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A Geometric Approach to Some Systems of Exponential Equations

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Zilber's Exponential Algebraic Closedness conjecture (also known as Zilber's Nullstellensatz) gives conditions under which a complex algebraic variety should intersect the graph of the exponential map of a semiabelian variety. We prove the special case of the conjecture where the variety has dominant projection to the domain of the exponential map, for abelian varieties and for algebraic tori. Furthermore, in the situation where the intersection is 0-dimensional, we exhibit structure in the intersection by parametrizing the sufficiently large points as the images of the period lattice under a (multivalued) analytic map. Our approach is complex geometric, in contrast to a real analytic proof given by Brownawell and Masser just for the case of algebraic tori.

1 Introduction

In his model-theoretic study of the complex exponential function, Zilber [20] asked what systems of equations built from polynomials and the exponential function have solutions in the complex field. The analogous question just for polynomials is solved by the fundamental theorem of algebra and the Hilbert Nullstellensatz.

The fact that the exponential map is a homomorphism of algebraic groups places some restrictions. For example, the following system of equations does not have a

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solution in $\mathbb{C}$:

$$\begin{cases} 2z_1 = z_2 + 1, \\ (e^{z_1})^2 = e^{z_2}, \end{cases}$$

because $(e^{z_1})^2 = e^{2z_1}$ and so having a solution would imply $e^1 = 1$.

Further strong restrictions are predicted by Schanuel's conjecture of transcendental number theory [15, p. 30], which asserts $\operatorname{tr.deg.}_{\mathbb{Q}}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geqslant n$ for any complex numbers $z_1, \ldots, z_n$ that are linearly independent over $\mathbb{Q}$. For example, a simple application of the conjecture gives the algebraic independence of e and π (an open question), so it would follow that for any non-zero rational polynomial $p(z, w)$, there is no solution to the system of equations

$$e^z = -1, \qquad p(z, e^1) = 0.$$

Zilber formulated a precise conjecture that captures the idea that every system of equations should have a solution unless that would contradict Schanuel's conjecture, which we call his *Exponential Algebraic Closedness* conjecture or *EAC* conjecture (sometimes also called *Zilber's Nullstellensatz* [8]).

The EAC conjecture is expressed in geometric terms. Let $\mathbb{G}_{\mathrm{m}}^n$ be the algebraic torus of dimension n. Since we are exclusively working over $\mathbb{C}$, we shall identify $\mathbb{G}_{\mathrm{m}}$ with its complex points, so $\mathbb{G}_{\mathrm{m}} = \mathbb{G}_{\mathrm{m}}(\mathbb{C}) = \mathbb{C}^\times$.

Conjecture 1.1 (EAC [20]). Let $V \subseteq \mathbb{C}^n \times \mathbb{G}_{\mathrm{m}}^n$ be a free and rotund variety. Then there is a point $\boldsymbol{z} \in \mathbb{C}^n$ such that $(\boldsymbol{z}, \exp(\boldsymbol{z})) \in V$.

Here $\boldsymbol{z} = (z_1, \ldots, z_n)$ and $\exp(\boldsymbol{z})$ means the tuple $(e^{z_1}, \ldots, e^{z_n})$. The *freeness* property in Conjecture 1.1 is related to the constraints from the exponential map being a group homomorphism, and *rotundity* is related to the constraints from Schanuel's conjecture. We shall omit the precise definitions of these properties, as they are slightly technical and we will not need them.

If both the Schanuel and the EAC conjectures are true, they would give a complete characterisation of the systems which have solutions. Zilber also showed in [20] (see also [3]) that, together, Schanuel's conjecture and a stronger form of EAC imply strong consequences for the model-theoretic structure $(\mathbb{C}, +, \times, \exp)$, in particular that it is quasiminimal, a previous conjecture by Zilber [19] that is still open to this day.

The number-theoretic part of Schanuel's conjecture seems out of reach. However, the functional part of Schanuel's conjecture was proved by Ax [2] and that implies that it is generically true [13, Theorem 1.4]. In particular, a positive solution to Zilber's EAC conjecture would characterize the systems for which the existence of solutions is essentially a number-theoretic transcendence problem, rather than a functional transcendence or geometric problem. Moreover, we now know that EAC directly implies the quasiminimality of $(\mathbb{C}, +, \times, \exp)$ [4, Theorem 1.5].

Apart from the classical exponential function, one can consider other periodic functions such as the Weierstrass $\wp$-functions and their derivatives. For example, in Section 2 of the paper, we will describe solutions to the following equation.

Example 1.2. Let $\wp$ be any Weierstrass $\wp$-function. Then there are $z \in \mathbb{C}$ such that $\wp'(\wp(z)^2) = z$, and indeed we can find 12 infinite families of solutions parametrized by the pairs $(\omega_1, \omega_2) \in \Lambda^2$, for $|\omega_1|, |\omega_2|$ sufficiently large, where Λ is the period lattice of $\wp$.

The $\wp$-functions are essentially the exponential maps of elliptic curves. Our method for $\wp$ also applies to a wide range of systems of equations, dealing with the exponential maps of abelian or semiabelian varieties.

In this generality, the EAC conjecture becomes the following.

Conjecture 1.3 (EAC for semiabelian varieties). Let S be a complex semiabelian variety of dimension n, and write $\exp_S : \mathbb{C}^n \to S$ for its exponential map. Let $V \subseteq \mathbb{C}^n \times S$ be a free and rotund subvariety. Then there is $\boldsymbol{z} \in \mathbb{C}^n$ such that $(\boldsymbol{z}, \exp_S(\boldsymbol{z})) \in V$.

Again, the notions of freeness and rotundity relate to $\exp_S$ being a homomorphism and to the semiabelian version of Schanuel's conjecture. We will not need them in this paper. We refer the interested reader to [12, Definition 2.26] or [4, Definition 7.1] for more details. At least when S is simple, EAC for S also implies that the structure $(\mathbb{C}, +, \times, \exp_S)$ is quasiminimal [4, Theorem 1.9].

As a notational convention, since all the algebraic varieties we will consider will be defined over $\mathbb{C}$, we will identify them with their sets of $\mathbb{C}$-points. For example, above we write S and V rather than $S(\mathbb{C})$ and $V(\mathbb{C})$. We will also write $\mathbb{P}_n$ for complex projective n-space rather than $\mathbb{P}_n(\mathbb{C})$. We denote points in affine and projective spaces by boldface letters such as $\boldsymbol{z}$ and their coordinates by standard letters with subscripts such as z_1. We say that a point in V of the form $(\boldsymbol{z}, \exp_S(\boldsymbol{z}))$ is an *exponential point of V*.

In this paper, we establish the following family of instances of Conjecture 1.3 in the case of abelian varieties.

Theorem 1.4. Let A be a complex abelian variety of dimension n. Let $V \subseteq \mathbb{C}^n \times A$ be an algebraic subvariety with dominant projection to $\mathbb{C}^n$, that is, its projection to $\mathbb{C}^n$ has dimension n. Then there is $\boldsymbol{z} \in \mathbb{C}^n$ such that $(\boldsymbol{z}, \exp_A(\boldsymbol{z})) \in V$.

It will follow easily from our proof that the set $\{\boldsymbol{z} \in \mathbb{C}^n : (\boldsymbol{z}, \exp_A(\boldsymbol{z})) \in V\}$ is in fact Zariski dense in $\mathbb{C}^n$ and actually that the points $(\boldsymbol{z}, \exp_A(\boldsymbol{z}))$ are Zariski dense in V. Moreover, we show that almost all of the large solutions are parametrized in terms of the period lattice Λ of $\exp_A$. (See Theorem 4.1 for the details.)

A subvariety with dominant projection as in this statement is automatically rotund and can be easily reduced to a free and rotund subvariety. Hence, Theorem 1.4 is indeed a special case of Conjecture 1.3.

The analogous theorem for algebraic tori was proven by Brownawell and Masser.

Theorem 1.5 ([6, Prop. 2]). Let $V \subseteq \mathbb{C}^n \times \mathbb{G}_{\mathrm{m}}^n$ be an algebraic subvariety with dominant projection to $\mathbb{C}^n$. Then there is a point $\boldsymbol{z} \in \mathbb{C}^n$ such that $(\boldsymbol{z}, \exp(\boldsymbol{z})) \in V$.

They used Newton's iterative method to approximate solutions, and in particular Kantorovich's theorem that gives criteria for these approximations to converge to an actual solution. Another account of the same proof is given in [7]. A similar theorem for the modular j-function was established in [9] using Rouché's theorem of complex analysis in place of Kantorovich's theorem. Using our methods, we also give a new proof of Theorem 1.5.

Unlike [6] and [7], in our proofs of Theorems 1.4 and 1.5, we exploit the geometry and topology of the system as much as possible; we do not use Kantorovich's theorem or Rouché's theorem.

To explain our approach, it is easiest to go back to the idea of the proof of Brownawell and Masser. If $(\boldsymbol{z}, \boldsymbol{w}) \in V$ with $\boldsymbol{z} \in \mathbb{C}^n$ and $\boldsymbol{w} \in \mathbb{G}_{\mathrm{m}}^n$, then the dominant projection assumption means that generically we can regard $\boldsymbol{w}$ as $\alpha(\boldsymbol{z})$, where α is an algebraic map. The problem reduces to finding a zero of

$$F(\boldsymbol{z}) := \exp(\boldsymbol{z}) - \alpha(\boldsymbol{z}).$$

They consider $\boldsymbol{z} \to \infty$ in a small complex neighborhood of a real straight line and prove (after some rescaling) that when $\boldsymbol{z} = \boldsymbol{\lambda} + \mathrm{Log}(\alpha(\boldsymbol{\lambda}))$ with $\boldsymbol{\lambda} \in (2\pi i\mathbb{Z})^n$ then $F(\boldsymbol{z})$ is small enough that Newton's method will converge to a zero of F near it. They then observe that the solutions they find are indexed by "sufficiently many" lattice points to give a Zariski dense set of solutions.

In the general abelian case, we face two technical challenges. The exponential and logarithmic maps of $\mathbb{G}_m$ are very well understood and are easy to differentiate explicitly to perform the necessary computations. This is less practical for abelian varieties. Moreover, whereas in the $\mathbb{G}_m$ setting both $\exp(\boldsymbol{z})$ and $\alpha(\boldsymbol{z})$ lie in $\mathbb{G}_m^n \subseteq \mathbb{C}^n$, in the abelian setting both quantities lie in A, which is a projective variety.

We choose a convenient affine chart by taking the logarithm, so we work in the covering space $\mathbb{C}^n$, and we define a new map F (at least locally) as

$$F(\boldsymbol{z}) := \boldsymbol{z} - \mathrm{Log}_A(\alpha(\boldsymbol{z})).$$

Instead of looking for zeros of F, we now want to find $\boldsymbol{z} \in \mathbb{C}^n$ such that $\exp_A(F(\boldsymbol{z})) = 0_A$, or equivalently, $F(\boldsymbol{z}) \in \Lambda$, the period lattice of $\exp_A$. Essentially from the compactness of A, we show that the 2nd term $\mathrm{Log}_A(\alpha(\boldsymbol{z}))$ is convergent as $\boldsymbol{z} \to \infty$ along most real lines, and so F is asymptotically a translation and hence is locally invertible. Writing S for the local inverse, the (sufficiently large) points $\boldsymbol{z}$ such that $\alpha(\boldsymbol{z}) = \exp_A(\boldsymbol{z})$ are then $S(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \Lambda$. So as well as finding solutions, we give an analytic parametrization of them by lattice points.

The algebraic map α, the logarithm Log_A, and the maps F and S are multivalued maps, so the above argument will be done locally around points at infinity, after embedding $\mathbb{C}^n$ into $\mathbb{P}_n$ in the usual way. By considering all the branches of α and Log_A, we can then parametrize the points $(\boldsymbol{z}, \exp_A(\boldsymbol{z})) \in V$ locally via the corresponding branches of S.

To summarize how S parametrizes the solutions globally, we shall ultimately describe S as a sheaf of analytic functions (in the sense of Remark 3.3) on an open subset Ω^* of $\mathbb{P}_n$, which is "large" in the sense that it contains a Zariski open dense subset of the hyperplane at infinity. We shall verify that every $\boldsymbol{z} \in \Omega^* \cap \mathbb{C}^n$ such that $(\boldsymbol{z}, \exp_A(\boldsymbol{z})) \in V$ is of the form $\boldsymbol{z} = S(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in \Lambda$.

Our method also works in the algebraic torus case. Since $\mathbb{G}_m$ is not compact, a little more analysis of growth rates is needed, although still less than in the Brownawell–Masser proof. In some final remarks at the end of the paper, we discuss how far this method might be pushed.

The *strong* EAC conjecture of Zilber, alluded to above, incorporates a transcendence condition: given any finitely generated subfield K of $\mathbb{C}$, one asks for $(\boldsymbol{z}, \exp_S(\boldsymbol{z}))$, which is generic in V over K. It seems likely that one could deduce transcendence results of this type for V with dominant projection to $\mathbb{C}^n$, assuming the appropriate form of Schanuel's conjecture, in the style of [7]. In fact, for $S = \mathbb{G}_m$, if one assumes both Schanuel's conjecture and the Zilber–Pink conjecture, then the strong EAC is equivalent to EAC ([14, Thm. 1.5]).

Overview of the paper Before developing the general theory, we outline the method of proof for Example 1.2 in Section 2. The multivalued nature of the maps becomes clear as we only have to take square roots and cube roots to describe the corresponding algebraic map α.

In Section 3, we explain how we extract the algebraic map α from the algebraic variety V. As we are interested in the behavior as we approach infinity, we take care in explaining how α can be extended continuously to these points at infinity (in the projective space $\mathbb{P}_n$) where it may fail to be analytic. The content of this section is folklore, but we give a self-contained account.

We state and prove Theorem 4.1 explaining the solution map S and its properties in Section 4. Theorem 1.4 follows, and this also covers Example 1.2. In Section 5, we indicate how to adapt this work for the algebraic torus setting, to give a new proof of Theorem 1.5. We close with some final remarks.

2 An Example

Consider the equation

$$\wp'(\wp(z)^2) = z, \tag{1}$$

from Example 1.2, where $\wp$ is a Weierstrass $\wp$-function. Our analysis will work uniformly for any $\wp$-function, but to be definite, let $\wp$ be the Weierstrass $\wp$-function associated to the lattice $\Lambda := \mathbb{Z} + i\mathbb{Z}$.

We want to determine whether (1) has any solutions in $\mathbb{C}$ and, if so, where those solutions are.

