Hopf index and the helicity of elliptically-polarised twisted light

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Here, we describe a systematic derivation of the general form of the optical helicity density of ellipticaly-polarised paraxial Laguerre-Gaussian modes LG$_{\ell,p,\sigma}$. The treatment incorporates the contributions of the longitudinal field components for both the paraxial electric E and magnetic B fields, which satisfy Maxwell's self-consistency condition in the sense that E is derivable from B and vice versa. Contributions to the helicity density, to leading order in $(k^2w_0^2)^{-1}$ (where k is the axial wavenumber and w_0 the beam waist), include terms proportional to optical spin σ, topological charge ℓ as well as a spin-orbit $\sigma|\ell|$ term. However, evaluations of the space integrals leading to the total helicity confirm that the space integral of the ℓ-dependent term in the density (which is due entirely to the longitudinal fields) vanishes identically for all ℓ and p, so that, in general, only σ determines the Hopf index, with the optical vortex LG$_{\ell p}$ character only featuring in the action constant.

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There is currently much interest in the emerging field of topological photonics, which deals with the topological aspects of light and how they can be put to good use in controlling its properties in diverse physical contexts [1–5]. Optical helicity is prominent among the properties of light that display topological features [6–14]. For monochromatic light of frequency ω in free space the form of the optical helicity density that has recently been investigated is the cycle-averaged form [15, 16] which is given by

$$\bar{\eta}(\mathbf{r}) = -\frac{\epsilon_0 c}{2\omega}\Im[\mathbf{E}^* \cdot \mathbf{B}] \tag{1}$$

where **E** and **B** are the electric and magnetic vector fields of the light and $\Im[...]$ indicates taking the imaginary part of [...]. The total cycle-averaged helicity $\bar{\mathcal{C}}$ is as defined by Ranada [6] as the volume integral of the helicity density. It is this property that displays explicit topological features characterised by the topological invariant $\mathcal{N}$ and an action constant $\mathcal{Q}$

$$\bar{\mathcal{C}} = \int d^3\mathbf{r}\, \bar{\eta}(\mathbf{r}) = \mathcal{N}\mathcal{Q} \tag{2}$$

$\mathcal{N}$ is called the Hopf index and is an invariant real number characteristic of the specific light field. By contrast, the constant $\mathcal{Q}$ is not invariant but depends on the parameters controlling the specific light field. Topological invariance here means that if the light field parameters are modified, for example by focusing, or by a change of frequency, or by a change of the applied field intensity, then the Hopf index does not change, while the action constant adjusts accordingly.

Recent research has sought to explore and clarify the role of the longitudinal (axial) component of the electric field of the light [17]. This component was previously regarded as insignificant and so invariably dropped from most treatments involving the optical beam properties and its interaction with matter. It has recently been established that the longitudinal electric field component and its associated magnetic field components can acquire magnitudes comparable to those of the transverse components. Its inclusion in the analysis has been shown to give rise to new effects, including the realisation of transverse angular momentum and its relation to the optical Spin Hall effect [18, 19] and has recently been shown to play a role in the trapping potential for atoms via a spin-orbit coupling mechanism [20]. Its role in quantised vortex beams has also been explored [21]. Among the explored beam properties of interest in the context of polarised paraxial light fields are the vector components of the volume densities as well as their space-integrals of the optical angular momentum and the Poynting vector [22–27].

The aim of this article is to find out what effect the inclusion of the longitudinal component has on the helicity of the light, its Hopf index, and its action integral. We consider the general realisable form of the light field, namely an elliptically-polarised paraxial Laguerre-Gaussian mode, labelled $\mathrm{LG}_{\ell,p,\sigma}$.

We assume that the beam has a waist w_0 and a Rayleigh range z_R, and we neglect all convergence phases, namely the Gouy and curvature phases. Lax et al. [28] have shown that such a beam is characterised by the diffraction length d in the longitudinal direction where $d = kw_0^2 \equiv 2z_R$ and it is anticipated that all problems of interest are such that $w_0 < d$, which means $kw_0 > 1$, Then to leading order in the expansion in powers of $1/kw_0$ such a beam can be represented by a transverse field, along with a longitudinal field [29] which emerges by use of the Lorentz gauge with a vector potential $\mathbf{A}$ and a scalar potential proportional to $\boldsymbol{\nabla}\cdot\mathbf{A}$. This leads first to the magnetic field $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ and from it we derive $\mathbf{E}$ using the Maxwell equation involving $\boldsymbol{\nabla}\times\mathbf{B}$. Self-consistency demands that once the form of $\mathbf{E}$ is determined from $\mathbf{B}$ it is crucial that we must obtain $\mathbf{B}$ from $\mathbf{E}$. We therefore focus on the derivation of vortex LG fields which starts from the Lorentz gauge

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}; \quad \mathbf{E} = i\omega\mathbf{A} - \boldsymbol{\nabla}\Phi \tag{3}$$

We assume an elliptically-polarised beam, so we write

$$\mathbf{A} = (\alpha\hat{\boldsymbol{x}} + \beta\hat{\boldsymbol{y}})\mathcal{F}_{\ell,p}(\rho,\phi)e^{ikz} \tag{4}$$

where $\mathcal{F}_{\ell,p}(\rho,\phi)$ is the amplitude function of the Laguerre-Gaussian mode. Explicitly for an $\mathrm{LG}_{\ell,p}$ mode $\mathcal{F}_{\ell,p}$ takes the following form [30]

$$\mathcal{F}_{\ell,p}(\rho,\phi) = \mathcal{E}_0\sqrt{\frac{p!}{(p+|\ell|)!}}e^{-\frac{\rho^2}{w_0^2}}\left(\frac{\sqrt{2}\rho}{w_0}\right)^{|\ell|}L_p^{|\ell|}\left(\frac{2\rho^2}{w_0^2}\right)e^{i\ell\phi} \tag{5}$$

where $L_p^{|\ell|}$ is the associated Laguerre polynomial of indices $|\ell|$ and p, with $\mathcal{E}_0$ an overall normalisation factor which is fixed by the requirement that its value is consistent with input power of known magnitude $\mathcal{P}$. The derivation of $\mathcal{E}_0$ is given in Appendix A.

