,. \tag{12}$$

It turns out that this is not valid here, because the leading-order critical layer matching would then yield a dispersiveless amplitude equation. Instead, with the scaling (11) we will use, the linear dispersive terms in the outer flow appear at the correct $O(\epsilon^2)$ to balance the leading-order nonlinearity.

III. OUTER FLOW

In the outer flow outside the critical layer, the perturbation streamfunction is expanded as follows:

$$\psi = \psi^{(0)} + \epsilon^{\frac{1}{2}}\, \psi^{(1)} + \epsilon\, \psi^{(2)} + \cdots\,, \tag{13}$$

with a similar expansion for the perturbation buoyancy. At the leading order, that is $O(\epsilon)$, we seek a solution in the separated form,

$$\psi^{(0)} = \phi(y) A^*(X, T)\,, \quad A^* = U_0^{'}(y_c)\, A\,, \tag{14}$$

such as

$$\mathcal{L}_0(\phi) = \partial_y^2 \phi + \frac{T_0^{'}(y)}{[U_0(y) - c_0]^2}\phi - \frac{U_0^{''}(y)}{U_0(y) - c_0}\phi = 0\,, \tag{15}$$

$$\text{while} \qquad \theta^{(0)} = \frac{T_0^{'}(y)\psi^{(0)}(y)}{U_0(y) - c_0}.$$

Equation (15) has two linearly independent solutions, whose Frobenius expansions in terms of $\eta = y - y_c$ (valid when $|\eta| \ll 1$), are given by

$$\phi_a = \eta + \sum_{n=2}^{\infty} a_{0,n}\eta^n\,, \tag{16}$$

$$\phi_b = 1 + \sum_{n=2}^{\infty} b_{0,n}\eta^n + b_0\phi_a(\eta)\ln|\eta^*|. \tag{17}$$

The first few coefficients are, recalling that $T_0^{'} = 0$,

$$a_{0,2} = \frac{1}{2}b_0, \quad a_{0,3} = \frac{1}{12}\Big(2\frac{U_0^{'''}}{U_0^{'}} + \frac{T_0^{''2}}{U_0^{'4}} - \frac{T_0^{'''}}{U_0^{'2}}\Big),$$

$$b_0 = \frac{U_0^{''}}{U_0^{'}} - \frac{T_0^{''}}{U_0^{'2}}, \quad \text{and} \quad b_{0,2} = \frac{1}{2}\Big(\frac{U_0^{'''}}{U_0^{'}} - 2b_0^2\Big) + \frac{1}{4}\Big(\frac{T_0^{''2}}{U_0^{'4}} - \frac{T_0^{'''}}{U_0^{'2}}\Big).$$

The solution ϕ_a is regular, but the solution ϕ_b has a logarithmic singularity. We denote $\bar{b}_0 = b_0/U_0^{'}$, and in general, in the sequel, we will introduce an overbar to define a division by $U_0^{'}$. The coordinate η is rescaled to $\eta^* = \eta/\eta_0$, where η_0 will be determined when matching the outer flow with the critical-layer flow. Using a superscript $\pm$ to distinguish the flow above or below the critical level, the mode is written on either side of y_c as

$$\begin{aligned} \phi^+(y) &= a^+ b_0\phi_a + \phi_b, \\ \phi^-(y) &= a^- b_0\phi_a + \phi_b\,. \end{aligned} \tag{18}$$

The constants $a^{\pm}$ are determined by the boundary conditions that $\phi^{\pm} = 0$ at $y = y_{\pm}$, respectively. Proceeding to the next order, that is $O(\epsilon^{3/2})$, the flow is affected by the advection (denoted by the subscript l) due to the additional mean flow U_1 and by dispersion (denoted by d)

$$\mathcal{L}_0(\psi^{(1)}) = -\Big[\mathcal{L}_1(\phi) - \frac{\hat{U}_1(y) - c_1}{U_0(y) - c_0}\mathcal{M}_0(\phi) + \phi\partial_X^2\Big]A^*\,, \tag{19}$$

where $\mathcal{L}_i(\phi) = \dfrac{U_i(y) - c_i}{U_0(y) - c_0}\partial_y^2\phi - \dfrac{U_i''(y)\phi}{U_0(y) - c_0} + \dfrac{T_i'(y)\phi}{[U_0(y) - c_0]^2}$, $\mathcal{M}_i(\phi) = \dfrac{T_i'(y)\phi}{[U_0(y) - c_0]^2}$, $i = 1, 2,$

$$\text{while}\quad \theta^{(1)} = \frac{T_0'(y)\psi^{(1)}}{U_0(y) - c_0} + \frac{T_1'(y)\psi^{(0)}}{U_0(y) - c_0} - T_0'(y)\frac{\hat{U}_1(y) - c_1}{[U_0(y) - c_0]^2}\psi^{(0)} + \lambda\frac{T_0'(y)\phi_v^{(1)}}{U_0(y) - c_0}X.$$

The behaviour of $\psi^{(1)}$ near the critical level is

$$\psi^{(1)} = \phi_l^{(1)}A^*(X, T) + \phi_d^{(1)}\partial_X^2 A^*(X, T) + \lambda\phi_v^{(1)}X + \phi\, B^*(X, T)\,, \tag{20}$$

$$\text{where}\quad \phi_l^{(1)} = \Big(\sum_{n=0}^{\infty}[b_{l,1,n}\ln|\eta^*| + c_{l,1,n}]\,\eta^n + \alpha_{l,1}\phi_a + \beta_{l,1}\phi_b\Big)\,,$$

$$\phi_d^{(1)} = \Big(\sum_{n=2}^{\infty}[c_{d,1,n}\eta^n] + \alpha_{d,1}\phi_a + \beta_{d,1}\phi_b\Big)\,,$$

$$\text{and}\quad \phi_v^{(1)} = (\alpha_{v,1}\phi_a + \beta_{v,1}\phi_b)\,,\qquad \phi = (ab_0\phi_a + \phi_b)\,.$$

$B^*(X, T)$ is the second-order amplitude of the mode which, like $A^*(X, T)$ obeys an amplitude equation derived at the next order. The coefficients α, β, respectively, are related to the regular, singular Frobenius solution, and are determined by the boundary conditions that $\phi_{l,d,v}^{(1)} = 0$ at $y = y_{\pm}$. Note that the strongest singularity is proportional to $b_0 T_1'/T_0''\ \ln\eta^*$, and it is stronger than for unstratified flow, where it is proportional to $\eta\ln\eta^*$. The necessity for the secular diffusive flow terms $\lambda\phi_v^{(1)}X$ appears later when matching the averaged vertical flux of the horizontal momentum.

At the next order, that is $O(\epsilon^2)$, $\psi^{(2)}$ and $\theta^{(2)}$ are governed by the equations

$$\mathcal{L}_0(\psi^{(2)}) = f_t(y)\partial_{TX^{-1}}A^* + f_l(y)A^* + \frac{1}{2}f_n(y)A^{*2} + f_d(y)\partial_X^2 A^* - \phi_d^{(1)}\partial_X^4 A^*$$
$$- \mathcal{L}_1(\phi)B^* - \phi\partial_X^2 B^* - \lambda\mathcal{L}_1(\phi_v^{(1)})X + \frac{\lambda}{Pr}\Big[\frac{PrU_1'''(y)}{U_0(y) - c_0} - \frac{T_1''(y)}{[U_0(y) - c_0]^2}\Big]X\,, \tag{21}$$

$$\text{with}\quad f_t(y) = \Big[\frac{T_0'(y)}{[U_0(y) - c_0]^3} - \frac{U_0''(y)}{[U_0(y) - c_0]^2}\Big]\phi\,,$$

$$f_l(y) = -\Big\{\mathcal{L}_1(\phi_l^{(1)}) + \mathcal{L}_2(\phi) - \frac{\hat{U}_1(y) - c_1}{U_0(y) - c_0}[\mathcal{M}_1(\phi) + \mathcal{M}_0(\phi_l^{(1)})]$$
$$+ \Big[\Big(\frac{\hat{U}_1(y) - c_1}{U_0(y) - c_0}\Big)^2 - \frac{U_2(y) - c_2}{U_0(y) - c_0}\Big]\mathcal{M}_0(\phi)\Big\}\,,$$

$$f_n(y) = \Big[\Big(\frac{U_0''(y)}{U_0(y) - c_0}\Big)'[U_0(y) - c_0] - \Big(\frac{T_0'(y)}{U_0(y) - c_0}\Big)'\Big]\frac{\phi^2}{[U_0(y) - c_0]^2}\,,$$

$$\text{and}\quad f_d(y) = -\Big[\frac{\hat{U}_1(y) - c_1}{U_0(y) - c_0}[\phi + \mathcal{M}_0(\phi_d^{(1)})] + \phi_l^{(1)} + \mathcal{L}_1(\phi_d^{(1)})\Big].$$

The buoyancy field is then given by

$$\theta^{(2)} = -\frac{T_0'(y)\phi}{[U_0(y)-c_0]^2}\partial_{TX^{-1}}A^* + \Big(\frac{T_0'(y)}{U_0(y)-c_0}\Big)'\frac{\phi^2 A^{*2}}{2[U_0(y)-c_0]}$$
$$+\frac{T_0'(y)\psi^{(2)}}{U_0(y)-c_0} + \frac{T_1'(y)\psi^{(1)}}{U_0(y)-c_0} + \frac{T_2'(y)\psi^{(0)}}{U_0(y)-c_0} - \frac{\hat{U}_1(y)-c_1}{U_0(y)-c_0}\theta^{(1)} - \frac{U_2(y)-c_2}{U_0(y)-c_0}\theta^{(0)}$$
$$+\lambda\Big[\frac{T_1'(y)\phi_v^{(1)}}{U_0(y)-c_0} - \frac{\hat{U}_1(y)-c_1}{U_0(y)-c_0}\theta_v^{(1)} + \frac{T_0'(y)\psi_v^{(2)}}{U_0(y)-c_0}\Big]X + \frac{\lambda}{Pr}\frac{T_1''(y)X}{U_0(y)-c_0}. \tag{22}$$

There are contributions from the time derivative of the modal amplitude A_T^* (denoted by the subscript *t*), the quadratic term A^{*2} (denoted by the subscript *n*) and the linear dispersive terms $\partial_X^2 A^*$ and $\partial_X^4 A^*$ (denoted by the subscript *f*) and a linear term A^* corresponding to the advection of $\psi^{(0)}$, $\theta^{(0)}$, $\psi^{(1)}$ and $\theta^{(1)}$ by the additional mean flow. Each contribution will affect the coefficients of the final amplitude equation (1) governing the evolution of A. Next, the solutions are Frobenius-expanded in the neighbourhood of y_c. Thus for instance, we find the corresponding terms in $\psi^{(2)}$ from (21),

$$\psi^{(2)} = g_t(y)\partial_{TX^{-1}}A^* + g_l(y)A^* + \frac{1}{2}g_n(y)A^{*2} + g_d(y)\partial_X^2 A^* + g_f(y)\partial_X^4 A^*$$
$$+(ab_0\,\alpha_{b,2}\phi_a + \phi_b)J_2[A] + \beta_{v,1}Pr(\alpha_{ind}\phi_a + \beta_{ind}\phi_b)A^* + \lambda g_v(y)X + \cdots, \tag{23}$$

$$\text{with}\quad g_t(y) = \sum_{n=0}^{\infty}[b_{t,2,n}\ln|\eta^*| + c_{t,2,n}]\eta^n + \alpha_{t,2}\phi_a + \beta_{t,2}\phi_b\,,$$

$$g_l(y) = \sum_{n=0}^{\infty}[a_{l,2,n}\ln^2|\eta^*| + b_{l,2,n}\ln|\eta^*| + \frac{c_{l,2,n}}{\eta}]\eta^n + \alpha_{l,2}\phi_a + \beta_{l,2}\phi_b\,,$$
$$g_n(y) = (\sum_{n=0}^{\infty}[a_{n,2,n}\ln^2|\eta^*| + b_{n,2,n}\ln|\eta^*| + \frac{c_{n,2,n}}{\eta}]\eta^n + \alpha_{n,2}\phi_a + \beta_{n,2}\phi_b\,,$$

$$g_d(y) = \sum_{n=0}^{\infty}[b_{d,2,n}\ln|\eta^*| + c_{d,2,n}]\eta^n + \alpha_{d,2}\phi_a + \beta_{d,2}\phi_b\,,$$

$$g_f(y) = \sum_{n=0}^{\infty}[c_{f,2,n}\eta^n] + \alpha_{f,2}\phi_a + \beta_{f,2}\phi_b\,,$$

$$\text{and}\quad g_v(y) = \sum_{n=0}^{\infty}[a_{v,2,n}\ln^2|\eta^*| + b_{v,2,n}\ln|\eta^*| + c_{v,2,n}]\eta^n + \alpha_{v,2}\phi_a + \beta_{v,2}\phi_b\,.$$

Note that $\psi^{(2)}$ has stronger singularities than $\psi^{(1)}$. There is an analogous expansion for $\theta^{(2)}$. The presence of a higher-order dispersive term leads to the possible existence in the evolution equation (1) of a term proportional to $\partial_X^5 A$. However, it will transpire that the coefficient of this term will be zero. $J_2[A]$ is a nonlinear term arising from the critical-layer induced flow and will be displayed at the end of Sec. IV E, but it does not affect the amplitude equation. The term proportional to $\beta_{v,1}A$ is induced by diffusion, corresponding to the secular diffusive terms at $O(\epsilon^{3/2})$ in the outer flow.

IV. NONLINEAR CRITICAL LAYER FLOW

In this section, we will integrate the vorticity and buoyancy inner equations and we will determine, up to the sixth order of the inner expansion, the vorticity, streamwise velocity, streamfunction, pressure and buoyancy in the critical-layer open and closed streamline flows. Matching with both flows will yield the mean flow jumps, the streamfunction phase jumps, and ultimately the searched evolution equation.

In the unstratified nonlinear critical layer, a balance between the linear and advective terms is achieved by the transformation,

$$\eta = y - y_c = \epsilon^{\frac{1}{2}} \mathcal{Y}\,.$$

The governing vorticity and buoyancy equations are now expressed in terms of the inner variable $\mathcal{Y}$, and so the critical-layer equations are

$$\{\epsilon^{\frac{3}{2}}\partial_T + (\Psi_{\mathcal{Y}}\partial_X - \Psi_X\partial_{\mathcal{Y}})\}(\partial_{\mathcal{Y}}^2 + \epsilon^{\frac{3}{2}}\partial_X^2)\Psi + \epsilon^{\frac{3}{2}}\Theta_X = \epsilon\,\lambda(\partial_{\mathcal{Y}}^4 + 2\epsilon^{\frac{3}{2}}\partial_X^2\partial_{\mathcal{Y}}^2 + \epsilon^3\partial_X^4)\Psi + \epsilon^3 F_q\,,$$

$$\{\epsilon^{\frac{3}{2}}\partial_T + (\Psi_{\mathcal{Y}}\partial_X - \Psi_X\partial_{\mathcal{Y}})\}\Theta = \epsilon\frac{\lambda}{Pr}(\partial_{\mathcal{Y}}^2 + \epsilon^{\frac{3}{2}}\partial_X^2)\Theta + \epsilon^2 F_b\,, \tag{24}$$

where the body force terms are now $F_q = -\lambda[U_0^{'''}(\mathcal{Y}) - \epsilon^{\frac{1}{2}}U_{u,1}^{'''}(\mathcal{Y})]$ and $F_b = -\lambda/Pr\,T_0^{''}(\mathcal{Y})$. The outer-flow expansion of Sec. III, when expressed using the inner variable $\mathcal{Y}$, shows the manner in which the inner expansion must proceed,

$$\Psi = \epsilon\left\{\Psi^{(0)} + \epsilon^{\frac{1}{2}}\ln\epsilon\,\Psi^{(1)} + \epsilon^{\frac{1}{2}}\Psi^{(2)} + \epsilon\ln^2\epsilon\,\Psi^{(3)} + \epsilon\ln\epsilon\,\Psi^{(4)} + \epsilon\,\Psi^{(5)} + \ldots\right\}, \tag{25}$$

while there is an analogous expansion for the buoyancy Θ. We can now determine explicitly the analytical expressions for the inner fields (Ψ, Θ) at each order.

A. $O(\epsilon)$ terms

The leading-order equations (24) reduce to the quasi-steady system,

$$(\Psi_{\mathcal{Y}}^{(0)}\partial_X - \Psi_X^{(0)}\partial_{\mathcal{Y}})\Psi_{\mathcal{Y}\mathcal{Y}}^{(0)} = \lambda\partial_{\mathcal{Y}}^2\Psi_{\mathcal{Y}\mathcal{Y}}^{(0)} \quad \text{and} \quad (\Psi_{\mathcal{Y}}^{(0)}\partial_X - \Psi_X^{(0)}\partial_{\mathcal{Y}})\Theta^{(0)} = \frac{\lambda}{Pr}(\Theta_{\mathcal{Y}\mathcal{Y}}^{(0)} - T_0^{''})\,. \tag{26}$$

We solve this system in the inviscid limit, $\lambda \to 0$, and so we search for a solution perturbed by diffusion terms in the form

$$\Psi^{(0)} = \Psi_i^{(0)} + \lambda\Psi_v^{(0)} + O(\lambda^2)\,, \quad \Theta^{(0)} = \Theta_i^{(0)} + \lambda\Theta_v^{(0)} + O(\lambda^2)\,. \tag{27}$$

As in Ref. 19, we employ the zero-order streamfunction as the cross-stream coordinate and so perform the following change of coordinates

$$(X, \mathcal{Y}, T) \to (\xi = X, S = \bar{\Psi}^{(0)}(X, \mathcal{Y}, T), \tau = T)\,. \tag{28}$$

The system (26) then becomes

$$\Psi_{i,\mathcal{Y}}^{(0)}\Psi_{i,\mathcal{Y}\mathcal{Y}\xi}^{(0)} = 0 \quad \text{and} \quad \Psi_{i,\mathcal{Y}}^{(0)}\Theta_{i,\xi}^{(0)} = 0\,. \tag{29}$$

Note that here, $U_0^{'}S = U_0^{'}\bar{\Psi}^{(0)} = \Psi^{(0)}$. The solutions for $\Psi_{i,\mathcal{Y}\mathcal{Y}}^{(0)}$ and $\Theta_i^{(0)}$ are expressed in terms of arbitrary functions

$$\Psi_{i,\mathcal{Y}\mathcal{Y}}^{(0)} = \mathcal{F}^{(0)}(S) \quad \text{and} \quad \Theta_i^{(0)} = \mathcal{G}^{(0)}(S)\,. \tag{30}$$

The related diffusive components are given by the equations

$$\Psi_{i,\mathcal{Y}}^{(0)}\Psi_{v,\mathcal{Y}\mathcal{Y}\xi}^{(0)} = \partial_{\mathcal{Y}}^2\Psi_{i,\mathcal{Y}\mathcal{Y}}^{(0)}\,, \qquad Pr\,\Psi_{i,\mathcal{Y}}^{(0)}\Theta_{v,\xi}^{(0)} = \partial_{\mathcal{Y}}^2\Theta_i^{(0)} - T_0^{''}\,.$$

Using Eqs. (30), we get

$$U_0^{'}\Psi_{v,\mathcal{Y}\mathcal{Y}\xi}^{(0)} = \{S_{\mathcal{Y}}\mathcal{F}_S^{(0)}\}_S\,, \qquad Pr\,U_0^{'}\Theta_{v,\xi}^{(0)} = \{S_{\mathcal{Y}}\mathcal{G}_S^{(0)}\}_S - T_0^{''}S_{\mathcal{Y}}^{-1}\,. \tag{31}$$

We now adapt Redekopp's averaging technique, see Ref. 34. That is, we require that the ξ-averaged inner-flow diffusive contribution match with its outer-flow diffusive counterpart. This is equivalent to matching the secular diffusive terms at the edges of the critical layer. These terms form the mean fields, the horizontally averaged result from the interaction of the internal wave with the basic fields (U_0, T_0). For a flow defined by a periodic wave train with wavelength $2\pi/k$, this condition reduces to an average over this spatial period. Here, we have then to take the solitary-wave limit $k \to 0$. The outcome of this procedure is that

$$\mathcal{F}_S^{(0)} = \frac{M^{(0)}}{\frac{k}{2\pi}\int_{-\pi/k}^{\pi/k} S_{\mathcal{Y}}\, d\xi} \quad \text{and} \quad \mathcal{G}_S^{(0)} = T_0^{''} + \frac{N^{(0)}}{\frac{k}{2\pi}\int_{-\pi/k}^{\pi/k} S_{\mathcal{Y}}\, d\xi},$$

where $M^{(0)}$ and $N^{(0)}$ are constants. When $\mathcal{Y} \to \infty$, $[\Psi_i^{(0)}, \Theta_i^{(0)}]$ are then matched with the outer expansions,

$$\Psi_i^{(0)} \to U_0^{'}\left[\frac{1}{2}\mathcal{Y}^2 + A(X,T)\right] + (\hat{U}_1 - c_1)\mathcal{Y}, \quad \Theta_i^{(0)} \to T_0^{''}\left[\frac{1}{2}\mathcal{Y}^2 + A(X,T)\right] + T_1^{'}\mathcal{Y} + T_2. \tag{32}$$

The matching is facilitated by putting

$$Y = \mathcal{Y} + \frac{\hat{U}_1 - c_1}{U_0^{'}}. \tag{33}$$

The outcome is that $\mathcal{F}^{(0)} = U_0^{'}$, $\mathcal{G}^{(0)} = T_0^{''} S +$ constant where $M^{(0)} = 0$ and $N^{(0)} = 0$. Thus, the leading-order inner streamfunction and buoyancy are given by

$$\Psi_i^{(0)} = \Psi^{(0)} = U_0^{'}\left[\frac{1}{2}Y^2 + A(X,T)\right] + \text{constant} = U_0^{'} S + \text{constant}, \tag{34}$$

$$\Theta_i^{(0)} = \Theta^{(0)} = T_0^{''} S + T_2 - \frac{1}{2}\frac{T_1^{'2}}{T_0^{''}}, \tag{35}$$

while the first-order wave speed is given by

$$c_1 = \hat{U}_1 - U_0^{'}\frac{T_1^{'}}{T_0^{''}}. \tag{36}$$

The validity of the cross-stream coordinate change (33) requires that the secondary mean buoyancy gradient has no jump, that is $T_1^{'+} = T_1^{'-}$, with the consequence that the leading-order rescaled Richardson number $J_c = \bar{\bar{T}}_1^{'}\epsilon^{1/2}$ is continuous through the critical level.

At this leading order, the critical layer is characterized by a zone of closed streamlines, commonly called the cat's eye, which is separated from the remaining part of the flow field by separatrices whose geometry differs according to the sign of A. Further analysis of the flow field within the cat's eye needs to be based on an extended form of the Prandlt-Batchelor theorem. This is described in Appendix A, and the outcome is that the leading-order vorticity and buoyancy fields in the cat's eye are given by the same expressions, that is $\Psi^{(0,\odot)} = \Psi^{(0)}$ and $\Theta^{(0,\odot)} = T_0^{''} S + \theta_0$. All quantities within the cat's eye are denoted by an additional superscript $\odot$. Further, continuity of the buoyancy field requires that $T_2^{+} = T_2^{-}$ and $\theta_0 = T_2 - T_1^{'2}/2T_0^{''}$. Note that the buoyancy in the cat's eye is not constant due to the presence of the body term F_b, unlike the usual situation, see Ref. 41 for instance. Indeed, this buoyancy field is convectively unstable in that half of the cat's eye where $T_0^{''} Y < 0$, an issue which we shall return to in the conclusion. The cat's eye has the classic structure, but is not centred vis-à-vis the critical level (cf. Fig. 2), as the symmetry axis is shifted to $y_a = y_c - \epsilon^{1/2} y_\delta$ with $y_\delta = T_1^{'}/T_0^{''}$. Technically speaking, the singularity located at $y = y_c$ in the outer expansions (16, 17, 20, 23) moves to the new level $y = y_a$. Importantly, here our assumption of an infinite period implies that the pattern depends on the sign of A. Both possible structures are shown in Fig. 2 in the reference frame moving with the nonlinear wave speed, that is in the coordinates $(\zeta = X - VT, Y)$.

