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Modular Fuss-Catalan Numbers

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Abstract

The modular Catalan numbers $C_{k,n}$, introduced by Hein and Huang in 2016 count equivalence classes of parenthesizations of $x_0 * \cdots * x_n$, where $*$ is a binary k-associative operation and k is a positive integer. The classical notion of associativity coincides with 1-associativity, in which case $C_{1,n} = 1$ and the single 1-equivalence class has size given by the Catalan number C_n. In this paper we introduce modular Fuss-Catalan numbers $C^m_{k,n}$ which count k-equivalence classes of parenthesizations of $x_0 * \cdots * x_n$ where $*$ is an m-ary k-associative operation for $m \geq 2$. Our main results are, an explicit formula for $C^m_{k,n}$, and a characterisation of k-associativity.

Keywords : Fuss-Catalan numbers, Modular Catalan numbers, m-Dyck paths, m-ary trees, Tamari lattice, k-associativity, m-ary operations.

Mathematics Subject Classification (2020) : 05A10 (Primary) 05A19 (Secondary)

1 Introduction

The Catalan numbers are a ubiquitous sequence of natural numbers with a rich mathematical history. They appear in mathematics in widely different contexts and count an ever growing list of sequences of combinatorial sets, see [22] for more on Catalan numbers. The n^{th} Catalan number C_n, where n is a non-negative integer is given by the closed formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Suppose $*$ is an m-ary operation for $m \geq 1$. An m-ary parenthesization of the m-ary product $x_0 * \cdots * x_n$ is a parenthesization where each product is m-ary. For example, when $m = 3$ the parenthesization $((x_0 * x_1 * x_2) * x_3 * x_4)$ is 3-ary whereas $((x_0 * x_1) * x_2 * x_3 * x_4)$ is not. The Fuss-Catalan numbers are a natural generalisation of Catalan numbers introduced by Fuss in [11]. They can be thought of as "higher-dimensional" Catalan numbers. For example, the Catalan number C_n counts the number of binary parenthesizations of the expression $x_0 * \cdots * x_n$, whereas the Fuss-Catalan number C^m_n counts the number of m-ary parenthesizations of the expression $x_0 * \cdots * x_n$, where m and n are non-negative integers, such that $m \geq 1$. The n^{th} Fuss-Catalan number with parameter m is given by the closed formula:

$$C^m_n = \frac{1}{(m-1)n+1}\binom{mn}{n}.$$

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When $m = 2$, we recover the Catalan numbers from the Fuss-Catalan numbers, that is to say $C_n^2 = C_n$. The modular Catalan numbers, introduced in [15], count equivalence classes of parenthesizations of $x_0 * \cdots * x_n$, where $*$ is a binary operation satisfying the k-associative law, which generalises the usual notion of associativity. In this paper we introduce and study a higher-dimensional version of the modular Catalan numbers, which we call modular Fuss-Catalan numbers.

Let X be a non-empty set with a binary operation $\star : X^2 \to X$, and let n be a positive integer. If $\star$ is associative, then the general associativity law states that the expression $x_1 \star \cdots \star x_n$ is unambiguous for all $x_1, \ldots, x_n \in X$. Which is to say, all possible parenthesizations of the expression result in the same evaluation. The order of operation of a binary operation $\star$ is *left-justified* if the order of operation is understood to be from left to right, in which case we write $x_1 \star \cdots \star x_n$ to mean $((\ldots((x_1 \star x_2) \star x_3) \cdots \star x_{n-1}) \star x_n)$. From this point onwards, it will be our convention to treat all binary operation as left-justified. Let $k \geq 1$ be a positive integer. There is a notion of k-associativity for binary operations which generalises the usual notion of associativity. A binary operation $\star$ is *k-associative* if

$$(x_1 \star x_2, \star \cdots \star x_{k+1}) \star x_{k+2} = x_1 \star (x_2 \star \cdots \star x_{k+1} \star x_{k+2}) \text{ for all } x_1, \ldots, x_{k+2} \in X.$$

By setting $k = 1$, we recover the classical notion of associativity for binary operations. In the case where $k > 1$, the general associativity law no longer holds, which is to say in general the evaluation of the expression $x_1 \star x_2 \star \cdots \star x_n$ depends on its parenthesization. The k-associative binary operations are studied in [15].

Fix a positive integer $m \geq 2$. An *m-ary operation* on X is a map $* : X^m \to X$. Another way to generalise associativity of binary operations is to consider associative m-ary operations. An m-ary operation $*$ is *associative* if for $1 \leq j \leq m-1$,

$$\begin{aligned} &x_1 * \cdots * x_{j-1} * (x_j * x_{j+1} * \cdots * x_{j+(m-1)}) * x_{j+(m-1)+1} * x_{j+(m-1)+2} * \cdots * x_{m+(m-1)} \\ &= x_1 * \cdots * x_{j-1} * x_j * (x_{j+1} * \cdots * x_{j+(m-1)} * x_{j+(m-1)+1}) * x_{j+(m-1)+2} * \cdots * x_{m+(m-1)} \end{aligned} \tag{1}$$

for all $x_1, \ldots, x_{m+(m-1)} \in X$, see for example [20, §1].

As in the case for associative binary operations, there is a general associativity law stating that the expression $x_1 * \cdots * x_n$ is independent of m-ary parenthesization (see for example [1, Theorem 2.1]). Which is to say that all possible parenthesizations of the expression result in the same evaluation. We note that n is not arbitrary in this case, but is of the form $n = m + g(m-1)$ for some integer $g \geq 1$. Associative m-ary operations are important for the study of m-semigroups and polyadic groups. These are generalisations of semigroups and groups where we consider associative m-ary operations instead of associative binary operations. The m-semigroups were introduced in [4] and polyadic groups were introduced in [20] and [21].

In this paper we will study m-ary k-associative operations, which are a further generalisation of associative binary operations that combines the two generalisations above. The order of operation of an m-ary operation $*$ is *left-justified* if the order of operation is understood to be from left to right, hence for an integer $g \geq 1$, we write $x_1 * \cdots * x_{m+g(m-1)}$ to mean

$$((\ldots((x_1 * \cdots * x_m) * x_{m+1} * \cdots * x_{m+(m-1)}) \cdots * x_{m+(g-1)(m-1)}) x_{m+(g-1)(m-1)+1} \cdots x_{m+g(m-1)}).$$

From this point onwards, it will be our convention to treat all m-ary operations as left-justified. An m-ary operation $*$ is *k-associative* if for $1 \leq j \leq m-1$, the following equality holds:

$$\begin{aligned} &x_1 * \cdots * x_{j-1} * (x_j * x_{j+1} * \cdots * x_{j+k(m-1)}) * x_{j+k(m-1)+1} * x_{j+k(m-1)+2} * \cdots * x_{m+k(m-1)} \\ &= x_1 * \cdots * x_{j-1} * x_j * (x_{j+1} * \cdots * x_{j+k(m-1)} * x_{j+k(m-1)+1}) * x_{j+k(m-1)+2} * \cdots * x_{m+k(m-1)}. \end{aligned} \tag{2}$$

We note that the terminology "k-associativity" is used by Wardlaw in [23] to mean associativity of k-ary operations. This is not to be confused with the notion of k-associativity we consider here, which is a generalisation of associativity for m-ary operations (and in the case of binary operations coincides with the notion of k-associativity as introduced in [15]).

Let $*$ be a k-associative m-ary operation, and $g \geq 1$ be a positive integer. For $n = m+g(m-1)$ and $k > 1$, the expression $x_1 * \cdots * x_n$ is ambiguous without a parenthesization to clarify the order of operation, which is to say the general associativity law no longer holds. Let p and p' be two parenthesizations of $x_1 * \cdots * x_n$. If we can obtain p' from p via a sequence of finitely many left side to right side applications of the k-associative property (2), then we write $p \preceq_k p'$. The *k-associative order* is the induced partial order on the set of parenthesizations of $x_1 * \cdots * x_n$. The k-components are the connected components of the k-associative order. Two parenthesizations of $x_1 * \cdots * x_n$ are *k-equivalent* if they lie in the same k-component. When $k = 1$ and $m = 2$, we recover the well-known *Tamari lattice* (see for example [13]). In general, determining whether two parenthesizations are k-equivalent is a non-trivial problem.

Cluster algebras are a class of commutative algebras defined combinatorially by a process of iterated mutation. They were first introduced in 2001 by S. Fomin and A. Zelevinsky in a series of seminal papers [8], [9], [2], [10] as an approach towards problems on total positivity [5] and canonical bases in quantum groups. Since their inception, cluster algebras have become an object of study in their own right. They find uses in many other areas including representation theory [19], Poisson geometry [12] and integrable systems [24].

To define a cluster algebra over the field $\mathbb{F} = \mathbb{Q}(u_1, \ldots, u_n)$ of rational functions in the indeterminates $u_1, \ldots, u_n$, one starts with a *seed*. A seed is a pair $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ which consists of a set of variables $\tilde{\mathbf{x}} = \{v_1, \ldots, v_n\}$, which freely generates the field $\mathbb{F}$, and an integer matrix $\tilde{\mathbf{B}}$ called the *exchange matrix*. By applying a certain mutation rule μ_k in a direction k, where $1 \leq k \leq n$, to the seed $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$, we obtain another seed $\mu_k(\tilde{\mathbf{x}}, \tilde{\mathbf{B}}) = (\tilde{\mathbf{x}}', \tilde{\mathbf{B}}^{'})$ consisting of a free generating set of variables $\tilde{\mathbf{x}}'$ and exchange matrix $\tilde{\mathbf{B}}^{'}$. Let $\mathcal{S}$ be the set of variables obtained from performing all possible finite sequences of mutations to the seed $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$. The cluster algebra with initial seed $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$, which we denote by $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ is the subring of $\mathbb{F}$ generated by all the variables in $\mathcal{S}$.

Of particular interest to us are the cluster algebras of type A_n (where n is a positive integer), and their combinatorics. The seeds of a cluster algebra of type A_n can be encoded as triangulations of an $(n+3)$-gon; see [6, Lemma 5.3.1]. Mutating a seed in a cluster algebra of type A_n turns out to be equivalent to performing a *flip* on a diagonal of the corresponding triangulation; see [6, Corollary 5.3.6]. The number of seeds of a cluster algebra of type A_n is equal to the number of triangulations of an $(n+3)$-gon, which is the Catalan number C_{n+1}; see [6, Corollary 5.3.6]. Consider the graph whose vertices are triangulations of $(n+3)$-gons, and where there is an edge between two triangulations T and T' if and only if we can obtain T' from T by performing a flip on a diagonal of T or vice versa. This graph is a regular graph where each vertex has degree n. In fact this graph is the 1-skeleton of the n-dimensional convex polytope known as the *associahedron*, see for example [7, §1.2]. This 1-skeleton is usually realised as follows: the vertices are binary parenthesizations of $x_1 \star \cdots \star x_{n+2}$, and the edges represent applications of the associativity rule. Because of this correspondence, k-associativity is of interest in the study of cluster algebras. In particular, viewing k-associativity as a mutation rule, it might be possible to generalise the definition of cluster algebras to a wider class of objects. If this is possible, then m-ary k-associativity could extend this generalisation to a higher-dimensional setting.

Let $A = (\mathbb{C}\langle u_1, \ldots, u_n\rangle, +, \cdot)$ be the free associative algebra over $\mathbb{C}$ in n indeterminates $u_1, \ldots, u_n$. Let ω be an element of order $k(m-1)$ in A, that is to say $\omega^{k(m-1)} = 1_A$, where 1_A is the multiplicative identity of A. We define an m-ary operation $\circ : A^m \to A$ in the following way for $a_1, \ldots, a_m \in A$:

$$a_1 \circ a_2 \circ \cdots \circ a_m = \omega^{m-1} \cdot a_1 + \omega^{m-2} \cdot a_2 + \cdots + \omega \cdot a_{m-1} + a_m.$$

It is easy to show that this operation is k-associative by direct calculation. Our first main result is a characterisation of k-equivalence.

