

This is a repository copy of *Deletion-restriction for sheaf homology of graded atomic lattices*.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/184798/>

Version: Accepted Version

---

**Article:**

Everitt, Brent orcid.org/0000-0002-0395-338X and Turner, Paul (2022) Deletion-restriction for sheaf homology of graded atomic lattices. *Advances in Mathematics*. 108354. ISSN 0001-8708

<https://doi.org/10.1016/j.aim.2022.108354>

---

**Reuse**

This article is distributed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivs (CC BY-NC-ND) licence. This licence only allows you to download this work and share it with others as long as you credit the authors, but you can't change the article in any way or use it commercially. More information and the full terms of the licence here: <https://creativecommons.org/licenses/>

**Takedown**

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing [eprints@whiterose.ac.uk](mailto:eprints@whiterose.ac.uk) including the URL of the record and the reason for the withdrawal request.

# Deletion-restriction for sheaf homology of graded atomic lattices

Brent Everitt and Paul Turner

*Dedicated to Marcia Everitt (1932-2018) and Ken Turner (1927-2014)*

**Abstract.** We give a long exact sequence for the homology of a graded atomic lattice equipped with a sheaf of modules, in terms of the deleted and restricted lattices. This is then used to compute the homology of the arrangement lattice of a hyperplane arrangement equipped with the natural sheaf. This generalises an old result of Luszti.

---

## Introduction

This paper is about graded atomic lattices, equipped with sheaves of modules, and their homology. Important examples in nature are the face lattices of polytopes, the intersection lattices of hyperplane arrangements, and the lattices of flats of matroids. This last family comprises precisely the geometric lattices. When studying lattices a key role is played by deletion-restriction, where the lattice  $L$  may be decomposed into two pieces with respect to some atom  $a$ : the deletion  $L_a$  and the restriction  $L^a$ . For example, the characteristic polynomial  $\chi_L(t)$  of a geometric lattice  $L$  may be expressed in terms of the characteristic polynomials of the deletion and restriction.

When  $L$  is equipped with constant coefficients – that is, the sheaf is the constant sheaf – then the homology reduces to the ordinary simplicial homology of the order complex  $|L|$  of  $L$ , and one can avail oneself of standard topological tools. For example, an argument using a Mayer-Vietoris sequence is enough to fully compute the homology of a geometric lattice [Fol66, Bjö82]. The long exact sequence used in the calculation is another manifestation of deletion-restriction, relating the homology of  $L$  with that of  $L_a$  and  $L^a$ ; see [OT92, §4.5] for details.

If the sheaf is non-constant then the topology of  $|L|$  can play a relatively minor role in homology – the space  $|L|$  can be contractible for example, but the sheaf homology may be highly non-trivial. This makes the calculation of homology for arbitrary sheaves less straightforward, and the techniques used for constant coefficients do not simply generalise.

Nevertheless, for an arbitrary sheaf it turns out there is a deletion-restriction long exact sequence, and this is the first main result of the paper:

**Main Theorem.** *Let  $L$  be a graded atomic lattice equipped with a sheaf  $F$ . Then for any atom  $a \in L$  there is a long exact sequence*

$$\begin{array}{c} \cdots \rightarrow H_i(L^a \setminus a; F) \rightarrow H_i(L_a \setminus \mathbf{0}; F) \rightarrow H_i(L \setminus \mathbf{0}; F) \rightarrow H_{i-1}(L^a \setminus a; F) \rightarrow H_{i-1}(L_a \setminus \mathbf{0}; F) \rightarrow \cdots \\ \left. \begin{array}{c} \cdots \rightarrow H_1(L \setminus \mathbf{0}; F) \rightarrow \widetilde{H}_0(L^a \setminus a; F) \rightarrow H_0(L_a \setminus \mathbf{0}; F) \rightarrow H_0(L \setminus \mathbf{0}; F) \rightarrow \text{coker}(\epsilon_*) \rightarrow 0 \end{array} \right\} \end{array}$$

---

BRENT EVERITT: Department of Mathematics, University of York, York YO10 5DD, United Kingdom. e-mail: brent.everitt@york.ac.uk. PAUL TURNER: Section de mathématiques, Université de Genève, 2-4 rue du Lièvre, CH-1211, Geneva, Switzerland. e-mail: paul.turner@unige.ch.

where  $\epsilon_*: H_0(L^a \setminus a; F) = \varinjlim^{L^a \setminus a} F \rightarrow F(a)$  is the map induced by the  $F_a^x: F(x) \rightarrow F(a)$ , for  $x \geq a$ , and the universality of the colimit.

Each lattice has had its minimum element  $\mathbf{0}$  removed, a necessary requirement for a lattice when considering its sheaf homology. If minima are not removed then, for general reasons, the homology will be concentrated in degree zero. When the coefficients are constant, both the minimum and the maximum  $\mathbf{1}$  have to be removed to avoid the homology completely collapsing. When the sheaf is non-constant there is no *a priori* reason to remove the maximum. We also warn the reader that at the generality of graded atomic lattices  $L$ , the restriction  $L^a$  is not itself atomic. To make inductive arguments therefore, one must start with an  $L$  carrying more structure: for example the face lattice of a polytope or a geometric lattice – see §§1.1-1.2.

In the case of a linear hyperplane arrangement, the associated arrangement lattice has elements the intersections of hyperplanes. As these intersections are again linear spaces this gives rise to a canonical sheaf on the lattice of intersections. We refer to this as the *natural sheaf*. Our main application of the deletion-restriction long exact sequence gives a complete calculation of the reduced homology in this case:

**Main Application.** *Let  $L$  be the intersection lattice of a hyperplane arrangement with  $\text{rk}(L) \geq 2$  and let  $F$  be the natural sheaf on  $L$ . Then  $\widetilde{H}_i(L \setminus \mathbf{0}; F)$  is trivial when  $i \neq \text{rk}(L) - 2$  and*

$$\dim \widetilde{H}_{\text{rk}(L)-2}(L \setminus \mathbf{0}; F) = (-1)^{\text{rk}(L)-1} \left. \frac{d}{dt} \chi(t) \right|_{t=1}$$

where  $\chi(t)$  is the characteristic polynomial of  $L$ .

The quantity appearing on the right-hand side is known (in the more general context of matroids) as the beta-invariant (see [Zas87, §7.3]). We note that Yuzvinsky [Yuz91] formulated the notion of a local sheaf to generate similar vanishing homology results, but these ideas are not readily applicable to the situation above.

Our original motivation was a result of Lusztig [Lus74, Theorem 1.12], where he proved that if  $V$  is a space over a finite field,  $A$  is the hyperplane arrangement consisting of *all* the hyperplanes in  $V$ , and  $F$  is the natural sheaf, then  $H_i(L \setminus \{\mathbf{0}, \mathbf{1}\}; F)$  vanishes in degrees  $0 < i < \text{rk}(L) - 2$ . Lusztig's interest in natural sheaves on arrangement lattices arose in his study of the discrete series representations of  $GL_n k$  for  $k$  a finite field. As a corollary to our main application we extend Lusztig's result to any arrangement:

**Corollary.** *Let  $L$  be the intersection lattice of a hyperplane arrangement  $A$  in the vector space  $V$  and let  $U = \bigcap_{a \in A} a$ . Suppose that  $\text{rk}(L) \geq 3$  and let  $F$  be the natural sheaf on  $L$ . Then  $H_i(L \setminus \{\mathbf{0}, \mathbf{1}\}; F)$  vanishes in degrees  $0 < i < \text{rk}(L) - 2$  with  $H_0(L \setminus \{\mathbf{0}, \mathbf{1}\}; F) \cong V \oplus U$  and*

$$\dim H_{\text{rk}(L)-2}(L \setminus \{\mathbf{0}, \mathbf{1}\}; F) = (-1)^{\text{rk}(L)-1} \left. \frac{d}{dt} \chi(t) \right|_{t=1} + |\mu(\mathbf{0}, \mathbf{1})| \dim U,$$

where  $\mu$  is the Möbius function of  $L$ .

We note that while our calculations involving hyperplane arrangements have homology vanishing in all but top degree, this behaviour is the exception rather than the rule. One can readily find lattices and sheaves whose homology is highly non-trivial. One example is the the Khovanov homology of a link diagram [Kho00] which may be interpreted in terms of sheaf homology (see

[ET15, ET14]). In this case there are many non-vanishing intermediate degrees, despite the underlying lattice being contractible. Even when the sheaf structure maps are all injections one easily finds non-trivial homology in intermediate degrees. A natural example is in the context of “sheaves on buildings”. Indeed, Lusztig’s result can be viewed as the case of the building of  $GL_n$  equipped with the fixed point sheaf of the natural representation, for which the structure maps are all inclusions. There are similar situations – the building of  $Sp_n$  for example – where the homology is non-vanishing in some intermediate degrees (see [RS85]).

