

RESEARCH ARTICLE

Towards a constructive simplicial model of Univalent Foundations

Nicola Gambino¹ | Simon Henry²¹ School of Mathematics, University of Leeds, Leeds, UK² Department of Mathematics and Statistics, University of Ottawa, Ottawa, Canada**Correspondence**Nicola Gambino, School of Mathematics, University of Leeds, Woodhouse Lane, Leeds LS2 9JT, United Kingdom.
Email: n.gambino@leeds.ac.uk**Funding information**

EPSRC, Grant/Award Number: EP/M01729X/1; AFOSR, Grant/Award Numbers: FA9550-17-1-0290, FA9550-21-1-0007; Czech Operational Programme “Research, Development and Education”, Grant/Award Number: CZ.02.2.69/0.0/0.0/16_027/0008360

Abstract

We provide a partial solution to the problem of defining a constructive version of Voevodsky’s simplicial model of Univalent Foundations. For this, we prove constructive counterparts of the necessary results of simplicial homotopy theory, building on the constructive version of the Kan-Quillen model structure established by the second-named author. In particular, we show that dependent products along fibrations with cofibrant domains preserve fibrations, establish the weak equivalence extension property for weak equivalences between fibrations with cofibrant domain and define a univalent fibration that classifies small fibrations between bifibrant objects. These results allow us to define a comprehension category supporting identity types, Σ -types, Π -types and a univalent universe, leaving only a coherence question to be addressed.

MSC (2020)

03G30 (primary), 03F50, 18N45 (secondary)

Contents

INTRODUCTION	1074
1. PRELIMINARIES	1077
2. THE COMPREHENSION CATEGORY OF COFIBRANT SIMPLICIAL SETS.	1080
3. IDENTITY TYPES AS PATH SPACES	1082

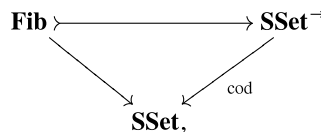
4. II-TYPES VIA COFIBRANT REPLACEMENTS 1085
 5. THE WEAK EQUIVALENCE EXTENSION PROPERTY 1087
 6. A CLASSIFYING SMALL FIBRATION BETWEEN BIFIBRANT OBJECTS 1090
 7. FIBRANCY AND UNIVALENCE OF THE UNIVERSE 1096
 8. CONCLUSIONS AND FUTURE WORK 1102
 APPENDIX: THE COFIBRANT REPLACEMENT FUNCTOR 1104
 ACKNOWLEDGEMENTS 1107
 REFERENCES 1107

INTRODUCTION

Context and motivation. This paper investigates Voevodsky’s model of Martin-Löf (ML)-type theory extended with the Univalence Axiom (UA) in the category of simplicial sets [30]. This model is of fundamental importance since it informs a new approach to the development of mathematics, known as Univalent Foundations, which takes the notion of a space (rather than that of a set) as the most primitive and is supported by a new approach to computer-assisted formalisation of mathematics [45]. In combination with the discovery of a close connection between identity types and Quillen’s homotopical algebra [5, 17], the simplicial model also provides inspiration for the development of Homotopy Type Theory [43].

The original definition of the model of $ML(UA) = ML + UA$ was carried out in Zermelo-Fraenkel set theory extended with the axiom of choice (ZFC) and two inaccessible cardinals [30, Theorem 3.4.3]. Since ML is a constructive theory, while ZFC is a classical one, the question of whether the simplicial model could be defined working constructively (that is, without the use of the law of excluded middle or the axiom of choice), so as to provide a constructive interpretation of UA, arose immediately after its discovery, around 2006 [31, 42, 44], and was one of the central issues investigated during the thematic programme on Univalent Foundations at the Institute for Advanced Study in 2012/13. In spite of significant efforts since then, the question is still open to date.

The aim of this paper is to provide a partial solution to this problem. In order to explain our results and the novel aspects of our work, let us recall from [30] that the main results of homotopy theory necessary to define the simplicial model of $ML(UA)$ are the existence of a Quillen model structure on the category of simplicial sets whose fibrations are the Kan fibrations, the fact that dependent product along a fibration preserves fibrations, the existence of a fibration $\pi : \bar{U} \rightarrow U$ that classifies small fibrations, the fibrancy of U and, finally, the univalence of the fibration π . In combination, these results allow us to define a comprehension category



where \mathbf{SSet} is the category of simplicial sets and \mathbf{Fib} is the category of Kan fibrations, and to show that this comprehension category supports the type constructors of ML (which we take here to be $0, 1, +, \mathbb{N}, Id, \Sigma, \Pi$ and U), in the sense of [32], and a univalent-type universe closed under the above type constructors, in the sense of [40]. Such a comprehension category is not quite a model of $ML(UA)$ because of well-known strictness issues [25], but it gives rise to one by an appropriate coherence result [30, 32].

A fundamental obstruction to a constructive development of the simplicial model was discovered in [10], where it was shown that it is not possible to prove constructively that for a simplicial set A and a Kan complex B , the exponential B^A is again a Kan complex. Because of this, Coquand and his collaborators defined homotopy-theoretic models of type theories with the UA in categories of cubical sets [7], opening a profitable new research direction, cf. [4, 14, 35] for example. Apart from switching from simplicial sets to cubical sets, they also switched from ordinary fibrations, defined by a right lifting property, to uniform fibrations, defined as maps equipped with additional structure which provides a choice of fillers for lifting problems satisfying uniformity conditions, as in algebraic weak factorisation systems [21, 22]. While categories of cubical sets considered in this line of work allow us to define constructively models of $\text{ML}(\text{UA})$ and admit a Quillen model structure [36], none of them is known to be Quillen equivalent to simplicial sets or topological spaces. Ongoing work of Awodey, Coquand, Riehl and Sattler aims at addressing this issue using equivariant presheaves in the cubical setting.

In simplicial sets, although it is possible to develop a constructive theory of uniform fibrations and prove that dependent product along a uniform fibration preserves uniform fibrations (and in particular that for a simplicial set A and an uniform Kan complex B , the simplicial set B^A is a uniform Kan complex), as shown in [18], the uniform fibrations are not as well-behaved as in cubical sets, since they do not admit a classifier, essentially because representables are not closed under products in simplicial sets, unlike in cubical sets [37]. In summary, to date there is no known model of $\text{ML}(\text{UA})$ that is definable in a constructive setting and based on a category homotopically equivalent to that of topological spaces (even assuming classical logic).

Main results. A breakthrough has been obtained by the second author in [24], where, building on his previous work on weak model structures [23], it is shown constructively that the category of simplicial sets admits a Quillen model structure in which the fibrations are the Kan fibrations. Moreover, this model structure is shown to be cartesian and proper. Crucially for our goals here, in this model structure not all monomorphisms are cofibrations, but only those monomorphisms $i : A \rightarrow B$ that are levelwise complemented and for which the degeneracy of n -simplices is decidable in $B_n \setminus A_n$, for every $[n] \in \Delta$. In particular, not all simplicial sets are cofibrant, but only those with decidable degeneracies. However, this model structure coincides with the standard one as soon as the law of excluded middle is assumed. Two other proofs of the existence of this constructive model structure are obtained in [19].

The present paper builds on this work to obtain constructive counterparts of all the other main results of homotopy theory necessary to define the simplicial model of $\text{ML}(\text{UA})$. In particular, our main results are the following:

- Theorem 3.4, asserting that, for a fibration with cofibrant domain $p : A \rightarrow X$, the mapping path space gives a factorisation of the diagonal $\delta_p : A \rightarrow A \times_X A$ as a trivial cofibration followed by a fibration;
- Theorem 4.1, asserting that dependent product along fibrations with cofibrant domain preserves fibrations, so that, for a cofibrant simplicial set A and a Kan complex B , the exponential B^A is a Kan complex;
- Theorem 6.5, asserting the existence of a small fibration $\pi_c : \bar{U}_c \rightarrow U_c$ that classifies small fibrations between cofibrant objects;
- Theorem 7.4, asserting that the simplicial set U_c is a cofibrant Kan complex;
- Theorem 7.6, asserting that the fibration $\pi_c : \bar{U}_c \rightarrow U_c$ is univalent.

These results allow us to define a comprehension category

$$\begin{array}{ccc} \mathbf{Fib}_{\text{cof}} & \xrightarrow{\quad} & \mathbf{SSet}_{\text{cof}}^{\rightarrow} \\ & \searrow & \swarrow \text{cod} \\ & \mathbf{SSet}_{\text{cof}} & \end{array}$$

where $\mathbf{SSet}_{\text{cof}}$ is the category of cofibrant simplicial sets and $\mathbf{Fib}_{\text{cof}}$ is the category of fibrations between cofibrant simplicial sets, and to show that this comprehension category supports the type constructors of ML and a univalent type universe (Theorem 8.1). In this comprehension category, type-theoretic contexts correspond to cofibrant objects, while dependent types correspond to fibrations $p : A \rightarrow X$ where A and X are cofibrant. This choice is informed by the fact that, for a simplicial set X , the slice category $\mathbf{SSet}_{/X}$ admits a model structure in which the bifibrant objects are the fibrations $p : A \rightarrow X$ with A cofibrant. Our main results above show how this comprehension category supports the type constructors of ML and a univalent type universe. For example, Theorem 3.4 shows that identity types can be interpreted as mapping path spaces.

Novel aspects. The key novelty of our approach is the use of the homotopy-theoretic notion of cofibrancy to encapsulate the logical notion of decidability of degeneracies, so as to work with it in a mathematically efficient way. Also, to the best of our knowledge, this paper is the first to make use of the cofibrant replacement functor in the study of models of dependent type theory.

We will use the cofibrant replacement functor to obtain Π -types and the type-theoretic universe in our comprehension category. For Π -types, given a fibration $p : A \rightarrow X$ with cofibrant domain, the result of applying the dependent product along p to a fibration $q : B \rightarrow A$ with cofibrant domain produces a map $\Pi_p(q) : \Pi_A(B) \rightarrow X$, which is a fibration by Theorem 4.1, but whose domain is not necessarily cofibrant. In order to remedy this, we define Π -types to be given by a cofibrant replacement of $\Pi_p(q) : \Pi_A(B) \rightarrow X$, which is now a fibration with cofibrant domain. This definition validates a judgemental β -rule and a propositional η -rule (see Remark 4.2 for details), a combination that arises naturally when ML-type theory is presented within the Logical Framework [20, 34], as well as in the versions of the Coq assistant used for the original development of Univalent Foundations library [45]. In particular, Voevodsky’s proof that the UA implies the principle of function extensionality uses the propositional, rather than judgemental, η -rule for Π -types.

For the definition of a type-theoretic universe satisfying the UA, we proceed in two steps. First, we construct a small fibration $\pi : \bar{\mathbf{U}} \rightarrow \mathbf{U}$ that classifies small fibrations with cofibrant fibers and prove that \mathbf{U} is fibrant. Then, we consider a cofibrant replacement \mathbf{U}_c of \mathbf{U} , which comes equipped with a trivial fibration $\varepsilon : \mathbf{U}_c \rightarrow \mathbf{U}$, so that we obtain a small fibration $\pi_c : \bar{\mathbf{U}}_c \rightarrow \mathbf{U}_c$ with bifibrant codomain and cofibrant domain by pullback along ε . Our final result, Theorem 7.6, shows that $\pi : \bar{\mathbf{U}} \rightarrow \mathbf{U}$ and $\pi_c : \bar{\mathbf{U}}_c \rightarrow \mathbf{U}_c$ are univalent fibrations.

We refrain from claiming that our work provides a complete solution to the problem of defining a constructive simplicial model of type theory since the comprehension category we construct is not split and therefore does not satisfy strictly the stability conditions governing the interaction of substitution and type-formation rules, often referred to as the Beck-Chevalley conditions. While there are several results to address similar issues [13, 25 30, 32 40], none of them seems to apply in our context.

However, our results here reduce the problem of defining a constructive version of the simplicial model of Univalent Foundations to that of proving appropriate coherence result for a particu-

lar kind of comprehension category defined, which is independent of simplicial homotopy theory and we leave to future work (see Section 8 for details).

Let us conclude these introductory remarks by mentioning that although our main results are inspired by the existing literature, especially those in [30], their proofs require systematic and careful cofibrancy considerations. In particular, in order to exhibit identity types as path objects (Theorem 3.4) and to prove the weak equivalence extension property (Proposition 5.2), we will need to establish the non-trivial and surprising fact that the appropriate dependent products preserve cofibrancy (see Lemma 2.5 and Lemma 5.1 for details), a fact that does not hold in general. Furthermore, when discussing a universe, we are not interested in defining a small fibration that classifies small fibrations as in [30], but rather a small fibration that classifies small fibrations between cofibrant objects.