It is well known that

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

for certain $g_2, g_3 \in \mathbb{C}$, and that the map

$$z \mapsto [1 : \wp(z) : \wp'(z)]$$

gives an embedding of $\mathbb{C}/\Lambda$ into the projective space $\mathbb{P}_2$. The image of $\mathbb{C}/\Lambda$ is an elliptic curve $E \subseteq \mathbb{P}_2$, and the above map is its exponential map $\exp_E : \mathbb{C} \to E$.

To exploit the geometry of elliptic curves, we consider $\wp$ and $\wp'$ in (1) as components of the exponential map. However, for simplicity, we will write equations and maps in affine coordinates. In particular, if $O := [0:0:1]$ is the point at infinity of E (which is also the identity element of the group structure of E), then the affine part of E is $E \setminus \{O\} \subseteq \mathbb{C}^2$.

Consider the following system of equations:

$$\begin{cases} x_k = \wp(z_k),\ y_k = \wp'(z_k) & \text{for } k = 1, 2, \\ y_k^2 = 4x_k^3 - g_2 x_k - g_3 & \text{for } k = 1, 2, \\ z_2 = x_1^2, \\ z_1 = y_2. \end{cases} \tag{2}$$

The equations on the 2nd line of (2) state that (x_1, y_1) and (x_2, y_2) lie on E. When combined with the last two equations, they define a subvariety V of $\mathbb{C}^2 \times E^2$. The solutions of (2) are the points $(\boldsymbol{z}, \exp_{E^2}(\boldsymbol{z})) \in V$. One can easily verify that the coordinate z_1 of such a point is a solution of (1) and that all solutions of (1) arise in this way.

We think of V as expressing a point $\boldsymbol{w} \in E^2$ as an algebraic function $\alpha = (\alpha_1, \alpha_2)$ of $\boldsymbol{z}$, that is, we have

$$\alpha_1(\boldsymbol{z}) = \left(\sqrt{z_2}, \sqrt{4z_2^{3/2} - g_2 z_2^{1/2} - g_3}\right),$$

$$\alpha_2(\boldsymbol{z}) = \left(\beta(z_1), z_1\right),$$

where

$$\beta(z_1) := \sqrt[3]{\frac{g_3 - z_1^2}{8} + \sqrt{\frac{(g_3 - z_1^2)^2}{64} - \frac{g_2^3}{1728}}} + \sqrt[3]{\frac{g_3 - z_1^2}{8} - \sqrt{\frac{(g_3 - z_1^2)^2}{64} - \frac{g_2^3}{1728}}}$$

is obtained by solving the cubic equation $4v^3 - g_2 v - g_3 = z_1^2$ with respect to v.

To be more precise, we have to choose single-valued branches of the square and cube roots, which we can do by restricting α to a suitable simply connected domain $D \subseteq \mathbb{C}^2$. Since α_1 and α_2 depend only on z_2 and z_1, respectively, we can define them separately. For $k = 1, 2$, let $N_k \subseteq \mathbb{C}$ be a closed disc around the origin containing the zeroes of the expressions appearing in the square and cube roots involved in α_k. Also, consider the line (a branch cut) $B := \mathbb{R}^{<0} \subseteq \mathbb{C}$. Now set $D_k := \mathbb{C} \setminus (N_k \cup B)$. Then D_1, D_2 are simply connected, and we find 4 branches of α_1 and 3 branches of α_2, respectively, on D_1 and D_2. Altogether, we get 12 branches of α on $D := D_1 \times D_2$, and we pick one of those.

We take a fundamental domain

$$M := \{x + iy : -1/2 < x, y \leqslant 1/2\}$$

for $\exp_E$. Let $\mathrm{Log}_{E^2} : E^2 \to M^2$ be the logarithmic map for this domain.

Now pick $\boldsymbol{z} \in D$ with $|z_1|, |z_2|$ sufficiently large. Thus, given some metric inducing the complex topology on $\mathbb{P}_2$, we can say that $\alpha(\boldsymbol{z})$ is close to the point at infinity (O, O). So $\mathrm{Log}_{E^2}\alpha(\boldsymbol{z}) \approx (0, 0) \in \mathbb{C}^2$.

Define a map

$$F : D \to \mathbb{C}^2 : \boldsymbol{z} \mapsto \boldsymbol{z} - \mathrm{Log}_{E^2}\alpha(\boldsymbol{z}).$$

Asymptotically, we have $F(\boldsymbol{z}) = \boldsymbol{z} + o(1)$ as $|z_1|, |z_2| \to \infty$. So F is locally invertible and indeed, shrinking D if necessary and staying away from the boundary, we find a connected open set $\tilde{D}$, which is in fact the image of D under F, and a map $S : \tilde{D} \to D$ which is the inverse of F. Moreover, we also have

$$S(\boldsymbol{z}) = \boldsymbol{z} + \mathrm{Log}_{E^2}\alpha(\boldsymbol{z}) + o(1) = \boldsymbol{z} + o(1) \tag{3}$$

for $|z_1|, |z_2| \to \infty$ with $\boldsymbol{z} \in \tilde{D}$.

Now for a lattice point $\boldsymbol{\lambda} \in \Lambda^2 \cap \tilde{D}$ we have

$$\boldsymbol{\lambda} = F(S(\boldsymbol{\lambda})) = S(\boldsymbol{\lambda}) - \mathrm{Log}_{E^2}\alpha(S(\boldsymbol{\lambda})), \tag{4}$$

hence $\alpha(S(\boldsymbol{\lambda})) = \exp_{E^2}(S(\boldsymbol{\lambda}))$. Therefore, the point $S(\boldsymbol{\lambda})$ is a solution to the equation $\alpha(\boldsymbol{z}) = \exp_{E^2}(\boldsymbol{z})$, and in fact $\{S(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \in \Lambda^2 \cap \tilde{D}\}$ is the set of all solutions of (4) in D.

Finally, one observes that since F is asymptotically the identity, the set $\tilde{D}$ almost contains D, in the sense that every point of D sufficiently far from the boundary of D must be in $\tilde{D}$. It follows that $\tilde{D} \cap \Lambda^2$ is not empty and in fact contains most points of $D \cap \Lambda^2$. This proves that (2) has solutions. In particular, (1) does too: if $S = (S_1, S_2)$ with $S_1, S_2 : \tilde{D} \to \mathbb{C}$, then for each $\boldsymbol{\lambda} \in \Lambda^2$ the element $S_1(\boldsymbol{\lambda})$ is a solution to (1).

By repeating the argument for all the possible branches of S, and by rotating the branch cuts of D_1, D_2, one can verify that all the solutions of (2) with z_1, z_2 sufficiently large arise in this way. Since D is a large subset of $\mathbb{C}^2$, these actually give almost all the solutions of (1) such that z and $\wp(z)$ are both sufficiently large.

Furthermore, we can use the parametrization of the large solutions of (2) by S to understand their geometric distribution. From (4), we have

$$\lambda_1 = S_1(\boldsymbol{\lambda}) - \wp^{-1}\left(\sqrt{S_2(\boldsymbol{\lambda})}\right),$$
$$\lambda_2 = S_2(\boldsymbol{\lambda}) - (\wp')^{-1}(S_1(\boldsymbol{\lambda})).$$

Therefore,

$$S_1(\boldsymbol{\lambda}) = \lambda_1 + \wp^{-1}\left(\sqrt{\lambda_2 + (\wp')^{-1}(S_1(\boldsymbol{\lambda}))}\right).$$

From this and from (3), we may conclude that

$$\begin{aligned} S_1(\boldsymbol{\lambda}) &= \lambda_1 + \wp^{-1}\left(\sqrt{\lambda_2 + (\wp')^{-1}(\lambda_1 + o(1))}\right) \\ &= \lambda_1 + \wp^{-1}\left(\sqrt{\lambda_2 + (\wp')^{-1}(\lambda_1)}\right) + o\left(\wp^{-1}\left(\sqrt{\lambda_2 + (\wp')^{-1}(\lambda_1)}\right)\right) \end{aligned}$$

as $|\lambda_1|, |\lambda_2| \to \infty$ with $\boldsymbol{\lambda} \in \Lambda^2 \cap \tilde{D}$.

Since $\wp$ and $\wp'$ are two to one and three to one, respectively, on the fundamental domain M, and $z \mapsto z^2$ is two to one, one can easily verify that S_1 takes 12 distinct values on each (λ_1, λ_2) according to the choice of the branches.

3 Algebraic Maps

The proof of Theorem 1.5 in [6] and in particular the account of [7] make use of *algebraic functions*. In [7], those are defined as analytic functions $\alpha : D \to \mathbb{C}$ over some domain $D \subseteq \mathbb{C}^n$ satisfying a non-trivial polynomial equation $P(\boldsymbol{z}, \alpha(\boldsymbol{z})) = 0$. In both papers, the authors restrict the choice of the domains D in order to have the appropriate asymptotic behavior at infinity.

We shall reduce our reliance on detailed asymptotic estimates at infinity in favor of topological and geometric considerations. In Theorems 1.4 and 1.5, we may assume that $\dim V = n$ after taking some intersections with generic hyperplanes. We then consider the Zariski closure $\overline{V}$ of V inside $\mathbb{P}_n \times A$, where A is either the given abelian variety or a suitable completion of $\mathbb{G}_{\mathrm{m}}^n$ and $\mathbb{C}^n$ is embedded into $\mathbb{P}_n$ is the usual way.

In this setting, the projection $\pi : \overline{V} \to \mathbb{P}_n$ is surjective, and by dimension considerations, all of its fibers are finite outside of a proper Zariski closed subset of $\mathbb{P}_n$.

We then work with continuous maps $\alpha : D^* \subseteq \mathbb{P}_n \to \overline{V} \to A$ such that $(\boldsymbol{z}, \alpha(\boldsymbol{z})) \in \overline{V}$ for all $\boldsymbol{z} \in D^*$, where D^* is some set to be specified later. We can create such a map by composing a continuous section of $\pi : \overline{V} \to \mathbb{P}_n$ with the projection $\overline{V} \to A$. We wish to understand the behavior of such an α at infinity, namely at the points of $\mathbb{P}_n \setminus \mathbb{C}^n$.

It is well known that the analytic local sections $D \subseteq \mathbb{P}_n \to \overline{V}$, where D are suitable open domains, form a sheaf of complex analytic maps. The domains, however, only cover a Zariski open dense subset of $\mathbb{P}_n$, which may well omit all of the points at infinity. We remedy this by taking continuous extensions to some $D^* \supseteq D$ containing points on the boundary of D.

We thus obtain maps $\alpha : D^* \to A$, which are continuous, but possibly not analytic, at the points at infinity. The continuity at infinity will encode the asymptotic information needed for the proof of Theorem 1.4. One could perhaps perform some local resolution of singularities in the style of Bierstone and Milman [5] to make the maps analytic everywhere, but it is not necessary.

The use of continuous extensions of analytic maps is classical, but for the sake of clarity, we state Proposition 3.2 below to pin down which maps we use, and we provide a self contained proof referring to elementary algebraic geometry and algebraic topology. We also make some definite choices of neighborhoods and sets to ensure we always deal with clearly defined single-valued functions.

Before going further into the technical details, let us work with an elementary example. Identify $\mathbb{P}_1$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Let $\rho : \mathbb{P}_1 \to \mathbb{P}_1$ be the map $z \mapsto z^2 + 1$. The fibers of ρ have cardinality 2, except over the branching points 1 and ∞, which have fibers $\{0\}$ and $\{\infty\}$ respectively. The restriction of ρ to $\mathbb{P}_1 \setminus \{0, \infty\} = \mathbb{C} \setminus \{0\}$ is a *covering map*: for each $w \in \mathbb{C} \setminus \{1\}$, there is an open neighborhood D of w in the complex topology such that $\rho^{-1}(D)$ splits into a disjoint union of open sets D_i where $\rho\restriction_{D_i}$ is a homeomorphism between D_i and D.

If we remove a branch cut, for instance by taking $D = \mathbb{C} \setminus \mathbb{R}_{\geqslant 0} \subseteq \mathbb{P}_1 \setminus \{0, \infty\}$, we obtain a simply connected domain, hence by standard topological arguments there are two analytic sections $\iota_1, \iota_2 : D \to \mathbb{P}_1$ of ρ, and $\rho^{-1}(D) = \iota_1(D) \cup \iota_2(D)$. On the other hand, it is clear that each section can be extended to a continuous section $\iota^* : D^* = D \cup \{0, \infty\} \to \mathbb{P}_1$ by setting $\iota^*(0) = 1$ and $\iota^*(\infty) = \infty$. Such an extension is unique, but it is not analytic at 0 and ∞.

We shall use a higher-dimensional version of the above construction.

Notation 3.1. For $\ell = 0, \dots, n$, let U_ℓ be the usual affine chart defined by

$$U_\ell := \{[z_0 : \cdots : z_n] \in \mathbb{P}_n : z_\ell = 1\}.$$

We identify $\mathbb{C}^n$ with U_0 via the embedding

$$\mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto [1 : z_1 : \cdots : z_n] \in U_0 \subseteq \mathbb{P}_n. \tag{5}$$

Now fix a chart U_ℓ with $\ell \geqslant 1$. Given a point $\boldsymbol{c} = [0 : c_1 : \cdots : c_n] \in U_\ell \subseteq \mathbb{P}_n$ (written with $c_\ell = 1$), a *polydisc* centered at $\boldsymbol{c}$ in the chart U_ℓ of radius $\varepsilon > 0$ takes the form

$$D^* = \left\{\left[x_0 : x_1 : \cdots : x_n\right] \in \mathbb{P}_n : |x_0| < \varepsilon, |x_i - c_i| < \varepsilon \text{ for } i = 1, \dots, n \text{ and } x_\ell = 1\right\}. \tag{6}$$

The intersection of D^* with $\mathbb{C}^n = U_0$ is then

$$D = \left\{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_\ell| > \varepsilon^{-1}, \left|\frac{z_i}{z_\ell} - c_i\right| < \varepsilon \text{ for } i = 1, \dots, n\right\}. \tag{7}$$

So the variable z_ℓ is going to infinity in the annulus given by $|z_\ell| > \varepsilon^{-1}$, and each other coordinate z_i lies in a disc around $c_i z_\ell$ of radius $\varepsilon |z_\ell|$.