The Lorentz gauge involves the continuity equation

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0 \tag{6}$$

This determines the scalar potential Φ in terms of $\mathbf{A}$. We obtain

$$\Phi = \frac{c^2}{i\omega}\boldsymbol{\nabla}\cdot\mathbf{A} \tag{7}$$

which yields

$$\Phi = \frac{c^2}{i\omega}\left(\alpha\frac{\partial\mathcal{F}}{\partial x} + \beta\frac{\partial\mathcal{F}}{\partial y}\right)e^{ikz} \tag{8}$$

where for ease of notation we have dropped the ℓ, p subscripts in $\mathcal{F}$. These will be restored when the need arises. The paraxial fields in this general case are as follows. The magnetic field emerges from the vector potential Eq.(4) directly using $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ in the form

$$\mathbf{B} = ik(\alpha\hat{\boldsymbol{y}} - \beta\hat{\boldsymbol{x}})\mathcal{F}e^{ikz} + \hat{\boldsymbol{z}}\left(\beta\frac{\partial\mathcal{F}}{\partial x} - \alpha\frac{\partial\mathcal{F}}{\partial y}\right)e^{ikz} \tag{9}$$

while the electric field, to the same level of paraxial approximation, is

$$\mathbf{E} = ick(\alpha\hat{\boldsymbol{x}} + \beta\hat{\boldsymbol{y}})\mathcal{F}e^{ikz} - c\left\{\alpha\frac{\partial\mathcal{F}}{\partial x} + \beta\frac{\partial\mathcal{F}}{\partial y}\right\}e^{ikz}\hat{\boldsymbol{z}} \tag{10}$$

Note that the fields $\mathbf{E}$ and $\mathbf{B}$ we are dealing with are in the paraxial regime [31] emerging from the vector potential $\mathbf{A}$ in the Lorentz gauge. We identify the z-components of the fields as the longitudinal components whose influence on the helicity properties of the light are of concern here. It is straightforward to show that once having found $\mathbf{E}$ we may proceed to derive $\mathbf{B}$ from the Maxwell equation $i\omega\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{E}$ to the same level of approximation. Similarly from $\mathbf{B}$ we should again derive $\mathbf{E}$. This self-consistency condition ensures that Maxwell's equations are satisfied in this paraxial regime. Note also that the case of a linearly-polarised beam corresponds to setting $\alpha = 1$ and $\beta = 0$ in Eqs.(9) and (10) which yields the corresponding expressions given by Haus [31]. However, our concern here is to consider the most general elliptically-polarised Laguerre-Gaussian light beam $\mathrm{LG}_{\ell,p,\sigma}$ for arbitrary ℓ and p and from this general case we can retrieve results for physically distinct beams including the simplest beams, namely Gaussian ($\ell = 0$) and doughnut (any ℓ, but $p = 0$) and both cases can be linearly or circularly-polarised.

Having now at our disposal expressions for the paraxial electric and the magnetic fields, our primary tasks here consist of the following (i) first evaluating the cycle-averaged helicity density $\bar{\eta}$ of the polarised Laguerre-Gaussian light beam; (ii) next evaluating the total helicity $\bar{\mathcal{C}}$ as the space integral of the helicity density $\bar{\eta}$; (iii) finally proceeding to discuss topological invariance in this context and determining the Hopf index.

We are interested in the most general cycle-averaged helicity density of the LG light field. This is now labelled as $\bar{\eta}_{\ell,p,\sigma}$ to emphasise the LG beam generality in terms of its winding number ℓ and radial number p along with the general elliptical polarisation σ. The helicity density is defined in Eq.(1) and from this we obtain the total helicity as the space integral.

$$\bar{\mathcal{C}}_{\ell,p,\sigma} = \int_{-d/2}^{d/2}dz\int_0^\infty\rho d\rho\int_0^{2\pi}d\phi\;\bar{\eta}_{\ell,p,\sigma}(\mathbf{r}) \tag{11}$$

We have assumed that the beam length is equal to the diffraction length $d = w_0^2 k$ [28]. Working out the dot product $\mathbf{E}^* \cdot \mathbf{B}$ we find

$$\mathbf{E}^* \cdot \mathbf{B} = ck^2(\alpha\beta^* - \beta\alpha^*)|\mathcal{F}|^2 - c\left(\beta\frac{\partial\mathcal{F}}{\partial x} - \alpha\frac{\partial\mathcal{F}}{\partial y}\right)\left\{\alpha^*\left(\frac{\partial\mathcal{F}}{\partial x}\right)^* + \beta^*\left(\frac{\partial\mathcal{F}}{\partial y}\right)^*\right\} \tag{12}$$

We identify the derivative terms as contributions to the helicity density due to the longitudinal field. Multiplying out in the second term we obtain

$$\mathbf{E}^* \cdot \mathbf{B} = ck^2(\alpha\beta^* - \beta\alpha^*)|\mathcal{F}|^2 - c\left\{\beta\alpha^*\left|\frac{\partial\mathcal{F}}{\partial x}\right|^2 - \alpha\beta^*\left|\frac{\partial\mathcal{F}}{\partial y}\right|^2 + |\beta|^2\left(\frac{\partial\mathcal{F}}{\partial x}\right)\left(\frac{\partial\mathcal{F}}{\partial y}\right)^* - |\alpha|^2\left(\frac{\partial\mathcal{F}}{\partial x}\right)^*\left(\frac{\partial\mathcal{F}}{\partial y}\right)\right\} \tag{13}$$

The evaluation of the partial derivatives is straightforward we have

$$\left(\frac{\partial\mathcal{F}}{\partial x}\right) = (\mathcal{R}\cos\phi - i\mathcal{T}\sin\phi)\,\mathcal{F} \tag{14}$$

and

$$\left(\frac{\partial\mathcal{F}}{\partial y}\right) = (\mathcal{R}\sin\phi + i\mathcal{T}\cos\phi)\,\mathcal{F} \tag{15}$$

where $\mathcal{R}$ and $\mathcal{T}$ depend on the form of the LG mode. For the general $\mathrm{LG}_{\ell,p}$ mode we have for $\mathcal{R}$

$$\mathcal{R} = \left\{\frac{-2\rho}{w_0^2} + \frac{|\ell|}{\rho} - \frac{4\rho}{w_0^2}\frac{L_{p-1}^{|\ell|+1}\left(\frac{2\rho^2}{w_0^2}\right)}{L_p^{|\ell|}\left(\frac{2\rho^2}{w_0^2}\right)}\right\} \tag{16}$$

and for $\mathcal{T}$ we have

$$\mathcal{T} = \frac{\ell}{\rho} \tag{17}$$

Note that the third term in $\mathcal{R}$ in Eq.(16) can have zeros in the denominator L_p^ℓ and so the third term on its own becomes infinite. However, as will be shown in what follows, this function will always appear multiplied by $|\mathcal{F}|^2$ which is proportional to $[L_p^\ell]^2$. The denominator of the third term then cancels with one of $[L_p^\ell]^2$ and the resulting expression then has no zeros in the denominator.