The paper is organised as follows. In Section 1 we set down the basics on lattices, and in particular discuss the notion of a dependent atom, that will play a key role in inductive arguments. In Section 2 we remind the reader about the basics of sheaf homology on posets – both unreduced and reduced. We also present the Leray-Serre spectral sequence arising from a poset map, which plays a key role. In Section 3 we present a deletion-restriction long exact sequence for arbitrary sheaves (Theorem 2) and also give a version using reduced homology (Corollary 3). In Section 4 we calculate, as an application, the sheaf homology of a hyperplane arrangement equipped with the natural sheaf (Theorem 3) and put this in a form which makes direct comparison to Lusztig’s result (Theorem 5). We end with a few remarks about the reduced broken circuit complex, whose homology also features the beta-invariant.

We are grateful to the referee for pointing out to us the literature concerning broken circuits and the beta-invariant.

## 1. Lattices

In §§1.1-1.2 we recall basic facts about posets, lattices, geometric lattices and arrangement lattices. Standard references for this material are [Bir79, Sta12, Sta07, OT92]. In §1.3 we set down facts about dependent atoms from [EF13] that will be useful in the inductive arguments of §4.

### 1.1. Basics

Let  $P = (P, \leq)$  be a finite poset. If  $x \leq y \in P$  and for any  $x \leq z \leq y$  we have either  $z = x$  or  $z = y$ , then  $y$  is said to cover  $x$ , and we write  $x < y$ .  $P$  is *graded* if there exists a function  $rk : P \rightarrow \mathbb{Z}$  such that (i)  $x < y$  implies  $rk(x) < rk(y)$ , and (ii)  $x < y$  implies  $rk(y) = rk(x) + 1$ . A minimum is an element  $\mathbf{0} \in P$  such that  $\mathbf{0} \leq x$  for all  $x \in P$  and a maximum is an element  $\mathbf{1} \in P$  such that  $x \leq \mathbf{1}$  for all  $x \in P$ . If  $P$  has a minimum  $\mathbf{0}$ , then the standard grading on  $P$  is defined by taking  $rk(x)$  to be the supremum of the lengths of all poset chains from  $\mathbf{0}$  to  $x$ . All the posets in this paper will be graded with the standard grading. The elements covering  $\mathbf{0}$  – those of rank 1 – are called *atoms*. A poset map  $f : Q \rightarrow P$  is a set map such that  $fx \leq fy \in P$  if  $x \leq y \in Q$ .

A subset  $K \subset P$  is *upper convex* if  $x \in K$  and  $x \leq y$  implies that  $y \in K$ . If  $x \leq y$ , the *interval*  $[x, y]$  consists of those  $z \in P$  such that  $x \leq z \leq y$ ; if  $x \in P$  the interval  $P_{\geq x}$  consists of those  $z \in P$  such that  $z \geq x$ ; one defines  $P_{\leq x}$ ,  $P_{> x}$  and  $P_{< x}$  similarly.

A lattice is a poset such that any two elements  $x$  and  $y$  have a unique supremum (or *join*)  $x \vee y$  and a unique infimum (or *meet*)  $x \wedge y$ . A finite lattice has minimum  $\mathbf{0}$  equal to the meet of all its elements and maximum  $\mathbf{1}$  equal to the join of all its elements. A graded lattice is *atomic* if every element can be expressed – not necessarily uniquely – as a join of atoms, and with the empty join taken to be  $\mathbf{0}$ . The rank,  $rk(L)$ , of a graded lattice  $L$  is  $rk(L) := rk(\mathbf{1})$ .

Examples of graded atomic lattices abound:

- If  $A$  is a (finite) set then the free, or *Boolean*, lattice  $B = B(A)$  has elements the subsets of  $A$  ordered by inclusion. It is a graded atomic lattice with  $rk(x) = |x|$ ,  $rk(B) = |A|$ , join  $x \vee y = x \cup y$ , meet  $x \wedge y = x \cap y$ , minimum  $\mathbf{0} = \emptyset$ , maximum  $\mathbf{1} = A$  and atoms the singletons – which we identify with  $A$ . Any element has a unique expression as a join of atoms.
- A (convex) polytope  $P$  in a real vector space  $V$  is the convex hull of a finite set of points – see [Grü03, Zie95]. The *face lattice*  $\mathcal{F}(P)$  has elements the faces of  $P$  ordered by reverse inclusion. This is a graded atomic lattice (graded by  $rk(f) = \dim P - \dim f$ ) with atoms the  $\dim P - 1$  faces, join  $f_1 \vee f_2 = f_1 \cap f_2$ , meet  $f_1 \wedge f_2$  the smallest face containing  $f_1$  and  $f_2$ , minimum  $P$  and maximum  $\emptyset$  (hence  $rk\mathcal{F}(P) = \dim P$ ).
- The *partition lattice*  $\Pi = \Pi(A)$  on the set  $A$  consists of all partitions  $\{X_1, X_2, \dots, X_n\}$  of  $A$  ordered by refinement:  $\{X_1, X_2, \dots, X_n\} \leq \{Y_1, Y_2, \dots, Y_m\}$  if each  $X_i$  is contained in some  $Y_j$ . The result is a graded lattice with  $rk\{X_1, X_2, \dots, X_n\} = \sum(|X_i| - 1)$ ;  $rk(\Pi) = |A| - 1$ , minimum the partition with all blocks singletons, maximum  $\{A\}$ , and atoms the partitions with just one block  $\{a, b\}$  not having size one.
- The intersections of a collection of hyperplanes ordered by reverse inclusion gives an *arrangement lattice* – see §1.2.

If  $L$  is a graded atomic lattice with atoms  $A$ , then for  $a \in A$ , define the *deletion* lattice  $L_a$  to be the elements of  $L$  that can be expressed as a join of the elements of  $A \setminus \{a\}$  (with the empty join taken to be  $\mathbf{0}$ ), and the *restriction* lattice  $L^a$  to be the interval  $L_{\geq a}$ . The deletion  $L_a$  is graded atomic with atoms  $A_a = A \setminus \{a\}$ , minimum  $\mathbf{0}_a = \mathbf{0}$ , maximum  $\mathbf{1}_a = \bigvee A_a$  and rank function  $rk_a = rk$ . The restriction  $L^a$  is graded but not in general atomic; if however  $L$  is the face lattice of a polytope [Zie95, Theorem 2.7] or an arrangement lattice (which includes the Boolean and partition examples) then  $L^a$  is atomic.

We finish our review of the basics with an important object in the theory of enumeration. If  $k$  is a field, then the *Möbius function*  $\mu = \mu_L$  of  $L$  is the  $k$ -valued function on the intervals  $[x, y]$  defined by

$$\mu(x, y) = - \sum_{x \leq z < y} \mu(x, z), \text{ for all } x < y \text{ in } L$$

and  $\mu(x, x) = 1$  for all  $x \in L$ .

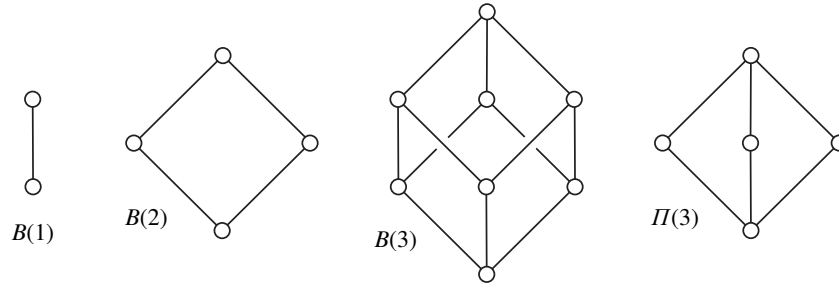
## 1.2. Geometric and arrangement lattices

A graded atomic lattice is *geometric* if the rank function satisfies

$$rk(x \vee y) + rk(x \wedge y) \leq rk(x) + rk(y) \tag{1}$$

for all  $x$  and  $y$ . The Boolean, partition and arrangement lattices above are geometric; the face lattices of polytopes are usually not. If  $L$  is geometric then for a given atom  $a$ , both  $L_a$  and  $L^a$  are again geometric lattices. The restriction  $L^a$  has atoms  $A^a = \{a \vee b : b \in A_a\}$ , minimum  $\mathbf{0}^a = a$ , maximum  $\mathbf{1}^a = \mathbf{1}$  and rank function  $rk^a = rk - 1$ .