Theorem 1.1. Let $* : X^m \to X$ be a k-associative m-ary operation on a set X where $m \geq 2$ and $k \geq 1$ are integers. Suppose that $p(x_1 * \cdots * x_n)$ and $p'(x_1 * \cdots * x_n)$ are two m-ary parenthesizations of the expression $x_1 * \cdots * x_n$, where $n = m + g(m-1)$ for some positive integer g. It then follows that $p(x_1 * \cdots * x_n)$ is k-equivalent to $p'(x_1 * \cdots * x_n)$ if and only if

$$p(u_1 \circ \cdots \circ u_n) = p'(u_1 \circ \cdots \circ u_n),$$

where the equality comes from evaluating the parenthesizations under $\circ$ in the algebra A.

We define the *modular Fuss-Catalan number* $C^m_{k,n}$ to be the number of k-equivalence classes of parenthesizations of $x_0 * \cdots * x_n$. By the general associativity law, $C^m_{1,n} = 1$ and there is a single 1-equivalence class whose size is given by the Fuss-Catalan number $C^m_n = \frac{1}{(m-1)n+1}\binom{mn}{n}$. The following is our second main result, which is a generalisation of [15, Proposition 2.10].

Theorem 1.2. The modular Fuss-Catalan number is given by the following explicit formula,

$$C^m_{k,n} = \sum_{\substack{1 \leq l \leq n \\ m-1 \mid l}} \frac{l}{n} \sum_{\substack{m_1 + \cdots + m_k = n \\ m_2 + 2m_3 + \ldots (k-1)m_k = \frac{n-l}{m-1}}} \binom{n}{m_1, \ldots m_k}.$$

This paper is organised as follows: we start in §2 by defining right k-rotations, left k-rotations and k-equivalence for m-ary trees. These are the corresponding notions to k-associativity and k-equivalence in the setting of m-ary trees from m-ary operations. In §3 we study k-equivalence by appealing to m-Dyck paths. By translating the notions of k-associativity and k-equivalence to m-Dyck paths, we were able to prove Theorem 3.15, a characterisation of k-equivalence for m-Dyck paths. As a consequence, we prove Theorem 3.16, a characterisation of k-equivalence. In §4, equipped with Theorem 3.16 we prove our first main result Theorem 4.4 (Theorem 1.1) using some other results obtained in this section and §1. Finally in §5 we derived the explicit formula in Theorem 1.2 using m-Dyck paths and a method adopted from [14, §5].

2 m-ary Trees

In studying k-equivalence, it is often more convenient to do so by appealing to other sequences of combinatorial sets counted by the Fuss-Catalan numbers. In this section we will study k-equivalence via m-ary trees. In order to do this, we use a known bijection between parenthesizations of m-ary expressions and m-ary trees outlined in [17, §0]. For the rest of this section, we fix integers $m \geq 2$, $g \geq 0$, $k \geq 1$, and $n = m + g(m-1)$.

Definition 2.1. **m-ary Tree** [22, §4, A14(b)]. An *m-ary tree* is a rooted tree with the property that each node either has 0 or m linearly ordered children. A *leaf* is a node with no children and the unique node without a parent is the *root* of the tree. For a node with m-children, the l^{th} child refers to the l^{th} node below when counting from left to right, and likewise the l^{th} subtree refers to the l^{th} subtree below when counting from left to right.

The objects we are calling m-ary trees in this paper are commonly referred to as *full m-ary trees* in the wider literature. There are multiple ways in which we can traverse (systematically examine the nodes of the tree so that each node is visited only once) the nodes of an m-ary tree. In this paper, it will be our convention to traverse m-ary trees by the pre-order traversal method. Recall that the *pre-order traverse* is defined recursively as follows.

Definition 2.2. Pre-order traverse[18, §2.3.1, page 319,336] If an m-ary tree is empty, then do nothing. Otherwise,

- Visit the root
- Traverse the 1^{st} subtree of the root
- Traverse the 2^{nd} subtree of the root
- ...
- Traverse the m^{th} subtree of the root.

It will be our convention in this paper to draw m-ary trees with the root at the top and leaves below the root. We shall denote the set of m-ary trees with n leaves by $\mathbf{B}_n^m$. We will enumerate the leaves by the order in which the leaves are visited in the pre-order traverse. Hence enumerating by 1 the first leaf to be visited in the pre-order traverse, by 2 the second leaf to be visited in the pre-order traverse, and so on up to n for the last leaf to be visited in the pre-order traverse. We will endow the m-ary trees with an additional edge labelling with labels from the set $\{l_1, \ldots, l_m\}$. An edge will be given the label l_i if it links a node with its i^{th} child. See the figure below for an example.

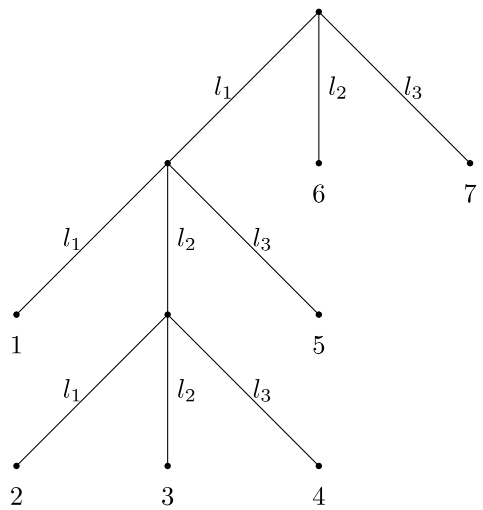


Figure 1: A labelled 3-ary tree.

Definition 2.3. Tag. Let $t_1, \ldots, t_m$ be m-ary trees. We define the *tag* of $t_1, \ldots, t_m$ to be the m-ary tree $t_1 \wedge \cdots \wedge t_m$, which has the tree t_i as the subtree rooted at the i^{th} child of the root for $1 \leq i \leq n$.

The following bijection is well-known, see for example [17, §0].

Proposition 2.4. [17, §0] Let X be a non-empty set and let $* : X^m \to X$ be an m-ary operation. Take $x_1, \dots, x_n$ in X. There is a bijection between the set of m-ary trees on n leaves and the set of m-ary parenthesizations of the expression $x_1 * \cdots * x_n$ which is defined in the following way. Let t be an m-ary tree with n leaves where the i^{th} leaf of t is labelled ε_i. Consider the tree t expressed as a bracketed tag of its leaves ε_i, where the ε_i are thought of as trees consisting of just a root. The bijection maps t to the parenthesization obtained by replacing $\wedge$ with $*$ and replacing ε_i with x_i. The inverse map from the set of m-ary parenthesizations of the expression $x_1 * \cdots * x_n$ to m-ary trees with n leaves acts in the naturally opposite way.

Example 2.5. Let t be the tree in Figure 1. Thinking of the leaves of t as 3-ary trees consisting of just a root, assign to each leaf i the label ε_i. We can write t as a tag of the leaves ε_i so $t = (\varepsilon_1 \wedge (\varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4) \wedge \varepsilon_5) \wedge \varepsilon_6 \wedge \varepsilon_7$. Under the bijection in Proposition 2.4 the tree t is assigned to the parenthesization of the x_i given by $(x_1 * (x_2 * x_3 * x_4) * x_5) * x_6 * x_7$.

Definition 2.6. Right k-rotation. Let $k \geq 1$ be a positive integer. Let $t_1, t_2, \dots, t_{(m-1)+k(m-1)}$ be m-ary trees. Let $1 \leq j \leq m-1$. Suppose that $t \in \mathbf{B}_n^m$ has a subtree,

$$s = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge (t_j \wedge t_{j+1} \wedge \cdots \wedge t_{j+k(m-1)}) \wedge t_{j+k(m-1)+1} \wedge t_{j+k(m-1)+2} \wedge \cdots \wedge t_{(m-1)+k(m-1)}$$

rooted at some node v in t. The *right k-rotation* of t at v is the operation of replacing s with the subtree

$$s' = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge t_j \wedge (t_{j+1} \wedge \cdots \wedge t_{j+k(m-1)} \wedge t_{j+k(m-1)+1}) \wedge t_{j+k(m-1)+2} \wedge \cdots \wedge t_{(m-1)+k(m-1)}.$$

Remark 2.7. It should be clear that under the bijection in Proposition 2.4, right k-rotation of m-ary trees corresponds to a left side to right side application of the k-associative rule in (2). We can also define a left k-rotation dually by switching the roles of s and s' in the definition above. In this case, a left k-rotation corresponds to a right side to left side application of the k-associative rule in (2).

Definition 2.8. Let t and t' be m-ary trees with n leaves. If we can obtain t' from t by applying finitely many right k-rotations to t, then we write $t \preceq_k t'$. The *k-associative order* is the induced partial order on $\mathbf{B}_n^m$. The *k-components* are the connected components (connected components of the Hasse diagram) of $\mathbf{B}_n^m$ under the k-associative order. Two m-ary trees with n leaves are *k-equivalent* if they belong to the same k-component of $\mathbf{B}_n^m$.

Example 2.9. The example below shows a right 2-rotation on a 3-ary tree. We apply the right 2-rotation at v.

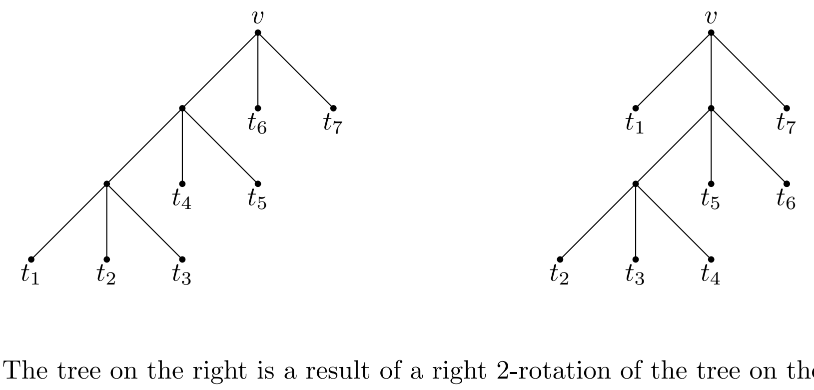


Figure 2: The tree on the right is a result of a right 2-rotation of the tree on the left at v.

The subtree rooted at v is $s = (t_1 \wedge t_2 \wedge t_3 \wedge t_4 \wedge t_5) \wedge t_6 \wedge t_7 = ((t_1 \wedge t_2 \wedge t_3) \wedge t_4 \wedge t_5) \wedge t_6 \wedge t_7$. The subtree s is then replaced by the subtree $s' = t_1 \wedge (t_2 \wedge t_3 \wedge t_4 \wedge t_5 \wedge t_6) \wedge t_7 = t_1 \wedge ((t_2 \wedge t_3 \wedge t_4) \wedge t_5 \wedge t_6) \wedge t_7$ at v.

The following proposition is a generalisation of [15, Proposition 2.5].

Proposition 2.10. Let t be an m-ary tree such that we can perform a right k-rotation of t at some node v. If $k = pk'$ for some positive integers p and k', then the right k-rotation at v can be decomposed into a sequence of p right k'-rotations of t. The same holds for left k-rotations

Proof. We argue by induction on p. The case for $p = 1$ is trivial. Suppose for induction that the statement is true for some $p \geq 1$. Suppose that $k = (p+1)k'$ for some positive integer k'.

Suppose we have a tree t which we can right k-rotate at some node v. Denote by s the subtree of t rooted at v. For some $1 \leq j \leq m-1$,

$s = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge (t_j \wedge t_{j+1} \wedge \cdots \wedge t_{j+pk'(m-1)} \wedge t_{j+pk'(m-1)+1} \wedge \cdots \wedge t_{j+k(m-1)}) \wedge t_{j+k(m-1)+1} \wedge \cdots \wedge t_{(m-1)+k(m-1)}$.

The right k-rotation replaces the subtree s with the subtree s' where

$s' = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge t_j \wedge (t_{j+1} \wedge \cdots \wedge t_{j+m-1} \wedge t_{j+m} \wedge \cdots \wedge t_{j+pk'(m-1)} \wedge t_{j+pk'(m-1)+1} \wedge t_{j+pk'(m-1)+2} \wedge \cdots \wedge t_{j+k(m-1)} \wedge t_{j+k(m-1)+1}) \wedge \cdots \wedge t_{(m-1)+k(m-1)}$.