The complex parameters α and β have the following properties

$$|\alpha|^2 + |\beta|^2 = 1; \quad \alpha\beta^* - \beta\alpha^* = 2i\Im[\alpha\beta^*] \tag{18}$$

So we then have

$$\sigma = i(\alpha\beta^* - \alpha^*\beta) = 2\alpha\beta' \tag{19}$$

where we have set $\beta = i\beta'$. Consider next the terms involving the modulus squares. We have

$$\begin{aligned}\beta\alpha^*\left|\frac{\partial\mathcal{F}}{\partial x}\right|^2 - \alpha\beta^*\left|\frac{\partial\mathcal{F}}{\partial y}\right|^2 &= i\alpha\beta'\left\{\left|\frac{\partial\mathcal{F}}{\partial x}\right|^2 + \left|\frac{\partial\mathcal{F}}{\partial y}\right|^2\right\} \\ &= i\alpha\beta'(\mathcal{R}^2 + \mathcal{T}^2)|\mathcal{F}|^2\end{aligned} \tag{20}$$

Consider finally the mixed terms. We have

$$\begin{aligned}|\beta|^2\left(\frac{\partial\mathcal{F}}{\partial x}\right)\left(\frac{\partial\mathcal{F}}{\partial y}\right)^* - |\alpha|^2\left(\frac{\partial\mathcal{F}}{\partial x}\right)^*\left(\frac{\partial\mathcal{F}}{\partial y}\right) &= \frac{1}{2}\left\{\left(\frac{\partial\mathcal{F}}{\partial x}\right)\left(\frac{\partial\mathcal{F}}{\partial y}\right)^* - \left(\frac{\partial\mathcal{F}}{\partial x}\right)^*\left(\frac{\partial\mathcal{F}}{\partial y}\right)\right\} \\ &= -i\mathcal{R}\mathcal{T}|\mathcal{F}|^2\end{aligned} \tag{21}$$

Collecting terms and using $2\alpha\beta' = \sigma$, we have for the dot product $\mathbf{E}^* \cdot \mathbf{B}$

$$\begin{aligned}\mathbf{E}^* \cdot \mathbf{B} &= -2i\alpha\beta' ck^2|\mathcal{F}|^2 - c\left\{i\alpha\beta'(\mathcal{R}^2 + \mathcal{T}^2) - i\mathcal{T}\mathcal{R}\right\}|\mathcal{F}|^2 \\ &= -i\sigma ck^2|\mathcal{F}|^2 - c\left\{\frac{1}{2}i\sigma(\mathcal{R}^2 + \mathcal{T}^2) - i\mathcal{T}\mathcal{R}\right\}|\mathcal{F}|^2\end{aligned} \tag{22}$$

Thus for a general beam $\mathrm{LG}_{\ell,p,\sigma}$ we find for the cycle averaged helicity density

$$\bar{\eta}_{\ell,p,\sigma} = \frac{\epsilon_0 c^2}{4\omega}\left\{\sigma(2k^2 + \mathcal{R}^2 + \mathcal{T}^2) - 2\mathcal{R}\mathcal{T}\right\}|\mathcal{F}_{\ell,p}|^2 \tag{23}$$

We can see that the derivative terms due to the longitudinal fields result in adding the terms $(\mathcal{R}^2+\mathcal{T}^2)|\mathcal{F}_{\ell,p}|^2$ to the sigma-dependent part of the helicity density and it introduces a σ-independent but ℓ-dependent contribution to the density as the term $-2\mathcal{R}\mathcal{T}|\mathcal{F}_{\ell,p}|^2$ It is convenient at this stage to substitute for $\mathcal{T}$ and obtain

$$\bar{\eta}_{\ell,p,\sigma}=\frac{\epsilon_0 c^2}{4\omega}\left\{\sigma\left[2k^2+\mathcal{R}^2+\left(\frac{\ell}{\rho}\right)^2\right]-\ell\left(\frac{2\mathcal{R}}{\rho}\right)\right\}|\mathcal{F}_{\ell,p}|^2 \tag{24}$$

With $\mathcal{R}$ given by Eq.(16), the result (24) is the first of three main results of this article. It represents the general form of the helicity density of the elliptically-polarised $LG_{\ell p,\sigma}$ beam, applicable for any winding number ℓ and radial number p. A close inspection of the helicity density shows that it receives two distinct contributions: the first, denoted $\bar{\eta}_\sigma$ is associated with polarisation and is proportional to σ. The second, denoted $\bar{\eta}_{\ell,p,0}$ given by the second term, is a the σ-independent contribution associated with the orbital angular momentum (OAM) vortex. Characteristically, it is proportional to the vortex winding number ℓ. Thus we can write the helicity density of the most general LG beam as the sum of two contributions

$$\bar{\eta}=\bar{\eta}_\sigma+\bar{\eta}_{\ell,p,0} \tag{25}$$

where $\bar{\eta}_\sigma$ is the expression in Eq.(24) that is proportional to σ. We have

$$\bar{\eta}_\sigma=\frac{\epsilon_0 c^2}{4\omega}\sigma\left[2k^2+\mathcal{R}^2+\left(\frac{\ell}{\rho}\right)^2\right]|\mathcal{F}_{\ell,p}|^2 \tag{26}$$

The rest of the expression in Eq.(24) defines $\bar{\eta}_{\ell,p}$

$$\bar{\eta}_{\ell,p,0}=-\ell\frac{\epsilon_0 c^2}{4\omega}\left(\frac{2\mathcal{R}}{\rho}\right)|\mathcal{F}_{\ell,p}|^2 \tag{27}$$

For a general mode LG mode the function $\mathcal{R}$ is as defined in Eq.(16).

