Article:

Huy Tuan, N, Lesnic, D orcid.org/0000-0003-3025-2770, Ngoc Thach, T et al. (1 more author) (2022) Regularization of the backward stochastic heat conduction problem. Journal of Inverse and Ill-Posed Problems, 30 (3). pp. 351-362. ISSN 0928-0219

https://doi.org/10.1515/jiip-2020-0013

REGULARIZATION OF THE BACKWARD STOCHASTIC HEAT CONDUCTION PROBLEM

NGUYEN HUY TUAN, DANIEL LESNIC, TRAN NGOC THACH, AND TRAN BAO NGOC

ABSTRACT. In this paper, we study the backward problem for the stochastic parabolic heat equation driven by a Wiener process. We show that the problem is ill-posed by violating the continuous dependence on the input data. In order to restore stability, we apply a filter regularization method which is completely new in the stochastic setting. Convergence rates are established under different *a priori* assumptions on the sought solution.

Keywords: Stochastic parabolic equations; Backward problems; Regularization; Error estimates.

2010 MSC: 26A33, 35B65, 35R11

1. INTRODUCTION

Ever since the first-time award of the Fields prize to Applied Mathematics in 2014, the study of stochastic PDEs has considerably increased in interest. While the subject is not at all new as far as direct problems are concerned, inverse problems for stochastic PDEs are in the early stages of development. Since in nature any physical phenomenon contains some elements of randomness, a model based on stochastic PDEs would be more realistic than a deterministic PDE model. The price to pay is that in this more realistic stochastic model, the noisy variables cannot be differentiated, and furthermore, the usual compactness embedding results do not hold for the solution belonging to appropriate spaces characteristic to stochastic PDEs, [14].

In this paper, we consider a backward problem for the stochastic parabolic heat equation, as follows:

$$\begin{cases} u_t(x,t) - \nu(t)\Delta u(x,t) = g(x,t) + f(t)\dot{\mathbb{W}}, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,T) = \xi(x), & x \in \Omega. \end{cases} \tag{1}$$

Here, $\nu : (0,T) \to \mathbb{R}$ is the diffusivity function satisfying the uniform ellipticity condition $0 < \nu^- \le \nu(t) \le \nu^+ < \infty$, where ν^- and ν^+ are two positive constants, and the noisy term $(g(x,t) + f(t)\dot{\mathbb{W}})$ represents a stochastic heat source. Typical examples of stochastic noise are standard Brownian motion, fractional Brownian motion, Poisson and Levy. In this paper, for simplicity, the standard Brownian motion $\mathbb{W}$ is considered, serving as guide for other more complicated types of noise. The stochastic nature of the first equation in (1) is of importance in modelling random perturbations coming from uncontrollable sources. The operator $-\Delta : D(-\Delta) := H_0^1(\Omega) \cap H^2(\Omega) \to L^2(\Omega)$ is the negative Laplacian with respect to the variable x, Ω is a C^2 bounded open set of $\mathbb{R}^m$, $m \ge 1$, and $\{\mathbb{W}(x,t)\}_{t\ge 0}$ is a $L^2(\Omega)$-Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ with normal filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying that all $\mathbb{P}$-null sets of $\mathcal{F}$ belong to $\mathcal{F}_0$. The formal time derivative $\dot{\mathbb{W}} := \partial\mathbb{W}/\partial t$ is called a white noise. Let $\mathbb{E}$ denote the expectation (with respect to $\mathbb{P}$). The final data ξ is a $\mathcal{F}_0$-measurable random variable and belongs to $L^2(\Omega)$. Our inverse problem here is to determine $u(x,t)$ from the input data ξ, g and f.

Now, we give some introduction to the history of Problem (1). The deterministic model of Problem (1), i.e., when $f(t)\dot{\mathbb{W}}$ is omitted, commonly known as the backward heat conduction problem (BHCP) has been extensively studied in the literature over the last few decades, see e.g. [2, 3, 4, 5, 9, 12, 13, 15, 16] to mention only a few. The BHCP arises in several practical areas such as heat transfer and image processing, [1, 9]. The problem is well-known to be severely ill-posed in the sense that a solution corresponding to the data ξ does not always exist, and in the case of existence, it does not depend continuously on the given data. In fact, from small noise in ξ, the corresponding solution at $t = 0$ will have large errors. Hence, one has to resort to regularization in order to restore stability.

To the best of our knowledge, there seems to be only a few works on inverse problems for the stochastic heat equation; here we can mention the inverse source problem, [10, 11, 14, 17], and the backward problem, [14, 17]. Continuing from [14] where only the uniqueness of solution has been investigated, in this paper we first study the ill-posedness of Problem (1). Then, we construct a regularized solution to

Problem (1) when the given data ξ, g and f are noisy. The rest of the paper is organized as follows. In Section 2, we introduce notation, present some preliminary results on Wiener processes and give the solution to Problem (1). In section 3, we give an example which shows the ill-posedness of our problem in stochastic setting. In section 4, we establish a filter regularization method for obtaining a stable approximate solution to our problem. Two examples of filters are proposed. Moreover, we obtain the convergence rate of the regularized solution towards the true solution, as the noise level tends to zero. Finally, section 5 presents and discusses numerical results.

2. Preliminaries

It is well-known that the eigenvalues λ_j of the negative Laplacian with Dirichlet boundary conditions satisfy that $0 < \lambda_j \leq \lambda_{j+1}$ and $\lim_{j\to\infty} \lambda_j = \infty$, [6]. The corresponding eigenfunctions φ_j satisfy $-\Delta\phi_j = \lambda_j \phi_j$ and form an orthonormal basis of $H := L^2(\Omega)$. Unlike the deterministic case in which u is regarded as a function of space and time, in the stochastic case we view u as an $H-$valued stochastic process. We denote by $\dot{H}^s$, $s > 0$, the Hilbert scale space equipped with the norm $\|\cdot\|_{\dot{H}^s}$ as follows:

$$\dot{H}^s := \{g \in H : \ \|g\|_{\dot{H}^s} := \|(-\Delta)^s g\|_H < \infty\},$$

where $(-\Delta)^s g := \sum_{j=1}^{\infty} \lambda_j^s \langle g, \varphi_j\rangle \varphi_j$, and $\langle\cdot,\cdot\rangle$ is the usual inner product in H. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a compete filtered probability space which satisfies that all $\mathbb{P}$-null sets of $\mathcal{F}$ belong to $\mathcal{F}_0$. We consider the Wiener process $\{\mathbb{W}(\cdot,t)\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ with a linear bounded covariance operator Q such that $\mathrm{Trace}(Q) < \infty$. Denoting by $(\beta_j^2)_{j\in\mathbb{N}^*}$ the eigenvalues of Q, the Wiener process has the Karhunen-Loeve expansion, [8],

$$\mathbb{W}(x,t) = Q^{1/2} \sum_{j=1}^{\infty} \gamma_j(t)\varphi_j(x) = \sum_{j=1}^{\infty} \beta_j \gamma_j(t) \varphi_j(x), \quad t \geq 0, \tag{2}$$

where $\{\gamma_j(t)\}_{j\in\mathbb{N}^*}$ is a sequence of one-dimensional Brownian motions.