We will show that the result of this right k-rotation can also be obtained by performing $(p+1)$ right k'-rotations.

Let r be the following subtree of s, which is rooted at the j^{th} child of the root of s,

$$r = (t_j \wedge t_{j+1} \wedge \cdots \wedge t_{j+pk'(m-1)} \wedge t_{j+pk'(m-1)+1} \wedge \cdots \wedge t_{j+k(m-1)}).$$

We can write

$$r = ((t_j \wedge t_{j+1} \wedge \cdots \wedge t_{j+m-1} \wedge t_{j+m} \wedge \cdots \wedge t_{j+pk'(m-1)}) \wedge t_{j+pk'(m-1)+1} \wedge \cdots \wedge t_{j+k(m-1)})$$

since the tag operation is left-justified. Performing a right pk'-rotation of t at the j^{th} child of the root of s, we replace r with

$$r' = (t_j \wedge (t_{j+1} \wedge \cdots \wedge t_{j+m-1} \wedge t_{j+m} \wedge \cdots \wedge t_{j+pk'(m-1)} \wedge t_{j+pk'(m-1)+1}) \wedge \cdots \wedge t_{j+k(m-1)}).$$

By the inductive hypothesis, this right pk'-rotation is the result of p right k'-rotations.

Set

$$u = (t_{j+1} \wedge \cdots \wedge t_{j+m-1} \wedge t_{j+m} \wedge \cdots \wedge t_{j+pk'(m-1)} \wedge t_{j+pk'(m-1)+1}),$$

then

$$r' = (t_j \wedge u \wedge \cdots \wedge t_{j+k(m-1)}).$$

Thus the right pk'-rotation of t at the j^{th} child of the root of s, replaces s with q at the node v in t where,

$$q = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge (t_j \wedge u \wedge t_{j+pk'(m-1)+2} \wedge \cdots \wedge t_{j+k(m-1)}) \wedge t_{j+k(m-1)+1} \wedge \cdots \wedge t_{(m-1)+k(m-1)}.$$

We then perform a right k'-rotation of t at v. This replaces q with the subtree,

$$q' = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge t_j \wedge (u \wedge t_{j+pk'(m-1)+2} \wedge \cdots \wedge t_{j+k(m-1)} \wedge t_{j+k(m-1)+1}) \wedge \cdots \wedge t_{(m-1)+k(m-1)}.$$

It is easy to see that $s' = q'$, therefore the result of performing the right $k = (p+1)k'$-rotation at v is precisely the result of performing $(p+1)$ right-k' rotations. The proof for left k-rotations is similar. □

Definition 2.11. Path. Let t be an m-ary tree and n a positive integer. A *path* p in t of length n from a node v to a node w is a sequence $p = (v_0, \dots, v_n)$ of nodes such that $v_0 = v, v_n = w$ and (v_i, v_{i+1}) is an edge in t for $1 \leq i \leq n-1$.

Definition 2.12. Depth. Let t be an m-ary tree with n leaves and edges labelled by labels from the set $\{l_1, \dots, l_m\}$. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let $\delta_j^{l_i}(t)$ be the number of edges labelled l_i in the unique path from the root to the j^{th} leaf. Let $\delta^{l_i}(t) = (\delta_1^{l_i}(t), \dots, \delta_n^{l_i}(t))$ and set $\delta(t) = (\delta^{l_1}(t), \dots, \delta^{l_n}(t))$. The *depth* of t is the n-tuple $\delta(t)$.

Example 2.13. Let t be the tree in Figure 1. The depth of tree t is given by

$$\delta(t) = ((2,2,1,1,1,0,0), (0,1,2,1,0,1,0), (0,0,0,1,1,0,1)).$$

The following lemmas are easy to verify.

Lemma 2.14. Suppose that t is an m-ary tree with n leaves and depth $(\delta^{l_1}(t), \dots, \delta^{l_m}(t))$. It then follows that $\delta_n^{l_m}(t) \neq 0$ and $\delta_n^{l_i}(t) = 0$ for $i \neq m$. Dually, $\delta_1^{l_1}(t) \neq 0$ and $\delta_1^{l_i}(t) = 0$ for $i \neq 1$.

This is because the unique path from the root to the n^{th} leaf involves choosing the m^{th} child at each stage. Similarly for the dual statement.

Lemma 2.15. Suppose that t is an m-ary tree with n leaves and depth $(\delta^{l_1}(t), \dots, \delta^{l_m}(t))$. It then follows that $\delta_{n-1}^{l_{m-1}}(t) = 1$, moreover for $1 \leq i \leq m-2$, we have that $\delta_{n-1}^{l_i}(t) = 0$.

This is because the unique path from the root to the $(n-1)^{\text{th}}$ leaf involves choosing the m^{th} child at every stage but one, in which case, we choose the $(m-1)^{\text{th}}$ child.

We shall prove the following result in the next section. This result is the m-ary tree analogue of Theorem 1.1.

Theorem 2.16. Suppose that t and t' are a pair of m-ary trees with n leaves with depths $(\delta^{l_1}(t), \dots, \delta^{l_m}(t))$ and $(\delta^{l_1}(t'), \dots, \delta^{l_m}(t'))$ respectively. It then follows that t and t' are k-equivalent if and only if

$$\sum_{i=1}^{m-1} (m-i)\delta^{l_i}(t) \equiv \sum_{i=1}^{m-1} (m-i)\delta^{l_i}(t') \bmod k(m-1)$$

where the addition on the n-tuples is componentwise.

The strength of the theorem is that it allows us to determine the k-equivalence of m-ary trees from simply reading their depths. The case $m = 2$ is known, see [15, Proposition 2.11]). We shall prove the case for general $m \geq 2$. To do this, we appeal to another sequence of combinatorial sets counted by the Fuss-Catalan numbers, the m-Dyck paths. The setting of m-Dyck path turns out to be a more natural setting for studying k-equivalence. We will first prove Theorem 3.15, which is the analogue of Theorem 1.1 for m-Dyck paths. Theorem 3.15 itself is a consequence of Proposition 3.13 and Proposition 3.14. Having proved Theorem 3.15, the theorem above is an easy corollary.

3 m-Dyck Paths

In this section we prove Theorem 2.16. In order to do so, we appeal to a generalisation of Dyck paths known as m-Dyck paths to further study k-equivalence. We prove the theorem by first proving an m-Dyck path version of it. For the rest of this section, we fix integers $m \geq 2$, $g \geq 0$, $k \geq 1$ and $n = m + g(m-1)$.

Definition 3.1. m-Dyck Path. An *m-Dyck path* is a lattice path in $\mathbb{Z}^2$ starting at $(0,0)$ consisting of up-steps (m,m) and down-steps $(1,-1)$, which remains above the x-axis and ends on the x-axis. The length of a Dyck path is defined to be the number of down-steps it has.

Definition 3.2. Translated m-Dyck Path. Let a, b be non-negative integers both not equal to 0. A *translated m-Dyck path* is a lattice path in $\mathbb{Z}^2$ starting at the point (a,b) consisting of up-steps (m,m) and down-steps $(1,-1)$, which remains above the line $y = b$ and ends on the line $y = b$.

We denote the set of m-Dyck paths of length n by $\mathbf{D}_n^m$. Where it is convenient, we refer to these paths as Dyck paths instead of m-Dyck paths. When referring to a translated m-Dyck path that is a sub path of a larger m-Dyck path, we will call it a *sub m-Dyck path* or just *sub-Dyck path*. The following lemma is straight forward, so we state it without proof.

Lemma 3.3. For every m-Dyck path D of length n, we can write

$$D = N^{d_1}S\dots SN^{d_n}S,$$

where N denotes the up-step $(1,1)$ and S denotes the down-step (1,-1). Note that when $m \neq 1$ the up-steps (1,1) are not steps on the path D since by definition up-steps of D are of the form (m,m). Here N^{d_i} is taken to mean a sequence of d_i consecutive up-steps N. The d_i are non-negative integer multiples of m such that $d_1 + \dots + d_n = n$, and $d_1 + \dots + d_j \geq j$ for $1 \leq j < n$. The latter conditions on the d_i are because the m-Dyck paths start at $(0,0)$ and end on the x-axis whilst remaining above the x-axis. Moreover the n-tuple $d(D) = (d_1, \dots, d_n)$ is unique to each m-Dyck path D.

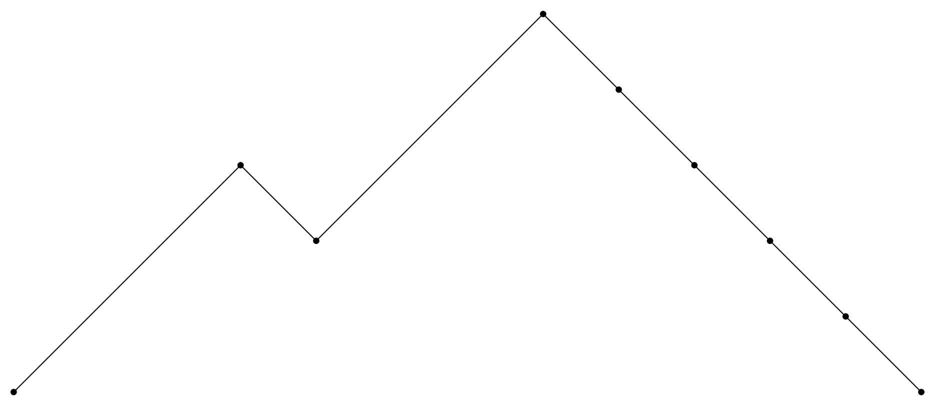


Figure 3: A 3-Dyck path with 2 (3,3) up-steps. $D = N^3SN^3SN^0SN^0SN^0SN^0S$.

Let $D = N^{d_1}S\dots SN^{d_n}S$ be an m-Dyck path of length n. When expressing D in this way, if $d_i = 0$, we will omit N^{d_i} from the expression. In this form we will also write S^l to mean a sequence of l consecutive S steps. For example, $D = N^3SN^3SN^0SN^0SN^0SN^0S = N^3SN^3S^5$.

We can express any m-ary tree as the tag of the m-ary sub-trees rooted at the children of the root. Therefore, for t an m-ary tree with n leaves, we write

$$t = t_1 \wedge \dots \wedge t_m,$$

where for $1 \leq i \leq m$ each t_i is an m-ary tree with n_i leaves and $n_1 + \dots + n_m = n$.

Let ε be the element of the singleton set $\mathbf{B}_0^m$, so ε is the m-ary tree which consists of just a root. Let $\mathbf{B}^m$ be the set m-ary trees with any appropriate number of leaves, and likewise let $\mathbf{D}^{m-1}$ be the set of $(m-1)$-Dyck paths of any appropriate length. We construct a map $\sigma_m \colon \mathbf{B}^m \to \mathbf{D}^{m-1}$ from the set of m-ary trees to the set of $(m-1)$-Dyck paths. We define σ_m inductively in the following way,

$$\sigma_m(t) = \begin{cases} N^0 S^0 & \text{if } t = \varepsilon; \\ N^{m-1}\sigma_m(t_1)S\sigma_m(t_2)S \dots S\sigma_m(t_m) & \text{otherwise.} \end{cases}$$

This construction generalises a well known map between binary trees (2-ary trees) and Dyck paths (1-Dyck paths); see for example [3, Page 58, Tamari Lattice, Paragraph 2].

Example 3.4. Consider the following 3-ary tree $t = \varepsilon \wedge \varepsilon \wedge (\varepsilon \wedge \varepsilon \wedge \varepsilon)$. We calculate $\sigma_3(t)$,

$$\begin{aligned} \sigma_3(t) &= N^2\sigma_3(\varepsilon)S\sigma_3(\varepsilon)S\sigma_3(\varepsilon \wedge \varepsilon \wedge \varepsilon) \\ &= N^2N^0S^0SN^0S^0SN^2\sigma(\varepsilon)S\sigma(\varepsilon)S\sigma(\varepsilon) \\ &= N^2S^2N^2S^2. \end{aligned}$$

See Figure 4 below.

