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Lagrangian Multiforms for Kadomtsev–Petviashvili (KP) and the Gelfand–Dickey Hierarchy

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We present, for the first time, a Lagrangian multiform for the complete Kadomtsev–Petviashvili hierarchy—a single variational object that generates the whole hierarchy and encapsulates its integrability. By performing a reduction on this Lagrangian multiform, we also obtain Lagrangian multiforms for the Gelfand–Dickey hierarchy of hierarchies, comprising, among others, the Korteweg–de Vries and Boussinesq hierarchies.

1 Introduction

A feature of integrable systems is the existence of hierarchies of mutually compatible equations. A significant limitation of using traditional Lagrangians for such hierarchies is that they do not capture this compatibility. This limitation was overcome by the Lagrangian multiform [7], which allows compatible Lagrangians (i.e., Lagrangians of compatible equations) to be combined into a single variational object. In recent years, numerous examples of Lagrangian multiforms for continuous one- and two-dimensional integrable hierarchies have been found (e.g., Calogero-Moser [16], Toda [9], potential KdV [14], and AKNS [2, 10, 12, 13]). It is natural to expect that there should exist a Lagrangian multiform for the most well-known three-dimensional integrable hierarchy, the Kadomtsev–Petviashvili (KP) hierarchy [6, 11]. A Lagrangian multiform for the discrete KP hierarchy (the first example of a Lagrangian 3-form) was given in [8], while a Lagrangian

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multiform for the first two flows of the continuous KP hierarchy was presented in [13]. This continuous KP Lagrangian multiform was limited in the sense that extending it to contain higher flows of the hierarchy would result in non-local terms in the multiform, and also there was no algorithmic method to perform such an extension.

In [3], Dickey gives a family of Lagrangians in terms of pseudodifferential operators for the individual equations of the KP hierarchy. In this paper, we assemble Dickey's KP Lagrangians, along with a new set of Lagrangians to create Lagrangian multiform for the full KP hierarchy. This is the first ever example of a continuous Lagrangian 3-form for a complete integrable hierarchy. Then, based on the reduction of KP to the Gelfand–Dickey hierarchy, we perform a reduction on the KP Lagrangian multiform to obtain Lagrangian multiforms for each of the integrable hierarchies that comprise the Gelfand–Dickey hierarchy.

We begin by giving a brief introduction to Lagrangian multiforms in Section 1.1 and then summarise key results relating to pseudodifferential operators in Section 1.2. In Section 2, we introduce the KP hierarchy in terms of pseudodifferential operators, and also its reduction to the Gelfand–Dickey hierarchy. In Section 3, we introduce Dickey's KP Lagrangian. Our main result, a Lagrangian multiform for the KP hierarchy, is given in Section 4, followed by its reduction to Gelfand–Dickey in Section 5.

1.1 Lagrangian multiforms

Lagrangian multiforms were first conceived of in [7] to allow a variational description of compatible systems of equations and have subsequently generated considerable research interest. The traditional variational approach involves a Lagrangian that is a volume form, that is,

$$\mathcal{L}(x, u^{(n)})\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_k, \tag{1}$$

on a k-dimensional base manifold. We use the notation $u^{(n)}$ to represent u and its derivatives with respect to the independent variables x_i, up to the n^{th} order. This can only give as many equations of motion as there are components of u. A *Lagrangian multiform*

$$\mathsf{M} = \sum_{1 \leq i_1 < \ldots < i_k \leq N} \mathcal{L}_{(i_1 \ldots i_k)}(x, u^{(n)})\, \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k} \tag{2}$$

is a k-form in an N dimensional base manifold with $k < N$, subject to the following variational principle. We require that any u that is a critical point of the action

$$S[u;\sigma] = \int_\sigma \mathsf{M}(x, u^{(n)}) \tag{3}$$

must be a critical point for all k-dimensional surfaces of integration σ. This results in the *multiform Euler–Lagrange equations*, given by $\delta \mathsf{dM} = 0$. Furthermore, we require that, on the equations given by $\delta \mathsf{dM} = 0$, any interior deformation of the surface σ must leave the critical action S unchanged (i.e., that on the equations defined by $\delta \mathsf{dM} = 0$, we require that $\mathsf{dM} = 0$). The multiform Euler–Lagrange equations can also be presented as a set of equations in terms of variational derivatives of the $\mathcal{L}_{(i_1 \ldots i_k)}$ that includes the usual Euler–Lagrange equations of each $\mathcal{L}_{(i_1 \ldots i_k)}$. In [15] and [13], proofs are given that show the equivalence of these two presentations of the multiform Euler–Lagrange equations. In Appendix A, we go further and show explicitly the link between these two presentations of the multiform Euler–Lagrange equations.

We shall use the convention that Lagrangians $\mathcal{L}_{(i \ldots j)}$ are anti-symmetric when permuting the sub-indices so, for example, $\mathcal{L}_{(123)} = \mathcal{L}_{(312)} = -\mathcal{L}_{(132)}$.

1.2 Pseudodifferential operators

The main results in this paper require the use of pseudodifferential operators. Here we give a brief summary based on [4, Chapter 1] and the references therein. We introduce the differential algebra $\mathcal{A}$ with generators $u_1, u_2, u_3, \ldots$ and derivation D_x, the total derivative with respect to x, such that $\mathrm{D}_x u_\alpha^{(i)} = (u_\alpha^{(i)})_x = u_\alpha^{(i+1)}$, where $u_\alpha^{(0)} = u_\alpha$. Also, D_x obeys the Leibnitz rule $\mathrm{D}_x u_\alpha^{(i)} u_\beta^{(j)} = u_\alpha^{(i+1)} u_\beta^{(j)} + u_\alpha^{(i)} u_\beta^{(j+1)}$. Elements of $\mathcal{A}$ are polynomials with real or complex coefficients in the generators u_α and their derivatives of arbitrary order. The operator ∂ is defined such that for $f \in \mathcal{A}$,

$$\partial^k f = f \partial^k + \binom{k}{1} f' \partial^{k-1} + \binom{k}{2} f'' \partial^{k-2} + \ldots \tag{4}$$

where $f \in \mathcal{A}$, $f' = \mathrm{D}_x f$ and

$$\binom{k}{i} = \frac{k(k-1)\ldots(k-i+1)}{i!}. \tag{5}$$

When $k > 0$ this sum naturally truncates, whereas when $k < 0$ the sum is infinite. Using these definitions for D_x and ∂, we note that for $f \in \mathcal{A}$, $\mathrm{D}_x f$ is also in $\mathcal{A}$, whereas ∂f is not, since $\partial f = \mathrm{D}_x f + f \partial$, which is an operator.

The ring of pseudodifferential operators $\mathcal{R}$ consists of elements

$$X = \sum_{i=-\infty}^{m} X_i \partial^i, \quad X_i \in \mathcal{A}. \tag{6}$$

Elements of $\mathcal{R}$ can be added (in the natural way) and multiplied term by term, moving all ∂s to the right hand side according to the commutation rule given in (4). Using the commutation rule (4), elements of $\mathcal{R}$ can also be written in the equivalent "left" form

$$X = \sum_{i=-\infty}^{m} \partial^i \tilde{X}_i, \quad \tilde{X}_i \in \mathcal{A}. \tag{7}$$

If the leading coefficient of X, X_m, is 1, then there exists a unique inverse X^{-1} also with leading coefficient 1, such that $XX^{-1} = X^{-1}X = 1$. There also exists a unique m^{th} root of X, $X^{1/m}$ starting with ∂. Then $X^{p/m} = (X^{1/m})^p$ and $(X^{1/m})^m = X$. We define $\mathcal{R}_+$ to be the set of all elements

$$X_+ = \sum_{i=0}^{m} X_i \partial^i \tag{8}$$

and $\mathcal{R}_-$ to be the set of all elements

$$X_- = \sum_{i=-\infty}^{-1} X_i \partial^i \tag{9}$$

The residue of a pseudodifferential operator, $\text{res}\{X\} = X_{-1}$, the coefficient of ∂^{-1} in X. We shall make use of two important properties relating to residues. Firstly,

$$\text{res}\big\{X_+ Y\big\} = \text{res}\big\{X_+ Y_-\big\} = \text{res}\big\{XY_-\big\}. \tag{10}$$

The second property we shall use is given on the following lemma.

Lemma 1.1. The residue of a commutator of two pseudodifferential operators X and Y,

$$\text{res}\{[X, Y]\} = \mathrm{D}_x h \tag{11}$$

for some $h \in \mathcal{A}$, so is a total x derivative.

This lemma is given in [4, Chapter 1], but the proof contains errors that are corrected here.

Proof. We verify this for single term pseudodifferential operators $S = s\,\partial^m$ and $T = t\,\partial^n$. We shall use the notation $s^{(k)} = \mathrm{D}_x^k s$ and similarly for t. We first note that $\text{res}\{[S, T]\}$ is only non-zero if one of m and n is greater than or equal to zero while the other is negative. Without loss of generality, we shall assume that $m \geq 0$ and $n < 0$. The product

$$ST = \sum_{k=0}^{\infty} \binom{m}{k} st^{(k)} \partial^{m+n-k}, \tag{12}$$

so

$$\text{res}\{ST\} = \binom{m}{m+n+1} st^{(m+n+1)} \tag{13}$$

when $m + n + 1 \geq 0$. Otherwise, $\text{res}\{ST\} = 0$ since $k \geq 0$ in (12). It follows that

$$\text{res}\{[S,T]\} = \binom{m}{m+n+1} st^{(m+n+1)} - \binom{n}{m+n+1} st^{(m+n+1)}. \tag{14}$$

We notice that

$$\binom{m}{m+n+1} = \frac{m(m-1)\dots(-n)}{(m+n+1)!} \quad \text{and} \quad \binom{n}{m+n+1} = \frac{n(n-1)\dots(-m)}{(m+n+1)!} \tag{15}$$

so

$$\binom{n}{m+n+1} = (-1)^{m+n+1} \binom{m}{m+n+1}. \tag{16}$$

Then

$$\begin{aligned} \text{res}\{[S,T]\} &= \binom{m}{m+n+1} (st^{(m+n+1)} + (-1)^{m+n} st^{(m+n+1)}) \\ &= \binom{m}{m+n+1} (st^{(m+n+1)} + s^{(1)} t^{(m+n)} - s^{(1)} t^{(m+n)} - s^{(2)} t^{(m+n-1)} + s^{(2)} t^{(m+n-1)} + \dots \\ &\quad \dots - (-1)^{m+n} t^{(1)} s^{(m+n)} + (-1)^{m+n} t^{(1)} s^{(m+n)} + (-1)^{m+n} ts^{(m+n+1)}) \end{aligned} \tag{17}$$

where, to get the expression on the second line, we have added and subtracted $\sum_{\alpha=1}^{m+n} s^{(\alpha)} t^{(m+n+1-\alpha)}$. We recognise this as a total x derivative, so

$$\text{res}\{[S,T]\} = \binom{m}{m+n+1} \mathrm{D}_x \sum_{\alpha=0}^{m+n} (-1)^{\alpha} s^{(\alpha)} t^{(m+n-\alpha)}. \tag{18}$$

It follows that, for general pseudodifferential operators X and Y, their residue, $\operatorname{res}\{[X, Y]\}$ can be expressed as the sum of total derivatives of the form given in (18) for pairs X_i and Y_j, so is a total x derivative. ■

2 The KP Hierarchy and Its Reduction to Gelfand–Dickey

2.1 The KP hierarchy

Here we give a brief summary of Sato's scheme [11] for the KP hierarchy [6]. We let

$$L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \ldots = \partial + \sum_{\alpha=1}^{\infty} u_\alpha \partial^{-\alpha}. \tag{19}$$

Using the notation L^i_+ to represent $(L^i)_+$, for $i > 0$

$$L_{x_i} = [L^i_+, L] \tag{20}$$

gives us the KP hierarchy. For each i, this produces an infinite set of partial differential equations (PDEs) containing derivatives with respect to x_i and x. From the case where $i = 1$, we see that $L_{x_1} = \mathrm{D}_x L$, allowing us to identify x_1 with x. A consequence of (20) is that

$$(L^n)_{x_i} = [L^i_+, L^n] \tag{21}$$

for all $n \geq 1$. This can be proved by induction on n. It follows that

$$\begin{aligned}
(L^j_+)_{x_i} - (L^i_+)_{x_j} &= [L^i_+, L^j]_+ - [L^j_+, L^i]_+ \\
&= [L^i_+ - L^i, L^j]_+ + [L^i, L^j_+]_+ \\
&= [-L^i_-, L^j]_+ + [L^i, L^j_+]_+ \\
&= [-L^i_-, L^j_+]_+ + [L^i, L^j_+]_+ \\
&= [L^i_+, L^j_+].
\end{aligned} \tag{22}$$

This gives us the "zero-curvature" equations for KP,

$$(L^j_+)_{x_i} - (L^i_+)_{x_j} = [L^i_+, L^j_+]. \tag{23}$$

For each $i, j > 0$, this produces a finite set of PDEs containing derivatives with respect to x_i, x_j and x. In the case where $i = 2$ and $j = 3$, (23) gives us

$$\begin{aligned} 3(u_1)_{x_2} &= 3u_1^{(2)} + 6u_2^{(1)} \\ 3(u_1^{(1)})_{x_2} + 3(u_2)_{x_2} - 2(u_1)_{x_3} &= u_1^{(3)} + 3u_2^{(2)} - 6u_1u_1^{(1)}. \end{aligned} \tag{24}$$

Letting $2u_1 = u$ and eliminating u_2, this gives us

$$3u_{x_2x_2} = (4u_{x_3} - u^{(3)} - 6uu^{(1)})_x, \tag{25}$$

the KP equation that gives its name to the hierarchy.

For a fixed choice of i and j, the PDEs given by (20) for i and j are not equivalent to the PDEs given by (23) for the same i and j, since (20) gives an infinite set of PDEs while (23) gives a finite one. However, the set of PDEs given by (20) for all $i > 0$ is equivalent to the set of PDEs given by (23) for all $i, j > 0$. We have already shown that we can obtain (23) from (20). The following lemma relates to the converse.

Lemma 2.1. The set of equations given by

$$(L_+^j)_{x_i} - (L_+^i)_{x_j} = [L_+^i, L_+^j] \tag{26}$$

for all $1 \leq i < j$ is equivalent to the set of equations given by

$$L_{x_i} = [L_+^i, L] \tag{27}$$

for all $i \geq 1$.