Outline of the paper. Section 1 recalls the constructive version of the Kan-Quillen model structure and some auxiliary results from [24]. Section 2 establishes some basic results on pullbacks. Section 3 shows how path spaces provide factorisations as trivial cofibration followed by fibration. Section 4 discusses the dependent product functor, proving the restricted Frobenius property. Section 5 establishes the weak equivalence extension property. Section 6 introduces a fibration $\pi_c : \bar{U}_c \rightarrow U_c$ that classifies small fibrations between cofibrant Kan complexes. Section 7 proves that U_c is bifibrant and that $\pi_c : \bar{U}_c \rightarrow U_c$ is univalent, using the weak equivalence extension property. We conclude in Section 8 by summarising our main results in terms of the comprehension category and stating some open problems for future research.

Remarks on constructivity. We work in Constructive Zermelo-Fraenkel (CZF) set theory, a set theory based on intuitionistic logic [2] which does not include the Power Set axiom. In CZF, the Exponentiation axiom (asserting that for any two sets A and B , the class of set functions from A to B is again a set) is provable, thanks to the Subset Collection axiom. No other application of Subset Collection axiom is used here. When discussing smallness of sets, simplicial sets and fibrations, we assume the existence of an inaccessible set u , as defined in [3, Definition 18.1.2]. We say that a set is small if it is an element of u . For the closure of the type universe under Π -types, we will assume a form of ‘propositional resizing’, asserting that every subset of a small set is again small, that is, $\forall a, b (a \subseteq b \wedge b \in u \rightarrow a \in u)$. A discussion of what can be done without this assumption is in Remark 8.3.

Throughout the paper, we use a slight abuse of language and say that a map $f : X \rightarrow Y$ has the right lifting property with respect to a given set of maps to mean that we have a pair consisting of $f : X \rightarrow Y$ and a function assigning diagonal fillers to the corresponding set of lifting problems, with no uniformity conditions. In particular, fibrations and trivial fibrations will always be treated as maps equipped with additional structure. Morphisms between such maps will always be taken to be mere commutative squares, with no compatibility conditions with respect to the chosen diagonal fillers. In contrast, cofibrations and trivial cofibrations will be simply maps satisfying suitable properties.

1 | PRELIMINARIES

We write Δ for the simplicial category. The objects of Δ are written as $[n]$, for $n \geq 0$. We write $\mathbf{SSet} =_{\text{def}} [\Delta^{\text{op}}, \mathbf{Set}]$ for the category of simplicial sets. For $n \geq 0$, $\Delta[n] \in \mathbf{SSet}$ is the representable

simplicial set associated to $[n] \in \Delta$. Given a map $f : Y \rightarrow X$ in \mathbf{SSet} , we write $f^* : \mathbf{SSet}_{/X} \rightarrow \mathbf{SSet}_{/Y}$ for the associated pullback functor. The functor f^* has a left adjoint, defined by composition, which we write $\Sigma_f : \mathbf{SSet}_{/Y} \rightarrow \mathbf{SSet}_{/X}$ and refer to as the *dependent sum* along f . Since \mathbf{SSet} is locally cartesian closed, pullback along f has also a right adjoint, which we write $\Pi_f : \mathbf{SSet}_{/Y} \rightarrow \mathbf{SSet}_{/X}$ and refer to as the *dependent product* along f . The action of these functors on a map $g : Z \rightarrow Y$ will be written $\Sigma_f(g) : \Sigma_Y(Z) \rightarrow X$ and $\Pi_f(g) : \Pi_Y(Z) \rightarrow X$, respectively. Since Σ_f is defined by composition, $\Sigma_Y(Z) =_{\text{def}} Z$ and $\Sigma_f(g) = fg$.

As a special case of a well-known result for presheaf categories, for every $[n] \in \Delta$ there is an equivalence of categories

$$\mathbf{SSet}_{/\Delta[n]} \simeq [\Delta_{/[n]}^{\text{op}}, \mathbf{Set}]. \tag{1.1}$$

For $F : \Delta_{/[n]}^{\text{op}} \rightarrow \mathbf{Set}$, we write $\pi_F : \int F \rightarrow \Delta[n]$ for the corresponding object of $\mathbf{SSet}_{/\Delta[n]}$. Here, $\int F$ is the simplicial set whose m -simplices are pairs (θ, x) where $\theta : [m] \rightarrow [n]$ is a map in Δ and $x \in F(\theta)$. The components of the natural transformation π_F are the first projections.

For $[n] \in \Delta$, we write $i^n : \partial\Delta[n] \rightarrow \Delta[n]$ for the boundary inclusion into the n -simplex and, for $1 \leq i \leq n$, $h_n^i : \Lambda^i[n] \rightarrow \Delta[n]$ for the i -th horn inclusion into the n -simplex. The simplicial set $\Delta[1]$ is an interval object in \mathbf{SSet} , with endpoint inclusions $\delta^k : \{k\} \rightarrow \Delta[1]$ defined by $\delta^k =_{\text{def}} h_k^1$. Throughout this paper, we shall work with the constructive Kan-Quillen model structure $(\text{Weq}, \text{Cof}, \text{Fib})$ on \mathbf{SSet} defined in [24]. For the convenience of the reader, we recall the main aspects of this model structure below. For this, let $\text{TrivCof} =_{\text{def}} \text{Weq} \cap \text{Cof}$ and $\text{TrivFib} =_{\text{def}} \text{Weq} \cap \text{Fib}$ be the classes of trivial cofibrations and trivial fibrations, respectively.

The weak factorisation system $(\text{Cof}, \text{TrivFib})$ of cofibrations and trivial fibrations is cofibrantly generated by the set $\mathcal{I} =_{\text{def}} \{i^n : \partial\Delta[n] \rightarrow \Delta[n] \mid n \geq 0\}$ of boundary inclusions, that is,

$$(\text{Cof}, \text{TrivFib}) = (\text{Sat}(\mathcal{I}), \mathcal{J}^{\text{th}}).$$

As shown in [23, Proposition 5.1.4] a map $f : Y \rightarrow X$ is a cofibration if and only if it is a levelwise complemented monomorphism and the degeneracy of the simplices of $X_n \setminus Y_n$ is decidable for every $[n] \in \Delta$. Thus, a simplicial set X is cofibrant if degeneracy of the simplices of X is decidable. Note that a map between cofibrant objects is a cofibration if and only if it is a levelwise complemented monomorphism. Cofibrant simplicial sets are of particular importance for our development because of their decidability property, which can be used to establish counterparts of classical results valid for all simplicial sets. An example is the Eilenberg-Zilber lemma [23, Lemma 5.1.2], asserting that in a cofibrant simplicial set X , any cell $x \in X$ can be written uniquely as $p^*(y)$, where y is a non-degenerate cell of X and p is a degeneracy.

The weak factorisation system $(\text{TrivCof}, \text{Fib})$ of trivial cofibrations and fibrations is cofibrantly generated by the set $\mathcal{J} =_{\text{def}} \{h_n^k : \Lambda^k[n] \rightarrow \Delta[n] \mid 0 \leq k \leq n\}$ of horn inclusions, that is,

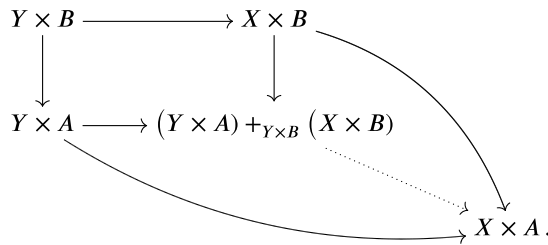
$$(\text{TrivCof}, \text{Fib}) = (\text{Sat}(\mathcal{J}), \mathcal{J}^{\text{th}}).$$

For a map $f : Y \rightarrow X$, we denote the action of the pullback f^* on a fibration $p : A \rightarrow X$ as

$$\begin{array}{ccc} A[f] & \longrightarrow & A \\ p[f] \downarrow & \lrcorner & \downarrow p \\ Y & \xrightarrow{f} & X. \end{array}$$

For $X \in \mathbf{SSet}$, we write $\mathbf{Fib}/_X$ for the category whose objects are fibrations with codomain X and whose maps are determined by saying that the functor $U : \mathbf{Fib}/_X \rightarrow \mathbf{SSet}/_X$ forgetting the fibration structure is full and faithful. We then write $\mathbf{BFib}/_X$ for the full subcategory of $\mathbf{Fib}/_X$ spanned by fibrations with cofibrant domain. For a simplicial set X , we write $\mathbb{L}(X)$ for its cofibrant replacement and $\mathbb{R}(X)$ for its fibrant replacement. These objects come equipped with a trivial fibration $\varepsilon_X : \mathbb{L}(X) \rightarrow X$ and a trivial cofibration $\eta_X : X \rightarrow \mathbb{R}(X)$, respectively. An explicit definition of the cofibrant replacement is given in the Appendix.

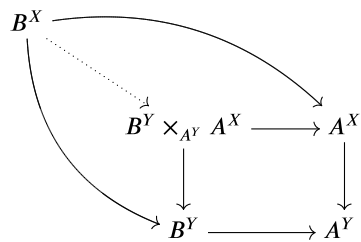
The model structure (Weq, Cof, Fib) is proper, that is, both left and right proper [24, Propositions 2.2.9 and 3.5.2] and the weak factorisation systems (Cof, TrivFib) and (TrivCof, Fib) satisfy the so-called pushout product property [23, Proposition 5.1.5 and Corollary 5.2.3]. Recall that, given two maps $f : Y \rightarrow X$ and $g : B \rightarrow A$, their *pushout product* $f \hat{\times} g$ is defined as the unique dotted map in the diagram



The pushout-product property is the statement that if f and g are cofibrations then so is $f \hat{\times} g$ and, if additionally either f and g is a weak equivalence, then so is $f \hat{\times} g$. Note that when f and g are monomorphisms, so in particular when they are cofibrations, the pushout in the diagram above is just an union of subobjects and the pushout product of f and g is just the inclusion

$$f \hat{\times} g : (Y \times A) \cup (X \times B) \rightarrow X \times A.$$

Dually, using exponentials instead of products and pullbacks instead of pushouts, for maps $f : Y \rightarrow X$ and $p : B \rightarrow A$, the *pullback exponential* $\langle f, p \rangle$ is defined as the unique dotted arrow in the diagram:



By adjointness (see [28] for details), $\langle p, q \rangle$ has the right lifting property against a map f if and only if q has the right lifting property against $f \hat{\times} p$. Therefore, the pushout-property implies its dual version, asserting that if $f : Y \rightarrow X$ is a cofibration and $p : B \rightarrow A$ a fibration then $\langle f, p \rangle$ is a fibration and if, additionally, either p or f is a weak equivalence, then so so is $\langle f, p \rangle$.

We also make use of another weak factorisation system (L, R) on \mathbf{SSet} introduced in [24, Section 3.1]. This is useful to establish decidability conditions; in particular, we will use it to

prove Proposition 3.3 and Lemma 5.1 below. By definition, (L, R) is the weak factorisation system cofibrantly generated by the set of degeneracy maps $\sigma : \Delta[m] \rightarrow \Delta[n]$, that is, the maps induced by surjections $[m] \rightarrow [n]$ in Δ . We refer to maps in L as the *degeneracy quotients* and the maps in R as the *degeneracy-detecting* maps. Since degeneracy maps are (split) epimorphisms in \mathbf{SSet} , the weak factorisation system (L, R) is actually a unique factorisation system. Since the degeneracy-detecting maps are the maps with the (unique) right lifting property against degeneracy map, they are exactly the simplicial morphisms $f : X \rightarrow Y$ such that for $x \in X$, $f(x)$ is degenerated if and only if x is degenerated. The degeneracy quotient maps are instead the pushouts of coproducts of degeneracy maps, that is, the maps of the form $X \rightarrow X[(x_i, \sigma_i)]$ where $x_i \in X_{n-1}$ are a family of cells, $\sigma : [n_i] \rightarrow [m_i]$ a family of degeneracies and $X[(x_i, \sigma_i)]$ is obtained from X by freely making x_i in the image of σ_i for each i . For the convenience of the reader, we recall from [24] the results about this factorisation system that will be used here.

Here and throughout the paper, a *finite set* means a set equipped with a bijection with the set $\{1, \dots, n\}$, for some $n \in \mathbb{N}$. Equality in such a set is always decidable, which we sometimes stress by saying ‘finite decidable set’, rather than just ‘finite set’, so as to distinguish this notion from other notions of finiteness, like Kuratowski finiteness. We say that a simplicial set X is finite if X_n is a finite decidable set for each n and X has a finite number of non-degenerate cells in total.