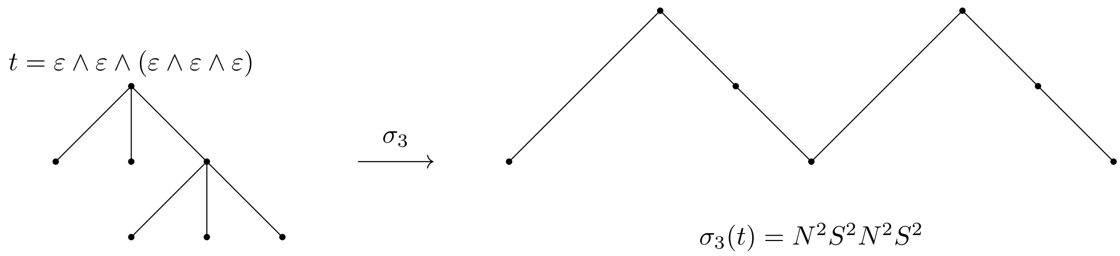
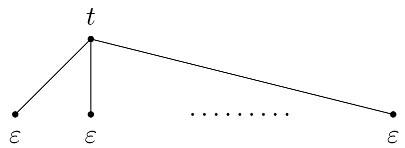


Figure 4: The image under σ_3 of the 3-ary tree $t = \varepsilon \wedge \varepsilon(\varepsilon \wedge \varepsilon \wedge \varepsilon)$.

Lemma 3.5. The map $\sigma_m \colon \mathbf{B}^m \to \mathbf{D}^{m-1}$ sends m-ary trees with n leaves to $(m-1)$-Dyck paths of length $n-1$.

Proof. We argue by induction. Recall that $n = m + g(m-1)$ for some integer $g \geq 0$. We prove the result by induction on g. When $g = 0$ there is only one tree to consider, namely $t = \varepsilon \wedge \cdots \wedge \varepsilon$.



It is easy to see that $\sigma_m(t) = N^{m-1}S^{m-1}$ which is an $(m-1)$-Dyck path of length $m-1$.

Now suppose that the result holds for $n = m + g'(m-1)$ with $0 \leq g' \leq g$. We consider the $g+1$ case. If t is an m-ary tree with $m + (g+1)(m-1)$ leaves, then we may write $t = t_1 \wedge \cdots \wedge t_m$ with the $t_i \in B_{n_i}^m$ and $n_1 + \cdots + n_m = m + (g+1)(m-1)$. By definition $\sigma_m(t) = N^{m-1}\sigma_m(t_1)S\sigma_m(t_2)S \dots S\sigma_m(t_m)$ and by the inductive hypothesis, each $\sigma(t_i)$ is an $(m-1)$-Dyck paths of length $n_i - 1$. In the expression for $\sigma_m(t)$ we have $m-1$ down-steps S following the

N^{m-1} inbetween the $\sigma_m(t_i)$. Therefore, the length of $\sigma_m(t)$ is $(n_1 - 1) + \cdots + (n_m - 1) + (m - 1)$ which is equal to $m + (g + 1)(m - 1) - 1$ as required.

What is left is to show that $\sigma_m(t)$ is weakly above the x-axis. By Lemma 3.3, each $\sigma_m(t_i)$ can be written in the form $\sigma_m(t_i) = N^{d_1^i} S \ldots SN^{d^i_{n_i}} S$, where $d_1^i + \cdots + d^i_{n_i} = n_i$, and $d_1^i + \cdots + d_r^i \geq r$ for $1 \leq r < n_i$. We can likewise write that $\sigma_m(t) = N^{d_1} S \ldots SN^{d_n} S$ where $d_1 = (m-1) + d_1^1$ and $d_n = 0$, and for $2 \leq h < n$ either $d_h = 0$ or $d_h = d_u^i$ for some appropriate i and u. Since there are $m - 1$ down-steps S following the N^{m-1} inbetween the $\sigma_m(t_i)$, and the $\sigma_m(t_i)$ are all weakly above the x-axis. It follows that $d_1^1 + \cdots + d_j \geq j - (m-1)$, therefore $d_1 + \cdots + d_j \geq j$, for $1 \leq j \leq n$, where there is equality if $j = n$. Thus $\sigma_m(t)$ is weakly above the x-axis, hence the map σ_m is indeed from $\mathbf{B}_n^m$ to $\mathbf{D}_{n-1}^{m-1}$. □

By the lemma above, σ_m induces a map $\sigma_{m,n} \colon \mathbf{B}_n^m \to \mathbf{D}_{n-1}^{m-1}$. This map is in fact a bijection between $\mathbf{B}_n^m$ and $\mathbf{D}_{n-1}^m$. For any tree $t = t_1 \wedge \cdots \wedge t_m$, where for $1 \leq i \leq m$ each t_i is an m-ary tree with n_i leaves and $n_1 + \cdots + n_m = n$. The map $\sigma_{m,n}$ is defined as follows:

$$\sigma_{m,n}(t) = N^{m-1}\sigma_{m,n_1}(t_1)S\sigma_{m,n_2}(t_2)S \ldots S\sigma_{m,n_m}(t_m).$$

Proposition 3.6. The map $\sigma_{m,n} \colon \mathbf{B}_n^m \to \mathbf{D}_{n-1}^{m-1}$ is a bijection.

Proof. It is well known that both the finite sets $\mathbf{B}_n^m$ and $\mathbf{D}_{n-1}^{m-1}$ have cardinality $\frac{1}{(m-1)n+1}\binom{mn}{n}$, see for example [16, §3]. Therefore, in order to show that $\sigma_{m,n}$ is a bijection, it suffices to show that it is a surjection. We argue by induction on n. When $n = 0$, it is trivial.

If $D \in \mathbf{D}_{n-1}^{m-1}$, then the first step of D is an up-step $N^{k_1(m-1)}$ where $k_1 \geq 1$ is an integer, so we can write $D = N^{m-1}N^{k_1(m-1)-(m-1)} \ldots S$ as in Lemma 3.3. Let $(x_1', m-2)$ be the first point on D with y-coordinate $m-2$ after the point $(m-1, m-1)$. The step in D ending at $(x_1', m-2)$ must be a down-step S starting at $(x_1, y_1) = (x_1' + 1, m-1)$. The part of D from $(m-1, m-1)$ to (x_1, y_1) is a translated $(m-1)$-Dyck path D_1, so we see that the path D starts as $N^{m-1}D_1S$. Let $(x_2', m-3)$ be the first point on D with y-coordinate $m-3$ after the point $(x_1', m-2)$. As above, the step in D ending at $(x_2', m-3)$ must be a down-step S starting at $(x_2, y_2) = (x_2' - 1, m-2)$. The part of D from $(x_1', m-2)$ to (x_2, y_2) is a translated $(m-1)$-Dyck path D_2. Therefore, $D = N^{m-1}D_1SD_2S \ldots S$, and continuing this argument we see that can be write $D = N^{m-1}D_1SD_2S \ldots D_mS$, where the D_i are translated $(m-1)$-Dyck paths for $1 \leq i \leq m$.

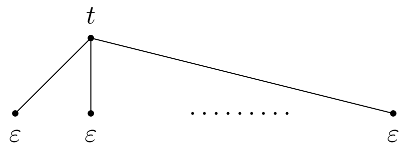
Regarding the translated $(m-1)$-Dyck paths D_i as $(m-1)$-Dyck paths, they each have length $n_i < n$ for $1 \leq i \leq m$. Hence by the inductive hypothesis, for each D_i there exists an m-ary tree t_i such that $D_i = \sigma_{m,n_i}(t_i)$. It then follows that $D = \sigma_{m,n}(t_1 \wedge t_2 \wedge \cdots \wedge t_m)$. □

Going forward, we drop the subscripts on $\sigma_{m,n}$ and just write σ when it is clear what is meant from the context.

Proposition 3.7. Suppose t is an m-ary tree with n leaves and depth $(\delta^{l_1}(t), \ldots, \delta^{l_m}(t))$. It follows that $\sigma(t) = N^{d_1}S \ldots SN^{d_{n-1}}SN^{d_n}$ where the d_i are given by

$$d_1 = (m-1)\delta_1^{l_1}(t),$$
$$d_j = \left(\sum_{i=1}^{m}(m-i)(\delta_j^{l_i}(t) - \delta_{j-1}^{l_i}(t))\right) + 1, \text{ for } 2 \leq j \leq n.$$

Proof. Recall that n satisfies the equation $n = m + g(m-1)$ for some integer $g \geq 0$. We prove the result by induction on g. When $g = 0$ there is only one tree to consider, namely $t = \varepsilon \wedge \cdots \wedge \varepsilon$.



For this tree, $\delta_j^{l_i} = \delta_{ij}$, where the right side is the usual Kronecker delta function. We also have that $\sigma(t) = N^{m-1}SN^0SN^0S\dots N^0SN^0 = N^{m-1}S^{m-1}$. We now need to verify that the exponents of the Ns satisfy the relations above. Indeed $d_1 = m-1 = (m-1)\delta_1^{l_1}$. Moreover $\sum_{i=1}^{m}(m-i)(\delta_j^{l_i} - \delta_{j-1}^{l_i}) + 1 = (m-j) - (m-(j-1)) + 1 = (m-m) + ((j-1)-j) + 1 = 0 = d_j$ for $2 \le j \le n$.

Now suppose that the result holds for $n = m + g'(m-1)$ for all $g' \le g$. We consider the $g+1$ case. Let t be an m-ary tree with $n = m + (g+1)(m-1)$ leaves. We may then write $t = t_1 \wedge \cdots \wedge t_m$ where each t_i is the subtree rooted at the i^{th} child of the root of t. Each subtree t_i has $n_i < n$ leaves and $n_1 + \cdots + n_m = n$. In writing t as the tag of its sub-trees at the root, we partition the leaves of t. We identify each leaf of t with a pair (h,j) if it lies in the subtree t_h and it is the j^{th} leaf in the pre-order traverse of t_h where $1 \le j \le n_h$. Therefore, for the leaf identified by (h,j),

$$\delta_{h,j}^{l_i}(t) = \begin{cases} \delta_{h,j}^{l_i}(t_h) + 1 & \text{if } i = h; \\ \delta_{h,j}^{l_i}(t_h) & \text{otherwise.} \end{cases} \tag{3}$$

By the inductive hypothesis $\sigma(t_h) = N^{d_{h,1}}SN^{d_{h,2}}\dots SN^{h,n_h}$, where

$$d_{h,1}(t_h) = (m-1)\delta_{h,1}^{l_1}(t_h)$$

and

$$d_{h,j}(t) = \sum_{i=1}^{m}(m-i)(\delta_{h,j}^{l_i}(t_h) - \delta_{h,j-1}^{l_i}(t_h)) + 1, \text{ for } 2 \le j \le n_h.$$

By definition, $\sigma(t) = N^{m-1}\sigma(t_1)S\sigma(t_2)\dots S\sigma(t_m)$, so

$$\begin{aligned} \sigma(t) &= N^{m-1}N^{d_{1,1}}SN^{d_{1,2}}S\dots SN^{d_{1,n_1}}SN^{d_{2,1}}SN^{d_{2,2}}S\dots N^{d_{2,n_2}}SN^{d_{3,1}}S\dots SN^{d_{m,n_m}} \\ &= N^{(m-1)+d_{1,1}}SN^{d_{1,2}}S\dots SN^{d_{1,n_1}}SN^{d_{2,1}}SN^{d_{2,2}}S\dots N^{d_{2,n_2}}SN^{d_{3,1}}S\dots SN^{d_{m,n_m}} \end{aligned}$$

Now we verify that the exponents of the Ns satisfy the required relations. We see that

$$(m-1) + d_{1,1} = (m-1) + (m-1)\delta_{1,1}^{l_1}(t_1) = (m-1)(\delta_{1,1}^{l_1}(t_1) + 1) = (m-1)\delta_{1,1}^{l_1}(t),$$

so the first exponent satisfies the required relation. We also see that

$$d_{h,j} = \sum_{i=1}^{m}(m-i)(\delta_{h,j}^{l_i}(t_h) - \delta_{h,(j-1)}^{l_i}(t_h)) + 1 = \sum_{i=1}^{m}(m-i)(\delta_{h,j}^{l_i}(t) - \delta_{h,(j-1)}^{l_i}(t)) + 1, \text{ for } 2 \le j \le n_h,$$

by (3), therefore the $d_{j,h}$ also satisfy the required relation for t.