We shall work with sectors of the annulus in order to have simply connected domains for our maps. So, for $\theta \in \mathbb{R}$ and $\eta \in (\theta, \theta + 2\pi]$, we define

$$D_{(\theta,\eta)} = \left\{(z_1, \dots, z_n) \in D : \theta < \arg(z_\ell) < \eta \text{ for some choice of } \arg(z_\ell)\right\} \tag{8}$$

and we extend to the points at infinity with

$$D^*_{(\theta,\eta)} = D_{(\theta,\eta)} \cup \left(D^* \setminus D\right). \tag{9}$$

We will call both $D_{(\theta,\eta)}$ and $D^*_{(\theta,\eta)}$ *sector domains.*

Note that indeed $D_{(\theta,\eta)}$ and $D^*_{(\theta,\eta)}$ are simply connected, and in fact contractible, as they are homeomorphic to, respectively, $\left(\varepsilon^{-1}, \infty\right) \times (\theta, \eta) \times \{z \in \mathbb{C} : |z| < \varepsilon\}^{n-1}$ and $\{z \in \mathbb{C} : |z| < \varepsilon, -\eta < \arg(z) < -\theta \text{ or } z = 0\} \times \{z \in \mathbb{C} : |z| < \varepsilon\}^{n-1}$.

We can now state the key result of this section, which will be used in the proofs of Theorems 1.4 and 1.5. In the following, recall that given a morphism $\rho : X \to Y$ of algebraic varieties, the *degree* of ρ, denoted $\deg(\rho)$, is the cardinality of the generic fibers, when it is finite.

Proposition 3.2. Let A be a complete variety and $\overline{V} \subseteq \mathbb{P}_n \times A$ be an irreducible variety of dimension n with surjective projection to $\mathbb{P}_n$ of degree d. Let $H := \mathbb{P}_n \setminus \mathbb{C}^n$ be the hyperplane at infinity (with $\mathbb{C}^n$ embedded as in (5)).

Then there is a Zariski open dense subset C of H with the following property: for all $\ell \geqslant 1$, $\boldsymbol{c} = [0 : c_1 : \cdots : c_n] \in C \cap U_\ell$, $\theta \in \mathbb{R}$, $\eta \in (\theta, \theta + 2\pi]$, and all sufficiently small polydiscs D^* at $\boldsymbol{c}$ in the chart U_ℓ, there are distinct continuous maps $\alpha_1, \ldots, \alpha_d : D^*_{(\theta,\eta)} \to A$ such that

1. for all $\boldsymbol{z} \in D^*_{(\theta,\eta)}$ we have $(\boldsymbol{z}, \alpha_i(\boldsymbol{z})) \in \overline{V}$;
2. for all $(\boldsymbol{z}, \boldsymbol{w}) \in \overline{V}$ with $\boldsymbol{z} \in D^*_{(\theta,\eta)}$ there is a k such that $\alpha_k(\boldsymbol{z}) = \boldsymbol{w}$;
3. each restriction $\alpha_i{\restriction}_{D_{(\theta,\eta)}}$ is complex analytic.

Note that the algebraic functions of [6, 7] are simply the coordinates of the maps α_i when restricted to $\mathbb{C}^n \to \mathbb{G}_{\mathrm{m}}^n$; our sector domains are definite instances of the "cones" mentioned in those papers. The additional precision is to avoid potential ambiguities. For instance, in [7, p. 1397], the authors claim that every algebraic function is asymptotically homogeneous, which however may be false if the cones include real lines pointing outside of C.

Remark 3.3. While Proposition 3.2 is fairly detailed in the use of specific polydiscs and sector domains, one may also read it as the construction of a particular sheaf.

Recall that the local analytic sections of the projection $\overline{V} \to \mathbb{P}_n$ with open domains form a sheaf of functions over $\mathbb{P}_n$. One can think of the sheaf as the collection of all continuations of any one local section and so as a multivalued analytic function. The collection of the maps α_i from Proposition 3.2, as θ, η vary, is essentially the composition of this multivalued function with the projection $\overline{V} \to A$.

First consider the map α_1 on the domain $D_{(0,2\pi)}$, which we now write as $\alpha^1_{(0,2\pi)}$ and all the d maps α_k on $D_{(\pi,3\pi)}$. For each k, the set $\left\{\boldsymbol{z} \in D_{(\pi,2\pi)} : \alpha^1_{(0,2\pi)}(\boldsymbol{z}) = \alpha_k(\boldsymbol{z})\right\}$ is clopen by the uniqueness of analytic continuation, so exactly one of these α_k must agree with $\alpha^1_{(0,2\pi)}$ on $D_{(\pi,2\pi)}$, and we write it as $\alpha^1_{(\pi,3\pi)}$.

Similarly, analytic continuation determines a unique branch of α on $D_{(j\pi,(j+2)\pi)}$ for each $j \in \mathbb{Z}$, which we write as $\alpha^1_{(j\pi,(j+2)\pi)}$. Continuity of each branch of α on $D^*_{(\theta,\eta)}$ ensures that α^1 extends uniquely to $D^* \setminus D$ and is single valued there. Since there are only d branches of α at each point, for some integer e with $0 < e < d$ we must have

$\alpha^1_{(2e\pi,(2e+2)\pi)} = \alpha^1_{(0,2\pi)}$. So α^1 is a multivalued map $D^* \to A$ that is e-valued on D and single-valued on $D^* \setminus D$. This ramification is exactly analogous to the function $z \mapsto z^{1/e}$ about the point at infinity in $\mathbb{P}_1$.

If $e < d$ then we can continue with the other branches of α_k to get connected multivalued maps $\alpha^1, \ldots, \alpha^{d'} : D^* \to A$, corresponding to the (at most) d' different values of $\alpha(\boldsymbol{c})$. We can then consider these maps together as a single d-valued map $\alpha : D^* \to A$.

We can also patch together the maps defined on sector domains around different points $\boldsymbol{c} \in C$. Overall, the sector domains cover an open subset Ω^* of $\mathbb{P}_n$ with $\Omega^* \setminus \mathbb{C}^n = C$, and the maps $\alpha : D_{(\theta,\eta)} \to A$ generate a sheaf $\boldsymbol{\alpha}$ representing a d-valued analytic map on $\Omega = \Omega^* \cap \mathbb{C}^n$. The additional information in Proposition 3.2 encodes how each branch extends continuously to C.

Ahlfors [1, pp. 284–308] explains the construction of algebraic functions in detail, although only in a single variable.

Later we will do something similar with other analytic maps.

The rest of this section offers a fairly detailed proof of Proposition 3.2, but we stress that the construction is folklore and that the techniques used here will not be relevant for the rest of the paper, and so the reader may well skip to the next section.

3.1 Covering maps

First, we recall how generically finite maps between irreducible varieties restrict to topological covering maps. These are classical facts, and we refer the reader to [11, Section IV.2] and [18, Section II.6] for more details. In the following, let X, Y be complete irreducible algebraic varieties of the same dimension, and $\rho : X \to Y$ be a *surjective* morphism.

Proposition 3.4. Let $Y_f = \{y \in Y : |\rho^{-1}(y)| < \infty\}$. Then Y_f is Zariski open dense in Y and $Y \setminus Y_f$ has codimension at least 2 in Y.

Proof. Since X, Y are complete, f is a closed morphism. Therefore, the conclusion is an immediate consequence of Chevalley's semi-continuity theorem [10, Cor. 13.1.5] (or [18, Thm. 1.25], after correcting the statement by adding the word "closed"): the set of points $y \in Y$ such that $\rho^{-1}(y)$ has dimension at least 1 is a proper and Zariski-closed subset of Y. By the same theorem, since $\rho^{-1}(Y \setminus Y_f)$ has dimension at most

$\dim(Y) - 1$, and all the fiberss over $Y \setminus Y_f$ have dimension at least 1, we must have $\dim(Y \setminus Y_f) < \dim(Y) - 1$. ■

Proposition 3.5. Let $Y_c = \{y \in Y_f : y \text{ is non-singular}, |\rho^{-1}(y)| = \deg(\rho)\}$. Then Y_c is Zariski open dense in Y_f and the restriction $\rho\restriction_{\rho^{-1}(Y_c)}$ is a covering map with respect to the complex topology.

Proof. Since X, Y are complete, ρ is proper, and so is $\rho\restriction_{\rho^{-1}(Y_f)}$ as Y_f is open. The latter map is quasi-finite (i.e., it has finite fibers); hence, it is finite in the sense of algebraic geometry [10, Thm. 8.11.1]. We also know that the non-singular points of Y form a Zariski open dense subset of Y, and they are normal. By [18, Theorem 2.29], the normal points of Y_f such that $|\rho^{-1}(y)| = \deg(\rho)$ form a Zariski open dense subset of Y_f, and in particular do so the non-singular ones. One can then verify that $\rho\restriction_{\rho^{-1}(Y_c)}$ is a covering map via the implicit function theorem (see [18, p. 143]). ■

Note in particular that $\rho\restriction_{\rho^{-1}(Y_c)}$ is an open map in both complex and Zariski topologies.

Corollary 3.6. Let $D \subseteq Y_c$ be simply connected. Let $d = \deg(\rho)$. Then there are exactly d complex analytic sections $\iota_1, \dots, \iota_d : D \to X$ of ρ, and $\rho^{-1}(D) = \bigcup_{i=1}^{d} \iota_i(D)$.

Proof. This is a standard algebraic topology result. Fixed $y \in D$ and $x \in \rho^{-1}(y)$, there is an analytic homeomorphism from a neighborhood of y to a neighborhood of x. Since D is simply connected, such a homeomorphism has a continuation to all of D. Any two sections ι, ι' such that $\iota(y) = \iota'(y) = x$ have the property that $\{y' \in D : \iota(y') = \iota'(y')\}$ is both open (by analytic continuation) and closed (by continuity of the map $\iota \times \iota' : D \times D \to X \times X$ and the fact that the diagonal of $X \times X$ is closed, since X is Hausdorff); thus, they coincide. In turn, there is exactly one section for every point in the fibers $\rho^{-1}(y)$. By repeating the argument on all $y \in D$, one sees that images of these sections cover all of $\rho^{-1}(D)$. ■

3.2 Extending sections of covering maps

We now wish to extend continuously the sections ι_i from a simply connected $\Delta \subseteq Y_c$ to some larger domain Δ^*, which may fall outside of Y_c. In Proposition 3.2, these will be the points at infinity in $D^*_{(\theta,\eta)} \setminus D_{(\theta,\eta)}$.

Whether this can be done depends on the topological properties of Δ^*, rather than the algebraic properties, so we will work in an abstract topological setting.

Lemma 3.7. Let X, Y be compact Hausdorff topological spaces, $\rho : X \to Y$ be a continuous function, and $\iota : \Delta \to X$ be a continuous section of ρ on some $\Delta \subseteq Y$. Let $\Delta^* \subseteq Y$ be such that $\Delta \subseteq \Delta^* \subseteq \overline{\Delta}$.

Suppose that for all $y \in \Delta^* \setminus \Delta$:

- $\rho^{-1}(y)$ is finite;
- there are arbitrarily small neighborhoods N of y such that $N \cap \Delta$ is connected.

Then ι extends (uniquely) to a continuous section $\iota^* : \Delta^* \to X$ of ρ.

Proof. Fix some $y \in \Delta^* \setminus \Delta$. Since X is Hausdorff, we may find an open neighborhood $B = B_y$ of $\rho^{-1}(y)$ where each connected component contains exactly one point of $\rho^{-1}(y)$.

Claim. For every neighborhood N' of y, there is a neighborhood $N \subseteq N'$ of y such that $\rho^{-1}(N) \subseteq B$ and $N \cap \Delta$ is connected.

Proof. Since Y is compact Hausdorff, N' contains closed neighborhoods $N'' \subseteq N'$ of y. Suppose by contradiction that $\rho^{-1}(N'') \nsubseteq B$ for all such N'''s. By compactness of X, the intersection of the (closed) sets $\rho^{-1}(N'') \setminus B$ contains some $x \notin B \supseteq \rho^{-1}(y)$; since Y is Hausdorff, we may pick a closed neighborhood of y not containing $\rho(x)$, a contradiction. By assumption, there is a neighborhood $N \subseteq N''$ of y such that $N \cap \Delta$ is connected, and of course $\rho^{-1}(N) \subseteq \rho^{-1}(N'') \subseteq B$. ■

Let N be any neighborhood of y given by the claim. Since $\Delta^* \subseteq \overline{\Delta}$ and N is a neighborhood of y, $N \cap \Delta$ is non-empty. Thus, $\iota(N \cap \Delta)$ is contained in exactly one connected component B_0 of B. By construction of B, we have $\overline{B_0} \cap (B \setminus B_0) = \emptyset$, thus $\overline{\iota(N \cap \Delta)} \cap \rho^{-1}(y) \subseteq \overline{B_0} \cap \rho^{-1}(y) = \{x_y\}$ for some $x_y \in \rho^{-1}(y)$.

If ι^* is a continuous extension of ι, since $y \in N \cap \overline{\Delta} \subseteq \overline{N \cap \Delta}$, we must have $\iota^*(y) \in \overline{\iota(N \cap \Delta)}$. In turn, if ι^* is a also section of ρ, we must have $\iota^*(y) \in \overline{\iota(N \cap \Delta)} \cap \rho^{-1}(y)$, hence $\iota^*(y) = x_y$.

Therefore, if continuous sections extending ι exist, they are unique. As for their existence, define the section $\iota^* \supseteq \iota$ by letting $\iota^*(y) = x_y$ for $y \in \Delta^* \setminus \Delta$. It remains to check that ι^* is continuous.

Let N' be an open neighborhood of a $y \in \Delta^* \setminus \Delta$. By the claim and the above argument, there is a neighborhood $N \subseteq N'$ such that $\iota^*(y) \in \overline{\iota(N \cap \Delta)} \subseteq \overline{\iota(N' \cap \Delta)}$. By repeating this for every $y' \in N' \cap (\Delta^* \setminus \Delta)$, we get $\iota^*(N' \cap \Delta^*) \subseteq \overline{\iota(N' \cap \Delta)}$.

Now fix $y \in \Delta^* \setminus \Delta$ and let B be a neighborhood of $\iota^*(y)$. Since X is compact Hausdorff, there is a closed neighborhood $B' \subseteq B$ of $\iota^*(y)$. By continuity of ι, we can find a neighborhood N' of y such that $\iota(N' \cap \Delta) \subseteq B'$. In turn, $\iota^*(N' \cap \Delta^*) \subseteq \overline{\iota(N' \cap \Delta)} \subseteq B' \subseteq B$. By definition, this means that ι^* is continuous at y. ■

The above can now be applied in the setting of maps between algebraic varieties. The following statement establishes a natural condition on pairs of domains $\Delta \subseteq \Delta^*$ guaranteeing the existence (and uniqueness) of such extensions.