- $A > 0$: the separatrix is defined by the value $S = S_c = A_0$, with a saddle point at $\zeta = 0$, $Y = 0$ where A attains its maximum A_0, while both centres are located on $A = S = 0$ as $\zeta = \pm\infty$.
- $A < 0$: there is a unique centre at $\zeta = 0$, $Y = 0$ corresponding to $A = S = A_0$, and the other two stagnation points are located on $A = S_c = 0$ as $\zeta = \pm\infty$.

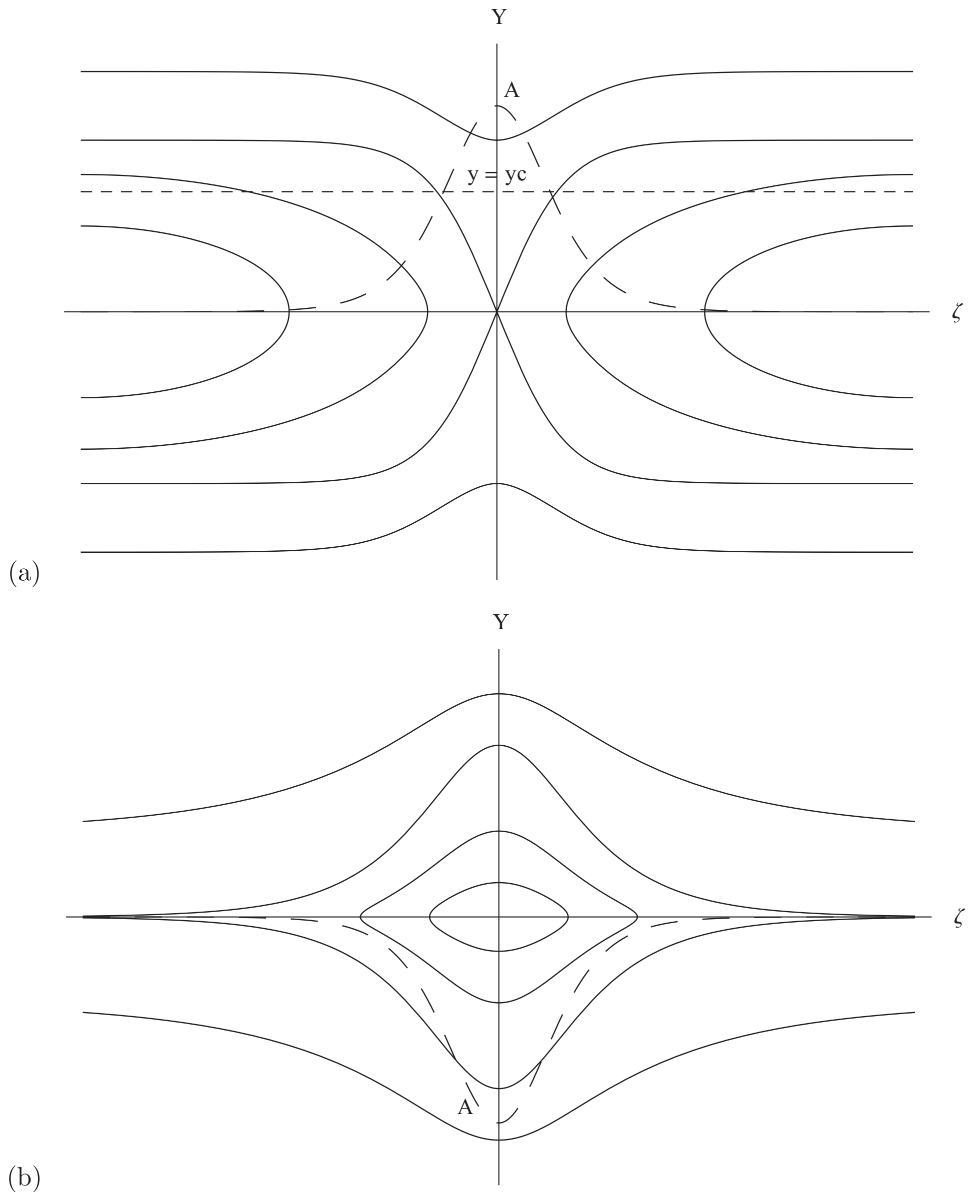


FIG. 2. Streamline (solid lines) structure in the critical layer according to the sign of A ($\zeta = X - VT$ is the streamwise coordinate on the axis $Y = 0$ in the frame moving with the nonlinear wave speed). The critical level $y = y_c$ is the dashed line and is located at $Y = y_\delta$. The wave $A(\zeta)$ is the long-dashed line for (a) D-wave: $A > 0$ and (b) E-wave: $A < 0$, $y_\delta = 0$.

Following the terminology,[34] the first critical layer is called a depression, while the second is an elevation. The leading-order pressure inside the critical layer is given by

$$P = \text{constant} - \epsilon^{\frac{1}{2}} \frac{gL}{\mathcal{U}^2} Y + \epsilon U_0'^2 A + \cdots .$$

Thus a depression (elevation) critical layer is characterized by a positive (negative) anomaly of pressure at $\zeta = 0$. The respective solitary waves are then denoted as D-waves, or E-waves, respectively. The D-wave streamfunction S increases from $S = 0$ (or $S = A_0$ for the E-wave) in the centre up to $S = S_c$, then $S \to +\infty$ at the edges of the critical layer. We will see later on, in Sec. VI, that for the E wave, $y_\delta = 0$.

B. New strained coordinates

In Subsection IV A, we have determined the flow within the critical layer by using the independent variable $S = \bar{\Psi}^{(0)}$ which gives a first approximation to the location of the dividing streamlines. But an examination of the next orders of the expansion shows that this definition is incomplete. The cat's eye is not strictly symmetric with respect to the level $y_a = y_c - y_\delta\,\epsilon^{1/2}$ owing to the distortions of the mean flow and a better independent variable $\tilde{S}$, a stressed coordinate, is needed in order to correctly describe the perturbed geometry of the critical layer. Also, it is necessary to ensure that the streamwise velocity is zero both at the core of the cat's eye and at the crossing point of the separatrices in the frame moving with the wave speed, properties which have previously been overlooked, see for instance.[35] Thus we rescale the order-ϵ streamfunction S using a strained coordinate $\tilde{S}$ of the following way:

$$S = \tilde{S} + \delta\varphi^{(1)}(\tilde{S}) + \epsilon^{\frac{1}{2}}\varphi^{(2)}(\tilde{S}) + \delta^2\varphi^{(3)}(\tilde{S}) + \epsilon^{\frac{1}{2}}\delta\,\varphi^{(4)}(\tilde{S}) + \epsilon\,\varphi^{(5)}(\tilde{S}) + \epsilon^{\frac{1}{2}}\delta^2\varphi^{(6)}(\tilde{S}) + \cdots, \tag{37}$$

where $\delta = \epsilon^{1/2}\ln\epsilon$. The deformation functions $\varphi^{(i)}$ are determined in sequence by ensuring that there is zero velocity at the stagnation points. It is useful to introduce an analogous strained coordinate $\tilde{Y}$, so that

$$\tilde{S} = \frac{1}{2}\tilde{Y}^2 + A\,, \tag{38}$$

$$Y = \tilde{Y} + \delta\,Y_1(\tilde{Y}) + \epsilon^{\frac{1}{2}}Y_2(\tilde{Y}) + \cdots. \tag{39}$$

The symmetry line is thus characterized by $\tilde{Y} = 0$ and the separatrix is then simply defined by

$$\tilde{Y}_s = [2(S_c - A)]^{\frac{1}{2}} \quad \text{where} \quad S_c = A_0,\ \text{D wave}\,, \quad \text{or} \quad S_c = 0,\ \text{E wave}\,. \tag{40}$$

In terms of these strained coordinates, we obtain the new expansion:

$$\tilde{\Psi} = \epsilon\,U_0^{'}\,\tilde{S} + \epsilon^{\frac{3}{2}}\,\tilde{\Psi}^{(2)} + \epsilon^2\ln^2\epsilon\,\tilde{\Psi}^{(3)} + \epsilon^2\ln\epsilon\,\tilde{\Psi}^{(4)} + \epsilon^2\tilde{\Psi}^{(5)} + \epsilon^{\frac{5}{2}}\ln^2\epsilon\,\tilde{\Psi}^{(6)} + \epsilon^{\frac{5}{2}}\ln\epsilon\,\tilde{\Psi}^{(7)} + \cdots. \tag{41}$$

At the next orders $O(\epsilon^{3/2}\ln\epsilon)$ and $O(\epsilon^2\ln\epsilon)$, using the same technique as that employed in Sec. IV A, we find that $\Psi^{(1)} = b_0/2YA^*$ and $\Theta^{(1)} = \bar{T}_0^{''}\Psi^{(1)}$. But then, this order vanishes in the $\tilde{S}$-expansion, that is, $\tilde{\Psi}^{(1)} = 0$ and $\tilde{\Theta}^{(1)} = 0$ since the first-order correction to S is given by $\varphi^{(1)} = -\bar{\Psi}^{(1)}$. At the next order $O(\epsilon^2\ln\epsilon)$, we find that $\tilde{\Psi}^{(3)} = -b_0^2U_0^{'}A^2/8$ and $\tilde{\Theta}^{(3)} = -b_0^2T_0^{''}A^2/8$ while $\varphi^{(3)} = 0$.

C. $O(\epsilon^{3/2})$

We next consider the terms of order $O(\epsilon^{3/2})$. Here, it is now necessary to consider the flow inside and outside the separatrices separately. For the flow outside the separatrices, the equations for $\Psi^{(2)}_{YY}$ and $\Theta^{(2)}$ are

$$U_0^{'}(S_Y\partial_X - S_X\partial_Y)\Psi^{(2)}_{YY} = -\Theta^{(0)}_X + \lambda\partial_Y^2\Psi^{(2)}_{YY}\,,$$

$$U_0^{'}(S_Y\partial_X - S_X\partial_Y)\Theta^{(2)} + J(\Psi^{(2)},\Theta^{(0)}) = -\partial_T\Theta^{(0)} + \frac{\lambda}{Pr}[\partial_Y^2\Theta^{(2)} - (S_Y - y_\delta)T_0^{'''}]\,, \tag{42}$$

where $J(\cdot\,,\,\cdot)$ is the operator Jacobian: $\partial_Y\partial_X - \partial_X\partial_Y$. Hence, the inviscid vorticity and buoyancy are expressed by

$$\Psi^{(2)}_{i,YY} = \bar{T}_0^{''}S_Y + Q(S) \quad \text{and} \qquad \Theta^{(2)}_i = \bar{T}_0^{''}\{\Psi^{(2)} - \Pi[S_Y]\} + \mathcal{G}^{(2)}(S)\,, \tag{43}$$

$$\text{where} \quad \Pi[S_Y] = \int_{x_c}^{\xi}\frac{\partial_T A(x,T)}{S_Y(S,x,T)}\,dx.$$

The lower integration bound x_c is the streamwise coordinate of the cat's eye core, the centre point for which $x_c = \text{sign}[\zeta]\,\infty$ for a D-wave and $x_c = V\ T\ (\zeta_c = 0)$ for a E-wave, when the solution is a V-speed travelling wave.

The secularity conditions and the matching when $S \to \infty$ yield,

$$Q(S) = sb_0 U_0^{'}\sqrt{2S} + \hat{U}_1^{'} - U_0^{''} y_\delta, \tag{44}$$

and

$$\mathcal{G}^{(2)}(S) = \frac{1}{6}s(T_0^{'''} - T_0^{''}\bar{U}_0^{''})(2S)^{\frac{3}{2}} + (T_1^{''} - T_0^{''}\hat{\bar{U}}_1^{'} - y_\delta T_0^{'''} + T_1^{'}\bar{U}_0^{''})S$$
$$+s(\frac{1}{2}T_0^{'''}y_\delta^2 + T_2^{'} - T_1^{''}y_\delta)\sqrt{2S} - \frac{1}{6}T_0^{'''}y_\delta^3 + \frac{1}{2}T_1^{''}y_\delta^2 - T_2^{'}y_\delta + T_3, \tag{45}$$

where $s = \text{sgn}[Y]$ ($S > 0$ for an open streamline). Further integrations with respect to Y and matching with the outer flow yield explicit expressions for the corresponding velocity, streamfunction and pressure fields. For instance, the streamwise velocity is given by

$$\psi_Y^{(2)} = \bar{T}_0^{''}S + b_0 G(A, S) + (\hat{U}_1^{'s} - U_0^{''}y_\delta)S_Y + \mathcal{U}^{(2)}(A)\,, \tag{46}$$

where $\quad G(A, S) = U_0^{'}\{A\ \ln[\Lambda(A, S)] + [S(S - A)]^{\frac{1}{2}}\}\,,\ \ \mathcal{U}^{(2)}(A) = [b_0 U_0^{'}(a^s + \frac{1}{2}) - \bar{T}_0^{''}]A,$

$$\Lambda(A, S) = \left|\frac{S - A}{A_0}\right|^{\frac{1}{2}} + \left|\frac{S}{A_0}\right|^{\frac{1}{2}}, \ \text{ and } \ S_Y = Y = s\,[2(S - A)]^{\frac{1}{2}}.$$

Using the extended Prandtl-Batchelor theorem (see Appendix A), the vorticity in the cat's eye is given by

$$Q^{(2,\odot)} = Q_2 + \bar{T}_0^{''}S_Y\,, \tag{47}$$

while the $\tilde{S}$-vorticity fields are unchanged, that is

$$\tilde{\Psi}_{YY}^{(2)} = Q(S) + \bar{T}_0^{''}\tilde{S}_Y\,, \qquad \tilde{\Psi}_{YY}^{(2,\odot)} = Q_2 + \bar{T}_0^{''}\tilde{S}_Y. \tag{48}$$

Since the vorticity should be continuous on the separatrices, by matching (48) on the separatrix $S = S_c$, this yields the two conditions

$$Q_2 = \underline{\hat{U}}_1^{'} - U_0^{''}y_\delta \quad \text{and} \quad [U_1^{'}]_-^+ = -2b_0 U_0^{'}Y_\infty. \tag{49}$$

The cross-stream coordinates of each separatrix (up $s = 1$, down $s = -1$) are given by $S_Y(S_c) = sY_s(\xi)$. We adopt the following notations, $\underline{f} = (f^+ + f^-)/2$, and $[f]_-^+ = f^+ - f^-$ which defines the jump of f across the critical layer. The constant vorticity Q_2 is the first-order additional mean vorticity evaluated at the core of the cat's eye at the level: $y_c - \epsilon^{1/2}y_\delta$. Using the appropriate matching conditions, the second-order mean flow thus possesses a velocity jump across the critical layer

$$[U_2]_-^+ = -2b_0 U_0^{'}y_\delta Y_\infty\,,$$

while the second-order wave speed is

$$c_2 = \underline{U}_2 + \left(\frac{1}{2}U_0^{''}y_\delta - \underline{\hat{U}}_1^{'}\right)y_\delta\,.$$

The separatrix cross-stream location when ξ tends to infinity, sY_∞ takes the following values: $Y_\infty = (2A_0)^{1/2}$, for the D wave and $Y_\infty = 0$, for the E wave. Consequently, the D-wave additional mean flow is distorted. The latter possesses a vorticity jump through y_c whose difference from the unstratified medium case is contained in the expression for b_0, which now depends on $T_0^{''}(y_c)$. The D-wave mean velocity has a $O(\epsilon)$ jump proportional to $J_c\,(\epsilon A_0)^{1/2}$.

The $\tilde{S}$-velocity fields are

$$\tilde{\Psi}_Y^{(2)} = \Psi_Y^{(2)} + \frac{U_0^{'}}{S_Y}\varphi^{(2)} = b_0 G(A, \tilde{S}) + (\hat{U}_1^{'} - U_0^{''}y_\delta)\tilde{S}_Y + \frac{1}{2}b_0 U_0^{'}Y_\infty\tilde{Y}_s,\ \ \tilde{\Psi}_Y^{(2,\odot)} = Q_2\tilde{S}_Y + \tilde{\mathcal{U}}_2(A),$$

and the second-order deformation of the streamlines is given by

$$\varphi^{(2,s)} = \bar{S}_Y \left(\frac{1}{2} b_0 U_0^{'} Y_\infty \tilde{Y}_s - \bar{T}_0^{''} \tilde{S} - \mathcal{U}^{(2)} \right).$$

We deduce from matching $\varphi^{(2)}$ with $\varphi^{(2,\odot)}$ on the separatrices that

$$\varphi^{(2,\odot)} = \bar{S}_Y \left(\frac{1}{2} b_0 U_0^{'} Y_\infty \tilde{Y}_s - \bar{T}_0^{''} \tilde{S} - \underline{\mathcal{U}}^{(2)} \right).$$

Importantly, we now note the absence of a^s-jump, that is $b_0[a]_-^+ = 0$, that is the logarithmic phase shift is zero, as expected, see Ref. 29. Next $\mathcal{U}^{(2+)} = \mathcal{U}^{(2-)}$, while the matching of $\tilde{\Psi}_Y^{(2)}$ with $\tilde{\Psi}_Y^{(2,\odot)}$ gives the integration function $\tilde{\mathcal{U}}_2$: $\tilde{\mathcal{U}}_2 = b_0 A^* \ln[\Lambda(A, S_c)]$. The $\tilde{S}$-streamfunctions are

$$\tilde{\Psi}^{(2)} = \frac{2}{3}\bar{T}_0^{''}\tilde{S}_Y(A-\tilde{S}) + b_0 U_0^{'}[A\{\tilde{S}_Y \ln[\Lambda(A,\tilde{S})] - s\sqrt{2\tilde{S}}\} + \frac{1}{6}s\sqrt{2\tilde{S}}^3 + \frac{1}{2}Y_\infty \tilde{Y}_s \tilde{S}_Y]$$
$$+ (\hat{U}_1^{'} - U_0^{''} y_\delta)\tilde{S} + \Phi^{(2)}(\xi, T),$$

$$\text{and} \qquad \tilde{\Psi}^{(2,\odot)} = \frac{2}{3}\bar{T}_0^{''}\tilde{S}_Y(A-\tilde{S}) + Q_2\tilde{S} + \tilde{\mathcal{U}}_2 \tilde{S}_Y + \tilde{\Phi}_2(\xi, T),$$

$$\text{with } \Phi^{(2)}(\xi, T) = (c_{l,1,0} + \beta_{l,1} - \hat{\bar{U}}_1^{'s} + \bar{\bar{T}}_1^{'} - a b_0 y_\delta)\, A^* + \beta_{d,1}\partial_X^2 A^* + B^*.$$

The matching of the streamfunction $\tilde{\Psi}^{(2)}$ on the separatrix $S = S_c$, to within an arbitrary constant, leads to

$$\tilde{\Psi}^{(2)}(S_c, \xi) + C^{(2)} = \tilde{\Psi}^{(2,\odot)}(S_c, \xi),$$

$$\text{and so} \quad \tilde{\Phi}_2(\xi, T) = \underline{\Phi}^{(2)} + \underline{C}^{(2)}, \qquad [C^{(2)}]_-^+ = -\frac{2}{3} b_0 U_0^{'} Y_\infty S_c,$$

$$[\beta_{d,1}]_-^+ = 0 \quad \text{and} \quad \frac{[\bar{\bar{T}}_1^{''}]_-^+}{b_0} - \frac{[\bar{U}_1^{''}]_-^+}{b_0} + [\beta_{l,1}]_-^+ = 2(2b_0 - \bar{\bar{T}}_0^{''})Y_\infty. \tag{50}$$

The buoyancy field in the cat's eye is, from Appendix A,

$$\tilde{\Theta}^{(2,\odot)} = \bar{T}_0^{''}(\tilde{\Psi}^{(2,\odot)} - \Pi[\tilde{S}_Y] + s\Pi_0[\tilde{S}]) - (\bar{Q}_2 T_0^{''} + T_0^{'''} y_\delta)\tilde{S} + \theta_2\,, \tag{51}$$

$$\text{where} \quad \Pi_0[S] = \int_{x_c}^{\xi(S)} \frac{\partial_T A(x, T)}{|S_Y(S, T, x)|}\, dx\,,$$

such as the upper integration bound $\xi(S)$ is given by $A[\xi, T] = S$. The function Π_0 appears in order to make the buoyancy continuous on the symmetry axis of the critical layer. We have now to match $\tilde{\Theta}^{(2)}$ on the separatrix. The outcome is that the separatrices are iso-buoyancy contours. Moreover, we can find the constant in (51)

$$\theta_2 = \frac{1}{2}\underline{T}_1^{''}(Y_\infty^2 + y_\delta^2) + \frac{1}{2}([T_2^{'}]_-^+ - [T_1^{''}]_-^+ y_\delta)Y_\infty - \bar{T}_0^{''}\underline{C}^{(2)} - \frac{1}{6}T_0^{'''} y_\delta^3 - \underline{T}_2^{'} y_\delta + \underline{T}_3\,, \tag{52}$$

and the jump $[\mathcal{G}^{(2)}(S_c)]_-^+$ from (43); this last condition provides with a relationship linking various mean distorted buoyancies at the critical level up to the third order and the nonlinear wave speed V:

$$\frac{1}{2}[T_1^{''}]_-^+\{Y_\infty^2 + y_\delta^2\} - [T_2^{'}]_-^+ y_\delta + [T_3]_-^+$$
$$= 2[(\frac{1}{3}\bar{\bar{T}}_0^{''} - b_0)T_0^{''} - \frac{1}{3}T_0^{'''}]S_c Y_\infty + 2(\underline{T}_1^{''} y_\delta - \underline{T}_2^{'})Y_\infty - T_0^{'''} y_\delta^2 Y_\infty - 2\bar{T}_0 V\sqrt{2|A_0|}. \tag{53}$$

The analytical expressions of buoyancy and vorticity in the cat's eye, obtained through the Prandtl-Batchelor extended theorem, show that, to the leading orders, the buoyancy is locked onto the streamfunction, and if we remove the body forces, the buoyancy and vorticity are constant. Our expressions are the consequence of the body force terms acting through the equations, plus constants of integration determined from matchings on the separatrix.