Now, let us recall some well-known functional spaces. For $r \geq 1$, we denote by $L^r(\Omega, H)$ the space of all H-valued random variables $\mathcal{X}$ on Ω equipped with the norm $\|\mathcal{X}\|_{L^r(\Omega,H)} := \left(\mathbb{E}\,\|\mathcal{X}\|_H^r\right)^{1/r} < \infty$, where $\mathbb{E}\,\|\mathcal{X}\|_H^r$ is the expectation of $\|\mathcal{X}\|_H^r$. Let V_1, V_2 be two Hilbert spaces and let $\mathcal{HS}(U_1, U_2)$ be the space of all Hilbert-Schmidt operators $\Psi : V_1 \to V_2$ endowed with the norm $\|\Psi\|^2_{\mathcal{HS}(U_1,U_2)} := \sum_{j=1}^{\infty} \|\Psi e_j\|^2_{V_2} < \infty$, where $\{e_j\}_{j\in\mathbb{N}^*}$ is an orthonormal basis of the space V_1. Denote by $L_0^2(V_1, V_2)$ the space of all linear bounded operators $f : V_1 \to V_2$ with the corresponding norm $\|f\|_{L_0^2(V_1,V_2)} = \left\|fQ^{1/2}\right\|_{\mathcal{HS}(V_1,V_2)}$. In the case of $V_1 \equiv V_2 \equiv V$, we set $L_0^2(V) := L_0^2(V,V)$ for the sake of convenience. For a Banach space $\mathfrak{B}$ and a given number $p \geq 1$, recall that

$$L^p(0,T;\mathfrak{B}) := \left\{ \chi : (0,T) \to \mathfrak{B} \quad \text{s.t.} \quad \|\chi\|_{L^p(0,T;\mathfrak{B})} := \left(\int_0^T \|\chi(t)\|^p_{\mathfrak{B}}\, dt \right)^{1/p} < \infty \right\}.$$

Next, we aim to find a representation for the solution in the form $u(x,t) = \sum_{j=1}^{\infty} u_j(t)\varphi_j(x)$, where $u_j(t) := \langle u(\cdot,t), \varphi_j\rangle$. Setting $g_j(t) := \langle g(\cdot,t), \varphi_j\rangle$ and $\xi_j := \langle \xi, \varphi_j\rangle$. It follows from Problem (1) that

$$\begin{cases} \dfrac{du_j}{dt}(t) + 2\lambda_j \nu(t) u_j(t) = g_j(t) + \beta_j f(t) \dfrac{d\gamma_j}{dt}(t), & t \in (0,T), \\ \langle u_j(T), \varphi_j\rangle = \xi_j. \end{cases}$$

By solving this, we obtain that the mild solution of Problem (1), which is an adapted H-valued stochastic process $u(x,t)$, is given by

$$u(x,t) = \mathcal{A}(t,T)\xi(x) - \int_t^T \mathcal{A}(t,s) g(x,s) ds - \int_t^T \mathcal{A}(t,s) f(s) d\mathbb{W}(x,s), \tag{3}$$

where $\mathcal{A}(t_1,t_2)h(x) := \sum_{j=1}^{\infty} \exp\left(\lambda_j \int_{t_1}^{t_2} \nu(\tau) d\tau\right) \langle h, \varphi_j\rangle \varphi_j(x)$ for $t_1, t_2 \in [0,T]$. Note that the last integral with respect to the Brownian motion is a so-called Itô's integral.

3. The ill-posedness of stochastic problem

This section is aimed to illustrate the instability of the solution to Problem (1), which leads to its ill-posedness. Supposing that $\xi \equiv 0$, $g \equiv 0$ and $f \equiv 0$, it is obvious that the solution is $u \equiv 0$. Next, we will give concrete sequences $\{\xi^k\}_{k\geq 1}, \{g^k\}_{k\geq 1}$ and $\{f^k\}_{k\geq 1}$ such that

$$\lim_{k\to\infty} \left\|\xi^k - \xi\right\|_{L^2(\Omega,H)} = \lim_{k\to\infty} \left\|g^k(\cdot,t) - g(\cdot,t)\right\|_{L^2(\Omega,H)} = \lim_{k\to\infty} \left\|f^k(t) - f(t)\right\|_{L^2_0(H)} = 0, \tag{4}$$

and $\|u^k(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega,H)}$ does not tend to zero as $k \to \infty$, where u^k is defined by

$$\begin{aligned} u^k(x,t) &= \mathcal{A}(t,T)\xi^k(x) - \int_t^T \mathcal{A}(t,s)g^k(x,s)ds - \int_t^T \mathcal{A}(t,s)f^k(s)d\mathbb{W}(x,s) \\ &=: u_1^k(x,t) - u_2^k(x,t) - u_3^k(x,t). \end{aligned} \tag{5}$$

This means that the errors in the solution (output data) are very large whereas the errors in the input data are not significant. For clarity, let us present the following illustration:

Lemma 3.1. *Let the input data $\xi^k(x)$, $g^k(x,t)$ and $f^k(t)$ for $k \in \mathbb{N}^*$, be given by*

$$\xi^k(x) = e^{-\nu^+\lambda_k T}\varsigma\varphi_k(x), \quad g^k(x,t) = e^{-\nu^+\lambda_k T}\sigma(t)\varphi_k(x), \quad f^k(t)h(x) = \frac{t}{\beta_k\sqrt[4]{\lambda_k}}\langle h, \varphi_k\rangle\varphi_k(x), \tag{6}$$

where ς is a random variable drawn from the standard normal distribution and $\sigma(t)$ is a Brownian motion. Then,

a) $\left\|\xi^k\right\|_{L^2(\Omega,H)} = e^{-\nu^+\lambda_k T}$, $\left\|g^k(\cdot,t)\right\|_{L^2(\Omega,H)} = \sqrt{t}e^{-\nu^+\lambda_k T}$ *and* $\left\|f^k(t)\right\|_{L^2_0(H)} = t\lambda_k^{-1/4}$;

b) $\left\|u^k(\cdot,t)\right\|_{L^2(\Omega,H)} \geq \left(\frac{3T^4-4T^3t+t^4}{6}\nu^-\right)^{1/2}\lambda_k^{1/4} - \frac{2}{3}\left(T^{3/2} - t^{3/2}\right) - 1.$

Remark 3.1. *Lemma 3.1 gives us a simple example showing that (4) holds, but $\left\|u^k(\cdot,t) - u(\cdot,t)\right\|_{L^2(\Omega,H)}$ tends to infinity as $k \to \infty$, implying that the solution is not stable.*

Proof. a) By using the properties of the random variable ς and the Brownian motion $\sigma(t)$, one obtains $\left\|\xi^k\right\|^2_{L^2(\Omega,H)} = e^{-2\nu^+\lambda_k T}\mathbb{E}\varsigma^2\left\|\varphi_k\right\|^2_{L^2(\Omega,H)} = e^{-2\nu^+\lambda_k T}$ and $\left\|g^k(\cdot,t)\right\|^2_{L^2(\Omega,H)} = te^{-2\nu^+\lambda_k T}$. For the third term, using that

$$f^k(t)\varphi_j(x) = \frac{t}{\beta_k\sqrt[4]{\lambda_k}}\langle\varphi_j, \varphi_k\rangle\varphi_k(x) = \begin{cases} \frac{t}{\beta_k\sqrt[4]{\lambda_k}}\varphi_k(x), & j = k, \\ 0, & j \neq k, \end{cases} \tag{7}$$

and $Q^{1/2}\varphi_k(x) = \beta_k\varphi_k(x)$, one obtains

$$\left\|f^k(t)\right\|^2_{L^2_0(H)} = \sum_{j=1}^\infty \left\|Q^{1/2}f^k(t)\varphi_j\right\|^2_H = \frac{t^2}{\beta_k^2\sqrt{\lambda_k}}\left\|Q^{1/2}\varphi_k\right\|^2_H = \frac{t^2}{\sqrt{\lambda_k}}.$$

b) The following is obtained by using the Itô's isometry

$$\left\|u_3^k(\cdot,t)\right\|^2_{L^2(\Omega,H)} = \left\|\int_t^T \mathcal{A}(t,s)f^k(s)d\mathbb{W}(\cdot,s)\right\|^2_{L^2(\Omega,H)} = \int_t^T \sum_{j=1}^\infty \left\|Q^{1/2}\mathcal{A}(t,s)f^k(s)\varphi_j\right\|^2_H ds. \tag{8}$$