The helicity density term $\bar{\eta}_{\ell,p,0}$ is the one that survives when $\sigma=0$, i.e.for a linearly-polarised beam. This term is entirely due to the inclusion of the longitudinal component. It is proportional to ℓ which changes sign when ℓ changes sign. For example the helicity density distribution $\bar{\eta}_{3,0,0}(\rho)$ and $\bar{\eta}_{-3,0,0}(\rho)$ of two linearly-polarised Laguerre-Gaussian beams which differ only in their winding number ℓ are such that $\bar{\eta}_{3,0,0}(\rho)=-\bar{\eta}_{-3,0,0}(\rho)$. This is a signature of chirality. However, although the helicity density distribution is non zero, its integration over all space yields the exact result for all ℓ and p. Substituting for $\mathcal{R}$ from Eq.(16), with $\mathcal{F}$ given by Eq.(5) we obtain the exact result that the total helicity for a linearly polarised $LG_{\ell,p}$ mode ($\sigma=0$) vanishes identically due to the radial integration. Thus

$$\bar{\mathcal{C}}_{\ell,p,0}=\int_{-d/2}^{d/2}dz\int_0^{2\pi}d\phi\int_0^\infty\rho\,d\rho\;\bar{\eta}_{\ell,p}=0;\quad(\sigma=0) \tag{28}$$

The steps leading to this result are displayed in Appendix B. Eq.(28) is the second of our main results showing that the total helicity of any linearly-polarised LG beam is zero for all ℓ and p. The corresponding density arises entirely from the inclusion of the longitudinal fields. Figure 1 displays the variations with ρ of the integrands $\rho\bar{\eta}_{\pm3,0,0}$. It is seen that the areas under and above the ρ axis add up to zero in each case, confirming that the total helicity vanishes in this case.

When both ℓ and σ are non-zero, the helicity density is given by the general form Eq.(24) and for illustration, Fig. 2 displays four cases involving the set $(\ell,\sigma)=(\pm3,\pm1)$. In each figure, the density is determined by the σ component with modifications arising due to the inclusion of the longitudinal field components.

The general expression for the total helicity is obtained as the space integral of the helicity density Eq.(24). We obtain, keeping in mind the null result in (28) which represents a vanishing OAM contribution to the total helicity,

$$\bar{\mathcal{C}}_{\ell,p,\sigma}=\int_{-d/2}^{d/2}dz\int_0^{2\pi}d\phi\int_0^\infty\rho\,d\rho\;\bar{\eta}_\sigma;\quad(\sigma\neq0) \tag{29}$$

Substituting for $\bar{\eta}_\sigma$ from Eq.(26) with $\mathcal{R}$ given by Eq.(16) and $\mathcal{F}$ given by Eq.(5) we obtain the result for the total helicity of an elliptically-polarised LG mode for arbitrary values of ℓ and p. The integrals in Eq.(29) are dealt with in Appendix C and the final result is as shown in Appendix C is

$$\bar{\mathcal{C}}_{\ell,p,\sigma}=\sigma\mathcal{Q}_0\left\{1+\frac{2p+|\ell|+1}{k^2w_0^2}\right\} \tag{30}$$

where $\mathcal{Q}_0$ (as shown in Appendix C) is given by

$$\mathcal{Q}_0=k^2w_0^2\left(\frac{\mathcal{P}}{\omega^2}\right) \tag{31}$$

Q_0 is thus a constant which has the dimensions of angular momentum (Js). Writing $\bar{\mathcal{C}}_{\ell,p,\sigma}$ as the product of the Hopf index $\mathcal{N}$ and the action constant $\mathcal{Q}$, we obtain at once

$$\mathcal{N}=\sigma;\qquad\mathcal{Q}=\mathcal{Q}_0\left(1+\frac{2p+|\ell|+1}{k^2w_0^2}\right) \tag{32}$$