Our main supply of geometric lattices will come from (linear) hyperplane arrangements. Let  $V$  be a finite dimensional vector space over a field  $k$ ; then an *arrangement* in  $V$  is a finite set  $A = \{a_i\}$  of linear hyperplanes in  $V$ . The *arrangement lattice*  $L = L(A)$  has elements all possible



**Fig. 1.** The arrangement lattices  $L(A)$  where  $|A| \leq 3$ : the Booleans  $B(n)$  for  $n = 1, 2, 3$  and the partition lattice  $\Pi(3)$ .

intersections of hyperplanes in  $A$  – with the empty intersection taken to be  $V$  – and is ordered by *reverse inclusion*. Then  $L$  is a geometric lattice with atoms the hyperplanes  $A$ , and

$$\mathbf{0} = V, \quad \mathbf{1} = \bigcap_{a \in A} a, \quad rk(x) = \text{codim } x, \quad x \vee y = x \cap y, \quad \text{and } x \wedge y = \bigcap \{z \in L : x \cup y \subseteq z\}$$

Given  $a \in A$ , the deletion lattice  $L_a$  is the arrangement lattice  $L(A_a)$ , and similarly the restriction lattice  $L^a$  is the arrangement lattice  $L(A^a)$ .

Some examples:

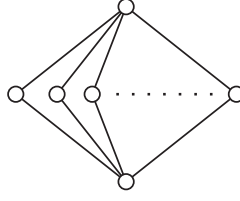
- Let  $v_1, v_2, \dots, v_n$  be a basis for  $V$  with corresponding coordinate functions  $x_1, x_2, \dots, x_n$ . The *coordinate arrangement*  $A$  consists of the hyperplanes having equations  $x_i = 0$ , for  $1 \leq i \leq n$ . The arrangement lattice  $L(A)$  is isomorphic to the Boolean lattice  $B(A)$  via the map  $x \in B(A) \mapsto \bigcap_{i \in x} \{x_i = 0\} \in L(A)$ .
- The symmetric group  $S_n$  acts on  $V$  by permuting basis vectors:  $\pi \cdot v_i = v_{\pi \cdot i}$  for  $\pi \in S_n$ . This realises  $S_n$  as a reflection group where the reflecting hyperplanes are those with equations  $x_i - x_j = 0$  for all  $1 \leq i \neq j \leq n$ . Collectively they form the *braid arrangement*  $A$  – so called, when  $k = \mathbb{C}$ , as the space  $V \setminus \bigcup_{a \in A} a$  has fundamental group the (pure) braid group on  $n$  strands. The arrangement lattice  $L(A)$  is isomorphic to the partition lattice  $\Pi(A)$  via the map induced by  $x_i - x_j = 0 \in L(A)$  maps to the partition with just one block  $\{i, j\}$  not having size one.
- More generally, if  $W \subset GL(V)$  is any finite reflection group, then the reflecting hyperplanes of  $W$  form a *reflectional arrangement*.

When  $|A| = 1$  or  $2$ , the only possibility for  $L$  is that it be Boolean of rank  $|A|$ . The arrangement lattices with 3 or fewer hyperplanes are shown in Figure 1. The first three are Boolean and the last is the partition lattice of a 3-element set. An arrangement lattice of rank 2 has the form shown in Figure 2, arising from an arrangement of lines in the plane.

An arrangement  $A$  in the space  $V$  is *essential* when  $rk(L) = \dim V$ , or equivalently,  $\bigcap_{a \in A} a$  is the zero space. The *characteristic polynomial*  $\chi = \chi_A$  of the arrangement  $A$  is defined by

$$\chi(t) = \sum_{x \in L} \mu(x) t^{\dim x}$$

where  $\mu(x)$  is the value of the Möbius function of the associated arrangement lattice  $L$  on the interval  $[\mathbf{0}, x]$ , i.e.  $\mu(x) = \mu(\mathbf{0}, x)$ .



**Fig. 2.** An arrangement lattice of rank 2 arising from an arrangement of lines in the plane.

### 1.3. Dependence

There is a notion of independence in a lattice that mimics linear algebra. Let  $L$  be a graded atomic lattice with atoms  $A$  and write  $\bigvee S$  for the join of the elements in a subset  $S \subseteq A$ . A set  $S \subset A$  of atoms is *independent* if  $\bigvee T < \bigvee S$  for all proper subsets  $T$  of  $S$ , and dependent otherwise. An atom  $a$  in a dependent set of atoms  $S$  with the property that  $\bigvee S \setminus \{a\} = \bigvee S$  is called a *dependent atom*. It is easy to show [EF13, §1.1] that if  $S$  is dependent then there is an independent  $T \subset S$  with  $\bigvee T = \bigvee S$ , and that any subset of an independent set is independent.

**Proposition 1.** *Let  $L$  be a graded atomic lattice with independent atoms  $A$ . Then  $L$  is isomorphic to the Boolean lattice  $B(A)$ .*

Birkhoff [Bir79, IV.4, Theorem 5] proves this for  $L$  a geometric lattice.

*Proof.* In  $B = B(A)$  any element has a unique expression as a join of atoms. Since  $B$  and  $L$  share the same set of atoms and each element in  $L$  may be written as a join of atoms, there is a canonical surjection  $f : B \rightarrow L$  given by

$$f : \bigvee_B a_i \mapsto \bigvee_L a_i.$$

We show that  $f$  is injective and that  $f^{-1}$  is a poset map, hence  $f$  is an isomorphism. Both follow from the following claim: if  $x, y \in L$  and  $x = a_1 \vee \cdots \vee a_k$ ,  $y = a'_1 \vee \cdots \vee a'_\ell$  are any expressions as joins of atoms, then  $x \leq y$  if and only if  $\{a_1, \dots, a_k\} \subseteq \{a'_1, \dots, a'_\ell\}$ . To prove the “only if” part of the claim, let  $x \leq y$  and suppose that  $a_i \notin \{a'_1, \dots, a'_\ell\}$  for some  $i$ . Then

$$a_1 \vee \cdots \vee a_k \vee a'_1 \vee \cdots \vee a'_\ell = x \vee y = y = a'_1 \vee \cdots \vee a'_\ell,$$

and after removing redundancies on the left (as joins commute and  $a \vee a = a$  for all  $a$ ) the right hand join of atoms is a proper subset of the left hand join of atoms. Taking the join of both sides with those atoms that are not any of the  $a_j$  or  $a'_j$  gives  $\bigvee A$  on the left and  $\bigvee A'$  on the right, for some  $A'$  a proper subset of  $A$ . This contradicts the independence of  $A$ . The “if” part of the claim is obvious.

Now let  $S = \{s_i\}$  and  $T = \{t_i\}$  be subsets of atoms with  $\bigvee f s_i = \bigvee f t_i$ . Then by the claim we have  $\{f s_i\} = \{f t_i\}$  and hence  $S = T$  as  $f$  is the identity map on  $A$ . Thus,  $f$  is injective. A similar argument shows that  $f^{-1}$  is a poset map.  $\square$

In a geometric lattice  $L$  with atom set  $A$  we have  $rk(L) = rk(\bigvee A) \leq |A|$ . Moreover,  $A$  is independent if and only if  $rk(\bigvee A) = |A|$ , so the above shows that  $rk(L) = |A|$  if and only if  $L$  is Boolean.

**Corollary 1.** *Let  $L$  be a non-Boolean geometric lattice with  $|A|$  hyperplanes. Then there exists a dependent atom  $a$  such that*

- *the deletion  $L_a$  has  $|A| - 1$  atoms and  $rk(L_a) = rk(L)$ ;*
- *the restriction  $L^a$  has at most  $|A| - 1$  atoms and  $rk(L^a) = rk(L) - 1$ .*

## 2. Sheaf homology

In §§2.1-2.2 we recall the basics of sheaves on posets and the resulting homologies – standard references are [GZ67, God73, Qui78, Qui73]. In §2.3 we recall a convenient tool for calculating homology: a Leray-Serre spectral sequence for which we reference [GZ67, Appendix II]. In §2.4 we recall the notion of reduced homology.

### 2.1. Sheaves

Let  $R$  be a commutative ring with 1. A *sheaf*<sup>1</sup> on a poset  $P$  is a *contravariant* functor

$$F : P \rightarrow {}_R\mathbf{Mod}$$

to the category of  $R$ -modules, where  $P$  is interpreted as a category in the usual way. The category of sheaves on  $P$  has objects the sheaves  $F$  and morphisms the natural transformations of functors  $\kappa : F \rightarrow G$ . We write  $F_x^y$  for the homomorphism, or *structure map*, of the sheaf given by  $F(x \leq y) : F(y) \rightarrow F(x)$ . Two important examples of sheaves are:

- For  $M \in {}_R\mathbf{Mod}$  the *constant sheaf*  $\Delta M$  is defined by  $\Delta M(x) = M$  for every  $x \in P$  and  $(\Delta M)_x^y = 1$  for every  $x \leq y$  in  $P$ .
- If  $L = L(A)$  is the intersection lattice of a hyperplane arrangement  $A$ , then the *natural sheaf* on  $L$  has  $F(x)$  just the space  $x$  itself, and for  $x \leq y$  in  $L$ , the structure map  $F_x^y$  is the inclusion of spaces  $y \hookrightarrow x$ .