On the other hand, from (7) and $Q^{1/2}\mathcal{A}(t,s)\varphi_k(x) = \exp\left(\lambda_k\int_t^s \nu(\tau)d\tau\right)\beta_k\varphi_k(x)$, we get

$$\left\|Q^{1/2}\mathcal{A}(t,s)f^k(s)\varphi_j\right\|^2_H = \begin{cases} \frac{s^2}{\sqrt{\lambda_k}}\exp\left(2\lambda_k\int_t^s \nu(\tau)d\tau\right), & j = k, \\ 0, & j \neq k. \end{cases} \tag{9}$$

Combining (8) and (9), we deduce that

$$\begin{aligned} \left\|u_3^k(\cdot,t)\right\|^2_{L^2(\Omega,H)} &= \lambda_k^{-1/2}\int_t^T s^2\exp\left(2\lambda_k\int_t^s \nu(\tau)d\tau\right)ds \geq 2\lambda_k^{1/2}\int_t^T s^2\left(\int_t^s \nu(\tau)d\tau\right)ds \\ &\geq 2\nu^-\lambda_k^{1/2}\int_t^T s^2(s-t)ds = \frac{3T^4 - 4T^3t + t^4}{6}\nu^-\lambda_k^{1/2}. \end{aligned} \tag{10}$$

The term $\left\|u_1^k(\cdot,t)\right\|_{L^2(\Omega,H)}$ can be estimated as follows:

$$\left\|u_1^k(\cdot,t)\right\|^2_{L^2(\Omega,H)} = \sum_{j=1}^\infty \exp\left(2\lambda_j\int_t^T \nu(\tau)d\tau\right)\mathbb{E}\langle\xi^k, \varphi_j\rangle^2 = \exp\left(2\lambda_k\int_t^T \nu(\tau)d\tau\right)e^{-2\nu^+\lambda_k T} \leq 1, \tag{11}$$

where we have used that $\mathbb{E}\langle \xi^k, \varphi_j\rangle^2 = 0$ if $j \neq k$ and $\mathbb{E}\langle \xi^k, \varphi_j\rangle^2 = e^{-2\nu^+\lambda_k T}$ if $j = k$. Similarly, one can check that

$$\left\|u_2^k(\cdot,t)\right\|_{L^2(\Omega,H)} \leq \int_t^T \left\|\mathcal{A}(t,s)g^k(\cdot,s)\right\|_{L^2(\Omega,H)} ds \leq \int_t^T \sqrt{s}\; ds = \frac{2}{3}\left(T^{3/2} - t^{3/2}\right). \tag{12}$$

Combining (5), (10)–(12) and $\left\|u^k(\cdot,t)\right\|_{L^2(\Omega,H)} \geq \left\|u_3^k(\cdot,t)\right\|_{L^2(\Omega,H)} - \left\|u_1^k(\cdot,t)\right\|_{L^2(\Omega,H)} - \left\|u_2^k(\cdot,t)\right\|_{L^2(\Omega,H)}$, we complete the proof of b). □

4. Regularization and error analysis

In reality, the exact data is not available and we only have measured data containing errors (denoted by ξ^δ, g^δ and f^δ). For this reason, henceforth, we suppose that

$$\left\|\xi^\delta - \xi\right\|_{L^2(\Omega,H)} + \left\|g^\delta - g\right\|_{L^1(0,T;L^2(\Omega,H))} + \left\|f^\delta - f\right\|_{L^2(0,T;L_0^2(H))} \leq \delta, \tag{13}$$

where $\delta \geq 0$ represents the noise level.

In this section, we propose a regularized solution and then give the convergence analysis. We will use a regularization method called the "filter method". The idea can be explained as follows. Since the exponential value $\exp\left(\lambda_j \int_t^s \nu(\tau)d\tau\right)$ tends to infinity rapidly as $j \to \infty$, one can see that the operator $\mathcal{A}(t,s)$ appearing in (3) is unbounded in $L^2(\mathcal{D})$ if $t < s$, and this makes the solution unstable. Therefore, our strategy is to use a new operator $\mathcal{B}_\alpha^\delta \mathcal{A}(t,s)$ to approximate $\mathcal{A}(t,s)$, where $\alpha = \alpha(\delta) > 0$ is a regularization parameter. Here, the operator $\mathcal{B}_\alpha^\delta$ is defined as follows:

$$\mathcal{B}_\alpha^\delta h(x) := \sum_{j=1}^{\infty} \mathbb{B}_{\alpha;j}^\delta \langle h, \varphi_j\rangle \varphi_j(x), \tag{14}$$

where the function $\mathbb{B}_{\alpha;j}^\delta$ is called a filter kernel (and will be defined later) satisfying

- for the given parameters δ and α, the operator $\mathcal{B}_\alpha^\delta$ is bounded in $L^2(\mathcal{D})$;
- if δ tends to zero then the kernel $\mathbb{B}_{\alpha;j}^\delta$ tends to 1 for all $j \in \mathbb{N}^*$.

According to the above explanations, we propose the regularized solution $w_\alpha^\delta(x,t) = \mathcal{B}_\alpha^\delta u^\delta(x,t)$, where

$$u^\delta(x,t) := \mathcal{A}(t,T)\xi^\delta(x) - \int_t^T \mathcal{A}(t,s)g^\delta(x,s)ds - \int_t^T \mathcal{A}(t,s)f^\delta(s)d\mathbb{W}(x,s). \tag{15}$$

Next, we state the main results of our paper in Subsection 4.1 and then prove it in Subsection 4.2.

4.1. The main results.

The following property of $\mathcal{B}_\alpha^\delta$ is needed to evaluate the error estimate.

Lemma 4.1. *For $\alpha > 0$ and $j \in \mathbb{N}^*$, assume that there exists a positive constant $\mathcal{L}_\alpha$ such that $\left|\mathbb{B}_{\alpha;j}^\delta\right| \exp\left(\lambda_j \int_t^s \nu(\tau)d\tau\right) \leq \mathcal{L}_\alpha$ for $0 \leq t < s \leq T$. Then, for $h \in L^2(\Omega,H)$, it holds that*

$$\left\|\mathcal{B}_\alpha^\delta \mathcal{A}(t,s)h\right\|_{L^2(\Omega,H)} \leq \mathcal{L}_\alpha \left\|h\right\|_{L^2(\Omega,H)}, \quad 0 \leq t < s \leq T.$$

Theorem 4.2 (Convergence analysis)**.** *Assume that $u(\cdot,t) \in L^2(\Omega,\dot{H}^\theta)$ for $t \in [0,T]$, with some $\theta > 0$. For $\alpha > 0$, assume that there exists a positive constant $\mathcal{L}_\alpha \geq \left|\mathbb{B}_{\alpha;j}^\delta\right| \exp\left(\lambda_j \int_t^s \nu(\tau)d\tau\right)$ for $0 \leq t < s \leq T$. Assume also that there exists a positive constant $\mathcal{L}_\alpha^*$ such that $\left|\mathbb{B}_{\alpha;j}^\delta - 1\right| \leq \mathcal{L}_\alpha^* \lambda_j^\theta$ for all $j \in \mathbb{N}^*$. Then, the following error estimate holds:*

$$\left\|w_\alpha^\delta(\cdot,t) - u(\cdot,t)\right\|_{L^2(\Omega,H)} \leq \mathcal{L}_\alpha \delta + \mathcal{L}_\alpha^* \left\|u(\cdot,t)\right\|_{L^2(\Omega,\dot{H}^\theta)}, \quad \forall t \in [0,T]. \tag{16}$$

If the constants $\mathcal{L}_\alpha$ and $\mathcal{L}_\alpha^*$ satisfy $\lim_{\delta\to 0^+} \mathcal{L}_\alpha \delta = \lim_{\delta\to 0^+} \mathcal{L}_\alpha^* = 0$, then from (16) it follows that the regularized solution $w_\alpha^\delta(\cdot,t)$ converges to the true solution $u(\cdot,t)$ in $L^2(\Omega,H)$ for all $t \in [0,T]$.