Lemma 1.1 [24, Lemma 3.1.8]. *Let $p : A \rightarrow B$ be a degeneracy quotient between finite decidable simplicial set, X a cofibrant simplicial set and $f : A \rightarrow X$ be any morphism. Then it is decidable whether f can be factored through p or not.*

The proof of this result above can be outlined in a few words. Since $B = A[(a_i, \sigma_i)]$ for a finite collection of cells a_i , f factors through p if and only if $f(a_i) = \sigma_i^* x_i$ for each i , but when X is cofibrant, one can decide for each i if this is the case or not. As there is only a finite number of such i , one can decide if it is the case for all i or not. See

Proposition 1.2 [24, Proposition 3.1.11]. *The class of degeneracy quotients is stable under pullback.*

By Proposition 1.2, if $f : A \rightarrow B$ is a degeneracy quotient then $f \times X : A \times X \rightarrow B \times X$ is again a degeneracy quotient for every X .

2 | THE COMPREHENSION CATEGORY OF COFIBRANT SIMPLICIAL SETS

The main goal of this section is to introduce the comprehension category of cofibrant simplicial sets. The results of simplicial homotopy theory that we obtain in the following sections will show how this comprehension category supports various type-theoretic constructions. We refer to [27] for the definition of a comprehension category and basic results and to [32] for the definitions of the categorical counterparts of the type-theoretic constructs.

Lemma 2.1.

- (i) *Let A, B be cofibrant simplicial sets. Then their product $A \times B$ is cofibrant.*

(ii) Let $f : A \rightarrow X$ and $g : B \rightarrow X$ be maps with cofibrant domain. Then their fiber product $A \times_X B$, fitting in the pullback diagram

$$\begin{array}{ccc}
 A \times_X B & \xrightarrow{q} & B \\
 \downarrow p & \lrcorner & \downarrow g \\
 A & \xrightarrow{f} & X,
 \end{array}$$

is cofibrant.

Proof. Part (i) follows immediately from the fact that the model structure on simplicial sets is cartesian, that is, it satisfies the pushout-product conditions.

For part (ii), $A \times_X B$ is a sub-simplicial set of $A \times B$ hence a cell of $A \times_X B$ is degenerate if and only if it is degenerate as a cell of $A \times B$, hence degeneracy in $A \times_X B$ is indeed decidable. \square

The next proposition introduces the comprehension category cofibrant simplicial sets. Recall that we write $\mathbf{SSet}_{\text{cof}}$ for the full subcategory of \mathbf{SSet} spanned by cofibrant simplicial sets. We then define $\mathbf{Fib}_{\text{cof}}$ to be the category that has fibrations between cofibrant objects as objects and commutative squares as maps. Here, recall from the Introduction that we take fibrations to be pairs consisting of a map and a function providing fillers for lifting problems against horn inclusions.

Proposition 2.2. *The category $\mathbf{SSet}_{\text{cof}}$ is the base of a comprehension category of the form*

$$\begin{array}{ccc}
 \mathbf{Fib}_{\text{cof}} & \xrightarrow{\chi} & \mathbf{SSet}_{\text{cof}}^{\rightarrow} \\
 \downarrow p & & \swarrow \text{cod} \\
 & & \mathbf{SSet}_{\text{cof}},
 \end{array}$$

where χ is the evident forgetful functor.

Proof. The category $\mathbf{SSet}_{\text{cof}}^{\rightarrow}$ is the arrow category of $\mathbf{SSet}_{\text{cof}}$, whose objects are maps between cofibrant objects. Then, the codomain functor

$$\text{cod} : \mathbf{SSet}_{\text{cof}}^{\rightarrow} \rightarrow \mathbf{SSet}_{\text{cof}}$$

is a Grothendieck fibration by part (ii) of Lemma 2.1. We then define $p = \text{cod} \circ \chi$, which is also a Grothendieck fibration, since the pullback of a fibration (with chosen lifts) is a fibration (with chosen lifts). Since χ is the forgetful functor, it clearly preserves pullbacks. \square

Remark 2.3. It is immediate to see that the comprehension category of cofibrant simplicial sets supports Σ -types. since for a fibration $p : A \rightarrow X$ between cofibrant objects, dependent sum along p functor $\Sigma_p : \mathbf{SSet}_{/A} \rightarrow \mathbf{SSet}_{/X}$, which is defined by composition with p , maps fibrations to fibrations.

We conclude this section with some results on cofibrant objects and cofibrations that will be useful in the following.

Proposition 2.4. *Let $p : A \rightarrow X$ be a map with cofibrant domain. Then pullback along p ,*

$$p^* : \mathbf{SSet}_{/X} \rightarrow \mathbf{SSet}_{/A},$$

preserves cofibrations.

Proof. Let $f : Y \rightarrow X$ be a cofibration and consider the pullback

$$\begin{array}{ccc} A[f] \twoheadrightarrow A & & \\ p[f] \downarrow \lrcorner & \square & \downarrow p \\ Y \twoheadrightarrow X & \xrightarrow{f} & \end{array}$$

We need to show that the monomorphism $A[f] \twoheadrightarrow A$ is a cofibration. Since A is cofibrant, it suffices to show that it is a levelwise complemented. For $a \in A_n$, we have that $a \in A[f]_n$ if and only if $p(a) \in Y_n$. Since $f : Y \twoheadrightarrow X$ is a levelwise complemented monomorphism, this is decidable. \square

Lemma 2.5.

- (i) *Let X be cofibrant and K finite decidable simplicial set. Then X^K is cofibrant.*
- (ii) *Let $f : A \rightarrow B$ be a degeneracy quotient of finite decidable simplicial sets and X cofibrant. Then $X^f : X^B \rightarrow X^A$ is a cofibration between cofibrant objects.*

Proof. For part (i), recall that an n -cell $\Delta[n] \rightarrow X^K$ is a morphism $\Delta[n] \times K \rightarrow X$. Now let $\sigma : \Delta[n] \rightarrow \Delta[m]$ be a degeneracy map. Then the map $\sigma \times K : \Delta[n] \times K \rightarrow \Delta[m] \times K$ is again degeneracy quotient. Since it is a map between finite decidable simplicial sets, the question of whether a map $\Delta[n] \times K \rightarrow X$ factors through $\Delta[n] \times K \rightarrow \Delta[m] \times K$ is decidable by Lemma 1.1. But this is exactly the question of degeneracy of cells in X^K . Similarly, for part (ii), since an n -cell of X^A is a morphism $\Delta[n] \times A \rightarrow X$, it belongs to X^B if and only if it can be factored as $\Delta[n] \times A \rightarrow \Delta[n] \times B \rightarrow X$. This is decidable by Lemma 1.1 because $\Delta[n] \times A \rightarrow \Delta[n] \times B$ is a degeneracy quotient between finite decidable simplicial sets by part (i). \square

3 | IDENTITY TYPES AS PATH SPACES

In this section we begin to show how the comprehension category of cofibrant simplicial sets introduced in Proposition 2.2 supports various type-theoretic constructs by considering identity types. Following the fundamental insight in [5], in order to equip the comprehension category with identity types it suffices to show that for every fibration $p : A \rightarrow X$ with cofibrant domain, there are a fibration with cofibrant domain $\partial : \text{Id}_A \rightarrow A \times_X A$ and a trivial cofibration $\text{refl}_A : A \rightarrow \text{Id}_A$ that provide a factorisation of the diagonal map $\delta_p : A \rightarrow A \times_X A$ [32]. While such a factorisation is guaranteed to exist by the model structure on \mathbf{SSet} , here we show that it can be obtained by letting Id_A be a mapping path space. This is useful in order to support the intuition of elements of identity types as paths, which is a central idea in Homotopy Type Theory [43], to ensure pullback stability of the construction, as well as preservation of (fibrewise) smallness.

Lemma 3.1.

- (i) Let X be cofibrant and A be fibrant. Then A^X is fibrant.
- (ii) Let $f : Y \rightarrow X$ be a cofibration and A be fibrant. Then $A^f : A^X \rightarrow A^Y$ is a fibration.
- (iii) Let $f : Y \rightarrow X$ be a trivial cofibration and A be fibrant. Then $A^f : A^X \rightarrow A^Y$ is a trivial fibration.
- (iv) Let X be cofibrant and $p : B \rightarrow A$ be a (trivial) fibration. Then $p^X : B^X \rightarrow A^X$ is also a (trivial) fibration.

Proof. The claims follow easily from the pushout product property of the model structure. □

Part (i) of Lemma 3.1 can be proved constructively also by following the combinatorial proof in [33], using the decidability of degeneracy in X given by the cofibrancy assumption to avoid appealing to the law of excluded middle.

For a simplicial set A , we define its path object by letting $\text{Path}(A) =_{\text{def}} A^{\Delta[1]}$. There is evident boundary map $\partial : \text{Path}(A) \rightarrow A \times A$, giving the endpoints of a path. We write $\partial_0, \partial_1 : \text{Path}(A) \rightarrow A$ for the compositions of ∂ with the two projections and $r : A \rightarrow \text{Path}(A)$ for the ‘constant path’ map.

Proposition 3.2. *Let A be fibrant.*

- (i) The object $\text{Path}(A)$ is fibrant.
- (ii) The boundary map $\partial : \text{Path}(A) \rightarrow A \times A$ is a fibration.
- (iii) The composite of $\partial : \text{Path}(A) \rightarrow A \times A$ with either projection is a trivial fibration.
- (iv) The map $r : A \rightarrow \text{Path}(A)$ induced by the unique map $\Delta[1] \rightarrow \Delta[0]$ is a weak equivalence.

Proof. Part (i) is just a special case of part (i) of Lemma 3.1. For part (ii), apply part (ii) of Lemma 3.1 to the cofibration $i^1 : \partial\Delta[1] \hookrightarrow \Delta[1]$. For part (iii), apply part (iii) of Lemma 3.1 to the horn inclusions $h_n^k : \Lambda^k[1] \rightarrow \Delta[1]$. Part (iv) follows from the 3-for-2 property for weak equivalences applied to $A \rightarrow A^{\Delta[1]} \rightarrow A$. Indeed, the composite is the identity and the second factor has just been proved to be a trivial fibration. □

In order to interpret identity types, we need $\text{Path}(A)$ to be cofibrant and the map $r : A \rightarrow \text{Path}(A)$ to be a trivial cofibration. This is achieved by the following proposition.

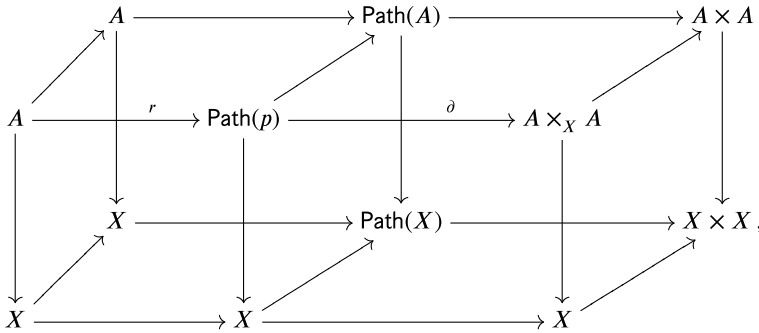
Proposition 3.3. *Let A be cofibrant. Then $\text{Path}(A)$ is cofibrant and the map $r : A \rightarrow \text{Path}(A)$ is a cofibration.*

Proof. This follows from part (iii) of Lemma 2.5, applied to the cofibrant simplicial set X , and the degeneracy map between finite simplicial sets $\Delta[1] \rightarrow \Delta[0]$. □

We now define mapping path spaces and extend Proposition 3.2 and Proposition 3.3. Given a map $p : A \rightarrow X$, we define $\text{Path}(p)$ via the pullback diagram

$$\begin{array}{ccc}
 \text{Path}(p) & \longrightarrow & \text{Path}(A) \\
 \downarrow & \lrcorner & \downarrow \\
 X & \xrightarrow{r_X} & \text{Path}(X).
 \end{array}$$

The structural maps $r_p : A \rightarrow \text{Path}(p)$ and $\partial : \text{Path}(p) \rightarrow A \times_X A$ are produced by the diagram:



where the three square in the vertical/diagonal direction are pullbacks. As before, the maps $\partial_0, \partial_1 : \text{Path}(A) \rightarrow A$ are defined as the composites of ∂ with the two projections.