Proof. We have already shown that (27) for i and j implies (26) for the same i and j. To show that (26) for all $1 \leq i, j$ implies (27) for all $i \geq 1$, we consider (23) in the form

$$(L_+^j)_{x_i} - (L_+^i)_{x_j} = [L_+^i, L^j]_+ - [L_+^j, L^i]_+, \tag{28}$$

and without loss of generality assume that $j > i$. The first $j - i$ terms of this (i.e., the coefficients of ∂^k for k from $i - 1$ to $j - 2$) are identical to the first $j - i$ terms of

$$L^j_{x_i} = [L_+^i, L^j]. \tag{29}$$

We now let $j = n+1$ in (29) and multiply from the left by L^{-n}, and from this we subtract (29) with $j = n$, multiplied on the left by L^{-n}, and on the right by L to obtain

$$L^{-n}(L^{n+1}_{x_i} - L^{n}_{x_i}L) = L^{-n}([L^i_+, L^{n+1}] - [L^i_+, L^n]L). \tag{30}$$

The left hand side of this is just L_{x_i}, while the right hand side simplifies to $[L^i_+, L]$. Therefore, two copies of (23) with $j = n$ and $j = n+1$ give us the first $n - i$ terms of

$$L_{x_i} = [L^i_+, L]. \tag{31}$$

Since n is arbitrary, we are able to obtain all terms of (20). ■

In [13], a Lagrangian multiform incorporating a re-scaled version of (25) and the corresponding equation arising from (23) with $i = 2$ and $j = 4$ was presented with the following Lagrangian coefficients:

$$\mathcal{L}_{(123)} = \frac{1}{2}v_{x_1x_1}v_{x_1x_3} - \frac{1}{2}v^2_{3x_1} - \frac{1}{2}v^2_{x_1x_2} + v^3_{x_1x_1} \tag{32a}$$

$$\mathcal{L}_{(412)} = \frac{1}{2}v_{x_1x_1}v_{x_1x_4} - 2v_{3x_1}v_{x_1x_1x_2} - \frac{2}{3}v_{x_1x_2}v_{x_2x_2} + 4v^2_{x_1x_1}v_{x_1x_2} \tag{32b}$$

$$\begin{aligned}\mathcal{L}_{(234)} = &-\frac{1}{2}v_{x_1x_3}v_{x_1x_4} - 4v_{x_1x_3}v_{3x_1x_2} + 2v_{x_1x_1x_3}v_{x_1x_1x_2} - \frac{2}{3}v_{x_2x_2}v_{x_2x_3} + v_{x_2x_2}v_{x_1x_4}\\ &+ 4v_{x_2x_2}v_{3x_1x_2} - \frac{8}{3}v_{x_1x_2x_2}v_{x_1x_1x_2} - v_{3x_1}v_{x_1x_1x_4} + \frac{4}{3}v_{3x_1}v_{3x_2} - 4v^2_{3x_1}v_{x_1x_2}\\ &+ 8v_{x_1x_1}v_{3x_1}v_{x_1x_1x_2} + 8v_{x_1x_1}v_{x_1x_2}v_{x_2x_2} + \frac{4}{3}v^3_{x_1x_2} - 8v_{x_1x_1}v_{x_1x_2}v_{x_1x_3} - 8v^3_{x_1x_1}v_{x_1x_2}\end{aligned} \tag{32c}$$

$$\begin{aligned}\mathcal{L}_{(341)} = &\frac{2}{3}v^2_{x_2x_2} + 2v^2_{4x_1} - 2v_{3x_1}v_{x_1x_1x_3} - \frac{4}{3}v_{x_2x_2}v_{x_1x_3} - \frac{2}{3}v_{x_1x_2}v_{x_2x_3} + v_{x_1x_2}v_{x_1x_4}\\ &- \frac{4}{3}v^2_{x_1x_1x_2} + \frac{4}{3}v_{3x_1}v_{x_1x_2x_2} + 12v^2_{x_1x_1}v_{4x_1} + 4v^2_{3x_1}v_{x_1x_1} - 4v^2_{x_1x_1}v_{x_2x_2}\\ &+ 4v_{x_1x_1}v^2_{x_1x_2} + 4v^2_{x_1x_1}v_{x_1x_3} + 10v^4_{x_1x_1},\end{aligned} \tag{32d}$$

where the dependent variable $v_{x_1x_1} = u$ has been used to eliminate non-local terms. These Lagrangians were found using the variational symmetries method outlined in the same paper. Although it is possible to extend this Lagrangian multiform to incorporate more flows of the hierarchy, the resultant Lagrangians become increasingly unwieldy. Also, as we progress up the hierarchy, an ever increasing number of non-local terms appear in the Lagrangians, and the Lagrangians grow very large very quickly. Expanding this multiform to include the x_5 flow results in Lagrangians that are many pages long. Also, this approach does not yield an explicit formula for all of the constituent Lagrangians of the multiform for the complete hierarchy, so in order to obtain a multiform for the entire hierarchy, a different approach is needed.

2.2 The Gelfand–Dickey hierarchy as a reduction of KP

The n^{th} Gelfand–Dickey hierarchy [5] can be formulated as follows. We let

$$L_{GD} = \partial^n + v_{n-2}\partial^{n-2} + v_{n-3}\partial^{n-3} + \ldots + v_0 \tag{33}$$

and let

$$P_m = (L_{GD}^{m/n})_+. \tag{34}$$

We note that while L_{GD} is not a pseudodifferential operator, in general a fractional power of L_{GD} will be. The n^{th} Gelfand–Dickey hierarchy is then given by

$$(L_{GD})_{x_m} = [P_m, L_{GD}]. \tag{35}$$

In the case where $n = 2$, this gives the KdV hierarchy, while for $n = 3$ we get the Boussinesq hierarchy. We now consider the KP equation (21)

$$L^n_{x_m} = [L^m_+, L^n]. \tag{36}$$

In order to reduce the KP hierarchy to the n^{th} Gelfand–Dickey hierarchy, we impose the constraint that $L^n_- = 0$. We note that

$$L^n_- = 0 \implies L^n = L^n_+, \tag{37}$$

an n^{th} order differential operator that we equate with L_{GD}. It follows that $L_{GD}^{1/n} = L$, so P_m is given by L^m_+, making (35) and the right hand expression in (36) equivalent. We also note that $L^n_- = 0 \implies L^{kn}_- = 0$ for all $k \in \mathbb{Z}_+$, so (36) gives $L^n_{x_m} = 0$ whenever n divides m. This is as expected since, by (35), $(L_{GD})_{x_m} = 0$ whenever P_m is an integer power of L_{GD}, which happens when n divides m.

3 A Lagrangian for the KP Hierarchy

In this section, we present a Lagrangian for the KP hierarchy that was originally given in [3]. We define $\mathcal{A}_\varphi$ to be the differential algebra analogous to $\mathcal{A}$ with generators $\varphi_0, \varphi_1, \varphi_2, \ldots$ (i.e., where elements of $\mathcal{A}_\varphi$ are differential polynomials in the generators φ_β), and we define $\mathcal{R}_\varphi$ to be the ring of pseudodifferential operators with coefficients in $\mathcal{A}_\varphi$. We define $\mathcal{R}_{\varphi+}$ and $\mathcal{R}_{\varphi-}$ analogously to $\mathcal{R}_+$ and $\mathcal{R}_-$. We make the dressing substitution

$$L = \phi \partial \phi^{-1} \tag{38}$$

where

$$\phi = 1 + \sum_{\beta=0}^{\infty} \varphi_\beta \partial^{-\beta-1}, \tag{39}$$

noting that because of the leading 1, a unique ϕ^{-1} exists. Expanding (38) we find that

$$L = \partial - \varphi_0' \partial^{-1} + (\varphi_0 \varphi_0' - \varphi_1')\partial^{-2} + (\varphi_1 \varphi_0' + \varphi_0 \varphi_1' - (\varphi_0')^2 - \varphi_0^2 \varphi_0' - \varphi_2')\partial^{-3} + \ldots, \tag{40}$$

where φ_β' denotes the x derivative of φ_β. Equating coefficients with (19), we see that $u_1 = -\varphi_0'$, $u_2 = \varphi_0 \varphi_0' - \varphi_1'$, $u_3 = \varphi_1 \varphi_0' + \varphi_0 \varphi_1' - (\varphi_0')^2 - \varphi_0^2 \varphi_0' - \varphi_2'$ etc., giving an injective map from $\mathcal{A}$ to $\mathcal{A}_\varphi$.

In order to determine the resulting KP equation in terms of ϕ, we invoke the idea of homogeneity in the sense of all terms of an expression carrying equal weight. Let us consider this in the case of the KP equation

$$3u_{x_2 x_2} = (4u_{x_3} - u^{(3)} - 6uu^{(1)})_x. \tag{41}$$

We begin by assigning a weight of 1 to the derivative with respect to x. On the left hand side of the equation, we see a $u_{x_2 x_2}$ term, which we compare to the $u^{(4)}$ term on the right hand side. In order for these terms to have equal weight, an x_2 derivative must have weight 2. Similarly, by comparing the $u_{x_3}^{(1)}$ and $u^{(4)}$ terms, it follows that an x_3 derivative has weight 3. Finally by comparing $u^{(3)}$ and $uu^{(1)}$ we see that u carries weight 2. Whenever it is possible to assign weights in this manner such that all terms of an expression carry equal weight, we say that the expression is *homogeneous*.

Homogeneity can also be introduced directly on the level of the pseudodifferential operators. Applying this to the KP operator

$$L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \dots, \tag{42}$$

we again assign a weight of 1 to the derivative with respect to x, so the leading ∂ carries weight 1. In order for all terms to carry equal weight, it follows that u_1 has weight 2, u_2 has weight 3, and in general u_α has weight $\alpha + 1$. Similarly, the leading 1 of the operator

$$\phi = 1 + \varphi_0 \partial^{-1} + \varphi_1 \partial^{-2} + \dots \tag{43}$$

tells us that ϕ has weight 0, so φ_0 has weight 1, φ_1 has weight 2, and φ_β has weight $\beta + 1$ in order that each term has weight 0. In this paper, we only deal with homogeneous equations. With this in mind, we have the following lemma.

Lemma 3.1. We let $L = \phi \partial \phi^{-1} \in \mathcal{R}_\varphi$. Then

$$L_{x_i} = [L^i_+, L] \iff \phi_{x_i} = -L^i_- \phi. \tag{44}$$

Proof. Using that $L = \phi \partial \phi^{-1}$, the equation

$$L_{x_i} = [L^i_+, L] \tag{45}$$

becomes

$$[\phi_{x_i} \phi^{-1} - L^i_+, L] = 0, \tag{46}$$

This is equivalent to the statement that

$$\phi_{x_i} \phi^{-1} - L^i_+ + f_i = 0 \tag{47}$$

for some f_i in $\mathcal{R}_\varphi$ such that $[L, f_i] = 0$. Letting $\tilde{f}_i = \phi^{-1} f_i \phi$, the requirement that $[L, f_i] = 0$ is equivalent to the requirement that $[\partial, \tilde{f}_i] = \mathrm{D}_x \tilde{f}_i = 0$. Therefore, $\tilde{f}_i$ is a constant in $\mathcal{R}_\varphi$, so

$$\tilde{f}_i = \sum_{j=-\infty}^{m} \gamma_j \partial^j \tag{48}$$

for some m, where each γ_j is a constant in $\mathcal{A}_\varphi$ (i.e., a real or complex number), and consequently

$$f_i = \sum_{j=-\infty}^{m} \gamma_j L^j \tag{49}$$

for the same constants γ_j. In (47) we see that both $\phi_{x_i}\phi^{-1}$ and L^i_+ are of weight i, so we require that f_i is also of weight i. Therefore, $\gamma_j = 0$ whenever $j \neq i$, so f_i is of the form $\gamma_i L^i$. When f_i takes this form, the coefficient of ∂^i in (47) is $\gamma_i - 1$, and setting this equal to zero gives us that $\gamma_i = 1$. Then (47) becomes

$$\phi_{x_i}\phi^{-1} + L^i_- = 0, \tag{50}$$

so the resulting equation for ϕ_{x_i} is

$$\phi_{x_i} = -L^i_-\phi. \tag{51}$$

That is,

$$L_{x_i} = [L^i_+, L] \implies \phi_{x_i} = -L^i_-\phi. \tag{52}$$

Conversely, we see that if (51) holds then

$$\begin{aligned} L_{x_i} &= (\phi\partial\phi^{-1})_{x_i} \\ &= \phi_{x_i}\partial\phi^{-1} - \phi\partial\phi^{-1}\phi_{x_i}\phi^{-1} \\ &= -Ł^i_-\phi\partial\phi^{-1} + \phi\partial\phi^{-1}Ł^i_- \\ &= [-L^i_-, L] \\ &= [L^i_+, L] \end{aligned} \tag{53}$$

so (51) implies (45). ■

Corollary 3.1. Lemmas 2.1 and 3.1 together tell us that the set of equations given by

$$(L^j_+)_{x_i} - (L^i_+)_{x_j} = [L^i_+, L^j_+] \tag{54}$$

in $\mathcal{R}$ for all $1 \leq i, j$ is equivalent to the set of equations given by

$$\phi_{x_i}\phi^{-1} + L^i_- = 0 \tag{55}$$

in $\mathcal{R}_\varphi$ for all $i \geq 1$.

We now consider a Lagrangian $\mathcal{L}_{(1ij)}\mathrm{d}x_1 \wedge \mathrm{d}x_i \wedge \mathrm{d}x_j$ with $\mathcal{L}_{(1ij)} \in \mathcal{A}_\varphi$. For such a Lagrangian, we can take variational derivatives $\dfrac{\delta\mathcal{L}_{(1ij)}}{\delta\varphi_\beta}$ (i.e., the Euler operator with respect to φ_β acting on $\mathcal{L}_{(1ij)}$) to obtain expressions in $\mathcal{A}_\varphi$. However, it is convenient to define the variational derivative with respect to the pseudodifferential operator ϕ,

$$\frac{\delta \mathcal{L}_{(1ij)}}{\delta \phi} = \sum_{\beta=0}^{\infty} \partial^\beta \frac{\delta \mathcal{L}_{(1ij)}}{\delta \varphi_\beta}. \tag{56}$$

According to this definition, $\frac{\delta \mathcal{L}_{(1ij)}}{\delta \phi}$ is a pseudodifferential operator in $\mathcal{R}_{\varphi+}$ that can be put in the usual form with all ∂s on the right using (4). The motivation for this definition is made clear by the following lemma.

Lemma 3.2. If there exist h_1, h_2, and h_3 such that

$$\delta \mathcal{L}_{(1ij)} = \mathrm{res}\{X\,\delta\phi\} + \mathrm{D}_x h_1 + \mathrm{D}_{x_i} h_2 + \mathrm{D}_{x_j} h_3 \tag{57}$$

for some $X \in \mathcal{R}_\varphi$, then the variational derivative of $\mathcal{L}_{(1ij)}$ with respect to ϕ,

$$\frac{\delta \mathcal{L}_{(1ij)}}{\delta \phi} = X_+ \tag{58}$$

Proof. Since $\delta\phi = \delta\varphi_0 \partial^{-1} + \delta\varphi_1 \partial^{-2} + \ldots$ has only negative powers of ∂, (57) is equivalent to

$$\delta \mathcal{L}_{(1ij)} = \mathrm{res}\{X_+\,\delta\phi\} + \mathrm{D}_x h_1 + \mathrm{D}_{x_i} h_2 + \mathrm{D}_{x_j} h_3. \tag{59}$$

We write X_+ in the "left" form described in equation (7), so

$$X_+ = \sum_{k=0}^{m} \partial^k \tilde{X}_k, \quad \tilde{X}_k \in \mathcal{A}_\varphi, \tag{60}$$

and consider the product of an arbitrary term in X_+ with an arbitrary term in $\delta\phi$. This will be of the form

$$\partial^n \tilde{X}_n\, \delta\varphi_m \partial^{-m-1} = \tilde{X}_n\, \delta\varphi_m \partial^{n-m-1} + \sum_{i=1}^{n} \binom{n}{i} \mathrm{D}_x^i (\tilde{X}_n\, \delta\varphi_m) \partial^{n-m-i-1} \tag{61}$$

and the only term on the right hand side that is not a total derivative is $\tilde{X}_n\,\delta\varphi_m\partial^{n-m-1}$. Therefore,

$$\delta\mathcal{L}_{(1ij)} = \operatorname{res}\{X_+\,\delta\phi\} + \mathrm{D}_x h_1 + \mathrm{D}_{x_i} h_2 + \mathrm{D}_{x_j} h_3 = \sum_{k=0}^{m} \tilde{X}_k\,\delta\varphi_k + \mathrm{D}_x \tilde{h}_1 + \mathrm{D}_{x_i} h_2 + \mathrm{D}_{x_j} h_3 \tag{62}$$

for some $\tilde{h}_1$, so the variational derivative

$$\frac{\delta\mathcal{L}_{(1ij)}}{\delta\varphi_k} = \tilde{X}_k \tag{63}$$

for $0 \le k \le m$ and is zero for $k > m$. It follows that

$$\frac{\delta\mathcal{L}_{(1ij)}}{\delta\phi} = \sum_{k=0}^{\infty} \partial^k \frac{\delta\mathcal{L}_{(1ij)}}{\delta\varphi_k} = \sum_{k=0}^{m} \partial^k \tilde{X}_k = X_+ \tag{64}$$

■

Following the formulation in [3], we introduce

$$\phi_p = 1 + p\sum_{\beta=0}^{\infty} \varphi_\beta \partial^{-\beta-1}. \tag{65}$$

where $p \in \mathbb{R}$.