Next, we present two examples of $\mathcal{B}_\alpha^\delta$, which plays the role of regularization filter in our method.

Corollary 4.3. *Assume that $u(\cdot,t) \in L^2(\Omega,\dot{H}^\theta)$ for $t \in [0,T]$, with some $\theta > 0$. Choosing the truncation filter operator as*

$$\mathcal{B}_\alpha^\delta h(x) = \sum_{j=1}^{\infty} \mathbb{B}_{\alpha;j}^\delta \langle h, \varphi_j\rangle \varphi_j(x), \quad \textit{where } \mathbb{B}_{\alpha;j}^\delta = \begin{cases} 1, & \textit{if } \lambda_j \leq \alpha, \\ 0, & \textit{otherwise}, \end{cases} \tag{17}$$

then, the following inequalities hold for all $j \in \mathbb{N}^*$:

i) $\left|\mathbb{B}_{\alpha;j}^\delta\right| \exp\left(\lambda_j \int_t^s \nu(\tau)d\tau\right) \leq \exp\left(\nu^+ \alpha T\right), \quad 0 \leq t < s \leq T,$
ii) $\left|\mathbb{B}_{\alpha;j}^\delta - 1\right| \leq \alpha^{-\theta} \lambda_j^\theta.$

Consequently,

$$\left\|w_\alpha^\delta(\cdot,t)-u(\cdot,t)\right\|_{L^2(\Omega,H)} \le \exp\left(\nu^+\alpha T\right)\delta + \alpha^{-\theta}\left\|u(\cdot,t)\right\|_{L^2(\Omega,\dot{H}^\theta)}, \quad \forall t\in[0,T]. \tag{18}$$

If we choose the regularization parameter $\alpha=\alpha(\delta)$ so as to satisfy that $\lim\limits_{\delta\to 0^+}\alpha=\infty$ and $\lim\limits_{\delta\to 0^+}\exp\left(\nu^+\alpha T\right)\delta=0$, then from (18) it follows that the regularized solution $w_\alpha^\delta(\cdot,t)$ converges to the true solution $u(\cdot,t)$ in $L^2(\Omega,H)$ for all $t\in[0,T]$.

Remark 4.1. *If we choose $\alpha=\lfloor\frac{\log(\delta^{-\eta})}{\nu^+T}\rfloor$, with some $\eta\in(0,1)$, where $\lfloor r\rfloor$ is the integer part of the real number r, then $\left\|w_\alpha^\delta(\cdot,t)-u(\cdot,t)\right\|_{L^2(\Omega,H)}$ is of order $\log^{-\theta}(\delta^{-\eta})$, which implies that this error tends to zero as δ tends to zero.*

Corollary 4.4. *Assume that $u(\cdot,t)\in L^2(\Omega,\dot{H}^1)$ for $t\in[0,T]$. Choosing the following filter operator:*

$$\mathcal{B}_\alpha^\delta h(x)=\sum_{j\ge 1}\mathbb{B}_{\alpha;j}^\delta\langle h,\varphi_j\rangle\varphi_j(x), \quad \textit{where } \mathbb{B}_{\alpha;j}^\delta=\left[1+\alpha\lambda_j\exp(\nu^+\lambda_j T)\right]^{-1}, \tag{19}$$

then the following inequalities hold for all $j\in\mathbb{N}^*$:

i) $\left|\mathbb{B}_{\alpha;j}^\delta\right|\exp\left(\lambda_j\int_t^s\nu(\tau)d\tau\right)\le\frac{\nu^+T/\alpha}{\log(\nu^+T/\alpha)}, \quad 0\le t<s\le T,$

ii) $\left|\mathbb{B}_{\alpha;j}^\delta-1\right|\le\frac{\nu^+T}{\log(\nu^+T/\alpha)}\lambda_j.$

Consequently, the following error estimate holds:

$$\left\|w_\alpha^\delta(\cdot,t)-u(\cdot,t)\right\|_{L^2(\Omega,H)}\le\frac{\nu^+T/\alpha}{\log\left(\nu^+T/\alpha\right)}\delta+\frac{\nu^+T}{\log\left(\nu^+T/\alpha\right)}\left\|u(\cdot,t)\right\|_{L^2(\Omega,\dot{H}^1)}. \tag{20}$$

If we choose the regularization parameter $\alpha=\alpha(\delta)$ so as to satisfy that $0<\alpha<\nu^+T$ and $\lim_{\delta\to 0^+}\alpha=\lim_{\delta\to 0^+}\frac{\delta}{\alpha}=0$, then from (20) it follows that the regularized solution $w_\alpha^\delta(\cdot,t)$ converges to the true solution $u(\cdot,t)$ in $L^2(\Omega,H)$ for all $t\in[0,T]$.

Remark 4.2. *If we choose $\alpha=\delta^\eta$, with some $\eta\in(0,1)$, then $\left\|w_\alpha^\delta(\cdot,t)-u(\cdot,t)\right\|_{L^2(\Omega,H)}$ is of order $\log^{-1}\left(\nu^+T\delta^{-\eta}\right)$, which implies that this error tends to zero as δ tends to zero.*

Remark 4.3. *For the deterministic problem obtained by omitting the stochastic term $f(t)\dot{\mathbb{W}}$, namely,*

$$\begin{cases} u_t(x,t)-\nu(t)\Delta u(x,t)=g(x,t), & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,T)=\xi(x), & x\in\Omega, \end{cases} \tag{21}$$

a similar regularized solution for Problem (21) can be constructed as

$$\widetilde{w}^\delta(x,t):=\mathcal{B}_\alpha^\delta\mathcal{A}(t,T)\xi^\delta(x)-\int_t^T\mathcal{B}_\alpha^\delta\mathcal{A}(t,s)g^\delta(x,s)ds. \tag{22}$$

Then, due to the fact that

$$\left\|w^\delta(\cdot,t)-\widetilde{w}^\delta(\cdot,t)\right\|^2_{L^2(\Omega,H)}=\int_t^T\left\|\mathcal{B}_\alpha^\delta\mathcal{A}(t,s)f^\delta(s)\right\|^2_{L_0^2(H)}ds\le\mathcal{L}_\alpha^2\int_t^T\left\|f^\delta(s)\right\|^2_{L_0^2(H)}ds,$$

it can be seen that the regularized solution $w^\delta(\cdot,t)$ of (1) tends the regularized solution $\widetilde{w}^\delta(\cdot,t)$ of (21) when the observation $f^\delta(t)\to 0$, i.e. $\|f^\delta\|_{L^2(0,T;L_0^2(H))}\to 0$.

4.2. **Proof of the main results.** Now, we begin to prove the results stated in the previous subsection.