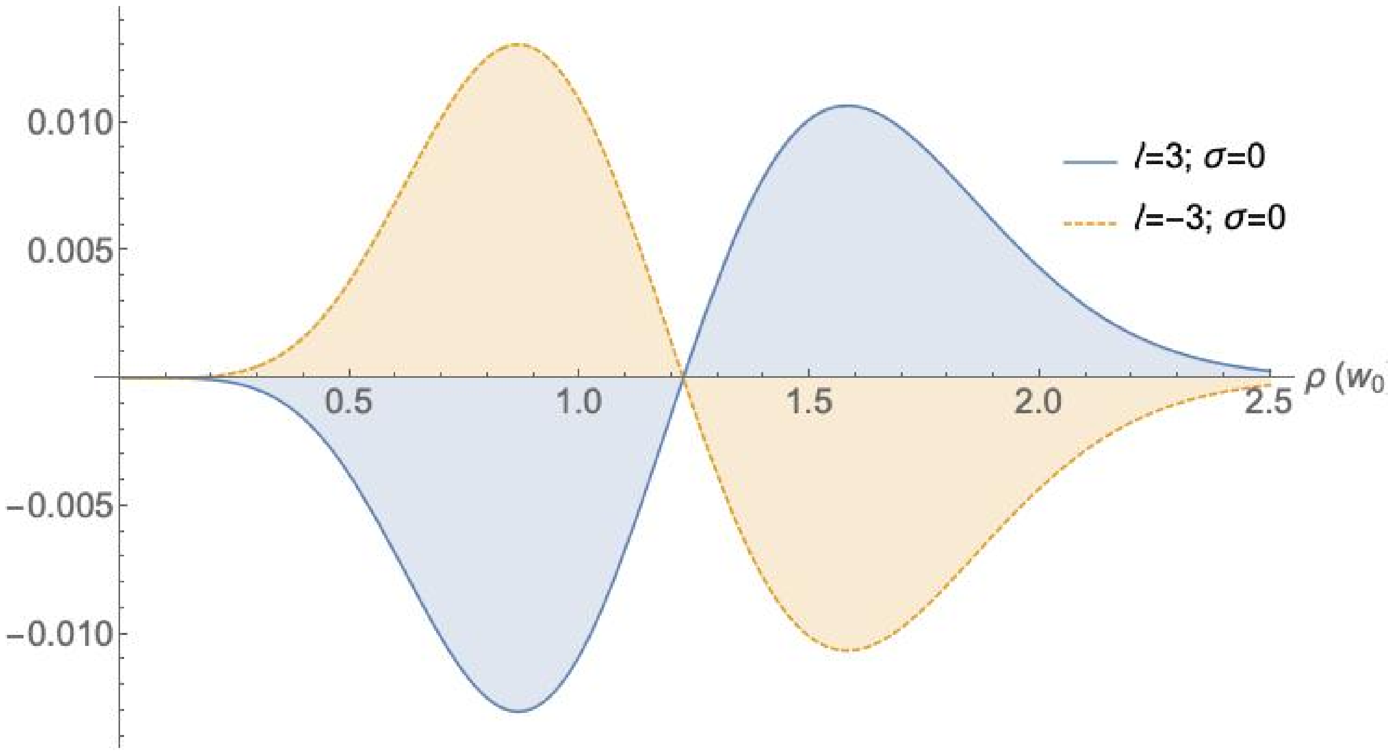

Fig. 1. Variations with ρ (in units of w_0) of the integrands $\rho\eta_{\pm 3,0,0}$ (arbitrary units) for two paraxial ($kw_0 > 1$) linearly-polarised doughnut beams ($\sigma = 0$). The areas enclosed by each curve add up to zero confirming the null result for the total helicity in these special cases. The figure also demonstrates the chirality displayed by these beams which differ only in the sign of their winding number. Here $kw_0 = 5$.

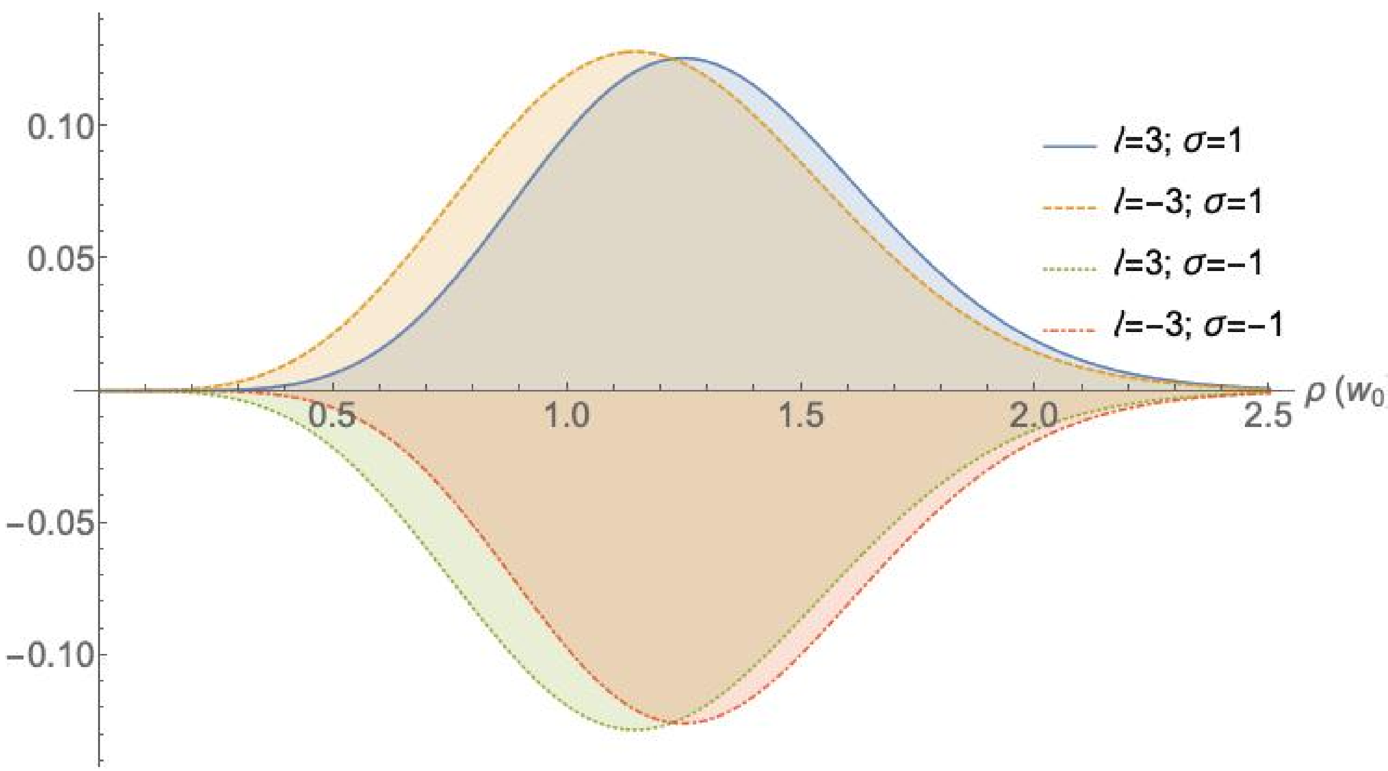

Fig. 2. Variations with ρ of the helicity densities based on Eq.(24) (arbitrary units) for four circularly-polarised paraxial ($kw_0 > 1$) Laguerre Gaussian (doughnut) beams. Here $kw_0 = 5$.