If  $f : Q \rightarrow P$  is a map of posets and  $F$  is a sheaf on  $P$ , then there is an induced sheaf on  $Q$  given by  $f^*F := F \circ f$ .

### 2.2. Homology

For any sheaf  $F$  on  $P$  the *colimit*  $\varinjlim^P F$  is constructed by taking the quotient of  $\bigoplus_{x \in P} F(x)$  by the submodule generated by all elements of the form  $a_y - F_x^y(a_y)$  where  $x \leq y$  and  $a_y \in F(y)$ . Taking colimits is right but not left exact, which earns them the privilege of left derived functors. These are called *higher colimits* and are denoted

$$\varinjlim_*^P := L_* \varinjlim^P .$$

If  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  is a short exact sequence of sheaves then there is a long exact sequence of modules:

$$\cdots \rightarrow \varinjlim_i^P F \rightarrow \varinjlim_i^P G \rightarrow \varinjlim_i^P H \rightarrow \cdots \rightarrow \varinjlim^P F \rightarrow \varinjlim^P G \rightarrow \varinjlim^P H \rightarrow 0 \quad (2)$$

<sup>1</sup> Strictly speaking we should say presheaf rather than sheaf, but as our posets are discrete (if one wishes to view them as topological objects) there is essentially no difference between presheaves and sheaves.



The homology of  $P$  with coefficients in the sheaf  $F$  are the higher colimits evaluated at the sheaf  $F$ .

Homology can be computed using an explicit chain complex in the following way (details may be found in [GZ67, Appendix II]). Recall that the *order complex* (or *nerve*)  $|P|$  of the poset  $P$  is the simplicial complex whose vertices are the elements of  $P$  and whose  $n$ -simplices are the chains

$$\sigma = x_n \leq \cdots \leq x_0. \quad (3)$$

Let  $S_*(P; F)$  be the chain complex whose group of  $n$ -chains is

$$S_n(P; F) = \bigoplus_{\sigma} F(x_0)$$

the direct sum over the  $n$ -simplices (3) of  $|P|$ . If  $\sigma$  is an  $n$ -simplex and  $s \in F(x_0)$ , then we will write  $s_{\sigma}$  for the element of  $S_n$  that has value  $s$  in the component indexed by  $\sigma$  and value 0 in all other components. The differential in  $S_*(P; F)$  is defined as follows. If

$$d_i \sigma = x_n \leq \cdots \leq \widehat{x}_i \leq \cdots \leq x_0 \text{ and } s_i \sigma = x_n \leq \cdots \leq x_i \leq x_i \leq \cdots \leq x_0 \quad (4)$$

for  $0 \leq i \leq n$ , then  $d : S_n(P; F) \rightarrow S_{n-1}(P; F)$  is given by

$$ds_{\sigma} = F_{x_1}^{x_0}(s)_{d_0 \sigma} + \sum_{i=1}^n (-1)^i s_{d_i \sigma}. \quad (5)$$

The higher colimits may be computed as the homology of this complex:

$$H_*(P; F) = \varinjlim^P F \cong H(S_*(P; F)).$$

In the special case of the constant sheaf  $F = \Delta M$ , the homology is just the ordinary simplicial homology of  $|P|$ :

$$H_*(P; \Delta M) \cong H_*(|P|, M). \quad (6)$$

If  $f : Q \rightarrow P$  is a map of posets and  $F$  is a sheaf on  $P$ , then there is a chain map  $S_*(Q; f^* F) \rightarrow S_*(P; F)$  induced by  $s_{\sigma} \mapsto s_{f\sigma}$ . In particular, if  $f : Q \hookrightarrow P$  is an inclusion, then  $f^* F$  is just the restriction of the sheaf  $F$  to the subposet  $Q$  (in which case we will simply write  $F$  for the restricted sheaf too) and  $S_*(Q; F)$  is a subcomplex of  $S_*(P; F)$ .

There is a variation on the complex  $S_*$  which uses only non-degenerate simplices. The group of  $n$ -chains is

$$T_n(P; F) = \bigoplus_{\sigma} F(x_0)$$

where this time the sum is over non-degenerate simplices  $\sigma = x_n < \cdots < x_0$ , and the differential is once again given by formula (5). Then  $T_*(P; F)$  is a subcomplex of  $S_*$ . If  $g : T_* \rightarrow S_*$  is the inclusion and  $f : S_* \rightarrow T_*$  the map with  $f(a_{\sigma}) = a_{\sigma}$  when  $\sigma$  is non-degenerate, and  $f(a_{\sigma}) = 0$  otherwise, then  $fg$  is the identity on  $T_*$ . Let  $h : S_n \rightarrow S_{n+1}$  be defined by

$$h(a_{\sigma}) = \begin{cases} (-1)^p a_{s_p \sigma}, & \text{when } \sigma = x_n \dots x_{p+\ell} x_p \dots x_p x_{p-1} \dots x_0 \\ 0, & \text{else} \end{cases}$$

where there are  $\ell \geq 2$  copies of  $x_p$  in the first case and  $s_p \sigma$  is given by (4). Then a standard check shows that  $h$  is a (chain) homotopy between  $gf$  and the identity on  $S_*$ . The complexes  $S_*$  and  $T_*$  are thus homotopy equivalent.

The following lemma gathers together some small results needed later.

- Lemma 1.** 1. If  $P$  is a finite graded poset, then  $H_i(P; F) \neq 0$  only if  $0 \leq i \leq \text{rk}(P)$ .  
 2. If  $P$  has a minimum or maximum, and  $\Delta M$  is a constant sheaf, then  $H_0(P; \Delta M) = A$  and  $H_i(P; \Delta M)$  vanishes for  $i > 0$ .  
 3. If  $P$  has a minimum  $\mathbf{0}$ , and  $F$  is any sheaf, then  $H_0(P; F) = F(\mathbf{0})$  and  $H_i(P; F)$  vanishes for  $i > 0$ .  
 4. If  $P$  has a maximum  $\mathbf{1}$ , and  $F$  is a sheaf on  $P$  such that  $F(\mathbf{1}) = 0$ , then  $H_*(P; F)$  is isomorphic to  $H_*(P \setminus \mathbf{1}; F)$ .

*Proof.* Part 1 follows immediately from the existence of  $T_*$ ; part 2 follows from (6) and the fact that  $|P|$  is contractible, as it is a cone on  $|P \setminus x|$ , where  $x$  is the maximum or minimum. In the presence of a minimum the colimit functor is naturally isomorphic to the evaluation functor  $F \mapsto F(\mathbf{0})$ , which is exact, hence part 3. Finally, the complexes  $S_*(P \setminus \mathbf{1}; F)$  and  $S_*(P; F)$  are identical when  $F(\mathbf{1}) = 0$ , hence part 4.  $\square$

*Remark:* If  $P$  has a maximum but no minimum then, according to the Lemma, homology with constant coefficients  $H_*(P; \Delta M)$  still vanishes in every non-zero degree. However, this is far from the case when one allows more interesting sheaves  $F$ . In general  $H_*(P; F)$  can be almost arbitrarily complicated.

### 2.3. The Leray-Serre spectral sequence

There is a spectral sequence for higher colimits given in [GZ67, Appendix II, Theorem 3.6]; the following is the specialisation of this result from small categories to posets.

Let  $f: P \rightarrow Q$  be a poset map and let  $F$  be a sheaf on  $P$ . For each  $q \geq 0$  define a sheaf  $H_q^{\text{fib}}$  on  $Q$  by

$$H_q^{\text{fib}}(x) = H_q(f^{-1}Q_{\geq x}; F)$$

where the sheaf denoted  $F$  on the right is the restriction of  $F$  to  $f^{-1}Q_{\geq x} \subset P$ . If  $x \leq y$  in  $Q$  then the structure map  $H_q^{\text{fib}}(y) \rightarrow H_q^{\text{fib}}(x)$  is induced by the inclusion  $Q_{\geq y} \hookrightarrow Q_{\geq x}$ .

**Theorem 1 (Leray-Serre).** *There is a spectral sequence*

$$E_{p,q}^2 = H_p(Q; H_q^{\text{fib}}) \Rightarrow H_{p+q}(P; F)$$

We warn the reader that the sheaves in [GZ67, Appendix II] are covariant, so the translation requires a number of headstands. The sequence is a special case of the results in [Gro57], where Grothendieck gives a spectral sequence that converges to the derived functors of a composite of two functors.

The following corollary is a homological version of the Quillen fibre lemma [Qui78], which states that if  $f: P \rightarrow Q$  is a poset map such that for all  $x \in Q$ , the fiber  $f^{-1}Q_{\geq x}$  is contractible, then  $f$  is a homotopy equivalence.