Proof of Lemma 4.1. For $h\in L^2(\Omega,H)$, it follows from the definition of $\mathcal{A}$, equation (14) and the hypothesis on that $\mathcal{L}_\alpha$ that

$$\left\|\mathcal{B}_\alpha^\delta\mathcal{A}(t,s)h\right\|^2_{L^2(\Omega,H)}=\sum_{j=1}^\infty\left|\mathbb{B}_{\alpha;j}^\delta\exp\left(\lambda_j\int_t^s\nu(\tau)d\tau\right)\right|^2\mathbb{E}\langle h,\varphi_j\rangle^2\le\mathcal{L}_\alpha^2\left\|h\right\|^2_{L^2(\Omega,H)}.$$

Lemma 4.1 has been proved. ∎

Proof of Theorem 4.2. Denote $\mathcal{E}_1(t) := \left\|\mathcal{B}_\alpha^\delta u^\delta(\cdot,t) - \mathcal{B}_\alpha^\delta u(\cdot,t)\right\|_{L^2(\Omega,H)}$ and $\mathcal{E}_2(t) := \left\|\mathcal{B}_\alpha^\delta u(\cdot,t) - u(\cdot,t)\right\|_{L^2(\Omega,H)}$. From (3) and (15), we have

$$\mathcal{E}_1(t) \le \left\|\mathcal{B}_\alpha^\delta \mathcal{A}(t,T)\big[\xi^\delta - \xi\big]\right\|_{L^2(\Omega,H)} + \left\|\int_t^T \mathcal{B}_\alpha^\delta \mathcal{A}(t,s)\Big[g^\delta(\cdot,s) - g(\cdot,s)\Big]ds\right\|_{L^2(\Omega,H)}$$
$$+ \left\|\int_t^T \mathcal{B}_\alpha^\delta \mathcal{A}(t,s)\left[f^\delta(s) - f(s)\right]d\mathbb{W}(\cdot,s)\right\|_{L^2(\Omega,H)} =: \mathcal{E}_{1;1}(t) + \mathcal{E}_{1;2}(t) + \mathcal{E}_{1;3}(t).$$

By using Lemma 4.1, one obtains $\mathcal{E}_{1;1}(t) \le \mathcal{L}_\alpha \left\|\xi^\delta - \xi\right\|_{L^2(\Omega,H)}$. For the next term, it is clear that $\mathcal{E}_{1;2}(t) \le \int_t^T \left\|\mathcal{B}_\alpha^\delta \mathcal{A}(t,s)\Big[g^\delta(\cdot,s) - g(\cdot,s)\Big]\right\|_{L^2(\Omega,H)} ds$. Using Lemma 4.1 one more time, one arrives at

$$\mathcal{E}_{1;2}(t) \le \mathcal{L}_\alpha \int_t^T \left\|g^\delta(\cdot,s) - g(\cdot,s)\right\|_{L^2(\Omega,H)} ds \le \mathcal{L}_\alpha \left\|g^\delta - g\right\|_{L^1(0,T;L^2(\Omega,H))}.$$

Next, by the Itô's isometry and the fact that $\left\|\mathcal{B}_\alpha^\delta \mathcal{A}(t,s)h\right\|_H \le \mathcal{L}_\alpha \|h\|_H$, we deduce that

$$\mathcal{E}_{1;3}^2(t) = \int_t^T \left\|\mathcal{B}_\alpha^\delta \mathcal{A}(t,s)\left[f^\delta(s) - f(s)\right]\right\|_{L_0^2(H)}^2 ds \le \mathcal{L}_\alpha^2 \int_t^T \left\|f^\delta(s) - f(s)\right\|_{L_0^2(H)}^2 ds,$$

which implies that $\mathcal{E}_{1;3}(t) \le \mathcal{L}_\alpha \left\|f^\delta - f\right\|_{L^2(0,T;L_0^2(H))}$. By the above arguments and using (13), we obtain that $\mathcal{E}_1(t) \le \mathcal{L}_\alpha \delta$. It follows from the condition $\left|\mathbb{B}_{\alpha,j}^\delta - 1\right| \le \mathcal{L}_\alpha^* \lambda_j^\theta$ that

$$\mathcal{E}_2^2(t) = \sum_{j=1}^\infty \left|\mathbb{B}_{\alpha,j}^\delta - 1\right|^2 \mathbb{E}\left|u_j(t)\right|^2 \le |\mathcal{L}_\alpha^*|^2 \sum_{j=1}^\infty \lambda_j^{2\theta} \mathbb{E}\left|u_j(t)\right|^2 = |\mathcal{L}_\alpha^*|^2 \left\|u(\cdot,t)\right\|_{L^2(\Omega,\dot{H}^\theta)}^2. \tag{23}$$

The latter estimate together with $\mathcal{E}_1(t) \le \mathcal{L}_\alpha \delta$ imply the desired result (16). ☐

Proof of Corollary 4.3. By choosing the operator $\mathcal{B}_\alpha^\delta$ as in (17), one can see that

$$\left|\mathbb{B}_{\alpha;j}^\delta\right| \exp\left(\lambda_j \int_t^s \nu(\tau)d\tau\right) \le \exp\left(\alpha \int_t^s \nu(\tau)d\tau\right) \le \exp\left(\nu^+ \alpha T\right).$$

For $\lambda_j \le \alpha = \alpha(\delta)$, it is easy to see that $\left|\mathbb{B}_{\alpha;j}^\delta - 1\right| = 0 \le \alpha^{-\theta}\lambda_j^\theta$. For $\lambda_j > \alpha = \alpha(\delta)$, we also have $\left|\mathbb{B}_{\alpha;j}^\delta - 1\right| = 1 = \lambda_j^{-\theta}\lambda_j^\theta \le \alpha^{-\theta}\lambda_j^\theta$. Now, applying Theorem 4.2 for $\mathcal{L}_\alpha = \exp\left(\nu^+\alpha T\right)$ and $\mathcal{L}_\alpha^* = \alpha^{-\theta}$, we obtain the estimate (18). ☐

Proof of Corollary 4.4. Using the inequality $\left[\exp(-a\lambda) + b\lambda\right]^{-1} \le \frac{a/b}{\log(a/b)}$ for $0 < b < a$ and $\lambda > 0$, one gets

$$\left|\mathbb{B}_{\alpha;j}^\delta\right| \exp\left(\lambda_j \int_t^s \nu(\tau)d\tau\right) \le \frac{\exp\left(\nu^+\lambda_j T\right)}{1 + \alpha\lambda_j \exp(\nu^+\lambda_j T)} = \left[\exp(-\lambda_j \nu^+ T) + \alpha\lambda_j\right]^{-1} \le \frac{\nu^+ T/\alpha}{\log\left(\nu^+ T/\alpha\right)}.$$

Furthermore, one has $\left|\mathbb{B}_{\alpha;j}^\delta - 1\right| = \alpha\lambda_j \left[\exp(-\lambda_j\nu^+ T) + \alpha\lambda_j\right]^{-1} \le \lambda_j \frac{\nu^+ T}{\log(\nu^+ T/\alpha)}$. Applying Theorem 4.2 for $\mathcal{L}_\alpha = \frac{\nu^+ T/\alpha}{\log(\nu^+ T/\alpha)}$ and $\mathcal{L}_\alpha^* = \frac{\nu^+ T}{\log(\nu^+ T/\alpha)}$, we obtain the estimate (20). ☐

5. Numerical results and discussion

In this section, we present a numerical example to illustrate the stability of the regularization fiter (19) (or (17)) in retriving the solution $u(x,t)$ backwards in time for various levels of noise δ. Let $\Omega = (0,1)$ and focus on the backward problem for the stochastic parabolic heat equation given by

$$\begin{cases} u_t(x,t) - \nu(t)u_{xx}(x,t) = g(x,t) + f(t)\dot{\mathbb{W}}, & (x,t) \in (0,1)\times(0,T), \\ u(0,t) = u(1,t) = 0, & t \in (0,T), \\ u(x,T) = \xi(x), & x \in (0,1). \end{cases} \tag{24}$$

We denote by λ_j be the eigenvalues of the negative Laplace operator $-\Delta$ and φ_j is the complete orthonormal system of eigenfunctions forming an orthogonal basis of $L^2(\Omega)$ such that

$$-\Delta\varphi_j = \lambda_j\varphi_j \text{ and } \varphi_j|_{\partial\Omega} = 0 \text{ for } j \in \mathbb{N}^*,$$

where $\lambda_j = j^2\pi^2,\ \varphi_j(x) = \sqrt{2}\sin(j\pi x)$.