Proposition 3.8. Let X be a complete irreducible algebraic variety and $\rho : X \to \mathbb{P}_n$ be a morphism of degree d. Let $\Delta \subseteq \Delta^* \subseteq \mathbb{P}_n$ be sets with Δ simply connected, $\Delta^* \subseteq \overline{\Delta}$ (where $\overline{\Delta}$ is the topological closure of Δ), such that for all $y \in \Delta^*$:

- $\rho^{-1}(y)$ is finite, and of cardinality d when $y \in \Delta$;
- there are arbitrarily small neighborhoods N of y such that $N \cap \Delta$ is connected.

Then there are exactly d continuous sections $\iota_1, \dots, \iota_d : \Delta^* \to X$ of ρ; moreover, they are complex analytic on Δ and $\rho^{-1}(\Delta^*) = \bigcup_{i=1}^{d} \iota_i(\Delta^*)$.

Proof. Since $\mathbb{P}_n$ is non-singular, on setting $Y = \mathbb{P}_n$ we have that $\Delta \subseteq Y_c$. By Corollary 3.6, there are sections $\iota'_1, \dots, \iota'_d : \Delta \to X$ satisfying the conclusion with Δ in place of Δ^*.

Since X and $\mathbb{P}_n$ are complete complex varieties, they are compact Hausdorff spaces, so by Lemma 3.7, such sections can be extended uniquely to continuous sections $\Delta^* \to X$.

It remains to verify that for every $y \in \Delta^*$, $\rho^{-1}(y) = \{\iota_1(y), \dots, \iota_d(y)\}$. Suppose by contradiction that there is x such that $\rho(x) = y \in \Delta^*$, but $\rho(x) \neq \iota_i(y)$ for all $i = 1, \dots, d$. By assumption, $\Delta^* \subseteq Y_f$, and Y_f is open by Proposition 3.4. By Remmert's open mapping theorem [16, Section V.6, Theorem 2], $\rho\restriction_{\rho^{-1}(Y_f)}$ is an open map. If $B \subseteq \rho^{-1}(Y_f)$ is an open neighborhood of x not containing any $\iota_i(y)$, then $\rho(B)$ is an open neighborhood of y disjoint from Δ, a contradiction since $\Delta^* \subseteq \overline{\Delta}$. ■

3.3 The algebraic maps

We return to the setting of Proposition 3.2, with the sets D^*, D, $D_{(\theta,\eta)}$, and $D^*_{(\theta,\eta)}$ as given in (6), (7), (8), and (9).

Proof of Proposition 3.2. Let π be the projection $\overline{V} \to \mathbb{P}_n$. Let $Y_c \subseteq \mathbb{P}_n$ be the set of the points y such that $\pi^{-1}(y)$ is finite of cardinality $d = \deg(\pi)$ (this coincides with the set Y_c of Proposition 3.5 on letting $Y = \mathbb{P}_n$ and $\rho = \pi$). We let

$$C := H \setminus \left(\{y \in H : \pi^{-1}(y) \text{ is infinite}\} \cup \overline{\mathbb{C}^n \setminus Y_c}^{\mathrm{Zar}} \right) \tag{10}$$

where $\overline{(\cdot)}^{\mathrm{Zar}}$ denotes the Zariski closure in $\mathbb{P}_n$.

By construction, $Y_c \cup C$ is a Zariski open subset of $\mathbb{P}_n$. Moreover, C is non-empty: $\mathbb{C}^n \setminus Y_c$ has dimension at most $n-1$ by Proposition 3.5, thus $\overline{\mathbb{C}^n \setminus Y_c}^{\mathrm{Zar}} \cap H \subseteq \overline{\mathbb{C}^n \setminus Y_c}^{\mathrm{Zar}} \setminus (\mathbb{C}^n \setminus Y_c)$ has dimension at most $n-2$, and likewise the set of points y such that $\theta^{-1}(y)$ is infinite has dimension at most $n-2$ by Proposition 3.4.

Let $\boldsymbol{c} \in C$, U_ℓ be a chart such that $\boldsymbol{c} \in U_\ell$, and let D^* be a polydisc centered at $\boldsymbol{c}$ in U_ℓ. When D^* is sufficiently small, $D^* \subseteq Y_c \cup C$, since $Y_c \cup C$ is open. We shall assume this to be the case.

We now wish to apply Corollary 3.6. Recall that the sector domains $D_{(\theta,\eta)}$, $D^*_{(\theta,\eta)}$ are simply connected. Moreover, for any $\boldsymbol{z} \in D^* \setminus D$, and any polydisc D' centered at $\boldsymbol{z}$ in U_ℓ and such that $D' \subseteq D^*$, we clearly have $D'_{(\theta,\eta)} = D' \cap D_{(\theta,\eta)}$, and that is connected. Since $D_{(\theta,\eta)}$ is open and locally connected, this shows that any $\boldsymbol{z} \in D^*_{(\theta,\eta)}$ has arbitrarily small neighborhoods N such that $N \cap D_{(\theta,\eta)}$ is connected, as desired.

Thus, we can apply Proposition 3.8 and obtain sections $\iota_1, \dots, \iota_d$ of π with domain $D^*_{(\theta,\eta)}$. Their composition with the projection from $\overline{V}$ to A are the desired maps $\alpha_1, \dots, \alpha_d$: since $\iota_i(\boldsymbol{z}) = (\boldsymbol{z}, \alpha_i(\boldsymbol{z}))$, conclusion (1) follows at once from ι_i being a section of π; (2) holds by $\pi^{-1}(D) = \bigcup_{i=1}^d \iota_i(D)$; (3) follows from Corollary 3.6 after noticing that $D_{(\theta,\eta)} \subseteq Y_c$. ∎

For comparison with Section 2, note that the sector domain $D^*_{(\theta,\theta+2\pi)}$ is effectively the polydisc D^* with a branch cut in the variable z_ℓ removed. A single branch cut is sufficient: since each coordinate z_i is close to a fixed multiple of z_ℓ, the branch cut in z_ℓ guarantees that the other coordinates also cannot make a loop around $\boldsymbol{c}$. One can easily verify that the 2nd branch cut in Section 2 becomes redundant if we add a restriction of the form $|z_1 - z_2| < \varepsilon$, as we do here with the polydiscs.

4 The Abelian Case

Recall that a complex abelian variety is an irreducible projective complex algebraic variety A with a commutative algebraic group structure, which makes it also a complex Lie group. Let $n := \dim A$. The exponential map $\exp_A : \mathbb{C}^n \to A$ is a surjective complex analytic homomorphism. Its kernel Λ, the *period lattice*, is isomorphic (as a topological group) to $\mathbb{Z}^{2n}$, with the discrete topology. We write Log_A for the local inverse of $\exp_A$.

In this section, we prove Theorem 1.4 stating that a variety $V \subseteq \mathbb{C}^n \times A$ with a dominant projection to $\mathbb{C}^n$ contains an exponential point. In fact, we prove a stronger result, not only showing the existence of exponential points on V but also locally describing the set of almost all sufficiently large such points.

Theorem 4.1. Let A be a complex abelian variety of dimension n. Let $V \subseteq \mathbb{C}^n \times A$ be an irreducible subvariety of dimension n with dominant projection $\pi : V \to \mathbb{C}^n$. Let $d := \deg \pi$. We embed $\mathbb{C}^n$ in projective space $\mathbb{P}_n$ in the usual way (5), identifying it with the chart U_0.

Then there is a subset $\Omega^* \subseteq \mathbb{P}_n$, which is open in the complex topology, such that $C := \Omega^* \setminus \mathbb{C}^n$ is Zariski open dense in $\mathbb{P}_n \setminus \mathbb{C}^n$, and there is a sheaf $\mathbf{S}$ of analytic maps on $\Omega := \Omega^* \cap \mathbb{C}^n$ taking values in $\mathbb{C}^n$ with the following properties:

1. The image $\mathbf{S}(\Omega)$ contains Ω except possibly for a bounded strip along the boundary $\partial\Omega$.
2. For $\boldsymbol{\lambda} \in \Omega \cap \Lambda$, each value of $\mathbf{S}(\boldsymbol{\lambda})$ satisfies $(\mathbf{S}(\boldsymbol{\lambda}), \exp_A(\mathbf{S}(\boldsymbol{\lambda}))) \in V$. Furthermore, these are the only exponential points $(\boldsymbol{z}, \exp_A(\boldsymbol{z}))$ of V with $\boldsymbol{z}$ in Ω (except possibly near the boundary).
3. These exponential points are locally in d-to-1 correspondence with the points of $\Lambda \cap \Omega$: $\mathbf{S}$ has d branches, possibly up to translation of the argument by elements of Λ.
4. The solutions $\mathbf{S}(\boldsymbol{\lambda})$ are asymptotically translates of the lattice: for each $\boldsymbol{c} \in C$ and branch S of $\mathbf{S}$, there is a $\boldsymbol{\gamma} \in \mathbb{C}^n$ such that $S(\boldsymbol{z}) = \boldsymbol{z} + \boldsymbol{\gamma} + o(1)$ for $\boldsymbol{z} \to \boldsymbol{c}$.
5. In particular, the set $\mathbf{S}(\Omega \cap \Lambda)$ is Zariski dense in $\mathbb{C}^n$, and the set of exponential points $\{(\boldsymbol{z}, \exp_A(\boldsymbol{z})) \in V\}$ is Zariski dense in V.

Further properties of the individual maps $S : D_{(\theta,\eta)} \to \mathbb{P}_n$ making up the sheaf, and their extensions to $D^*_{(\theta,\eta)}$, are given in Proposition 4.8.

Theorem 1.4 can easily be deduced from Theorem 4.1.

Proof of Theorem 1.4. Let $V \subseteq \mathbb{C}^n \times A$ be an algebraic subvariety with dominant projection to $\mathbb{C}^n$. If $\dim(V) = n$, we are done by Theorem 4.1.

In general, let W be a proper subvariety of V. We can choose a subvariety $H \subseteq A$ of codimension $\dim(V) - n$, such that $V' := V \cap (\mathbb{C}^n \times H)$ is irreducible and not contained in W. (For example, H can be taken to be an intersection of sufficiently generic hyperplanes.) In particular, $V' \cap W$ is not Zariski dense in V'. Then $\dim V' = n$, and V' has dominant projection to $\mathbb{C}^n$, so by Theorem 4.1, V' contains an exponential point outside of W. Therefore, the exponential points are Zariski dense in V, as desired. ∎

The proof of Theorem 4.1 will run through the rest of this section. We give a brief summary of the key steps first.

Proof outline. The proof proceeds in five steps.

1. We use Proposition 3.2 to find the set C and extract a algebraic maps $\alpha : D^*_{(\theta,\eta)} \to A$ from V, where $D^*_{(\theta,\eta)}$ ranges over sector domains around points $\boldsymbol{c} \in C$. For simplicity, in this summary, we will work with the restrictions of α to $D_{(\theta,\eta)}$, the affine part of $D^*_{(\theta,\eta)}$.
2. We show that a bounded holomorphic branch of $\mathrm{Log}_A(\alpha(\boldsymbol{z}))$ can be defined on $D_{(\theta,\eta)}$, which we denote by $G : D_{(\theta,\eta)} \to \mathbb{C}^n$. Then we consider the map $F(\boldsymbol{z}) := \boldsymbol{z} - G(\boldsymbol{z}) : D_{(\theta,\eta)} \to \mathbb{C}^n$. A point $\boldsymbol{z} \in D_{(\theta,\eta)}$ satisfies $\exp_A(\boldsymbol{z}) = \alpha(\boldsymbol{z})$ if and only if $F(\boldsymbol{z}) \in \Lambda$.
3. We prove, possibly after shrinking $D_{(\theta,\eta)}$, that F is injective on $D_{(\theta,\eta)}$ and its Jacobian matrix of 1st partial derivatives is non-singular. This implies that F has a holomorphic inverse S.
4. We show that each sector domain $D_{(t,s)}$ is covered by the images $F(D_{(\theta,\eta)})$ as (θ, η) varies. In particular, the image of F contains all the lattice points in a neighborhood of $\boldsymbol{c}$, and so the solutions we want are the images under S of the lattice points. We also describe the asymptotic behavior of S.
5. Finally, we explain how the local maps $S : D_{(\theta,\eta)} \to \mathbb{C}^n$ are patched together and complete the proof of the theorem.

We remark that Steps 1–4 are already sufficient to prove Theorem 1.4, namely the existence of the exponential points. Step 5 yields the additional distribution of the exponential points toward infinity as described in Theorem 4.1.

Step 1: the algebraic maps. We begin by applying Proposition 3.2 and extracting algebraic maps from $\overline{V}$, the projective closure of V in $\mathbb{P}_n \times A$. Let us fix the following data:

- $\boldsymbol{c}$ an arbitrary point of C, where C is as in Proposition 3.2;
- $1 \leqslant \ell \leqslant n$ such that $c_\ell \neq 0$;
- D^* a small polydisc at $\boldsymbol{c}$ in the chart U_ℓ as defined in (6).

Then for each $\theta \in \mathbb{R}$ and each $\eta \in (\theta, \theta + 2\pi]$ we have the sector domains $D_{(\theta,\eta)} \subseteq \mathbb{C}^n$ and their extensions $D^*_{(\theta,\eta)}$ to $\mathbb{P}_n$ as given in (8) and (9). By Proposition 3.2, there are (unique) algebraic maps

$$\alpha_1, \ldots, \alpha_d : D^*_{(\theta,\eta)} \to A \tag{11}$$

whose graphs cover the points of $\overline{V}$ over $D^*_{(\theta,\eta)}$.

Through the proof, we may shrink the ε used to define D^* in (6) to ensure certain properties hold for certain maps. When a statement begins with "For small enough D^*", the following statements will implicitly assume that ε is sufficiently small to make that statement true.

For the sake of readability, we will drop the subscript and just write $\alpha : D^*_{(\theta,\eta)} \to A$ for the algebraic map.

Step 2: mapping the solutions to the lattice. The goal of this step is to define a map F, which maps the solutions of the equation $\exp_A(\boldsymbol{z}) = \alpha(\boldsymbol{z})$ to lattice points.

Since $D^*_{(\theta,\eta)}$ is simply connected, we can choose a continuous branch on $D^*_{(\theta,\eta)}$ of the (multivalued) composite $\mathrm{Log}_A \circ \alpha$. We pick one such branch and call it G. Then we have the following:

$$\begin{gathered} G : D^*_{(\theta,\eta)} \to \mathbb{C}^n \text{ continuous on } D^*_{(\theta,\eta)} \text{ and holomorphic on } D_{(\theta,\eta)}, \\ \text{such that } \exp_A(G(\boldsymbol{z})) = \alpha(\boldsymbol{z}) \text{ for all } \boldsymbol{z} \in D^*_{(\theta,\eta)}. \end{gathered} \tag{12}$$

As with α, we can patch together the maps G as (θ, η) varies, and this patching is uniquely determined by analytic continuation on $D_{(\theta,\eta)}$ and then by continuity on $D^*_{(\theta,\eta)}$. The union of their graphs yields a multivalued map $\mathbf{G} : D^* \to \mathbb{C}^n$, of which the maps G are single-valued branches; the restrictions of the maps to D and their continuations yield a sheaf of analytic maps as in Remark 3.3.