Theorem 3.4. *Assume that $p : A \rightarrow X$ is a fibration between cofibrant objects Then,*

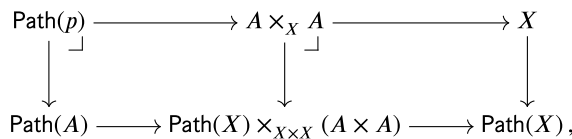
- (i) $\text{Path}(p)$ is cofibrant,
- (ii) the map $\text{Path}(p) \rightarrow X$ is a fibration,
- (iii) the map $\partial : \text{Path}(p) \rightarrow A \times_X A$ is a fibration,
- (iv) $\partial_k : \text{Path}(p) \rightarrow A \times_X A$ is a trivial fibration, for $k \in \{0, 1\}$,
- (v) the map $r_p : A \rightarrow \text{Path}(p)$ is a trivial cofibration.

Proof. The map $\text{Path}(p) \rightarrow X$ is a pullback of the maps $\text{Path}(A) \rightarrow \text{Path}(X)$ along $X \rightarrow \text{Path}(X)$. Hence, since $\text{Path}(A) \rightarrow \text{Path}(X)$ is a fibration (by part (iv) of Lemma 3.1), the map $\text{Path}(p) \rightarrow X$ is a fibration. Since X is cofibrant by assumption and $\text{Path}(A)$ is cofibrant by Proposition 3.3, we have that $\text{Path}(p)$ is also cofibrant by Lemma 2.1.

By the dual of the pushout-product property, the map $\langle \partial \Delta[n] \hookrightarrow \Delta[n], A \rightarrow X \rangle$ is a fibration. This map is

$$\text{Path}(A) \rightarrow (A \times A) \times_{X \times X} \text{Path}(X).$$

Moreover, in the diagram



the right-hand square is easily seen to be a pullback and the total rectangle is the pullback defining $\text{Path}(p)$, hence the left-hand square is also a pullback. Since the bottom left map is a fibration, $\text{Path}(p) \rightarrow A \times_X A$ is a fibration as well.

By a similar argument, for $k \in \{0, 1\}$, the map $\langle \Lambda^k[n] \hookrightarrow \Delta[n], A \rightarrow X \rangle$ is a trivial fibration. Indeed, this is the map $q : \text{Path}(A) \rightarrow A \times_X \text{Path}(X)$ which fits into the pullback diagrams

$$\begin{array}{ccccc}
 \text{Path}(p) & \longrightarrow & A & \longrightarrow & X \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
 \text{Path}(A) & \xrightarrow{q} & A \times_X & \longrightarrow & \text{Path}(X)
 \end{array}$$

This shows that the canonical maps $\partial_k : \text{Path}(p) \rightarrow A$ are trivial fibrations.

We conclude by showing that the map $A \rightarrow \text{Path}(p)$ is levelwise complemented. Indeed, it fits into a factorisation

$$A \rightarrow \text{Path}(p) \rightarrow \text{Path}(A)$$

of a map which has been proved to be a levelwise complemented inclusion in Proposition 3.3. Therefore, for any cell of $\text{Path}(p)$ one can decide if it is in A or not by considering it as a cell of $\text{Path}(A)$. Since A and $\text{Path}(p)$ are cofibrant, this shows that $A \rightarrow \text{Path}(p)$ is a cofibration. The 3-for-2 property for weak equivalences applied to $A \rightarrow \text{Path}(p) \rightarrow A$ shows that $A \rightarrow \text{Path}(p)$ is also a weak equivalence, and hence a trivial cofibration. \square

4 | Π -TYPES VIA COFIBRANT REPLACEMENTS

The aim of this section is to prove the results of simplicial homotopy theory necessary to show that the comprehension category of cofibrant simplicial sets supports Π -types. In order to do this, we should consider a fibration with cofibrant domain $p : A \rightarrow X$ and define an operation mapping a fibration with cofibrant domain $q : B \rightarrow A$ to a new fibration with cofibrant domain $\tilde{\Pi}_p(q) : \tilde{\Pi}_A(B) \rightarrow X$ together with additional data [32]. Given such a map $q : B \rightarrow A$, we proceed in two steps. First, we apply the dependent product along p ,

$$\Pi_p : \mathbf{SSet}_{/A} \rightarrow \mathbf{SSet}_{/X},$$

to $q : B \rightarrow A$ and obtain a map $\Pi_p(q) : \Pi_A(B) \rightarrow X$, which we will show to be again a fibration. Since this fibration does not seem to have cofibrant domain in general, we then apply a cofibrant replacement in $\mathbf{SSet}_{/X}$ to $\Pi_p(q) : \Pi_A(B) \rightarrow X$ so as to obtain a map with all the desired properties. The resulting map will support a categorical counterpart of the propositional η -rule for Π -types.

We begin by showing that, for a fibration $p : A \rightarrow X$ with cofibrant domain, the dependent product along p preserves fibrations. By adjointness, this is equivalent to showing that its left adjoint, that is, pullback along f , preserves trivial cofibrations. This amounts to proving a restricted version of the Frobenius property [8], obtained by considering pullbacks along fibrations with cofibrant domain rather than general fibrations. By the obstruction results in [10] these cofibrancy assumptions are essential in our constructive setting.

Theorem 4.1. *Let $p : A \rightarrow X$ be a fibration with cofibrant domain,*

- (i) *The pullback functor $p^* : \mathbf{SSet}_{/X} \rightarrow \mathbf{SSet}_{/A}$ preserves trivial cofibrations.*
- (ii) *The dependent product $\Pi_p : \mathbf{SSet}_{/A} \rightarrow \mathbf{SSet}_{/X}$ preserves fibrations.*

Proof. Part (i) follows from Proposition 2.4, which shows that p^* preserves cofibrations, together with the right properness of the simplicial model structure which shows that it preserves weak equivalences. Part (ii) follows from part (i) as Π_p is right adjoint to p^* . \square

Another proof of Theorem 4.1 is given in [19, Section 4] modifying appropriately the arguments in [18].

Remark 4.2. We define explicitly how the Π -types of the comprehension category of cofibrant simplicial sets are defined. For this, let us recall that, for maps $p : A \rightarrow X$ and $q : B \rightarrow A$, the dependent product $\Pi_p(q) : \Pi_A(B) \rightarrow X$ is equipped with a map

$$\text{app} : \Pi_A(B) \times_A A \rightarrow B$$

in $\mathbf{SSet}/_A$ which is universal in the sense that, for every $Y \rightarrow X$, the function

$$\begin{aligned} \mathbf{SSet}/_X[Y, \Pi_A(B)] &\longrightarrow \mathbf{SSet}/_A[Y \times_A A, B] \\ f &\longmapsto \text{app}(f \times_A 1_A) \end{aligned}$$

is a bijection. This means that we have a function λ in the opposite direction such that

$$\text{app}(\lambda(b) \times_A 1_A) = b, \quad \lambda(\text{app}(f \times_A 1_A)) = f, \tag{4.1}$$

for every $b : Y \times_A A \rightarrow B$ and $f : Y \rightarrow \Pi_A(B)$. These equations correspond to the well-known judgemental β -rule and η -rule for Π -types, respectively.

When p and q are fibrations and A is cofibrant, the map $\Pi_p(q) : \Pi_A(B) \rightarrow X$ is a fibration by Theorem 4.1 but $\Pi_A(B)$ is not cofibrant, in general. Thus, we interpret Π -types as the cofibrant replacement of $\Pi_A(B)$, which is given by a cofibrant simplicial set $\mathbb{L}(\Pi_A(B))$ equipped with a trivial fibration $\varepsilon : \mathbb{L}(\Pi_A(B)) \rightarrow \Pi_A(B)$. We then define $\widetilde{\text{app}} : \mathbb{L}(\Pi_A(B)) \times_A A \rightarrow B$ as the composite

$$\widetilde{\text{app}} =_{\text{def}} \text{app} \circ (\varepsilon \times_A 1_A).$$

For a bifibrant simplicial set Y and maps $Y \rightarrow X, b : Y \times_A A \rightarrow B$, we define $\widetilde{\lambda}(b) : Y \rightarrow \mathbb{L}(\Pi_A(B))$ to be the diagonal filler

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{L}(\Pi_A(B)) \\ \downarrow & \nearrow & \downarrow \varepsilon \\ Y & \xrightarrow{\lambda(b)} & \Pi_A(B), \end{array} \tag{4.2}$$

which exists since Y is cofibrant and ε is a trivial fibration. It follows that $\widetilde{\text{app}}(\widetilde{\lambda}(b) \times_A 1_A) = b$, so the β -rule holds as an equality. Instead, for $f : X \rightarrow \mathbb{L}(\Pi_A(B))$, we have a homotopy

$$\eta_f : \widetilde{\lambda}(\widetilde{\text{app}}(f \times_A 1_A)) \sim f,$$

which is constructed as the diagonal filler in the following diagram:

$$\begin{array}{ccc} \partial\Delta[1] \times Y & \xrightarrow{[f, \widetilde{\lambda}(\widetilde{\text{app}}(f \times_A 1_A))]} & \mathbb{L}(\Pi_A(B)) \\ \downarrow & \nearrow & \downarrow \varepsilon \\ \Delta[1] \times Y & \longrightarrow & \Pi_A(B), \end{array}$$

where the bottom map is the constant homotopy given by the equality in the η -rule in (4.1).

This result will allow us to equip the comprehension category of cofibrant simplicial sets Proposition 2.2 with weakly stable Π -types. We will take the Π -types in the comprehension category to be given by the objects $\mathbb{L}(\Pi_A(B))$. These will satisfy stability properties that are weaker than those satisfied by $\Pi_A(B)$ in the classical simplicial model, essentially the same as those enjoyed by Id -types in homotopy-theoretic models [5], which are weaker even than those considered in [11].

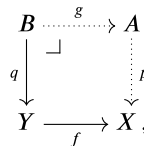
Let us also note that the requirement of cofibrancy of Y is used in an essential way to construct the diagonal filler in (4.2). For this reason, it seems impossible to obtain an analogous result for the comprehension category with base the category of all simplicial sets and total category that of Kan fibrations with cofibrant domain.

5 | THE WEAK EQUIVALENCE EXTENSION PROPERTY

The main goal of this section is to prove the so-called weak equivalence extension property, which will be the key to prove the univalence of the classifying fibrations considered in Sections 6, 7. For this, we follow closely the approach in [30], but exploiting crucially the cofibrancy requirements that are part of our set-up.

Lemma 5.1. *Let $f : Y \rightarrow X$ be a fibration between cofibrant objects.*

- (i) *The dependent product along f , $\Pi_f : \mathbf{SSet}_{/Y} \rightarrow \mathbf{SSet}_{/X}$, preserves trivial fibrations.*
- (ii) *The counit of the adjunction $f^* \dashv \Pi_f$ is a natural isomorphism.*
- (iii) *If $q : B \rightarrow Y$ is a map with cofibrant domain, then $\Pi_f(q) : \Pi_Y(B) \rightarrow X$ is so.*
- (iv) *Trivial fibrations extend along f , that is, given a trivial fibration $q : B \rightarrow Y$ as in the solid diagram*



then there exists a trivial fibration $p : A \rightarrow X$ which fits in the dotted pullback square above. Moreover if B is cofibrant then A can be chosen to be cofibrant as well.

Proof. We prove the different parts separately.

For part (i), since the dependent product functor Π_f is right adjoint to the pullback functor f^* and trivial fibrations are the maps with the right lifting property with respect to cofibrations, Π_f preserves trivial fibrations if and only if f^* preserves cofibrations. But this follows from Proposition 2.4.

For part (ii), since f is a monomorphism, $\Sigma_f : \mathbf{SSet}_{/X} \rightarrow \mathbf{SSet}_{/Y}$ is fully faithful and hence the unit $\eta : 1 \rightarrow f^* \Sigma_f$ is an isomorphism. By adjointness, the counit $\varepsilon : f^* \Pi_f \rightarrow 1$ is also an isomorphism.