Proposition 3.2. The Lagrangian density

$$\mathcal{L}_{(1ij)} = \operatorname{res}\left\{ -\int_0^1 p^{-1}[(\phi_p\partial^i\phi_p^{-1})_+, (\phi_p\partial^j\phi_p^{-1})_+]\phi_p^{-1}\mathrm{d}p + \partial^j\phi^{-1}\phi_{x_i} - \partial^i\phi^{-1}\phi_{x_j} \right\} \tag{66}$$

gives Euler–Lagrange equations that are equivalent to the KP equation

$$(L_+^i)_{x_j} - (L_+^j)_{x_i} + [L_+^i, L_+^j] = 0. \tag{67}$$

It is important to note that where ∂ appears in this Lagrangian, it signifies an operator that acts on everything to its right, rather than the x derivative of whatever is immediately to its right. Also, even though ϕ consists of an infinite number of components, because this Lagrangian is a residue, only a finite number of these components actually feature. A proof that (66) gives the KP equation as its Euler–Lagrange equations is given in [3] and repeated here. We shall require the following lemma:

Lemma 3.3. The following formula holds:

$$\delta \mathrm{res}\left\{\int_0^p \tilde{p}^{-1}[(\phi_{\tilde{p}}\partial^i\phi_{\tilde{p}}^{-1})_+,(\phi_{\tilde{p}}\partial^j\phi_{\tilde{p}}^{-1})_+]\phi_{\tilde{p}}^{-1}\mathrm{d}\tilde{p}\right\}=-\mathrm{res}\left\{[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+]\delta\phi_p\,\phi_p^{-1}\right\}+\mathrm{D}_x h_1 \tag{68}$$

with

$$h_1=\int\int_0^p \tilde{p}^{-1}\mathrm{res}\big\{[T[V,S],U]+[[T,U]_+S,V]+[U[V,S]_+,T]+[UT,[V,S]_+]+[T[S,U],V] + [U,[T,V]_+S]+[V[S,U]_+,T]+[VT,[S,U]_+]+[[U,V],TS]+[T,[U,V]S]\big\}\mathrm{d}\tilde{p}\,\mathrm{d}x. \tag{69}$$

where $S=\phi_{\tilde{p}}^{-1}$, $T=\delta\phi_{\tilde{p}}\,\phi_{\tilde{p}}^{-1}$, $U=(\phi_{\tilde{p}}\partial^i\phi_{\tilde{p}}^{-1})_+$, and $V=(\phi_{\tilde{p}}\partial^j\phi_{\tilde{p}}^{-1})_+$. This h_1 is local.

The first part of this result is essentially the same as the one given by Dickey in [3]. However, Dickey does not give an explicit expression for h_1, since when considering a single Lagrangian, it is only necessary to show that it is a total x derivative. In the Lagrangian multiform case, we will require an expression for h_1, so it is included here.

Proof of Lemma 3.3. We proceed by taking the p derivative of

$$\delta\mathrm{res}\left\{\int_0^p \tilde{p}^{-1}[(\phi_{\tilde{p}}\partial^i\phi_{\tilde{p}}^{-1})_+,(\phi_{\tilde{p}}\partial^j\phi_{\tilde{p}}^{-1})_+]\phi_{\tilde{p}}^{-1}\mathrm{d}\tilde{p}\right\}+\mathrm{res}\left\{[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+]\delta\phi_p\,\phi_p^{-1}\right\}, \tag{70}$$

multiplying by p, and using that $p\dfrac{\partial\phi_p}{\partial p}=\phi_p-1$. This gives us

$$\begin{aligned}&\delta\mathrm{res}\left\{[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+]\phi_p^{-1}\right\}+\mathrm{res}\left\{[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+]\delta\phi_p\,\phi_p^{-2}\right\}\\&+\mathrm{res}\left\{\left(p\frac{\partial}{\partial p}[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+]\right)\delta\phi_p\,\phi_p^{-1}\right\}.\end{aligned} \tag{71}$$

Again using $p\dfrac{\partial\phi_p}{\partial p}=\phi_p-1$ we find that

$$p\frac{\partial}{\partial p}(\phi_p\partial^i\phi_p^{-1})_+=-[\phi_p^{-1},(\phi_p\partial^i\phi_p^{-1})_+]_+. \tag{72}$$

We shall also use that

$$\delta(\phi_p\partial^i\phi_p^{-1})_+ = [\delta\phi_p\ \phi_p^{-1}, (\phi_p\partial^i\phi_p^{-1})_+]_+. \tag{73}$$

Letting $S = \phi_p^{-1}$, $T = \delta\phi_p\ \phi_p^{-1}$, $U = (\phi_p\partial^i\phi_p^{-1})_+$, and $V = (\phi_p\partial^j\phi_p^{-1})_+$, (71) is equivalent to

$$\mathrm{res}\Big\{[[T,U]_+,V]S + [U,[T,V]_+]S + [U,V]TS - [U,V]ST - [[S,U]_+,V]T - [U,[S,V]_+]T\Big\} \tag{74}$$

In order to show that this is a total x derivative, we make use of (11), the property that the residue of a commutator is a total x derivative. We consider (74) two terms at a time. Firstly,

$$\begin{aligned}
&\mathrm{res}\{[[T,U]_+,V]S - [U,[S,V]_+]T\}\\
=&\mathrm{res}\{[T,U]_+[V,S] + [[T,U]_+S,V] + [T,U][V,S]_+ + [U[V,S]_+,T] + [UT,[V,S]_+]\}\\
=&\mathrm{res}\{[T,U][V,S] + [[T,U]_+S,V] + [U[V,S]_+,T] + [UT,[V,S]_+]\}\\
=&\mathrm{res}\{T[U,[V,S]] + [T[V,S],U] + [[T,U]_+S,V] + [U[V,S]_+,T] + [UT,[V,S]_+]\}.
\end{aligned} \tag{75}$$

Then

$$\begin{aligned}
&\mathrm{res}\{[U,[T,V]_+]S - [[S,U]_+,V]T\}\\
=&\mathrm{res}\{[T,V]_+[S,U] + [U,[T,V]_+S] + [T,V][S,U]_+ + [V[S,U]_+,T] + [VT,[S,U]_+]\}\\
=&\mathrm{res}\{[T,V][S,U] + [U,[T,V]_+S] + [V[S,U]_+,T] + [VT,[S,U]_+]\}\\
=&\mathrm{res}\{T[V,[S,U]] + [T[S,U],V] + [U,[T,V]_+S] + [V[S,U]_+,T] + [VT,[S,U]_+]\}.
\end{aligned} \tag{76}$$

Finally,

$$\begin{aligned}
&\mathrm{res}\{[U,V]TS - [U,V]ST\}\\
=&\mathrm{res}\{[U,V][T,S]\}\\
=&\mathrm{res}\{T[S,[U,V]] + [[U,V],TS] + [T,[U,V]S]\}.
\end{aligned} \tag{77}$$

Adding (75), (76), and (77) together, we notice that

$$\mathrm{res}\{T([U,[V,S]] + [V,[S,U]] + [S,[U,V]])\} = 0 \tag{78}$$

by the Jacobi identity, so (74) is equal to

$$\operatorname{res}\{[T[V,S],U]+[[T,U]_+S,V]+[U[V,S]_+,T]+[UT,[V,S]_+]+[T[S,U],V]+[U,[T,V]_+S]$$
$$+[V[S,U]_+,T]+[VT,[S,U]_+]+[[U,V],TS]+[T,[U,V]S]\}. \tag{79}$$

Since every term is the residue of a commutator, this is a total x derivative. We set h_1 equal to the local expression obtained by letting $p \to \tilde{p}$ in (79), integrating with respect to $\tilde{p}$ from 0 to p, integrating with respect to x, and setting the constant of integration equal to zero (i.e., the expression given in (69)). It follows that, for this choice of h_1, (68) holds. ∎

Proof of Proposition 3.2. We use Lemma 3.3 with $p=1$ to obtain

$$\delta\operatorname{res}\left\{\int_0^1 p^{-1}[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+]\phi_p^{-1}\mathrm{d}p\right\}=-\operatorname{res}\left\{[(\phi\partial^i\phi^{-1})_+,(\phi\partial^j\phi^{-1})_+]\delta\phi\,\phi^{-1}\right\}+\mathrm{D}_x(h_1|_{p=1}). \tag{80}$$

Variation of the rest of the Lagrangian (66) gives us

$$\begin{aligned}
&\delta\operatorname{res}\{\partial^j\phi^{-1}\phi_{x_i}-\partial^i\phi^{-1}\phi_{x_j}\}\\
=&\mathrm{D}_{x_i}\operatorname{res}\{\partial^j\phi^{-1}\delta\phi\}-\mathrm{D}_{x_j}\operatorname{res}\{\partial^i\phi^{-1}\delta\phi\}\\
&+\operatorname{res}\{\phi\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}\delta\phi\,\phi^{-1}\}-\operatorname{res}\{\phi\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}\delta\phi\,\phi^{-1}\}\\
&-\operatorname{res}\{\phi_{x_i}\partial^j\phi^{-1}\delta\phi\,\phi^{-1}\}+\operatorname{res}\{\phi_{x_j}\partial^i\phi^{-1}\delta\phi\,\phi^{-1}\}+\partial h_2\\
=&\mathrm{D}_{x_i}\operatorname{res}\{\partial^j\phi^{-1}\delta\phi\}-\mathrm{D}_{x_j}\operatorname{res}\{\partial^i\phi^{-1}\delta\phi\}\\
&+\operatorname{res}\{((L_+^i)_{x_j}-(L_+^j)_{x_i})\delta\phi\,\phi^{-1}\}+\mathrm{D}_x h_2,
\end{aligned} \tag{81}$$

where we have made use of (10) and the fact that $\delta\phi\,\phi^{-1}\in\mathcal{R}_-$ to obtain the the final expression. Combining (80) and (81) we get

$$\begin{aligned}
\delta\mathcal{L}_{(1ij)}=&\operatorname{res}\{\big((L_+^i)_{x_j}-(L_+^j)_{x_i}+[L_+^i,L_+^j]\big)\delta\phi\,\phi^{-1}\}\\
=&\operatorname{res}\{\phi^{-1}\big((L_+^i)_{x_j}-(L_+^j)_{x_i}+[L_+^i,L_+^j]\big)\delta\phi\}+\mathrm{D}_x h_3,
\end{aligned} \tag{82}$$

so

$$\frac{\delta \mathcal{L}_{(1ij)}}{\delta \phi} = \{\phi^{-1}\big((L_+^i)_{x_j} - (L_+^j)_{x_i} + [L_+^i, L_+^j]\big)\}_+, \tag{83}$$

and when set equal to zero, this is equivalent to (23). ■

Example 3.3. The explicit form of $\mathcal{L}_{(123)}$ given by (66) is

$$\begin{aligned}\mathcal{L}_{(123)} =& - U_{xxx_3} + X_{x_2} - VU_{xx_2} - WU_{x_2} - VV_{x_2} - U^2U_{x_3} + VU_{x_3} + UU_{xx_3} + U^2U_{xx_2} \\ &+ UV_{x_3} + U^2V_{x_2} - UU_{xxx_2} - U^3U_{x_2} - UW_{x_2} - 2UV_{xx_2} - 3V_xU_{x_2} - 3U_{xx}U_{x_2} + 2U_xU_{x_3} \\ &- 3U_xV_{x_2} - 3U_xU_{xx_2} - W_{x_3} + U_{xxxx_2} - \frac{3}{2}UV_{xxx} - \frac{3}{2}U_{xxx}V - 3V_{xx}V - \frac{3}{2}{U_x}^2U^2 \\ &+ 2U_{xxx}U^2 + 2V_{xx}U^2 + 2{U_x}^2V - \frac{1}{2}UU_{xxxx} - \frac{3}{2}U_xU_{xxx} - 3U_xV_{xx} - \frac{3}{2}U_{xx}U^3 + 2{U_x}^3 \\ &+ 3W_{xx_2} - 2V_{xx_3} + 3V_{xxx_2} + 5UU_xU_{x_2} + 2UVU_{x_2} + 3U_{xx}U_xU + 2U_{xx}VU,\end{aligned} \tag{84}$$

where $U = \varphi_0$, $V = \varphi_1$, $W = \varphi_2$, and $X = \varphi_3$. This was calculated using Maple and PSEUDO [1]. Note that although X and Y appear in this Lagrangian, their presence is trivial in that they do not contribute to or feature in the resulting Euler–Lagrange equations. We can simplify $\mathcal{L}_{(123)}$ considerably by subtracting total derivatives to obtain the equivalent Lagrangian

$$\tilde{\mathcal{L}}_{(123)} = 3U_x^2U^2 - \frac{3}{2}U_{xx_2}U^2 + 3V_{xx}U^2 + \frac{5}{2}U_x^3 + U_xU_{x_3} + U_{xx}^2 - 3U_xV_{x_2} - 3U_xV_{xx} + 3V_x^2 \tag{85}$$

that gives identical Euler–Lagrange equations. The variational derivatives with respect to U and V are

$$\begin{aligned}\frac{\delta \mathcal{L}_{(123)}}{\delta U} =& - 6U^2U_{xx} - 6UU_x^2 - 6UU_{xx_2} + 6UV_{xx} - 3U_xU_2 - 15U_xU_{xx} - 2U_{xx_3} + 2U_{xxxx} \\ &+ 3V_{xx_2} + 3V_{xxx} \\ \frac{\delta \mathcal{L}_{(123)}}{\delta V} =& 6UU_{xx} + 6U_x^2 - 3U_{xxx} + 3U_{xx_2} - 6V_{xx},\end{aligned} \tag{86}$$

giving us that

$$\begin{aligned}\frac{\delta\mathcal{L}_{(123)}}{\delta\phi} =& \partial\frac{\delta\mathcal{L}_{(123)}}{\delta V}+\frac{\delta\mathcal{L}_{(123)}}{\delta U}\\ =& \frac{\delta\mathcal{L}_{(123)}}{\delta V}\partial+\mathrm{D}_x\frac{\delta\mathcal{L}_{(123)}}{\delta V}+\frac{\delta\mathcal{L}_{(123)}}{\delta U}\\ =& (6UU_{xx}+6U_x^2-3U_{xxx}+3U_{xx_2}-6V_{xx})\partial-U_{xxxx}+6UU_{xxx}+3U_{xxx_2}-3V_{xxx}\\ &+-6U^2U_{xx}+3U_xU_{xx}-6UU_x^2-6UU_{xx_2}+6UV_{xx}-3U_{x_2}U_x-2U_{xx_3}+3V_{xx_2}\end{aligned} \tag{87}$$

Since the Euler–Lagrange equations (83) have a pre-factor of ϕ^{-1}, we calculate

$$\begin{aligned}\left(\phi\frac{\delta\mathcal{L}_{(123)}}{\delta\phi}\right)_+ =& (6UU_{xx}+6U_x^2-3U_{xxx}+3U_{xx_2}-6V_{xx})\partial-3U_{x_2}U_x-3UU_{xx_2}\\ &+3U_{xxx_2}+3V_{xx_2}-2U_{xx_3}+3UU_{xxx}+3U_xU_{xx}-U_{xxxx}-3V_{xxx}.\end{aligned} \tag{88}$$

Making the substitution $u_1=-U_x$, $u_2=UU_x-V_x$ (based on the expansion (40)), this becomes

$$(3u_1^{(2)}-3(u_1)_{x_2}+6u_2^{(1)})\partial+2(u_1)_{x_3}-3(u_1^{(1)})_{x_2}-3(u_2)_{x_2}-6u_1u_1^{(1)}+u_1^{(3)}+3u_2^{(2)}. \tag{89}$$

Setting this equal to zero gives us equations that are equivalent to (24).