D. $O(\epsilon^2 \ln \epsilon)$

The equations for the third-order vorticity and buoyancy are

$$U_0'(S_Y\partial_X - S_X\partial_Y)\Psi^{(4)}_{YY} = J(\Psi^{(2)}_{YY},\Psi^{(1)}) - \partial_X\Theta^{(1)} + \lambda\partial_Y^2\Psi^{(4)}_{YY},$$
$$U_0'(S_Y\partial_X - S_X\partial_Y)\Theta^{(4)} + J(\Psi^{(4)},\Theta^{(0)})$$
$$= J(\Theta^{(2)},\Psi^{(1)}) + J(\Theta^{(1)},\Psi^{(2)}) - \partial_T\Theta^{(1)} + \frac{\lambda}{Pr}\partial_Y^2\Theta^{(4)}. \tag{54}$$

The inviscid fields are given, outside the separatrices, by

$$\Psi^{(4)}_{i,YY} = \Psi^{(1)}\bar{Q}_S + \mathcal{F}^{(4)}(S), \tag{55}$$

and

$$\Theta^{(4)}_i = \bar{T}_0''\left\{\Psi^{(4)} - \frac{1}{2}b_0A_{TX^{-1}} - \Pi_S[S_Y]\bar{\Psi}^{(1)}\right\} + \bar{\Psi}^{(1)}\mathcal{G}^{(2)}_S(S) + \mathcal{G}^{(4)}(S).$$

Matching with the outer flow shows that $\mathcal{F}^{(4)} = 0$ and $\mathcal{G}^{(4)} = 0$. The E-wave vorticity and buoyancy are singular on the separatrix since $S_c = 0$ but, as in the unstratified case, the singularity is removed by the replacement of the cross-stream coordinate S with $\tilde{S}$, which better models the distortion of the streamlines in the nonlinear critical layer. Further integrations give the streamwise velocity and the streamfunction. The $\tilde{S}$-fields are

$$\tilde{\Psi}^{(4)}_{YY} = -\frac{1}{2}b_0\bar{T}_0''A,\quad \tilde{\Psi}^{(4)}_{i,Y} = -\frac{1}{2}b_0\bar{T}_0''\tilde{S}_YA,\quad \tilde{\Psi}^{(4)}_i = \frac{1}{4}b_0^2Y_\infty A^*Y_s - b_0(\bar{T}_0''\tilde{S} + \mathcal{U}^{(2)})A + \Phi^{(4)}(\xi,T),$$

$$\tilde{\Theta}^{(4)} = \bar{T}_0''\left(\tilde{\Psi}^{(4)} - \frac{1}{2}b_0A_{TX^{-1}}\right),$$

$$\text{with}\quad \varphi^{(4)} = -S_Y\bar{\mathcal{U}}^{(4)}(\xi,T),\quad \mathcal{U}^{(4)}(\xi,T) = \frac{1}{2}b_0[(\beta_{l,1} - \hat{U}_1' + \bar{T}_1')A^* + \beta_{d,1}\partial_X^2A^* + B^*],$$

$$\text{and}\quad \Phi^{(4)}(\xi,T) = \frac{1}{2}\{(b_{l,2,0} + \frac{1}{2}b_0^2y_\delta^2 - \beta_{l,1}b_0y_\delta)A^* + (b_{n,2,0} - \frac{1}{2}\bar{b}_0^2)A^{*2}$$
$$+(b_{d,2,0} - \beta_{d,1}b_0y_\delta)\partial_X^2A^* + b_{t,2,0}\partial_{TX^{-1}}A^* - b_0y_\delta B^*\}.$$

The vorticity and buoyancy fields within the cat's eye are given by (see Appendix A),

$$\tilde{\Psi}^{(4,\odot)}_{i,YY} = Q_4 - \frac{1}{2}b_0\bar{T}_0''A,$$

and

$$\tilde{\Theta}^{(4,\odot)} = \bar{T}_0''\left(\tilde{\Psi}^{(4,\odot)} - \frac{1}{2}b_0\partial_{TX^{-1}}A\right) + \mathcal{G}^{(4,\odot)}(\tilde{S}).$$

Next, the streamwise velocity is given by

$$\tilde{\Psi}^{(4,\odot)}_{i,Y} = Q_4\tilde{S}_Y - \frac{1}{2}b_0\bar{T}_0''\tilde{S}_YA,$$

where $\varphi^{(4,\odot)} = \varphi^{(4)}$. Matching of the velocity yields

$$Q_4 = 0,\qquad [\beta_{l,1}]^+_- = -2b_0Y_\infty,\ \ (\mathcal{U}^{(4)+} = \mathcal{U}^{(4)-}). \tag{56}$$

Hence, we find a new jump for the distorted mean flow,

$$[U_1^{''} - \bar{T}_1^{''}]_-^+ = 2b_0(\bar{T}_0^{''} - 3b_0 U_0^{'})Y_\infty. \tag{57}$$

Matching of the streamfunctions $\tilde{\Psi}_i^{(4)}$ with

$$\tilde{\Psi}^{(4,\odot)} = \frac{1}{2}b_0^2\{Y_\infty Y_s - A\ \ln[\Lambda(A, S_c)]\}A^* - b_0(\bar{T}_0^{''}\tilde{S} + \mathcal{U}^{(2)})A + \tilde{\Phi}_4(\xi, T)$$

$$\text{yields}\quad [\Phi^{(4)}]_-^+ = 0\,, \quad \text{and}\quad \tilde{\Phi}_4(\xi, T) = \frac{1}{2}b_0^2\{A\ \ln[\Lambda(A, S_c)] - \frac{1}{2}Y_s Y_\infty\}A^* + \Phi^{(4)} + \underline{C}^{(4)}\,,$$

from which we can establish that

$$[T_2^{'}]_-^+ = -y_\delta [T_1^{''}]_-^+\,, \tag{58}$$

and that $C^{(4)s} = 0$. Matching of the buoyancy is possible if we choose $\mathcal{G}^{(4,\odot)}(S_c) = 0$. From Appendix A, we then find that the constant $\theta_4 = 0$ in Eq. (A7), so that $\mathcal{G}^{(4,\odot)}(S)$ is just the zero function in the case of a *D*-wave, but has a complex expression for an *E*-wave.

The assumption of weak stratification in the critical layer has led to lesser importance of buoyancy effects *vis-à-vis* the vorticity dynamics since the buoyancy intervenes in the vorticity equation at a lower order than the vorticity. The vorticity and buoyancy are thus effectively uncoupled, since, at each order; the vorticity is calculated from the vorticity equation, and then, after a double *S*-integration, the streamfunction is determined, and then the buoyancy is calculated from the buoyancy equation through the newly found streamfunction. Terming the critical layer as a weakly stratified region must be here thus interpreted as a zone where the vorticity dynamics is weakly coupled with buoyancy, but not a zone where the local Richardson number is small. The latter is indeed defined as the product of the overall Richardson number *Ri* by *J*. As *Ri* may have a large value, then $Ri\ J_c$ may reach values larger than 0.25.

E. $O(\epsilon^2)$ terms

We next consider the terms of order $O(\epsilon^2)$. The equations for $\Psi_{YY}^{(5)}$ and $\Theta^{(5)}$ are

$$U_0^{'}(S_Y\partial_X - S_X\partial_Y)\Psi_{YY}^{(5)} = J(\Psi_{YY}^{(2)}, \Psi^{(2)}) - \Theta_X^{(2)} - \Psi_{YYT}^{(2)} + \lambda(\partial_Y^2\Psi_{YY}^{(5)} - U_0^{'''})\,,$$

$$U_0^{'}(S_Y\partial_X - S_X\partial_Y)\Theta^{(5)} + J(\Psi^{(5)}, \Theta^{(0)}) = J(\Theta^{(2)}, \Psi^{(2)}) - \partial_T\Theta^{(2)} + \frac{\lambda}{Pr}(\partial_Y^2\Theta^{(5)} - \frac{1}{2}T_0^{IV}\mathcal{Y}^2)\,.$$

Using the variables *S* and ξ, in the inviscid limit and outside the separatrices, they reduce to

$$\Psi_{i,YY}^{(5)} = \bar{Q}_S\{\Psi^{(2)} - \Pi[S_Y]\} + \bar{S}_Y(\mathcal{G}_S^{(2)}(S) - \bar{T}_0^{''}\Pi_S[S_Y]) + \mathcal{F}^{(5)}(S), \tag{59}$$

and

$$\Theta_i^{(5)} = \bar{T}_0^{''}\Psi^{(5)} + \bar{\mathcal{G}}_S^{(2)}(\Psi^{(2)} - \Pi[S_Y]) - \bar{\bar{T}}_0^{''}\int_{x_c}^{\xi}\Pi_S[S_Y]\Big(\Psi_\xi^{(2)} - \frac{\partial_T A}{S_Y}\Big)\,dx$$
$$- \bar{\bar{T}}_0^{''}\int_{x_c}^{\xi}\frac{\Psi_\tau^{(2)} - \Pi_\tau[S_Y]}{S_Y}\,dx + \mathcal{G}^{(5)}(S)\,. \tag{60}$$

For the *E*-wave, these $O(\epsilon^2)$ vorticity and buoyancy expressions are singular, on the separatrices as $S \to S_c = 0$. As described in Ref. 20, using the cross-stream coordinate $\tilde{S} \to \hat{S} = \tilde{S}/(\epsilon^{1/2}\ \ln\epsilon)$ enables this singularity to be removed in a zone along the separatrices of width $\delta y = \epsilon\ln\epsilon$. The velocity field $\tilde{\Psi}_Y^{(5)}$, the stream function $\tilde{\Psi}^{(5)}$ and hence $\tilde{\Theta}^{(5)}$ can be found explicitly, where constants of integration are determined by matching with the outer flow and by using the secularity condition. In particular, matching the buoyancy with the outer flow yields the diffusion-induced mean buoyancy

$$T_{ind}(y) \sim \epsilon\frac{1}{2}\beta_{v,1}\beta_{ind}Pr\bar{T}_0^{''}(y - y_c)^2$$

$$+\epsilon^{\frac{3}{2}}\beta_{v,1}Pr[\beta_{ind}\bar{T}_0^{''}y_\delta+\frac{1}{6}T_0^{'''}y_\delta^3-\frac{1}{2}T_1^{''}y_\delta^2+T_2^{'}y_\delta-T_3](y-y_c)+\cdots\,, \quad (61)$$

with $\beta_{ind}=(T_1^{''}y_\delta-T_0^{'''}y_\delta^2/2-T_2^{'})/\bar{T}_0^{''}$, the singular-Frobenius-solution coefficient in Eq. (20). The vorticity within the closed streamlines is given by

$$\Psi_{YY}^{(5,\odot)}=Q^{(5,\odot)}(S)-\bar{\bar{T}}_0^{''}\{\Pi_S[S_Y]-s\Pi_{0,S}[S]+Q_2\}S_Y-\bar{T}_0^{'''}y_\delta S_Y\,. \quad (62)$$

Application of the Prandtl-Batchelor theorem (see Appendix A) determines

$$Q^{(5,\odot)}(S)=U_0^{'''}S+\omega_5(S)+Q_5\,.$$

For the D-wave, $\omega_5(S)=0$, while the expression $\omega_5(S)$ for the E-wave is given in Appendix A. One integration yields the $\tilde{S}$-velocity

$$\tilde{\Psi}_Y^{(5,\odot)}=\frac{1}{3}U_0^{'''}\tilde{S}_Y(2A+\tilde{S})+Q_5\tilde{S}_Y-\bar{\bar{T}}_0^{''}\{\Pi[\tilde{S}_Y]-s\Pi_0[\tilde{S}]-\Pi[sY_s]+s\Pi_0[S_c]\}+\bar{T}_0^{''}\varphi^{(2)}(\tilde{S})$$
$$+\{[b_0(\underline{\beta}_{l,1}+\bar{\bar{T}}_1^{'})-\bar{U}_0^{''}\bar{\bar{T}}_1^{'}+y_\delta(\bar{\bar{T}}_0^{''2}-b_0^2)]A^*+b_0(\underline{\beta}_{d,1}\partial_X^2A^*+B^*)\}\ \ln[\Lambda(A,Sc)]\,, \quad (63)$$

while the buoyancy field within the separatrices is given by

$$\tilde{\Theta}^{(5,\odot)}=\bar{T}_0^{''}\{\tilde{\Psi}^{(5,\odot)}-\Pi_{\tilde{S}}[\tilde{S}_Y]\varphi^{(2)}(\tilde{S})-s\Pi_{0,\tilde{S}}[\tilde{S}]Q_2\tilde{S}\}-s(\bar{Q}_2\bar{T}_0^{''}+\bar{T}_0^{'''}y_\delta)\Pi_0[\tilde{S}]$$
$$+\{s\bar{\bar{T}}_0^{''}\Pi_{0,S}[\tilde{S}]-\bar{Q}_2\bar{T}_0^{''}-\bar{T}_0^{'''}y_\delta\}\{\tilde{\Psi}^{(2,\odot)}-\Pi[\tilde{S}_Y]\}-\bar{\bar{T}}_0^{''}\int_{x_c}^{\xi}\Pi_{\tilde{S}}[\tilde{S}_Y]\Big(\Psi_\xi^{(2,\odot)}-\frac{\partial_T A}{\tilde{S}_Y}\Big)dx$$
$$-\bar{\bar{T}}_0^{''}\int_{x_c}^{\xi}(\Psi_\tau^{(2,\odot)}-\Pi_\tau[\tilde{S}_Y])\frac{dx}{\tilde{S}_Y}+\mathcal{G}^{(5,\odot)}(\tilde{S})\,. \quad (64)$$

To show that $\tilde{\Theta}^{(5,\odot)}$ is continuous on the axis of the critical layer at $\tilde{Y}=0$ does not seem obvious because of the presence of two integrals. However, both of them are continuous except $-\Pi_{0,\tilde{S}}[\tilde{S}]\,\Phi_2(\xi)$ which emerges from the first integral, but is cancelled by $s\bar{\bar{T}}_0^{''}\Pi_{0,S}[\tilde{S}]\tilde{\Psi}^{(2,\odot)}$. The steady-travelling assumption moreover yields a lot of cancellations between both of integrals and only $-s\bar{\bar{T}}_0^{''}\Pi_{0,\tilde{S}}[\tilde{S}]\,\{\Psi^{(2,\odot)}(\tilde{S},A)-\Pi[\tilde{S}_Y]-\Psi^{(2,\odot)}(\tilde{S},0)\}$ remains, one can then show that the regularity is assured for $\tilde{S}\to 0$.

The expressions outside the separatrices (given in Appendix B) are then matched with the above expressions accross the separatrices. Matching the vorticity fields yields the relation

$$U_0^{'}\Big(\tilde{\Psi}_{YY}^{(5)}(S_c,\xi)-\tilde{\Psi}_{YY}^{(5,\odot)}(S_c,\xi)\Big)=\Big(\tilde{\Psi}_{YY\tilde{S}}^{(2,\odot)}(S_c,\xi)-\tilde{\Psi}_{YY\tilde{S}}^{(2)}(S_c,\xi)\Big)\Big(\tilde{\Psi}^{(2,\odot)}(S_c,\zeta_s)-\tilde{\Psi}^{(2,\odot)}(S_c,\xi)\Big)\neq 0.$$

The moving coordinate ζ_s of the stagnation point is $\zeta_s=0$ for a D-wave or $\zeta_s=\pm\infty$ for a E-wave. It transpires that this cannot be satisfied, as it would yield a dispersiveless amplitude equation. However, the horizontal velocity, streamfunction, pressure, buoyancy fields and deformation function $\varphi^{(5)}$ can all be matched continuously, leading to a complete determination of all unknowns, provided that a certain compatibility condition is satisfied. This, in turn, then yields the desired evolution equation for the amplitude A:

$$\frac{2}{9}b_0\bar{T}_0^{''}[(S_c+6A)Y_\infty-(S_c+2A)Y_s]-\frac{2}{3}(\bar{T}_0^{'''}-\bar{T}_0^{''}\bar{U}_0^{''})[(S_c+2A)Y_s-S_cY_\infty]$$
$$-\frac{1}{2}\{2b_0Y_\infty\bar{T}_0^{''}+[\bar{T}_1^{''}]_-^+\}(Y_s-Y_\infty)Y_\infty+2(b_0\mathcal{U}^{(2)}+\underline{\bar{T}}_2^{'}-\underline{\bar{T}}_1^{''}y_\delta+\frac{1}{2}\bar{T}_0^{'''}y_\delta^2)Y_\infty+\frac{3}{2}[\bar{T}_1^{''}]_-^+y_\delta^2+[\mathcal{U}^{(5)}]_-^+$$
$$+2\,[\,\frac{1}{2}(U_0^{'''}-\bar{T}_0^{'''})y_\delta^2+\frac{1}{4}[U_1^{''}]_-^+Y_\infty-\underline{\hat{U}}_1^{''}y_\delta+\underline{U}_2^{'}-\underline{\bar{T}}_2^{'}-Q_5-2b_0^2A^*]\,Y_s$$
$$=\mathcal{R}(A)=b_0\int_\infty^{S_c}\frac{\Pi[(S-A[X,T])^{\frac{1}{2}}]}{[2S(S-A[X,T])]^{\frac{1}{2}}}\,dS+2\bar{\bar{T}}_0^{''}\Pi_0[S_c]\,, \quad (65)$$

$$\text{with}\quad \bar{\mathcal{U}}^{(5)}(\xi,T) = [\alpha_{l,1} + b_0\beta_{l,1} + \frac{1}{2}(\bar{\bar{T}}_0^{''}\hat{U}_1^{'} - \hat{U}_1^{''} - \bar{\bar{T}}_1^{''}) + \frac{1}{2}(b_0^2 - \bar{U}_0^{'''} - 2ab_0^2 - \bar{\bar{T}}_0^{''}\bar{U}_0^{''} + 2\bar{\bar{T}}_0^{'''} - \bar{\bar{T}}_0^{''2})y_\delta]A + (\alpha_{d,1} + b_0\beta_{d,1})\partial_X^2 A + (1+a)b_0 B\,. \tag{66}$$

This is an integro-differential equation containing two integrals, where we note that the stratification leads to a second integral compared to the homogeneous case.[19] The calculation of the term on the right-hand side yields two different expressions according to the wave type,

$$\mathcal{R}(A) = -Y_0\left\{b_0\int_{x_c}^{X} F\left[\hat{A}(X,T), \frac{\hat{A}(x,T)}{\hat{A}(X,T)}\right]\frac{\partial_T\hat{A}(x,T)}{\hat{Y}_s(X,T)}\,dx - \bar{\bar{T}}_0^{''}I_0[S_c]\right\}\qquad \text{for a } D\text{-wave},$$

$$\text{or}\qquad \mathcal{R}(A) = Y_0\left\{b_0\int_{x_c}^{X} K^{'}\left[\frac{\hat{A}(X,T)}{\hat{A}(x,T)}\right]\frac{\partial_T\hat{A}(x,T)}{\hat{Y}_s(x,T)}\,dx - \bar{\bar{T}}_0^{''}I_0[S_c]\right\}\qquad \text{for a } E\text{-wave},$$

$$\text{with}\quad I_0[S_c] = \int_{x_c}^{X(S_c)}\frac{\partial_T\hat{A}(x,T)}{\hat{Y}_s(x,T)}\,dx, \tag{67}$$

where the rescaled amplitude $\hat{A}$ and separatrix location $\hat{Y}_s$ are $\hat{A} = A/A_0$, $\hat{Y}_s = \sqrt{1-\hat{A}}$ (D wave) and $\hat{Y}_s = \sqrt{\hat{A}}$ (E wave), and $Y_0 = \sqrt{2|A_0|}$ is the maximum half-width of the cat's eye. The function $F(\cdot, m)$ is the incomplete elliptic integral of the first kind, with parameter m ($0 \le m \le 1$) being the square of the elliptic modulus (see Ref. 42), and $K^{'}(m)$ is the complete complementary elliptic integral of the first kind, with parameter m ($0 \le m \le 1$). We take up a detailed examination of the evolution equation in Sec. V, and close this section with a brief summary of the determination of the remaining unknown constants.

The matching of the buoyancy on the separatrix yields the relationship

$$U_0^{'}\Big(\tilde{\Theta}^{(5)}(S_c,\xi) - \tilde{\Theta}^{(5,\odot)}(S_c,\xi)\Big) = \Big(\tilde{\Theta}_{\tilde{S}}^{(2,\odot)}(S_c,\xi) - \tilde{\Theta}_{\tilde{S}}^{(2)}(S_c,\xi)\Big)\Big(\tilde{\Psi}^{(2,\odot)}(S_c,\zeta_s) - \tilde{\Psi}^{(2,\odot)}(S_c,\xi)\Big) \neq 0$$

which gives two conditions on $\mathcal{G}^{(5)}(S_c)$ and $\mathcal{G}^{(5,\odot)}(S_c)$:

$$[\mathcal{G}^{(5)}]_-^+(S_c) = \bar{T}_0^{''}[C^{(5)}]_-^+ - (\bar{Q}_2\bar{T}_0{''} + \bar{T}_0^{'''}y_\delta)([C^{(2)}]_-^+ + 2\Pi_0[S_c]) - \Big[\Big(\bar{\mathcal{G}}_S^{(2)}(S_c) + \bar{Q}_2\bar{T}_0{''} + \bar{T}_0^{'''}y_\delta\Big)\tilde{\Psi}^{(2)}(S_c,\zeta_s)\Big]_-^+ + 2T_0^{''}\Pi_{0,\tilde{S}}[S_c]\Phi_2(\zeta_s), \tag{68}$$

and

$$\mathcal{G}^{(5,\odot)}(S_c) = \underline{\mathcal{G}}^{(5)}(S_c) - \bar{T}_0^{''}\underline{C}^{(5)} + (\bar{Q}_2\bar{T}_0^{''} + \bar{T}_0^{'''}y_\delta)\underline{C}^{(2)} + \underline{\Big(\bar{\mathcal{G}}_S^{(2)}(S_c) + \bar{Q}_2\bar{T}_0{''} + \bar{T}_0^{'''}y_\delta\Big)\tilde{\Psi}^{(2)}(S_c,\zeta_s)},$$

which enables the determination of the constant θ_5 in (A9) and (A10) in Appendix A. $\mathcal{G}^{(5,\odot)}$ is thus completely determined. We also get two conditions on $\mathcal{G}_{\tilde{S}}^{(2)}(S_c)$

$$\underline{\mathcal{G}}_{\tilde{S}}^{(2)}(S_c) = -\bar{Q}_2T_0{''} - T_0^{'''}y_\delta, \quad [\mathcal{G}_{\tilde{S}}^{(2)}(S_c)]_-^+ = 2\bar{T}_0^{''}\Pi_{0,\tilde{S}}[S_c],$$

which come down to two relations involving the distorted derivatives of D-wave buoyancy to the second order and the wave speed V:

$$\underline{T}_1^{''} = [T_1^{''}]_-^+\frac{y_\delta}{Y_\infty}, \tag{69}$$

$$[T_1^{''}]_-^+\Big(1-2\frac{y_\delta^2}{Y_\infty^2}\Big)=(T_0^{''}\bar{U}_0^{''}-2b_0T_0^{''}-T_0^{''''})Y_\infty-(2\underline{T}_2^{'}+T_0^{''''}y_\delta^2+2V\bar{T}_0^{''})Y_\infty^{-1}. \tag{70}$$

The jumps in $\alpha_{l,\,1}$ and $\alpha_{d,\,1}$ are determined by a compatibility condition applied to Eq. (19):

$$[\alpha_{l,1}]_-^+=\mathcal{P}\int_{y_1}^{y_2}\Big(2\frac{(\hat{U}_1(y)-c_1)}{[U_0(y)-c_0]^3}T_0^{'}(y)+\frac{\hat{U}_1^{''}(y)}{U_0(y)-c_0}-\frac{U_0^{''}(y)[\hat{U}_1(y)-c_1]+T_1^{'}(y)}{[U_0(y)-c_0]^2}\Big)\phi^2(y)\,dy$$
$$+ab_0\Big([\beta_{l,1}]_-^+-[b_{l,1,0}]_-^+\Big)+(a+1)b_0[c_{l,1,0}]_-^+\,, \tag{71}$$

$$[\alpha_{d,1}]_-^+=\int_{y_1}^{y_2}\phi^2(y)\,dy+ab_0[\beta_{d,1}]_-^+\,, \tag{72}$$

where $\mathcal{P}$ is the Cauchy principal part of the integral. Note that these jumps $[\alpha_{l,1}]_-^+$, $[\alpha_{d,1}]_-^+$ appear in the coefficients of the resulting evolution equation. On the other hand, the jumps $[b_{l,1,0}]_-^+$, $[\beta_{l,1}]_-^+$, $[c_{l,1,0}]_-^+$ and $[\beta_{d,1}]_-^+$ are determined by matching the streamfunction. Importantly, since all these jumps are non-zero generally, the term $\epsilon^{1/2}\psi^{(1)}$ in the outer expansion (13) is not continuous across the critical layer, and is one reason why the present theory differs from those[26,34] which assumed that this term was continuous.