For part (iii), let $y : \Delta[n] \rightarrow \Pi_Y(B)$ be a k -cell. We will show that for a given degeneracy $\sigma : [n] \rightarrow [k]$ it is decidable if y is ‘ σ -degenerated’, that is, if y factors through $\sigma : \Delta[n] \rightarrow \Delta[k]$. As degeneracy is decidable in X , one can freely assume that the image of y in X via $\Pi_f(q)$ is σ -degenerated (as if it is not the case, y is not σ -degenerated), hence one has the solid diagram

$$\begin{array}{ccc}
 \Delta[n] & \longrightarrow & \Pi_Y(B) \\
 \sigma \downarrow & \nearrow ? & \downarrow \\
 \Delta[k] & \longrightarrow & X .
 \end{array}$$

Because of the adjunction between dependent product and the pullback along f , the existence of a lift as above is equivalent to the existence of a lift in the diagram

$$\begin{array}{ccc}
 f^* \Delta[n] & \xrightarrow{x} & B \\
 f^* \sigma \downarrow & \nearrow ? & \downarrow \\
 f^* \Delta[k] & \longrightarrow & Y .
 \end{array}$$

The objects $f^* \Delta[n]$ and $f^* \Delta[k]$ are decidable simplicial subset of $\Delta[n]$ and $\Delta[k]$ because f is itself a levelwise complemented monomorphism, hence both of them are finite decidable simplicial sets. The map $f^* \sigma$ is degeneracy quotient (by Proposition 1.2), hence Lemma 1.1 shows that the existence of such a lift is decidable (as degeneracy quotients are epimorphisms, the existence of lift making the upper triangle commutes is equivalent to the existence of lift making the square commute).

For part (iv), given a trivial fibration $q : B \rightarrow Y$, define $p : A \rightarrow X$ to be $\Pi_f(q) : \Pi_Y(B) \rightarrow Y$. This map is a trivial fibration by part (i) and the square is a pullback by part (ii). The final remark about the cofibrancy of A follows from part (iii). □

Proposition 5.2 (Weak equivalence extension property). *Let*

$$\begin{array}{ccc}
 B & \xrightarrow{g} & A \\
 q \downarrow & & \downarrow p \\
 Y & \xrightarrow{f} & X
 \end{array}$$

be a commutative diagram where $p : A \rightarrow X$ and $q : B \rightarrow Y$ are fibrations with cofibrant domains and $f : Y \rightarrow X$ is a cofibration between cofibrant objects. Assume that the map $u : B \rightarrow A[f]$ defined by $u =_{\text{def}} (q, g)$, fitting the diagram of solid maps

$$\begin{array}{ccccc}
 B & \xrightarrow{\quad} & \bar{B} & \xrightarrow{\quad} & A \\
 q \downarrow & \searrow u & \downarrow \bar{q} & \searrow v & \downarrow p \\
 & & A[f] & \xrightarrow{\quad} & A \\
 & & \downarrow & & \downarrow \\
 Y & \xrightarrow{f} & X & & X
 \end{array}$$

is a weak equivalence in $\mathbf{SSet}_{/Y}$. Then there exist a fibration $\bar{q} : \bar{B} \rightarrow X$, a weak equivalence $v : \bar{B} \rightarrow A$ in $\mathbf{SSet}_{/X}$ and a map $B \rightarrow \bar{B}$ such that both of the other squares in the diagram above are pullbacks.

Proof. We define the required object \bar{B} as the following pullback:

$$\begin{array}{ccc} \bar{B} & \longrightarrow & \Pi_Y(B) \\ \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{\eta_A} & \Pi_Y(A[f]), \end{array}$$

where η_A is a component of the unit of adjunction $f^* \dashv \Pi_f$. An application of the pullback $f^* : \mathbf{SSet}_{/X} \rightarrow \mathbf{SSet}_{/Y}$ to this square gives the commutative square

$$\begin{array}{ccc} \bar{B}[f] & \longrightarrow & B \\ \downarrow & & \downarrow \\ A[f] & \xlongequal{\quad} & A[f], \end{array}$$

which is a pullback since $f^*\Pi_f \cong 1$ by part (ii) Lemma 5.1. Hence $B \cong \bar{B}[f]$, as required.

Since B is cofibrant, $\Pi_Y(B)$ is cofibrant by part (iii) of Lemma 5.1. Hence, the simplicial set \bar{B} is also cofibrant by Lemma 2.1. Furthermore, the maps $B \rightarrow \bar{B}$ and $A[f] \rightarrow A$ are cofibrations by Proposition 2.4, as they are pullback of the cofibration $f : Y \rightarrow X$.

It remains to prove that $v : \bar{B} \rightarrow A$ is a weak equivalence and that $\bar{q} : \bar{B} \rightarrow X$ is a fibration. The map u can be factored into a trivial cofibration followed by a trivial fibration, and it is sufficient to prove these claims for each half of the factorisation separately, that is, when u is a trivial fibration and when it is a trivial cofibration.

If u is a trivial fibration, then its image under Π_f is a trivial fibration by part (i) of Lemma 5.1. Since the map $\bar{B} \rightarrow A$ is a pullback of this map, it is also a trivial fibration. This also implies that the composite $\bar{B} \rightarrow A \rightarrow X$ is a fibration.

We now assume that $u : B \rightarrow A[f]$ is a trivial cofibration. Using that the maps from \bar{B} and $A[f]$ to Y are fibrations, we can show that u is a strong deformation retract over Y , that is, there is a retraction $r : A[f] \rightarrow B$ of u in $\mathbf{SSet}_{/Y}$ and a homotopy

$$H : \Delta[1] \times A[f] \rightarrow A[f]$$

between $u \circ r$ and $1_{A[f]}$, whose composite with $A[f] \rightarrow Y$ is the trivial homotopy. Indeed, r and H are, respectively, constructed as the dotted diagonal liftings in the squares:

$$\begin{array}{ccc} B \xlongequal{\quad} B & & A[f] +_B (B \times \Delta[1]) +_B A[f] \longrightarrow A[f] \\ \downarrow & \nearrow r & \downarrow \\ A[f] \longrightarrow Y, & & A[f] \times \Delta[1] \longrightarrow Y, \end{array}$$

We want to show that $\bar{B} \rightarrow A$ is also a deformation retract by constructing a similar homotopy

$$H' : \Delta[1] \times A \rightarrow A.$$

This homotopy will be constructed so that it is H on $\Delta[1] \times A[f]$, it is the map

$$\Delta[1] \times \bar{B} \rightarrow \Delta[0] \times \bar{B} \cong \bar{B} \rightarrow A$$

on $\Delta[0] \times \bar{B}$ (indeed they agree on $\Delta[1] \times B$) and it is the identity on $\Delta[0] \times A$. This is achieved by taking a diagonal filling in the square:

$$\begin{array}{ccc}
 (\Delta[1] \times (\bar{B} \cup A[f])) \cup (\Delta[0] \times A) & \xrightarrow{\quad} & A \\
 \downarrow & \searrow^{H'} & \downarrow \\
 \Delta[1] \times A & \xrightarrow{\quad} & X.
 \end{array}$$

Such a diagonal filler exists since the map on the left-hand side is a trivial cofibration, being the pushout-product of $Y_0 : \Delta[0] \rightarrow \Delta[1]$ and the cofibration $\bar{B} \cup A[f] \rightarrow A$, and the map on the right-hand side is a fibration by assumption.

It remains to see that the map $H_1 : A \rightarrow A$ is indeed a retraction of $\bar{B} \rightarrow A$. We already know that the restriction of H_1 to \bar{B} is the inclusion of \bar{B} in A , so it is enough to show that H_1 has values in \bar{B} . We also know that H_1 restricted to $A[f]$ takes values in $B \subseteq \bar{B}$. By definition of \bar{B} , the map H_1 factors into \bar{B} if and only if it takes values in $\Pi_Y(B)$ when seen as a map to $\Pi_Y(A[f])$, and by adjunction this is the case if and only if the map corresponding to $H_1, A[f] = f^*(A) \rightarrow A[f]$ takes values in B , but this is indeed the case, as already mentioned above.

Since $\bar{B} \rightarrow A$ is a deformation retract, it is invertible in the homotopy category and so it is a weak equivalence. The construction above also shows that \bar{B} is retract of A in $\mathbf{SSet}_{/X}$ and hence $\bar{q} : \bar{B} \rightarrow X$ is a fibration because $p : A \rightarrow X$ is. □

Another proof of Theorem 4.1 is obtained in [19, Section 3.2] modifying the argument in [36].

6 | A CLASSIFYING SMALL FIBRATION BETWEEN BIFIBRANT OBJECTS

The aim of this section is to begin establishing the results necessary to have a pseudo Tarski universe satisfying the UA in the comprehension category of cofibrant simplicial sets. For this, we need to define a cofibrant Kan complex U_c and a Kan fibration with cofibrant domain $\pi_c : \bar{U}_c \rightarrow U_c$. The closure of the type-theoretic universe under various type-formation operations corresponds to the closure of the class of maps arising as pullbacks of π_c under the operations necessary to interpret the corresponding types. For this, it is convenient to consider π_c to be a fibration that classifies small fibrations between cofibrant simplicial sets, that is, such that for every such fibration $p : A \rightarrow X$ there exists a map $a : X \rightarrow U_c$ fitting in a pullback diagram of the form

$$\begin{array}{ccc}
 A & \xrightarrow{\quad} & \bar{U}_c \\
 p \downarrow & \lrcorner & \downarrow \pi_c \\
 X & \xrightarrow{a} & U_c.
 \end{array}$$

Here, note that we make no requirement for the map a to be unique, in contrast for example with the situation of the subobject classifier in an elementary topos. Indeed, the map a is not unique, but only unique up to a contractible space of choices, a fact that will be expressed by showing that π_c is univalent.

In this section, we limit ourselves to define $\pi_c : \bar{U}_c \rightarrow U_c$ and prove that it classifies small fibrations between cofibrant simplicial sets. We will then show that U_c is bifibrant, that \bar{U}_c is cofibrant

and that π_c is univalent in Section 7. For the goal of this section, we proceed in two steps. First, we modify the construction of the weak classifier for small fibrations in [30] to obtain a small fibration $\pi : \bar{U} \rightarrow U$ which classifies small fibrations with cofibrant fibers. Since the base of this fibration does not appear to be cofibrant, we need to consider a suitable cofibrant replacement of U and obtain the required fibration $\pi_c : \bar{U}_c \rightarrow U_c$ via a pullback.

As a preliminary step, let us recall that a simplicial set A is *small* if A_n is a small set (in the sense specified in the Introduction) for every $[n] \in \Delta$ and that a map $p : A \rightarrow X$ of simplicial sets is *small* if for every $x : \Delta[n] \rightarrow X$ the simplicial set $A[x]$ given by the pullback square

$$\begin{array}{ccc} A[x] & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow p \\ \Delta[n] & \xrightarrow{x} & X, \end{array}$$

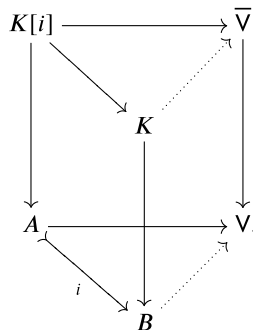
is small. Let us also recall the construction of a map of simplicial sets $\rho : \bar{V} \rightarrow V$ that classifies small maps of simplicial sets, which is a special case of the results in [26] for arbitrary presheaf categories. For this, we use the equivalence in (1.1) and the notation associated to it. The simplicial set V is defined by letting

$$V_n =_{\text{def}} \{F : \Delta/[n]^{\text{op}} \rightarrow \mathbf{Set}\},$$

for $[n] \in \Delta$, where \mathbf{Set} denotes the category of small sets. More generally, if X is a simplicial set then maps from X to V corresponds to functors $F : \Delta/X^{\text{op}} \rightarrow \mathbf{Set}$, where Δ/X denotes the category of elements of X . The object \bar{V} is then defined in exactly the same way by replacing the category of sets by the category of pointed sets and the map $\bar{V} \rightarrow V$ is induced by forgetting the marked point.

The next lemma is essentially [12, Lemma 3.9] and [38, Property (2')], but it is proved here constructively by exploiting the decidability property of cofibrations. The lemma will be used to ‘rectify’ the choice of lifts in the proofs of Theorem 7.4 and Theorem 7.6.

Lemma 6.1. *Given a solid diagram*



where both solid squares are pullbacks and $i : A \rightarrow B$ is a cofibration, there exists dotted arrows that makes the whole diagram commutative and the third square also a pullback.

Proof. Since the map $K \rightarrow B$ is classified by some map $\chi : B \rightarrow V$, there are two dotted arrow as in the diagram above that make the third square a pullback. However, the whole diagram obtained

in this way is not necessarily commutative: the map χ corresponds to a presheaf G on $\Delta_{/B}$, the map $A \rightarrow V$ corresponds to a presheaf F on $\Delta_{/A}$ and the commutativity of the given solid diagram means that the composite

$$\Delta_{/A}^{\text{op}} \longrightarrow \Delta_{/B}^{\text{op}} \xrightarrow{G} \mathbf{Set}$$

is isomorphic to F . But the whole diagram commutes, as required, only if they are not just isomorphic, but equal. Since $i : A \rightarrow B$ is a cofibration, it is levelwise decidable, hence the functor $\Delta_{/A} \rightarrow \Delta_{/B}$ is a decidable inclusion at the level of objects and fully faithful. We can then redefine G to be equal to F on object that are in the image of $\Delta_{/A}$ and equal to the previously defined G on the other objects. There is an evident way to make this into a functor, using the natural isomorphism between F and the restriction of G , and this gives the required commutative diagram. \square

We now come to our first step, in which we define a small fibration $\pi : \bar{U} \rightarrow U$ which classifies the class of small fibrations with cofibrant fibers.