4 Lagrangian Multiforms for the KP Hierarchy

In this section, we present two closely related Lagrangian multiform structures for the KP hierarchy. Let

$$\mathsf{M}=\sum_{1\le i<j<k}\mathcal{L}_{(ijk)}\mathrm{d}x_i\wedge\mathrm{d}x_j\wedge\mathrm{d}x_k. \tag{90}$$

be a differential 3-form. We shall define the coefficients $\mathcal{L}_{(ijk)}$ such that the PDEs defined by $\delta\mathrm{d}\mathsf{M}=0$ are the full set of equations of the KP hierarchy, and we shall show that on these equations $\mathrm{d}\mathsf{M}=0$. We define $P_{(ijkl)}$ such that

$$\mathrm{d}\mathsf{M}=\sum_{1\le i<j<k<l}P_{(ijkl)}\mathrm{d}x_i\wedge\mathrm{d}x_j\wedge\mathrm{d}x_k\wedge\mathrm{d}x_l, \tag{91}$$

and will show that each $P_{(1ijk)}$ has a double zero on the equations of the KP hierarchy, so the coefficients $P_{(1ijk)}$ will be of the form

$$\sum_{\gamma=1}^{n} A_\gamma B_\gamma \tag{92}$$

where each A_γ and B_γ is zero on the equations of the KP hierarchy. More specifically, the A_γ will be of the form

$$(L^i_+)_{x_j} - (L^j_+)_{x_i} + [L^i_+, L^j_+] \tag{93}$$

while the B_γ will be of the form

$$\phi_{x_i}\phi^{-1} + L^i_-, \tag{94}$$

giving us the required double zero. Then

$$\delta P_{(1ijk)} = \sum_{\gamma=1}^{n} \delta A_\gamma B_\gamma + A_\gamma \delta B_\gamma \tag{95}$$

so the equations given by $\delta P_{(1ijk)} = 0$ will be a subset of the equations of the KP hierarchy. In order for the equations given by $\delta P_{(1ijk)} = 0$ for all $1 < i, j, k$ to be the full set of equations of the KP hierarchy, we require that the factors A_γ and B_γ span the set of equations of the KP hierarchy, and also that the A_γ and B_γ are non-degenerate. Rather than show this directly, we will instead show the equivalent result that the full set of equations of the KP hierarchy arise from the Euler–Lagrange equations of the $\mathcal{L}_{(1ij)}$ Lagrangians. Then, for the $P_{(ijkl)}$ where $1 < i, j, k, l$ we will show that $\delta P_{(ijkl)} = 0$ on the equations of the KP hierarchy. Together, these results will show that the multiform Euler–Lagrange equations given by $\delta \mathsf{dM} = 0$ are a subset of the equations of the KP hierarchy and include the entire KP hierarchy. It follows that the multiform Euler–Lagrange equations are precisely the equations of the KP hierarchy.

The factorised form of $P_{(1ijk)}$ in terms of the A_γ and B_γ would suggest that as well as giving us equations in the form

$$(L^i_+)_{x_j} - (L^j_+)_{x_i} + [L^i_+, L^j_+] = 0, \tag{96}$$

the multiform Euler–Lagrange equations should also include KP equations of the type

$$\phi_{x_i}\phi^{-1} + L^i_- = 0. \tag{97}$$

However, Corollary 3.1 tells us that the set of equations of the form of (96) for all $i, j > 0$ is equivalent to the set of equations of the form of (97) for all $i > 0$, so we are free to view either of these equivalent sets of equations as the complete set of multiform Euler–Lagrange equations for M.

4.1 A Lagrangian multiform for KP based on Dickey's Lagrangian

We define

$$\begin{aligned}\Gamma_{ijk} :=& \frac{1}{2}([\phi\partial^k\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_j}, \phi^{-1}] + [\phi\partial^j\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_i}, \phi^{-1}] + [\phi\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_k}, \phi^{-1}] \\ &- [\phi\partial^k\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_i}, \phi^{-1}] - [\phi\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_k}, \phi^{-1}] - [\phi\partial^i\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_j}, \phi^{-1}] \\ &+ [\phi_{x_j}, \partial^k\phi^{-1}\phi_{x_i}\phi^{-1}] + [\phi_{x_i}, \partial^j\phi^{-1}\phi_{x_k}\phi^{-1}] + [\phi_{x_k}, \partial^i\phi^{-1}\phi_{x_j}\phi^{-1}] \\ &- [\phi_{x_i}, \partial^k\phi^{-1}\phi_{x_j}\phi^{-1}] - [\phi_{x_k}, \partial^j\phi^{-1}\phi_{x_i}\phi^{-1}] - [\phi_{x_j}, \partial^i\phi^{-1}\phi_{x_k}\phi^{-1}]),\end{aligned} \tag{98}$$

$$\begin{aligned}\Delta_{ij,k} := -\int_0^1 p^{-1}\big(&[T[V,S],U] + [[T,U]_+S,V] + [U[V,S]_+,T] + [UT,[V,S]_+] + [T[S,U],V] \\ &+ [U,[T,V]_+S] + [V[S,U]_+,T] + [VT,[S,U]_+] + [[U,V],TS] + [T,[U,V]S]\big)\mathrm{d}p\end{aligned} \tag{99}$$

where $S = \phi_p^{-1}$, $T = (\phi_p)_{x_k}\phi_p^{-1}$, $U = (\phi_p\partial^i\phi_p^{-1})_+$, and $V = (\phi_p\partial^j\phi_p^{-1})_+$,

$$\Theta_{ij,k} := \frac{1}{2}\big([\phi_{x_k}\phi^{-1}, L^i_+L^j_-] + [L^j_-, L^i_+\phi_{x_k}\phi^{-1}] + [L^j_+\phi_{x_k}\phi^{-1}, L^i_-] + [L^j_+L^i_-, \phi_{x_k}\phi^{-1}]\big) \tag{100}$$

and

$$\Lambda_{ijk} := \frac{1}{2}\big([L^i_+L^j_- - L^j_+L^i_-, L^k] + [L^k_+L^i_-, L^j_+] + [L^i_+, L^k_+L^j_-] + [L^i_-, L^{j+k}] + [L^{i+k}, L^j_-]\big). \tag{101}$$

In these definitions, L is used as an abbreviation of $\phi\partial\phi^{-1}$, so all of the above are pseudodifferential operators whose coefficients are in terms of φ_β and their derivatives.

Theorem 4.1. The 3-form

$$\mathsf{M} = \sum_{1\le i<j<k} \mathcal{L}_{(ijk)}\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k \tag{102}$$

with coefficients

$$\mathcal{L}_{(1jk)} = \mathrm{res}\Big\{-\int_0^1 p^{-1}[(\phi_p\partial^j\phi_p^{-1})_+, (\phi_p\partial^k\phi_p^{-1})_+]\phi_p^{-1}\mathrm{d}p + \partial^k\phi^{-1}\phi_{x_j} - \partial^j\phi^{-1}\phi_{x_k}\Big\} \tag{103}$$

and

$$\mathcal{L}_{(ijk)} = \int \mathrm{res}\{\Gamma_{ijk} + \Delta_{ij,k} + \Delta_{jk,i} + \Delta_{ki,j} + \Theta_{ij,k} + \Theta_{jk,i} + \Theta_{ki,j} + \Lambda_{ijk}\}\mathrm{d}x \tag{104}$$

(with the constant of integration set to zero) when $i > 1$ is a Lagrangian multiform for the KP hierarchy. Each $\mathcal{L}_{(ijk)}$ is a local expression in the fields φ_β and their derivatives. The multiform Euler–Lagrange equations given by $\delta \mathrm{dM} = 0$ are the full set of equations of the KP hierarchy and consequences thereof. On the equations of the KP hierarchy, $\mathrm{dM} = 0$.

We have constructed $\mathcal{L}_{(ijk)}$ in this way so that

$$\mathrm{dM} = \sum_{1\le i<j<k<l} P_{(ijkl)} \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k \wedge \mathrm{d}x_l \tag{105}$$

has a double zero on the equations of the KP hierarchy. In particular, this $\mathcal{L}_{(ijk)}$ is such that

$$\begin{aligned} P_{(1ijk)} =& - \mathrm{D}_{x_k}\mathcal{L}_{(1ij)} - \mathrm{D}_{x_i}\mathcal{L}_{(1jk)} + \mathrm{D}_{x_j}\mathcal{L}_{(1ik)} + \mathrm{D}_{x_1}\mathcal{L}_{(ijk)} \\ =& - \mathrm{res}\Big\{\frac{1}{2}((L^i_+)_{x_j} - (L^j_+)_{x_i} + [L^i_+, L^j_+])(\phi_{x_k}\phi^{-1} + L^k_-) \\ &+ \frac{1}{2}((L^j_+)_{x_k} - (L^k_+)_{x_j} + [L^j_+, L^k_+])(\phi_{x_i}\phi^{-1} + L^i_-) \\ &+ \frac{1}{2}((L^k_+)_{x_i} - (L^i_+)_{x_k} + [L^k_+, L^i_+])(\phi_{x_j}\phi^{-1} + L^j_-)\Big\}. \end{aligned} \tag{106}$$

Before we can show this to be the case, we shall require a number of lemmas. Lemmas 4.1 and 4.2 are closely related to Dickey's computations to obtain the Euler–Lagrange equations of his KP Lagrangian that we reproduced in Section 3. Lemma 4.3 then re-arranges some of the resulting terms to get us closer to (106), while Lemma 4.4 gives us the terms in (106) that do not contain any x_i, x_j, or x_k derivatives. Also, it is important to note that each of Γ_{ijk}, $\Delta_{ij,k}$, $\Theta_{ij,k}$, and Λ_{ijk} are expressed in terms of the residue of commutators. Therefore, they are all total x derivatives so can be integrated with respect to x to obtain a local expression for $\mathcal{L}_{(ijk)}$.

Lemma 4.1. The Γ_{ijk} defined in (98) is such that

$$\begin{aligned} &\mathrm{D}_{x_i}(\partial^k\phi^{-1}\phi_{x_j} - \partial^j\phi^{-1}\phi_{x_k}) + \mathrm{D}_{x_j}(\partial^i\phi^{-1}\phi_{x_k} - \partial^k\phi^{-1}\phi_{x_i}) + \mathrm{D}_{x_k}(\partial^j\phi^{-1}\phi_{x_i} - \partial^i\phi^{-1}\phi_{x_j}) \\ &= \frac{1}{2}(-(L^k)_{x_j}\phi_{x_i} + (L^j)_{x_k}\phi_{x_i} - (L^i)_{x_k}\phi_{x_j} + (L^k)_{x_i}\phi_{x_j} - (L^j)_{x_i}\phi_{x_k} + (L^i)_{x_j}\phi_{x_k})\phi^{-1} + \Gamma_{ijk}. \end{aligned} \tag{107}$$

Proof of Lemma 4.1.

$$
\begin{aligned}
&\mathrm{D}_{x_i}(\partial^k\phi^{-1}\phi_{x_j}-\partial^j\phi^{-1}\phi_{x_k})+\mathrm{D}_{x_j}(\partial^i\phi^{-1}\phi_{x_k}-\partial^k\phi^{-1}\phi_{x_i})+\mathrm{D}_{x_k}(\partial^j\phi^{-1}\phi_{x_i}-\partial^i\phi^{-1}\phi_{x_j})\\
&=\partial^k\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_i}+\partial^i\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_j}+\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_k}\\
&\quad-\partial^k\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_j}-\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_k}-\partial^j\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_i}.
\end{aligned}
\tag{108}
$$

We now use commutators to get this in the form $(L^i)_{x_j}\phi_{x_k}\phi^{-1}$:

$$
\begin{aligned}
=&\frac{1}{2}(-\phi\partial^k\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_j}\phi^{-1}+\phi\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_k}\phi^{-1}-\phi\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_k}\phi^{-1}\\
&+\phi\partial^k\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_i}\phi^{-1}-\phi\partial^j\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_i}\phi^{-1}+\phi\partial^i\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_j}\phi^{-1})\\
&+\frac{1}{2}(-\phi_{x_j}\partial^k\phi^{-1}\phi_{x_i}\phi^{-1}+\phi_{x_k}\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}-\phi_{x_k}\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}\\
&+\phi_{x_i}\partial^k\phi^{-1}\phi_{x_j}\phi^{-1}-\phi_{x_i}\partial^j\phi^{-1}\phi_{x_k}\phi^{-1}+\phi_{x_j}\partial^i\phi^{-1}\phi_{x_k}\phi^{-1})\\
&+\frac{1}{2}([\phi\partial^k\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_j},\phi^{-1}]+[\phi\partial^j\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_i},\phi^{-1}]+[\phi\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_k},\phi^{-1}]\\
&-[\phi\partial^k\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_i},\phi^{-1}]-[\phi\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_k},\phi^{-1}]-[\phi\partial^i\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_j},\phi^{-1}]\\
&+[\phi_{x_j},\partial^k\phi^{-1}\phi_{x_i}\phi^{-1}]+[\phi_{x_i},\partial^j\phi^{-1}\phi_{x_k}\phi^{-1}]+[\phi_{x_k},\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}]\\
&-[\phi_{x_i},\partial^k\phi^{-1}\phi_{x_j}\phi^{-1}]-[\phi_{x_k},\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}]-[\phi_{x_j},\partial^i\phi^{-1}\phi_{x_k}\phi^{-1}])\\
=&\frac{1}{2}(-(L^k)_{x_j}\phi_{x_i}+(L^j)_{x_k}\phi_{x_i}-(L^i)_{x_k}\phi_{x_j}+(L^k)_{x_i}\phi_{x_j}-(L^j)_{x_i}\phi_{x_k}+(L^i)_{x_j}\phi_{x_k})\phi^{-1}+\Gamma_{ijk}.
\end{aligned}
\tag{109}
$$

■

Lemma 4.2. The $\Delta_{ij,k}$ defined in (99) is such that

$$
\begin{aligned}
&\mathrm{D}_{x_k}\mathrm{res}\left\{-\int_0^1 p^{-1}[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+]\phi_p^{-1}\mathrm{d}p\right\}\\
&=\mathrm{res}\left\{[(\phi\partial^i\phi^{-1})_+,(\phi\partial^j\phi^{-1})_+]\phi_{x_k}\,\phi^{-1}\right\}+\mathrm{res}\{\Delta_{ij,k}\}.
\end{aligned}
\tag{110}
$$

Proof of Lemma 4.2. Since each $\mathcal{L}_{(1ij)}$ is autonomous, we notice that $\mathrm{D}_{x_k}\mathcal{L}_{(1ij)}=\delta\mathcal{L}_{(1ij)}|_{\delta\phi=\phi_{x_k}}$. It follows from Lemma 3.3 that the left hand side of (110) is equal to

$$
\mathrm{res}\left\{[(\phi\partial^i\phi^{-1})_+,(\phi\partial^j\phi^{-1})_+]\phi_{x_k}\,\phi^{-1}\right\}-\mathrm{D}_x h_1|_{\delta\phi_{\tilde{p}}=(\phi_{\tilde{p}})_{x_k}}
\tag{111}
$$

evaluated at $p=1$. We note that $\mathrm{res}\{\Delta_{ij,k}\}$ as defined in (99) is precisely $-\mathrm{D}_x h_1|_{\delta\phi_{\tilde{p}}=(\phi_{\tilde{p}})_{x_k}}$ evaluated at $p=1$. That is,

$$\Delta_{ij,k} := -\int_0^1 p^{-1}\big([T[V,S],U] + [[T,U]_+S,V] + [U[V,S]_+,T] + [UT,[V,S]_+] + [T[S,U],V]$$
$$+ [U,[T,V]_+S] + [V[S,U]_+,T] + [VT,[S,U]_+] + [[U,V],TS] + [T,[U,V]S]\big)\mathrm{d}p \quad (112)$$

with $S=\phi_p^{-1}$, $T=(\phi_p)_{x_k}\phi_p^{-1}$, $U=(\phi_p\partial^i\phi_p^{-1})_+$ and $V=(\phi_p\partial^j\phi_p^{-1})_+$. ■

Lemma 4.3. The $\Theta_{ij,k}$ defined in (100) is such that

$$\mathrm{res}\{[L_+^i,L_+^j]\phi_{x_k}\phi^{-1}\} = \frac{1}{2}\mathrm{res}\{[L_+^i,L_+^j]\phi_{x_k}\,\phi^{-1} + (L_+^j)_{x_k}L_-^i - (L_+^i)_{x_k}L_-^j\} + \mathrm{res}\{\Theta_{ij,k}\}. \quad (113)$$

Proof of Lemma 4.3. Using the identity

$$0 = [L^i,L^j]_+ = [L_+^i,L_+^j] + [L_+^i,L_-^j]_+ + [L_-^i,L_+^j]_+, \quad (114)$$

we see that

$$\begin{aligned}
\mathrm{res}\{[L_+^i,L_+^j]\phi_{x_k}\phi^{-1}\} =& \frac{1}{2}\mathrm{res}\{[L_+^i,L_+^j]\phi_{x_k}\phi^{-1}\} - \frac{1}{2}\mathrm{res}\{[L_+^i,L_-^j]\phi_{x_k}\phi^{-1} + [L_-^i,L_+^j]\phi_{x_k}\phi^{-1}\}\\
=& \frac{1}{2}\mathrm{res}\{[L_+^i,L_+^j]\phi_{x_k}\phi^{-1}\} + \frac{1}{2}\mathrm{res}\{L_+^i\phi_{x_k}\phi^{-1}L_-^j - \phi_{x_k}\phi^{-1}L_+^iL_-^j\\
&+ \phi_{x_k}\phi^{-1}L_+^jL_-^i - L_+^j\phi_{x_k}\phi^{-1}L_-^i + [\phi_{x_k}\phi^{-1},L_+^iL_-^j] + [L_-^j,L_+^i\phi_{x_k}\phi^{-1}]\\
&+ [L_+^j\phi_{x_k}\phi^{-1},L_-^i] + [L_+^jL_-^i,\phi_{x_k}\phi^{-1}]\}\\
=& \frac{1}{2}\mathrm{res}\{[L_+^i,L_+^j]\phi_{x_k}\,\phi^{-1} + (L_+^j)_{x_k}L_-^i - (L_+^i)_{x_k}L_-^j\}\\
&+ \frac{1}{2}\mathrm{res}\{[\phi_{x_k}\phi^{-1},L_+^iL_-^j] + [L_-^j,L_+^i\phi_{x_k}\phi^{-1}] + [L_+^j\phi_{x_k}\phi^{-1},L_-^i]\\
&+ [L_+^jL_-^i,\phi_{x_k}\phi^{-1}]\}\\
=& \frac{1}{2}\mathrm{res}\{[L_+^i,L_+^j]\phi_{x_k}\,\phi^{-1} + (L_+^j)_{x_k}L_-^i - (L_+^i)_{x_k}L_-^j\} + \mathrm{res}\{\Theta_{ij,k}\},
\end{aligned} \quad (115)$$

where

$$\Theta_{ij,k} := \frac{1}{2}\big([\phi_{x_k}\phi^{-1},L_+^iL_-^j] + [L_-^j,L_+^i\phi_{x_k}\phi^{-1}] + [L_+^j\phi_{x_k}\phi^{-1},L_-^i] + [L_+^jL_-^i,\phi_{x_k}\phi^{-1}]\big). \quad (116)$$

■

Lemma 4.4. The identity

$$\mathrm{res}\{[L_+^i,L_+^j]L_-^k + [L_+^j,L_+^k]L_-^i + [L_+^k,L_+^i]L_-^j\} = -2\mathrm{res}\{\Lambda_{ijk}\}, \quad (117)$$

holds.