The only exponents left to verify are the $d_{h,1}$ for $2 \le h \le m$. In this case, the leaf $(h-1, n_{h-1})$ is the rightmost leaf in the subtree t_{h-1}, so by Lemma 2.14, $\delta_{h-1,n_{h-1}}^{l_i}(t_{h-1}) = 0$ when $i \ne m$. Therefore, by (3), $\delta_{h-1,n_{h-1}}^{l_i}(t) = 0$ when $i \ne m, h-1$, so $\delta_{h-1,n_{h-1}}^{l_{h-1}}(t) = 1$. The leaf $(h,1)$ is the leftmost leaf in the subtree t_h, so by a dual statement of Lemma 2.14, $\delta_{h,1}^{l_i}(t_h) = 0$ when $i \ne 1$. Therefore, by (3), $\delta_{h,1}^{l_i}(t) = 0$ when $i \ne 1, h$ and $\delta_{h,1}^{l_h}(t) = 1$. It follows that,

$$
\begin{aligned}
\sum_{i=1}^{m}(m-i)(\delta_{h,1}^{l_i}(t)-\delta_{h-1,n_{h-1}}^{l_i}(t))+1 &= (m-1)\delta_{h,1}^{l_1}(t_h)+(m-h)-(m-(h-1)) \\
&\quad -(m-m)\delta_{h-1,n_{h-1}}^{l_m}(t_{h-1})+1 \\
&= (m-1)\delta_{h,1}^{l_1}(t_h) \\
&= d_{h,1}
\end{aligned}
$$

therefore the $d_{h,1}$ also satisfy the required relations for $2 \leq h \leq m$. This completes the proof. □

Remark 3.8. It is important to note that $d_n = 0$ in Proposition 3.7 since otherwise D is not a Dyck path. We can observe that $d_n = 0$ by referencing Lemma 2.14 and Lemma 2.15. Hence in the proposition above, $\sigma(t)$ is indeed a Dyck path. Also note that since $d_n = 0$, this form of $\sigma(t)$ is the same as that given in Lemma 3.3.

The bijection between m-ary trees with n leaves and $(m-1)$-Dyck paths of length $n-1$ induces an operation corresponding to k-rotation on Dyck paths, which we shall call a *k-compression*. Recall in the definition of a right k-rotation we replace a sub-tree of the form,

$$s = t_1 \wedge t_2 \wedge t_3 \wedge \cdots \wedge t_{j-1} \wedge (t_j \wedge t_{j+1} \wedge \cdots \wedge t_{j+k(m-1)}) \wedge t_{j+k(m-1)+1} \wedge \cdots \wedge t_{(m-1)+k(m-1)}$$

by a subtree of the form,

$$s' = t_1 \wedge t_2 \wedge \cdots \wedge t_j \wedge (t_{j+1} \wedge t_{j+2} \cdots \wedge t_{j+k(m-1)+1}) \wedge t_{j+k(m-1)+2} \wedge \cdots \wedge t_{(m-1)+k(m-1)}.$$

It is easy to see that

$$\sigma(s) = N^{m-1}D_1SD_2S\ldots D_{j-1}SN^{k(m-1)}D_jSD_{j+1}S\ldots SD_{m+k(m-1)},$$

and

$$\sigma(s') = N^{m-1}D_1SD_2S\ldots D_{j-1}SD_jSN^{k(m-1)}D_{j+1}S\ldots SD_{m+k(m-1)},$$

where $D_i = \sigma(t_i)$.

Definition 3.9. Right k-Compression. Let $k \geq 1$ and $1 \leq j \leq m-1$ be positive integers. Let D be an $(m-1)$-Dyck path of length $n-1$. Suppose D contains a sub-Dyck path of the form

$$X = N^{m-1}D_1SD_2S\ldots D_{j-1}SN^{k(m-1)}D_jSD_{j+1}S\ldots SD_{m+k(m-1)},$$

where the D_i are (possibly translated) Dyck paths which may be empty. A *right k-compression* at X is the operation of replacing X with the sub-Dyck path

$$X' = N^{m-1}D_1SD_2S\ldots D_{j-1}SD_jSN^{k(m-1)}D_{j+1}S\ldots SD_{m+k(m-1)}.$$

A *left k-compression* is the inverse operation of replacing X' with X. Let D, D' be $(m-1)$-Dyck paths of length $n-1$. Write $D \preceq_k D'$ to mean that D' can be obtained from D by applying finitely many right k-compressions. The *k-associative order* is the induced partial order on $\mathbf{D}_{n-1}^{m-1}$. The *k-components* are the connected components of $\mathbf{D}_{n-1}^{m-1}$ under the k-associative order. Two $(m-1)$-Dyck paths of length $n-1$ are *k-equivalent* if they belong to the same k-component.

Let $\mathbb{M} \subset \mathbb{N}^n$ be the set of n-tuples of non-negative integers $(e_1, \ldots, e_n)$ satisfying the following relations,

$$e_1 + \cdots + e_n = n-1,$$

$$(m-1)|e_i \text{ for } 1 \le i \le n,$$

$$e_1 + \cdots + e_{j-1} \ge j-1 \text{ for all } 2 \le j \le n.$$

Notice that it follows from the first and last relation that $e_n = 0$.

Proposition 3.10. The map $d\colon \mathbf{D}_{n-1}^{m-1} \to \mathbb{M}$ maps an $(m-1)$-Dyck path of length $(n-1)$ $D = N^{d_1}S \dots SN^{d_n}$ to the n-tuple $d(D) = (d_1, \dots, d_n)$. This map is a bijection.

Proof. Let $D = N^{d_1}S \dots SN^{d_n}$ be an $(m-1)$-Dyck path. Let $d(D) = (d_1, \dots, d_n)$. Note that $d_n = 0$ by Remark 3.8, so the form of D is precisely as in Lemma 3.3. By Lemma 3.3 the tuple $(d_1, \dots, d_{n-1})$ is unique, so the map d is well-defined. Since D is an $(m-1)$-Dyck path, by definition $(m-1)|d_i$. All Dyck paths start and end on the x-axis, therefore they must go up the same number of times as they go down. Hence if a path has length $n-1$, which is the number of down-steps S, then $d_1 + \cdots + d_n = n-1$. By definition, Dyck paths cannot go below the x-axis, this is to say that $d_1 + \cdots + d_{j-1} \ge j-1$ for all $j \ge 2$.

Let $f\colon \mathbb{M} \to \mathbf{D}_{n-1}^{m-1}$ be the map given by $f(e_1, \dots, e_n) = N^{e_1}S \dots SN^{e_n}$. This is a $(m-1)$-dyck path by the arguments similar to those above. It is easy to see that $f(d(D)) = D$, and $d(f((e_1, \dots, e_n))) = (e_1, \dots, e_n)$. Therefore, d is indeed a bijection. □

Proposition 3.11. Let D, D' be $(m-1)$-Dyck paths of length $n-1$ with $d(D) = (d_1, \dots, d_n)$ and $d(D') = (d'_1, \dots, d'_n)$. Suppose that we can obtain D' from D by applying a right k-compression to D. There then exist $1 \le j < i \le n$ such that $d'_i = d_i + k(m-1)$, $d'_j = d_j - k(m-1)$ and $d'_h = d_h$ for $h \ne i, j$.

Proof. Recall from the definition of right k-compression, there exists a sub-dyck path

$$X = N^{m-1}D_1SD_2S \dots D_{a-1}SN^{k(m-1)}D_aSD_{a+1}S \dots SD_{m+k(m-1)}$$

in D which we then replace with the sub-dyck path

$$X' = N^{m-1}D_1SD_2S \dots D_{a-1}SD_aSN^{k(m-1)}D_{a+1}S \dots SD_{m+k(m-1)}$$

to get D'. In replacing X with X' we are simply moving the substring $N^{k(m-1)}$ from the immediate left of the (possibly translated) Dyck path D_a to the immediate left of (possibly translated) Dyck path D_{a+1}. Write $D = N^{d_1}S \dots SN^{d_n}$. Since we have the sub-dyck path X in D, we have the sub-strings $N^{k(m-1)}D_a = N^{d_j}S \dots S$ and $D_{a+1} = N^{d_i} \dots S$ in D for some $1 \le j < i \le n$. Therefore, in replacing X with X' to get D' (moving the $N^{k(m-1)}$ up-steps) we can observe that we have the sub-strings $D_a = N^{d_j - k(m-1)}S \dots S$ and $N^{k(m-1)}D_{a+1} = N^{d_i + k(m-1)}S \dots S$ in the Dyck path D'. This proves the statement of the proposition. □

Remark that if we replace right k-compression with left k-compression in the proposition above, we get that $j > i$ instead.

Corollary 3.12. Let D, D' be $(m-1)$-Dyck paths of length $n-1$. If D and D' are k-equivalent, then $d(D) \equiv d(D') \bmod k(m-1)$.

Proof. This is an immediate consequence of Proposition 3.11. □

Let D be an m-Dyck path of length n. A dyck path D is *k-minimal* if it is minimal in its k-equivalence class. That is to say there does not exist a Dyck path $D' \in \mathbf{D}_n^m$ such that $D' \preceq_k D$. Let $p = (x, y)$ in $\mathbb{Z}^2$ be a point on the m-Dyck path D. The *level* of the point p is the integer y, and we say that p is on the y^{th} level.

Proposition 3.13. An $(m-1)$-Dyck path D is minimal if and only if for $d(D) = (d_1, \dots, d_n)$, we have that $d_i < k(m-1)$ for all $i \neq 1$.

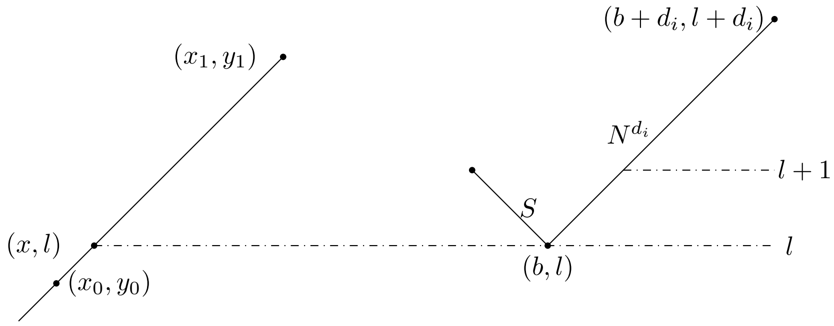
Proof. Suppose that $d_i < k(m-1)$ for all $i \neq 1$ and D is not minimal. We can then left k-compress D to obtain another dyck path D'. By Proposition 3.11 there is some $j > 1$ such that the j-th entry of $d(D')$ is $d'_j = d_j - k(m-1)$. By the assumption that $d_i < k(m-1)$ for $i \neq 1$, we must have that $d'_j < 0$, a contradiction. Thus D must be minimal.

Recall D is of the form $D = N^{d_1}S \dots SN^{d_i}S \dots SN^{d_n}$. Suppose that D is minimal and there exists some $i \neq 0$ such that $d_i \geq k(m-1)$. We will show that D is not minimal by demonstrating that we can left k-compress D. That is to say we will show that there is a sub-Dyck path X' in D which required to perform a left k-compression, where

$$X' = N^{m-1}D_1SD_2S \dots D_{j-1}SD_jSN^{k(m-1)}D_{j+1}S \dots SD_{(m-1)+k(m-1)}$$

for $1 \leq j \leq (m-1)$.