Definition 6.2. We say that a map $p : A \rightarrow X$ has cofibrant fibers if for every $x : \Delta[n] \rightarrow X$ the simplicial set $A[x]$ given by the pullback square

$$\begin{array}{ccc} A[x] & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow p \\ \Delta[n] & \xrightarrow{x} & X, \end{array}$$

is cofibrant.

Lemma 6.3.

- (i) If a map $p : A \rightarrow X$ has cofibrant domain then it has cofibrant fibers.
- (ii) If X is cofibrant and $p : A \rightarrow X$ has cofibrant fibers then A is cofibrant.

Proof. Part (i) follows from Lemma 2.1. For part (ii), let $[n] \in \Delta$, $a \in A_n$. Since X is cofibrant, by the constructive version of the Eilenberg-Zilber lemma [23, Lemma 5.1.2], we can write $p(a) \in X$ in a unique way as $p(a) = s^*(x)$, where $s : [n] \rightarrow [k]$ is a degeneracy and $x \in X_k$ is a non-degenerate cell. Let $x : \Delta[k] \rightarrow X$ be the corresponding map. We now form the pullback

$$\begin{array}{ccc} A[x] & \xrightarrow{w} & A \\ \downarrow & \lrcorner & \downarrow p \\ \Delta[k] & \xrightarrow{x} & X. \end{array}$$

By the universal property of the pullback, there is a unique cell $e \in A[x]_n$ such that $w(e) = a$, and the image of e in $\Delta[k]$ is the cell $s : [n] \rightarrow [k]$, whose image in X are both equal to $p(a) = s^*(x)$.

By the assumption that p has cofibrant fibers, the simplicial set $A[x]$ is cofibrant and hence it is decidable whether e is degenerate or not. We claim that a is degenerate if and only if e is, which implies that it is decidable whether a is degenerate.

Indeed as $a = w(e)$ then if e is degenerate so a is. Conversely, assume that $a = \sigma^*(y_1)$ for a non-trivial degeneracy σ . Then $p(a) = \sigma^*(x_1)$, hence by the uniqueness part of the Eilenberg-Zilber lemma for X one has that $s = s_1 \circ \sigma$ for some degeneracy s_1 , and $x_1 = s_1^*(x')$. In particular, we get a unique cell e_1 of $A[x]$ whose image in $\Delta[n]$ and X are s_1 and a_1 , respectively, whose images in X are both equal to $x_1 = s_1^*(x')$. Finally, the image of $p^*(e_1)$ in $\Delta[n]$ and A are $p^*y_1 = a$ and $s_1 \circ p = s$, respectively, and hence $p^*(e_1) = e$, which proves that e is degenerate as soon as a is. \square

Define an object U by letting, for $[n] \in \Delta$,

$$U_n = \{F \in V_n \mid \pi_F : \int F \rightarrow \Delta[n] \text{ is a small fibration and } \int F \text{ is cofibrant}\}.$$

Here, recall that by a fibration we mean a function together with a fibration structure, that is, a choice of lifts against the horn inclusions $\Lambda^k[n] \rightarrow \Delta[n]$. There is then an evident forgetful map $U \rightarrow V$, which is not injective. Here, U is a presheaf because V is a presheaf and the pullback of a fibration (with chosen lifts) is again a fibration (with chosen lifts), using Lemma 2.1 to obtain the required cofibrancy conditions.

We then define the map $\pi : \bar{U} \rightarrow U$ via the pullback

$$\begin{array}{ccc} \bar{U} & \longrightarrow & \bar{V} \\ \downarrow \pi & \lrcorner & \downarrow \rho \\ U & \longrightarrow & V. \end{array} \tag{6.1}$$

Proposition 6.4.

- (i) The map $\pi : \bar{U} \rightarrow U$ is a small fibration with cofibrant fibers.
- (ii) The map $\pi : \bar{U} \rightarrow U$ classifies small fibrations with cofibrant fibers. More precisely, a map $v : X \rightarrow V$ can be lifted to a map $X \rightarrow U$ if and only if it classifies a fibration with cofibrant fibers. Moreover, choice of a lift to U corresponds (functorially in X and V) to the choice of a fibration structure on the map classified by v .
- (iii) A lifting problem

$$\begin{array}{ccc} A & \longrightarrow & U \\ \downarrow i & & \downarrow \\ B & \xrightarrow{t} & V, \end{array}$$

where i is a cofibration, has a solution if and only if the map $K \rightarrow B$ classified by t is a fibration with cofibrant fibers.

Proof. For part (i), consider a map $a : \Delta[n] \rightarrow U$ and the pullbacks

$$\begin{array}{ccccc} A & \longrightarrow & \bar{U} & \longrightarrow & \bar{V} \\ \downarrow p & \lrcorner & \downarrow \pi & \lrcorner & \downarrow \rho \\ \Delta[n] & \xrightarrow{a} & U & \longrightarrow & V. \end{array}$$

This shows that $p : A \rightarrow \Delta[n]$ is isomorphic to $\pi_F : \int F \rightarrow \Delta[n]$ in $\mathbf{SSet}_{/\Delta[n]}$, where F corresponds under the equivalence in (1.1) to $a : \Delta[n] \rightarrow \mathbb{V}$. Therefore, by definition of \mathbb{U} , A is cofibrant and $p : A \rightarrow \Delta[n]$ is a small fibration. This implies that $\pi : \bar{\mathbb{U}} \rightarrow \mathbb{U}$ has cofibrant fibers. To show it is a fibration, we rewrite a general lifting problem against a horn inclusion $h_n^k : \Lambda^k[n] \rightarrow \Delta[n]$ as follows:

$$\begin{array}{ccccc}
 \Lambda^k[n] & \longrightarrow & A & \longrightarrow & \bar{\mathbb{U}} \\
 \downarrow h_n^k & & \downarrow p & & \downarrow \pi \\
 \Delta[n] & \xlongequal{\quad} & \Delta[n] & \xrightarrow{a} & \mathbb{U},
 \end{array}$$

and then use that $p : A \rightarrow \Delta[n]$ is a fibration, with chosen lifts, which produces a chosen lift in the diagram above.

For part (ii), if a map $v : X \rightarrow \mathbb{V}$ lifts to \mathbb{U} , then the map $A \rightarrow X$ classified by v is a pullback of $\bar{\mathbb{U}} \rightarrow \mathbb{U}$ and therefore it is a small fibration with cofibrant fibers. Conversely, let $p : A \rightarrow X$ be a small fibration with cofibrant fibers. Being a small map, p fits into a pullback of the form

$$\begin{array}{ccc}
 A & \longrightarrow & \bar{\mathbb{V}} \\
 \downarrow p & \lrcorner & \downarrow \rho \\
 X & \xrightarrow{a} & \mathbb{V}.
 \end{array}$$

For each cell of X , $x : \Delta[n] \rightarrow X$ the pullback $A[x] \rightarrow \Delta[n]$ is a small fibration with cofibrant domain by the assumptions on p , which means, by the definition of \mathbb{U} that the image of the cell x in \mathbb{V} lifts to \mathbb{U} . As one can easily check that these lifts to \mathbb{U} are functorial in $[n]$, the map $X \rightarrow \mathbb{V}$ lifts to \mathbb{U} , and hence $A \rightarrow X$ is actually the pullback of $\bar{\mathbb{U}} \rightarrow \mathbb{U}$. One also easily checks that there is a correspondence between fibration structures on the map $A \rightarrow X$ and compatible choices of fibration structures on each pullback along a map $\Delta[n] \rightarrow X$, so that the two constructions explained above are inverse of each other.

For part (iii), the existence of diagonal lift implies that t classifies a fibration with cofibrant fibers by part (ii). Conversely, assuming that t classifies a fibration with cofibrant fibers, it follows from part (ii) that constructing a diagonal lift is the same as choosing lifts for the map $K \rightarrow B$ classified by t such that the choice of lift induced on the pullback to A is the one induced by the map $A \rightarrow \mathbb{U}$. But this is easily achieved as follows. Given a lifting problem of $\Lambda^k[n] \rightarrow \Delta[n]$ against $K \rightarrow B$, since $A \rightarrow B$ is a cofibration it is decidable whether the map $\Delta[n] \rightarrow B$ factors through A or not. One then chooses the lift given by the fibration structure of $K \rightarrow B$ if $\Delta[n] \rightarrow B$ does not factor through A and by the fibration structure on the pullback $K[i] \rightarrow A$ of $K \rightarrow B$ along $i : A \rightarrow B$ otherwise, noting that $K[i] \rightarrow A$ is a fibration classified by the composite $ti : A \rightarrow B \rightarrow \mathbb{U}$. \square

We now construct a fibration $\pi_c : \bar{\mathbb{U}}_c \rightarrow \mathbb{U}_c$ that classifies small fibrations between cofibrant objects. In particular, the base \mathbb{U}_c of this fibration will be cofibrant. In order to do this, let \mathbb{U}_c be a cofibrant replacement of \mathbb{U} , which comes with a trivial fibration

$$\tau : \mathbb{U}_c \rightarrow \mathbb{U}. \tag{6.2}$$

We then define \bar{U}_c via the pullback

$$\begin{array}{ccc}
 \bar{U}_c & \longrightarrow & \bar{U} \\
 \pi_c \downarrow & \lrcorner & \downarrow \pi \\
 U_c & \xrightarrow{\tau} & U.
 \end{array} \tag{6.3}$$

We now prove that $\pi_c : \bar{U}_c \rightarrow U_c$ has the required properties.

Theorem 6.5.

- (i) $\pi_c : \bar{U}_c \rightarrow U_c$ is a small fibration with cofibrant fibers and cofibrant codomain.
- (ii) The map $\pi_c : \bar{U}_c \rightarrow U_c$ classifies small fibrations between cofibrant objects, that is, if a map $p : A \rightarrow X$, with X cofibrant, is a small fibration between cofibrant objects if and only there exists a pullback diagram of the form

$$\begin{array}{ccc}
 A & \longrightarrow & \bar{U}_c \\
 p \downarrow & \lrcorner & \downarrow \pi_c \\
 X & \xrightarrow{a} & U_c.
 \end{array}$$

Proof. For part (i), U_c is cofibrant by construction and the rest of the claim follows from part (ii) of Proposition 6.4.

For part (ii), let $p : A \rightarrow X$ be a small fibration between cofibrant objects, by part (i) of Lemma 6.3, it has cofibrant fibers and hence by part (ii) of Proposition 6.4 there is a pullback diagram of the form

$$\begin{array}{ccc}
 A & \longrightarrow & \bar{U} \\
 p \downarrow & \lrcorner & \downarrow \pi \\
 X & \xrightarrow{a} & U.
 \end{array}$$

Since X is cofibrant, we have a diagonal filler in the diagram

$$\begin{array}{ccc}
 0 & \longrightarrow & U_c \\
 \downarrow & \nearrow a_c & \downarrow \tau \\
 X & \xrightarrow{a} & U.
 \end{array}$$

We then obtain the diagram

$$\begin{array}{ccccc}
 A & \longrightarrow & \bar{U}_c & \longrightarrow & \bar{U} \\
 p \downarrow & & \downarrow \pi_c & & \downarrow \pi \\
 X & \xrightarrow{a_c} & U_c & \xrightarrow{a} & U.
 \end{array}$$

Here, the right-hand side square and the rectangle are pullbacks and therefore the left-hand side square is also a pullback. Hence p is indeed a pullback of $\bar{U}_c \rightarrow U_c$. Conversely, any pullback $p : A \rightarrow X$ of $\bar{U}_c \rightarrow U_c$ is a small fibration with cofibrant fibers, hence by part (ii) of Lemma 6.3, if X is cofibrant, p is small fibration between cofibrant objects. \square

7 | FIBRANCY AND UNIVALENCE OF THE UNIVERSE

The aim of this section is to show that U_c is a cofibrant Kan complex and that $\pi_c : \bar{U}_c \rightarrow U_c$ is univalent. For this, let us first return to consider the fibration $\pi : \bar{U} \rightarrow U$ defined via the pullback in (6.1). Let U^\rightarrow be the simplicial set whose n -simplices are triples of the form (F_0, F_1, ϕ) , where F_0 and F_1 are n -simplices of U , that is, functors

$$F_0, F_1 : \Delta_{/[n]}^{op} \rightarrow \mathbf{Set}$$

and $\phi : F_0 \Rightarrow F_1$ is a natural transformation. By the equivalence in (1.1), such triples correspond to commutative diagrams of the form

$$\begin{array}{ccc}
 A_1 & \xrightarrow{f} & A_2 \\
 p_1 \searrow & & \swarrow p_2 \\
 & \Delta[n] &
 \end{array}
 \tag{7.1}$$

where p_1 and p_2 are fibrations with cofibrant domain.