**Corollary 2.** *Let  $f: P \rightarrow Q$  be a surjective poset map, let  $G$  be a sheaf on  $Q$  and let  $F = f^*G$  be the induced sheaf on  $P$ . Suppose that for all  $x \in Q$  the homology  $H_*(f^{-1}Q_{\geq x}; F)$  vanishes outside degree 0 and  $H_0(f^{-1}Q_{\geq x}; F) \cong G(x)$ . Then there is an isomorphism*

$$H_*(P; F) \cong H_*(Q; G)$$





We finish the proof by showing that  $H_i(L_1; F) \cong H_i(L_a \setminus \mathbf{0}; F)$  for all  $i$ . For this we apply the Leray-Serre spectral sequence to the map  $t : L_1 \rightarrow L_a \setminus \mathbf{0}$  of Lemma 2. The spectral sequence is of the form

$$E_{p,q}^2 = H_p(L_a \setminus \mathbf{0}; H_q^{\text{fib}}) \Rightarrow H_{p+q}(L_1; F)$$

where for  $x \in L_a \setminus \mathbf{0}$ ,

$$H_q^{\text{fib}}(x) = H_q(t^{-1}((L_a \setminus \mathbf{0})_{\geq x}); F).$$

By Lemma 2, the poset  $t^{-1}((L_a \setminus \mathbf{0})_{\geq x})$  has a minimum, so Lemma 1 part 3 then gives

$$H_q^{\text{fib}}(x) = \begin{cases} F(x) & q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the spectral sequence has a single row ( $q = 0$ ) on which  $E_{p,0}^2 = H_p(L_a \setminus \mathbf{0}; F)$ . The sequence thus collapses at the  $E^2$ -page, and we conclude that  $H_p(L_a \setminus \mathbf{0}; F) \cong H_p(L_1; F)$ .  $\square$

The final step in the proof is a direct use of the Leray-Serre spectral sequence, rather than an application of Corollary 2, because the induced sheaf  $t^*F$  on  $L_1$  is not the one we want.

We state as a corollary a special case that we will use on hyperplane arrangements in the next section.

**Corollary 3 (Reduced Deletion-Restriction Long Exact Sequence).** *Let  $L$  be a graded atomic lattice equipped with a sheaf  $F$ . Let  $a \in L$  be an atom such that  $\epsilon_* : \varinjlim^{L^a \setminus a} F \rightarrow F(a)$  is a surjection. Then, there is a long exact sequence*

$$\begin{array}{ccccccc} \dots & \rightarrow & \widetilde{H}_i(L^a \setminus a; F) & \rightarrow & \widetilde{H}_i(L_a \setminus \mathbf{0}; F) & \rightarrow & \widetilde{H}_i(L \setminus \mathbf{0}; F) & \rightarrow & \widetilde{H}_{i-1}(L^a \setminus a; F) & \rightarrow & \widetilde{H}_{i-1}(L_a \setminus \mathbf{0}; F) & \rightarrow & \dots \\ \curvearrowright & & & & & & & & & & & & & \curvearrowleft \\ \dots & \rightarrow & \widetilde{H}_1(L \setminus \mathbf{0}; F) & \rightarrow & \widetilde{H}_0(L^a \setminus a; F) & \rightarrow & \widetilde{H}_0(L_a \setminus \mathbf{0}; F) & \rightarrow & \widetilde{H}_0(L \setminus \mathbf{0}; F) & \rightarrow & 0 & & \end{array}$$

*Proof.* Consider the long exact sequence (\*) in the proof of Theorem 2, and let

$$f : \widetilde{H}_0(L^a \setminus a; F) \rightarrow H_0(L_1; F) \text{ and } g : H_0(L_1; F) \rightarrow H_0(L \setminus \mathbf{0}; F).$$

One can then show that  $\text{im } f \subseteq \widetilde{H}_0(L_1; F) \subset H_0(L_1; F)$ . Now restrict  $g$  to  $\tilde{g} : \widetilde{H}_0(L_1; F) \rightarrow H_0(L \setminus \mathbf{0}; F)$ . One then gets that  $\text{im } f = \ker \tilde{g}$  and so  $H_0(L_1; F)$  can be replaced by  $\widetilde{H}_0(L_1; F)$  in the long exact sequence. Similarly  $\tilde{g}$  maps  $\widetilde{H}_0(L_1; F)$  onto  $\widetilde{H}_0(L \setminus \mathbf{0}; F)$ , so we can also replace the last term in the sequence with its reduced version (the final  $\text{coker}(\epsilon_*)$  is already 0 by the assumption in the Corollary). Then continue as in the proof of Theorem 2, replacing  $\widetilde{H}_0(L_1; F)$  by  $\widetilde{H}_0(L_a \setminus \mathbf{0}; F)$ . All the other terms in the sequence (\*) are automatically equal to their reduced versions.  $\square$

#### 4. Application to hyperplane arrangements

In this section  $L = L(A)$  is the intersection lattice of a hyperplane arrangement  $A$  in the vector space  $V$ , and  $F$  is the natural sheaf on  $L$  (see §2.1).

#### 4.1. Reduced homology

Our goal is to compute  $\widetilde{H}_i(L \setminus \mathbf{0}; F)$ , and our main tool is Corollary 3, the reduced deletion-restriction long exact sequence. To apply it we need the following small result.

**Lemma 3.** *Let  $L$  be the intersection lattice of a hyperplane arrangement with  $\text{rk}(L) \geq 2$  and let  $F$  be the natural sheaf on  $L$ . Then the map  $\epsilon_*: \varinjlim^{L \setminus \mathbf{0}} F \rightarrow F(\mathbf{0})$  induced by the  $F_x^x: F(x) \rightarrow F(\mathbf{0})$ , for  $x \in L \setminus \mathbf{0}$ , is surjective.*

*Proof.* Since  $\text{rk}(L) \geq 2$ , the arrangement has at least two distinct hyperplanes, whose vector space sum is  $F(\mathbf{0})$ . The result follows immediately from the definition of colimit.  $\square$

For any atom  $a$  in an arrangement lattice  $L$ , the restriction  $L^a$  is also an arrangement lattice with minimum  $a$ ; in particular  $L^a$  is graded atomic and  $\epsilon_*: \varinjlim^{L^a \setminus a} F \rightarrow F(a)$  is a surjection, so we can use the long exact sequence of Corollary 3 to make inductive arguments. Throughout this section we will therefore use reduced homology.

We begin with the special cases of rank 2 lattices and of Boolean lattices.

**Proposition 2.** *Let  $L = L(A)$  be the intersection lattice of a hyperplane arrangement with  $\text{rk}(L) = 2$  and let  $F$  be the natural sheaf on  $L$ . Then  $\widetilde{H}_i(L \setminus \mathbf{0}; F)$  is trivial when  $i \neq 0$  and  $\dim \widetilde{H}_0(L \setminus \mathbf{0}; F) = |A| - 2$ .*

*Proof.* The homology is concentrated in degrees 0 and 1. The complex  $T_*$  of §2.2 can be written out explicitly, from which it is easily seen that  $d: T_1 \rightarrow T_0$  is injective, hence  $H_1(T) = 0$ . Moreover

$$\dim T_0 = |A|(\dim V - 1) + (\dim V - 2) \text{ and } \dim T_1 = |A|(\dim V - 2)$$

so that

$$\dim H_0(T) = \dim T_0 - \dim(\text{im } d) = \dim T_0 - \dim T_1 = \dim V + |A| - 2$$

The augmentation  $\epsilon_*: H_0(T) \rightarrow V$  is surjective by Lemma 3, so that

$$\dim \widetilde{H}_0 = \dim \ker \epsilon_* = \dim H_0(T) - \dim V = |A| - 2. \quad \square$$

**Proposition 3.** *Let  $B$  be a Boolean lattice that is the intersection lattice of a hyperplane arrangement with  $\text{rk}(B) \geq 2$ , and let  $F$  be the natural sheaf on  $B$ . Then  $\widetilde{H}_*(B \setminus \mathbf{0}; F)$  is trivial.*

*Proof.* We use induction on the number  $|A|$  of hyperplanes, which in the Boolean case equals the rank  $\text{rk}(B)$ .