Suppose the up-step N^{d_i} starts at some point (b, l) and ends at $(b+d_i, l+d_i)$. The immediately preceding down-step S starts at $(b-1, l+1)$ and ends at (b, l). Let $0 \leq x \leq b-1$ be maximal such that the point (x, l) is on the Dyck path D. By the maximality, the point (x, l) is part of an up-step. Let U to be the up-step in D beginning at (x, l) if (x, l) is at the start of an up-step; otherwise let U to be the up-step containing (x, l). Let (x_1, y_1) be the end point of the up-step U. Let $(x_0, y_0) = (x_1 - (m-1), y_1 - (m-1))$, this is the start point of the up-step U. See the figure below.



Let (x_2, y_2) be the point at which it is the first time the Dyck path goes below the level y_1 after the point (x_1, y_1). That is $y_2 = y_1 - 1$. We can then observe that subpath D_1 starting from (x_1, y_1) and ending at $(x_2 - 1, y_2 + 1)$ is a translated $(m-1)$-Dyck path. Note that it could happen that $(x_1, y_1) = (x_2 - 1, y_2 + 1)$, in this case D_1 is just the empty $(m-1)$-Dyck path.

Let (x_3, y_3) be the point at which it is the first time the Dyck path goes below the level $y_1 - 1$ after the point (x_2, y_2), that is $y_3 = y_1 - 2$. We define D_2 to be the path starting from (x_2, y_2) to $(x_3 - 1, y_3 + 1)$. As before D_2 is a translated m-Dyck path which starts and ends on the $(y_1 - 1)^{\text{th}}$ level.

Let $j = y_1 - l$. We can repeat this procedure to define translated m-Dyck paths $D_3, D_4, \dots, D_j$. Here each m-Dyck path D_r starts at the point (x_r, y_r) and ends at the point $(x_{r+1} - 1, y_{r+1} + 1)$, where the start and end points are defined as above and $y_{r+1} = y_r - 1 = y_1 - r$. Note that the translated m-Dyck path D_j begins on the level $y_j = y_1 - (j-1) = l + 1$, so the last point of D_j is $(x_{j+1} - 1, l+1)$ for some $x_{j+1} - 1 \leq b$. We claim that $(x_{j+1} - 1, l+1) = (b-1, l+1)$. The point $(b-1, l+1)$ is the last time we are on the $(l+1)^{\text{th}}$ level before the N^{d_i} up-step. By construction, there is a down-step from $(x_{j+1} - 1, l+1)$ to (x_{j+1}, l). By the maximality of x we have that $x_{j+1} = b$ or $x_{j+1} = x$. Note that $x_{j+1} \geq x_1 > x$, so we have that $x_{j+1} = b$.

So far we have constructed a subpath from (x_0, y_0) to $(b, l+1)$ given by

$$X'' = N^{m-1} D_1 S D_2 S \dots S D_j S,$$

where the S down-steps are the down steps from $(x_i - 1, y_i + 1)$ to (x_i, y_i). Note that $y_i = y_1 - (i-1)$ for $2 \leq i < j$ and the S after D_j is the one from $(b-1, l+1)$ to (b, l). The N^{m-1} is the up-step U from (x_0, y_0) to (x_1, y_1).

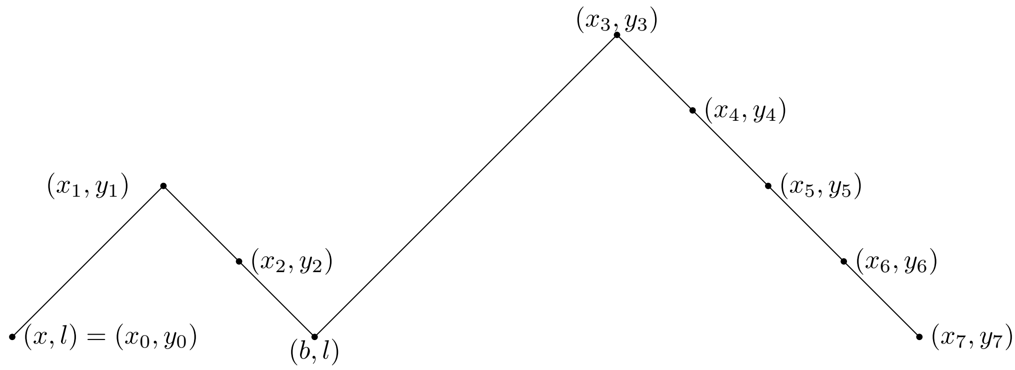
Since $d_i \geq k(m-1)$, there is an up-step $N^{k(m-1)}$ from (b, l) to $(x_{j+1}, y_{j+1}) = (b + k(m-1), l + k(m-1))$. We define D_{j+1} to be the path from (x_{j+1}, y_{j+1}) to $(x_{j+2} - 1, y_{j+2} + 1)$ where (x_{j+2}, y_{j+2}) is the point at which the Dyck path first sits on level $l + k(m-1) - 1$ after (x_{j+1}, y_{j+1}). In the same fashion we define the $(m-1) - j + k(m-1)$ sub paths $D_{j+2}, D_{j+3} \dots D_{(m-1)+k(m-1)}$. These are all translated m-Dyck paths by the same arguments as above. By how we construct the Dyck paths, we see that the path $D_{(m-1)+k(m-1)}$ ends on level $y_0 = y_1 - (m-1)$.

We have thus successfully constructed the sub-Dyck path of D,

$$X' = N^{m-1} D_1 S D_2 S \dots D_{j-1} S D_j S N^{k(m-1)} D_{j+1} S \dots S D_{(m-1)+k(m-1)}.$$

As before the S are the intermediate down steps between the D_i and the D_i may also be empty. □

We illustrate the constructive proof above with an example for the case where $k = 2$ and $m = 3$.



The $N^{m-1} = N^2$ up-step is the one from (x_0, y_0) to (x_1, y_1). The translated 2-Dyck paths D_1 and D_2 are the empty paths $N^0 S^0$ at (x_1, y_1) and (x_2, y_2) respectively. The $N^{k(m-1)} = N^{2(2)}$ up-step is the one from (b, l) to (x_3, y_3). The rest of the translated 2-Dyck paths $D_3, \dots, D_6$ are the empty paths at $(x_4, y_4), \dots, (x_7, y_7)$ respectively. Therefore, X' in this case is the whole path above.

We now show that minimal m-Dyck paths do exist and that they are unique in each k-equivalence class.

Proposition 3.14. Every k-equivalence class contains a unique minimal Dyck path.

Proof. To show existence, we consider the bijection $d\colon \mathbf{D}_{n-1}^{m-1} \to \mathbb{M}$ from Proposition 3.10 where $d(D) = (d_1, \dots, d_n)$. Endow $\mathbb{N}^n$ with the standard lexicographic order. By Proposition 3.11, d is order reversing. That is $D \prec_k D'$ implies that $d(D') <_{\text{lex}} d(D)$. Recall that the lexicographic order is a partial order therefore it has no cycles because of anti-symmetry. Hence suppose that there is a k-equivalence class with no minimal Dyck path. Take D belonging to such a class and repeatedly left k-compress it. Since there is no minimal element in this class, we can do this indefinitely. As a result we obtain the descending chain.

$$\cdots \prec_k D^{(a)} \prec_k \cdots \prec_k D^1 \prec_k D.$$

Applying d to the descending chain, we get the ascending chain.

$$d(D) <_{\text{lex}} d(D^1) <_{\text{lex}} \cdots <_{\text{lex}} d(D^a) <_{\text{lex}} \cdots$$

Since $\mathbf{D}_{n-1}^{m-1}$ is finite, this ascending chain must be a cycle. Since the lexicographic order is anti-symmetric, this cycle must contain only one element. Therefore, if a k-equivalence class does not have a minimal element, it only contains one Dyck path in which case that Dyck path is trivially minimal. This is a contradiction to our assumption. Thus every k-equivalence class has a minimal Dyck path.

Suppose we have two minimal Dyck paths D and D' in an equivalence class. By Proposition 3.13 all but the first entries of $d(D)$ and $d(D')$ are strictly less than $k(m-1)$. But since D and D' are k-equivalent, $d(D) \equiv d(D') \bmod k(m-1)$. This means all but the first entries of $d(D)$ and $d(D')$ are equal. The equality of these entries forces the first entries to also be equal since clearly it cannot be the case otherwise. Therefore, $d(D) = d(D')$ which implies $D = D'$. Therefore, the minimal Dyck paths are unique in their equivalence classes. □

Theorem 3.15. Suppose that D and D' are m-Dyck paths of length n. The m-Dyck paths D and D' are k-equivalent if and only if $d(D) \equiv d(D') \bmod k(m-1)$.

Proof. Suppose that D and D' are k-equivalent. Furthermore, suppose without loss of generality that we obtain D' from D by application of a finite sequence of k-compressions. From Proposition 3.11, we see that a k-compression maps $d(D)$ to an n-tuple which is congruent to $d(D)$ modulo $k(m-1)$. Therefore, since $d(D')$ an n-tuple which is a result of a finite sequence of k-compressesions on D, then $d(D) \equiv d(D') \bmod k(m-1)$.

Suppose now that $d(D) \equiv d(D') \bmod k(m-1)$. Consider their respective minimal representatives in their k-equivalence classes D_{min} and D'_{min} respectively. It then follows that

$$d(D_{min}) \equiv d(D) \equiv d(D') \equiv d(D'_{min}) \bmod k(m-1).$$

Therefore, $d(D_{min}) \equiv d(D'_{min}) \bmod k(m-1)$, hence by Proposition 3.13 we obtain that $d(D_{min}) = d(D'_{min})$ which means $D_{min} = D'_{min}$. Therefore, D and D' belong to the same k-equivalence class. Therefore, we conclude D and D' are k-equivalent. □

Theorem 3.16. Suppose that t and t' are a pair of m-ary trees with n leaves and depth $(\delta^{l_1}(t), \dots, \delta^{l_m}(t))$, $(\delta^{l_1}(t'), \dots, \delta^{l_m}(t'))$ respectively. The trees t and t' are k-equivalent if and only if

$$\sum_{i=1}^{m-1} (m-i)\delta^{l_i}(t) \equiv \sum_{i=1}^{m-1} (m-i)\delta^{l_i}(t') \bmod k(m-1),$$

where the addition on the n-tuples is componentwise.

Proof. Suppose that t and t' are k-equivalent, then their corresponding Dyck paths $D = \sigma(t)$ and $D' = \sigma(t')$ respectively are also k-equivalent. Therefore, by Theorem 3.15, $d(D) \equiv d(D')$ mod $k(m-1)$. By Proposition 3.7,

$$d_1 = (m-1)\delta_1^{l_1}(t),$$

$$d_j = \sum_{i=1}^{m}(m-i)(\delta_j^{l_i}(t) - \delta_{j-1}^{l_i}(t)) + 1, \text{ for } j > 1.$$

Since $d_1 \equiv d_1' \bmod k(m-1)$, we have that $(m-1)\delta_1^{l_1}(t) \equiv (m-1)\delta_1^{l_1}(t') \bmod k(m-1)$. Furthermore, we observe that from the structure of the of m-ary trees that $\delta_1^{l_i}(t) = 0$ and $\delta_1^{l_i}(t') = 0$ for $i \neq 1$. Thus

$$(m-1)\delta_1^{l_i}(t) \equiv (m-1)\delta_1^{l_i}(t') \bmod k(m-1) \text{ for } 1 \leq i \leq m,$$

hence

$$\sum_{i=1}^{m}(m-1)\delta_1^{l_i}(t) \equiv \sum_{i=1}^{m}(m-1)\delta_1^{l_i}(t') \bmod k(m-1).$$

From the fact that,
$d_2 = \sum_{i=1}^{m}(m-i)(\delta_2^{l_i}(t) - \delta_1^{l_i}(t)) + 1 \equiv d_2' = \sum_{i=1}^{m}(m-i)(\delta_2^{l_i}(t') - \delta_1^{l_i}(t')) + 1 \bmod k(m-1)$,
we conclude that,

$$\sum_{i=1}^{m}(m-i)\delta_2^{l_i}(t) \equiv \sum_{i=1}^{m}(m-i)\delta_2^{l_i}(t') \bmod k(m-1).$$