Finally, matching the streamfunction yields the relationship

$$\tilde{\Psi}^{(5,\odot)}(S_c,\xi)-\tilde{\Psi}^{(5)}(S_c,\xi)=C^{(5)},$$

which gives the following jumps:

$$[\beta_{n,2}]_-^+=0\,,\quad[\beta_{d,2}]_-^+=-2\beta_{d,1}b_0Y_\infty\,,\quad[\beta_{f,2}]_-^+=0\,,\quad[\beta_{c,2}]_-^+=0\,, \tag{73}$$

$$[\beta_{B,d,1}]_-^+=0\,,\quad[\beta_{B,l,1}]_-^+=-2b_0Y_\infty\,, \tag{74}$$

$$[\beta_{l,2}]_-^+=[\bar{U}_2^{'}-\frac{1}{2}\bar{\bar{T}}_2^{'}-c_{l,2,1}]_-^++2b_0(\underline{\beta}_{l,1}-b_0y_\delta)Y_\infty+[\frac{1}{2}\bar{\bar{T}}_1^{''}-\bar{U}_1^{''}+\alpha_{l,1}]_-^+y_\delta-2\frac{\beta_{v,1}}{\bar{T}_0^{''}}Pr[T_1^{''}]_-^+y_\delta\,,$$

and

$$[C^{(5)}]_-^+=\{\frac{4}{3}[\underline{\hat{U}}_1^{''}-\underline{\bar{T}}_1^{''}+\bar{Q}_2\bar{T}_0^{''}+(\bar{T}_0^{''''}-U_0^{''''})y_\delta]Y_\infty+b_0(3b_0-\bar{\bar{T}}_0^{''})y_\delta U_0^{'}Y_\infty+[U_2^{'}]_-^+\}S_c\,.$$

Using (68), we can get another relation involving the third-order mean buoyancy gradients

$$[\mathcal{G}^{(5)}]_-^+(S_c)=\bar{T}_0^{''}\Bigg\{\Bigg[\frac{4}{3}(\underline{\hat{U}}_1^{''}-U_0^{''''}y_\delta-\underline{\bar{T}}_1^{''})+2b_0(3b_0-\bar{\bar{T}}_0^{''})y_\delta U_0^{'}$$
$$+\frac{2}{3}\Big(\bar{Q}_2+\frac{T_0^{''''}}{T_0^{''}}y_\delta\Big)\Big(2\bar{T}_0^{''}+b_0U_0^{'}+3\frac{V}{S_c}\Big)\Bigg]Y_\infty+[U_2^{'}]_-^+\Bigg\}S_c+2T_0^{''}\Pi_{0,\bar{s}}[S_c]\Phi_2(\zeta_s). \tag{75}$$

Finally, we display the jump of the new nonlinear contributions $J_2[A]$

$$\Big[J_2[A]\Big]_-^+=[\bar{T}_2^{'}]_-^+\{Y_\infty Y_s-2A\ln[\Lambda(A,S_c)]\}+\frac{4}{3}\underline{\bar{T}}_1^{''}(S_c-A)Y_s\,,$$

and so this critical-layer induced nonlinear term can be expressed as

$$J_2[A]=\bar{T}_2^{'}\{Y_\infty Y_s-2A\ln[\Lambda(A,S_c)]\}+\frac{2}{3}s\bar{T}_1^{''}(S_c-A)Y_s.$$

V. THE AMPLITUDE EQUATION

We now relate the evolution equation (65) to the simpler-looking evolution equation (1). To achieve this aim, we look for a travelling wave solution where $A = A(\zeta)$, $\zeta = X - VT$ where $A \to 0$ as $\zeta \to \pm\infty$, so that (1) becomes

$$\partial_T A = \mathcal{D}_X[A] \equiv VA = -\mathcal{D}[A]. \tag{76}$$

On the other hand, the right-hand side of Eq. (65) then becomes

$$\mathcal{R}(A) = 2V[b_0(Y_\infty - Y_s) - \bar{\bar{T}}_0^{''} Y_\infty], \qquad \text{for a } D\text{-wave}\,, \tag{77}$$

$$\mathcal{R}(A) = VY_0\Big(b_0\{K^{'}[\hat{A}(\zeta)] - E^{'}[\hat{A}(\zeta)]\} - 2\bar{\bar{T}}_0^{''}\Big), \qquad \text{for a } E\text{-wave}, \tag{78}$$

where $E^{'}(m)$ is the complete complementary elliptic integral of the second kind with modulus $m = \hat{A}(\zeta)$.

A. *D*-wave

Thus we find that, for the *D*-wave,

$$\mathcal{D}[A] = -\frac{1}{4}\frac{(Y_s + Y_\infty)[\mathcal{L}_H(A) + V\bar{\bar{T}}_0^{''} Y_s]}{b_0 - \bar{\bar{T}}_0^{''}} = -V_0 A - \frac{1}{2}R_0 A^2 - S_0\partial_X^2 A + \mathcal{D}_0[A], \tag{79}$$

where $\mathcal{L}_H(A)$ is the left-hand side of Eq. (65). Since this must hold when $A = 0$, we can now determine the constant vorticity Q_5

$$Q_5 = \underline{U}_2^{'} - \hat{\underline{U}}_1^{''} y_\delta + \frac{1}{2}U_0^{'''} y_\delta^2 + \frac{1}{4}\left(1 - \frac{y_\delta^2}{Y_\infty^2}\right)[\bar{T}_1^{''}]_-^+ Y_\infty - \frac{1}{2}b_0(3b_0 - \bar{\bar{T}}_0^{''})U_0^{'} Y_\infty^2 + \bar{\bar{T}}_0^{''} V.$$

The coefficients and terms in Eq. (79) are given by

$$\begin{aligned} V_0 = &\left(a - 1 - \frac{1}{3}\frac{\bar{\bar{T}}_0^{'''}}{b_0^2} + \frac{1}{3}\bar{U}_0^{''}\frac{\bar{\bar{T}}_0^{''}}{b_0^2} - \frac{4}{9}\frac{\bar{\bar{T}}_0^{''}}{b_0}\right)\frac{b_0^2 U_0^{'} A_0}{b_0 - \bar{\bar{T}}_0^{''}} + \frac{1}{4}\frac{[\alpha_{l,1}]_-^+}{b_0 - \bar{\bar{T}}_0^{''}}U_0^{'} Y_\infty \\ &+ \frac{\underline{\bar{T}}_2^{'}}{b_0 - \bar{\bar{T}}_0^{''}} + \frac{1}{2}\left(\bar{T}_0^{'''} - \frac{1}{2}\frac{\underline{\bar{T}}_1^{''}}{y_\delta}\right)\frac{y_\delta^2}{b_0 - \bar{\bar{T}}_0^{''}}\,, \end{aligned} \tag{80}$$

$$R_0 = 4\frac{b_0^2 U_0^{'}}{b_0 - \bar{\bar{T}}_0^{''}}\left(1 + \frac{1}{9}\frac{\bar{\bar{T}}_0^{''}}{b_0} + \frac{1}{3}\frac{\bar{\bar{T}}_0^{'''}}{b_0^2} - \frac{1}{3}\bar{U}_0^{''}\frac{\bar{\bar{T}}_0^{''}}{b_0^2}\right), \tag{81}$$

$$S_0 = \frac{1}{4}\frac{[\alpha_{d,1}]_-^+}{b_0 - \bar{\bar{T}}_0^{''}}U_0^{'} Y_\infty\,, \tag{82}$$

and

$$\begin{aligned} \mathcal{D}_0[A] = &\frac{1}{2}\frac{b_0^2 Y_\infty}{b_0 - \bar{\bar{T}}_0^{''}}\left(1 - a + \frac{14}{9}\frac{\bar{\bar{T}}_0^{''}}{b_0} + \frac{2}{3}\frac{\bar{\bar{T}}_0^{'''}}{b_0^2} - \frac{2}{3}\frac{\bar{\bar{T}}_0^{''}}{b_0^2}\bar{U}_0^{''} + \frac{1}{2}\frac{[\bar{\bar{T}}_1^{''}]_-^+}{b_0^2 Y_\infty}\right) A^* Y_s \\ &- \frac{1}{4}\{([\alpha_{l,1}]_-^+ A + [\alpha_{d,1}]_-^+\partial_X^2 A\}\frac{U_0^{'} Y_s}{b_0 - \bar{\bar{T}}_0^{''}}. \end{aligned} \tag{83}$$

Then, using (76) and (79) with (81)–(83), we find that

$$(V_0 - V)A + 4\mu r_0 A^2 - \mu\Omega A Y_\infty Y_s = \chi_d\,(Y_s + Y_\infty)\partial_X^2 A\,, \tag{84}$$

$$\text{where} \quad Y_s = [2(A_0 - A)]^{\frac{1}{2}}\,, \quad Y_\infty = \sqrt{2A_0}\,,$$

$$\text{and} \quad \Omega = 1 - a + \frac{14}{9}\frac{\bar{\bar{T}}_0''}{b_0} + \frac{2}{3}\frac{\bar{\bar{T}}_0'''}{b_0^2} - \frac{2}{3}\frac{\bar{\bar{T}}_0''}{b_0^2}\bar{U}_0'' + \frac{1}{2}\frac{[\bar{\bar{T}}_1'']_-^+}{b_0^2 Y_\infty} - \frac{[\alpha_{l,1}]_-^+}{2b_0^2 Y_\infty}, \tag{85}$$

$$\text{while} \quad \chi_d = -\frac{1}{4}U_0'[\alpha_{d,1}]_-^+/(b_0 - \bar{\bar{T}}_0''),\ \mu = \frac{1}{2}b_0^2 U_0'/(b_0 - \bar{\bar{T}}_0''),\ \text{and } R_0 = 8\mu r_0.$$

Here, we recall that $A_0 = A(0)$ and $\hat{A} = A/A_0$. This can readily be integrated once more to yield

$$\chi_d(\partial_X \hat{A})^2 = \mu Y_\infty (1-\hat{A})\left\{\frac{(V_0 - V)}{\mu A_0}\left[\frac{2}{3}(1-\hat{A})^{\frac{1}{2}} - 1\right] - 2r_0\left[1 + \hat{A} - \frac{4}{5}(1-\hat{A})^{\frac{1}{2}}\left(\frac{2}{3} + \hat{A}\right)\right]\right.$$
$$\left. - \Omega(1-\hat{A})^{\frac{1}{2}}\left[(1-\hat{A})^{\frac{1}{2}} - \frac{4}{3}\right]\right\}. \tag{86}$$

The speed V is obtained from (86) by letting $\hat{A} \to 0$

$$V = V_0 + \mu(\frac{14}{5}r_0 - \Omega)A_0.$$

The case $b_0 = \bar{\bar{T}}_0''$ corresponds to a nonlinear wave whose speed V is arbitrary and for which the evolution equation does not contain the time derivative. The parameter V_0 in Eq. (80) is then linked to r_0 and Ω, by the relation $2V_0(b_0 - \bar{\bar{T}}_0'') = b_0^2 U_0'(\Omega - 14/5 r_0)$. The rhs of (86) behaves like $\hat{A}^2$ around $A \to 0$, which leads to an exponential decay of the amplitude at $|\zeta| = \infty$. We can transform (86) by introducing the new variable $\hat{Y}_s = (1-\hat{A})^{1/2}$, so that

$$(\partial_X \hat{Y}_s)^2 = \frac{4}{5}r_0\frac{b_0^2 Y_\infty}{[\alpha_{d,1}]_-^+}\left(\hat{Y}_s^3 + \frac{5}{8}(\frac{\Omega}{r_0} - 2)\hat{Y}_s^2 - \frac{5}{4}(\frac{2}{5} + \frac{\Omega}{r_0})\hat{Y}_s + \frac{3}{4} + \frac{5}{8}\frac{\Omega}{r_0}\right). \tag{87}$$

This leads to a degenerate elliptic integral

$$\int_0^{\hat{Y}_s}\frac{dr}{\sqrt{s_0(r - r_1)}(1-r)} = \pm\frac{\zeta}{\zeta_0}, \tag{88}$$

$$\text{where} \quad \zeta_0 = \left|\frac{5}{4}\frac{[\alpha_{d,1}]_-^+}{b_0^2 r_0 Y_\infty}\right|^{\frac{1}{2}}, \qquad r_1 = -\frac{3}{4} - \frac{5}{8}\frac{\Omega}{r_0},\ \text{and } s_0 = \text{sign}[r_0].$$

Note that $[\alpha_{d,1}]_-^+$ is always strictly positive (cf. Eq. (72)). If $s_0 = 1$, then we require that $r_1 \le 0$, or $\Omega/r_0 \ge -6/5$, and then

$$\hat{A} = 1 - \left(1 - (1 - r_1)\text{sech}^2\left\{\text{argcosh}[(1-r_1)^{\frac{1}{2}}] + \frac{1}{2}(1-r_1)^{\frac{1}{2}}|\frac{\zeta}{\zeta_0}|\right\}\right)^2. \tag{89}$$

On the other hand if $s_0 = -1$, then we require that $r_1 > 1$, or $\Omega/r_0 < -14/5$ and then

$$\hat{A} = 1 - \left(1 - (r_1 - 1)\text{cosech}^2\left\{\text{argsinh}[(r_1 - 1)^{\frac{1}{2}}] + \frac{1}{2}(r_1 - 1)^{\frac{1}{2}}|\frac{\zeta}{\zeta_0}|\right\}\right)^2. \tag{90}$$

Note here that the term in the argument of cosech is never zero, and so this solution is not singular. Also, this case holds when $r_0 < 0$, which in turn requires the condition $s_0 = -1$ or

$$3U_0''^2 + \frac{4}{3}T_0''(2\bar{\bar{T}}_0'' - 5\bar{U}_0'') + T_0''' < 0. \tag{91}$$

This condition is not realizable for an unstratified flow, but for the present weakly stratified case, it is not as stringent. For both waves, the wavelength is inversely proportional to $A_0^{1/4}$. From (86) when $\hat{A} = 1$ (the maximum amplitude $A = A_0$), or directly from the solutions (89) and (90), we see that there is a slope discontinuity at $\zeta = 0$, that is $d\hat{A}/dX(\zeta = 0^+) \neq d\hat{A}/dX(\zeta = 0^-)$, unless $r_1 = 0$ when the solution reduces to

$$\hat{A} = 1 - \tanh^4\left\{\frac{1}{2}\frac{\zeta}{\zeta_0}\right\}. \tag{92}$$

We note that if Eq. (87) is twice differentiated with respect to X, and we replace $V\partial_X A$ with $-\partial_T A$ then we obtain the KdV equation

$$\hat{Y}_{s,T} + V_{0,D}\hat{Y}_{s,X} + R_{0,D}\hat{Y}_s\hat{Y}_{s,X} + S_{0,D}\hat{Y}_{s,XXX} = 0, \tag{93}$$

with

$$R_{0,D} = \frac{3}{5}R_0 A_0\,,\quad S_{0,D} = -2S_0\,,\ \text{and}\ V_{0,D} = V_0 + \frac{4}{5}\mu r_0 A_0.$$

However, this KdV equation cannot be obtained directly from the unsteady Equation (65). Instead is a "toy"-model equation which can only be used to infer the effects of unsteady behaviour.

B. *E*-wave

For the E-wave, we use (78) to characterize the amplitude Equation (1) by

$$\mathcal{D}[A] = -\frac{A\,\mathcal{L}_H(A)}{Y_0(b_0\{K'[\hat{A}] - E'[\hat{A}]\} - 2\bar{\bar{T}}''_0)}. \tag{94}$$

Again, assuming a travelling-wave form, we get

$$\frac{1}{2}V\{K'[\hat{A}] - E'[\hat{A}]\} - V\frac{\bar{\bar{T}}''_0}{b_0} - V_0\hat{Y}_s + 2b_0U'_0 r_0 A_0\hat{A}\hat{Y}_s - \chi_l Y_0\hat{A} = \chi_d Y_0\partial_X^2\hat{A}\,, \tag{95}$$

$$\text{where here}\quad \hat{Y}_s = [\hat{A}]^{\frac{1}{2}}\,,\quad \text{while}\quad \chi_d = -\frac{1}{4}[\alpha_{d,1}]^+_-/\bar{b}_0\,,\ \ \chi_l = -\frac{1}{4}[\alpha_{l,1}]^+_-/\bar{b}_0\,,$$

$$\text{and } V_0 = \frac{1}{\bar{b}_0}\left[\frac{1}{2}(U'''_0 - \bar{T}'''_0)y_\delta^2 + U'_2 - \bar{T}'_2 - Q_5\right].$$

This can readily be integrated once more to yield

$$(\partial_X\hat{A})^2 = \frac{\hat{A}}{[\alpha_{d,1}]^+_-}\left[\frac{16}{5}b_0^2 r_0 Y_0\hat{Y}_s\hat{A} - [\alpha_{l,1}]^+_-\hat{A} + \frac{16}{3}\frac{\bar{b}_0}{Y_0}V_0\hat{Y}_s + 8\frac{\bar{\bar{T}}''_0}{Y_0}\bar{V}\right.$$
$$\left. + \frac{8}{3}\frac{\bar{b}_0 V}{Y_0\hat{A}}\Big((1+\hat{A})E'[\hat{A}] - 2\hat{A}K'[\hat{A}] - 1\Big)\right], \tag{96}$$

where we recall that $Y_0 = (-2A_0)^{1/2}$. The fifth term inside the large square bracket is $O(\hat{A}\ln\hat{A})$ when $\hat{A}\to 0$, which is consistent when the nonlinear wave has a finite X-period. But for a solitary wave, where $\partial_X\hat{A}\sim O(\hat{A})$ as $\hat{A}\to 0$ is needed for exponential decay in the tail, we must set the nonlinear wave speed $V = 0$. The same argument leads to $V_0 = 0$, which thus determines the constant vorticity Q_5

$$Q_5 = \frac{1}{2}(U'''_0 - \bar{T}'''_0)y_\delta^2 + U'_2 - \bar{T}'_2. \tag{97}$$

The requirement that $|A|$ has a maximum at $\zeta = 0$ leads to

$$[\alpha_{l,1}]^+_- = \frac{16}{5}b_0^2 r_0 Y_0. \tag{98}$$

Next, we rewrite (96) with the variable $\hat{Y}_s$ to get

$$\int_1^{\hat{Y}_s}\frac{dr}{\sqrt{s_0(r-1)}\,r} = \pm\frac{\zeta}{\zeta_0}, \tag{99}$$

$$\text{where}\quad \zeta_0 = 2\sqrt{\frac{[\alpha_{d,1}]^+_-}{|[\alpha_{l,1}]^+_-|}}\quad \text{and}\quad s_0 = \text{sign}[r_0]\,.$$

A solitary-wave solution requires that $s_0 = -1$ ($r_0 \leq 0$), and then we get that

$$\hat{A} = \text{sech}^4\Big[\frac{1}{2}\frac{\zeta}{\zeta_0}\Big]. \tag{100}$$

Note that this is a more localized profile than the KdV-solitary-wave profile sech^2, and it has a wavelength inversely proportional to $|A_0|^{1/4}$.

As for the D-wave, Eq. (96) can rewritten with the variable $\hat{Y}_s$, with the time derivative restored, but with $V_0 = 0$,

$$\hat{Y}_{s,T}\Big(2\{E^{'}[\hat{A}] - 1\} - \hat{Y}_s^2 K^{'}[\hat{A}]\Big) + S_{0,E}\hat{Y}_s^4\hat{Y}_{s,XXX} + R_{0,E}\hat{Y}_s^5\hat{Y}_{s,X} + V_{0,E}\hat{Y}_s^4\hat{Y}_{s,X} = 0, \tag{101}$$

with

$$S_{0,E} = \frac{[\alpha_{d,1}]_-^+ U_0^{'}}{b_0 - \bar{\bar{T}}_0^{''}}\sqrt{2|A_0|} = 4S_0, \quad R_{0,E} = \frac{6}{5}R_0\,A_0,\; V_{0,E} = \frac{1}{4}\frac{[\alpha_{l,1}]_-^+ U_0^{'}}{b_0 - \bar{\bar{T}}_0^{''}}\sqrt{2|A_0|} = -\frac{2}{5}R_0 A_0\,.$$

Here, R_0 and S_0 here refer to the values of the D-wave coefficients (81) and (82). Again, this equation cannot be derived directly from (65), and is a "toy"-model which could be used to study unsteady effects.

The travelling-wave assumption leads to a stationary KdV equation whose solution is (100). The condition $s_0 = -1$ requires that the constraint (91) be satisfied. It is a less restrictive condition than the equivalent Rossby wave inequality ($8/9\,\beta < U_0^{''} < \beta$). We examined this condition for various velocity and buoyancy profiles for which $T_0'(y_c) = 0$:

(1) Consider a mixing-layer buoyancy profile of the form

$$T_0(y) = \frac{1}{2}\{(1+\beta_T) + (1-\beta_T)\tanh[(y-y_c)^3 + \gamma(y-y_c)^2]\}, \tag{102}$$

with a buoyancy scaled by its value at $y \to +\infty$, β_T being the buoyancy ratio between both streams. With $\beta_T < 1$, the buoyancy gradient $T_0^{'}(y)$ is positive for y sufficiently far from y_c. The related velocity profile is then

$$U_0(y) = \frac{1}{2}[(1+\beta_U) + (1-\beta_U)\tanh(y-y_0)].$$

The condition (91) is then equivalent to

$$(1-\beta_U)^2\tanh^2(y_c-y_0)\text{sech}^4(y_c-y_0)$$
$$+\frac{32}{9}\gamma^2\frac{(1-\beta_T)^2}{(1-\beta_U)^2}\cosh^4(y_c-y_0) + (1-\beta_T)[\frac{40}{9}\gamma\tanh(y_c-y_0)+1] < 0,$$

which implies that necessarily

$$\gamma\tanh(y_c-y_0) < -\frac{9}{40}\,.$$

If y_c is above the inflexion point y_0 of the velocity field, then $\gamma < 0$ and y_c is a local buoyancy minimum, and if y_c is below y_0, vice versa. Not all buoyancy profiles are possible; the existence of a real parameter γ satisfying the preceding relation leads to the condition

$$7(1-\beta_U)^2\tanh^2(y_c-y_0) > 18(1-\beta_T)\cosh^4(y_c-y_0), \tag{103}$$

which restricts the (β_U, β_T) domain to $(1-\beta_U)^2/(1-\beta_T) \gtrsim 17.357$, characterizing small buoyancy differential streams. Figure 3 displays the evolution of the buoyancy with the height y. When the above inequality is true, a γ-range exists for which the profile (102) allows for the propagation of E-waves and D-waves of the second kind. Here, the figure gives the profiles $T_0(y)$ for $\beta_U = -1$, $\beta_T = 0.9$, $y_c - y_0 = -1.1$ and for several values of γ, the range that allows for $s_0 = -1$ is located around $\gamma = 1.94$ and 3.23. An adverse shear is favorable to their propagation; indeed, as β_U increases, β_T approaches 1. For example, when $\beta_U = -1$, (103) is satisfied as soon as $\beta_T \gtrsim 0.770$ and when $\beta_U = 0$, only for $\beta_T \gtrsim 0.943$.