The base case,  $\text{rk}(B) = 2$ , follows from Proposition 2, so suppose  $\text{rk}(B) > 2$ . For any hyperplane  $a \in A$  the deletion  $B_a$  and restriction  $B^a$  are again Boolean, and of rank  $\text{rk}(B) - 1$ . Thus  $\widetilde{H}_*(B_a \setminus \mathbf{0}; F) = 0$  and  $\widetilde{H}_*(B^a \setminus \mathbf{0}; F) = 0$  by induction. The result then follows from the reduced deletion-restriction long exact sequence.  $\square$

We now state and prove our main application:

**Theorem 3.** *Let  $L$  be the intersection lattice of a hyperplane arrangement with  $\text{rk}(L) \geq 2$  and let  $F$  be the natural sheaf on  $L$ . Then  $\widetilde{H}_i(L \setminus \mathbf{0}; F)$  is trivial when  $i \neq \text{rk}(L) - 2$  and*

$$\dim \widetilde{H}_{\text{rk}(L)-2}(L \setminus \mathbf{0}; F) = (-1)^{\text{rk}(L)-1} \left. \frac{d}{dt} \chi(t) \right|_{t=1}$$

where  $\chi(t)$  is the characteristic polynomial of  $L$ .

*Proof.* If  $L$  has rank 2 and  $\dim V = n$  then the characteristic polynomial is

$$\chi(t) = \sum_{x \in L} \mu(\mathbf{0}, x) t^{\dim x} = t^n - |A|t^{n-1} + (|A| - 1)t^{n-2}$$

and we easily calculate

$$(-1)^{rk(L)-1} \frac{d}{dt} \chi(t) \Big|_{t=1} = |A| - 2.$$

This, and Proposition 2, proves the theorem for rank 2 lattices.

If  $L$  is Boolean of rank  $r > 2$  and  $\dim V = n \geq r$ , then the characteristic polynomial is

$$\chi(t) = t^{n-r}(t-1)^r.$$

The derivative of  $\chi(t)$  vanishes at  $t = 1$ , so this and Proposition 3 prove the theorem for Boolean lattices.

We now proceed by induction on the number  $|A|$  of hyperplanes, where we assume that  $rk(L) \geq 3$ . If  $|A| = 2$  then  $rk(L) \leq 2$ , so we take as our base case  $|A| = 3$ :

– *The base case  $|A| = 3$ .* As  $rk(L) \geq 3$ , then §1.2 shows that the only possibility for  $L$  is that it be Boolean of rank 3, and the theorem has already been proved in this case.

– *The vanishing degrees when  $|A| > 3$ .* We may assume that  $L$  is non-Boolean of rank  $\geq 3$  and  $|A| > 3$  – though being non-Boolean is not part of the inductive hypothesis.

Corollary 1 guarantees that the non-Boolean  $L$  has a dependent atom  $a \in A$ , so the deletion  $L_a$  is an arrangement lattice with  $|A| - 1$  hyperplanes and  $rk(L_a) = rk(L) \geq 3$ . Thus, the inductive hypothesis, and hence the result, holds for  $L_a$ .

Corollary 1 again gives the restriction  $L^a$  is an arrangement lattice with at most  $|A| - 1$  hyperplanes and  $rk(L^a) = rk(L) - 1$ . If  $rk(L) = 3$  then the result holds for  $L^a$  by Proposition 2. If  $rk(L) > 3$  then  $rk(L^a) \geq 3$ , and  $L^a$  must contain at least 3 hyperplanes; the result then holds for  $L^a$  by induction.

The reduced deletion-restriction long exact sequence

$$\cdots \rightarrow \widetilde{H}_i(L_a \setminus \mathbf{0}; F) \rightarrow \widetilde{H}_i(L \setminus \mathbf{0}; F) \rightarrow \widetilde{H}_{i-1}(L^a \setminus a; F) \rightarrow \cdots$$

then has  $\widetilde{H}_i(L_a \setminus \mathbf{0}; F)$  trivial for  $i \neq rk(L) - 2$  and  $\widetilde{H}_{i-1}(L^a \setminus a; F)$  trivial for  $i - 1 \neq rk(L^a) - 2$ , or equivalently, for  $i \neq rk(L) - 2$ . Thus,  $\widetilde{H}_i(L \setminus \mathbf{0}; F) = 0$  for  $i \neq rk(L) - 2$ .

– *The dimension in degree  $rk(L) - 2$ .* Let  $\theta$  be an integer-valued function, defined on arrangement lattices of rank  $\geq 2$ , that satisfies the following three properties:

- (1)  $\theta(L) = |A| - 2$ , if  $L$  is a rank 2 lattice with  $|A|$  atoms;
- (2)  $\theta(L) = 0$ , if  $L$  is Boolean;
- (3)  $\theta(L) = \theta(L_a) + \theta(L^a)$ , where  $a$  is a dependent atom in  $L$ .

If such a function exists it is unique: indeed by Corollary 1 we may continue to apply the recursive relation (3) until we find Boolean lattices – whose values are given by (2) – or rank 2 lattices, whose values are given by (1).

Let

$$\Phi(L) = \dim \widetilde{H}_{rk(L)-2}(L \setminus \mathbf{0}; F).$$

We claim that  $\Phi$  satisfies (1), (2) and (3) above. Courtesy of Proposition 2, we have  $\Phi(L) = |A| - 2$  when  $L$  has rank 2 – hence (1) – and Proposition 3 gives  $\Phi(L) = 0$  for Booleans, so (2) is also satisfied. The vanishing degrees above leaves only the short exact fragment:

$$0 \rightarrow \widetilde{H}_{rk(L_a)-2}(L_a \setminus \mathbf{0}; F) \rightarrow \widetilde{H}_{rk(L)-2}(L \setminus \mathbf{0}; F) \rightarrow \widetilde{H}_{rk(L^a)-2}(L^a \setminus a; F) \rightarrow 0$$

of the deletion-restriction long exact sequence, where  $rk(L_a) = rk(L)$  and  $rk(L^a) = rk(L) - 1$ . We immediately see that  $\Phi$  satisfies (3).

Now define

$$\Psi(L) = (-1)^{rk(L)-1} \left. \frac{d}{dt} \chi(t) \right|_{t=1}$$

We have already calculated  $\Psi(L)$  at the beginning of the proof for rank two lattices and for Booleans, showing  $\Psi$  satisfies (1) and (2) above. Furthermore, the characteristic polynomial satisfies the relation:

$$\chi_L(t) = \chi_{L_a}(t) - \chi_{L^a}(t)$$

from which it follows that

$$(-1)^{rk(L)-1} \chi_L(t) = (-1)^{rk(L_a)-1} \chi_{L_a}(t) + (-1)^{rk(L^a)-1} \chi_{L^a}(t).$$

Differentiating and specialising to  $t = 1$  shows that  $\Psi$  also satisfies (3). By uniqueness we conclude that  $\Phi = \Psi$ , giving the dimension in degree  $rk(L) - 2$  to be as claimed.  $\square$

#### 4.2. Unreduced homology

It is easy to compute unreduced homology from the above. Reduced and unreduced only differ in degree zero where we have a short exact sequence

$$0 \rightarrow \widetilde{H}_0(L \setminus \mathbf{0}; F) \rightarrow H_0(L \setminus \mathbf{0}; F) \rightarrow V \rightarrow 0.$$

We immediately get

**Proposition 4.** *Let  $L$  be the intersection lattice of a hyperplane arrangement with  $rk(L) \geq 2$  and let  $F$  be the natural sheaf on  $L$ . Then  $H_i(L \setminus \mathbf{0}; F)$  is trivial when  $i \neq 0$  or  $rk(L) - 2$ . Moreover,*

- *If  $rk(L) > 2$  we have  $H_0(L \setminus \mathbf{0}; F) \cong V$  and the potentially non-trivial group in degree  $rk(L) - 2$  has the dimension given in Theorem 3.*
- *If  $rk(L) = 2$  we have  $\dim H_0(L \setminus \mathbf{0}; F) = |A| - 2 + \dim V$ .*

#### 4.3. Generalising a result of Lusztig

When using constant coefficients, the homology of a poset with a maximum is concentrated in degree zero for general reasons (see Lemma 1). To avoid this collapse the maximum is normally removed before taking homology. The same is true when the poset has a minimum. For a more general sheaf the presence of a maximum does not *a priori* concentrate the homology in this way. Nonetheless, for consistency it is tempting to remove the maximum in this case too, as in the following celebrated result of Lusztig [Lus74, Theorem 1.12].



**Theorem 4. (Lusztig)** *Let  $V$  be a vector space over a finite field of dimension  $\geq 3$  and let  $A$  be the arrangement consisting of all the hyperplanes in  $V$ . Let  $L = L(A)$  be the associated arrangement lattice and  $F$  be the natural sheaf. Then  $H_i(L \setminus \{\mathbf{0}, \mathbf{1}\}; F)$  vanishes in the degrees  $0 < i < \text{rk}(L) - 2$  and  $H_0(L \setminus \{\mathbf{0}, \mathbf{1}\}; F) \cong V$ .*

In this section we make explicit the connection between our Theorem 3 and Lusztig's result. In particular we describe  $H_*(L \setminus \{\mathbf{0}, \mathbf{1}\}; F)$  for any arrangement lattice  $L$  equipped with the natural sheaf  $F$ .