From this congruence and the congruence

$$d_3 = \sum_{i=1}^{m}(m-i)(\delta_3^{l_i}(t) - \delta_2^{l_i}(t)) + 1 \equiv d_3' = d_3 = \sum_{i=1}^{m}(m-i)(\delta_3^{l_i}(t') - \delta_2^{l_i}(t')) + 1,$$

we conclude that,

$$\sum_{i=1}^{m}(m-i)\delta_3^{l_i}(t) \equiv \sum_{i=1}^{m}(m-i)\delta_3^{l_i}(t') \bmod k(m-1).$$

Continuing in this manner we obtain the following,

$$\sum_{i=1}^{m}(m-i)\delta_j^{l_i}(t) \equiv \sum_{i=1}^{m}(m-i)\delta_j^{l_i}(t') \bmod k(m-1), \text{ for } 1 \leq j \leq n.$$

This is the same as saying,

$$\sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t) \equiv \sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t') \bmod k(m-1).$$

Now for the converse, suppose that

$$\sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t) \equiv \sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t') \bmod k(m-1).$$

This implies that,

$$\sum_{i=1}^{m}(m-i)\delta_j^{l_i}(t) \equiv \sum_{i=1}^{m}(m-i)\delta_j^{l_i}(t') \bmod k(m-1), \text{ for } 1 \leq j \leq n.$$

This further implies that

$$d_1 = (m-1)\delta_1^{l_1}(t) \equiv d_1' = (m-1)\delta_1^{l_1}(t) \bmod k(m-1)$$

$$d_j = \sum_{i=1}^{m}(m-i)(\delta_j^{l_i}(t) - \delta_{j-1}^{l_i}(t)) + 1 \equiv d_j' = \sum_{i=1}^{m}(m-i)(\delta_j^{l_i}(t') - \delta_{j-1}^{l_i}(t')) + 1 \bmod k(m-1).$$

Therefore, by Theorem 3.15, $D = \sigma(t)$ and $D' = \sigma(t')$ are k-equivalent which implies that t and t' are k-equivalent. □

4 An Application to m-ary operations

The main aim of this section is to prove Theorem 1.1. To do so, we introduce a particular k-associative m-ary operation which will be denoted by $\circ$. This operation will be used to evaluate m-ary parenthesizations and we will show that this operation characterises k-equivalence. This is to say that two parenthesizations will be k-equivalent (k-associative) if and only if their evaluations under this operation are equal. For the rest of this section, we fix integers $m \geq 2$, $g \geq 0$, $k \geq 1$ and $n = m + g(m-1)$.

Let $A = \mathbb{C}\langle u_1, \ldots, u_n \rangle$ be the free unital associative algebra over $\mathbb{C}$ in n indeterminates $u_1, u_2, \ldots, u_n$. We define a binary operation $\circ$ on A as follows. Let ω be an element of A of order $k(m-1)$, for example $\omega = e^{\frac{2\pi i}{k(m-1)}}$. For a, b in A, we define $a \circ b = \omega \cdot a + b$, where $\cdot$ and $+$ are the multiplication and addition operations in A respectively. This is taken to be a left-justified operation. Sometimes we will omit the $\cdot$ for convenience. The binary operation $\circ$ on A induces an m-ary operation on A^m defined in the following way,

$$a_1 \circ a_2 \cdots \circ a_m = \omega^{m-1} \cdot a_1 + \omega^{m-2} \cdot a_2 + \cdots + \omega \cdot a_{m-1} + a_m. \tag{4}$$

It is easy to see by direct calculation that the following two lemmas are true.

Lemma 4.1. The binary operation $\circ$ on A is $k(m-1)$-associative.

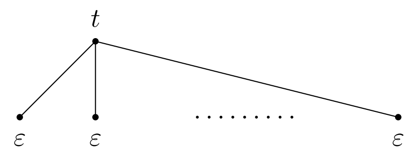
Lemma 4.2. The m-ary operation on A^m induced by the binary operation $\circ$ on A is k-associative.

Let X be a non-empty set and let $* : X^m \to X$ be an m-ary operation. Take $x_1, \ldots, x_n$ in X. Recall that there is a bijection between the set of m-ary trees on n leaves and the set of m-ary parenthesizations of the expression $x_1 * \cdots * x_n$, see Proposition 2.4. We will write $p_t = p(x_1 * \cdots * x_n)_t$ to be the m-ary parenthesization of the expression $x_1 * \cdots * x_n$ corresponding to the m-ary tree t. We denote the evaluation of p_t with respect to $\circ$ by $p(u_1 \circ \cdots \circ u_n)_t$. When there is no risk of confusion, we omit the subscript t.

Lemma 4.3. Suppose that $p(x_1 * \cdots * x_n)_t$ is an m-ary parenthesization of $x_1 * \cdots * x_n$ corresponding to the m-ary tree on n leaves t, which has depth $\delta(t) = (\delta^{l_1}(t), \ldots, \delta^{l_m}(t))$ be the depth of t. It follows that

$$p(u_1 \circ \cdots \circ u_n)_t = \omega^{\sum_{i=1}^{m}(m-i)\delta_1^{l_i}(t)} \cdot u_1 + \omega^{\sum_{i=1}^{m}(m-i)\delta_2^{l_i}(t)} \cdot u_2 + \cdots + \omega^{\sum_{i=1}^{m}(m-i)\delta_n^{l_i}(t)} \cdot u_n.$$

Proof. Recall that n satisfies the equation $n = m + g(m-1)$ for some integer $g \geq 0$. We prove the result by induction on g. When $g = 0$ there is only on tree to consider, namely $t = \varepsilon \wedge \cdots \wedge \varepsilon$.



For this tree, $\delta^{l_i}_j = \delta_{ij}$, where the right side is the usual Kronecker delta function. it is easy to see that the statement holds in this case by the definition of $u_1 \circ \cdots \circ u_m$ in (4).

Now suppose that the result holds for $n = m + g'(m-1)$ for all $g' \leq g$. We consider the $g+1$ case. Let t be an m-ary tree with $n = m + (g+1)(m-1)$ leaves. We may write $t = t_1 \wedge \cdots \wedge t_m$ where each t_i is the subtree rooted at the i^{th} child of the root of t. Each subtree t_i has $n_i < n$ leaves and $n_1 + \cdots + n_m = n$. In writing t as the tag of its sub-trees at the root, we partition the leaves of t. We identify each leaf of t with a tuple (h, j) if it lies in the subtree t_h and it is the j^{th} leaf in the pre-order traverse of t_h, where $1 \leq j \leq n_h$. Therefore, for the leaf identified with (h, j),

$$\delta^{l_i}_{h,j}(t) = \begin{cases} \delta^{l_i}_{h,j}(t_h) + 1 & \text{if } i = h; \\ \delta^{l_i}_{h,j}(t_h) & \text{otherwise.} \end{cases} \tag{5}$$

From the equation above, it follows that,

$$(m-i)\delta^{l_i}_{h,j}(t) = \begin{cases} (m-i)\delta^{l_i}_{h,j}(t_h) + (m-i) & \text{if } i = h; \\ (m-i)\delta^{l_i}_{h,j}(t_h) & \text{otherwise.} \end{cases} \tag{6}$$

The identification of the leaves with the tuples (h, j) gives another labelling of the variables u_s, where $1 \leq s \leq n$. Since the variable u_s corresponds to the s^{th} leaf of t, and the s^{th} leaf is identified with (h, j), we write $u_{(h,j)}$ for u_s. Hence

$$p(u_1 \circ u_2 \circ \cdots \circ u_n)_t = p(u_{(1,1)} \circ u_{(1,2)} \ldots u_{(m,n_m)})_t.$$

It is then easy to see that,

$$\begin{aligned} p(u_{(1,1)} \circ u_{(1,2)} \circ \cdots \circ u_{(m,n_m)})_t =& p(u_{(1,1)} \circ \cdots \circ u_{(1,n_1)})_{t_1} \circ p(u_{(2,1)} \circ \cdots \circ \cdots \circ u_{(2,n_2)})_{t_2} \circ \\ & \cdots \circ p(u_{(m,1)} \circ \cdots \circ u_{(m,n_m)})_{t_m} \end{aligned}$$

$$= \omega^{m-1} p(u_{(1,1)} \circ \cdots \circ u_{(1,n_1)})_{t_1} + \omega^{m-2} p(u_{(2,1)} \circ \cdots \circ \cdots \circ u_{(2,n_2)})_{t_2} + \cdots + p(u_{(m,1)} \circ \cdots \circ u_{(m,n_m)})_{t_m}.$$

By in the inductive assumption,

$$\begin{aligned} p(u_{(h,1)} \circ u_{(h,2)} \circ \cdots \circ u_{(h,n_h)})_{t_h} =& \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,1)}(t_h)} \cdot u_{(h,1)} + \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,2)}(t_h)} \cdot u_{(h,2)} + \\ & \cdots + \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,n_h)}(t_h)} \cdot u_{(h,n_h)} \end{aligned}$$

from which it follows that,

$$\begin{aligned} \omega^{m-h} p(u_{(h,1)} \circ u_{(h,2)} \circ \cdots \circ u_{(h,n_h)})_{t_h} = \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,1)}(t_h) + (m-h)} \cdot u_{(h,1)} + \\ \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,2)}(t_h) + (m-h)} \cdot u_{(h,2)} + \cdots + \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,n_h)}(t_h) + (m-h)} \cdot u_{(h,n_h)}. \end{aligned}$$

By equation (6),

$$\left(\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,j)}(t_h)\right)+(m-h)=\left(\sum_{\substack{i=1\\i\neq h}}^{m}(m-i)\delta^{l_i}_{(h,j)}(t_h)\right)+(m-h)\delta^{l_h}_{(h,j)}(t_h)+(m-h)$$
$$=\left(\sum_{\substack{i=1\\i\neq h}}^{m}(m-i)\delta^{l_i}_{(h,j)}(t)\right)+(m-h)\delta^{l_h}_{(h,j)}(t)$$
$$=\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,j)}(t)$$

Therefore,

$$p(u_{(1,1)}\circ u_{(1,2)}\circ\cdots\circ u_{(m,n_m)})_t=\omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(1,1)}(t)}\cdot u_{(1,1)}+\omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(1,2)}(t)}\cdot u_{(1,2)}+\cdots+\omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(m,n_m)}(t)}\cdot u_{(m,n_m)},$$

as required. This completes the proof.

□

We are now able to prove our first main result of the paper.

Theorem 4.4. Suppose that $p=p(x_1*\cdots*x_n)_t$ and $p'=p'(x_1*\cdots*x_n)_{t'}$ are two m-ary parenthesizations of $x_1*\cdots*x_n$ corresponding to the m-ary trees on n leaves t and t' respectively. It then follows that p and p' are k-equivalent with respect to k-associativity if and only if,

$$p(u_1\circ\cdots\circ u_n)_t=p'(u_1\circ\cdots\circ u_n)_{t'}.$$

Proof. Suppose the parenthesizations p and p' are k-equivalent. It follows that the trees t and t' are also k-equivalent. By Theorem 3.16,

$$\sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t)\equiv\sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t')\bmod k(m-1).$$

Therefore,

$$p(u_1\circ u_2\cdots\circ u_n)_t=p'(u_1\circ u_2\cdots\circ u_n)_{t'}$$

by Lemma 4.3.