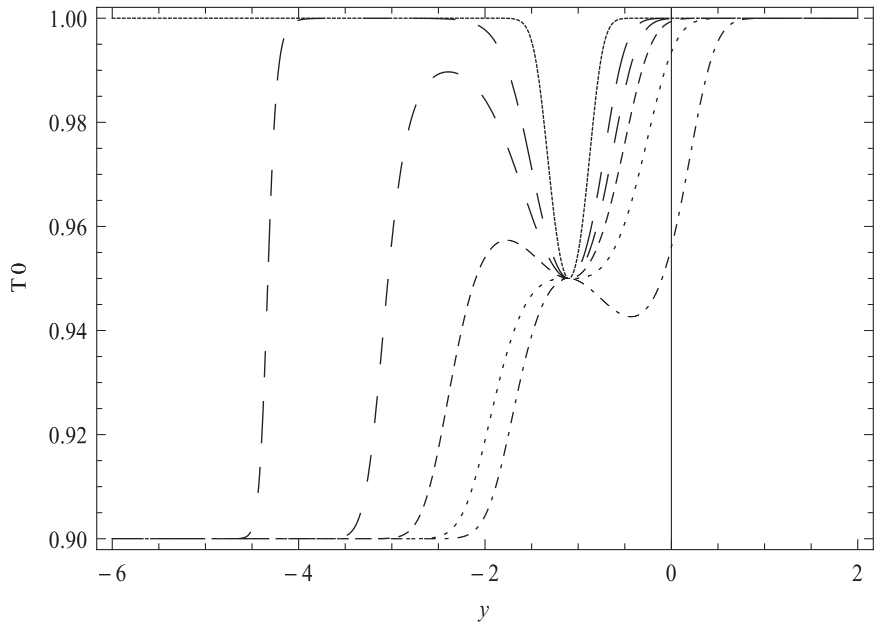


FIG. 3. Profile $T_0(y)$, $y_c = -1.1$, $y_0 = 0$, $\beta_U = -1$, $\beta_T = 0.9$. (−· −) $\gamma = -1$, (· ·) $\gamma = 0$, (−−) $\gamma = 1$, (— —) $\gamma = 1.94$, (— — —) $\gamma = 3.23$, (−−) $\gamma = 10$.

(2) Consider the finite-depth waveguide buoyancy profile of the form

$$T_0(y) = [y - \tanh(y - y_c)]/[y_2 - \tanh(y_2 - y_c)],\ \ \beta_T = [y_1 - \tanh(y_1 - y_c)]/[y_2 - \tanh(y_2 - y_c)],$$

with the same velocity profile as above. This second profile only satisfies the case $s_0 = 1$ for $y_2 - \tanh(y_2 - y_c) > 0$ ($T_0^{'}(y) \geq 0$, for all $y_1 \leq y_0 \leq y_2$).

(3) However, the profiles that satisfy locally around y_c

$$T_0(y) = \frac{1}{2}\bar{T}_0^{''}\eta^2, \quad U_0(y) = c_0 + U_0^{'}\eta + \frac{1}{2}U_0^{''}\eta^2$$

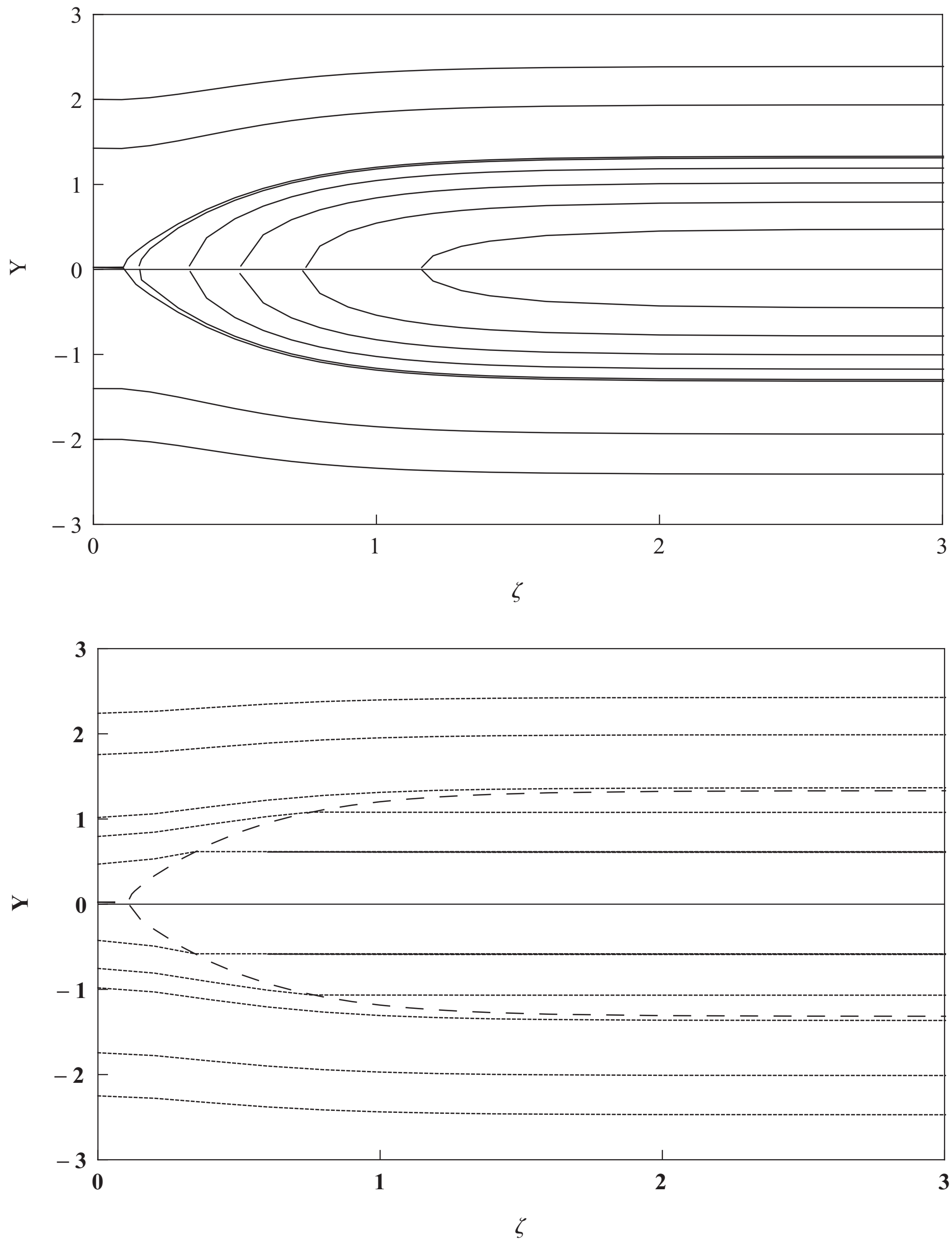
satisfy (91) provided that

$$9U_0^{''2} + 8\bar{T}_0^{''2} < 20\bar{T}_0^{''}U_0^{''},$$

which implies that $U_0^{''}\bar{T}_0^{''} > 0$. This corresponds to the case where buoyancy opposes to dynamics in the coefficient $b_0 = (U_0^{''} - \bar{T}_0^{''})/U_0^{'}$. More exactly, the ratio $U_0^{''}/\bar{T}_0^{''}$ is bounded by

$$\frac{1}{9}(10 - 2\sqrt{7}) < \frac{U_0^{''}}{\bar{T}_0^{''}} < \frac{1}{9}(10 + 2\sqrt{7}).$$

Figures 4–7 display the streamlines, isovorticity, isobar, and iso-kinetic energy contours for the *D*-wave and *E*-wave critical layers computed with expansions to the second-order. To the two first orders of the expansion, an iso-buoyancy contour is identical to a streamline, highlighting the lock of buoyancy onto the streamlines. The local Richardson number is always negative in half of the critical layer. Indeed in the core, at the leading order, the buoyancy varies as $\Theta = T_0^{''}S + \theta_0$. In these examples, the buoyancy decreases in the lower half of the critical layer. The *D*-wave separatrix possesses a very flat slope at $\zeta = 0$ as observed in a finite-period wave critical layer.[17,23] The D-wave speed V is chosen zero. The streamlines vary with the nonlinear wave speed V only from the order ϵ^2. The buoyancy varies with V from the order $\epsilon^{3/2}$. If $V = 1$, the buoyancy value on each contour in Fig. 4(a) slightly increases of a few percents with respect to V=0 and becomes: 6.19, 4.22, 2.25, 2.16, 1.87, 1.49, 1.10, 0.728, 2.25, 4.20, and 6.17. The *D*-wave cat's eye possesses a positive pressure anomaly around the saddle point whereas the *E*-wave cat's eye possesses a



(a)

(b)

FIG. 4. (a) Streamlines and (b) iso-vorticity lines for a *D*-wave nonlinear critical layer whose numerical values of the parameters are: $U_0^{'} = 2, U_0^{''} = T_0^{''} = 1,\ T_0^{'''} = 0.5, U_{u,1}^{'} = 0.2, U_1^{'+} = -0.5, y_\delta = 0.8,\ \beta_{d,1}^{+} = 0.3,\ c_{l,1,0}^{+} = 0.4, T_1^{''+} = 0.1,\ T_2^{'+} = T_3^{+} = 0,\ \ \theta_0 = 0.2,\ r_1 = -1,\ \zeta_0 = 1/2,\ V = 0$ (solitary wave-like solution parameters, see Sec. V), and $\epsilon = 0.002$. The streamlines are iso-buoyancy contours whose buoyancy on each contour is, from the bottom to the top: $1000 \times \Theta = 6.08, 4.12, 2.18, 2.11, 1.82, 1.47, 1.08, 0.697, 2.18, 4.17$, and 6.15. The cat's eye is thus cooled with respect to the zone outside the separatrices. The vorticity value on each contour is from the bottom to the top: 1.89, 1.91, 1.935, 1.943, 1.95, 1.96, 1.98, 1.986, 1.994, 2.02, and 2.04.

negative pressure anomaly around the centre point. The iso-vorticity lines within the cat's eyes satisfy $\tilde{Y} =$ constant; so they are quasi-horizontal for both waves. We can do the same remark for the iso-kinetic energy lines; indeed, they are at the leading-order given by $\tilde{Y} =$ constant outside and inside the cat's eye.

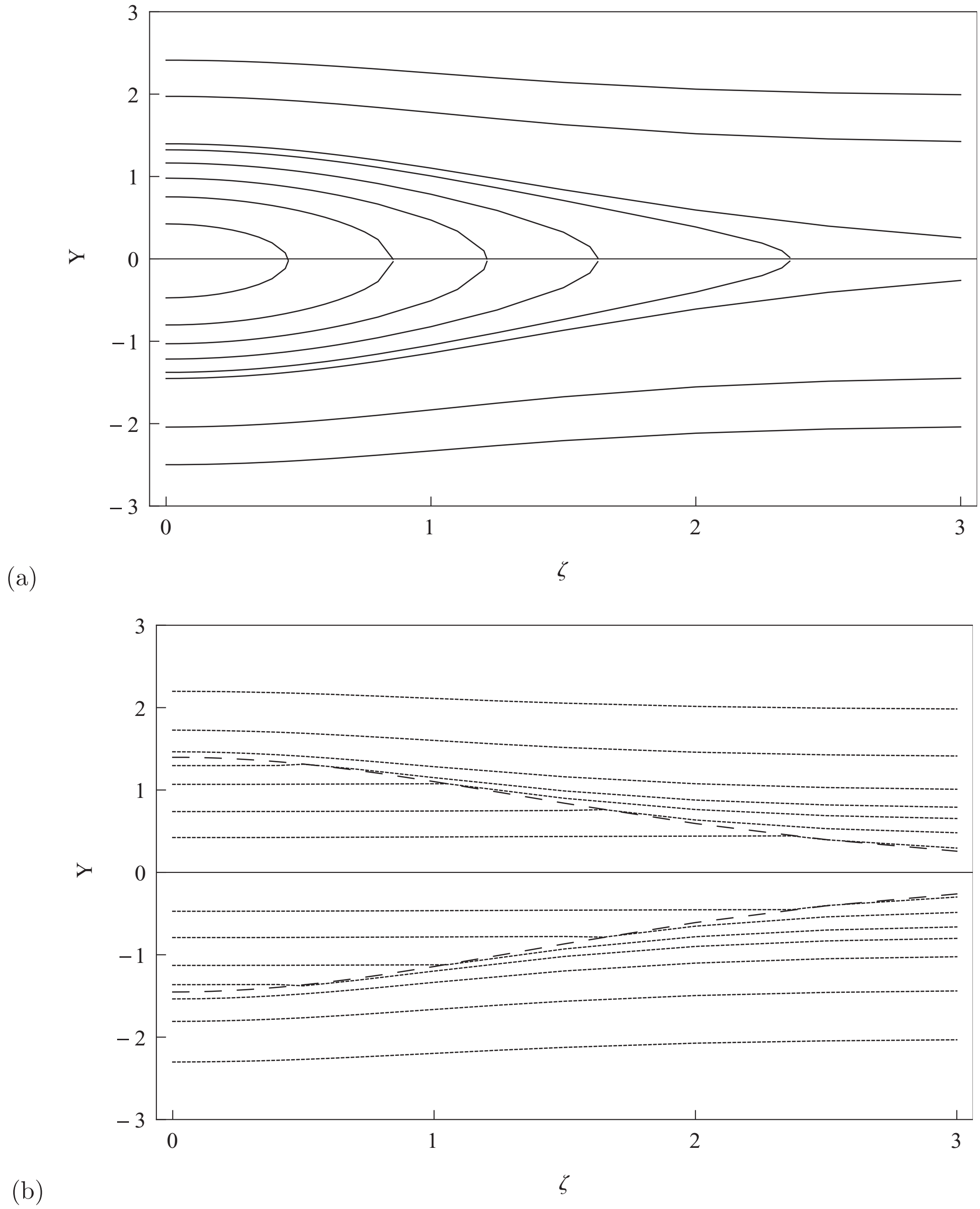


FIG. 5. (a) Streamlines and (b) iso-vorticity lines for an E-wave nonlinear critical layer whose numerical values of the parameters are: $U_0' = 2$, $U_0'' = T_0'' = 1$, $T_0''' = 0.5$, $U_{u,1}' = U_1' = 0.2$, $y_\delta = 0$, $\beta_{d,1} = 0.3$, $c_{l,1,0} = 0.4$, $T_1'' = 0.1$, $T_2' = T_3 = 0$, $\theta_0 = 0.2$, $\zeta_0 = 2$ (solitary wave-like solution parameter, see Sec. V), and $\epsilon = 0.002$. The buoyancy on each contour, from the bottom to the top is: $1000 \times \Theta = 4.36, 2.373, 0.399, 0.197, -0.203, -0.603, -1.00, -1.40, 0.399, 2.372$, and 4.33. The cat's eye is also thus cooled with respect to the zone outside the separatrices. The vorticity value on each contour is from the bottom to the top: 1.87, 1.90, 1.92, 1.93, 1.936, 1.944, 1.95, 1.96, 1.97, 1.98, 1.99, 2.00, 2.01, 2.03, and 2.05.

C. Constant-shear flow

We conclude this section with the case of a constant shear flow, $U_0(y) = c_0 + U_0'\eta$, and a quadratic buoyancy profile, $T_0(y) = T_0''\eta^2/2$. As for the unstratified flow, the second-type D-wave and the E-wave are not possible here since the condition $s_0 = -1$ is not satisfied. In this special case, all coefficients can be nevertheless found explicitly. We thus obtain for the $s_0 = 1$-D-wave:

$$b_0 = -T_0''/U_0'^2$$

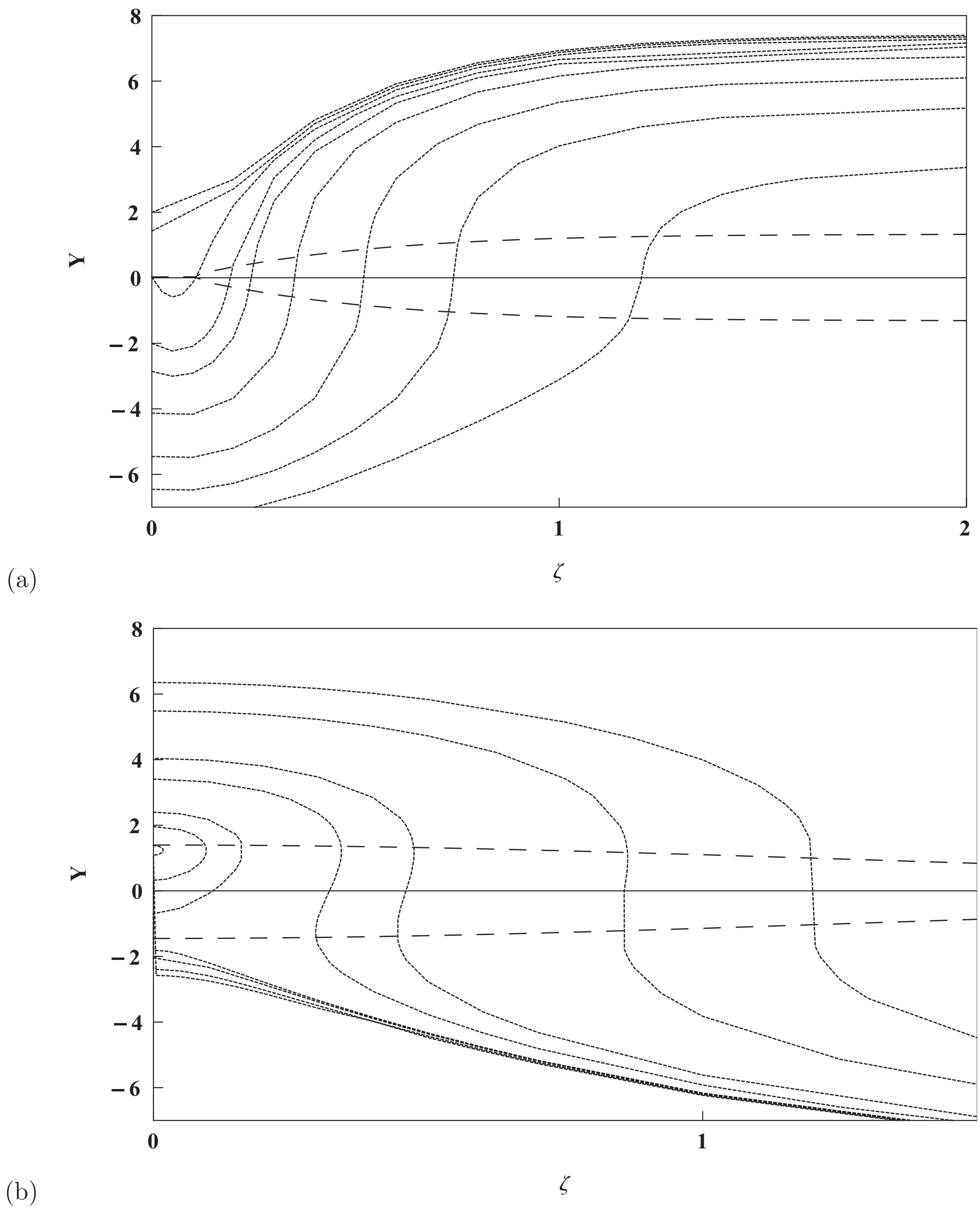


FIG. 6. Isobar lines $P^{'} = P + gL/\mathcal{U}^2\eta =$ constant for (a) D-wave and (b) E-wave with the same numerical values as for Figs. 4 and 5. The pressures are from bottom to top: $1000 * P^{'} =$, D-wave: 0.742, 2.44, 3.94, 5.32, 6.11, 6.41, 6.75, 6.94, and 7.08; E-wave: −7.89, −7.85, −7.81, −7.79, −7.44, −7.06, −5.50, and −3.99. The three closed contours have the following pressures: −7.79, −7.85, and −7.89.

and

$$V_0 = \frac{1}{2}(\frac{5}{9} - a)\bar{T}_0^{''}\, A_0 - \frac{1}{8}\frac{[\alpha_{l,1}]_-^+}{T_0^{''}}U_0^{'3}Y_\infty - \frac{1}{2}\Big(\underline{T}_2^{'} + \frac{1}{4}\underline{T}_1^{''}y_\delta\Big)\,\frac{U_0^{'}}{T_0^{''}}\,, \quad R_0 = -\frac{16}{9}\bar{T}_0^{''},$$

$$S_0 = -\frac{1}{8}\frac{[\alpha_{d,1}]_-^+}{T_0^{''}}U_0^{'3},$$

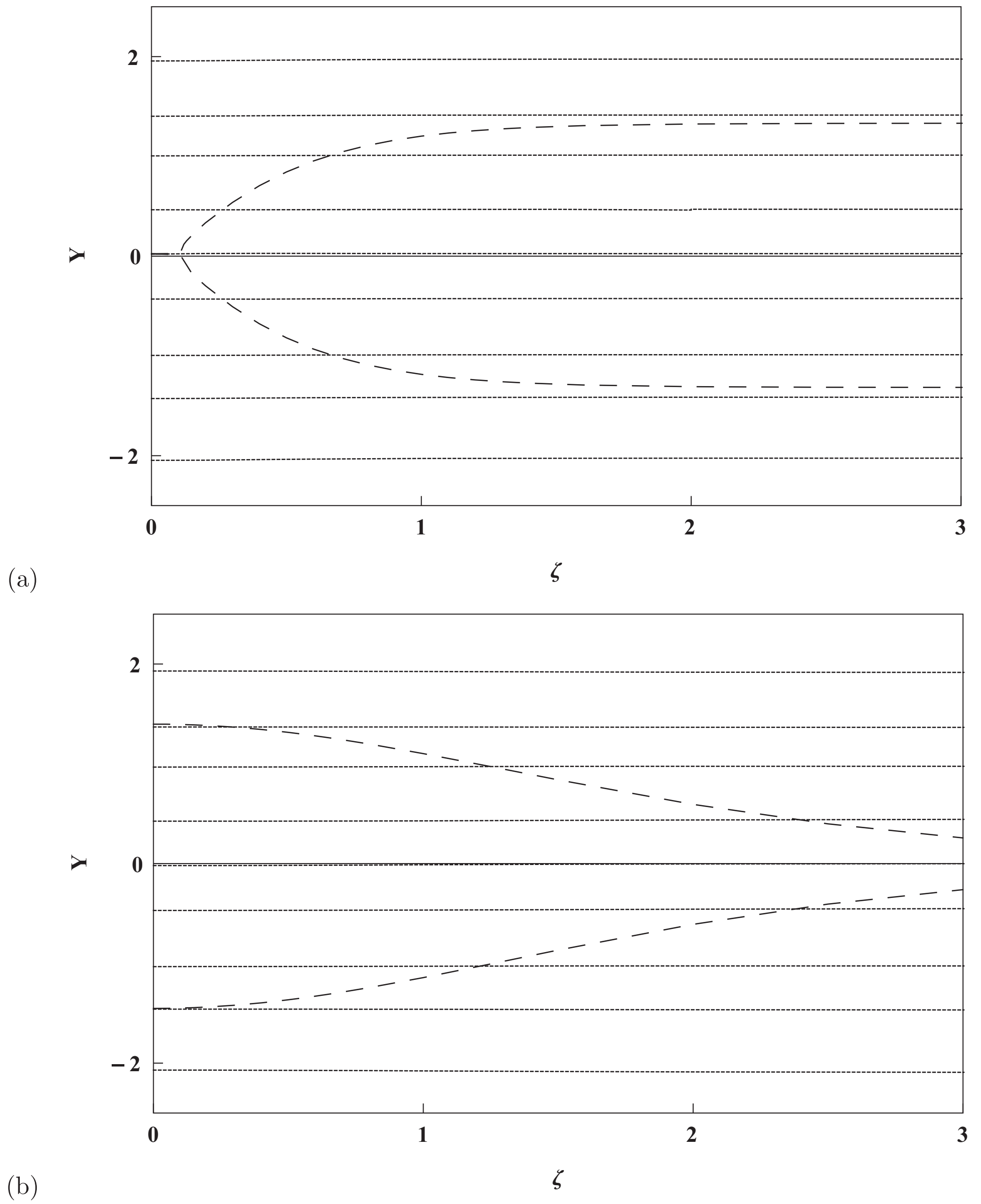


FIG. 7. Iso-kinetic energy contours in the plane (ξ, Y) for (a) a D-wave and(b) an E-wave with the same numerical values as for Figs. 4 and 5, except that $\epsilon = 0.01$. The energy ($\times 100$) are from bottom to top: D-wave: 7.35, 3.69, 1.84, 0.369, 0, 0.369, 1.84, 3.69, and 7.41; E-wave: 7.27, 3.69, 1.84, 0.369, 0, 0.369, 1.84, 3.69, and 7.48. The patterns are not symmetric with respect to the ζ-axis. The motionless particles are located on the axis $\tilde{Y} = 0$.