Let μ be some fixed positive real number.

Proposition 4.2. For small enough D^*, the image $G(D^*)$ is bounded in $\mathbb{C}^n$.

Moreover, for small enough D^*, every branch G of $\mathbf{G}$ on every sector domain has image contained in an open ball centered at $G(\boldsymbol{c})$ of radius at most μ.

Proof. By continuity, $G(\boldsymbol{z}) \to G(\boldsymbol{c})$ as $\boldsymbol{z} \to \boldsymbol{c}$, so for each (θ, η), by shrinking D^* we may assume that the image $G(D^*_{(\theta,\eta)})$ falls into an open ball around $G(\boldsymbol{c})$, with radius at most μ. We can shrink D^* sufficiently so that this bound holds simultaneously for G on all d branches of α and all sector domains $D^*_{(\theta,\eta)}$. ■

We now use G to define a new map F, which will take the solutions to lattice points. Define $F : D^*_{(\theta,\eta)} \to \mathbb{P}_n$ by

$$F(\boldsymbol{z}) := \begin{cases} \boldsymbol{z} - G(\boldsymbol{z}), & \text{when } \boldsymbol{z} \in D_{(\theta,\eta)}, \\ \boldsymbol{z}, & \text{when } \boldsymbol{z} \in D^*_{(\theta,\eta)} \setminus D_{(\theta,\eta)}. \end{cases} \tag{13}$$

Proposition 4.3. The map F has the following properties.

- F is continuous on $D^*_{(\theta,\eta)}$ and holomorphic on $D_{(\theta,\eta)}$;
- $F(D_{(\theta,\eta)}) \subseteq \mathbb{C}^n$;
- A point $\boldsymbol{z} \in D_{(\theta,\eta)}$ is a solution to the equation $\exp_A(\boldsymbol{z}) = \alpha(\boldsymbol{z})$ if and only if $F(\boldsymbol{z}) \in \Lambda$.

Proof. Since G is holomorphic on $D_{(\theta,\eta)}$, it follows that F is also holomorphic there. To prove continuity of F at a point $\boldsymbol{a} \in D^*_{(\theta,\eta)} \setminus D_{(\theta,\eta)}$, it suffices to observe that G is bounded in $\mathbb{C}^n$, that is, in the chart U_0, and so in the natural metric of the chart U_ℓ around $\boldsymbol{a}$, we have $F(\boldsymbol{z}) - \boldsymbol{z} \to 0$ as $\boldsymbol{z} \to \boldsymbol{a}$.

The 2nd statement is evident and the 3rd statement follows immediately from (12) and (13). ■

Just as for $\mathbf{G}$, the union of the maps F yields a multivalued function $\mathbf{F} : D^* \to \mathbb{P}_n$, which we may suggestively write as $\mathbf{F}(\boldsymbol{z}) = \boldsymbol{z} - \mathbf{G}(\boldsymbol{z})$.

Step 3: local injectivity of F. In this step, we show that F is injective when $D^*_{(\theta,\eta)}$ is small enough, both in terms of shrinking the polydisc D^* and of moving η closer to θ. Hence, as a multivalued function, $\mathbf{F}$ is locally invertible on D.

First, we recall Cauchy's estimate from the theory of complex functions. See, for example, [17, Chapter 1, Section 2.6, Theorem 4].

Fact 4.4. (Cauchy estimate). Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on an open domain $\Omega \subseteq \mathbb{C}^n$ containing a closed polydisc T of radius r centered at a point $\boldsymbol{w} \in \Omega$. Then for any k we have

$$\left| \frac{\partial f}{\partial z_k}(\boldsymbol{w}) \right| \leqslant \frac{\max_{\boldsymbol{z} \in T} |f(\boldsymbol{z})|}{r}.$$

Proposition 4.5. For all $\nu > 0$, there is D^* small enough such that for all intervals (θ, η), for all $\boldsymbol{z} \in D_{(\theta,\eta)}$, the norm of the Jacobian matrix of the 1st partial derivatives $\mathrm{d}G(\boldsymbol{z})$ is less than ν.

In particular, for small enough D^*, for all branches F, G of, respectively, $\mathbf{F}$, $\mathbf{G}$ on every sector domain $D^*_{(\theta,\eta)}$, for all $\boldsymbol{z} \in D_{(\theta,\eta)}$ we have

- $\|\mathrm{d}G(\boldsymbol{z})\| < 1/2$,
- $\det(\mathrm{d}F(\boldsymbol{z})) \neq 0$.

To be more precise, here we use the ℓ^∞-norm on $\mathbb{C}^{n^2}$ as the matrix norm and denote it by $\|\cdot\|$.

Proof. By continuity, we may shrink D^* so that for all $\boldsymbol{z} \in D$ and for all branches G of $\mathbf{G}$ at $\boldsymbol{z}$, we have $|G(\boldsymbol{z}) - G(\boldsymbol{c})| < \nu$. Here $|\cdot|$ denotes the ℓ^2-norm on $\mathbb{C}^n$.

Furthermore, we may shrink D^* further and assume that for every $\boldsymbol{z} \in D$, every polydisc T of radius 1 around $\boldsymbol{z}$, and every $\boldsymbol{z}' \in T$, we have $|G(\boldsymbol{z}') - G(\boldsymbol{c})| < \nu$. Now choose $\boldsymbol{z} \in D$ and a branch of $\mathbf{G}$ defined on $D_{(\arg(z_\ell)-\pi, \arg(z_\ell)+\pi)}$. Then the polydisc T of radius 1 around $\boldsymbol{z}$ is entirely contained in $D_{(\arg(z_\ell)-\pi, \arg(z_\ell)+\pi)}$.

Then we apply the Cauchy estimate (Fact 4.4) to the coordinate functions of $G(\boldsymbol{z}) - G(\boldsymbol{c})$ and deduce that their partial derivatives at $\boldsymbol{z}$, which are equal to those of $G(\boldsymbol{z})$, are bounded by ν.

Since $\mathrm{d}F(\boldsymbol{z}) = I - \mathrm{d}G(\boldsymbol{z})$, when $\|\mathrm{d}G(\boldsymbol{z})\|$ is sufficiently small, $\mathrm{d}F(\boldsymbol{z})$ is close to the identity matrix; hence, it is non-singular. ∎

We can summarize the above statements with $\|\mathrm{d}\mathbf{G}(\boldsymbol{z})\| < \frac{1}{2}$, $\det(\mathbf{F}(\boldsymbol{z})) \neq 0$ for all $\boldsymbol{z} \in D$, where the inequalities implicitly apply to all values of $\mathbf{G}$ and $\mathbf{F}$.

We now show that F is injective when its domain $D^*_{(\theta,\eta)}$ is sufficiently small, in the sense that η is sufficiently close to θ. Recall that by Proposition 4.2 the set $G(D^*_{(\theta,\eta)})$ is bounded. Indeed, given any $\boldsymbol{x}, \boldsymbol{y} \in D^*_{(\theta,\eta)}$, and any branch G of $\mathbf{G}$ with that domain, we have $|G(\boldsymbol{x}) - G(\boldsymbol{y})| < 2\mu$.

Proposition 4.6. For small enough D^*, there is a small $\delta > 0$ such that for all $\theta \in \mathbb{R}$, the map F is injective on $D^*_{(\theta,\theta+2\pi-\delta)}$.

Proof. Write η for $\theta + 2\pi - \delta$, with δ to be determined later. Since F is the identity on $D^*_{(\theta,\eta)} \setminus D_{(\theta,\eta)}$, it is injective there. Also, F maps $D_{(\theta,\eta)}$ to $\mathbb{C}^n$ which is disjoint from $D^*_{(\theta,\eta)} \setminus D_{(\theta,\eta)}$, so it suffices to show that F is injective on $D_{(\theta,\eta)}$. Suppose $\boldsymbol{x}, \boldsymbol{y} \in D_{(\theta,\eta)}$ are

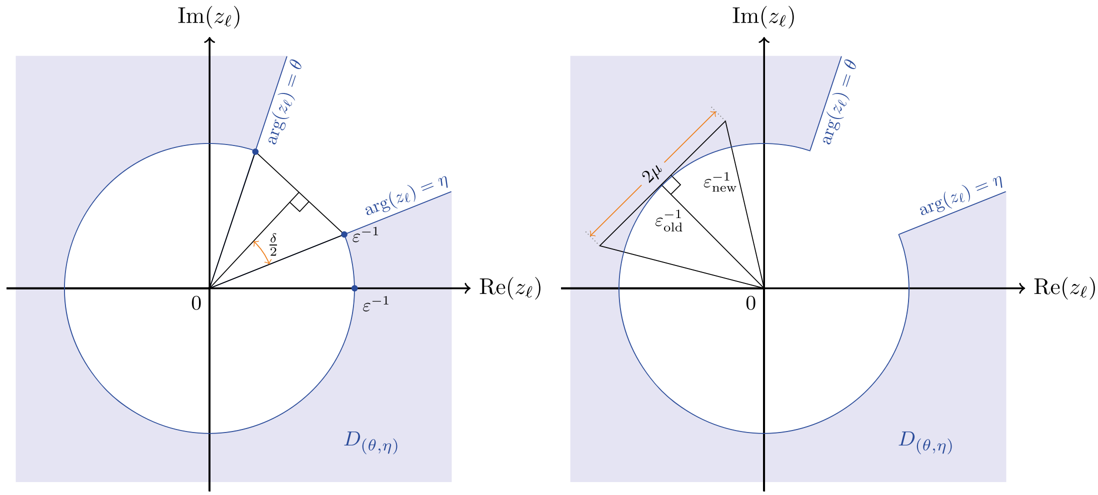

Fig. 1. Some distances between points in the sector domain $D_{(\theta,\eta)}$.

such that $F(\boldsymbol{x}) = F(\boldsymbol{y})$. Recall that $G(\boldsymbol{z}) = \boldsymbol{z} - F(\boldsymbol{z})$. So we have

$$|\boldsymbol{x} - \boldsymbol{y}| = |G(\boldsymbol{x}) - G(\boldsymbol{y})| < 2\mu.$$

If the line segment $[\boldsymbol{x}, \boldsymbol{y}]$ is entirely contained in $D_{(\theta,\eta)}$ then, by the mean value inequality,

$$|\boldsymbol{x} - \boldsymbol{y}| = |G(\boldsymbol{x}) - G(\boldsymbol{y})| \leqslant \max_{\boldsymbol{z}\in[\boldsymbol{x},\boldsymbol{y}]} \|\mathrm{d}G(\boldsymbol{z})\| \cdot |\boldsymbol{x} - \boldsymbol{y}|.$$

Then by Proposition 4.5 we have $\max_{\boldsymbol{z}\in[\boldsymbol{x},\boldsymbol{y}]} \|\mathrm{d}G(\boldsymbol{z})\| < 1/2$, so $\boldsymbol{x} = \boldsymbol{y}$.

Now assume that $[\boldsymbol{x}, \boldsymbol{y}] \nsubseteq D_{(\theta,\eta)}$. First, suppose that the segment is contained in D and crosses the region $D \setminus D_{(\theta,\eta)}$. In particular, it will contain (at least) two points $\boldsymbol{z}'$, $\boldsymbol{z}''$ on the boundary of $D_{(\theta,\eta)}$ with $\arg(z'_\ell) = \theta$ and $\arg(z''_\ell) = \eta$; moreover, we must have $\delta < \pi$. Therefore, $|\boldsymbol{z}' - \boldsymbol{z}''| \geqslant |z'_\ell - z''_\ell| \geqslant \varepsilon^{-1} \cdot 2\sin(\frac{\delta}{2})$. See the 1st image in Figure 1.

By choosing δ large enough, and possibly shrinking ε, we get $|\boldsymbol{z}' - \boldsymbol{z}''| \geqslant |z'_\ell - z''_\ell| > 2\mu$, a contradiction.

If the above does not happen, we observe that $[\boldsymbol{x}, \boldsymbol{y}]$ is contained in D, and thus in $D_{(\theta,\eta)}$, as soon as $\boldsymbol{x}$, $\boldsymbol{y}$ lie in a polydisc at $\boldsymbol{c}$ of slightly smaller radius; to be precise, as soon as $|\boldsymbol{x}|, |\boldsymbol{y}| \geqslant \sqrt{\varepsilon^{-2} + \mu^2}$. It then suffices to shrink D^* a little further to reach the desired conclusion. See the 2nd image in Figure 1, where $\varepsilon_{\mathrm{old}}$ represents the starting value of ε, and $\varepsilon_{\mathrm{new}}$ the new one. ■

Step 4: mapping the lattice to the solutions. In this step, we describe the inverse of F, which maps lattice points to solutions of the equation $\exp_A(\boldsymbol{z}) = \alpha(\boldsymbol{z})$.

We fix a sector domain $D^*_{(\theta,\eta)}$ with $\eta = \theta + 2\pi - \delta$ as given by the previous step, so that F is injective on that domain, and hence has an inverse. Now we show that the image of F differs from its domain by at most a strip of bounded width around the boundary. Let $B \subseteq \mathbb{C}^n$ be a closed ball centered at $\mathbf{0}$ containing $G(D^*_{(\theta,\eta)})$. Then by the definition of F, for all $\boldsymbol{z} \in D_{(\theta,\eta)}$ we have $F(\boldsymbol{z}) - \boldsymbol{z} \in B$.

Proposition 4.7. The image $E := F(D_{(\theta,\eta)})$ is open and contains

$$E' := \{\boldsymbol{z} \in D_{(\theta,\eta)} : \boldsymbol{z} + B \subseteq D_{(\theta,\eta)}\}.$$

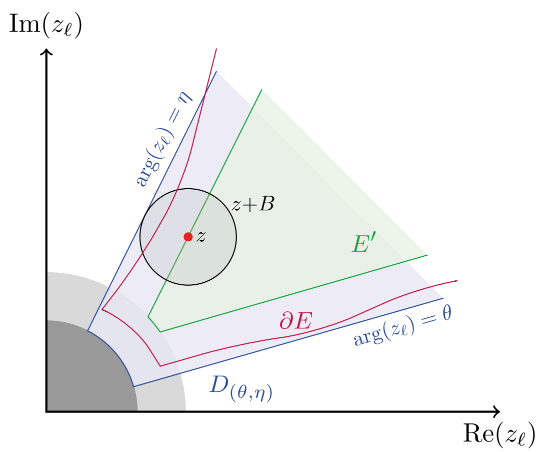

Fig. 2. A pictorial representation of how $D_{(\theta,\eta)}, E$ and ∂E might look.