Recall §1.2 that an arrangement is essential when  $\bigcap_{a \in A} a = 0$ . In particular, for  $F$  the natural sheaf on  $L$ , we have  $F(\mathbf{1}) = 0$ , and so by Lemma 1 part 4 we get  $H_*(L \setminus \{\mathbf{0}, \mathbf{1}\}; F) \cong H_*(L \setminus \mathbf{0}; F)$ . As the arrangement in Lusztig's result is essential, Theorem 4 follows immediately from Theorem 3 and Proposition 4. In fact we get more than is claimed in Theorem 4 as we give the dimension of the top degree homology as well.

We are also interested in non-essential hyperplane arrangements, where  $\bigcap_{a \in A} a \neq 0$ . The following recasts our Theorem 3 in a way that it can be directly seen as a generalisation of Lusztig's result.

**Theorem 5.** *Let  $L$  be the intersection lattice of a hyperplane arrangement  $A$  in the vector space  $V$  and let  $U = \bigcap_{a \in A} a$ . Suppose that  $\text{rk}(L) \geq 3$  and let  $F$  be the natural sheaf on  $L$ . Then  $H_i(L \setminus \{\mathbf{0}, \mathbf{1}\}; F)$  vanishes in degrees  $0 < i < \text{rk}(L) - 2$  with  $H_0(L \setminus \{\mathbf{0}, \mathbf{1}\}; F) \cong V \oplus U$  and*

$$\dim H_{\text{rk}(L)-2}(L \setminus \{\mathbf{0}, \mathbf{1}\}; F) = (-1)^{\text{rk}(L)-1} \left. \frac{d}{dt} \chi(t) \right|_{t=1} + |\mu(\mathbf{0}, \mathbf{1})| \dim U.$$

*Proof.* Define a new sheaf  $F'$  on  $L \setminus \mathbf{0}$  by

$$F'(x) = \begin{cases} \mathbf{0}, & x = \mathbf{1} \\ F(x), & x \neq \mathbf{1} \end{cases}$$

with obvious structure maps induced from  $F$ . As  $F'$  is essential, Lemma 1 part 4 gives

$$H_*(L \setminus \mathbf{0}; F') \cong H_*(L \setminus \{\mathbf{0}, \mathbf{1}\}; F') = H_*(L \setminus \{\mathbf{0}, \mathbf{1}\}; F)$$

To prove the result we must therefore compute  $H_*(L \setminus \mathbf{0}; F')$ . There is a short exact sequence of sheaves

$$0 \rightarrow F' \rightarrow F \rightarrow G \rightarrow 0$$

where  $G$  is the sheaf on  $L \setminus \mathbf{0}$  defined by  $G(\mathbf{1}) = U$  and  $G(x) = 0$  otherwise. By (2) this gives a long exact sequence of homology groups

$$\cdots \rightarrow H_{i+1}(L \setminus \mathbf{0}; G) \rightarrow H_i(L \setminus \mathbf{0}; F') \rightarrow H_i(L \setminus \mathbf{0}; F) \rightarrow H_i(L \setminus \mathbf{0}; G) \rightarrow H_{i-1}(L \setminus \mathbf{0}; F') \rightarrow \cdots$$

We can identify the complex  $S_*(L \setminus \mathbf{0}; G)$  with the complex  $S_{*-1}(L \setminus \{\mathbf{0}, \mathbf{1}\}; \Delta U)$ , and we have  $H_i(L \setminus \mathbf{0}; G) = H_{i-1}(L \setminus \{\mathbf{0}, \mathbf{1}\}; \Delta U)$ , so that in particular  $H_0(L \setminus \mathbf{0}; G) = 0$ . The homology groups  $H_*(L \setminus \{\mathbf{0}, \mathbf{1}\}; \Delta U)$  are well known ([Fol66, Bjö82, OT92]) and it follows that

$$H_i(L \setminus \mathbf{0}; G) \cong H_{i-1}(L \setminus \{\mathbf{0}, \mathbf{1}\}; \Delta U) \cong \begin{cases} |U|^{\mu(\mathbf{0}, \mathbf{1})}, & i = \text{rk}(L) - 1 \\ U, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

From this, Proposition 4 and the long exact sequence above, we immediately get  $H_i(L \setminus \mathbf{0}; F')$  vanishes in the degrees  $0 < i < rk(L) - 2$ . In low degree and top degree we get short exact sequences from which the homology in degree zero and  $rk(L) - 2$  are easily shown to be as claimed.  $\square$

#### 4.4. Cellular homology and broken circuits

To a geometric lattice  $L$  one can associate a simplicial complex  $BC(L)$ , the *broken circuit complex* (see [Wil76, Bry77, Sta07]) that encodes some of the combinatorial geometry of the lattice. In this section we outline some connections between  $BC(L)$  and the sheaf homology of  $L$ . Our description of  $BC(L)$  follows [Sta07, Lectures 3-4].

Let  $L$  be a geometric lattice and fix a total ordering  $\preceq$  of the atoms  $A$ . Label the covering relation  $\prec$ , using the atoms, by defining

$$\lambda(x \prec y) = a \in A \quad (9)$$

where  $a$  is the maximum atom, with respect to the total order  $\preceq$ , with the property that  $x \vee a = y$ . The function (9) is an example of a  $\lambda$ -labelling; see [Sta07, Definition 4.11] for the definition of  $\lambda$ -labelling. The *broken circuit complex*  $BC(L)$  has vertices  $A$  and  $r$ -simplices the  $\{a_{i_0}, \dots, a_{i_r}\} \subseteq A$  whenever there is a saturated chain

$$\sigma = \mathbf{0} \prec x_0 \prec x_1 \prec \dots \prec x_r = x \quad (10)$$

with  $\lambda(\mathbf{0}, x_0) = a_{i_0}$ ,  $\lambda(x_0, x_1) = a_{i_1}, \dots, \lambda(x_{r-1}, x_r) = a_{i_r}$  and  $a_{i_0} \preceq a_{i_1} \preceq \dots \preceq a_{i_r}$ . Such a chain is said to be  $\lambda$ -increasing. The resulting complex  $BC(L)$  depends on the choice of total order  $\preceq$ , but it turns out that its homotopy type does not. Figure 3 illustrates these ideas for the partition lattice  $\Pi(4)$ .

The number of  $\lambda$ -increasing chains (10) is equal to  $(-1)^{rk(x)} \mu(\mathbf{0}, x)$  and  $\{a_{i_0}, \dots, a_{i_r}\}$  is a so-called “no-broken-circuit base” for the interval  $L_{\leq x}$ . It is not hard to see that  $BC(L)$  is a pure  $(rk(L) - 1)$ -dimensional simplicial complex, and that any maximal dimensional simplex contains the vertex that is maximal in the total ordering  $\preceq$  of  $A$ . In particular,  $BC(L)$  is a cone whose base is called the *reduced broken circuit complex*  $\widetilde{BC}(L)$ .

Since  $BC(L)$  is a cone, it is contractible, and thus has trivial reduced homology. On the other hand the reduced broken circuit complex  $\widetilde{BC}(L)$  has reduced homology

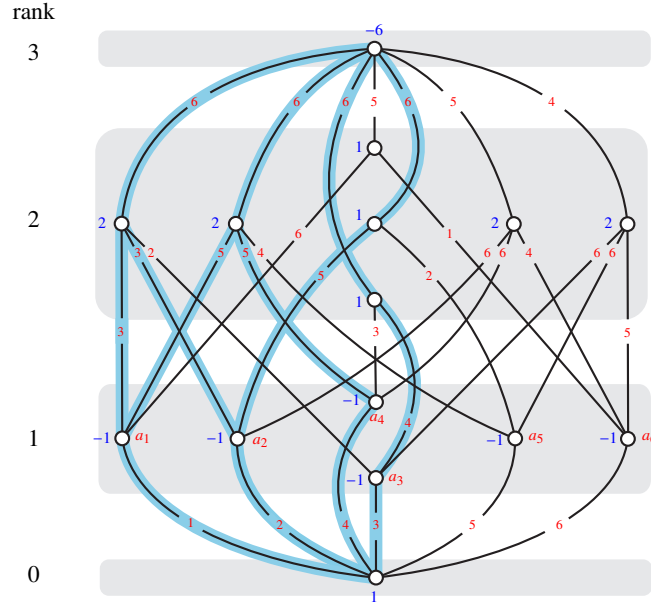
$$\dim \widetilde{H}_i(\widetilde{BC}(L)) = \begin{cases} \beta(L), & i = rk(L) - 2 \\ 0, & \text{otherwise} \end{cases}$$

where

$$\beta(L) = (-1)^{rk(L)-1} \left. \frac{d}{dt} \chi(t) \right|_{t=1}$$

is the beta-invariant; see [Sta77, Bjö92]. Comparing this to Theorem 3, it is then very natural to ask what, if any, is the relationship between the sheaf homology of  $L$  equipped with the natural sheaf and the reduced broken circuit complex? We are grateful to the referee for pointing this out.