Proof of Lemma 4.4. We consider $\mathrm{res}\{[L^i, L^j]L^k\}$, (which is clearly zero) and express this in terms of the positive and negative parts of the powers of L:

$$0 = \mathrm{res}\{[L^i, L^j]L^k\} = \mathrm{res}\big\{[L^i_+, L^j_+]L^k_- + [L^i_-, L^j_+]L^k_+ + [L^i_+, L^j_-]L^k_+ \\ +[L^i_-, L^j_-]L^k_+ + [L^i_+, L^j_-]L^k_- + [L^i_-, L^j_+]L^k_-\big\}. \tag{118}$$

The first three terms on the right hand side of (118) can be written as

$$\mathrm{res}\big\{[L^i_+, L^j_+]L^k_- + [L^j_+, L^k_+]L^i_- + [L^k_+, L^i_+]L^j_- \\ +[L^i_-, L^j_+L^k_+] + [L^k_+, L^j_+L^i_-] + [L^i_+L^j_-, L^k_+] + [L^i_+L^k_+, L^j_-]\big\} \tag{119}$$

while the final three terms on the right hand side of (118) can be written as

$$\mathrm{res}\Big\{\frac{1}{2}([L^j_-, L^k_+] + [L^j_+, L^k_-])L^i_- + \frac{1}{2}([L^k_-, L^i_+] + [L^k_+, L^i_-])L^j_- + \frac{1}{2}([L^i_-, L^j_+] + [L^i_+, L^j_-])L^k_- \\ +\frac{1}{2}([L^i_-, L^j_-L^k_+] + [L^k_+, L^j_-L^i_-] + [L^i_-L^j_-, L^k_+] + [L^i_-L^k_+, L^j_-] + [L^i_+L^j_-, L^k_-] + [L^i_+L^k_-, L^j_-] \\ +[L^i_-, L^j_+L^k_-] + [L^k_-, L^j_+L^i_-]\Big\}. \tag{120}$$

By (114), this is equal to

$$\frac{1}{2}\mathrm{res}\big\{ - [L^j_+, L^k_+]L^i_- - [L^k_+, L^i_+]L^j_- - [L^i_+, L^j_+]L^k_- + [L^i_-, L^j_-L^k_+] + [L^k_+, L^j_-L^i_-] \\ + [L^i_-L^j_-, L^k_+] + [L^i_-L^k_+, L^j_-] + [L^i_+L^j_-, L^k_-] + [L^i_+L^k_-, L^j_-] + [L^i_-, L^j_+L^k_-] + [L^k_-, L^j_+L^i_-]\big\}. \tag{121}$$

Since (119) and (121) sum to zero, it follows that

$$\mathrm{res}\big\{[L^i_+, L^j_+]L^k_- + [L^j_+, L^k_+]L^i_- + [L^k_+, L^i_+]L^j_-\big\} \\ = - \mathrm{res}\big\{2[L^i_-, L^j_+L^k_+] + 2[L^k_+, L^j_+L^i_-] + 2[L^i_+L^j_-, L^k_+] + 2[L^i_+L^k_+, L^j_-] + [L^i_-, L^j_-L^k_+] \\ + [L^k_+, L^j_-L^i_-] + [L^i_-L^j_-, L^k_+] + [L^i_-L^k_+, L^j_-] + [L^i_+L^j_-, L^k_-] + [L^i_+L^k_-, L^j_-] + [L^i_-, L^j_+L^k_-] \\ + [L^k_-, L^j_+L^i_-]\big\}, \tag{122}$$

which simplifies to

$$-\mathrm{res}\big\{[L^i_+L^j_- - L^j_+L^i_-, L^k] + [L^k_+L^i_-, L^j_+] + [L^i_+, L^k_+L^j_-] + [L^i_-, L^{j+k}] + [L^{i+k}, L^j_-]\big\} \\ = -2\mathrm{res}\{\Lambda_{ijk}\} \tag{123}$$

where

$$\Lambda_{ijk} := \frac{1}{2}\left([L_+^i L_-^j - L_+^j L_-^i, L^k] + [L_+^k L_-^i, L_+^j] + [L_+^i, L_+^k L_-^j] + [L_-^i, L^{j+k}] + [L^{i+k}, L_-^j]\right). \quad (124)$$

■

Proof of Theorem 4.1. Since Γ_{ijk}, $\Delta_{ij,k}$, $\Theta_{ij,k}$, and Λ_{ijk} are composed entirely of commutators, it follows from Lemma 1.1 that

$$\mathcal{L}_{(ijk)} = \int \operatorname{res}\left\{\Gamma_{ijk} + \Delta_{ij,k} + \Delta_{jk,i} + \Delta_{ki,j} + \Theta_{ij,k} + \Theta_{jk,i} + \Theta_{ki,j} + \Lambda_{ijk}\right\}\mathrm{d}x \quad (125)$$

is local. Since the multiform Euler–Lagrange equations arising from $\delta\mathrm{dM} = 0$ include the Euler–Lagrange equations of the $\mathcal{L}_{1ij}$, we know that the set of equations given by $\delta\mathrm{dM} = 0$ includes all KP equations of the form

$$(L_+^i)_{x_j} - (L_+^j)_{x_i} + [L_+^i, L_+^j] = 0. \quad (126)$$

By Corollary 3.1, $\delta\mathrm{dM} = 0$ also gives us KP equations of the form

$$\phi_{x_i} + L_-^i\phi = 0. \quad (127)$$

In order to proceed, we again use the notation $P_{(ijkl)}$ such that

$$\mathrm{dM} = \sum_{1\le i<j<k<l} P_{(ijkl)}\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k \wedge \mathrm{d}x_l. \quad (128)$$

Combining the results of Lemmas 4.1 to 4.4, we see that

$$\begin{aligned} P_{(1ijk)} &= -\mathrm{D}_{x_k}\mathcal{L}_{(1ij)} - \mathrm{D}_{x_i}\mathcal{L}_{(1jk)} + \mathrm{D}_{x_j}\mathcal{L}_{(1ik)} + \mathrm{D}_{x_1}\mathcal{L}_{(ijk)} \\ &= -\operatorname{res}\Big\{\frac{1}{2}((L_+^i)_{x_j} - (L_+^j)_{x_i} + [L_+^i, L_+^j])(\phi_{x_k}\phi^{-1} + L_-^k) \\ &\qquad + \frac{1}{2}((L_+^j)_{x_k} - (L_+^k)_{x_j} + [L_+^j, L_+^k])(\phi_{x_i}\phi^{-1} + L_-^i) \\ &\qquad + \frac{1}{2}((L_+^k)_{x_i} - (L_+^i)_{x_k} + [L_+^k, L_+^i])(\phi_{x_j}\phi^{-1} + L_-^j)\Big\}, \end{aligned} \quad (129)$$

and since equations of the form $(L_+^i)_{x_j} - (L_+^j)_{x_i} + [L_+^i, L_+^j] = 0$ and $\phi_{x_i}\phi^{-1} + L_-^i = 0$ are both equations of the KP hierarchy, P_{1ijk} has a double zero on the hierarchy.

In order to complete the proof, we must show that for

$$P_{(ijkl)} = \mathrm{D}_{x_i}\mathcal{L}_{(jkl)} - \mathrm{D}_{x_j}\mathcal{L}_{(ikl)} + \mathrm{D}_{x_k}\mathcal{L}_{(ijl)} - \mathrm{D}_{x_l}\mathcal{L}_{(ijk)}, \tag{130}$$

$\delta P_{(ijkl)} = 0$ and $P_{(ijkl)} = 0$ on the equations of the KP hierarchy. We require that $\delta P_{(ijkl)} = 0$ on the equations of the KP hierarchy in order to confirm that $\delta P_{(ijkl)} = 0$ does not define any equations that are not part of the KP hierarchy, and we require that $P_{(ijkl)} = 0$ in order that $\mathsf{dM} = 0$ on the equations of the hierarchy. To show this, we first note that from its definition in terms of the $\mathcal{L}_{(ijk)}$, $P_{(ijkl)}$ is a polynomial with no constant term, in $(\varphi_\beta^{(n)})_I$ where n gives the order of derivative with respect to x and I is a multi-index representing derivatives with respect to x_i for $i > 1$. Also, since $\mathsf{d}^2\mathsf{M}$ is identically zero,

$$D_x P_{(ijkl)} = \mathrm{D}_{x_i}P_{(1jkl)} - \mathrm{D}_{x_j}P_{(1ikl)} + \mathrm{D}_{x_k}P_{(1ijl)} - \mathrm{D}_{x_l}P_{(1ijk)}. \tag{131}$$

This is an identity, so we do not require the φ_β to satisfy the equations of the KP hierarchy for this to hold. Since each of $P_{(1ijk)}$, $P_{(1ikl)}$, $P_{(1ijl)}$, and $P_{(1jkl)}$ has a double zero on the equations of the KP hierarchy, it follows that $\mathrm{D}_x P_{(ijkl)}$ also has a double zero on the equations of the KP hierarchy, and therefore that

$$\frac{\partial}{\partial(\varphi_\beta^{(n)})_I}\mathrm{D}_x P_{(ijkl)} = 0 \tag{132}$$

for all I and n. Using the identity

$$\frac{\partial}{\partial(\varphi_\beta^{(n+1)})_I}\mathrm{D}_x P_{(ijkl)} = \mathrm{D}_x\frac{\partial}{\partial(\varphi_\beta^{(n+1)})_I}P_{(ijkl)} + \frac{\partial}{\partial(\varphi_\beta^{(n)})_I}P_{(ijkl)} \tag{133}$$

we see that for a fixed choice of I, if n is the largest such that $(\varphi_\beta^{(n)})_I$ appears in $P_{(ijkl)}$, then

$$\frac{\partial}{\partial(\varphi_\beta^{(n)})_I}P_{(ijkl)} = 0 \tag{134}$$

on the equations of the KP hierarchy. It also follows from (133) that, on the equations of the KP hierarchy, if

$$\frac{\partial}{\partial(\varphi_\beta^{(n)})_I}P_{(ijkl)} = 0 \quad \text{then} \quad \frac{\partial}{\partial(\varphi_\beta^{(n-1)})_I}P_{(ijkl)} = 0. \tag{135}$$

Therefore, on the equations of the KP hierarchy,

$$\frac{\partial}{\partial(\varphi_\beta^{(n)})_I} P_{(ijkl)} = 0 \tag{136}$$

for all I and n, so $\delta P_{(ijkl)} = 0$. Since $P_{(ijkl)}$ is autonomous, (136) tells us that

$$\mathrm{D}_{x_i} P_{(ijkl)} = 0 \quad \forall i > 0 \tag{137}$$

so $P_{(ijkl)}$ is constant, and since the KP hierarchy admits the zero solution, we conclude that this constant is zero, and $P_{(ijkl)} = 0$ on the equations of the KP hierarchy.

Thus, the set of equations defined by $\delta \mathsf{dM} = 0$ is precisely the full set of equations of the KP hierarchy, and on these equations, $\mathsf{dM} = 0$, so M is a Lagrangian multiform for the KP hierarchy. ■

4.2 An alternative KP Lagrangian multiform

In the KP Lagrangian multiform of Theorem 4.1, we used Dickey's KP Lagrangian for the $\mathcal{L}_{(1ij)}$, and the Lagrangian defined in (104) for the $\mathcal{L}_{(ijk)}$ when $1 < i, j, k$. Here we present an alternative version of the KP Lagrangian multiform in which every Lagrangian is of the same type.

Theorem 4.2. The differential 3-form

$$\widetilde{\mathsf{M}} = \sum_{1 \le i < j < k} \widetilde{\mathcal{L}}_{(ijk)} \, \mathsf{d}x_i \wedge \mathsf{d}x_j \wedge \mathsf{d}x_k \tag{138}$$

where

$$\widetilde{\mathcal{L}}_{(ijk)} = \int \operatorname{res}\{\Gamma_{ijk} + \Delta_{ij,k} + \Delta_{jk,i} + \Delta_{ki,j} + \Theta_{ij,k} + \Theta_{jk,i} + \Theta_{ki,j} + \Lambda_{ijk}\} \mathsf{d}x \tag{139}$$

(i.e., the Lagrangian defined in (104)), is a Lagrangian multiform for the KP hierarchy.

Proof. We recall that in Section 2 we identified x_1 with x. For now we choose not to do so and treat them as separate co-ordinates. This allows us to consider a 3-form M_1 such that the coefficient of $\mathsf{d}x \wedge \mathsf{d}x_i \wedge \mathsf{d}x_j$ with $1 \le i < j$ is Dickey's KP Lagrangian $\mathcal{L}_{(xij)}$, while the coefficient of $\mathsf{d}x_i \wedge \mathsf{d}x_j \wedge \mathsf{d}x_k$ with $1 \le i < j < k$ is the Lagrangian $\mathcal{L}_{(ijk)}$ defined in (104). It then follows from the proof of Theorem 4.1 that this is also a Lagrangian multiform for the KP hierarchy. The multiform Euler–Lagrange equations for M_1 will be the multiform Euler–Lagrange equations of M plus an additional set of equations that tell us to equate derivatives with respect to x_1 with derivatives with respect to x, arising from equations of the form

$$(L_+)_{x_j} - (L_+^j)_{x_1} + [L_+, L_+^j] = 0, \tag{140}$$

and dM_1 will have a double zero on these equations. We now define M_2 to be the restriction of M_1 to a submanifold with co-ordinates $x_1, x_2, x_3, \ldots$, obtained by fixing $x = c$, a constant. It follows that dM_2 still has a double zero on this same set of equations. If we then equate x_1 with x in M_2, we get $\widetilde{\mathsf{M}}$ and it follows that $\mathsf{d}\widetilde{\mathsf{M}}$ has a double zero on the equations of the KP hierarchy. Therefore, the equations defined by $\delta\mathsf{d}\widetilde{\mathsf{M}} = 0$ are a subset of the equations of the KP hierarchy.

To complete the proof that $\widetilde{\mathsf{M}}$ is a Lagrangian multiform for the KP hierarchy, we must show that the equations defined by $\delta\mathsf{d}\widetilde{\mathsf{M}} = 0$ are precisely the full set of equations of the KP hierarchy. We shall do this by showing that the Euler–Lagrange equations of the $\mathcal{L}_{(1jk)}$ Lagrangians give us these equations.