Proof. Since $\mathrm{d}F(\boldsymbol{z})$ is non-singular on $D_{(\theta,\eta)}$, by the inverse function theorem, F is a local homeomorphism, hence an open map. So the image E is open and connected in $\mathbb{C}^n$. (See Figure 2.)

Clearly, $E \cap E'$ is a non-empty open subset of E'. Hence, if $E' \setminus E \neq \emptyset$ then $\partial E \cap E' \neq \emptyset$ where ∂ denotes the boundary. Take a point $\boldsymbol{x} \in \partial E \cap E'$ and a small closed neighborhood $\boldsymbol{x} \in N \subseteq E'$. Then we have

$$\overline{F^{-1}(N)} \subseteq \overline{N + B} = N + B \subseteq D_{(\theta,\eta)}.$$

Now pick a sequence $\boldsymbol{x}_k \in N \cap E$ with $\boldsymbol{x}_k \to \boldsymbol{x}$ as $k \to \infty$. Since $F^{-1}(N)$ is a non-empty bounded subset of $D_{(\theta,\eta)}$, if we set $\boldsymbol{z}_k := F^{-1}(\boldsymbol{x}_k)$, then a subsequence of $\boldsymbol{z}_k$ has a limit point $\boldsymbol{z} \in \overline{F^{-1}(N)} \subseteq D_{(\theta,\eta)}$. Then by continuity of F we conclude that $\boldsymbol{x} = F(\boldsymbol{z}) \in E$, which is a contradiction. ■

Proposition 4.8. For small enough D^*, there is a small $\delta' > 0$ such that for all $\theta \in \mathbb{R}$ and $\eta = \theta + 2\pi - \delta'$, there is a map $S : D^*_{(\theta,\eta)} \to \mathbb{P}_n$ with the following properties:

1. For all $\boldsymbol{z} \in D^*_{(\theta,\eta)}$, we have $F(S(\boldsymbol{z})) = \boldsymbol{z}$, where we take the branch F of $\mathbf{F}$ defined on $D^*_{(\theta,\eta)}$ or its analytic continuation to a slightly larger domain containing the image of S.
2. In particular, for $\boldsymbol{z} \in D^* \setminus D$, we have $S(\boldsymbol{z}) = \boldsymbol{z}$ and for $\boldsymbol{z} \in D_{(\theta,\eta)}$ we have $S(\boldsymbol{z}) - \boldsymbol{z} \in B$, the bounded ball defined above.
3. For $\boldsymbol{z} \in D_{(\theta,\eta)}$, we have $\exp_A(S(\boldsymbol{z})) = \alpha(S(\boldsymbol{z}))$ if and only if $\boldsymbol{z} \in \Lambda$.
4. S is continuous on $D^*_{(\theta,\eta)}$ and holomorphic on $D_{(\theta,\eta)}$.
5. The restriction of S to the finite part of the domain $D_{(\theta,\eta)}$ is asymptotically a translation. More precisely,

$$S(\boldsymbol{z}) - \boldsymbol{z} \to G(\boldsymbol{c}) \text{ as } \boldsymbol{z} \to \boldsymbol{c} \text{ with } \boldsymbol{z} \in D_{(\theta,\eta)}.$$

Proof. For a suitable small δ', the set E' from the previous proposition contains the sector domain $D_{(\theta+\delta'/2,\eta-\delta'/2)}$, except for a strip of bounded width near the part of the boundary given by $|z_\ell| = \varepsilon^{-1}$. We shrink ε to remove this bounded strip, and then the image of F (analytically continued from the new $D^*_{(\theta,\eta)}$ back to the original domain with larger ε) contains $D^*_{(\theta+\delta'/2,\eta-\delta'/2)}$. Since F is injective on that domain, we can define S to be its set-theoretic inverse map with domain $D^*_{(\theta+\delta'/2,\eta-\delta'/2)}$. Relabeling $\theta + \delta'/2$ as θ and $\theta + 2\pi - \delta'$ as η, we get the S of the statement of the proposition satisfying point 1. Points 2 and 3 follow from the properties of F. It follows from the inverse function theorem that S is holomorphic on $D_{(\theta,\eta)}$. Continuity of S on $D^*_{(\theta,\eta)}$ follows the same way as continuity of F using the fact that $S(\boldsymbol{z}) - \boldsymbol{z}$ is bounded.

For point 5, observe that $\boldsymbol{z} = F(S(\boldsymbol{z})) = S(\boldsymbol{z}) - G(S(\boldsymbol{z}))$, and so $S(\boldsymbol{z}) - \boldsymbol{z} = G(S(\boldsymbol{z}))$. By continuity of S, we get that $S(\boldsymbol{z}) \to S(\boldsymbol{c}) = \boldsymbol{c}$ for $\boldsymbol{z} \to \boldsymbol{c}$, hence $S(\boldsymbol{z}) - \boldsymbol{z} = G(S(\boldsymbol{z})) \to G(\boldsymbol{c})$ for $\boldsymbol{z} \to \boldsymbol{c}$ by continuity of G. ■

Step 5: analytic continuation of the solution map S. In this final step, we finish the proof of Theorem 4.1 by considering the maps S from the domains $D^*_{(\theta,\eta)}$ as a sheaf giving a multivalued map. Just as we did for $\mathbf{F}$ and $\mathbf{G}$, the union of the maps S yields a multivalued map $D^* \to \mathbb{P}_n$. We now push this further by allowing $\boldsymbol{c}$ to vary along the set $C \subseteq \mathbb{P}_n \setminus \mathbb{C}^n$ of Proposition 3.2. We shall also restrict the domains from $D^* \subseteq \mathbb{P}_n$ to $D \subseteq \mathbb{C}^n$, in order to get *analytic* maps, without the potential singularities at infinity, which become irrelevant in our final conclusion.

For each $\boldsymbol{c} \in C$, we have an open polydisc D^* around $\boldsymbol{c}$. Since C is open in $H = \mathbb{P}_n \setminus \mathbb{C}^n$, we may always assume, after shrinking D^*, that $D^* \cap H$ is a subset of C. We now write this D^* as $D^*_{\boldsymbol{c}}$ and define $\Omega^* = \bigcup_{\boldsymbol{c} \in C} D^*_{\boldsymbol{c}}$, an open subset of $\mathbb{P}_n$. Then $\Omega^* \setminus \mathbb{C}^n$ is indeed C since obviously $\boldsymbol{c} \in D^*_{\boldsymbol{c}}$, so by construction $C \subseteq \Omega^* \cap H \subseteq C$.

For each $\boldsymbol{c}$ and for each interval $(\theta, \eta) \subseteq \mathbb{R}$ with $\eta \leq \theta + 2\pi - \delta'$ we have a map $S : D^*_{\boldsymbol{c},(\theta,\eta)} \to \mathbb{P}_n$. It is clear that such maps, when restricted to $D_{\boldsymbol{c},(\theta,\eta)}$ so to become analytic, are continuations of each other, in the following sense: for any two maps S, $\tilde{S}$ as above, the set $\{S(\boldsymbol{z}) = \tilde{S}(\boldsymbol{z})\}$ is both closed and open in the intersection of their domains. Thus, as in Remark 3.3, their restrictions to $\Omega = \Omega^* \cap \mathbb{C}^n$ generate a sheaf $\mathbf{S}$ of analytic maps, the union of which is an analytic multivalued map $\Omega \to \mathbb{C}^n$.

This gives us the data of Theorem 4.1. We can now prove that $\mathbf{S}$ has the required properties.

1 Note that $\mathbf{S}$ is a local homeomorphism because its local inverses are by Proposition 4.5. Thus, its image $\mathbf{S}(\Omega)$ is open.
Fix a fundamental domain of $\mathbb{C}/\Lambda$, and let ν be its diameter. We claim that for every $\boldsymbol{a} \in \Omega$, there is a branch S of $\mathbf{S}$ such that $|S(\boldsymbol{a}) - \boldsymbol{a}| < \mu + \nu$. Indeed, if we pick $\boldsymbol{c}$ and a sector domain of $D_{\boldsymbol{c}}$ containing $\boldsymbol{a}$, we can choose a branch G of $\mathbf{G}$ on that sector domain with $|G(\boldsymbol{c})| < \nu$, and find $|S(\boldsymbol{a}) - \boldsymbol{a}| = |G(S(\boldsymbol{a}))| \leqslant |G(S(\boldsymbol{a})) - G(\boldsymbol{c})| + |G(\boldsymbol{c})| < \mu + \nu$ by Proposition 4.2. Moreover, we may assume that $|S(\boldsymbol{z}) - \boldsymbol{z}| < \mu + \nu$ for every $\boldsymbol{z}$ in a neighborhood of $\boldsymbol{a}$.
Given this, it suffices to reason as in Proposition 4.7. Suppose that $\Omega' = \{\boldsymbol{z} \in \Omega : \boldsymbol{z} + B \subseteq \Omega\}$ is not contained in Ω, where B is the closed ball at $\mathbf{0}$ of radius $\mu + \nu$. Then there exists $\boldsymbol{a} \in \Omega'$ on the boundary of $\mathbf{S}(\Omega)$. Pick a local branch S on a small neighborhood of $\boldsymbol{a}$, all contained in Ω', satisfying $|S(\boldsymbol{z}) - \boldsymbol{z}| < \mu + \nu$, and a sequence $\boldsymbol{x}_k$ in the image of S such that $\boldsymbol{x}_k \to \boldsymbol{a}$. Now observe that the preimages $S^{-1}(\boldsymbol{x}_k)$ must converge to some $\boldsymbol{x} \in \Omega$ such that $S(\boldsymbol{x}) = \boldsymbol{a}$, a contradiction.
It follows that every point of Ω not in $\mathbf{S}(\Omega)$ has distance at most $\mu + \nu$ from the boundary $\partial\Omega$ in $\mathbb{C}^n$.

2 This follows from the 3rd condition of Proposition 4.8, together with the fact that the branches of α cover V (which is point 2 of Proposition 3.2), with the same proviso as above about the strip of bounded width at the boundary.

3 For each sector domain $D_{(\theta,\eta)}$ of Ω, there are d distinct branches $G_1, \ldots, G_d$ of $\mathbf{G}$ on $D_{(\theta,\eta)}$, corresponding to the d distinct branches of α, such that every other branch is of the form $G_i + \boldsymbol{\lambda}$ for some $\boldsymbol{\lambda} \in \Lambda$. Thus, the same is true for $\mathbf{F}$ for some branches $F_1, \ldots, F_d$.
Let $S_1, \ldots, S_d$ be the corresponding branches obtained in Proposition 4.8. Then every branch of $\mathbf{S}$ is of the form $S_i(\boldsymbol{z} + \boldsymbol{\lambda})$ for some $1 \leqslant i \leqslant d$ and $\boldsymbol{\lambda} \in \Lambda$.

To conclude, we observe that such branches are all distinct. Suppose that $S_i(\boldsymbol{z}+\boldsymbol{\lambda}) \equiv S_{i'}(\boldsymbol{z}+\boldsymbol{\lambda}')$ on $D_{(\theta,\eta)}$. Then also $F_i - \boldsymbol{\lambda} \equiv F_{i'} - \boldsymbol{\lambda}'$ on $D_{(\theta,\eta)}$, as the branches of $\mathbf{F}$ are local inverses of the branches of $\mathbf{S}$. In turn, $i = i'$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$, as desired.

4 This is the asymptotic condition from Proposition 4.8, with the points $\boldsymbol{\gamma}$ being the values of $G(\boldsymbol{c})$.

5 It is now clear that the set $\mathbf{S}(\Omega \cap \Lambda)$ is Zariski dense in $\mathbb{C}^n$ and, since $\dim V = n$, it follows at once that $\{(\boldsymbol{z}, \exp_A(\boldsymbol{z})) \in V\}$ is Zariski dense in V.

That completes the proof of Theorem 4.1. ■

Remark 4.9. For each $\boldsymbol{c}$, the polydisc $D^*_{\boldsymbol{c}}$ is given with a radius $\varepsilon = \varepsilon_{\boldsymbol{c}}$, but we do not have any uniformity in ε as $\boldsymbol{c}$ varies. Indeed, as $\boldsymbol{c}$ approaches the boundary of C, we may have $\varepsilon_{\boldsymbol{c}} \to 0$. Furthermore, the index ℓ could vary as well. For points $\boldsymbol{a} \in \mathbb{P}_n \setminus (\mathbb{C}^n \cup C)$, the algebraic map α may have worse singularities than the ramification points we have dealt with. For these reasons, we do not have a complete description of all the exponential points $(\boldsymbol{z}, \exp_A(\boldsymbol{z})) \in V$, which are large, that is, such that $|\boldsymbol{z}|$ is larger than some given ε^{-1}. However, each exponential point is known to be isolated, so they cannot accumulate anywhere in $\mathbb{C}^n$, so the points we have found should be a large proportion of the total in any meaningful sense.

Remark 4.10. Our method can be used to prove the existence of solutions of any exponential equations of the form $\exp_A(\boldsymbol{z}) = \beta(\boldsymbol{z})$, where $\beta : D^*_{(\theta,\eta)} \to \mathbb{C}^n$ is holomorphic on $D_{(\theta,\eta)}$ and continuous on $D^*_{(\theta,\eta)}$, and θ, η are given. In steps 2–4, one can simply omit all references to the uniformity in θ, η, and also replace θ, η with suitable values θ', η' satisfying $\theta < \theta' < \eta' < \eta$ when necessary.

If one has a sufficiently rich understanding of the analytic continuations of β around the points at infinity, the arguments of step 5 could be used to give a global description of the solutions in the style of Theorem 4.1.

5 The Case of Algebraic Tori

Let $\exp : \mathbb{C} \to \mathbb{G}_{\mathrm{m}}$ be the usual exponential map. We will also let $\exp : \mathbb{C}^n \to \mathbb{G}_{\mathrm{m}}^n$ denote the exponential map of $\mathbb{G}_{\mathrm{m}}^n$ for any n, given by coordinate-wise action of the former map. The lattice of periods of exp is $\Lambda := (2\pi i\mathbb{Z})^n$. We will write Log for the logarithmic map corresponding to exp, and $\log : \mathbb{R}_{>0} \to \mathbb{R}$ for the real logarithm.