This is the third of the main results of this work. The first term (unity) inside the brackets is such that $\mathcal{Q} = \mathcal{Q}_0$. We have checked by explicit evaluations that $\mathcal{Q}_0$ is the action constant associated with a transverse elliptically-polarised Gaussian beam in the absence of the longitudinal fields, while the second term is due entirely to the inclusion of the longitudinal field components. The numerator in this second term is, by coincidence, reminiscent of the factor involved in the form of the Gouy phase of the LG light beam, while the magnitude of the denominator $k^2w_0^2$ signifies the extent of the paraxiality of the beam. As explained above in most cases we are concerned with the case $kw_0 > 1$. The second term between the brackets in $\mathcal{Q}$ becomes equal to or greater than unity for a beam waist w_0 given by

$$w_0 \leq \bar{\lambda}\sqrt{2p + |\ell| + 1} \tag{33}$$

Thus the second term in $\mathcal{Q}$ can become significant relative to unity for smaller beam waists which indicates strong focusing. For a Gaussian beam $\ell = 0$, $p = 0$ and the total helicity becomes

$$\bar{\mathcal{C}}_{0,0,\sigma} = \sigma\mathcal{Q}_0\left(1 + \frac{1}{k^2w_0^2}\right) \tag{34}$$

in which case the condition $w_0 = \bar{\lambda}$ leads to the second term equal to unity. For $p \gg 0$ and/or $\ell \gg 0$, the second term inside the brackets becomes greater than unity even for larger $w_0 \gg \bar{\lambda}$, so there is no need for tight focusing.

In conclusion, we have evaluated the helicity density and the total helicity of circularly-polarised paraxial Laguerre-Gaussian beams and determined from that the topologically invariant Hopf integer index $\mathcal{N}$. In our analysis, we have made use of electromagnetic fields in the paraxial limit which incorporate the longitudinal components and to the same level of the paraxial regime the fields conform with the requirement of duality in the sense that **E** can be derived from **B** and **B** can be derived from **E** using Maxwell's equations. We proceeded to evaluate the cycle-averaged helicity density, as Eq.(24) and shown that it receives separate contributions, one proportional to σ and a second proportional to ℓ and there is also a spin-orbit term proportional to $\sigma|\ell|^2$ within the σ contribution.

Our general result in Eq.(24) for the helicity density includes the contributions arising entirely from the inclusion of the longitudinal components. This general form applies to the various cases, namely linearly-polarised as well as circularly-polarised doughnut beams; circularly-polarised Gaussian beams as well as linearly-polarised LG beams for which $\sigma = 0$. In this latter case only the ℓ-dependent helicity density (which derives entirely from the longitudinal field) survives and this displays chirality in the sense that a change of the sign of ℓ changes the sign of the helicity density. Although a linearly-polarised LG beam has a non-zero helicity distribution, it gives rise to a null contribution to the total helicity, which continues to be determined solely by the elliptical polarisation. The vanishing of the total helicity for a linearly polarised LG beam ($\sigma = 0$) is consistent with the vanishing of the total optical spin (SAM) of the same linearly-polarised beam, despite the fact that the spin density distribution is non-zero. The spin density and SAM of the linearly-polarised LG mode is discussed in Appendix D. Here we have shown that in all cases the Hopf index is conserved as $\mathcal{N} = \sigma$ and that the effect of the optical vortex is only to enhance the action constant $\mathcal{Q}$ by the additive factor $(2p + |\ell| + 1)/(k^2w_0^2)$. This factor can equal or exceed unity even for moderate focusing for which $\bar{\lambda} < w_0 \leq s\bar{\lambda}$ where s can be much greater than unity, depending on the choice of ℓ and p.

In short, we have demonstrated that an elliptically-polarised Laguerre-Gaussian beam $\mathrm{LG}_{\ell,p,\sigma}$ displays topological invariance in that it always has the Hopf index σ and we have clarified the roles of the winding number ℓ and the radial number p in the presence and absence of σ. The finding that the Hopf index is determined solely by σ is significant, with ℓ-dependent terms originating from the longitudinal fields playing no role in the Hopf index. The only topological feature of the inclusion of the longitudinal components is a modification of the action constant $\mathcal{Q}$ of the total helicity. Another outstanding problem arising from this work is to determine what topological properties one would expect to find when considering vortex beams with fractional orbital angular momentum [32, 33]. However, this matter will not be pursued any further here.

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DISCLOSURES

The authors declare no conflicts of interest.

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APPENDIX A

The normalisation factor $\mathcal{E}_0$

The overall normalisation factor which appears in the form of the paraxial LG beam in Eq.(5) is such that the average power of the beam $\mathcal{P}$ (average energy per unit area per second) is conserved. The formal definition of the power is the surface integral of the average Poynting vector $\mathbf{E}^*\times\mathbf{B}/2\mu_0$ over a plane for which the surface element is $\mathbf{d\Sigma}=d\Sigma\hat{\mathbf{z}}$, so only the z-component of the Poynting vector enters the integration. The power $\mathcal{P}$ is the integral of the z-component of the Poynting vector over the beam cross-section

$$\mathcal{P}=\frac{1}{2\mu_0}\int_0^{2\pi}d\phi\int_0^{\infty}|(\mathbf{E}^*\times\mathbf{B})_z|\rho d\rho \tag{35}$$

The evaluation of the z-component of the Poynting vector is straightforward using Eq.(10)and (9) and yields $|(\mathbf{E}^*\cdot\mathbf{B})_z|=ck^2|\mathcal{F}_{\ell,p}|^2$. Thus we have

$$\begin{aligned}\mathcal{P} &= \left(\frac{\pi ck^2}{\mu_0}\right)\int_0^{\infty}|\mathcal{F}_{\ell,p}|^2\rho d\rho\\ &= \left(\frac{\pi ck^2}{\mu_0}\right)\frac{1}{4}w_0^2\mathcal{E}_0^2\\ &= \left(\frac{\pi\omega^2\epsilon_0 cw_0^2}{4}\right)\mathcal{E}_0^2\end{aligned} \tag{36}$$

where we have substituted for $\mathcal{F}_{\ell,p}$ from eq.(5) and made use of the standard integral

$$\int_0^{\infty}x^{|\ell|}e^{-x}[L_p^{|\ell|}(x)]^2dx=\frac{(p+|\ell|)!}{p!} \tag{37}$$

We obtain finally for the normalisation constant $\mathcal{E}_0$

$$\mathcal{E}_0^2 = \frac{4\mathcal{P}}{\epsilon_0 c\pi\omega^2 w_0^2} \tag{38}$$

APPENDIX B

VERIFICATION OF EQ.(29)