$$\bar{\mathcal{D}}_0[A] = \frac{1}{4}\Big(\frac{5}{9} + a\Big)\bar{\bar{T}}_0'' A Y_s Y_\infty - \frac{1}{8}\frac{[T_1'']_-^+}{T_0''} A Y_s + \frac{1}{8}([\alpha_{l,1}]_-^+ A + [\alpha_{d,1}]_-^+ \partial_X^2 A)\frac{Y_s}{\bar{\bar{T}}_0''},$$

$$\text{and } V = V_0 - \frac{1}{4}\Big(a + \frac{137}{45} + \frac{1}{2}\frac{[\alpha_{l,1}]_-^+}{T_0''^2} U_0'^4 Y_\infty - \frac{1}{2}\frac{[T_1'']_-^+}{T_0''^2 Y_\infty} U_0'^2\Big)\bar{T}_0'' A_0,$$

$$r_1 \le 0 \ \text{ gives } \ [\alpha_{l,1}]_-^+ \le 2\Big(\frac{23}{45} - a\Big)(\bar{\bar{T}}_0'')^2 Y_\infty + [\bar{\bar{T}}_1'']_-^+ .$$

The singular mode can be then analytically expressed in terms of the dimensionless coordinate $\varrho = 2\,|\bar{\bar{T}}_0^{''}\,\eta|^{1/2}$ and the first-kind Bessel functions $J_1(\,\cdot\,)$ and $Y_1(\,\cdot\,)$:

$$\phi^+(y) = \varrho\,[\mathcal{A}^+\,J_1(\varrho) + \mathcal{B}^+ Y_1(\varrho)]\,, \qquad \text{if} \quad T_0^{''} > 0\,,$$
$$\phi^-(y) = \varrho\,[\mathcal{A}^-\,J_1(i\varrho) + \mathcal{B}^- Y_1(i\varrho)]\,, \tag{104}$$

$$\phi^+(y) = \varrho\,[\mathcal{A}^+\,J_1(i\varrho) + \mathcal{B}^+ Y_1(i\varrho)]\,, \qquad \text{if} \quad T_0^{''} < 0\,,$$
$$\phi^-(y) = \varrho\,[\mathcal{A}^-\,J_1(\varrho) + \mathcal{B}^- Y_1(\varrho)]\,. \tag{105}$$

The dispersion relation (between $\varrho(y_1)$ and $\varrho(y_2)$) can then be found straightforwardly[19] since the Rossby modes and internal modes are in this simplified field, strictly identical. The outer-flow streamfunction can be also expressed analytically.[19]

VI. CRITICAL-LAYER INDUCED WAVE MEAN FLOW INTERACTION

In this section, we will determine the growth rates of the mean streamwise and cross-stream momenta, the mean buoyancy, the mean kinetic energy, and the mean potential energy at the critical level. This is achieved by deriving the x-averaged momentum and energy equations in the outer and inner flows, and then equating the jumps of the mean vertical flux of these different quantities in both flows. We will thus obtain the time scales of the critical-layer induced mean flow and mean buoyancy. The outer-flow approach is valid for $\lambda = O(1)$ but the calculation using the inner flow assumes $\lambda \ll 1$.

A. Mean-flow acceleration

We recall that, to shorten the notation, the mean flow is defined at its critical level, so that for instance, $T_1 = T_1(y_c)$ and $T_1^{'} = T_1^{'}(y_c)$. The outer-flow averaged x-momentum equation is, in the reference frame moving with speed c,

$$\partial_t \langle u \rangle = -\partial_y \langle uw \rangle + \frac{1}{R}\partial_y^2 U + F_v - \langle p_x \rangle - 2\langle u u_x \rangle, \tag{106}$$

$$\text{where} \quad \frac{1}{R}\partial_y^2 U + F_v = \epsilon^{\frac{1}{2}}/R\,[U_1^{''}(y) + \,\epsilon^{\frac{1}{2}} U_2^{''}(y) + \cdots],$$

and in this section, $\langle\,\cdot\,\rangle$ denotes the mean over the streamwise coordinate x for constant y. Usually, in wave mean flow interaction theories, the two last terms are omitted because they involve derivatives with respect to x. However here, the flow contains secular contributions proportional to $\lambda\, X$, generated by weak viscous and thermal diffusions. Thus, their averaged related X-derivatives $\langle \partial_X \rangle$ are small but not zero. In the critical-layer theory, all leading-order averages are proportional to λ. The two last terms in (106) cannot be thus neglected and are in general of the same order as the cross-stream gradient of the averaged vertical flux of the horizontal momentum. The calculations are performed using the deformed critical layer in the strained coordinate $\tilde{S}$ system, but the tilde is omitted further for simplicity. For the same reason, most details are omitted, and the results are then as follows.

Taking the integral of (106) over the critical-layer width δ_{cl} between y_{B1} and y_{B2} where

$$y_{B1} = y_c - y_\delta \epsilon^{\frac{1}{2}} - \delta_{cl}/2 + \epsilon\langle Y_2^- \rangle + \cdots \quad \text{and} \quad y_{B2} = y_c - y_\delta \epsilon^{\frac{1}{2}} + \delta_{cl}/2 + \epsilon\langle Y_2^+ \rangle + \cdots,$$

corresponding to $\tilde{Y}_{B_{1,2}} = \pm\delta_{cl}/(2\epsilon^{1/2})$. Noting that it can be shown using (39) that $\langle Y_1 \rangle = 0$, $\langle Y_2^{\pm} \rangle = b_0 S_c - \bar{\bar{T}}_0^{''}\,\delta_{cl}^2/(8\epsilon)$, and assuming that $\epsilon^{1/2} \ll \delta_{cl} \ll 1$, we get that

$$\int_{y_{B1}}^{y_{B2}} \partial_t \langle u \rangle\, dy = -[\langle uw \rangle]_{y_{B1}}^{y_{B2}} + \frac{\epsilon^{\frac{1}{2}}}{R}[U_1^{'}(y) + \epsilon^{\frac{1}{2}} U_2^{'}(y) + \ldots]_{y_{B1}}^{y_{B2}}$$
$$- \lambda\epsilon^{\frac{1}{4}} \int_{y_{B1}}^{y_{B2}} \langle p_{v,X} \rangle\, dy - 2\lambda\epsilon^{\frac{1}{4}} \int_{y_{B1}}^{y_{B2}} \langle u u_{v,X} \rangle\, dy.$$

(107)

The jump $[q]_{y_{B1}}^{y_{B2}}$ of the quantity q is defined by

$$[q]_{y_{B1}}^{y_{B2}} = q(y_{B2}) - q(y_{B1}) - [q]_{y_a^-}^{y_a^+},$$

while the integral of the same inner-flow quantity Q

$$\int_{Y_{B1}}^{Y_{B2}} Q_Y dY = [Q]_{Y_{B1}}^{Y_{B2}} - [Q]_{S1^-}^{S1^+} - [Q]_{S2^-}^{S2^+},$$

where $[q]_{y_a^-}^{y_a^+}$ is the q-jump through the critical-layer axis of symmetry located in $y_a = y_c - y_\delta \epsilon^{1/2} + o(\epsilon)$. The two last terms at the r.h.s are the jumps of Q through both lower and upper separatrices S_1 and S_2. Each leading-order contribution in (107) of order $\epsilon^{9/4}$ cancels except the x-momentum jump, and the result for a D-wave is

$$\int_{y_{B1}}^{y_{B2}} \partial_t \langle u \rangle \, dy = \lambda\, \beta_{v,1} U_0^{'} \delta_{cl} \epsilon^{\frac{7}{4}} + O\Big(\lambda \epsilon^{\frac{11}{4}}\Big), \tag{108}$$

$$\text{where} \quad -[\langle uw \rangle]_{y_{B1}}^{y_{B2}} = \lambda \epsilon^{\frac{7}{4}} U_0^{'} \Big[\beta_{v,1}(\eta + y_\delta \epsilon^{\frac{1}{2}})\Big]_{y_{B1}}^{y_{B2}} = \lambda \beta_{v,1} U_0^{'} \delta_{cl} \epsilon^{\frac{7}{4}}.$$

The matching of $\Psi_{v,X}^{(2)} = \beta_{v,1}$ on the separatrix with $\Psi_{v,X}^{(2,\odot)} = \beta_{v,1}^{\odot}$ yields: $\beta_{v,1}^{\odot} = \beta_{v,1}^{+} = \beta_{v,1}^{-}$. Introducing $\int_{Y_{B1}}^{Y_{B2}} \langle uw \rangle_Y dY$ derived from the averaged inner-flow x-momentum equation into (107), we obtain a new expression for the growth rate

$$\begin{aligned}\int_{y_{B1}}^{y_{B2}} \partial_t \langle u \rangle \, dy = {} & [\langle uw \rangle]_{y_a^-}^{y_a^+} - [\langle uw \rangle]_{S1^-}^{S1^+} - [\langle uw \rangle]_{S2^-}^{S2^+} + \frac{\epsilon^{\frac{1}{2}}}{R}[U_1^{'}(y) + \epsilon^{\frac{1}{2}} U_2^{'}(y) + \ldots]_{y_{B1}}^{y_{B2}} \\ & - \lambda \epsilon^{\frac{3}{4}} [\langle \Psi_{YY} \rangle]_{Y_{B1}}^{Y_{B2}} - \epsilon^{\frac{1}{2}} \int_{Y_{B1}}^{Y_{B2}} F_v \, dY - \lambda \epsilon^{\frac{1}{4}} \int_{y_{B1}}^{y_{B2}} \langle p_{v,X} \rangle \, dy \\ & + \lambda \epsilon^{\frac{3}{4}} \int_{Y_{B1}}^{Y_{B2}} \langle p_{v,X} \rangle \, dY - 2\lambda \epsilon^{\frac{1}{4}} \int_{y_{B1}}^{y_{B2}} \langle u u_{v,X} \rangle \, dy \\ & + 2\lambda \epsilon^{-\frac{1}{4}} \int_{Y_{B1}}^{Y_{B2}} \langle \Psi_Y \Psi_{v,XY} \rangle \, dY. \end{aligned} \tag{109}$$

Using the continuity of $w_v = -\Psi_{v,x}$ in the viscous boundary layer along the separatrices, we find no mean x-momentum flux jumps through the separatrices, so that $[\langle uw \rangle]_{S1^-}^{S1^+} = [\langle uw \rangle]_{S2^-}^{S2^+} = 0$. The calculation of each contribution in (109) can now be found explicitly, and the final outcome for a D-wave is

$$\int_{y_{B1}}^{y_{B2}} \partial_t \langle u \rangle \, dy = -\lambda\, [U_1^{'}]_-^+ \epsilon^{\frac{9}{4}} + O\Big(\lambda \epsilon^{\frac{11}{4}}\Big) = 2\lambda\, b_0 U_0^{'} Y_\infty \epsilon^{\frac{9}{4}} + O\Big(\lambda \epsilon^{\frac{11}{4}}\Big). \tag{110}$$

As $\langle u \rangle$ is given by $\epsilon^{1/2}(U_1 - c_1)$ at the leading order, the time scale τ_1 is

$$\tau_1 = \lambda\, \epsilon^{\frac{5}{4}} t = \lambda T\,, \quad \partial_{\tau_1}[U_1(y_c, \tau_1) - c_1(\tau_1)] = \kappa_U = \frac{b_0}{\sigma} U_0^{'}\,, \tag{111}$$

where $1 \ll \sigma = \delta_{cl}/[2(2|A_0|\epsilon)^{1/2}] \ll \epsilon^{-1/2}$, σ is a geometrical constant whose value remains undetermined by the theory of the nonlinear critical layer. This slow time is the time scale given in Refs. 18–20. However, the outer-flow diffusion, the horizontal gradients of pressure and kinetic energy were omitted in those papers, but are taken into account here. For $c_0 > 0$, the mean flow is accelerated if $U_0^{''} > \bar{T}_0^{''}$; this is essentially the same criterion as for a Rossby-wave nonlinear critical layer, the ratio of the second derivative of the buoyancy over the velocity shear at the critical level playing the same stabilizing rôle as the rotation gradient β. The time τ_1 is faster than the viscous time scale t/R; introducing body terms related to vorticity and buoyancy prevents the mean flow from damping over the latter time. In the slow process of the critical layer formation, as the stratification level is weakening around the critical layer, the modal equation simplifies into $\phi''(\eta) + (\bar{T}_0'' - U_0'')/(U_0'\eta)\phi(\eta) = 0$. By analogy with the stability of a shear flow in the β plane, the analysis of the flow closed by the separatrices yields that this remains linearly stable since the constant $\bar{T}_0'' - U_0''$ does not have an inflexion point. The mean flow within the diffusion boundary layers evolves with the even smaller time scale $\tau_{DB} = \epsilon\,\tau_1 \ll \tau_1$ in the quasi-steady assumption where $\epsilon \ll \tau_{DB} \ll 1$ as shown in Ref. 19. Moreover, the leading-order buoyancy gradient $T_1^{'}(y_c)$ also evolves with time τ_1, see (36). We can now define the critical-layer thickness δ_{cl} so that the growth rates defined by (108) and (110) are equal, and so

$$\sigma = \frac{b_0}{\beta_{v,1}}\,. \tag{112}$$

We here see all the interest of having a $O(\epsilon^{3/2})$ viscous outer flow. The coefficient $\beta_{v,1}$ related to the singular Frobenius solution is a free parameter of the problem which cannot be taken equal to zero, whose sign is the same as b_0's but whose magnitude must be far less than $|b_0|$ so that the viscous critical layer may not interfere with the nonlinear one. We easily deduce that the velocity growth rate $\kappa_U = \beta_{v,1} U_0^{'}$; the acceleration only depends on the shear and the amplitude of the leading viscous flow.

For the E wave, the cat's eye quantities, for instance $\langle P^{\odot}_{v,X}\rangle$, $\langle \Psi^{\odot}_{i,YY}\rangle$ and $\langle \Psi^{\odot}_{v,X}\rangle$ are very complex but are not needed, since on average the cat's eye thickness is zero, that is $\langle Y_s\rangle = Y_\infty = 0$. Moreover, the different mean-flow jumps are zero as well, that is $[U_2^{'}]_-^+ = [U_1^{''}]_-^+ = 0$ ... (with $U_1(y) = U_{u,\,1}(y)$), so that for an E-wave

$$\int_{y_{B1}}^{y_{B2}} \partial_t \langle u\rangle \, dy \equiv \partial_t U_2\, \epsilon\, \delta_{cl} = 0\,. \tag{113}$$

Thus for an E-wave, the mean flow is not accelerated. This result seems to be general for E-waves, since we also found the same result for a Rossby solitary wave.[20] Moreover, we deduce from (108) that $\beta_{v,1} = 0$.

Next, it can be shown that the mean-vertical-velocity growth rate given by

$$\partial_t\langle w\rangle = \langle\theta\rangle - \partial_y\langle p + w^2\rangle - \langle (uw)_x\rangle + \frac{1}{R}\langle \partial_y^2 w\rangle,$$

is $O(\lambda^2)$, the two last terms being $O(\lambda^2)$, that is

$$\int_{y_{B1}}^{y_{B2}} \partial_t\langle w\rangle\, dy = O\left(\lambda^2\epsilon^4\right) \quad\text{and}\quad \int_{y_{B1}}^{y_{B2}} \langle\theta_i\rangle\, dy = [\langle (p + w^2)_i\rangle]_{y_{B1}}^{y_{B2}}. \tag{114}$$

Since $\langle w(y)\rangle \simeq -\lambda\beta_{v,1}\phi_b(y)\epsilon^{7/4} + O(\lambda\epsilon^{9/4})$, if $\beta_{v,1}$ is chosen constant, σ will vary as slowly as b_0, the leading viscous term then evolves with τ_{DB}, but the time scale is given by the second-order term which evolves faster with τ_1, so $\tau_w = \tau_1$. The vertical mean motion is then negligible with respect to the horizontal mean motion since $\langle u(y_a)\rangle/\langle w(y_a)\rangle = O(\epsilon^{1/2} R)$.

Finally, the x-averaged mean vorticity outer equation

$$\partial_t\langle\omega\rangle = -\langle\theta_x\rangle - \partial_y\langle w\omega\rangle - \langle (u\omega)_x\rangle + \frac{1}{R}\langle\partial_y^2\omega\rangle$$

yields the mean-vorticity growth rate

$$\int_{y_{B1}}^{y_{B2}} \partial_t\langle\omega\rangle\, dy = -\frac{\lambda}{Pr}[\bar{T}_1^{''}]_-^+ \ln|\frac{1}{2}\delta^*_{cl}|\,\epsilon^{\frac{9}{4}}. \tag{115}$$

Though the mean-vorticity evolution is favored by the buoyancy/dynamics coupling through the mean-buoyancy distortion $[\bar{T}_1^{''}]_-^+$, this growth rate is slowly evolving at the slow time scale $\lambda/Pr\,\epsilon^{5/4}|\ln\epsilon|t$, which is far slower than the critical-layer homogenization time[43] $(\lambda/Pr\,\epsilon^{3/4}\,t)$.

B. Mean-flow heating

The outer-flow x-averaged buoyancy equation is

$$\partial_t \langle \theta \rangle = -\partial_y \langle \theta w \rangle - \langle (\theta u)_x \rangle + \frac{1}{Pr\, R} \partial_y^2 T + F_b. \tag{116}$$

Integrating this equation over the critical layer between y_{B1} and y_{B2}, we get that

$$\int_{y_{B1}}^{y_{B2}} \partial_t \langle \theta \rangle \, dy = -[\langle w\theta \rangle]_{y_{B1}}^{y_{B2}} - \lambda \epsilon^{\frac{1}{4}} \int_{y_{B1}}^{y_{B2}} \langle (u\theta)_{v,X} \rangle \, dy + \frac{\epsilon^{\frac{1}{2}}}{PrR} [T_1^{'}(y) + \epsilon^{\frac{1}{2}} T_2^{'}(y) + \ldots]_{y_{B1}}^{y_{B2}}. \tag{117}$$

The critical-layer induced heating given by the outer flow is therefore

$$\int_{y_{B1}}^{y_{B2}} \partial_t \langle \theta \rangle \, dy = -2 \frac{\lambda}{Pr} \bar{\bar{T}}_0^{''} \Big([T_1^{''}]_-^+ + \frac{1}{3} \beta_{v,1} Pr T_0^{''} \sigma Y_\infty | \Big) \sigma^2 Y_\infty^2 \epsilon^{\frac{13}{4}}. \tag{118}$$

We deduce that the buoyancy time scale is $\lambda \epsilon^{9/4} t$ if $\langle \theta(y \sim y_a) \rangle \sim \epsilon^{1/2}\, T_1(y_c)$. However, this disagrees with the corresponding calculation for the $O(\epsilon^{11/4})$ potential energy (see Sec. VI D), yielding a time scale of order $\lambda \epsilon^{7/4} t$. We deduce therefore that $T_1(y_c) = 0$, and consequently $\langle \theta(y \sim y_a) \rangle \sim (\theta_0 + T_0^{''} Y^2/2)\, \epsilon$, while the integrated mean buoyancy growth rate is hence $\int_{y_{B1}}^{y_{B2}} \partial_t \langle \theta \rangle \, dy \equiv \epsilon \partial_t T_2\, \delta_{cl}$, where the contributions due to $T_1^{'}$ cancel, and then the potential-energy growth rate scales like $\langle \partial_t e_p \rangle = O(\epsilon^{15/4})$. The buoyancy time scale is then the diffusion time

$$\tau_2 = \frac{t}{PrR}, \qquad \text{with} \quad \partial_{\tau_2} T_2 = \kappa_T = -\, \bar{\bar{T}}_0^{''} \Big([T_1^{''}]_-^+ + \frac{1}{3} \beta_{v,1} Pr T_0^{''} \sigma Y_\infty \Big) \sigma Y_\infty. \tag{119}$$

This is slower than the velocity time scale, and so the $O(\epsilon)$ mean buoyancy plays a weaker part in the critical-layer dynamics, and evolves over the thermal-diffusion time scale like a passive tracer, whereas the $O(\epsilon^{1/2})$ buoyancy gradient $T_1^{'}(y_c)$ evolves with τ_1. The growth rate of $\epsilon\, T_2(y_c)$ is proportional to the wave maximum amplitude $\epsilon\, A_0$ if we assume a jump $[T_1^{''}]_-^+ \equiv Y_\infty$ as we see later on. Thus, the critical layer heats if the jump $[T_1^{''}]_-^+$ satisfies the condition

$$\frac{[T_1^{''}]_-^+}{T_0^{''}} + \frac{1}{3} Prb_0 Y_\infty < 0. \tag{120}$$

Equating the heating rate (118) with this given by the inner-flow buoyancy vertical flux

$$\int_{y_{B1}}^{y_{B2}} \partial_t \langle \theta \rangle \, dy = \Big\{ \bar{\bar{T}}_0^{''} \{ [T_1^{''}]_-^+ [\frac{1}{2}(1 - \ln 2) - 2\sigma^2] - [T_0^{'''} - (\bar{T}_0^{''})^2] Y_\infty \} Y_\infty - Pr \beta_{v,1} \Big(\frac{1}{2} [T_1^{''}]_-^+ Y_\infty + \frac{1}{3} [T_0^{'''} - (\bar{T}_0^{''})^2] Y_\infty^2 + T_0^{'''} y_\delta^2 + 2(\theta_0 b_0 \ln 2 + \underline{T}_2^{'} - \underline{T}_1^{''} y_\delta) + \frac{3}{2} \frac{[T_1^{''}]_-^+}{Y_\infty} y_\delta^2 \Big) \Big\} Y_\infty \epsilon^{\frac{13}{4}}, \tag{121}$$

gives a relationship between $\underline{T}_1^{''}$ and $\underline{T}_2^{'}$. The background fields (U_0, T_0) being given, we now need to determine the jump of the second derivative $T_1^{''}$ (which depends on $\underline{T}_1^{''}$). The jumps of the mean fields $[q]_-^+$ are given by the matching on the separatrix but the means $\underline{q}$ are determined by studying the diffusion boundary layers, similarly to our study of the Rossby-wave nonlinear critical layer.[19] We do not then know so far the values of $\underline{T}_1^{''}$ and $\underline{T}_2^{'}$. The diffusion boundary layers emerge as a slow diffusion of the critical layer. Their width then follows the law of the viscous diffusion length, that is, $\delta_{DB} \sim \epsilon^{1/2} \sqrt{\tau_1}$, see Fig. 1. We can hence assume that the critical level slowly moves within the diffusion boundary layers according to the relation $y_c = y_{c,0} + 2\epsilon^{1/2} \sqrt{\tau_1}\, \kappa(\epsilon, \tau_1)$, see Refs. 18 and 19, in the quasi-steady régime. As a result, the critical-layer flow also evolves with a third time scale, a very slow one that characterizes the evolution of y_c, that is τ_{DB}, since all Frobenius series terms: $b_0, b_{0,2}, a_{0,2}, \ldots$ are functions of the background flow and buoyancy evaluated at y_c. Finally, the actual axis of the critical layer located at $y_a = y_c - \epsilon^{1/2} y_\delta + o(\epsilon)$ varies with the time scale $\epsilon^{1/2} \tau_1 \equiv \tau_2$.