Let $\mathbb{W}(t)$ be an $L^2(0,1)$-valued Q-Wiener process, where $Q = -\Delta^{-1}$. Then, $Q\varphi_j = Q\lambda_j^{-1}(-\Delta)\varphi_j = \lambda_j^{-1}\varphi_j$. It follows that $\{\lambda_j^{-1}\}_{n=1}^{\infty}$ and $\{\varphi_j\}_{j=1}^{\infty}$ are the eigenvalues and eigenfunctions of Q, respectively. Moreover, we have

$$\mathrm{Tr}(Q) = \sum_{j=1}^{\infty} \lambda_j^{-1} = \sum_{j=1}^{\infty} \frac{1}{j^2\pi^2} = \frac{1}{6} \text{ and } \beta_j = \frac{1}{j^2\pi^2}.$$

The solution of Problem (24) is given by expression (3). Let us take $V = L^2(0,1)$ and the operator $f\colon L^2(0,1) \longrightarrow L^2(0,1)$ be defined by

$$t \longmapsto f(t)(\cdot) := \sum_{k=1}^{\infty} \frac{t}{\lambda_k} \langle \cdot, \varphi_k \rangle \varphi_k, \tag{25}$$

where the scalar product is in $L^2(0,1)$, and its noisy correspondent $f^\delta\colon L^2(0,1) \longrightarrow L^2(0,1)$ given by

$$t \longmapsto f^\delta(t)(\cdot) := \sum_{k=1}^{\infty} \frac{t}{\lambda_k} \langle \cdot + \delta, \varphi_k \rangle \varphi_k. \tag{26}$$

Then,

$$\begin{aligned}
&\int_t^T \mathcal{A}(t,s) f(s) d\mathbb{W}(x,s) \\
&= \sum_{j=1}^{\infty} \varphi_j(x) \left[\int_t^T \exp\left(\lambda_j \int_t^s \nu(\tau)d\tau\right) \sum_{k=1}^{\infty} \frac{s}{\lambda_k} \langle d\mathbb{W}(\cdot,s), \varphi_k \rangle \varphi_k \right] \\
&= \sum_{j=1}^{\infty} \varphi_j(x) \exp\left(\lambda_j \int_t^s \nu(\tau)d\tau\right) \frac{s\beta_j}{\lambda_j} d\gamma_j(s),
\end{aligned} \tag{27}$$

and the solution (3) can be rewritten by

$$\begin{aligned}
u(x,t) = \sum_{j=1}^{\infty} \varphi_j(x) \Bigg[&\exp\left(\lambda_j \int_t^T \nu(\tau)d\tau\right) \int_0^1 \xi(z)\varphi_j(z)dz \\
&- \int_t^T \exp\left(\lambda_j \int_t^s \nu(\tau)d\tau\right) \left(\int_0^1 g(z,s)\varphi_j(z)dz\right) ds \\
&- \int_t^T \exp\left(\lambda_j \int_t^s \nu(\tau)d\tau\right) \frac{s\beta_j}{\lambda_j} d\gamma_j(s) \Bigg].
\end{aligned} \tag{28}$$

Next, let $t = \tau_0 < \tau_1 < ... < \tau_n = T$ be a partition of the interval $[t,T]$ and by using a stochastic explicit single step method, we consider the increments $\Delta\gamma_j(\tau_i) = \gamma_j(\tau_{i+1}) - \gamma_j(\tau_i)$ and $\Delta\tau_i = \tau_{i+1} - \tau_i$ for $i = 0, ..., n-1$, where $\Delta\gamma_j(\tau_i)$ are independent $\mathcal{N}(0, \Delta\tau_i)$ normally distributed random variables. A typical Brownian motion γ_j for a fixed j, for $n = 1000$ is shown in Figure 1.

We can formally calculate, by the definition of Stratonovich integral,

$$\int_t^T \Phi(t,s) \circ d\mathbb{W}(s) = \lim_{\Delta\tau_i \to 0} \sum_{i=0}^{n-1} \Phi\left(t, \frac{\tau_i + \tau_{i+1}}{2}\right) \Delta\gamma_j(\tau_i). \tag{29}$$

Finally, we use the finite difference method (FDM) with the following partitions of temporal and spatial variables. For $x \in [0,1]$ and $t \in [0,T]$, let us consider the partition $\Omega_X \times \Omega_T$, where

$$\begin{aligned}
\Omega_X &:= \left\{ x_p = (p-1)\varepsilon_x, \text{ for } p = 1,2,...,\mathbf{N}_x, \mathbf{N}_x + 1 \text{ and } \varepsilon_x = \frac{1}{\mathbf{N}_x} \right\}, \\
\Omega_T &:= \left\{ t_q = (q-1)\varepsilon_t, \text{ for } q = \mathbf{N}_t + 1, \mathbf{N}_t, ..., 1 \text{ and } \varepsilon_t = \frac{T}{\mathbf{N}_t} \right\}.
\end{aligned}$$

Let us take $T = 1$ and $\nu = 1$. Then, equation (28) simplifies as

$$\begin{aligned}
u(x_p,t_q) \approx \sum_{j=1}^{\infty} \Bigg[&\exp\left(j^2\pi^2(1-t_q)\right) \int_0^1 \xi(z)\varphi_j(z)dz - \int_{t_q}^1 \exp\left(j^2\pi^2(s-t_q)\right) \left(\int_0^1 g(z,s)\varphi_j(z)dz\right) ds \\
&- \sum_{i=0}^{n-1} \exp\left(j^2\pi^2(\tau_i - t_q)\right) \frac{\tau_i}{j^4\pi^4} \Delta\gamma_j(\tau_i) \Bigg] \varphi_j(x_p).
\end{aligned} \tag{30}$$

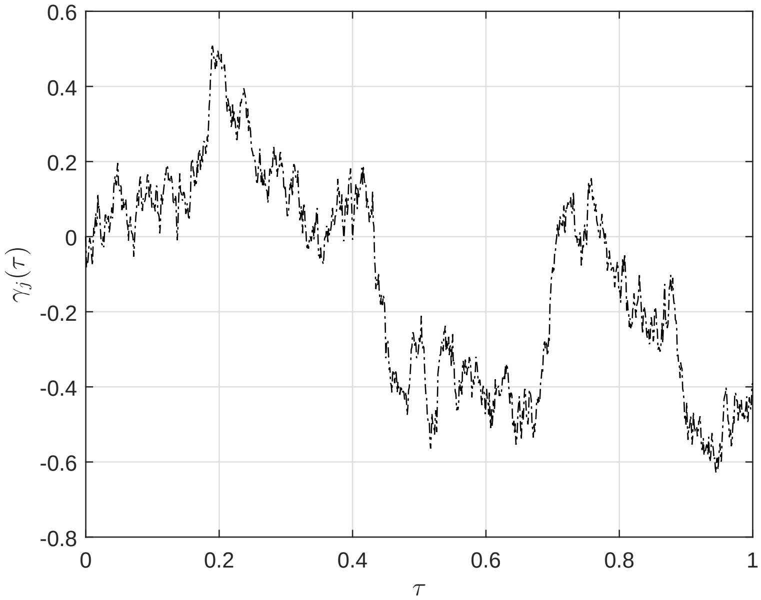


FIGURE 1. Example of a typical Brownian motion $\gamma_j(\tau)$, as a function of τ, for a fixed j, for $n = 1000$.