In this appendix we show that the space integral of $\bar{\eta}_{\ell,p,0}$ which is the same as total helicity of the linearly-polarised LG beam ($\sigma = 0$) is identically zero. On substituting for $\mathcal{F}$ from Eq.(5) in Eq.(27), we find that the relevant expression for radial integration is

$$\begin{aligned}\mathcal{I}_{\ell,p} &= \int_0^\infty \frac{\mathcal{R}}{\tilde{\rho}} e^{-2\tilde{\rho}^2}(2\tilde{\rho}^2)^{|\ell|}\left[L_p^{|\ell|}(2\tilde{\rho}^2)\right]^2 \tilde{\rho}d\tilde{\rho} \\ &= \int_0^\infty \left\{-\tilde{\rho}^2 + \frac{|\ell|}{\tilde{\rho}} - 4\tilde{\rho}^2\frac{L_{p-1}^{\ell+1}(2\tilde{\rho}^2)}{L_p^{|\ell|}(2\tilde{\rho}^2)}\right\}\frac{1}{\tilde{\rho}}e^{-2\tilde{\rho}^2}(2\tilde{\rho}^2)^{|\ell|}\left[L_p^{|\ell|}(2\tilde{\rho}^2)\right]^2\tilde{\rho}d\tilde{\rho}\end{aligned} \tag{39}$$

where $\tilde{\rho} = \rho/w_0$ and we have inserted the expression for $\mathcal{R}$ from Eq.(16). We are left with three integrals to deal with. The denominator $L_p^{|\ell|}(2\tilde{\rho}^2)$ in the third term between the brackets cancels with one of $\left[L_p^{|\ell|}(2\tilde{\rho}^2)\right]^2$, rendering the integrand well-behaved with no zeros in the denominator. It is then convenient to change variable using the substitution $x = 2\tilde{\rho}^2$, we now have

$$\begin{aligned}\mathcal{I}_{\ell,p} &= -\int_0^\infty e^{-x}x^{|\ell|}\left[L_p^{|\ell|}(x)\right]^2 dx \\ &+ |\ell|\int_0^\infty e^{-x}x^{|\ell|-1}\left[L_p^{|\ell|}(x)\right]^2 dx - 2\int_0^\infty L_{p-1}^{|\ell|+1}(x)L_p^{|\ell|}(x)e^{-x}x^{|\ell|}dx\end{aligned} \tag{40}$$

The last two integrals can be combined into one integral on making use of the identity

$$\frac{d}{dx}L_p^{|\ell|}(x) = -L_{p-1}^{|\ell|+1}(x) \tag{41}$$

We thus have

$$\mathcal{I}_{\ell,p} = -\int_0^\infty e^{-x}x^{|\ell|}\left[L_p^{|\ell|}(x)\right]^2 dx + \int_0^\infty e^{-x}\frac{d}{dx}\left\{x^{|\ell|}\left[L_p^{|\ell|}(x)\right]^2\right\}dx \tag{42}$$

The last integral can be integrated by parts to yield

$$\begin{aligned}\mathcal{I}_{\ell,p} &= -\int_0^\infty e^{-x}x^{|\ell|}\left[L_p^{|\ell|}(x)\right]^2 dx + \left\{e^{-x}x^{|\ell|}\left[L_p^{|\ell|}(x)\right]^2\right|_0^\infty \\ &+ \int_0^\infty e^{-x}x^{|\ell|}\left[L_p^{|\ell|}(x)\right]^2 dx\end{aligned} \tag{43}$$

The second term vanishes at both limits and we are left with the final result $\mathcal{I}_{\ell,p} = 0$. This proves that the space integral of the helicity density $\bar{\eta}_{\ell,p}$ vanishes on the basis of a vanishing radial integral and we can write

$$\bar{\mathcal{C}}_{\ell,p,0} = 0; \quad (\sigma = 0) \tag{44}$$

APPENDIX C

EVALUATION OF TOTAL HELICITY OF LG$_{\ell,p}$ FOR $\sigma \neq 0$

The total helicity in this general case is given by the volume integral of the helicity density, Eq.(26). We have after performing the ϕ and the z integrals

$$\begin{aligned}\bar{\mathcal{C}}_{\ell,p,\sigma} &= 2\pi d\int_0^\infty \rho d\rho\;\bar{\eta}_\sigma \\ &= \left(\frac{\epsilon_0 c^2\pi d}{2\omega}\right)\sigma\int_0^\infty\left[2k^2 + \mathcal{R}^2 + (\ell/\rho)^2\right]|\mathcal{F}_{\ell,p}|^2\rho d\rho\end{aligned} \tag{45}$$

where $\mathcal{F}_{\ell,p}$ is as is given by Eq.(5) and $\mathcal{R}$ by Eq.(16). On inserting these functions we find that we have to deal with seven ρ integrals as follows

$$\bar{\mathcal{C}}_{\ell,p,\sigma} = \left(\frac{\epsilon_0 c^2\pi d}{2\omega}\right)\sigma\sum_{j=1}^{7} I_j \tag{46}$$

We have the following results, some of which require considerable manipulations

$$I_1 = \int_0^\infty 2k^2|\mathcal{F}_{\ell,p}|^2\rho d\rho = \frac{1}{2}\mathcal{E}_0^2 k^2 w_0^2 \tag{47}$$

$$I_2 = \frac{4|\ell|}{w_0^2}\int_0^\infty |\mathcal{F}_{\ell,p}|^2\rho d\rho = \mathcal{E}_0^2|\ell| \tag{48}$$

$$I_3 = \frac{4}{w_0^2}\int_0^\infty \tilde{\rho}^2|\mathcal{F}_{\ell,p}|^2\rho d\rho = \frac{1}{2}\mathcal{E}_0^2|(2p+|\ell|+1) \tag{49}$$

$$I_4 = \frac{8p!}{(p+|\ell|)!}\mathcal{E}_0^2\int_0^\infty \tilde{\rho}^2 x^{|\ell|}e^{-x}[L_{p-1}^{|\ell|+1}(x)]^2 d\tilde{\rho} = 2\mathcal{E}_0^2 p \tag{50}$$

$$I_5 = |\ell|^2\int_0^\infty \frac{1}{\tilde{\rho}^2}|\mathcal{F}_{\ell,p}|^2\tilde{\rho}d\tilde{\rho} = \frac{1}{2}|\ell|\mathcal{E}_0^2 \tag{51}$$

$$I_6 = \frac{16}{w_0^2}\int_0^\infty \tilde{\rho}^2\frac{L_{p-1}^{|\ell|+1}(x)}{L_p^{|\ell|}(x)}|\mathcal{F}_{\ell,p}|^2\rho d\rho = -2p\mathcal{E}_0^2 \tag{52}$$

$$I_7 = -\frac{8|\ell|}{w_0^2}\int_0^\infty \frac{L_{p-1}^{|\ell|+1}(x)}{L_p^{|\ell|}(x)}|\mathcal{F}_{\ell,p}|^2\rho d\rho = 0 \tag{53}$$