In this section, we adapt the ideas of the previous section to prove a theorem for algebraic tori, analogous to Theorem 4.1. The only difference is the asymptotic behavior of the solutions, which are no longer asymptotically translations of lattice points.

Theorem 5.1. Let $V \subseteq \mathbb{C}^n \times \mathbb{G}_m^n$ be a subvariety of dimension n with dominant projection $\pi : V \to \mathbb{C}^n$. Let $d := \deg \pi$. We embed $\mathbb{C}^n$ in projective space $\mathbb{P}_n$ in the usual way (5).

Then there is a subset $\Omega^* \subseteq \mathbb{P}_n$, which is open in the complex topology, such that $C := \Omega^* \setminus \mathbb{C}^n$ is Zariski open dense in $\mathbb{P}_n \setminus \mathbb{C}^n$, and there is a sheaf $\mathbf{S}$ of analytic maps on $\Omega := \Omega^* \cap \mathbb{C}^n$ taking values in $\mathbb{C}^n$ with the following properties:

1. The image $\mathbf{S}(\Omega)$ contains Ω except possibly for a narrow strip along the boundary $\partial\Omega$.
2. For $\boldsymbol{\lambda} \in \Omega \cap \Lambda$, each value of $\mathbf{S}(\boldsymbol{\lambda})$ satisfies $(\mathbf{S}(\boldsymbol{\lambda}), \exp(\mathbf{S}(\boldsymbol{\lambda}))) \in V$. Furthermore, these are the only exponential points $(\boldsymbol{z}, \exp(\boldsymbol{z}))$ of V with $\boldsymbol{z} \in \Omega$ (except possibly near the boundary).
3. These exponential points are locally in d-to-1 correspondence with the points of $\Lambda \cap \Omega$: $\mathbf{S}$ has d branches, possibly up to translation of the argument by elements of Λ.
4. The solutions $\mathbf{S}(\boldsymbol{\lambda})$ are asymptotically close to lattice points: for each $\boldsymbol{c} \in C$ and each branch S of $\mathbf{S}$ we have $S(\boldsymbol{z}) = \boldsymbol{z} + O(\log|\boldsymbol{z}|)$ for $\boldsymbol{z} \to \boldsymbol{c}$.
5. In particular, the set $\mathbf{S}(\Omega \cap \Lambda)$ is Zariski dense in $\mathbb{C}^n$, and the set of exponential points $\{(\boldsymbol{z}, \exp(\boldsymbol{z})) \in V\}$ is Zariski dense in V.

The proof follows that of Theorem 4.1 closely, so we will focus on the differences. The two essential differences are that abelian varieties are compact whereas algebraic tori are not and (relatedly) that the lattice $(2\pi i\mathbb{Z})^n$ does not accumulate to every point at infinity in the complex topology (although it does in the Zariski topology). In the case of algebraic tori, we have to deal with two extra points 0 and ∞ (in dimension 1). Furthermore, any branch of the logarithmic map of an abelian variety is bounded, which is not true for algebraic tori but we are able to make do with logarithmic growth instead.

As in the abelian case, we split the proof into several steps.

Step 1: the algebraic maps. We embed $\mathbb{G}_m$ into $\mathbb{P}_1$ identified with $\mathbb{C} \cup \{\infty\}$ and consider the Zariski closure $\overline{V}$ of V in $\mathbb{P}_n \times \mathbb{P}_1^n$. We shall apply Proposition 3.2 and extract algebraic maps from the $\overline{V}$, but we want the images of those maps to be contained in $\mathbb{G}_m^n$. To this end, let $Z := \{\boldsymbol{z} \in \mathbb{C}^n : (\boldsymbol{z}, 0) \in \overline{V} \text{ or } (\boldsymbol{z}, \infty) \in \overline{V}\}$. Then Z has codimension $\geqslant 1$ in $\mathbb{C}^n$, and the set Z^* of its limit points in $H := \mathbb{P}_n \setminus \mathbb{C}^n$ is a lower dimensional Zariski closed subset of H. So we shrink the set C given by Proposition 3.2 by removing Z^*. To get our growth estimates later, we shrink C further and assume that for any point $[0 : t_1 : \ldots : t_n] \in C$ none of the t_i is 0. One consequence is that we can work with the fixed chart U_1 rather than a varying chart U_ℓ.

Thus, we end up with the following data:

- C, a Zariski open dense subset of H such that for any point $[0 : t_1 : \ldots : t_n] \in C$ none of the t_i is 0;
- $\boldsymbol{c} := [0 : 1 : c_2 : \ldots : c_n]$ an arbitrary point of C;
- D^* a small polydisc at $\boldsymbol{c}$ in the chart U_1 as defined in (6), chosen small enough that it does not meet the set Z given above.

As in the abelian case, we will shrink D^* (by reducing ε) to ensure certain properties of certain maps hold, which will be explicitly stated every time. In particular, we choose the ε defining D^* to be at most $\min\left\{\frac{|c_i|}{2}, \frac{1}{2|c_i|} : i = 1, \ldots, n\right\}$ so that for any $\boldsymbol{z} = (z_1, \ldots, z_n) \in D$ we have $|z_1| > 2$, all other $|z_i| > 1$, and the coordinates of $\boldsymbol{z}$ are roughly proportional to each other:

$$\frac{1}{2} < \frac{|z_i|}{|c_i z_1|} < \frac{3}{2}. \tag{14}$$

Of course, this assumption cannot be made uniformly as $\boldsymbol{c}$ varies, since we can have $|c_i|$ arbitrarily small.

For each $\theta \in \mathbb{R}$ and each $\eta \in (\theta, \theta + 2\pi]$, we have the sector domains $D_{(\theta,\eta)} \subseteq \mathbb{C}^n$ and their extensions $D^*_{(\theta,\eta)}$ to $\mathbb{P}_n$ as given in (8) and (9), and the algebraic maps

$$\alpha^1, \ldots, \alpha^d : D^*_{(\theta,\eta)} \to (\mathbb{P}_1)^n. \tag{15}$$

As before, we drop the indices and write α to denote one of these maps.

Step 2: mapping the solutions to the lattice. Since $D \cap Z = \emptyset$, the restriction of α to $D_{(\theta,\eta)}$ takes values in $\mathbb{G}_{\mathrm{m}}^n$. So, as in Section 4, we can choose a holomorphic branch G of $\mathrm{Log} \circ \alpha$ on $D_{(\theta,\eta)}$:

$$G : D_{(\theta,\eta)} \to \mathbb{C}^n \text{ holomorphic on } D_{(\theta,\eta)},$$
$$\exp(G(\boldsymbol{z})) = \alpha(\boldsymbol{z}) \text{ for all } \boldsymbol{z} \in D_{(\theta,\eta)}. \tag{16}$$

In this case, we cannot necessarily continue G to a map $D^*_{(\theta,\eta)} \to \mathbb{C}^n$, since as $\boldsymbol{z} \to \boldsymbol{c} \in C$ we may have some coordinate $\alpha_i(\boldsymbol{z}) \to 0$ or ∞, where the logarithm is not defined.

We also remark that, as in the abelian case, the different choices of maps G together yield a multivalued map $\mathbf{G} : D \to \mathbb{C}^n$ with associated sheaf of analytic maps as in Remark 3.3.

By our choice of ε, all coordinates are roughly proportional on D and are larger than 1 in absolute value. Since α is an algebraic map, for each coordinate function α_i, there is a positive integer $q_i \in \mathbb{N}$ such that for all $\boldsymbol{z} \in D$ we have

$$|\boldsymbol{z}|^{-q_i} < |\alpha_i(\boldsymbol{z})| < |\boldsymbol{z}|^{q_i}. \tag{17}$$

Let $q := \max\{q_i : 1 \leqslant i \leqslant n\}$. Then for each coordinate-function G_i of G, we have

$$\left|\mathrm{Re}(G_i(\boldsymbol{z}))\right| = \left|\log|\alpha_i(\boldsymbol{z})|\right| \leqslant q \log|\boldsymbol{z}|.$$

Similarly, the argument of $\alpha_i(\boldsymbol{z})$ is bounded on the sector domain $D_{(\theta,\eta)}$, hence so is the imaginary part of $G_i(\boldsymbol{z})$. So the ratio

$$\frac{G_i(\boldsymbol{z})}{\log|\boldsymbol{z}|}$$

is bounded. Thus, $|G(\boldsymbol{z})| = O(\log|\boldsymbol{z}|)$ for $\boldsymbol{z} \in D_{(\theta,\eta)}$, that is, G has logarithmic growth as $|\boldsymbol{z}| \to \infty$.

Now define a map $F : D^*_{(\theta,\eta)} \to \mathbb{C}^n$ by

$$F(\boldsymbol{z}) := \begin{cases} \boldsymbol{z} - G(\boldsymbol{z}), & \text{for } \boldsymbol{z} \in D_{(\theta,\eta)}, \\ \boldsymbol{z}, & \text{for } \boldsymbol{z} \in D^*_{(\theta,\eta)} \setminus D_{(\theta,\eta)}. \end{cases} \tag{18}$$

Proposition 5.2. The map F enjoys the following properties.

- F is continuous on $D^*_{(\theta,\eta)}$ and holomorphic on $D_{(\theta,\eta)}$;
- $F(D_{(\theta,\eta)}) \subseteq \mathbb{C}^n$;
- A point $\boldsymbol{z} \in D_{(\theta,\eta)}$ is a solution to the equation $\exp(\boldsymbol{z}) = \alpha(\boldsymbol{z})$ if and only if $F(\boldsymbol{z}) \in \Lambda$.

Proof. As in Proposition 4.3, this is almost immediate except for showing that F is continuous at any point $\boldsymbol{a} \in D^*_{(\theta,\eta)} \setminus D_{(\theta,\eta)}$. In the abelian case, this followed since $G(\boldsymbol{z})$

was bounded. Here we have $|G(\boldsymbol{z})|$ growing logarithmically in $|\boldsymbol{z}|$, so again as we take the limit going to infinity in projective space, it becomes negligible compared to $|\boldsymbol{z}|$. ■

As for $\mathbf{G}$, the maps F and their continuations yield a sheaf of analytic maps from D to $\mathbb{C}^n$, which we write as $\mathbf{F}(\boldsymbol{z}) = \boldsymbol{z} - \mathbf{G}(\boldsymbol{z})$.

Step 3: local injectivity of F. Now we want to show that F is injective on suitable domains $D_{(\theta,\eta)}$. As in the abelian case, we need to estimate the partial derivatives of G.

Lemma 5.3. There is $K > 0$ such that, for any $\boldsymbol{w} \in D_{(\theta,\eta)}$ and any $r > 0$ such that the closed polydisc T of radius r at $\boldsymbol{w}$ is contained in $D_{(\theta,\eta)}$, we have

$$\|\mathrm{d}G(\boldsymbol{w})\| \leqslant K\frac{\log|\boldsymbol{w}|}{r}. \tag{19}$$

Proof. It suffices to establish such estimates for all partial derivatives of all coordinates of G. Let g be a coordinate function of G. Then g has logarithmic growth in $\boldsymbol{z}$, so applying the Cauchy estimate (Fact 4.4) yields

$$\frac{\partial g(\boldsymbol{w})}{\partial z_k} = O\left(\frac{\max_{\boldsymbol{z}\in T}\log|\boldsymbol{z}|}{r}\right). \tag{20}$$

If $T \subseteq D_{(\theta,\eta)}$ is a polydisc of radius r, since $|z_m| \geqslant 1$ on $D_{(\theta,\eta)}$ for every m, we must have $r < |w_m|$ for every m, hence $r < |\boldsymbol{w}|$. Therefore, if $\boldsymbol{z} \in T$ then $|\boldsymbol{z}| \leqslant |\boldsymbol{w}| + |\boldsymbol{z} - \boldsymbol{w}| \leqslant |\boldsymbol{w}| + r < 2|\boldsymbol{w}|$. This then implies that $\log|\boldsymbol{z}| = O(\log|\boldsymbol{w}|)$ for $\boldsymbol{z} \in T$. So the desired bound follows from (20). ■

We can now prove the analogue of Proposition 4.5, with a similar proof.

Proposition 5.4. For all $\nu > 0$, there is D^* small enough such that for all intervals (θ,η), for all $\boldsymbol{z} \in D_{(\theta,\eta)}$, the norm of the Jacobian matrix of the 1st partial derivatives $\mathrm{d}G(\boldsymbol{z})$ is less than ν.

In particular, for small enough D^*, for all branches F, G of $\mathbf{F}$, $\mathbf{G}$, respectively, on every sector domain $D^*_{(\theta,\eta)}$, for all $\boldsymbol{z} \in D_{(\theta,\eta)}$ we have

- $\|\mathrm{d}G(\boldsymbol{z})\| < 1/2$,
- $\det(\mathrm{d}F(\boldsymbol{z})) \neq 0$.

Proof. Let K be as in Lemma 5.3, and let $\boldsymbol{z} \in D$. If the polydisc of radius $r > \nu^{-1}K \log |\boldsymbol{z}|$ around $\boldsymbol{z}$ is entirely contained in D then it is contained in $D_{(\arg(z_1)-\pi, \arg(z_1)+\pi)}$, and then by Lemma 5.3 we get $\|\mathrm{d}G(\boldsymbol{z})\| < \nu$. Otherwise, $|\boldsymbol{z}|$ is small, and we can shrink D^* to remove this case. The rest follows easily. ■

Proposition 5.5. For small enough D^*, there is a small $\delta > 0$ such that the map F with domain $D^*_{(\theta,\theta+2\pi-\delta)}$ is injective.

Proof. The argument of Proposition 4.6 goes through, except that we need to show that if D^* is sufficiently small and $\boldsymbol{x}, \boldsymbol{y} \in D$ with $\arg(x_1) - \arg(y_1) = \delta$ then $|\boldsymbol{x} - \boldsymbol{y}| > |G(\boldsymbol{x}) - G(\boldsymbol{y})|$. This can easily be deduced from the observation that $|\boldsymbol{x} - \boldsymbol{y}|$ is bounded below linearly in $|x_1|$ as $\boldsymbol{x}$ and $\boldsymbol{y}$ approach $\boldsymbol{c}$, while $|G(\boldsymbol{x}) - G(\boldsymbol{y})|$ grows logarithmically in $|x_1|$. ■

Step 4: mapping the lattice to the solutions. We fix a small δ as in Proposition 5.5.

Proposition 5.6. For small enough D^*, there is a small δ' such that for any $\theta \in \mathbb{R}$, writing $\eta = \theta + 2\pi - \delta$, the image $F(D^*_{(\theta,\eta)})$ contains $D^*_{(\theta+\delta'/2,\eta-\delta'/2)}$, except for a strip of bounded width near the part of the boundary given by $|z_1| = \varepsilon^{-1}$.