Next step, based on (13) and the filter kernel $\mathbb{B}^{\delta}_{\alpha,j} = \mathbb{B}^{\delta}_{\alpha;j} = [1 + \alpha\lambda_j \exp(\lambda_j)]^{-1}$ in (19), we have the following regularized solution:

$$w^{\delta}_{\alpha}(x_p, t_q) = \mathcal{B}^{\delta}_{\alpha} u^{\delta}(x_p, t_q) = \sum_{j=1}^{\infty} \Bigg[\frac{\exp\left(j^2\pi^2(1-t_q)\right)}{1+\alpha j^2\pi^2 \exp(j^2\pi^2)} \int_0^1 \xi^{\delta}(z)\varphi_j(z)dz - \int_{t_q}^1 \frac{\exp\left(j^2\pi^2(s-t_q)\right)}{1+\alpha j^2\pi^2\exp(j^2\pi^2)} \left(\int_0^1 g^{\delta}(z,s)\varphi_j(z)dz \right) ds - \sum_{i=0}^{n-1} \frac{\exp\left(j^2\pi^2(\tau_i - t_q)\right)}{1+\alpha j^2\pi^2\exp(j^2\pi^2)} \frac{\tau_i}{j^4\pi^4} \Delta\gamma^{\delta}_j(\tau_i) \Bigg] \varphi_j(x_p), \tag{31}$$

where $\Delta\gamma^{\delta}_j(\tau_i) = \gamma^{\delta}_j(\tau_{i+1}) - \gamma^{\delta}_j(\tau_i)$ and $\gamma^{\delta}_j(\tau_i) = \gamma_j(\delta + \tau_i) - \gamma_j(\tau_i)$. Note that (31) becomes the same as (30) for exact data, i.e. $\delta = 0$, in which case $\alpha = 0$. Similarly, for the filter (17), we have the following regularized solution:

$$w^{\delta}_{\alpha}(x_p, t_q) = \mathcal{B}^{\delta}_{\alpha} u^{\delta}(x_p, t_q) = \sum_{j=1}^{J} \Bigg[\exp\left(j^2\pi^2(1-t_q)\right) \int_0^1 \xi^{\delta}(z)\varphi_j(z)dz - \int_{t_q}^1 \exp\left(j^2\pi^2(s-t_q)\right) \left(\int_0^1 g^{\delta}(z,s)\varphi_j(z)dz \right) ds - \sum_{i=0}^{n-1} \exp\left(j^2\pi^2(\tau_i - t_q)\right) \frac{\tau_i}{j^4\pi^4} \Delta\gamma^{\delta}_j(\tau_i) \Bigg] \varphi_j(x_p), \tag{32}$$

where, according to (17), $J = J(\alpha)$ is taken as the maximum positive integer for which $\lambda_J \leq \alpha$.

By fixing $t = t_{\text{obs}}$, we define the mean square error values between the solutions (30) and (31) (or (32)), as follows:

$$\mathcal{E}^{\delta}(t_{\text{obs}}) = \mathbb{E}|u^{\delta}(\cdot, t_{\text{obs}}) - u(\cdot, t_{\text{obs}})|^2 \approx \frac{1}{\mathbf{N}_x + 1} \sum_{p=1}^{\mathbf{N}_x+1} \left| \mathcal{S}^{\delta}(x_p, t_{\text{obs}}) - \mathcal{S}(x_p, t_{\text{obs}}) - \mathbb{E}|\mathcal{P}^{\delta}(x_p, t_{\text{obs}}) - \mathcal{P}(x_p, t_{\text{obs}})| \right|^2, \tag{33}$$

where

$$\mathcal{S}(x_p,t_q)=\sum_{j=1}^{\infty}\Bigg[\exp\big(j^2\pi^2(1-t_q)\big)\int_0^1\xi(z)\varphi_j(z)dz$$

$$-\int_{t_q}^1\exp\big(j^2\pi^2(s-t_q)\big)\left(\int_0^1 g(z,s)\varphi_j(z)dz\right)ds\Bigg]\varphi_j(x_p),$$

$$\mathcal{S}^\delta(x_p,t_q)=\sum_{j=1}^{\infty}\Bigg[\exp\big(j^2\pi^2(1-t_q)\big)\int_0^1\xi^\delta(z)\varphi_j(z)dz$$

$$-\int_{t_q}^1\exp\big(j^2\pi^2(s-t_q)\big)\left(\int_0^1 g^\delta(z,s)\varphi_j(z)dz\right)ds\Bigg]\varphi_j(x_p),$$

$$\mathcal{P}(x_p,t_q)=\sum_{j=1}^{\infty}\left[\sum_{i=0}^{n-1}\frac{\exp\big(j^2\pi^2(\tau_i-t_q)\big)}{1+\alpha j^2\pi^2\exp(j^2\pi^2)}\frac{\tau_i}{j^4\pi^4}\Delta\gamma_j(\tau_i)\right]\varphi_j(x_p),$$

$$\mathcal{P}^\delta(x_p,t_q)=\sum_{j=1}^{\infty}\left[\sum_{i=0}^{n-1}\frac{\exp\big(j^2\pi^2(\tau_i-t_q)\big)}{1+\alpha j^2\pi^2\exp(j^2\pi^2)}\frac{\tau_i}{j^4\pi^4}\Delta\gamma_j^\delta(\tau_i)\right]\varphi_j(x_p),$$

$$\mathbb{E}|\mathcal{P}^\delta(x_p,t_q)-\mathcal{P}(x_p,t_q)|=\frac{1}{n}\sum_{j=1}^{\infty}\left[\sum_{i=0}^{n-1}\frac{\exp\big(j^2\pi^2(\tau_i-t_q)\big)}{1+\alpha j^2\pi^2\exp(j^2\pi^2)}\frac{\tau_i}{j^4\pi^4}\big(\gamma_j(\tau_i+\delta)-\gamma_j(\tau_i)\big)\right]\varphi_j(x_p) \quad (34)$$

for (31), and

$$\mathcal{P}(x_p,t_q)=\sum_{j=1}^{J}\left[\sum_{i=0}^{n-1}\exp\big(j^2\pi^2(\tau_i-t_q)\big)\frac{\tau_i}{j^4\pi^4}\Delta\gamma_j(\tau_i)\right]\varphi_j(x_p),$$

$$\mathcal{P}^\delta(x_p,t_q)=\sum_{j=1}^{J}\left[\sum_{i=0}^{n-1}\exp\big(j^2\pi^2(\tau_i-t_q)\big)\frac{\tau_i}{j^4\pi^4}\Delta\gamma_j^\delta(\tau_i)\right]\varphi_j(x_p),$$

$$\mathbb{E}|\mathcal{P}^\delta(x_p,t_q)-\mathcal{P}(x_p,t_q)|=\frac{1}{n}\sum_{j=1}^{J}\left[\sum_{i=0}^{n-1}\exp\big(j^2\pi^2(\tau_i-t_q)\big)\frac{\tau_i}{j^4\pi^4}\big(\gamma_j(\tau_i+\delta)-\gamma_j(\tau_i)\big)\right]\varphi_j(x_p)$$

for (32).

We choose the input data:

$$g(x,t)=2\pi^2\exp(t-1)\sin(\pi x)+\sum_{i=0}^{n-1}\exp(\pi^2(t-1))\frac{\tau_i\big(\gamma(\tau_{i+1})-\gamma(\tau_i)\big)}{\pi^4},\quad (x,t)\in(0,1)\times(0,1),$$

$$\xi(x)=\sin(\pi x),\quad x\in(0,1)$$

and its noisy perturbations given by

$$g^\delta(x,t)=2\pi^2\exp(t-1)\sin(\pi x)+\frac{\delta\,\mathrm{randn}(\cdot)}{\pi^2}$$

$$+\sum_{i=0}^{n-1}\exp(\pi^2(t-1))\frac{\tau_i\big(\gamma(\tau_{i+1})-\gamma(\tau_i)\big)}{\pi^4},\quad (x,t)\in(0,1)\times(0,1),$$

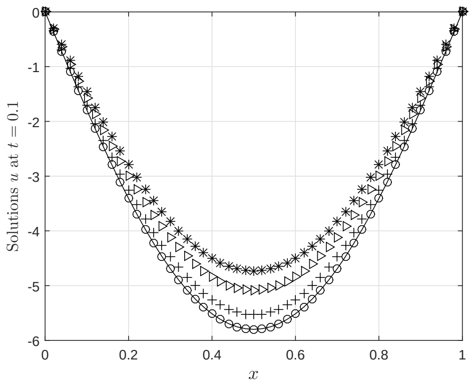
$$\xi^\delta(x)=\sin(\pi x)+\frac{\delta\,\mathrm{randn}(\cdot)}{\pi^2},\quad x\in(0,1).$$

Numerical results are presented for $\mathbf{N}_x=\mathbf{N}_t=50$. We also take ten terms in the series (30), (31) and (34). Various levels of noise $\delta\in\in\{0.001,0.005,0.01\}$ are tested. Numerical results are illustrated only for the filter (19) since for (17) the truncation filter is simply zero (hence, it has an oversmoothing effect) for the considered levels of noise δ if α is chosen according to the Remark 4.1.