We first consider the coefficient $P_{(xijk)}$ from dM_1.

$$\begin{aligned} P_{(xijk)} = &- \mathrm{D}_{x_k}\mathcal{L}_{(xij)} - \mathrm{D}_{x_i}\mathcal{L}_{(xjk)} + \mathrm{D}_{x_j}\mathcal{L}_{(xik)} + \mathrm{D}_{x}\mathcal{L}_{(ijk)} \\ = &- \mathrm{res}\Big\{\frac{1}{2}((L_+^i)_{x_j} - (L_+^j)_{x_i} + [L_+^i, L_+^j])(\phi_{x_k}\phi^{-1} + L_-^k) \\ &+ \frac{1}{2}((L_+^j)_{x_k} - (L_+^k)_{x_j} + [L_+^j, L_+^k])(\phi_{x_i}\phi^{-1} + L_-^i) \\ &+ \frac{1}{2}((L_+^k)_{x_i} - (L_+^i)_{x_k} + [L_+^k, L_+^i])(\phi_{x_j}\phi^{-1} + L_-^j)\Big\}, \end{aligned} \tag{141}$$

so in the case where $i = 1$ this becomes

$$\begin{aligned} P_{(x1jk)} = &- \mathrm{D}_{x_k}\mathcal{L}_{(x1j)} - \mathrm{D}_{x_1}\mathcal{L}_{(xjk)} + \mathrm{D}_{x_j}\mathcal{L}_{(x1k)} + \mathrm{D}_{x}\mathcal{L}_{(1jk)} \\ = &- \mathrm{res}\Big\{\frac{1}{2}(-(L_+^j)_{x_1} + (L_+^j)_{x})(\phi_{x_k}\phi^{-1} + L_-^k) \\ &+ \frac{1}{2}((L_+^j)_{x_k} - (L_+^k)_{x_j} + [L_+^j, L_+^k])(\phi_{x_1}\phi^{-1} + L_-) \\ &+ \frac{1}{2}((L_+^k)_{x_1} - (L_+^k)_{x})(\phi_{x_j}\phi^{-1} + L_-^j)\Big\} \end{aligned} \tag{142}$$

since $L_+ = \partial$. If we equate x_1 and x in this expression then this becomes zero. This is obvious in the first and third line; for the second line, we note that $L_- = (\phi\partial\phi^{-1})_- =$

$(\partial - \phi_x\phi^{-1})_- = -\phi_x\phi^{-1}$. We now define

$$\bar{\mathcal{L}}_{(xij)} = \mathcal{L}_{(xij)}|_{x\to x_1} \tag{143}$$

and consider the 2-form

$$\mathsf{L} = \bar{\mathcal{L}}_{(x1j)}\mathsf{d}x_1 \wedge \mathsf{d}x_j + \bar{\mathcal{L}}_{(x1k)}\mathsf{d}x_1 \wedge \mathsf{d}x_k + (\bar{\mathcal{L}}_{(xjk)} - \bar{\mathcal{L}}_{(1jk)})\mathsf{d}x_j \wedge \mathsf{d}x_k. \tag{144}$$

By construction, $\mathsf{dL} = -P_{(x1jk)}|_{x\to x_1} = 0$. Then, by Corollary A.2, the variational derivative of each of the Lagrangian coefficients in L is zero. Therefore,

$$\frac{\delta}{\delta\phi}(\bar{\mathcal{L}}_{(xjk)} - \bar{\mathcal{L}}_{(1jk)}) = 0 \tag{145}$$

so

$$\frac{\delta\bar{\mathcal{L}}_{(1jk)}}{\delta\phi} = \frac{\delta\bar{\mathcal{L}}_{(xjk)}}{\delta\phi} = \{\phi^{-1}\big((L^i_+)_{x_j} - (L^j_+)_{x_i} + [L^i_+, L^j_+]\big)\}_+. \tag{146}$$

Since $\bar{\mathcal{L}}_{(1jk)} = \widetilde{\mathcal{L}}_{(1jk)}$, all equations of the KP hierarchy are consequences of $\delta\mathsf{d}\widetilde{\mathsf{M}} = 0$, so $\widetilde{\mathsf{M}}$ is a Lagrangian multiform for the KP hierarchy. ■

5 Reduction to Multiforms for the Gelfand–Dickey Hierarchy

In order to reduce KP to the n^{th} Gelfand–Dickey hierarchy, we imposed the constraint that $L^n_- = 0$. Since, by (51), $\phi_{x_n} = -L^n_-\phi$, we can achieve this in the Lagrangian multiform by setting $\phi_{x_n} = 0$. A simple way to obtain a Lagrangian multiform for the n^{th} Gelfand–Dickey hierarchy is to leave the KP multiform obtained in Section 4 unchanged and impose this constraint on the Euler–Lagrange equations. A more satisfactory approach involves setting $\phi_{x_n} = 0$ in (129) to obtain

$$\begin{aligned} &\mathrm{D}_{x_n}\hat{\mathcal{L}}_{(1ij)} + \mathrm{D}_{x_i}\hat{\mathcal{L}}_{(1jn)} - \mathrm{D}_{x_j}\hat{\mathcal{L}}_{(1in)} - \mathrm{D}_{x_1}\hat{\mathcal{L}}_{(ijn)} \\ &= \mathrm{res}\Big\{\frac{1}{2}((L^i_+)_{x_j} - (L^j_+)_{x_i} + [L^i_+, L^j_+])L^k_- \\ &\quad + \frac{1}{2}(-(L^n_+)_{x_j} + [L^j_+, L^n_+])(\phi_{x_i}\phi^{-1} + L^i_-) \\ &\quad + \frac{1}{2}((L^n_+)_{x_i} + [L^n_+, L^i_+])(\phi_{x_j}\phi^{-1} + L^j_-)\Big\}. \end{aligned} \tag{147}$$

If we can find Lagrangians $\hat{\mathcal{L}}_{(ijk)}$ such that (147) holds, then the constraint $L^n_- = 0$ will be naturally incorporated into the multiform Euler–Lagrange equations, giving us the n^{th} Gelfand–Dickey hierarchy. The $\hat{\mathcal{L}}$ are not uniquely defined by this expression, but a natural choice would be

$$\hat{\mathcal{L}}_{(1ij)} = 0, \tag{148a}$$

$$\hat{\mathcal{L}}_{(1in)} = \mathrm{res}\Big\{ -\int_0^1 p^{-1}[(\phi_p\partial^i\phi_p^{-1})_+, (\phi_p\partial^n\phi_p^{-1})_+]\phi_p^{-1}\mathrm{d}p + \partial^n\phi^{-1}\phi_{x_i}\Big\}, \tag{148b}$$

$$\hat{\mathcal{L}}_{(1jn)} = \mathrm{res}\Big\{ -\int_0^1 p^{-1}[(\phi_p\partial^j\phi_p^{-1})_+, (\phi_p\partial^n\phi_p^{-1})_+]\phi_p^{-1}\mathrm{d}p + \partial^n\phi^{-1}\phi_{x_j}\Big\}, \tag{148c}$$

and

$$\hat{\mathcal{L}}_{(ijn)} = \int \{\hat{\Gamma}_{ijn} + \Delta_{jn,i} + \Delta_{ni,j} + \Theta_{jn,i} + \Theta_{ni,j} + \Lambda_{ijn}\}\mathrm{d}x \tag{148d}$$

with the constant of integration set to zero, where

$$\begin{aligned}\hat{\Gamma}_{ijn} = \frac{1}{2}\mathrm{res}\{&[\phi\partial^n\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_j}, \phi^{-1}] - [\phi\partial^n\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_i}, \phi^{-1}] \\ &+[\phi_{x_j}, \partial^n\phi^{-1}\phi_{x_i}\phi^{-1}] - [\phi_{x_i}, \partial^n\phi^{-1}\phi_{x_j}\phi^{-1}]\}\end{aligned} \tag{149}$$

is equal to Γ_{ijn} with $\phi_{x_n} = 0$. The KP multiform (90) reduces to

$$\mathsf{M}_{(n)} = \sum_{1\le i<j} \hat{\mathcal{L}}_{(ijn)}\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_n. \tag{150}$$

This multiform does not contain any derivatives with respect to x_n, so does not allow any motion in the x_n direction, and is equivalent (i.e., produces identical multiform Euler–Lagrange equations) to

$$\hat{\mathsf{M}}_{(n)} = \sum_{1\le i<j} \hat{\mathcal{L}}_{(ijn)}\mathrm{d}x_i \wedge \mathrm{d}x_j, \tag{151}$$

a Lagrangian 2-form for the n^{th} Gelfand–Dickey hierarchy. As was the case for the KP Lagrangian multiform, a Lagrangian multiform with all coefficients in the form of (148d) is also a Lagrangian multiform for the n^{th} Gelfand–Dickey hierarchy.

6 Conclusion

The Lagrangian multiforms we have presented constitute, in our view, the first instance of establishing the integrability of the KP hierarchy at the Lagrangian level. In contrast to the Lagrangian multiform for KP hierarchy (up to the x_4 flow) that was presented in [13], we now have explicit formulae for the constituent Lagrangians of the Lagrangian multiform for the complete hierarchy, and the constituent Lagrangians are fully local. In addition, while for the Lagrangian multiform in [13] the x_1 and x_2 co-ordinates held a special status (i.e., were treated differently to the other co-ordinates), for the Lagrangian multiform presented here, only x_1 holds a special status. Aspirations for future work include obtaining a Lagrangian multiform for KP that treats every co-ordinate (including x) on an equal footing, and also to connect the continuous KP Lagrangian multiform from this paper with the discrete KP Lagrangian multiform given in [8].

A Multiform Euler–Lagrange Equations in Terms of Variational Derivatives

It was first shown in [14] that $\delta \mathsf{dM} = 0$ on critical points of a differential form

$$\mathsf{M} = \sum_{1\le i_1<\ldots<i_k\le N} \mathcal{L}_{(i_1\ldots i_k)}\, \mathsf{d}x_{i_1} \wedge \ldots \wedge \mathsf{d}x_{i_k}. \tag{A.1}$$

In [15] and [13], different proofs are given of how the equations given by $\delta\mathsf{dM} = 0$ can be expressed in terms of variational derivatives of the coefficients $\mathcal{L}_{(i_1\ldots i_k)}$. In this section, we shall present an alternative proof of this that also gives explicitly the link between the equations in terms of variational derivatives of the $\mathcal{L}_{(i_1\ldots i_k)}$ and the $P_{(i_1\ldots i_{k+1})}$ defined by

$$\mathsf{dM} = \sum_{1\le i_1<\ldots<i_{k+1}\le N} P_{(i_1\ldots i_{k+1})} \mathsf{d}x_{i_1} \wedge \ldots \wedge \mathsf{d}x_{i_{k+1}}. \tag{A.2}$$

In terms of the $\mathcal{L}_{(i_1\ldots i_k)}$,

$$P_{(i_1\ldots i_{k+1})} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \mathrm{D}_{x_{i_\alpha}} \mathcal{L}_{(i_1\ldots i_{\alpha-1} i_{\alpha+1}\ldots i_{k+1})}. \tag{A.3}$$

We recall that the multiform Euler–Lagrange equations are given by $\delta\mathsf{dM} = 0$. We introduce the notation I to represent the N component multi-index $(i_1, \ldots, i_N)$ such that

$$u_I := \left(\prod_{\alpha=1}^{p} (\mathrm{D}_{x_\alpha})^{i_\alpha}\right) u. \tag{A.4}$$

We shall write Ik^r to denote $(i_1, \ldots, i_k + r, \ldots, i_N)$, $I\backslash k^r$ to denote $(i_1, \ldots, i_k - r, \ldots, i_N)$ and $|I|$ to denote the sum $i_1 + \ldots + i_N$. This allows us to express the multiform Euler–Lagrange equations are given by $\delta \mathsf{dM} = 0$ in the form

$$\frac{\partial}{\partial u_I} P_{(i_1 \ldots i_{k+1})} = 0 \tag{A.5}$$

for all $1 \leq i_1 < \ldots < i_{k+1}$ and all multi-indices I. For a fixed choice of $i_1 \ldots i_{k+1}$, we shall write $\mathcal{L}_{(\bar{\alpha})}$ to denote $\mathcal{L}_{(i_1 \ldots i_{\alpha-1} i_{\alpha+1} \ldots i_{k+1})}$. We then define

$$\frac{\delta \mathcal{L}_{(\bar{\alpha})}}{\delta u_I} = \sum_{\substack{J \\ j_{i_\alpha}=0}} (-\mathrm{D})_J \frac{\partial \mathcal{L}_{(\bar{\alpha})}}{\partial u_{IJ}}, \tag{A.6}$$

where the multi-index J is such that components $j_\alpha = 0$ whenever $\alpha \neq i_1, \ldots, i_{k+1}$, that is, J represents derivatives with respect to $x_{i_1}, \ldots, x_{i_{k+1}}$ only. We define that $\frac{\delta \mathcal{L}_{(\bar{\alpha})}}{\delta u_I} = 0$ in the case where any component of the multi-index I is negative. Note that by this definition, the variational derivative of the Lagrangian $\mathcal{L}_{(i_1 \ldots i_{\alpha-1} i_{\alpha+1} \ldots i_{k+1})}$ with respect to u_I only sees derivatives of u_I with respect to the variables $x_{i_1}, \ldots, x_{i_{\alpha-1}}, x_{i_{\alpha+1}}, \ldots, x_{i_{k+1}}$, even though derivatives with respect to other variables may appear in the Lagrangian. This corresponds with only being able to perform integration by parts with respect to variables that are integrated over in the action.

Using the identity

$$\frac{\partial}{\partial u_I} \mathrm{D}_{x_i} = \frac{\partial}{\partial u_{I \setminus i}} + \mathrm{D}_{x_i} \frac{\partial}{\partial u_I} \tag{A.7}$$

tells us that

$$\frac{\partial}{\partial u_I} P_{(i_1 \ldots i_{k+1})} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \left(\frac{\partial \mathcal{L}_{(\bar{\alpha})}}{\partial u_{I \setminus i_\alpha}} + \mathrm{D}_{x_{i_\alpha}} \frac{\partial \mathcal{L}_{(\bar{\alpha})}}{\partial u_I} \right) \tag{A.8}$$

so

$$\begin{aligned} \frac{\delta}{\delta u_I} P_{(i_1 \ldots i_{k+1})} &= \sum_J (-\mathrm{D})_J \frac{\partial}{\partial u_{IJ}} P_{(i_1 \ldots i_{k+1})} \\ &= \sum_J (-\mathrm{D})_J \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \left(\frac{\partial \mathcal{L}_{(\bar{\alpha})}}{\partial u_{IJ \setminus i_\alpha}} + \mathrm{D}_{x_{i_\alpha}} \frac{\partial \mathcal{L}_{(\bar{\alpha})}}{\partial u_{IJ}} \right). \end{aligned} \tag{A.9}$$

Whenever $j_{i_\alpha} \neq 0$ in this sum, so J is of the form $K i_\alpha$ for some multi-index K, then

$$\pm (-\mathrm{D})_J \frac{\partial \mathcal{L}_{(\bar{\alpha})}}{\partial u_{IJ \setminus i_\alpha}} = \mp \mathrm{D}_{x_{i_\alpha}} (-\mathrm{D})_K \frac{\partial \mathcal{L}_{(\bar{\alpha})}}{\partial u_{IK}} \tag{A.10}$$

will appear in this sum. When $J = K$, the term

$$\pm (-\mathrm{D})_K \mathrm{D}_{x_{i_\alpha}} \frac{\partial \mathcal{L}_{(\bar{\alpha})}}{\partial u_{IK}} \tag{A.11}$$

will appear. These two terms cancel, so (A.9) simplifies to

$$\begin{aligned}\frac{\delta}{\delta u_I} P_{(i_1\ldots i_{k+1})} &= \sum_{\alpha=1}^{k+1} \sum_{\substack{J \\ j_{i_\alpha}=0}} (-1)^{\alpha+1} (-\mathrm{D})_J \frac{\partial \mathcal{L}_{(\bar{\alpha})}}{\partial u_{IJ\backslash i_\alpha}} \\ &= \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \frac{\delta \mathcal{L}_{(\bar{\alpha})}}{\delta u_{I\backslash i_\alpha}}.\end{aligned} \tag{A.12}$$

It follows that if (A.5) holds, then

$$\frac{\delta}{\delta u_I} P_{(i_1\ldots i_{k+1})} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \frac{\delta \mathcal{L}_{(\bar{\alpha})}}{\delta u_{I\backslash i_\alpha}} = 0. \tag{A.13}$$

We have shown that

$$\delta \mathrm{dM} = 0 \implies \frac{\delta}{\delta u_I} P_{(i_1\ldots i_{k+1})} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \frac{\delta \mathcal{L}_{(\bar{\alpha})}}{\delta u_{I\backslash i_\alpha}} = 0 \tag{A.14}$$

for all $1 \le i_1 \le \ldots \le i_{k+1} \le N$ and I. Since

$$\frac{\partial P_{(i_1\ldots i_{k+1})}}{\partial u_I} = \sum_{\substack{J \\ j_i \le 1}} \mathrm{D}_J \frac{\delta P_{(i_1\ldots i_{k+1})}}{\delta u_{IJ}} \tag{A.15}$$

(a proof of this identity is given in [13]) it follows that the converse also holds. We summarise this result in the following theorem:

Theorem A.1. For a differential k-form M as given in (A.1), and $P_{(i_1\ldots i_{k+1})}$ as defined in (A.3),

$$\frac{\delta}{\delta u_I} P_{(i_1\ldots i_{k+1})} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \frac{\delta \mathcal{L}_{(\bar{\alpha})}}{\delta u_{I\backslash i_\alpha}}. \tag{A.16}$$

The set of equations defined by

$$\frac{\delta}{\delta u_I} P_{(i_1\ldots i_{k+1})} = 0 \tag{A.17}$$

for all $1 \le i_1 \le \ldots \le i_{k+1} \le N$ and I is equivalent to the set of equations defined by $\delta \mathrm{dM} = 0$.