C. Mean kinetic energy

In dimensionless coordinates, the total energy density of the system: wave + mean flow, is

$$e_t = e_k + e_p = \frac{1}{2}(|\mathbf{u}|^2 + \frac{\Theta^2}{Ri}),$$

where the first and second terms are the kinetic and potential energies, respectively. The averaged kinetic-energy equation is, in the outer flow,

$$\partial_t \langle e_k \rangle = \langle \theta w \rangle - \partial_y \langle (p + e_k) w \rangle - \langle [(p + e_k) u]_x \rangle + \frac{1}{R} \langle u \partial_y^2 u \rangle + \langle u F_u \rangle. \tag{122}$$

The kinetic-energy growth rate derived from (122) is

$$\int_{y_{B1}}^{y_{B2}} \partial_t \langle e_k \rangle \, dy = \lambda \Big(\frac{1}{2} [U_1^{''}]_-^+ \sigma + \beta_{v,1} b_0 (\frac{1}{3}\sigma^2 + \sigma \ln 2 + 1) U_0^{'} Y_\infty \Big) U_0^{'} \sigma Y_\infty^2 \, \epsilon^{\frac{13}{4}}, \tag{123}$$

while the kinetic-energy growth rate resulting from the inner-flow energy vertical flux jump is

$$\int_{y_{B1}}^{y_{B2}} \partial_t \langle e_k \rangle \, dy = \lambda \Big(\frac{1}{2} [U_1^{''}]_-^+ + \frac{1}{3} \beta_{v,1} (b_0 + 2 \bar{\bar{T}}_0^{''}) U_0^{'} Y_\infty \Big) U_0^{'} Y_\infty^2 \, \epsilon^{\frac{13}{4}}. \tag{124}$$

This can now be equated with (123), so that we may obtain the jump of $T_1^{''}$ after using (57), as a function of the background field

$$[T_1^{''}]_-^+ = \frac{2 b_0 Y_\infty}{\sigma^2 - 1} \Big(T_0^{''} (1 - \sigma^2 + \frac{2}{3}\sigma^{-1}) + b_0 U_0^{'2} (\frac{8}{3}\sigma^2 - \sigma \ln 2 - 4 + \frac{1}{3}\sigma^{-1}) \Big). \tag{125}$$

Note that

$$\text{as} \quad y \simeq y_a \,, \quad \int_{y_{B1}}^{y_{B2}} \partial_{\tau_1} \langle e_k \rangle \, dy \sim O(\epsilon^2) \,.$$

For a solitary-wave perturbed flow, the mean-flow kinetic energy is equal to the mean kinetic energy, so that the mean perturbation kinetic energy is zero and

$$\int_{y_{B1}}^{y_{B2}} U \partial_{\tau_1} U \, dy = \int_{y_{B1}}^{y_{B2}} \partial_{\tau_1} \langle e_k \rangle \, dy \,.$$

Equation (125) shows that $[T_1^{''}]_-^+$ depends on τ_{DB}, consequently $\underline{T}_1^{''}$ and $[T_2^{'}]_-^+$ vary with τ_1 and τ_{DB} (cf. Eqs. (58) and (69)). Eq. (121) then reveals that $T_2^{'}(y_c)$ depends on τ_1, τ_2 and τ_{DB}. The buoyancy profiles $T_1(y)$ and $T_2(y)$ thus obey complex equations in the diffusion boundary layers, since $T_1(y)$ is forced in the critical layer over the time scales τ_1 and τ_{DB} through $T_1^{'}(y_c)$ and $T_1^{''}(y_c)$, and evolves on τ_{DB} in the diffusion boundary layers, whereas $T_2(y)$ is forced by the critical layer over the time scales τ_2 and τ_{DB} ($T_2(y_c)$) and, τ_1, τ_2 and τ_{DB} ($T_2^{'}(y_c)$), and diffuses on the time scale τ_{DB}. We will not display these evolution equations, being beyond the scope of this present study.

D. Mean potential energy

The averaged potential energy equation is, in the outer flow,

$$\partial_t \langle e_p \rangle = -\partial_y \langle e_p w \rangle - \langle (e_p u)_x \rangle + \; Ri^{-1} \Big(\frac{1}{PrR} \langle \theta \partial_y^2 \theta \rangle + \langle \theta F_b \rangle \Big). \tag{126}$$

This leads to a critical-layer induced potential-energy growth rate

$$\int_{y_{B1}}^{y_{B2}} \partial_t \langle e_p \rangle \, dy = \lambda \beta_{v,1} \Big\{ T_0^{''} [\frac{1}{6} T_0^{'''} y_\delta^2 - \frac{1}{3} (\underline{T}_1^{''} y_\delta - \underline{T}_2^{'}) - \frac{1}{5} (\bar{T}_0^{''})^2 \sigma^2 Y_\infty^2] \sigma^2 Y_\infty^2$$
$$+ \theta_0 [T_0^{'''} y_\delta^2 - \frac{2}{3} (\bar{T}_0^{''})^2 \sigma^2 Y_\infty^2 - 2 (\underline{T}_1^{''} y_\delta - \underline{T}_2^{'})] \Big\} \sigma Y_\infty \frac{\epsilon^{\frac{17}{4}}}{Ri}, \tag{127}$$

which leads to a potential-energy growth rate such as $\partial_{\tau_2}\langle e_p\rangle = \kappa_P\,\epsilon^2$ with

$$\kappa_P = Pr\frac{\beta_{v,1}}{2Ri}\Big\{T_0^{''}[\frac{1}{6}T_0^{''''}y_\delta^2 - \frac{1}{3}(\underline{T}_1^{''}y_\delta - \underline{T}_2^{'}) - \frac{1}{5}(\bar{T}_0^{''})^2\sigma^2 Y_\infty^2]\sigma^2 Y_\infty^2 + \theta_0[T_0^{''''}y_\delta^2 - \frac{2}{3}(\bar{T}_0^{''})^2\sigma^2 Y_\infty^2 - 2(\underline{T}_1^{''}y_\delta - \underline{T}_2^{'})]\Big\}.$$

The expression for κ_P is too complex to be exploited analytically since it nonlinearly involves the mean flow and buoyancy $U_0^{'}$, $U_0^{''}$, $T_0^{''}$, $T_0^{''''}$, $T_1^{'}$ and T_2, the parameters σ, Pr and the wave amplitude Y_∞. For $\sigma \to \infty$, the growth rate greatly simplifies $\kappa_P = -2b_0/(5Ri)PrT_0^{''}(\bar{T}_0^{''})^2\sigma^3 A_0^2$, and becomes proportional to the square of the wave maximum amplitude and whose sign is that of $-b_0\,T_0^{''}$. Equating (127) with the growth rate obtained by the inner flow gives a relationship between $\underline{T}_1^{''''}$, $\underline{T}_2^{''}$, $\underline{T}_3^{'}$, $\underline{U}_1^{''}$, and $\underline{U}_1^{'}$. We will not display this second expression of the growth rate because it is too long.

The rate of transmitted energy is, at the leading order, the jump of the averaged vertical flux of the total energy and pressure

$$\tau_E = [\langle(e_k + e_p + p)w\rangle]_-^+.$$

The sign of τ_E usually yields the energy balance inside the critical layer, but τ_E cannot be used here due to the presence of the secular terms in the nonlinear critical-layer theory, where instead we use

$$\kappa_E = \int_{y_{B1}}^{y_{B2}} \partial_t\left\langle e_k + e_p\right\rangle dy\,.$$

If $\kappa_E > 0$, then the critical layer absorbs the total energy carried by the solitary wave and transmits it to the mean flow and buoyancy. As the potential energy is negligible, indeed

$$\int_{y_{B1}}^{y_{B2}} \partial_t\left\langle e_p\right\rangle dy \Big/ \int_{y_{B1}}^{y_{B2}} \partial_t\left\langle e_k\right\rangle dy = O\left(\frac{\epsilon}{Pr\,Ri}\right),$$

we can predict an absorption of energy using (123) alone, so that for absorption, we come down to an inequality involving only $U_0^{'}$, $U_0^{''}$, $T_0^{''}$ and σ, independently on the wave amplitude:

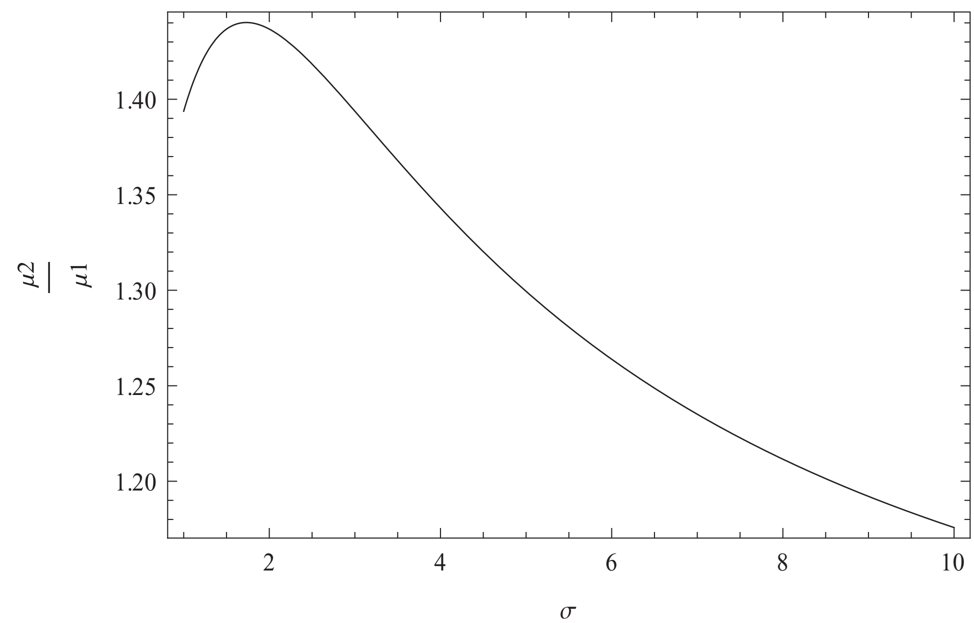


FIG. 8. Ratio μ_1/μ_2 as a function of σ.

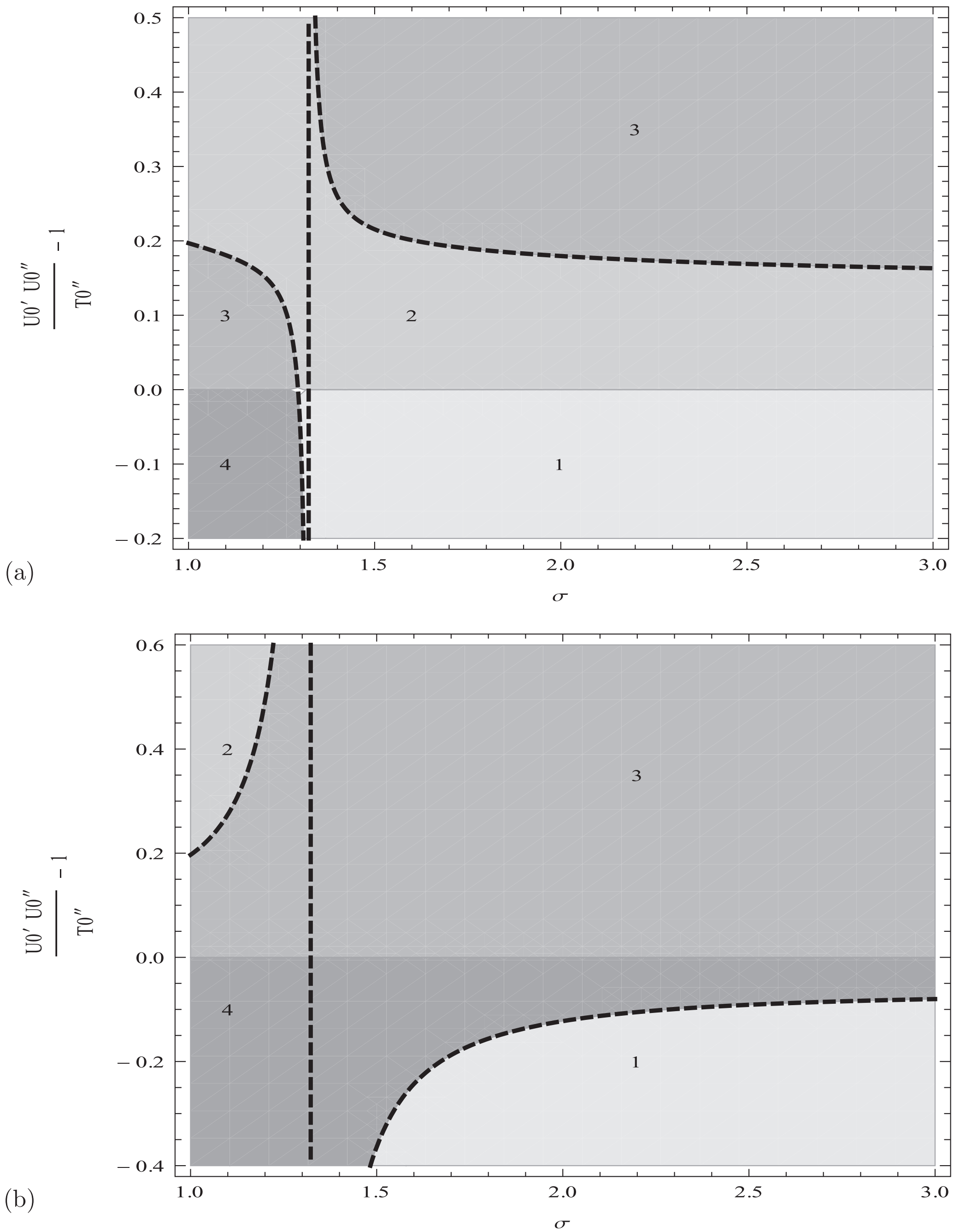


FIG. 9. Diagrams of the locations in the space $(U_0^{'}U_0^{''}/T_0^{''}-1, \sigma)$ where the critical layer heating occurs through the relation (129). The different zones correspond to: (1) $b_0 > 0$, $T_0^{''} < 0$, (2) $b_0 > 0$, $T_0^{''} > 0$, (3) $b_0 < 0$, $T_0^{''} < 0$, and (4) $b_0 < 0$, $T_0^{''} > 0$. Figure (a) displays the case $Pr = 0.71$ and Fig. (b) $Pr = 4$; the critical value of Pr which divides both diagrams is $Pr = 0.982$. The vertical line passes by $\sigma = 1.323$.

$$(U_0^{''} - \bar{T}_0^{''})\,[\mu_1(\sigma)U_0^{''} - \mu_2(\sigma)\bar{T}_0^{''}] < 0, \tag{128}$$

with

$$\mu_1(\sigma) = \frac{1}{3}\sigma^2 + (\ln 2 - \frac{1}{3})\sigma + 1, \quad \mu_2(\sigma) = \frac{1}{3}\sigma^2 + (\ln 2 + \frac{1}{3})\sigma + 1, \quad 1 < \mu_1 < \mu_2.$$

The inequality (128) yields the basic condition for a kinetic-energy absorption:

$$1 < \frac{U_0''}{\bar{T}_0''} < \frac{\mu_2}{\mu_1}(\sigma),$$

that is to have a small parameter b_0, or equivalently a weakly singular critical layer. Indeed, Fig. 8 displays the ratio $\mu_2/\mu_1(\sigma)$. As σ increases from one, μ_2/μ_1 varies relatively little, the bounding range slowly shrinks to zero: for $1 < \sigma \lesssim 10$, $1.176 \lesssim \mu_2/\mu 1 \lesssim 1.441$. The value of σ giving a maximum-range is $\sigma \simeq 1.732$. This kind of critical layer therefore favors a loss of kinetic energy by the mean flow. In the same way, the heating criterium (120) yields a similar inequality containing in addition the Prandtl number

$$b_0\Big[2\Big(\frac{U_0'U_0''}{T_0''} - 1\Big)(\frac{8}{3}\sigma^2 - \sigma\ln 2 - 4 + \frac{1}{3}\sigma^{-1}) + (\sigma^2 - 1)(\frac{1}{3}Pr - 1) + \frac{2}{3}\sigma^{-1}\Big] < 0. \tag{129}$$

Figure 9 shows that there still exists an infinite region in the plane $(b_0U_0'^2/T_0'', \sigma)$, whatever Pr and the signs of b_0 and T_0'' may be, so that the critical layer may heat or cool. However, this region changes shape and size according to these parameters. Both curves in the diagrams change curvature at the critical Prandtl number $Pr = 0.982$. If we are in the case, for example, where $b_0 > 0$ and $T_0'' < 0$ for $Pr < 0.982$, then the critical layer heats if the point $(b_0U_0'^2/T_0'', \sigma)$ is in the region 1 in the Figure 9(a). However, if we are in the case where $b_0 > 0$ and $T_0'' > 0$, then this point must be in the region 2.

The evolutions of the mean flow and buoyancy are thus stringently related to how high streamline and iso-buoyancy line distortions are, for the above growth rates depend on the jumps: $[U_1']^+_-$, $[T_1'']^+_-$ and $[U_1'']^+_-$. Though E-wave distortion is higher than D-wave distortion, the former is restricted to the zone within and around the cat's eye, and thus concerns only the wave-like flow, function of $A[X, T]$, but not the induced mean flow. Indeed, rapidly away from the cat's eye, E-wave distortion vanishes (as A decreases) and on space average, the E-wave critical-layer induced mean flow is not distorted and the shifting $y_\delta = 0$ (for the E wave, after Eq. (36), $c_1 = 0$). As a result, E-wave mean flow and buoyancy are steady in the asymptotic quasi-steady state régime and equate the basic profiles of velocity and buoyancy, $U(y) = U_0(y)$ and $T(y) = T_0(y)$ $(U_1(y) = U_{u,1})$. Finally, the E-wave local rescaled Richardson number is $O(\epsilon)$, namely $J_c = T_2'/U_0'^2\,\epsilon$.

The long-wavelength neutral and singular internal D-mode has thus an opposite effect on the mean flow with respect to the internal neutral mode[16,17] for $J_c > 1/4$. Mean flow kinetic energy decays. The solitary wave conserves its energy, as is gradually lost by dissipation over very long times. This particular neutral mode behaviour can as well be brought out in the interaction between a vortex and a vortex Rossby neutral mode. When a vortex supports neutral or nearly neutral asymmetric modes, the circular vortex is weakened. The energy is trapped in these modes, then is gradually lost by diffusion. On the other hand, stable asymmetric modes decay, which leads to an axisymmetrization and intensification of the vortex.[44] In another example of wave/mean flow interaction, the variability in the global upper-tropospheric flow fields can be explained by the presence of a tropical westerly anomaly: the tropical axisymmetric mode (TAM). TAM is essentially identified as a near-neutral mode of the zonally symmetric atmosphere and is associated with the weakened Hadley cell.[45]

VII. CONCLUSION

In this study, we have described explicitly the strong interaction of a free internal wave with a stratified shear flow when a critical level singularity appears in the linearized formulation. The critical layer dynamics is fully nonlinear and analyzed in the inviscid limit, appropriate for geophysical flows with a high Reynolds number. We assumed that the flow is quasi-steady, and made a long-wave assumption, Crucially, we have assumed that the stratification is weak at the critical level. As noted, nevertheless the basic buoyancy field $T_0(y)$ is technically weakly convectively unstable, and in particular, the buoyancy field within the cat's eye has a negative buoyancy gradient in one half of the cat's eye. This has come about because we have assumed that the buoyancy gradient vanishes at the critical level, that is $T_0^{'}(y_c) = 0$. However, because the basic buoyancy field is maintained by a forcing term $F_b(y)$, we expect that any potential convective instability will be suppressed. Of course, if we also suppose that $T_0^{''}(y_c) = 0$, then the basic buoyancy field is stable, and in particular, the buoyancy within the cat's eye is then constant at the leading order. The Prandtl-Batchelor theorem used to determine the stationary and laminar flow inside a closed two-dimensional domain is extended here to allow for the slowing evolving and weakly forced flow due to weak viscous and thermal diffusions within the separatrices. In our asymptotic procedure, the vorticity and buoyancy are determined at each order of the perturbation expansion within the cat's eye. As expected, at the leading order, the vorticity is constant within the separatrices of the cat's eye. However, due to the buoyancy-flow coupling, the vorticity is no longer constant within the cat's eye at the order $O(\epsilon^{1/2})$, whereas owing to the buoyancy forcing, the leading-order, $O(\epsilon)$-buoyancy is not constant. Here, ϵ is the small, dimensionless wave amplitude.

The depression wave critical layer is not symmetrical with respect to the critical level, since it is y-shifted by a value proportional to the local rescaled Richardson number of order $O(\epsilon^{1/2})$. Other adjustments of the streamline pattern must be introduced, using a strained-coordinate parametrization of each streamline inside the critical layer, similar to those found for Rossby-wave critical-layer dynamics.[19,20] The distortion which is then induced on the y-axis is $O(\epsilon \ln \epsilon)$. The addition of a stratified buoyancy, though small, to the wave mean flow interaction process leads to a stronger deformation of the streamlines than for the unstratified case, and also a stronger distortion of the critical-layer induced mean flow. This distortion of the mean flow occurs at the $O(\epsilon^{1/2})$-first order for the vorticity, and then, at the second order for the velocity and buoyancy gradient, and at the third order for the buoyancy in agreement with Ref. 46. For the elevation wave (the E-wave), a singularity around the separatrices is encountered for both the $O(\epsilon)$ vorticity and the $O(\epsilon^2)$ buoyancy. It is removed, as in the β-plane critical-layer case,[20] with a new scaling $\hat{S} \equiv \tilde{S}/(\epsilon^{1/2} \ln \epsilon)$. It seems that this is a general outcome as the same issue appears in the works.[19,20,47,48] That is, it is always possible to remove the singularities which emerge in a critical-layer nonlinear dynamics through the use of new scalings which preserve the regular-quantity jumps.

The pressure matching condition on the separatrices of the cat's eye leads to a compatibility condition on the streamfunction amplitude $A(X, T)$, which is an integro-differential equation containing two integral functionals of $\partial_T A$. For a steady travelling solitary wave, we can obtain explicit solutions, which are nevertheless more complex than those for a modified Korteweg-de-Vries equation. The additional terms arise from the critical-layer dynamics. In particular, the assumption of a nonlinear critical layer implies a specific long-wave scaling which provides stronger dispersion through a smaller wavelength than in the usual derivation of the KdV equation.[26] However, for steady travelling waves, the related evolution equation for the cross-stream location of the separatrix, rather than the amplitude of the streamfunction, is a (steady) Korteweg-de-Vries equation. Thus, we have established that there are solitary-wave solutions for both depression waves (D-waves) and elevation waves (E-waves) with critical-layer configurations. The conditions on the mean flow for the appearance of a solitary E-wave are more stringent than for the D-wave case but can be realized in a stratified medium. The E-wave found here is stationary in the frame moving with the linear wave. However, it is possible that unsteady solitary E-waves could exist as solutions of the integro-differential equation (65). Numerical solutions of this latter equation with simplified unsteady equations (93) and (101) are planned in a future study. Away from the cat's eye, the E-wave critical layer is not distorted, all mean flow jumps such as $[T_1^{''}]_-$, $[U_1^{''}]_-^+ \ldots$ are zero and subsequently the critical-layer induced flow and buoyancy which spread out in both diffusion boundary layers on either side after averaging are the basic background $U_0(y)$ and $T_0(y)$. The result is an absence of E-wave/mean flow interaction.