Suppose that

$$p(u_1\circ u_2\cdots\circ u_n)_t=p'(u_1\circ u_2\cdots\circ u_n)_{t'},$$

then

$$\omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_1(t)}\cdot u_1+\omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_2(t)}\cdot u_2+\cdots+\omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_n(t)}\cdot u_n$$

$=$

$$\omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_1(t')}\cdot u_1+\omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_2(t')}\cdot u_2+\cdots+\omega^{\sum_{r=1}^{m}(m-i)\delta^{l_i}_n(t')}\cdot u_n.$$

Since $u_1, \dots, u_n$ are algebraically independent and hence linearly independent in A, the coefficients of the u_i on each side of the equation must be equal.
Hence

$$\omega^{\sum_{i=1}^{m}(m-i)\delta_j^{l_i}(t)} = \omega^{\sum_{i=1}^{m}(m-i)\delta_j^{l_i}(t')} \text{ for } 1 \le j \le n.$$

Since ω has order $k(m-1)$ this implies,

$$\sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t) \equiv \sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t') \bmod k(m-1).$$

Hence t and t' are k-equivalent by Theorem 3.16 which implies that p and p' are also k-equivalent by Remark 2.7. □

Example 4.5. In example 2.9 we saw that the 3-ary parenthesization $((x_1x_2x_3)x_4x_5)x_6x_7$ is 2-equivalent to $x_1((x_2x_3x_4)x_5x_6)x_7$. Let us check the theorem above for this example.
The depth of the first tree is

$$(\delta^{l_1} = (3,2,2,1,1,0,0), \delta^{l_2} = (0,1,0,1,0,1,0), \delta^{l_3} = (0,0,1,0,1,0,1)).$$

Therefore, the valuation of $((x_1x_2x_3)x_4x_5)x_6x_7$ with respect to $\circ$ is

$$\omega^6 x_1 + \omega^5 x_2 + \omega^4 x_3 + \omega^3 x_4 + \omega^2 x_5 + \omega x_6 + x_7.$$

The depth of $x_1((x_2x_3x_4)x_5x_6)x_7$ is

$$(\delta^{l_1} = (1,2,1,1,0,0,0), \delta^{l_2} = (0,1,2,1,2,1,0), \delta^{l_3} = (0,0,0,1,0,1,1)),$$

hence the valuation of $x_1((x_2x_3x_4)x_5x_6)x_7$ with respect to $\circ$ is

$$\omega^2 x_1 + \omega^5 x_2 + \omega^4 x_3 + \omega^3 x_4 + \omega^2 x_5 + \omega x_6 + x_7.$$

Since ω has order 4 the valuations are equal.

5 Modular Fuss-Catalan Number

Recall that we define the modular Fuss-Catalan number $C_{k,n}^m$ to be the number of k-equivalence classes of parenthesizations of $x_0 * \cdots * x_n$. In the previous sections we saw that k-associativity corresponds to k-rotation and k-compression. Therefore, $C_{k,n}^m$ also counts the k-equivalence classes of $(m-1)$-Dyck paths of length n. In this section we follow the strategy of [14, §5] to derive an explicit formula for $C_{k,n}^m$, see Theorem 1.2. By Proposition 3.14, each k-equivalence class has a unique minimal element. Therefore, to count the number of k-equivalence classes, we just need to count the number of minimal elements. For the rest of this section, we fix integers $m \ge 2$, $g \ge 0$, $k \ge 1$ and $n = m + g(m-1)$.

Assume that l is a positive integer in $\{1, \dots, n\}$ such that $(m-1)$ divides l. Let N denote the up-step $(1,1)$ and S denote the down-step $(1,-1)$ in $\mathbb{Z}^2$. Denote by $\mathcal{C}_{k,n,l}'^m$ the set of all strings (lattice paths) of the form $N^lSN^{i_1}SN^{i_2}\dots SN^{i_n}$ such that $i_1 + i_2 + \cdots + i_n = n - l$ where $(m-1)|i_p$ for all $1 \le p \le n$ and $0 \le i_1, \dots, i_n < k(m-1)$. Thus $\mathcal{C}_{k,n,l}'^m$ is a set of lattice paths of length n where the first up-step is of size l. For integers $1 \le j \le k$, denote by m_j the number of $(j-1)(m-1)$s appearing among the $i_1, \dots, i_n \in \{0, m-1, 2(m-1), \dots, (k-1)(m-1)\}$.

It is easy to see that

$$|\mathcal{C}'^m_{k,n,l}| = \sum_{\substack{m_1+\cdots+m_k=n \\ m_2+2m_3+\cdots+(k-1)m_k=\frac{n-l}{m-1}}} \binom{n}{m_1, m_2, \ldots, m_k}.$$

For a string $w = N^l SN^{i_1} SN^{i_2} \ldots SN^{i_n}$ in $\mathcal{C}'^m_{k,n,l}$ and j in $\{0, 1, \ldots, n-1\}$ we define

$$w^{\bullet j} = N^l SN^{i_{j+1}} S \ldots SN^{i_n} SN^{i_1} S \ldots SN^{i_j}.$$

Let $\mathcal{C}^m_{k,n,l}$ be the subset of strings in $\mathcal{C}'^m_{k,n,l}$ which are $(m-1)$-Dyck paths of length n. Since $(m-1)$-Dyck paths can be thought of as 1-Dyck paths where up-steps come in multiples of $m-1$, the following lemmas follow by similar arguments to Lemma 5.5 and Lemma 5.6 from [14], which can be thought of as the $m = 2$ case. Thus we will state the followings lemmas without proof.

Lemma 5.1. For a string w in $\mathcal{C}'^m_{k,n,l}$ the set $\{0 \leq j \leq n-1 : w^{\bullet j} \in \mathcal{C}^m_{k,l,n}\}$ has cardinality l.

Let ϕ be the following map,

$$\phi : \mathcal{C}^m_{k,n,l} \times \{0, 1, \ldots, n-1\} \to \mathcal{C}'^m_{k,n,l},$$

$$(w, j) \mapsto w^{\bullet j}.$$

Lemma 5.2. For a string w in $\mathcal{C}'^m_{k,n,l}$, the fibre $\phi^{-1}(w)$ of ϕ over w has cardinality $|\phi^{-1}(w)| = l$.

By Proposition 3.14, the modular Fuss-Catalan number counts the number of minimal $(m-1)$-Dyck paths. Moreover by Proposition 3.13, minimal Dyck paths D satisfy $d(D) = (d_1, \ldots, d_n)$ where $d_i < k(m-1)$ for $i \neq 1$. Combining the results of Proposition 3.13, Proposition 3.14 and Lemma 5.2,

$$|\mathcal{C}^m_{k,n,l}| = \frac{l}{n}|\mathcal{C}'^m_{k,n,l}|.$$

Let $\mathcal{C}^m_{k,n}$ be the set of minimal $(m-1)-$Dyck paths, then

$$|\mathcal{C}^m_{k,n}| = \sum_{\substack{1 \leq l \leq n \\ (m-1)|l}} |\mathcal{C}^m_{k,n,l}|.$$

Therefore,

$$C^m_{k,n} = |\mathcal{C}^m_{k,n}| = \sum_{\substack{1 \leq l \leq n \\ m-1|l}} \frac{l}{n} \sum_{\substack{m_1+\cdots+m_k=n \\ m_2+2m_3+\cdots+(k-1)m_k=\frac{n-l}{m-1}}} \binom{n}{m_1, \ldots m_k}$$

is the number of minimal $(m-1)$-Dyck paths of length n, so by Proposition 3.6, it is the number of minimal m-ary trees of length $n+1$. This completes the proof of Theorem 1.2.

Example 5.3. In this example, we will count the number of 2-equivalence classes of 3-ary trees with 7 leaves. There are twelve 3-ary trees altogether. See the figure below for the complete list.

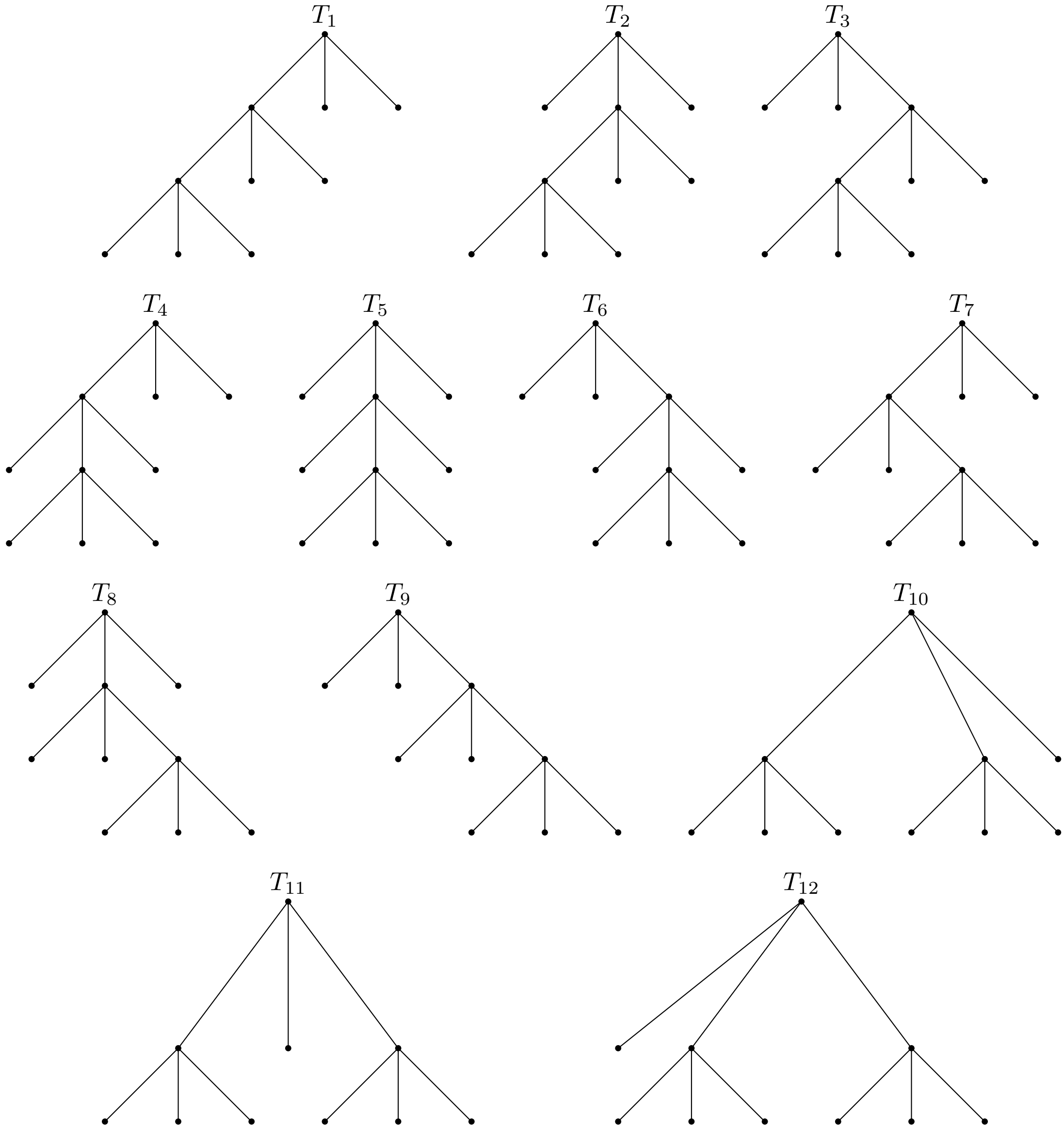


Figure 5: The complete list of the 3-ary trees with 7 leaves.

Observe that the trees T_1, T_2 and T_3 in the top row correspond to the following parenthesizations

$$((x_1 * x_2 * x_3) * x_4 * x_5) * x_6 * x_7,$$

$$x_1 * ((x_2 * x_3 * x_4) * x_5 * x_6) * x_7,$$

$$x_1 * x_2 * ((x_3 * x_4 * x_5) * x_6 * x_7)$$

respectively. We can see that we get the tree T_2 from the tree T_1 by a 2-rotation at the root of T_1, and likewise we get the tree T_3 from the tree T_2 by a 2-rotation at the root of T_2. Therefore

T_1, T_2 and T_3 belong to the same 2-equivalence class. Further observe that the other trees cannot be 2-rotated because they do not contain a subtree of form required to perform a 2-rotation. We conclude that $C^3_{2,6} = 10$. Let us check this against the explicit formula of Theorem 1.2.

$$\begin{aligned} C^3_{2,6} &= \sum_{\substack{1\leq l\leq 6\\ 2|l}} \frac{l}{6} \sum_{\substack{m_1+m_2=6\\ m_2=\frac{6-l}{2}}} \binom{6}{m_1, m_2} \\ &= \frac{2}{6}\binom{6}{4,2} + \frac{4}{6}\binom{6}{5,1} + \frac{6}{6}\binom{6}{6,0} \\ &= \frac{1}{3}(15) + \frac{2}{3}(6) + 1(1) \\ &= 10. \end{aligned}$$

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