Proof. The proof is as in Proposition 4.7, except we use the logarithmic growth of G in place of boundedness. More precisely, one should use $E' = \{\boldsymbol{z} \in D_{(\theta,\eta)} : B_{\boldsymbol{z}} \subseteq D_{(\theta,\eta)}\}$, where $B_{\boldsymbol{z}}$ is the closed ball centered at $\boldsymbol{z}$ of radius $\mu \left|\log |\boldsymbol{z}|\right|$, and μ is chosen such that $|S(\boldsymbol{z}) - \boldsymbol{z}| < \mu \left|\log |z|\right|$ on $D_{(\theta,\eta)}$. Then the argument of Proposition 4.7 shows that $F(D^*_{(\theta,\eta)})$ contains E'. It is then clear that $D^*_{(\theta+\delta'/2,\eta-\delta'/2)}$ is contained in E', except for a strip of bounded width near the part of the boundary given by $|z_1| = \varepsilon^{-1}$. ■

As before, we find a sheaf of maps S, which are local inverses to F.

Proposition 5.7. For small enough D^*, for all $\theta \in \mathbb{R}$ and $\eta = \theta + 2\pi - \delta - \delta'$, there is a map $S : D^*_{(\theta,\eta)} \to \mathbb{P}_n$ with the following properties:

1. For all $\boldsymbol{z} \in D^*_{(\theta,\eta)}$, we have $F(S(\boldsymbol{z})) = \boldsymbol{z}$, where we take the branch of F defined on $D^*_{(\theta,\eta)}$ or its analytic continuation to a slightly larger domain.
2. In particular, for $\boldsymbol{z} \in D^* \setminus D$, we have $S(\boldsymbol{z}) = \boldsymbol{z}$.
3. For $\boldsymbol{z} \in D_{(\theta,\eta)}$, we have $\exp(S(\boldsymbol{z})) = \alpha(S(\boldsymbol{z}))$ if and only if $\boldsymbol{z} \in \Lambda$.
4. S is continuous on $D^*_{(\theta,\eta)}$ and holomorphic on $D_{(\theta,\eta)}$.

5. The restriction of S to the finite part of the domain $D_{(\theta,\eta)}$ is approximately given by $S(\boldsymbol{z}) \approx \boldsymbol{z} + G(\boldsymbol{z})$. More precisely,

$$S(\boldsymbol{z}) - \boldsymbol{z} - G(\boldsymbol{z}) \to \boldsymbol{0} \text{ as } \boldsymbol{z} \to \boldsymbol{c} \text{ with } \boldsymbol{z} \in D_{(\theta,\eta)}.$$

Proof. The proof of points 1–4 is the same as in the abelian case. For point 5 observe, as in the abelian case, that $S(\boldsymbol{z}) - \boldsymbol{z} = G(S(\boldsymbol{z}))$ and

$$|(S(\boldsymbol{z}) - \boldsymbol{z}) - G(\boldsymbol{z})| = |G(S(\boldsymbol{z})) - G(\boldsymbol{z})| \leqslant \max_{\boldsymbol{w}\in[\boldsymbol{z},S(\boldsymbol{z})]} \|\mathrm{d}G(\boldsymbol{w})\| \cdot |S(\boldsymbol{z}) - \boldsymbol{z}|.$$

Since $\|\mathrm{d}G(\boldsymbol{w})\| \to 0$ by Proposition 5.4, we have $|G(S(\boldsymbol{z})) - G(\boldsymbol{z})| = o(|G(S(\boldsymbol{z}))|)$ as $\boldsymbol{z} \to \boldsymbol{c}$. By the triangle inequality $||G(S(\boldsymbol{z}))| - |G(\boldsymbol{z})|| = o(|G(S(\boldsymbol{z}))|)$, which implies that $\frac{|G(S(\boldsymbol{z}))|}{|G(\boldsymbol{z})|} \to 1$ and so $|S(\boldsymbol{z}) - \boldsymbol{z}| \sim |G(\boldsymbol{z})| = O(\log|\boldsymbol{z}|)$.

Further, there is a constant $\gamma > 0$ such that for all sufficiently large $\boldsymbol{z} \in D_{(\theta,\eta)}$ there are θ', η' such that the polydisc of radius $\gamma|\boldsymbol{z}|$, centered at $\boldsymbol{z}$, is contained in $D_{(\theta',\eta')}$. Then by Lemma 5.3 we have $\|\mathrm{d}G(\boldsymbol{z})\| \leqslant \gamma^{-1}K\frac{\log|\boldsymbol{z}|}{|\boldsymbol{z}|}$. When $\boldsymbol{z}$ is sufficiently large, so is $\boldsymbol{w} \in [\boldsymbol{z}, S(\boldsymbol{z})]$, hence

$$\max_{\boldsymbol{w}\in[\boldsymbol{z},S(\boldsymbol{z})]} \|\mathrm{d}G(\boldsymbol{w})\| = O\left(\frac{\log|\boldsymbol{z}|}{|\boldsymbol{z}|}\right)$$

and so

$$\max_{\boldsymbol{w}\in[\boldsymbol{z},S(\boldsymbol{z})]} \|\mathrm{d}G(\boldsymbol{w})\| \cdot |S(\boldsymbol{z}) - \boldsymbol{z}| = O\left(\frac{(\log|\boldsymbol{z}|)^2}{|\boldsymbol{z}|}\right)$$

so it tends to 0. Thus, $|(S(\boldsymbol{z}) - \boldsymbol{z}) - G(\boldsymbol{z})| \to 0$ as $\boldsymbol{z} \to \boldsymbol{c}$. ■

Step 5: analytic continuation of S. The patching together of the maps S on the domains $D^*_{(\theta,\eta)}$, yielding the desired sheaf $\mathbf{S}$, is done exactly as in the abelian case.

Point 1 of Theorem 5.1 can also be proven as in the abelian case, with the following changes. One observes that for every point $\boldsymbol{a} \in \Omega$, there is a branch S of $\mathbf{S}$ around $\boldsymbol{a}$ satisfying the inequality $|S(\boldsymbol{z}) - \boldsymbol{z}| < \mu\left|\log|\boldsymbol{z}|\right|$ for some μ, by the estimates of Step 2. Furthermore, one can choose the branches S so that the imaginary part of $S(\boldsymbol{a})$ is bounded uniformly in $\boldsymbol{a}$. We can then assume that μ is chosen uniformly in $\boldsymbol{a}$. It now suffices to take $\{\boldsymbol{z} \in \Omega : B_{\boldsymbol{z}} \subseteq \Omega\}$ as the set Ω', where $B_{\boldsymbol{z}}$ is the closed ball centered at $\boldsymbol{z}$ of radius $\mu\left|\log|\boldsymbol{z}|\right|$ as in Proposition 5.6, and continue the proof as in the abelian case.

One finds that the image $\mathbf{S}(\Omega)$ covers Ω except possibly for points $\boldsymbol{z}$ of distance at most $\mu\left|\log|\boldsymbol{z}|\right|$ from the boundary.

Point 2 follows from point 3 of Proposition 5.7.

Point 3 can be proven as in the abelian case by working with d branches of $\mathbf{F}$ and $\mathbf{G}$.

Point 4 follows from Proposition 5.7 (5).

For point 5, observe that $\Omega \cap \Lambda \neq \emptyset$ when all the ratios $\frac{c_k}{c_j}$ are rational. Moreover, even though the lattice Λ has rank n (as opposed to $2n$ in the abelian case), the set $\Omega \cap \Lambda$ is Zariski dense in $\mathbb{C}^n$, and so is $\mathbf{S}(\Omega \cap \Lambda)$. That completes the proof of Theorem 5.1, and Theorem 1.5 follows. ■

Remark 5.8. As in the abelian case (Remark 4.10), our method yields solutions of $\exp(\boldsymbol{z}) = \beta(\boldsymbol{z})$ for any holomorphic map $\beta : D_{(\theta,\eta)} \to \mathbb{C}^n$, provided we have control on its growth rate as in (17). In particular, this gives an alternative proof of [7, Rem. 2.8].

6 Final Remarks

Our methods can be adapted to solve more general systems of exponential equations, for example those combining exponential and $\wp$-functions. We indicate below that the analogue of Theorem 1.4 holds for split semiabelian varieties, that is, complex algebraic groups that are isomorphic to a product of an abelian variety and an algebraic torus.

Theorem 6.1. Let $S = A \times \mathbb{G}_{\mathrm{m}}^q$ be a complex split semiabelian variety of dimension n. Let $V \subseteq \mathbb{C}^n \times S$ be an algebraic subvariety with dominant projection to $\mathbb{C}^n$. Then there is $\boldsymbol{z} \in \mathbb{C}^n$ such that $(\boldsymbol{z}, \exp_S(\boldsymbol{z})) \in V$.

The proof is simply a combination of the proofs of Theorems 4.1 and 5.1, so we just present a brief outline.

Proof sketch. Let $p = \dim A$ so that $p + q = n$. The exponential map of S then can be written as $\exp_S = (\exp_A, \exp_{\mathbb{G}_{\mathrm{m}}^q})$. If $\Lambda \subseteq \mathbb{C}^p$ is the lattice of periods of $\exp_A$ then $\Lambda \times (2\pi i\mathbb{Z})^q$ is the lattice of periods of $\exp_S$.

We extract a multivalued algebraic map $\alpha : D^* \to S$ from V as before, and we can write $\alpha(\boldsymbol{z})$ as $(\beta(\boldsymbol{z}), \gamma(\boldsymbol{z}))$ where $\beta : D^* \to A, \gamma : D^* \to \mathbb{G}_{\mathrm{m}}^q$.

Then locally on sector domains we can define

$$F(\boldsymbol{z}) := \boldsymbol{z} - \mathrm{Log}_S\alpha(\boldsymbol{z}) = \boldsymbol{z} - (\mathrm{Log}_A\beta(\boldsymbol{z}), \mathrm{Log}_{\mathbb{G}_{\mathrm{m}}^q}\gamma(\boldsymbol{z})).$$

The previous arguments now show that the total derivative of F tends to the identity as $|\boldsymbol{z}| \to \infty$, and the rest of the proof follows as before. ■

It seems likely that this method can be adapted to work for any semiabelian variety, without the split assumption. The issue in the semiabelian case is that we have neither the compactness of abelian varieties nor the explicit formulas for logarithmic maps as in the case of tori. So a better geometric or analytic understanding of the logarithmic maps would be needed. For instance, one needs an appropriate estimate on the growth of the semiabelian logarithm, so as to ensure that the total derivative of F tends to the identity.

Relaxing the assumption on V that the projection to $\mathbb{C}^n$ is dominant seems more difficult. Our proof ultimately depends on the same good asymptotic behavior of a suitable function F as $|\boldsymbol{z}| \to \infty$ in $\mathbb{C}^n$ as was used in [6] to apply Newton's method. When V does not project dominantly to $\mathbb{C}^n$, any analogous function F we define can oscillate or grow too fast. Nonetheless, we hope that our approach using geometric and topological methods to show that the image of F contains lattice points will be more robust to such issues than Newton's method is.

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References

[1] Ahlfors, L. V. *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*, 3rd ed. International Series in Pure and Applied Mathematics. New York: McGraw-Hill Book Co., 1978.

[2] Ax, J. "On Schanuel's conjectures." *Ann. of Math. (2)* 93 (1971): 252–68.

[3] Bays, M. and J. Kirby. "Excellence and uncountable categoricity of Zilber's exponential fields." 2013: arXiv:1305.0493.

[4] Bays, M. and J. Kirby. "Pseudo-exponential maps, variants, and quasiminimality." *Algebra Number Theory* 12, no. 3 (2018): 493–549.

[5] Bierstone, E. and P. D. Milman. "Local Resolution of Singularities." In *Real Analytic and Algebraic Geometry (Trento, 1988)*, 42–64. Lecture Notes in Math. 1420. Berlin: Springer, 1990.

[6] Brownawell, W. D. and D. W. Masser. "Zero estimates with moving targets." *J. London Math. Soc.* 95, no. 2 (2017): 441–54.

[7] D'Aquino, P., A. Fornasiero, and G. Terzo. "Generic solutions of equations with iterated exponentials." *Trans. Amer. Math. Soc.* 370, no. 2 (2018): 1393–407.

[8] D'Aquino, P., A. Macintyre, and G. Terzo. "Comparing $\mathbb{C}$ and Zilber's exponential fields: zero sets of exponential polynomials." *J. Inst. Math. Jussieu* 15, no. 1 (2016): 71–84.

[9] Eterović, S. and S. Herrero. "Solutions of equations involving the modular j function." *Trans. Amer. Math. Soc.* 374, no. 6 (2021): 3971–98.

[10] Grothendieck, A. and J. Dieudonné. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III." *Inst. Hautes Études Sci. Publ. Math.* 28 (1966): 255.

[11] Hartshorne, R. *Algebraic Geometry*. Grad. Texts in Math. 52. New York, NY: Springer, 1977.

[12] Kirby, J. "The theory of the exponential differential equations of semiabelian varieties." *Selecta Math. (N. S.)* 15, no. 3 (2009): 445–86.

[13] Kirby, J. "Exponential algebraicity in exponential fields." *Bull. Lond. Math. Soc.* 42, no. 5 (2010): 879–90.

[14] Kirby, J. and B. Zilber. "Exponentially closed fields and the conjecture on intersections with tori." *Ann. Pure Appl. Logic* 165, no. 11 (2014): 1680–706.

[15] Lang, S. *Introduction to Transcendental Numbers*. Reading, Mass.–London–Don Mills, Ont.: Addison-Wesley Publishing Co., 1966.

[16] Łojasiewicz, S. *Introduction to Complex Analytic Geometry*. Basel: Birkhäuser, 1991. Translated from the Polish by Maciej Klimek.

[17] Shabat, B. V. *Introduction to Complex Analysis. Part II. Functions of Several Variables*. Transl. Math. Monogr. 110. Providence, RI: American Mathematical Society, 1992. Translated from the Third (1985) Russian edition by J. S. Joel.

[18] Shafarevich, I. R. *Basic Algebraic Geometry 1: Varieties in Projective Space*, 3rd ed. Berlin, Heidelberg: Springer, 2013.

[19] Zilber, B. "Generalized analytic sets." *Algebra Logika* 36, no. 4 (1997): 387–406, 478.

[20] Zilber, B. "Pseudo-exponentiation on algebraically closed fields of characteristic zero." *Ann. Pure Appl. Logic* 132, no. 1 (2005): 67–95.