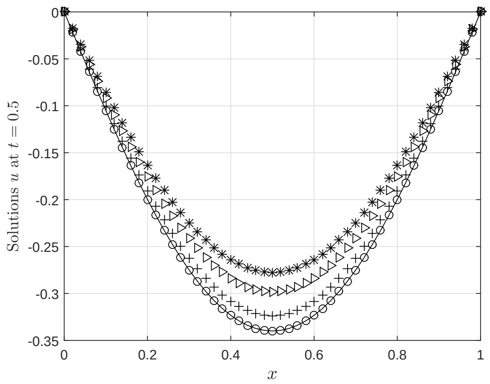
Table 1 and Figure 2 illustrate the numerical results for the filter (19) with the regularization parameter $\alpha=\delta^{1/2}$ chosen according to the Remark 4.2 with $\eta=1/2$. Details are as follows: the errors $\mathcal{E}^\delta(t_{\text{obs}})$ at $t_{\text{obs}}\in\{0.1,0.5,0.9\}$ between the solution (30) and the regularized solution (31) are shown in Table 1 and the behaviours of the numerical solutions are presented in Figure 2. Overall, we can conclude that the smaller the noise δ, the smaller the error $\mathcal{E}^\delta(t_{\text{obs}})$, which confirms the stability of the filter (19).

Table 1. The mean square error values $\mathcal{E}^{\delta}(t_{\text{obs}})$ for $\delta \in \{0.001, 0.005, 0.01\}$ and $t_{\text{obs}} \in \{0.1, 0.5, 0.9\}$ for the filter (19) with $\alpha = \delta^{1/2}$.

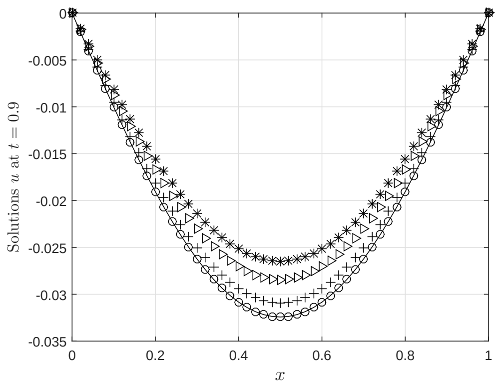
	$\delta = 0.001$	$\delta = 0.005$	$\delta = 0.01$
$t_{\text{obs}} = 0.1$	0.1889	0.4990	0.7449
$t_{\text{obs}} = 0.5$	0.0110	0.0292	0.0436
$t_{\text{obs}} = 0.9$	0.0010	0.0027	0.0041



(a) The solution u at $t = 0.1$



(b) The solution u at $t = 0.5$



(c) The solution u at $t = 0.9$

Figure 2. Graphs of the solution u at $t \in \{0.1, 0.5, 0.9\}$ for various levels of noise $\delta = 0$ $(-\circ-)$, $\delta = 0.001$ $(-+-)$, $\delta = 0.005$, $(-\triangleright-)$ and $\delta = 0.01$ $(-\star-)$, obtained using the filter (19) with $\alpha = \delta^{1/2}$.

References

[1] H. Amann, *Time-delayed Perona–Malik type problems*, Acta Mathematica Universitatis Comenianae 76(1) (2007) 15–38.

[2] Dinh Nho Hao, N. Van Duc and H. Sahli, *A non-local boundary value problem method for parabolic equations backward in time*, Journal of Mathematical Analysis and Applications 345(2) (2008) 805-815.

[3] Dinh Nho Hao, N. Van Duc and D. Lesnic, *Regularization of parabolic equations backward in time by a non-local boundary value problem method*, IMA Journal of Applied Mathematics 75(2) (2010) 291-315.

[4] Dinh Nho Hao, N. Van Duc and N.V. Thang, *Backward semi-linear parabolic equations with time-dependent coefficients and local Lipschitz source*, Inverse Problems 34(2) (2018) 05510.

[5] M. Denche and K. Bessila, *A modified quasi-boundary value method for ill-posed problems*, Journal of Mathematical Analysis and Applications 301 (2005) 419–426.

[6] L.C. Evans, *Partial Differential Equation*, American Mathematical Society, Providence, Rhode Island, 1997.

[7] L.C. Evans, *An Introduction to Stochastic Differential Equations*, American Mathematical Society, Providence, Rhode Island, 2006.

[8] Pao Liu Chow, *Stochastic Partial Differential Equations*, Chapman and Hall/CRC, 2007.

[9] S. Hapuarachchi and Y. Xu, *Backward heat equation with time dependent variable coefficient*, Mathematical Methods in the Applied Sciences 40(4) (2017) 928–938.
[10] B.T. Johansson, *A procedure for the reconstruction of a stochastic stationary temperature field*, IMA Journal of Applied Mathematics 73 (2008) 641–650.
[11] B.T. Johansson and M. Pricop, *A method for identifying a spacewise-dependent heat source under stochastic noise interference*, Inverse Problems in Science and Engineering 18(1) (2010) 51–63.
[12] B.T. Johansson, D. Lesnic and T. Reeve, *A method of fundamental solutions for radially symmetric and axisymmetric backward heat conduction problems*, International Journal of Computer Mathematics 89(11) (2012) 1555–1568.
[13] M.V. Klibanov and A.G. Yagola, *Convergent numerical methods for parabolic equations with reversed time via a new Carleman estimate*, Inverse Problems 35 (2019) 115012 (30pp).
[14] Q. Lu, *Carleman estimate for stochastic parabolic equations and inverse stochastic parabolic problems*, Inverse Problems 28 (2012) 045008 (18pp).
[15] N.H. Tuan and P.H. Quan, *Some extended results on a nonlinear ill-posed heat equation and remarks on a general case of nonlinear terms*, Nonlinear Analysis, Real World Applications 12(6) (2011) 2973–2984.
[16] N.H. Tuan, D. Lesnic and P.T.K. Van, *Identification of the initial population of a nonlinear predator-prey system backwards in time*, Journal of Mathematical Analysis and Applications 479(1) (2019) 1195–1225.
[17] G. Yuan, *Inverse problems for stochastic parabolic equations with additive noise*, Journal of Inverse and Ill-Posed Problems 29(1) (2021) 93–108.

(N.H. Tuan) Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam
Email address: nguyenhuytuan@tdmu.edu.vn

(D. Lesnic) Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK
Email address: amt5ld@maths.leeds.ac.uk

(T.N. Thach) Department of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam

(T.N. Thach) Vietnam National University, Ho Chi Minh City, Vietnam
Email address: ngocthachtnt@gmail.com

(T.B. Ngoc) Department of Mathematics, Faculty of Science, Nong Lam University, Ho Chi Minh City, Viet Nam
Email address: tranbaongoc@hcmuaf.edu.vn