In the above integrals the variables $\tilde{\rho} = \rho/w_0$ and $x = 2\tilde{\rho}^2$. The evaluations make use of a number of standard integrals involving associated Laguerre polynomials. In particualar:

$$\int_0^\infty x^{|\ell|+1}e^{-x}[L_p^{\ell}(x)]^2 = \frac{(p+|\ell|)!}{p!}(2p+|\ell|+1) \tag{54}$$

and

$$\int_0^\infty x^{|\ell|}e^{-x}L_{p-1}^{|\ell|+1}(x)L_p^{|\ell|}(x)dx = 0 \tag{55}$$

Collecting terms from the results of the integrals we have

$$\begin{aligned}
\bar{C}_{\ell,p} &= \left(\frac{\epsilon_0 c^2\pi d}{2\omega}\right)\sigma\left\{\frac{1}{2}(k^2w_0^2\mathcal{E}_0^2) + \frac{1}{2}\mathcal{E}_0^2(2p+|\ell|+1) + |\ell|\mathcal{E}_0^2 + 2p\mathcal{E}_0^2 - |\ell|\mathcal{E}_0^2 - 2p\mathcal{E}_0^2 + 0\right\} \\
&= \left(\frac{k^2w_0^2\epsilon_0 c^2\pi d}{4\omega}\right)\mathcal{E}_0^2\left\{1 + \frac{2p+|\ell|+1}{k^2w_0^2}\right\}\sigma \\
&= \left(\frac{k^2w_0^2\epsilon_0 c^2\pi d}{4\omega}\right)\left(\frac{4\mathcal{P}}{\epsilon_0 c\pi\omega^2 w_0^2}\right)\left\{1 + \frac{2p+|\ell|+1}{k^2w_0^2}\right\}\sigma \\
&= \frac{k^2 cd\mathcal{P}}{\omega^3}\left\{1 + \frac{2p+|\ell|+1}{k^2w_0}\right\}\sigma \\
&= k^2w_0^2\left(\frac{\mathcal{P}}{\omega^2}\right)\left\{1 + \frac{2p+|\ell|+1}{k^2w_0^2}\right\}\sigma
\end{aligned} \tag{56}$$

where we have substituted for the diffraction length $d = w_0^2 k$. Thus we have

$$Q_0 = k^2w_0^2\left(\frac{\mathcal{P}}{\omega^2}\right) \tag{57}$$

APPENDIX D

OPTICAL SPIN AND SPIN DENSITY OF A LINEARLY-POLARISED LG

The cycle-averaged optical spin angular momentum (SAM) density is given by

$$\bar{\mathbf{s}} = \frac{\epsilon_0}{\omega}\Im[\mathbf{E}^*\times\mathbf{E}] \tag{58}$$

Consider a linearly-polarised Laguerre-Gaussian beam $\mathrm{LG}_{\ell,p}$ which has the following electric field

$$\mathbf{E} = ick\hat{\mathbf{x}}\mathcal{F}e^{ikz} - c\left\{\frac{\partial\mathcal{F}}{\partial x}\right\}e^{ikz}\hat{\mathbf{z}} \tag{59}$$

where for a linearly-polarised mode we set $\beta = 0$ and $\alpha = 1$ in Eq.(10) and dropped the labels ℓ, p in $\mathcal{F}$. The x-derivative of $\mathcal{F}$ is as follows

$$\left(\frac{\partial \mathcal{F}}{\partial x}\right) = \left(\mathcal{R}\cos\phi - i\mathcal{T}\sin\phi\right)\mathcal{F} \tag{60}$$

where $\mathcal{R}$ and $\mathcal{T}$ are as given in Eqs.(16) and (17).

The cycle averaged z-component s_z of the spin density is proportional to $E_x^*E_y - E_y^*E_x$ and since $E_y = 0$ this component is zero. Similarly the x-component s_x is proportional to $E_y^*E_z - E_z^*E_y$ and so this is also zero. The only surviving component of the optical spin density is thus the y-component which is proportional to $E_xE_z^* - E_zE_x^*$. Thus we have

$$s_y = \Im\left\{\frac{i\epsilon_0 c^2 k}{\omega}\left(\mathcal{F}^*\left[\frac{\partial \mathcal{F}}{\partial x}\right] + \mathcal{F}\left[\frac{\partial \mathcal{F}}{\partial x}\right]^*\right)\right\} \tag{61}$$

We obtain on substituting from Eq.(60)

$$\begin{aligned} s_y &= \frac{\epsilon_0 c^2 k}{\omega}\left(\mathcal{F}^*\mathcal{R}\mathcal{F} + \mathcal{F}R\mathcal{F}^*\right)\cos\phi \\ &= \frac{2\epsilon_0 c^2 k}{\omega}\mathcal{R}|\mathcal{F}|^2\cos\phi \end{aligned} \tag{62}$$

This spin density is in the direction of y, orthogonal to both the linear polarisation along x and the longitudinal component along z. It has a spatial distribution, which can be displayed.

The volume integral of the density s_y is zero because of a vanishing angular integration. Hence the total spin **S** vector is clearly zero since also s_x and s_z are both zero We then have

$$\mathbf{S} = \int_{-d/2}^{d/2} dz \int_0^{2\pi} d\phi \int_0^{\infty} \rho d\rho \mathbf{s} = \mathbf{0} \tag{63}$$

which confirms that the total cycle-averaged spin angular momentum is zero despite the fact that there is a finite spin density distribution.