Corollary A.2. A corollary of Theorem A.1 is that

$$\frac{\delta}{\delta u_{x_{i_\alpha}}} P_{(i_1\ldots i_{k+1})} = (-1)^{\alpha+1} \frac{\delta \mathcal{L}_{(i_1\ldots i_{\alpha-1} i_{\alpha+1}\ldots i_{k+1})}}{\delta u}, \tag{A.18}$$

so the usual Euler–Lagrange equations of each Lagrangian coefficient in M can be expressed in terms of variational derivatives of the coefficients of dM.

B Explicit Form of the KP Lagrangian Multiform

Here we present the first four Lagrangians of the KP Lagrangian multiform M and $\tilde{\mathsf{M}}$, expressed in terms of the φ_β that constitute ϕ. In order to avoid notational confusion over the use of subscripts, we let $U = \varphi_0$, $V = \varphi_1$, $W = \varphi_2$, and $X = \varphi_3$. The following Lagrangians were found using Maple and PSEUDO [1]. In order to obtain $\mathcal{L}_{(234)}$, a Maple procedure based on (18) was used.

$$\begin{aligned}\mathcal{L}_{(123)} = & -U_{xxx_3} + X_{x_2} - VU_{xx_2} - WU_{x_2} - VV_{x_2} - U^2U_{x_3} + VU_{x_3} + UU_{xx_3} + U^2U_{xx_2} + UV_{x_3} \\ & + U^2V_{x_2} - UU_{xxx_2} - U^3U_{x_2} - UW_{x_2} - 2UV_{xx_2} - 3V_xU_{x_2} - 3U_{xx}U_{x_2} + 2U_xU_{x_3} \\ & - 3U_xV_{x_2} - 3U_xU_{xx_2} - W_{x_3} + U_{xxxx_2} - \frac{3}{2}UV_{xxx} - \frac{3}{2}U_{xxx}V - 3V_{xx}V - \frac{3}{2}{U_x}^2U^2 \\ & + 2U_{xxx}U^2 + 2V_{xx}U^2 + 2{U_x}^2V - \frac{1}{2}UU_{xxxx} - \frac{3}{2}U_xU_{xxx} - 3U_xV_{xx} - \frac{3}{2}U_{xx}U^3 \\ & + 2{U_x}^3 + 3W_{xx_2} - 2V_{xx_3} + 3V_{xxx_2} + 5UU_xU_{x_2} + 2UVU_{x_2} + 3U_{xx}U_xU + 2U_{xx}VU,\end{aligned} \tag{B.1}$$

$$\begin{aligned}\widetilde{\mathcal{L}}_{(123)} = & \, 2U^2U_{xxx} + 3UU_xU_{xx} + 2U_x^3 + \frac{1}{2}U_{x_2}V_x - \frac{1}{2}U_xV_{x_2} - 2U^2U_{xx_2} + \frac{3}{2}VU_{xx_2} + \frac{3}{2}UU_{xxx_2} \\ & + 2U_xU_{xx_2} - \frac{3}{2}VU_{xxx} - \frac{3}{2}UV_{xxx} - \frac{3}{2}U^3U_{xx} - 3U_xV_{xx} - \frac{3}{2}U_xU_{xxx} - \frac{1}{2}U_{x_2}U_{xx} \\ & - \frac{1}{2}UU_{xxxx} - UU_xU_{x_2} - UU_{xx_3} + 2VU_x^2 + 2U^2V_{xx} - 3VV_{xx} - \frac{3}{2}U^2U_x^2 \\ & + \frac{3}{2}UV_{xx_2} + 2UU_{xx}V,\end{aligned} \tag{B.2}$$

$$\begin{aligned}\mathcal{L}_{(134)} = & -6U_{xx}V_{xxx} - \frac{3}{2}U_xU_{xxxxx} - 5U_xV_{xxxx} - 6U_xW_{xxx} - 4V_xU_{xxxx} + U_{xxxxx_2} + 40V_xU_xU_{xx} \\ & - 6WV_{xxx}12V_xV_{xxx} - 4U_xW_{x_2} + Y_{x_2} + UW_{x_3} - 4V_{x_2}V_x - 6V_{x_2}U_{xx} + 8U_{x_2}{U_x}^2 \\ & - 4U_{x_2}W_x - 6U_{x_2}V_{xx} - \frac{21}{2}U^2{U_{xx}}^2 - 6U^3V_{xxx} - U_{xxxx_3} - 3U^3W_{xx} - 6W_{xx}W \\ & - 6U_{xx}W_{xx} - 2U_{xx}U_{xxxx} - \frac{3}{2}UV_{xxxxx} - \frac{9}{2}U^3U_{xxxx} + UU_{xxx_3} + 2UV_{xx_3} + V_{x_3}V \\ & + U_{xx_3}V + WU_{x_3} - U_{xx_3}U^2 - V_{x_3}U^2 - 3UW_{xx_2} + U_{x_3}U^3 - UX_{x_2} - VU_{xxx_2} \\ & - U_{xx_2}W + U^2U_{xxx_2} - 8U_xV_{xx_2} - 4U_{xx_2}V_x - 6U_{xx_2}U_{xx} - 4U_xU_{xxx_2} + 3U_{x_3}V_x \\ & + 3U_{x_3}U_{xx} + 3V_{x_3}U_x + 3U_{xx_3}U_x - 3UV_{xxx_2} - UU_{xxxx_2} + U^2W_{x_2} - VW_{x_2}\end{aligned}$$

$$- V_{x_2}U^3 - V_{x_2}W + U_{x_2}U^4 + U_{x_2}V^2 - U_{x_2}X + 2U^2V_{xx_2} - 2VV_{xx_2} - U_{xx_2}U^3$$
$$+ \frac{24}{5}U^3U_xV_x + 24U^3U_xU_{xx} - 5U_{x_3}U_xU - 2U_{x_3}UV - 12V_xW_{xx} + 20U_xU_{xx}{}^2$$
$$+ 16U_xV_x{}^2 + \frac{34}{3}U_x{}^2U_{xxx}8U_x{}^2V_{xx} - 2UW_{xxxx} - 3U_x{}^4 + 2U_{xx_2}UV + 7U_{xx_2}U_xU$$
$$+ 9U_{x_2}UU_{xx} + 7U_{x_2}UV_x + 2U_{x_2}WU + 6U_{x_2}U_xV - 9U_{x_2}U_xU^2 - 3U_{x_2}U^2V - X_{x_3}$$
$$+ 16UU_xW_{xx} + \frac{46}{3}U_{xxx}U_xV + 7V_{x_2}U_xU + 2V_{x_2}UV + \frac{70}{3}UU_xV_{xxx} + 8U_xVV_{xx}$$
$$+ \frac{41}{3}UU_xU_{xxxx} + 4U_{xx}WV + 12U_xU_{xx}W - 12UVU_xV_x - 42UVU_xU_{xx}$$
$$- 6VW_{xxx} - 2U_{xxxx}W + 12U_{xx}{}^2V + 6UU_{xx}V_{xx} + 12UU_{xxx}U_{xx} - 60UU_x{}^2U_{xx}$$
$$+ 8U_xWV_x + 16U_{xx}VV_x + 4UV_xV_{xx} + \frac{28}{3}UU_{xxx}V_x - 33UU_x{}^2V_x + 12UVV_{xxx}$$
$$- \frac{1}{2}UU_{xxxxxx} + 4UV_{xx}W + \frac{22}{3}UVU_{xxxx} + 8UVW_{xx} + 4UU_{xxx}W - 6UV^2U_{xx}$$
$$- 6UU_x{}^2W - 27U^2U_{xxx}U_x + \frac{36}{5}U^3VU_{xx} + \frac{48}{5}U^2VU_x{}^2 + 6W_{xxx_2} + 4V_{xxxx_2}$$
$$- 3W_{xx_3} - 3V_{xxx_3} + 4X_{xx_2} - 9U^2V_xU_{xx} - 6U^2V_{xx}V - 3U^2WU_{xx} - 15U^2V_{xx}U_x$$
$$- 12U^2U_{xxx}V + 4U^2W_{xxx} - 3U_x{}^3V + 4U_{xxx}V^2 - 5VV_{xxxx} - \frac{3}{2}VU_{xxxxx}, \tag{B.3}$$

$$\widetilde{\mathcal{L}}_{(134)} = -3U_x^3V - 4U_x^2U^4 + 16U_{xx}V_xV - 5VV_{xxxx} + 2UU_{xxxx_3} + 8UVW_{xx} - 6VW_{xxx}$$
$$- 6U_xW_{xxx} - 6U_x^2UW - \frac{9}{2}U^3U_{xxxx} - 6U^3V_{xxx} + 2U_{xx_3}W + 24U_{xx}U_xU^3$$
$$- 2V_{x_3}U_{xx} + \frac{24}{5}U_{xxx}U^4 + \frac{28}{3}UV_xU_{xxx} - 6U_{xx}V_{xxx} + 8U_x^2V_{xx} + 16U_xUW_{xx}$$
$$- 3U^3W_{xx} - 2U_{xx}U_{xxxx} - 3U^2WU_{xx} - \frac{3}{2}VU_{xxxxx} + 20U_xU_{xx}^2 - 6U_{xx}W_{xx}$$
$$- 2UW_{xxxx} - U_xW_{x_3} + 2U^2U_{xxxxx} + \frac{24}{5}U_xV_xU^3 - 42U_{xx}U_xUV - 2U_xU_{xx_4}$$
$$+ 3U^3U_{xx_3} + 3UV_{xxx_3} + 3VU_{xxx_3} + 4U_{xxx_3}U_x + 4U^2W_{xxx} - 4U^2U_{xxx_3}$$
$$+ 2V_{xx_3}U_x + 2VV_{xx_3} - \frac{3}{2}U_xU_{xxxxx} - U_{xx}UU_{x_3} - 2WU_{xxxx} + 2U^2U_{xx_4}$$
$$- 12U_xUVV_x + 6U_{xx_3}V_x - 2U^5U_{xx} + 16V_x^2U_x + 2UW_{xx_3} + 2U_{xx_3}U_{xx}$$
$$+ 4U_{xxx}V^2 + 12U_{xx}^2V + U_{x_3}W_x - 12V_xW_{xx} - 4U_{xxxx}V_x - \frac{1}{2}V_xU_{x_4}$$

$$
\begin{aligned}
&+8U_xVV_{xx}-27U^2U_xU_{xxx}+12V_{xxx}UV+\frac{14}{3}U^2V_{xxxx}+\frac{96}{5}U^2U_x^3+4UWV_{xx}\\
&+\frac{46}{3}U_{xxx}U_xV-U_{x_3}U_{xxx}+12U_{xx}U_xW-5U_xV_{xxxx}-33UV_xU_x^2+\frac{22}{3}U_{xxxx}UV\\
&-6WV_{xxx}-\frac{21}{2}U^2U_{xx}^2-60U_x^2UU_{xx}+UU_xU_{x_4}+3U^2U_xU_{x_3}+8U_xV_xW\\
&+\frac{34}{3}U_x^2U_{xxx}-\frac{3}{2}UV_{xx_4}-\frac{3}{2}UU_{xxx_4}+4U_{xxx}UW-\frac{7}{3}UU_{x_3}V_x-12V_{xxx}V_x\\
&+4UV_xV_{xx}-\frac{16}{3}UU_{xx_3}V-\frac{3}{2}VU_{xx_4}+\frac{70}{3}U_xV_{xxx}U-\frac{4}{3}VU_xU_{x_3}-12U_{xxx}VU^2\\
&-3U_x^4-6U^2VV_{xx}-15U^2U_xV_{xx}+\frac{48}{5}U_x^2VU^2-\frac{35}{3}UU_xU_{xx_3}-\frac{1}{3}UU_xV_{x_3}\\
&-6UV^2U_{xx}-\frac{4}{3}U_x^2U_{x_3}-9U_{xx}V_xU^2-\frac{8}{3}U^2V_{xx_3}+\frac{36}{5}U^3VU_{xx}+40U_xU_{xx}V_x\\
&-6WW_{xx}+4VWU_{xx}+12UU_{xxx}U_{xx}+6U_{xx}UV_{xx}+\frac{41}{3}UU_xU_{xxxx}\\
&+\frac{12}{5}U^4V_{xx}+\frac{1}{2}U_{xx}U_{x_4}-\frac{1}{2}UU_{xxxxxx}+\frac{1}{2}U_xV_{x_4}-\frac{3}{2}UV_{xxxxx},
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
\mathcal{L}_{(142)}=\;&6U^3U_{xxx}+4U^3V_{xx}-\frac{24}{5}U^3{U_x}^2-\frac{16}{5}U^4U_{xx}+2U_{xx}U_{xxx}-U_{xxxxx_2}+4U_{xx}V_{xx}\\
&-16VU_{xx}U_x-\frac{20}{3}UU_{xx}V_x-\frac{16}{3}VV_xU_x-16UU_{xxx}U_x-\frac{44}{3}UV_{xx}U_x+4U_xW_{x_2}\\
&+U_{xxx_3}+U^2U_{x_3}-VU_{x_3}-UU_{xx_3}-UV_{x_3}-2U_xU_{x_3}+W_{x_3}-Y_{x_2}+4V_{x_2}V_x\\
&+6V_{x_2}U_{xx}-8U_{x_2}{U_x}^2+4U_{x_2}W_x+6U_{x_2}V_{xx}+4U_{x_2}U_{xxx}+2V_{xx_3}+3VU_{xxxx}\\
&+8VV_{xxx}+4VW_{xx}-\frac{8}{3}W{U_x}^2+12U{U_x}^3-6U{U_{xx}}^2+4V_{xx}W-4U^2U_{xxxx}\\
&-\frac{20}{3}U^2V_{xxx}-\frac{8}{3}U^2W_{xx}+2U_{xxx}W-\frac{8}{3}V^2U_{xx}-\frac{8}{3}UU_{xx}W-\frac{28}{3}UU_{xxx}V\\
&-8UV_{xx}V+8UV{U_x}^2+4U^2V_xU_x+3UW_{xx_2}+UX_{x_2}+VU_{xxx_2}+U_{xx_2}W\\
&-U^2U_{xxx_2}+8U_xV_{xx_2}+4U_{xx_2}V_x+6U_{xx_2}U_{xx}+4U_xU_{xxx_2}+3UV_{xxx_2}\\
&+UU_{xxxx_2}-U^2W_{x_2}+VW_{x_2}+V_{x_2}U^3+V_{x_2}W-U_{x_2}U^4-U_{x_2}V^2+U_{x_2}X\\
&-2U^2V_{xx_2}+2VV_{xx_2}+U_{xx_2}U^3+3U_xU_{xxxx}+4U_xW_{xx}+8U_xV_{xxx}+4U_{xxx}V_x\\
&+8V_{xx}V_x-\frac{32}{3}V_x{U_x}^2-16{U_x}^2U_{xx}-2U_{xx_2}UV-7U_{xx_2}U_xU-9U_{x_2}UU_{xx}\\
&-7U_{x_2}UV_x-2U_{x_2}WU-6U_{x_2}U_xV+9U_{x_2}U_xU^2+3U_{x_2}U^2V-7V_{x_2}U_xU
\end{aligned}
$$