Lemma 7.1. *The map $(\text{dom}, \text{cod}) : U^\rightarrow \rightarrow U \times U$ is a fibration*

Proof. Observe that U^\rightarrow is exactly $\Pi_p(\bar{U} \times \bar{U})$, where $p : \bar{U} \times \bar{U} \rightarrow U \times U$ is the evident map. It follows from Theorem 4.1 that $U^\rightarrow \rightarrow U \times U$ is a fibration. More precisely, Theorem 4.1 implies that any pullback of $U^\rightarrow \rightarrow U \times U$ to a cofibrant $X \rightarrow U \times U$ is a fibration (due to the cofibrancy assumption of Theorem 4.1), but this is sufficient to prove that $U^\rightarrow \rightarrow U \times U$ is a fibration, as in the argument for part (i) of Proposition 6.4. \square

We recall that $\mathbf{BFib}_{/X}$ denotes the full subcategory of $\mathbf{SSet}_{/X}$ of fibrations with cofibrant domain. Given $f : A \rightarrow B$ an arrow in $\mathbf{BFib}_{/X}$ we now construct the object $\text{Inv}(f)$ in $\mathbf{BFib}_{/X}$ of weak equivalences structures on f . We will use the characterisation of weak equivalences as maps that have both a left inverse and a right inverse, that is, what are called bi-invertible maps in [43, Section 4.3]. For this, we essentially follow [39] and [41, Section 1.4], which in turn is a category-theoretic counterpart of the type-theoretic treatment in [43]. We cannot use the results in [39] and [41, Section 1.4] directly since they both work under assumptions on path objects that would not be satisfied for us without restricting to cofibrant objects and assume that Π -types are given by dependent products, that is, right adjoints to pullback functors, which we do not have if we restrict to cofibrant objects. For these reasons, we need to prove Proposition 7.2 explicitly. However, since this is very similar to the references cited above, we will not give all details.

Given a map $f : A \rightarrow B$ in $\mathbf{BFib}_{/X}$, a right inverse for f , is a map $h : B \rightarrow A$ in $\mathbf{BFib}_{/X}$ and a homotopy $f \circ h \sim 1_B$, given by an arrow $B \rightarrow \text{Path}_{/X}(B)$ in $\mathbf{BFib}_{/X}$. Similarly, a left inverse for f

is an arrow $g : B \rightarrow A$ together with a homotopy $g \circ f \sim 1_A$, given by an arrow $A \rightarrow \text{Path}_{/X}(A)$ in $\text{BFib}_{/X}$.

Using dependent products (without taking cofibrant replacements), we can construct fibrant objects $\text{LInv}(f) \rightarrow X$ and $\text{RInv}(f) \rightarrow X$, that are, respectively, the objects of left and right inverse of f . Explicitly, for an object $Y \in \mathbf{SSet}_{/X}$ a map $Y \rightarrow \text{LInv}(f)$ is the same as a left inverse to the map $A \times_X Y \rightarrow B \times_X Y$ in $\text{BFib}_{/Y}$.

We then define $\text{Inv}(f) = \text{LInv}(f) \times_X \text{RInv}(f)$. It is also fibrant over X , and an arrow $Y \rightarrow \text{Inv}(f)$ in $\mathbf{SSet}_{/X}$ is the same as the data of both a left inverse and a right inverse of the pullback of f to $\mathbf{SSet}_{/Y}$.

Proposition 7.2. *Let $f : U \rightarrow V$ be an arrow in $\text{BFib}_{/X}$ between two fibrations with cofibrant fibers $U \twoheadrightarrow X$ and $V \twoheadrightarrow X$. Let $A \hookrightarrow B$ a cofibration between cofibrant objects in $\mathbf{SSet}_{/X}$, then a lifting problem*

$$\begin{array}{ccc} A & \longrightarrow & \text{Inv}(f) \\ \downarrow i & & \downarrow \\ B & \xrightarrow{p} & X \end{array}$$

has a diagonal filling if and only if the pullback of f to $\mathbf{SSet}_{/B}$ is a weak equivalence.

Proof. Let $p^* : \mathbf{SSet}_{/X} \rightarrow \mathbf{SSet}_{/B}$ be the pullback along p . By construction, if we have a lift $B \rightarrow \text{Inv}(f)$ then we get both a left and a right inverse to $p^*(f) : p^*(U) \rightarrow p^*(V)$, which is an arrow between bifibrant objects in $\mathbf{SSet}_{/B}$. This implies that $p^*(f)$ is a weak equivalence, and concludes the proof of one implication.

Conversely, we assume that $p^*(f) : p^*(U) \rightarrow p^*(V)$ is a weak equivalence. Note that as U and V are assumed to have cofibrant fibers over X and A and B are cofibrant, all the objects p^*U , i^*U , p^*V and i^*V are cofibrant.

Because $\text{Inv}(f)$ is constructed as $\text{RInv}(f) \times_X \text{LInv}(f)$, we only need to construct lifts to $\text{LInv}(f)$ and $\text{RInv}(f)$ separately. We start with the lift to $\text{RInv}(f)$. A right inverse to $f : U \rightarrow V$ in $\mathbf{SSet}_{/X}$ can be described as a section to a fibration $\pi_f : U \times_V \text{Path}_{/X}(V) \rightarrow V$. Indeed, in terms of generalised elements it should associate to each $v \in V$ an element $u \in U$ together with a homotopy between $f(u)$ and v . The construction $f \mapsto \pi_f$ is stable under pullback in the sense that $\pi_{p^*f} \cong p^*(\pi_f)$. Moreover, if f is an equivalence π_f also is, and hence it is a trivial fibration, in particular $p^*(\pi_f)$ and $i^*(\pi_f)$ are trivial fibrations. Now constructing a lift to $\text{RInv}(f)$ is the same as constructing a section of $p^*(\pi_f)$ to B (which is a trivial fibration) that extends the given section of $i^*(\pi_f)$. By our cofibrancy assumption, and Proposition 2.4, the horizontal maps in the square:

$$\begin{array}{ccc} i^*(U \times_V \text{Path}_{/X}(V)) & \longrightarrow & p^*(U \times_V \text{Path}_{/X}(V)) \\ \downarrow i^*\pi_f & & \downarrow p^*(\pi_f) \\ i^*V & \longrightarrow & p^*V \end{array}$$

are cofibrations, and so the desired section on $p^*\pi_f$ extending the one on $i^*\pi_f$ can be constructed using the right lifting property of the trivial fibration $p^*\pi_f \simeq \pi_{p^*f}$ against $p^*V \rightarrow i^*V$.

We now move to the lift to $\text{LInv}(f)$. A left inverse of f , can be described dually as a retraction of the map $\sigma_f : U \hookrightarrow \Delta[1] \times_U \coprod_U V$ in \mathbf{SSet}_X . Similarly to the previous case, the construction

$f \mapsto \sigma_f$ is also stable under pullback (that is, $p^*\sigma_f \cong \sigma_{p^*f}$), the map σ_f is cofibration when U and V are cofibrant and is a trivial cofibration if in addition f is a weak equivalence. So, as before, $p^*(\sigma_f)$ and $i^*(\sigma_f)$ are trivial cofibrations. Thus, the lift that we are trying to construct is equivalent to a retraction in $\mathbf{SSet}_{/B}$ of the right map in the square below that extends the retraction already existing in $\mathbf{SSet}_{/A}$ of the left map:

$$\begin{array}{ccc}
 i^*U & \longrightarrow & p^*U \\
 \downarrow i^*\sigma_f & & \downarrow p^*(\sigma_f) \\
 i^*(\Delta[1] \times U \amalg_U V) & \longrightarrow & p^*(\Delta[1] \times U \amalg_U V) .
 \end{array}$$

This can be achieved by choosing a lift in:

$$\begin{array}{ccc}
 p^*U \amalg_{i^*U} i^*(\Delta[1] \times U \amalg_U V) & \longrightarrow & p^*U \\
 \downarrow & \nearrow & \downarrow \\
 p^*(\Delta[1] \times U \amalg_U V) & \longrightarrow & B.
 \end{array}$$

Here, the map on the right is a fibration. The left arrow is a weak equivalence by 2-out-of-3, since both maps $p^*(\sigma_f)$ and (the pushout of) $i^*\sigma_f$ are trivial cofibration. It is also a cofibration since it is a monomorphism and the decidability condition hold because all the maps in the previous square are cofibrations. □

Finally, we define $\text{Weq}(U) \rightarrow U^\rightarrow$ by applying this construction $\text{Inv}(f)$ to the universal arrow between small fibrations with cofibrant fibers, which lives over U^\rightarrow . By Proposition 7.2, we have the following result.

Proposition 7.3. *The map $\text{Weq}(U) \rightarrow U^\rightarrow$ is a fibration and given $A \twoheadrightarrow B$ a cofibration between cofibrant objects, a lifting problem*

$$\begin{array}{ccc}
 A & \longrightarrow & \text{Weq}(U) \\
 \downarrow & & \downarrow \\
 B & \xrightarrow{p} & U^\rightarrow
 \end{array}$$

has a dotted diagonal filling if and only if the map classified by p is a weak equivalence.

Theorem 7.4.

- (i) *The simplicial set U is fibrant.*
- (ii) *The simplicial set U_c is bifibrant.*

Proof. We prove part (i). Since $(s, t) : \text{Weq}(U) \rightarrow U \times U$ is a fibration, for any cofibrant simplicial set X , maps $a_1, a_2 : X \rightrightarrows U$ and homotopy $h : \Delta[1] \times X \rightarrow U$ from a_1 to a_2 , there is a weak equivalence in $\mathbf{SSet}_{/X}$ between the objects classified by a_1 and a_2 , constructed as follows. For this, we first consider a diagonal filler in the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{i_1} & \text{Weq}(U) \\
 \delta^0 \downarrow & \nearrow & \downarrow \\
 \Delta[1] \times X & \xrightarrow{(a_1, h)} & U \times U
 \end{array}$$

Here, i_1 denotes a map classifying the identify of the object classified by a_1 . By a_1 in the first component of the bottom arrow we mean the composite $\Delta[1] \times X \rightarrow X \rightarrow U$. Composing the dotted arrow with δ^1 gives us a map $X \rightarrow \text{Weq}(U)$ whose projection to $U \times U$ is (a_1, a_2) , that is, it classifies a weak equivalence between the objects classified by a_1 and a_2 . One can do the same thing with δ^0 and δ^1 exchanged to get a weak equivalence in the other direction.

Using this fact, we can now prove that U is fibrant. A map $h_n^k : \Lambda^k[n] \rightarrow U$ classifies a fibration $q : B \rightarrow \Lambda^k[n]$ with cofibrant domain. The horn inclusion $h_n^k : \Lambda^k[n] \rightarrow \Delta[n]$ fits into the retract diagram

$$\begin{array}{ccccc}
 \Lambda^k[n] & \hookrightarrow & (\Delta[1] \times \Lambda^k[n]) \cup \Delta[n] & \longrightarrow & \Lambda^k[n] \\
 \downarrow & & \downarrow & & \downarrow \\
 \Delta[n] & \hookrightarrow & \Delta[1] \times \Delta[n] & \longrightarrow & \Delta[n],
 \end{array}$$

where the map $\Delta[0] \rightarrow \Delta[1]$ can be either δ^0 or δ^1 depending on whether $0 < k$ or $k < n$. See, for example, the last part of the proof of [29, Theorem 3.2.3].