It is possible to make a very explicit connection between the sheaf homology of  $L$  equipped with the *constant* sheaf and the (un-reduced) broken circuit complex. In order to do this it is most



**Fig. 3.** The partition lattice  $\Pi(4)$  – adapted from a picture by Tilman Piesik [JB] – with the total ordering  $a_1 \prec a_2 \prec \dots \prec a_6$  and the resulting  $\lambda$ -labelling in red; the values of the Möbius function  $\mu(\mathbf{0}, x)$  and the  $\lambda$ -increasing chains (10) that give the six 2-simplices of the  $BC$ -complex are in blue.

convenient to pass via the “cellular homology” of  $L$  with coefficients in a sheaf (see [ET15] – suitably adapted to be homological rather than cohomological) and we now briefly explain these ideas.

Let  $L_0 = L \setminus \mathbf{0}$  and  $F$  be a sheaf on  $L_0$ . Filter  $L_0$  by rank, defining  $L_0^r = \{x \in L_0 : rk(x) \leq r+1\}$ ; the “+1” is because we are using the  $L$ -rank function for  $L_0$ . Then,  $S_*(L_0^{r-1}; F)$  is a subcomplex of  $S_*(L_0^r; F)$  with quotient complex that we denote by  $S_*(L_0^r, L_0^{r-1}; F)$ . The *cellular chain complex*  $C_*(L_0; F)$  has chains

$$C_r(L_0, F) = H_r(L_0^r, L_0^{r-1}; F)$$

and differential  $C_r \rightarrow C_{r-1}$  provided by the boundary map that arises in the long exact sequence of the triple of subposets  $(L_0^r, L_0^{r-1}, L_0^{r-2})$ . By [ET15, Theorem 2 and §4.4] the cellular chain complex computes sheaf homology:

$$H_*(L_0; F) \cong HC_*(L_0; F).$$

After a little analysis, the arguments of [ET15, §2 and §4.4] can be massaged to show that

$$C_r(L_0; F) \cong \bigoplus_{rk(x)=r} A_x \otimes F(x)$$

where

$$A_x \cong \widetilde{H}_{r-1}((L_0)_{<x}; \mathbb{Z}) \cong \mathbb{Z}^{|\mu(\mathbf{0}, x)|}$$

with the middle term the ordinary reduced homology of the nerve of  $(L_0)_{<x}$ . This is in turn free of rank the Möbius function of  $(L_0)_{<x}$  by [Fol66]. It is also possible to give an explicit presentation

for the abelian group  $A_x$ . Let  $b_0, b_1, \dots, b_r$  be a set of linearly independent (in the sense of §1.3) atoms for  $(L_0)_{\leq x}$  such that  $\bigvee b_i = x$ , and consider the saturated chain

$$\sigma = b_0 < b_0 \vee b_1 < \dots < b_0 \vee b_1 \vee \dots \vee b_r.$$

Any other saturated chain has the form

$$\pi(b_0) < \pi(b_0) \vee \pi(b_1) < \dots < \pi(b_0) \vee \pi(b_1) \vee \dots \vee \pi(b_r)$$

for a unique  $\pi \in S_r$ . The group  $A_x$  is freely generated by elements of the form

$$\alpha_\sigma = \sum_{\pi \in S_r} (-1)^{\text{sgn}(\pi)} \sigma_\pi \quad (11)$$

where  $\sigma$  is  $\lambda$ -increasing [Bjö82].

**Proposition 5.** *If  $\sigma$  is a  $\lambda$ -increasing chain as above with  $\lambda(\mathbf{0}, x_0) = a_{i_0}, \dots, \lambda(x_{r-1}, x_r) = a_{i_r}$ , then the map*

$$\alpha_\sigma \mapsto r\text{-simplex } \{a_{i_0}, a_{i_1}, \dots, a_{i_r}\} \text{ of } BC(L)$$

*induces an isomorphism from  $C_*(L \setminus \mathbf{0}; \Delta k)$  to the simplicial chain complex of  $BC(L)$ .*

The homology itself is uninteresting but using cellular homology in this way, makes a fairly direct link between the different chain complexes computing it.

Proposition 5 is an encouraging start. To explain the appearance of the beta-invariant in Theorem 3 we need to replace the constant sheaf  $\Delta k$  in  $C_*(L \setminus \mathbf{0}; \Delta k)$  by the natural sheaf  $F$ , and we need to replace the unreduced broken circuit complex  $BC(L)$  by the reduced complex  $\widetilde{BC}(L)$ . These are modifications of quite a different nature: the reduced broken circuit complex may be defined for any lattice (but no sheaf is present) and the natural sheaf is a construction only defined for hyperplane arrangements. It may be possible though, to choose bases for the hyperplanes in such a way that the combinatorics can be pushed to give an isomorphism of chain complexes similar to the one in Proposition 5.

## References

- [JB] John Baez, *Lattice of partitions*, American Mathematical Society Blogs, available at [blogs.ams.org](http://blogs.ams.org).
- [Bir79] Garrett Birkhoff, *Lattice theory*, 3rd ed., American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, Providence, R.I., 1979.
- [Bjö92] Anders Björner, *The homology and shellability of matroids and geometric lattices*, Matroid applications, Encyclopedia Math. Appl., vol. 40, Cambridge Univ. Press, Cambridge, 1992, pp. 226–283.
- [Bjö82] Anders Björner, *On the homology of geometric lattices*, Algebra Universalis **14** (1982), no. 1, 107–128.
- [Bry77] Tom Brylawski, *The broken-circuit complex*, Trans. Amer. Math. Soc. **234** (1977), no. 2, 417–433.
- [EF13] Brent Everitt and John Fountain, *Partial mirror symmetry, lattice presentations and algebraic monoids*, Proc. London Math. Soc. **107** (2013), no. 3, 414–450.
- [ET15] Brent Everitt and Paul Turner, *Cellular cohomology of posets with local coefficients*, J. Algebra **439** (2015), 134–158.
- [ET14] ———, *The homotopy theory of Khovanov homology*, Algebr. Geom. Topol. **14** (2014), no. 5, 2747–2781.
- [Fol66] Jon Folkman, *The homology groups of a lattice*, J. Math. Mech. **15** (1966), 631–636.
- [GZ67] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967.

- [God73] Roger Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1973 (French). Troisième édition revue et corrigée; Publications de l'Institut de Mathématique de l'Université de Strasbourg, XIII; Actualités Scientifiques et Industrielles, No. 1252.
- [Gro57] Alexander Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221 (French).
- [Grü03] Branko Grünbaum, *Convex polytopes*, 2nd ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.
- [Kho00] Mikhail Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426.
- [Lus74] George Lusztig, *The discrete series of  $GL_n$  over a finite field*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 81.
- [OT92] Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992.
- [Qui78] Daniel Quillen, *Homotopy properties of the poset of nontrivial  $p$ -subgroups of a group*, Adv. in Math. **28** (1978), no. 2, 101–128.
- [Qui73] ———, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.
- [RS85] Mark A. Ronan and Stephen D. Smith, *Sheaves on buildings and modular representations of Chevalley groups*, J. Algebra **96** (1985), no. 2, 319–346.
- [Sta12] Richard P. Stanley, *Enumerative combinatorics. Volume 1*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.
- [Sta07] ———, *An introduction to hyperplane arrangements*, Geometric combinatorics, IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2007, pp. 389–496.
- [Sta77] ———, *Cohen-Macaulay complexes*, Higher combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976), Reidel, Dordrecht, 1977, pp. 51–62. NATO Adv. Study Inst. Ser., Ser. C: Math. and Phys. Sci., 31.
- [Wac07] Michelle L. Wachs, *Poset topology: tools and applications*, Geometric combinatorics, IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2007, pp. 497–615.
- [Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.
- [Wil76] Herbert S. Wilf, *Which polynomials are chromatic?*, Colloquio Internazionale sulle Teorie Combinatorie (Roma, 1973), Accad. Naz. Lincei, Rome, 1976, pp. 247–256. Atti dei Convegni Lincei, No. 17 (English, with Italian summary).
- [Yuz91] Sergey Yuzvinsky, *Cohomology of local sheaves on arrangement lattices*, Proc. Amer. Math. Soc. **112** (1991), no. 4, 1207–1217.
- [Zas87] Thomas Zaslavsky, *The Möbius function and the characteristic polynomial*, Combinatorial geometries, Encyclopedia Math. Appl., vol. 29, Cambridge Univ. Press, Cambridge, 1987, pp. 114–138. MR921071
- [Zie95] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.