$$- 2V_{x_2}UV + UU_{xxxxx} + 3UV_{xxxx} - 6W_{xxx_2} - 4V_{xxxx_2} - 4X_{xx_2} + 2UW_{xxx} + 22U^2U_{xx}U_x + 8U^2VU_{xx}, \tag{B.5}$$

$$\begin{aligned}\widetilde{\mathcal{L}}_{(142)} = {} & 6U^3U_{xxx} + \frac{1}{3}UU_xV_{x_2} + \frac{7}{3}UU_{x_2}V_x - \frac{16}{3}U_xV_xV - \frac{20}{3}U_{xx}UV_x - 2UU_{xxxx_2} - \frac{8}{3}V^2U_{xx} \\
& + 8U^2VU_{xx} - 2U_{xx_2}W + 2V_{x_2}U_{xx} - 16U_x^2U_{xx} + \frac{8}{3}U^2V_{xx_2} + 3V_{xxxx}U + 2W_{xxx}U \\
& + \frac{4}{3}VU_xU_{x_2} - \frac{8}{3}U_x^2W + 4U^3V_{xx} + 12UU_x^3 + U_xW_{x_2} + \frac{4}{3}U_x^2U_{x_2} + 4WV_{xx} + 3VU_{xxxx} \\
& + 8VV_{xxx} + 4VW_{xx} - 3U^3U_{xx_2} - 3UV_{xxx_2} - 3VU_{xxx_2} - 4U_{xxx_2}U_x + 4U^2U_{xxx_2} \\
& - 2V_{xx_2}U_x - 2VV_{xx_2} + U_{xx}UU_{x_2} - 6U_{xx_2}V_x - 2UW_{xx_2} - 2U_{xx_2}U_{xx} + 2U_{xx}U_{xxx} \\
& - U_{x_2}W_x + 8U_xV_{xxx} - 4U^2U_{xxxx} + 2WU_{xxx} + 4U_{xx}V_{xx} + 4U_{xxx}V_x + U_{x_2}U_{xxx} \\
& + 3U_xU_{xxxx} + 8V_xV_{xx} - 8UVV_{xx} - 16U_{xx}U_xV - 3U^2U_xU_{x_2} - \frac{32}{3}U_x^2V_x \\
& - \frac{8}{3}UWU_{xx} + \frac{35}{3}UU_xU_{xx_2} - \frac{28}{3}U_{xxx}UV - 16U_{xxx}UU_x + UU_{xx_4} + 4U_xW_{xx} \\
& + \frac{16}{3}UU_{xx_2}V - \frac{44}{3}UU_xV_{xx} - \frac{24}{5}U^3U_x^2 + 22U^2U_xU_{xx} + 4U_xV_xU^2 - \frac{16}{5}U^4U_{xx} \\
& + UU_{xxxxx} - \frac{8}{3}U^2W_{xx} - 6U_{xx}^2U - \frac{20}{3}U^2V_{xxx} + 8UU_x^2V,\end{aligned} \tag{B.6}$$

$$\begin{aligned}\mathcal{L}_{(234)} = {} & -12U_{x_2}UVU_{xx} - 9U^2U_{xx_2}V_x + 14U_xV_xV_{xx} + 4UU_xW_{xxx} + U_xVV_{xxx} + 2UU_{xx}^2V \\
& + 3UU_xV_{xxxx} + UU_xU_{xxxxx} + 12U_xV_xW_x + 4U_xU_{xx}U_{xxx} + 6U_xVW_{xx} + UU_{xx_2}U_{x_3} \\
& + U_{x_2}V_{x_3}U - U_{x_2}U_{xx_3}U - UV_{x_2}U_{x_3} - U_{xxxx_2}U_{xx} + \frac{14}{3}U^2V_{xxxx_2} - 3U_{xx_2}X_x \\
& + 8U_xW_{xx_2}U + \frac{18}{5}U_{x_2}U^3V_x + 6V_xV_{xx_3} - 5UV_{xx}U_{xxx} - \frac{8}{3}UU_{xx_3}W + \frac{2}{3}U_x^2V_{x_3} \\
& + 2U_{xxx_3}U_{xx} - 6U_x^2U_{xx_3} - 3UV_{x_2}U_x^2 + 8UU_x^2U_{x_3} + 6U^2U_{x_3}U_{xx} - 8U_{xx_3}UU_{xx} \\
& + 4U_{x_3}V_xU^2 - 2U_{xxx}U_{x_3}U + 6U_{x_2}V_xU_{xx} + \frac{23}{3}U_{xxx_2}U_x^2 - 2W_{x_2}UU_{xx} \\
& + 3W_{x_2}U^2U_x - \frac{1}{2}U_{xx}U_{xxxxx} - VU_{xx}U_{xxx} - \frac{1}{2}U_{xx_4}U_{xx} + \frac{6}{5}U^3V_{x_2}U_x \\
& - \frac{8}{3}U_xWU_{x_3} - \frac{1}{2}U_{x_2}U_{xx_4} - \frac{3}{2}VU_{xxx_4} + 11U_{xx_2}VV_x + U_{x_2}U_{xxx_3} - 2V_{x_3}U_{xx_2} \\
& + 2U_{xx_3}V_{x_2} + W_{x_2}U_{x_3} - U_{xxx_2}U_{x_3} - U_{x_2}W_{x_3} + 8UVW_{xx_2} - \frac{7}{2}U_xV_{xxxx_2}\end{aligned}$$

$$
\begin{aligned}
&- 2UU_{xx}W_{xx} - 6WW_{xx_2} - \frac{4}{3}U_{x_3}W_xU + 6UU_{xx}U_xW + UU_{xxx}U_xV + 6UV_{xx}U_xV \\
&- 29UU_xV_xU_{xx} + 16U^2U_xU_{xx_3} - 12UU_xU_{xxx_3} - 8UV_{xx_3}V + 8U^2U_{xx_3}V \\
&- 6U_xV^2U_{xx} + \frac{25}{3}U_{xxx_2}UU_{xx} - 6UV_{x_2}U_xV - 33UU_{xx_2}U_xV - 6U_{x_2}VUV_x \\
&- 4U_{x_2}U^4U_x - \frac{3}{2}V_{xx_4}U_x + 6U_x^3W + 11U_{xx_2}U_xW + 2V_{x_2}VU_{xx} - \frac{3}{2}UV_{xxx_4} \\
&- 6V_{xx_2}U^2U_x + 5V_{xx_2}U_xV + V_{xx_2}UV_x + 13U_{xx_2}VU_{xx} + V_{xx_2}UU_{xx} \\
&+ 8UU_{xx_2}W_x - \frac{8}{3}U_{xx_3}V^2 + 2U_{x_2}VW_x - \frac{9}{2}U^3U_{xxxx_2} - 3VV_{xx_4} + 4V_{xx_3}W \\
&+ 4W_{xx_3}V + 8V_{xxx_3}V + 2U^2V_{xx_4} + 3U_{xxxx_3}V - U_{xxx_4}U_x + 2U^2U_{xxx_4} + U_{x_4}U_x^2 \\
&- \frac{22}{3}U_xU_{x_3}V_x - \frac{1}{2}V_{xx_2}U_{xxx} - \frac{5}{2}U_{xx}V_{xxx_2} - 4U_{xxx_2}W_x - 3U_{x_2}U^2W_x \\
&- \frac{8}{3}U^2W_{xx_3} - 2V_{xx_2}V_{xx} + 3V_{xx_2}U_x^2 + UU_{xx}U_{xxxx} - 3U_x^3U_{x_2} - 4U_xU_{xx}U_{x_3} \\
&- \frac{3}{2}V_xU_{xx_4} - 2U^5U_{xx_2} - 6U^3V_{xxx_2} + 3V_x^2U_{xx} - 7U_x^2V_{xxx} - 2U_x^2U_{xxxx} \\
&+ 8U_x^3U_{xx} - V_{x_4}U_xU - 27UU_{xx_2}U_x^2 + 2U_{x_4}VU_x + U_{x_4}V_xU + 2U_{xx_4}U_xU \\
&+ 2U_{xx_4}UV - \frac{1}{2}U_{xxxx_4}U + 10UU_{xx_2}V_{xx} + U_{x_4}U_{xx}U + 14U_{xxx_2}U_xV \\
&+ 8U_{xxx_2}UV_x + 2U_{x_2}UW_{xx} + 6U_{x_2}VV_{xx} - 2W_{x_2}U_xV - 2W_{x_2}UV_x - 6U_{x_2}U_x^2V \\
&+ U_{x_2}UU_{xxxx} + 12U_x^2U^2V_x - 6UU_xV_x^2 - 2W_{xxxx_2}U - 6U_x^2V_xV + 6VU_{xx}W_x \\
&- 18UU_xU_{xx}^2 + V_xVU_{xxx} - 2U_{xxx}U^2V_x + \frac{2}{3}U_{xx}^2U_{x_2} + \frac{20}{3}U_{xxx}UU_{xx_2} \\
&- 6U^2U_{x_2}V_{xx} + 4U_{xx}U^2V_{xx} - 6V_xVV_{xx} + 3UV_xU_{xxxx} - 4U_xW_{xxx_2} + 4U_{xx}^3 \\
&- 2V_x^3 + 4U^3V_{xx_3} + 2U_{xxx_3}W - 3V_{xx_2}W_x - \frac{28}{3}U_{xx_3}UV_x + \frac{2}{3}V_{x_3}UU_{xx} \\
&- \frac{24}{5}U^3U_{x_3}U_x - 2WU_{xxxx_2} + 2U^2U_{xxxxx_2} - 4UU_x^2U_{xxx} - 5V_{xxx}U^2U_x \\
&- \frac{16}{5}U^4U_{xx_3} + 5U_{x_2}U_xW_x - \frac{3}{2}U^3U_{xx_4} + 6U_{x_2}U_xV_{xx} + 8U_xU_{xx}V_{xx} \\
&+ 11U_xV_xU_{xxx} + 9U_xU_{xx}W_x + 4U_{x_2}V_xW - \frac{3}{2}UV_{xxxxx_2} - 2W_{xx_2}U_{xx} \\
&- \frac{9}{2}U_{xxx}U^2U_{x_2} + 8U_{x_3}UU_xV + \frac{36}{5}U^3U_{xx_2}V + 3UV_{xxxx_3} + 2UW_{xxx_3} \\
&- 12U_{xxx_2}U^2V + 4U^2W_{xxx_2} + \frac{24}{5}U_{xxx_2}U^4 - \frac{1}{2}U_{xxxxxx_2}U - \frac{4}{3}U_xVV_{x_3}
\end{aligned}
$$

$$
\begin{aligned}
&+2UV_{x_2}W_x+14UU_xV_{xxx_2}-6W_{xx}U^2U_x+5UV_xV_{xxx}+6UV_xW_{xx}+4UV_{xx_2}W\\
&+5U_{x_2}V_x^2-U_{xxx_2}U_{xxx}-\frac{3}{2}U_{x_4}U_xU^2+\frac{29}{3}UU_xU_{xxxx_2}+4U_xV_{x_2}W-U_x^2VU_{xx}\\
&+4U_{xx_2}VW+4U_{xx}U_{x_2}W+UU_{xxxxx_3}-4U^2U_{xxxx_3}+\frac{31}{3}U_{xx_2}U_xU_{xx}-2W_{x_2}U_x^2\\
&-7V_xV_{xxx_2}+4U_{xxx_2}UW-2VU_{xx}V_{xx}-6WV_{xxx_2}-5U_{xx_2}W_{xx}+6U^3U_{xxx_3}\\
&+5U_{xxx_3}V_x+2V_{x_2}U_xV_x+\frac{84}{5}U^3U_{xx_2}U_x+4W_xU_{xx_3}+2U_xU_{xxxx_3}+2U_xW_{xx_3}\\
&+4V_{xx}U_{xx_3}+U_{xx_3}U_{xxx}+2V_{xx_3}U_{xx}+5V_{xxx_3}U_x-6UU_{xx_2}V^2-\frac{5}{2}U_{xx_2}V_{xxx}\\
&-\frac{7}{2}U_{xxx_2}V_{xx}+\frac{1}{3}UV_{x_2}U_{xxx}-3U_{xxxx}U^2U_x+15U_{xx_2}U_xV_x-6UU_x^2V_{xx}\\
&+\frac{48}{5}U_{x_2}U^2U_xV-3U_xWU_{xxx}-6U_xWV_{xx}-15U_{xx}UU_xU_{x_2}-15U_{x_2}UU_xV_x\\
&-5VV_{xxxx_2}-6VW_{xxx_2}-8U_x^2W_{xx}-W_xV_{xxx}+2V_{xx}W_{xx}+2V_{xx}V_{xxx}+3W_{xx}U_{xxx}\\
&-\frac{5}{2}U_{xxxx_2}V_x-\frac{16}{3}U_{x_3}VU_{xx}-\frac{20}{3}U^2V_{xxx_3}-3U^3W_{xx_2}+4U_{xxx_2}V^2+\frac{1}{2}U_{x_2}V_{x_4}\\
&+12V_{xxx_2}UV-2UW_xU_{xxx}-6UU_{xx}X_x+2U_{xxx}U_xU^3+3U_{xxx}U^2U_{xx}+\frac{12}{5}U^4V_{xx_2}\\
&-\frac{39}{2}U_{xxx_2}U^2U_x-6W_{xx_2}V_x-U_{xxxxx_2}U_x-\frac{38}{3}U_xU_{xx_3}V+\frac{66}{5}U^2U_x^2U_{x_2}\\
&+\frac{36}{5}U_{xx}U^3U_{x_2}-6U^2V_{xx_2}V-2U_{xx}W_{xxx}+10V_xU_{xx}^2+6X_xV_{xx}+3X_xU_{xxx}\\
&-4V_xW_{xxx}-3V_xV_{xxxx}-V_xU_{xxxxx}-6W_xW_{xx}-6U_x^2X_x+2V_{xxx}U_{xxx}-2UU_{xxx}^2\\
&-2UV_{xx}^2+18U_{xx}U_x^2U^2-4U_x^3U^3-2U_{xx}^2U^3-3U^2U_{xx_2}W-6UU_x^2W_x\\
&+\frac{4}{3}U_xW_{x_3}U+\frac{7}{3}U_{x_2}UV_{xxx}+\frac{1}{2}U_{xxx}U_{xxxx}-\frac{28}{3}U_{xxx_3}UV-4U_{x_3}VV_x\\
&-\frac{3}{2}VU_{xxxxx_2}-\frac{1}{2}V_{x_2}U_{x_4}-6U_{x_2}U_xUW-\frac{10}{3}U_{x_3}UV_{xx}+\frac{22}{3}U_{xxxx_2}UV\\
&-\frac{22}{3}UU_xV_{xx_3}-\frac{3}{2}U_{xx}V_{xxxx}-\frac{27}{2}U^2U_{xx_2}U_{xx}+\frac{5}{3}U_{x_2}U_xU_{xxx}-\frac{1}{2}U_{xx_2}U_{xxxx}\\
&+\frac{10}{3}U_{x_2}VU_{xxx}+\frac{1}{2}V_{xx}U_{xxxx}+\frac{1}{2}U_{xx_2}U_{x_4}. \qquad \text{(B.7)}
\end{aligned}
$$

The Lagrangian $\widetilde{\mathcal{L}}_{(234)}$ is identical to $\mathcal{L}_{(234)}$. From the Lagrangians given here for $1 < i$, $j \leq 4$, we see that $\widetilde{\mathcal{L}}_{(1ij)}$ gives a shorter Lagrangian than $\mathcal{L}_{(1ij)}$. In general, the difference between $\widetilde{\mathcal{L}}_{(1ij)}$ and $\mathcal{L}_{(1ij)}$ can be expressed as the sum of a total x_i derivative and a total x_j derivative.

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