The D-wave mean-flow change induced by the presence of the critical layer occurs over the viscous time scale $\tau_1 = \lambda T$ yet derived.[19,20] Its magnitude is function of only two parameters, the shear at the critical level and the amplitude of the leading-order viscous flow, and it is independent of the wave amplitude. The presence of this critical layer mostly results in a loss of kinetic energy for the mean flow while the buoyancy of the latter can increase or decrease. In this critical-layer theory, the averaged vertical flux of horizontal momentum is proportional to the rescaled dimensionless viscosity λ and is, generally of same order as the outer viscous diffusion of the mean horizontal momentum, and as the x-derivatives of the secular terms. It is therefore no longer possible to neglect the viscous and thermal diffusions in this present wave mean-flow interaction study, as in previous wave mean-flow interaction studies where the motion was essentially periodic and did not involve secular terms. These secular terms are therefore essential in the critical-layer interaction. The leading-order secular-streamfunction amplitude ($O(\epsilon^{3/2})$) is inversely proportional to the D-wave critical-layer thickness, assuming this amplitude small favors the nonlinear-critical-layer régime with respect to the viscous-critical-layer one. As the diffusion boundary layers spread slowly with a $O(\sqrt{\tau_{DB}})$ thickness, where $\epsilon \ll \tau_{DB} \ll 1$, at either side of the critical layer, they induce a small displacement of the critical level with the same time scale τ_{DB} which affects the critical-layer dynamics. As a result, as the flow remains weakly stratified around the critical layer, the $O(\epsilon^{1/2})$ mean horizontal velocity and the $O(\epsilon^{1/2})$ mean-buoyancy gradient evolves on the fast time scale τ_1 and on τ_{DB} as well, whereas the leading-order $O(\epsilon)$ mean buoyancy evolves more slowly, with the diffusion time scale $\tau_2 = t/(Pr\,R)$ and with τ_{DB}. Consequently, the mean heating and mean potential energy also evolve over the time scales τ_2 and τ_{DB}. Reciprocally, the mean dynamics and buoyancy evolve with the time scale τ_{DB} in the diffusive boundary layers but the presence of the critical layer introduces a complex forcing, coupling thermal and dynamical effects whose time scales are τ_1 and τ_2 at leading orders, and which is located at the distorted level $y_c - \epsilon^{1/2} T_1^{'}(y_c)/T_0^{''}(y_c)$.

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APPENDIX A: EXTENDED PRANDTL-BATCHELOR THEOREM WITHIN THE CAT'S EYE

In this appendix, we find the expressions for the vorticity and buoyancy within the separatrices using two integral constraints. We begin with the equations governing the motion of a uniform incompressible fluid

$$\boldsymbol{\Omega} \times \mathbf{u} - \nabla H + \Theta\, \mathbf{y} + \nu \nabla^2 \mathbf{u} + \mathbf{F}_u = \partial_t \mathbf{u}, \tag{A1}$$

$$\chi \nabla^2 \Theta - \mathbf{u} \cdot \nabla \Theta + \mathbf{F}_b = \partial_t \Theta, \tag{A2}$$

where $\boldsymbol{\Omega} = -\nabla \times \mathbf{u}$, $H = p/\rho + |u|^2/2$, $\nu = 1/R$ and $\chi = 1/(Pr\,R)$. The procedure we will employ is an extension of the method[49] used for a homogeneous fluid, and extended to a stratified fluid.[41]

We assume a solution which is a steady travelling wave. Then, in the frame moving with the velocity of this wave, the motion is steady and the streamlines are closed in the E-wave case. For the D-wave, we close the flow by putting the left half of the cat's eye at the right-hand end. We use the variable $\tilde{S}$, which makes the calculations easier since the pattern is symmetric vis-à-vis the curve $\tilde{Y} = 0$. Then, we integrate Eq. (A1) around the enclosing streamline defined by its value of $\tilde{S}$, to get

$$\oint \boldsymbol{\Omega} \times \mathbf{u} \cdot d\mathbf{l} - \oint \nabla H \cdot d\mathbf{l} + \oint \Theta\, \mathbf{y} \cdot d\mathbf{l} + \nu \oint \{\nabla \times \boldsymbol{\Omega} - [U_0^{''}(Y) - \epsilon^{\frac{1}{2}} U_{u,1}^{''}(Y)]\mathbf{x}\} \cdot d\mathbf{l}$$
$$+ V\epsilon^{\frac{5}{4}} \oint \partial_X \mathbf{u} \cdot d\mathbf{l} = 0. \tag{A3}$$

The two first terms on the left-hand side vanish. The third and last inviscid term are also zero due to the symmetry of the pattern with respect to the axis $\zeta = 0$. Thus we get that

$$\epsilon^{\frac{7}{4}} \oint \{\nabla \times \boldsymbol{\Omega} - [U_0^{''}(Y) - \epsilon^{\frac{1}{2}} U_{u,1}^{''}(Y)]\mathbf{x}\} \cdot d\mathbf{l} + \oint \Theta_v \mathbf{y} \cdot d\mathbf{l} + V\epsilon^{\frac{5}{4}} \oint \partial_X \mathbf{u}_v \cdot d\mathbf{l} = 0. \tag{A4}$$

The third term is negligible at the leading orders. For the buoyancy equation (A2), we perform a curvilinear integration around the enclosing streamline to get

$$-\oint \frac{\mathbf{u}}{|\mathbf{u}|} \cdot \nabla \Theta\, dl + \chi \oint \frac{\nabla^2 \Theta - T_0^{''}(Y)}{|\mathbf{u}|} dl + V\epsilon^{\frac{5}{4}} \oint \frac{\partial_X \Theta}{|\mathbf{u}|} dl = 0. \tag{A5}$$

The first and the third inviscid terms vanish by symmetry, and the viscous part of the latter term is also negligible at the leading orders, so that

$$\epsilon^{\frac{7}{4}} \oint \frac{\nabla^2\Theta_i - T_0^{''}(Y)}{|\mathbf{u}|} dl + V Pr \epsilon^{\frac{5}{4}} \oint \frac{\partial_X \Theta_v}{|\mathbf{u}|} dl = 0. \tag{A6}$$

The E-wave motion is even more simplified due to its stationarity, $\partial_T A = 0$ after Sec. V. Note that as the path of integration is infinite for the D-wave case, we divide each integral by the length of the X-range involved, and thus we obtain an average of the integrand over a contour characterized by a value of $\tilde{S}$. The X-average $\langle f\rangle$ of a field f within the cat's eye, along the $\tilde{S}$-contour, is given by

$$\langle f \rangle = \frac{1}{x_0(\tilde{S})} \int_0^{x_0(\tilde{S})} f \, d\zeta \,, \quad \text{for an } E\text{-wave},$$

$$\langle f \rangle = \frac{1}{L - x_0(\tilde{S})} \int_{x_0(\tilde{S})}^{L} f \, d\zeta \,, \quad L \to \infty \,, \quad \text{for a } D\text{-wave} \,, \quad \text{where } \tilde{S} = A(\zeta = x_0).$$

We then expand the vorticity. For instance, for the vorticity $\Omega = -(\nabla \times \mathbf{u}) \cdot \mathbf{e}_z = \epsilon^{-1}\partial_Y^2 \Psi + \epsilon^{1/2}\partial_X^2 \Psi$, we get that $\Omega = \Omega^{(0,\odot)} + \epsilon^{\frac{1}{2}}\Omega^{(2,\odot)} + \epsilon \ln\epsilon \, \Omega^{(4,\odot)} + \epsilon \, \Omega^{(5,\odot)} + \dots$.

At $O(\epsilon)$, Eqs. (A1) and (A4) yield, respectively,

$$\Omega^{(0,\odot)} = Q^{(0,\odot)}(\tilde{S}) = U_0^{'},$$

while Eqs. (A2) and (A6) yield

$$\Theta^{(0,\odot)} = T_0^{''} \tilde{S} + \theta_0.$$

At $O(\epsilon^{3/2})$, Eq. (A1) projected onto the cross-stream and then the streamwise axes yields, $\Omega^{(2,\odot)} = \bar{T}_0^{''} \tilde{S}_Y + Q^{(2,\odot)}(\tilde{S})$. Then using $\Omega^{(2,\odot)}$ in Eq. (A4) yields $Q^{(2,\odot)}(\tilde{S}) = \text{constant} = Q_2$. Equation (A2) implies a general solution for $\tilde{\Theta}^{(2,\odot)}$ of the form

$$\tilde{\Theta}^{(2,\odot)} = \bar{T}_0^{''}(\tilde{\Psi}^{(2,\odot)} - \Pi[\tilde{S}_Y] + s\Pi_0[S]) + \mathcal{G}^{(2,\odot)}(\tilde{S}) \,.$$

Then, Eq. (A6) yields the condition

$$(\mathcal{G}_{\tilde{S}}^{(2,\odot)}(\tilde{S}) + \bar{T}_0^{''} Q_2 + T_0^{'''} y_\delta) \left\langle \frac{1}{|\tilde{S}_Y|} \right\rangle = 0 \,,$$

$$\text{and so} \quad \mathcal{G}^{(2,\odot)}(\tilde{S}) = -(\bar{T}_0^{''} Q_2 + T_0^{'''} y_\delta)\tilde{S} + \theta_2 \,.$$

The function $\Pi_0[S]$ assures the continuity of the buoyancy on the axis $\tilde{Y} = 0$ at any point $\zeta = \xi - VT$.

Then in the same way, at $O(\epsilon^2 \ln\epsilon)$, Eq. (A1) gives the general form of the vorticity

$$\Omega^{(4,\odot)} = Q^{(4,\odot)}(\tilde{S}) - \frac{1}{2} b_0 \bar{T}_0^{''} A,$$

while Eq. (A4) then implies that

$$Q^{(4,\odot)}(\tilde{S}) = Q_4 = \text{constant}.$$

Equation (A2) gives the general form of the buoyancy,

$$\tilde{\Theta}^{(4,\odot)} = \bar{T}_0^{''}[\tilde{\Psi}^{(4,\odot)} - \frac{1}{2} b_0 \partial_{TX^{-1}} A] + \mathcal{G}^{(4,\odot)}(\tilde{S}),$$

with $\mathcal{G}^{(4,\odot)}(\tilde{S})$ given by (A6)

$$\mathcal{G}^{(4,\odot)}(\tilde{S}) = \frac{1}{2} b_0 (\bar{T}_0^{''2} - T_0^{'''}) \int_{S_c}^{\tilde{S}} \frac{\langle A|S_Y|\rangle}{\langle |S_Y|\rangle} \, dS + \theta_4. \tag{A7}$$

For a D-wave, $\mathcal{G}^{(4,\odot)}(\tilde{S}) = \theta_4 = \text{constant}$, because the integrand numerator is zero. Note again that the variable $\tilde{S}$ enables us to simplify the $O(\epsilon^2 \ln\epsilon)$ equations.

Finally at $O(\epsilon^2)$, Eq. (A1) yields the expression of vorticity displayed in Eq. (62). Then, the unknown function $Q^{(5,\odot)}(\tilde{S})$ is obtained from Eq. (A4) leading to the integral constraint,

$$\left\langle |\tilde{S}_Y|(Q^{(5,\odot)}_{\tilde{S}}(\tilde{S})-U_0^{'''})-\theta_v^{(2,\odot)}\frac{A_X}{|S_Y|}-\bar{T}_0^{''}|\tilde{S}_Y|[\tilde{S}_Y(\Pi_{\tilde{S}}[|\tilde{S}_Y|]-\Pi_{0,\tilde{S}})]_{\tilde{S}}+sV(\Psi^{(2,\odot)}_{v,Y\xi}+A_X\Psi^{(2,\odot)}_{v,YS})\right\rangle = 0.$$

Then, taking account of the symmetries $X \to -X$ and $\tilde{Y} \to -\tilde{Y}$, we get that

$$Q^{(5,\odot)}_{\tilde{S}}(\tilde{S})\langle|\tilde{S}_Y|\rangle = U_0^{'''}\langle|\tilde{S}_Y|\rangle + \bar{T}_0^{''}\langle|\tilde{S}_Y|[|\tilde{S}_Y|(\Pi_{\tilde{S}}[|\tilde{S}_Y|]-\Pi_{0,\tilde{S}})]_{\tilde{S}}\rangle$$
$$+[\frac{1}{Pr}(\bar{\bar{T}}_0^{''}\bar{T}_0^{''}-\bar{\bar{T}}_0^{'''})+\beta_{v,1}\bar{T}_0^{''}]\left\langle\zeta\frac{A_X}{|\tilde{S}_Y|}\right\rangle-\frac{\bar{T}_0^{''}}{Pr}\left\langle\frac{A_X}{|\tilde{S}_Y|}[\int_{x_c}^{\xi}(|\tilde{S}_Y|\Pi_{\tilde{S}}[|\tilde{S}_Y|])_{\tilde{S}}\,dx]\right\rangle-\theta_{2,v}(\tilde{S})\left\langle\frac{A_X}{|\tilde{S}_Y|}\right\rangle. \quad \text{(A8)}$$

Only the first term at the rhs has a no zero D-wave average, the third term remains for the E-wave too. Hence, $Q^{(5,\odot)}(\tilde{S}) = U_0^{'''}\tilde{S} + Q_5$ for a *D*-wave, and $Q^{(5,\odot)}(\tilde{S}) = U_0^{'''}\tilde{S} + \omega_5(S) + Q_5$ for an *E*-wave with

$$\omega_5(\tilde{S}) = \frac{1}{Pr}(\bar{T}_0^{''}\bar{\bar{T}}_0^{''} - \bar{T}_0^{'''})\int_{S_c}^{\tilde{S}}\frac{\left\langle\frac{\zeta A_X}{|S_Y|}\right\rangle}{\langle|S_Y|\rangle}\,dS.$$

The buoyancy $\theta_v^{(2,\odot)}$ is obtained from the viscous part of Eq. (A2)

$$\theta_v^{(2,\odot)}(\xi,\tilde{S}) = \frac{1}{Pr}\Big[\bar{\bar{T}}_0^{''}\Big((\bar{T}_0^{''}+\beta_{v,1}PrU_0^{'})\xi - \int_{x_c}^{\xi}[\tilde{S}_Y\Pi_{\tilde{S}}]_{\tilde{S}}\,dx\Big) - \bar{T}_0^{'''}\xi\Big] + \theta_{2,v}(\tilde{S}).$$

For a *D*-wave, $\mathcal{G}^{(5,\odot)}(\tilde{S})$ is given by

$$\mathcal{G}^{(5,\odot)}(\tilde{S}) = [(\bar{Q}_2T_0^{''}+T_0^{'''}y_\delta)\bar{Q}_2-\bar{T}_0^{''}Q_5+b_0S_c(T_0^{'''}-\bar{T}_0^{''2})+\frac{1}{2}T_0^{IV}y_\delta^2-V(\bar{\bar{T}}_0^{''}\bar{T}_0^{''}+\beta_{v,1}PrU_0^{'}-\bar{T}_0^{'''})]\tilde{S}$$
$$+\frac{1}{6}(T_0^{IV}+T_0^{''}\bar{\bar{T}}_0^{''2}-\bar{T}_0^{'''}\bar{T}_0^{''}-\bar{U}_0^{'''}T_0^{''})\tilde{S}^2+\frac{4}{3}V\bar{T}_0^{''2}(\tilde{S}-S_c)+\theta_5\,, \quad \text{(A9)}$$

whereas for the *E*-wave, we have

$$\mathcal{G}^{(5,\odot)}(\tilde{S}) = [\bar{Q}_2^2T_0^{''}-T_0^{''}\bar{Q}_5]\tilde{S}+(\bar{T}_0^{''2}-T_0^{'''})\int_{S_c}^{\tilde{S}}\frac{\langle\bar{\mathcal{U}}^{(2)}|\tilde{S}_Y|\rangle}{\langle|\tilde{S}_Y|\rangle}\,dS$$
$$+\frac{1}{6}T_0^{IV}\int_{S_c}^{\tilde{S}}\frac{\langle|S_Y|^3\rangle}{\langle|S_Y|\rangle}\,dS-(\bar{T}_0^{'''}\bar{T}_0^{''}+\bar{U}_0^{'''}T_0^{''}-T_0^{''}\bar{\bar{T}}_0^{''2})\int_{S_c}^{\tilde{S}}\frac{S\langle|S_Y|\rangle-\frac{1}{3}\langle|S_Y|^3\rangle}{\langle|S_Y|\rangle}\,dS$$
$$-\bar{T}_0^{''}\int_{S_c}^{\tilde{S}}\frac{\int_{S_c}^{S}\omega_5(S_1)\langle|\tilde{S}_Y(S_1)|\rangle_{\tilde{S}}\,dS_1}{\langle|\tilde{S}_Y(S)|\rangle}\,dS+\theta_5\,. \quad \text{(A10)}$$

APPENDIX B: ORDER-ϵ^2 INNER FLOW

This section gives the expressions of the related $\tilde{S}$-velocity, buoyancy, and pressure outside the cat's eye which, due to their length, could not be inserted in the main text of the paper,

$$\tilde{\Psi}_Y^{(5)} = \frac{1}{3}U_0^{'''}\tilde{S}_Y(2A+\tilde{S})+\frac{1}{9}b_0\bar{T}_0^{''}[s(\tilde{S}+6A)\sqrt{2\tilde{S}}-(\tilde{S}+2A)\tilde{S}_Y-s(S_c+6A)Y_\infty+s(S_c+2A)Y_s]$$
$$-\frac{1}{3}(\bar{T}_0^{'''}-\bar{T}_0^{''}\bar{U}_0^{''})[(\tilde{S}+2A)\tilde{S}_Y-s\tilde{S}\sqrt{2\tilde{S}}-s(S_c+2A)Y_s+sS_cY_\infty]$$
$$+(\bar{T}_0^{''}\hat{\bar{U}}_1^{'}+\bar{T}_0^{'''}y_\delta-\bar{T}_1^{''}-\bar{U}_0^{''}\bar{T}_1^{'})[A\ln[\Lambda(A,\tilde{S})]+[\tilde{S}(\tilde{S}-A)]^{\frac{1}{2}}-\frac{1}{2}Y_\infty Y_s-\tilde{S}+S_c]$$
$$+s(b_0\mathcal{U}^{(2)}+\frac{1}{2}\bar{T}_0^{'''}y_\delta^2-\bar{T}_1^{''}y_\delta+\bar{T}_2^{'})(\sqrt{2\tilde{S}}-Y_\infty)+sb_0^2A^*\ln[\Lambda(A,\tilde{S})]\sqrt{2\tilde{S}}$$

$$+ (\hat{U}_1'' - U_0''' y_\delta)[\tilde{S}(\tilde{S} - A)]^{\frac{1}{2}} - \frac{1}{2}(\underline{\hat{U}}_1'' - U_0''' y_\delta) Y_\infty Y_s + [\frac{1}{2}(U_0''' - \bar{T}_0''') y_\delta^2 - \hat{U}_1'' y_\delta + U_2' - \bar{T}_2'] \tilde{S}_Y$$
$$- \frac{1}{2}[U_2' - U_1'' y_\delta - \bar{T}_2']_-^+ Y_s - s\frac{1}{2} b_0 \int_{S_c}^{\tilde{S}} \frac{\Pi[(w - A)^{\frac{1}{2}}]}{[2w(w - A)]^{\frac{1}{2}}} dw - \bar{\bar{T}}_0'' \{\Pi[\tilde{S}_Y] - \Pi[s Y_s]\}$$
$$+ \{[\bar{\bar{T}}_1'' - \hat{U}_1' \bar{T}_0'' + b_0(\beta_{l,1} + \bar{\bar{T}}_1') - y_\delta \bar{\bar{T}}_0''' + y_\delta(\bar{\bar{T}}_0''^2 - b_0^2)] A^* + b_0(\beta_{d,1} \partial_X^2 A^* + B^*)\} \ln[\Lambda(A, \tilde{S})]$$
$$+ \alpha \tilde{S}_Y + \bar{T}_0'' \varphi^{(2)}(\tilde{S}),$$

with α an unknown constant to determine while matching, while the buoyancy is

$$\tilde{\Theta}^{(5)} = \bar{T}_0''(\tilde{\Psi}^{(5)} - \varphi^{(2)} \Pi_{\tilde{S}}[\tilde{S}_Y]) + \bar{\mathcal{G}}_{\tilde{S}}^{(2)}(\tilde{\Psi}^{(2)} - \Pi[\tilde{S}_Y])$$
$$- \bar{\bar{T}}_0'' \int_{x_c}^{\xi} \Pi_S[S_Y]\Big(\Psi_\xi^{(2)} - \frac{\partial_T A}{S_Y}\Big) dx - \bar{\bar{T}}_0'' \int_{x_c}^{\xi} (\Psi_\tau^{(2)} - \Pi_\tau[\tilde{S}_Y]) \frac{dx}{\tilde{S}_Y} + \mathcal{G}^{(5)}(\tilde{S}). \qquad \text{(B1)}$$

The matching of the pressures

$$\tilde{P}^{(5)} = U_0' \tilde{\Psi}^{(5)} + Q\, \tilde{\Psi}^{(2)} - \frac{1}{2} \tilde{\Psi}_Y^{(2)2} + \tilde{S}_Y(\tilde{\Theta}^{(2)} - U_0' \tilde{\Psi}_Y^{(5)}) + \bar{T}_0'' A_{TX^{-1}}$$
$$- \int_{x_c}^{\xi} \tilde{S}_Y(\Psi_{\tilde{S}T}^{(2)} + A_T \Psi_{\tilde{S}\tilde{S}}^{(2)}) dx + U_0' \int_{S_c}^{\tilde{S}} \mathcal{F}^{(5)} dS + \tilde{\Theta}^{(0)} \varphi^{(2)} \tilde{S}_Y^{-1} + \mathcal{P}^{(5)}, \qquad \text{(B2)}$$

$$\tilde{P}^{(5,\odot)} = U_0' \tilde{\Psi}^{(5,\odot)} + Q_2 \tilde{\Psi}^{(2,\odot)} - \frac{1}{2} \Psi_Y^{(2,\odot)2} + \tilde{S}_Y(\tilde{\Theta}^{(2,\odot)} - U_0' \tilde{\Psi}_Y^{(5,\odot)}) + \bar{T}_0'' A_{TX^{-1}}$$
$$- \int_{x_c}^{\xi} \tilde{S}_Y(\Psi_{\tilde{S}T}^{(2,\odot)} + A_T \Psi_{\tilde{S}\tilde{S}}^{(2,\odot)}) dx + U_0' \int_{S_c}^{\tilde{S}} Q^{(5,\odot)}(S)\, dS + \tilde{\Theta}^{(0)} \varphi^{(2)} \tilde{S}_Y^{-1} + \mathcal{P}^{(5,\odot)}, \quad \text{(B3)}$$

comes down to the matching of

$$\int_{x_c}^{\xi} \tilde{S}_Y(\Psi_{\tilde{S}T}^{(2)} + A_T \Psi_{\tilde{S}\tilde{S}}^{(2)}) dx + U_0' \tilde{S}_Y \tilde{\Psi}_Y^{(5)}.$$

However, the integral is continuous through the separatrices, so the streamwise velocity must be continuous, which leads to two conditions giving $\tilde{\Psi}_Y^{(5,\odot)}$ and α. The matching of the deformation functions $\varphi^{(5)}$ yields the evolution Eq. (65),

$$\alpha = Q_5 - \frac{1}{2}(U_0''' - \bar{T}_0''') y_\delta^2 - \frac{1}{4}[U_1'']_-^+ Y_\infty + \underline{\hat{U}}_1'' y_\delta - \underline{U}_2' + \underline{\bar{T}}_2'.$$

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