By the observation above, the composite map $(\Delta[1] \times \Lambda^k[n]) \cup \Delta[n] \rightarrow \Lambda^k[n] \rightarrow U$ gives a solid diagram of the form

$$\begin{array}{ccccc}
 B & \xrightarrow{\dots} & \bar{B} & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & A' & \xrightarrow{\dots} & A \\
 \downarrow & \swarrow & \downarrow & \swarrow & \\
 \Lambda^k[n] & \xrightarrow{h_n^k} & \Delta[n] & &
 \end{array}$$

So we can construct a fibration $\bar{q} : \bar{B} \rightarrow \Delta[n]$ with cofibrant domain whose pullback along h_n^k is isomorphic to $q : B \rightarrow \Lambda^k[n]$. The map $b : \Delta[n] \rightarrow U$ classifying \bar{q} , if chosen carefully, gives the lift we are looking for. More precisely, by Lemma 6.1 one can choose (in the diagram below) a map $a : \Delta[n] \rightarrow V$ that classifies \bar{q} and extends the map $\Lambda^k[n] \rightarrow U \rightarrow V$ classifying q . Finally, using part (iii) of Proposition 6.4 one can (also in the diagram below) lift this map to map $b : \Delta[n] \rightarrow U$ that gives the lift we are looking for,

$$\begin{array}{ccc}
 \Lambda^k[n] & \longrightarrow & U \\
 \downarrow & \nearrow b & \downarrow \\
 \Delta[n] & \xrightarrow{a} & V
 \end{array}$$

Part (ii) follows immediately from part (i) since $\tau : U_c \rightarrow U$ of (6.2) is a trivial fibration. □

We now wish to define what it means for a small fibration with cofibrant fibers, and in particular for a small fibration between cofibrant objects, to be univalent. For this, we fix such a fibration $p : A \rightarrow X$ and construct an object $\text{Weq}(p) \rightarrow X \times X$ that represents weak equivalences between fibers of p , in the sense that maps $Y \rightarrow \text{Weq}(p)$ in $\mathbf{SSet}_{/X \times X}$ are in bijective correspondence with tuples (x_1, x_2, w, e) consisting of two maps $x_1, x_2 : Y \rightarrow X$, a map $w : A[x_1] \rightarrow A[x_2]$ in $\mathbf{SSet}_{/Y}$, and e a weak equivalence structure on w . The required object $\text{Weq}(p)$ can be constructed as the pullback

$$\begin{array}{ccc} \text{Weq}(p) & \longrightarrow & \text{Weq}(U) \\ \downarrow \lrcorner & & \downarrow (s,t) \\ X \times X & \xrightarrow{a \times a} & U \times U, \end{array}$$

where $a : X \rightarrow U$ is a classifying map for the small fibration $p : A \rightarrow X$, which exists by our assumption that A and X are cofibrant and part (iii) of Proposition 6.4. The verification that $\text{Weq}(p)$ has the required universal property is an easy calculation, which we leave to the readers. There is an evident map $i : X \rightarrow \text{Weq}(p)$ corresponding via the universal property of $\text{Weq}(p)$ to the tuple of identity maps $(1_X, 1_X, 1_A, e)$ where e is just the ‘trivial’ weak equivalence structure on the identity, using the identity as both a left and a right inverse.

Definition 7.5. Let $p : A \rightarrow X$ be a small fibration with cofibrant fibers. We say that p is *univalent* if the map $i : X \rightarrow \text{Weq}(p)$ is a weak equivalence.

For a small fibration with cofibrant fibers $p : A \rightarrow X$, being univalent is equivalent to either $s : \text{Weq}(p) \rightarrow X$ or $t : \text{Weq}(p) \rightarrow X$ being a trivial fibration. Also note that, when X is cofibrant, we have a map $j : \text{Path}(X) \rightarrow \text{Weq}(p)$ fitting in the diagram

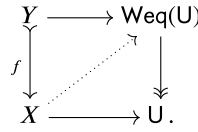
$$\begin{array}{ccc} X & \xrightarrow{i} & \text{Weq}(p) \\ \downarrow r & \nearrow j & \downarrow (s,t) \\ \text{Path}(X) & \xrightarrow{\partial_X} & X \times X. \end{array}$$

In this case, p is univalent if and only if j is a weak equivalence, mirroring the type-theoretic definition of univalence. This will be the case for $\pi_c : \bar{U}_c \rightarrow U_c$, for example.

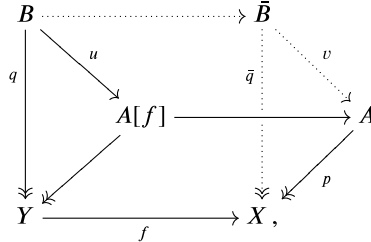
Theorem 7.6.

- (i) *The fibration $\pi : \bar{U} \rightarrow U$ is univalent.*
- (ii) *The fibration $\pi_c : \bar{U}_c \rightarrow U_c$ is univalent.*

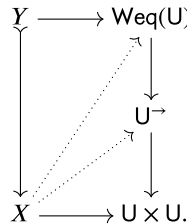
Proof. For part (i), we prove that $t : \text{Weq}(U) \rightarrow U$ has the right lifting property with respect to all cofibrations between cofibrant objects. So let $f : Y \rightarrow X$ be a cofibration between cofibrant objects and consider the diagonal filling problem



By construction of $\text{Weq}(U)$, and using Proposition 7.3, this induces a diagram of solid arrows as in the equivalence extension property of Proposition 5.2. Indeed, the map $X \rightarrow U$ gives us $p : A \rightarrow X$, the composite of $Y \rightarrow \text{Weq}(U)$ with the first projection gives us $q : B \rightarrow Y$, while the rest of the data and the commutativity of the square gives us a weak equivalence u between B and $A[f]$ over X . The completion of this diagram given by Proposition 5.2,

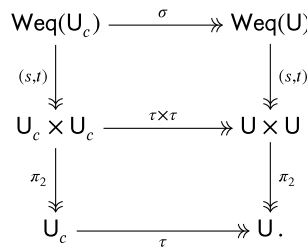


is what one needs to produce the required diagonal filler, at least if one chooses the classifying maps correctly. Indeed, by Lemma 6.1 one can choose of map $X \rightarrow V$ classifying $\bar{B} \rightarrow X$ whose restriction along $Y \rightarrow X$ is the composite $Y \rightarrow \text{Weq}(U) \rightarrow U \rightarrow V$ that classifies B , and by part (iii) of Proposition 6.4, one can lift it to a classifying map $X \rightarrow U$ that restricts to the composite $Y \rightarrow \text{Weq}(U) \rightarrow U$. At this point we have a solid commutative diagram



The dotted lift $X \rightarrow U^\rightarrow$ is obtained from the map $\bar{B} \rightarrow A$ using the definition of U^\rightarrow as an exponential type, and the dotted lift $X \rightarrow \text{Weq}(U)$ is constructed from Proposition 7.3 as the map $\bar{B} \rightarrow A$ is a weak equivalence. Post-composing all this with the appropriate projection $U \times U \rightarrow U$ gives the solution to our lifting problem and concludes the proof of part (i).

For part (ii), we prove that $t : \text{Weq}(\pi_c) \rightarrow U_c$ is a trivial fibration. For this, let us observe that we have a diagram



The composite on the left-hand side is the map t that we wish to show to be a trivial fibration. First, using part (i), observe that it is a fibration since it is the composite of two fibrations. Secondly, observe that the top square is a pullback and so σ is a trivial fibration since $\tau \times \tau$ is so. Thus, applying 3-for-2 to the outer square, we obtain that t is a weak equivalence and hence a trivial fibration, as required. \square

8 | CONCLUSIONS AND FUTURE WORK

We can now summarize our results using the comprehension category of cofibrant simplicial sets introduced in Proposition 2.2,

$$\begin{array}{ccc}
 \mathbf{Fib}_{\text{cof}} & \xrightarrow{\chi} & \mathbf{SSet}_{\text{cof}}^{\rightarrow} \\
 & \searrow p & \swarrow \text{cod} \\
 & \mathbf{SSet}_{\text{cof}} &
 \end{array}$$

For this, we use the terminology of [32] regarding the stability conditions and that of [40, Appendix A] regarding the type-theoretic universe. When we say that the universe is closed under some constructions, we simply mean that the fibrations classified by the universe are closed under these constructions.

Theorem 8.1. *The comprehension category of cofibrant simplicial sets has:*

- a pseudo-stable empty type 0, unit type 1 and natural numbers type \mathbb{N} ,
- pseudo-stable $+$ -types,
- partially pseudo-stable Id-types,
- pseudo-stable Σ -types,
- weakly stable Π -types,
- a pseudo Tarski universe $\pi_c : \overline{\mathcal{U}}_c \rightarrow \mathcal{U}_c$, containing 0, 1, \mathbb{N} and closed under the Σ -types, Id-types, Π -types and $+$ -types constructions.

Furthermore, the fibration $\pi_c : \overline{\mathcal{U}}_c \rightarrow \mathcal{U}_c$ is univalent.

Proof. We have checked that this is indeed a comprehension category in Proposition 2.2. For 0, 1 and \mathbb{N} , these are simply given by the discrete simplicial sets, which are cofibrant (as every cell of dimension > 0 is degenerated) and fibrant. For $+$ types, we use coproducts observing that $A + B$ is fibrant and cofibrant if A and B are so.

Identity types are constructed in Section 3. In particular, see Theorem 3.4. The construction of Id-types themselves involves only categorical dependent products and pullbacks, which are all pseudo-stable, but the construction of the J-eliminator involves lifting properties which is not pseudo-stable in general. Thus, we have partially pseudo-stable identity types, as in [32, Definition 2.3.4]. The Σ -types have been constructed in Remark 2.3 and are clearly stable under pullback up to isomorphism. The Π -types have been constructed in Section 4 as cofibrant replacement $\mathbb{L}\Pi_A B$ of the categorical dependent products $\Pi_A B$. The categorical dependent products are pseudo-stable, so given such a Π -type $\mathbb{L}\Pi_A B$, in context Γ its pullback $f^*(\mathbb{L}\Pi_A B)$ along a context morphism $f : \Gamma' \rightarrow \Gamma$ might be different from $\mathbb{L}\Pi_{f^*A} f^*B$ but is also a cofibrant replacement of $\Pi_{f^*A} f^*B$ hence also has the property of being a Π -type.

Finally, in Section 6 we constructed the universe $\overline{U}_c \rightarrow U_c$ as a classifier for all small fibrations between cofibrant objects. As small fibrations are stable under all the constructor above, this gives the stability of the universe. It has been shown in Theorem 7.6 in that this fibration is indeed univalent. Regarding the closure of the universe, the only delicate point concerns Π -types because of the use of the cofibrant replacement. The explicit definition of the cofibrant replacement functor in the Appendix shows that the trivial fibration $\varepsilon_X : \mathbb{L}X \rightarrow X$ is a small fibration for every X , independently of whether X is small or not (see Remark 8.3 for a comment on the use of a ‘resizing rule’ in the proof of this claim and part (i) of Corollary A.9 for details). Hence the construction of Π -types preserves small fibrations, even between non-small objects. \square

Remark 8.2. The Π -types constructed in the present paper have stronger properties than being weakly stable Π -types as stated above. First, as explained in Remark 4.2, they satisfy a propositional η -rule, while [32] require no η -rule. Secondly, given our Π -type $\mathbb{L}(\Pi_A B)$ over X in the sense of Theorem 8.1, and $f : X' \rightarrow X$, then both $f^*(\mathbb{L}(\Pi_A B))$ and Π -type $\mathbb{L}(\Pi_{f^*A} f^*B)$ are both cofibrant replacements of $f^*(\Pi_A B) \cong \Pi_{f^*A} f^*B$. Hence, they are homotopy equivalent.

Remark 8.3. Our proof that the universe is closed under Π -types relies on the assumption that the inaccessible set u satisfies a form of propositional resizing, as observed in Corollary A.9. If one wishes to avoid this additional assumption, there are at least two alternatives. The first alternative arises if one does not make the assumption that subsets of small sets are small and keeps all the definitions the same. Then, we still have a comprehension category with all the structures mentioned in Theorem 8.1, but do not have anymore that the universe is closed under the construction of arbitrary Π -types. Instead, one has (in type-theoretic notation) that if $\Gamma, a : A \vdash B(a) : U_c$, then $\Gamma \vdash \Pi_A B : U_c$ if Γ is a context involving only small types. The second alternative arises if one works instead in CZF without assuming any inaccessible sets, and changes to an interpretation where contexts are ‘simplicial classes’ (where classes are formulas), general types are given by Kan fibrations between simplicial classes and a small fibration is one whose fibers are sets. In this case, part (ii) of Corollary A.9 provides the smallness property that ensures that the cofibrant replacement preserves small type even in non-small context. We then have that the universe is closed under Π -type, but we cannot form general Π -types, but only those of the form $\Pi_A B$ where A is a small type.

Remark 8.4. A lack of strict stability for Π -types is also present in the semisimplicial model defined in [6]. This is of interest since cofibrant simplicial sets are closely connected to semisimplicial sets [24].

Remark 8.5. The main result of [32] asserts that if a contextual category \mathcal{C} , satisfying the additional condition (LF) of [32, Definition 3.1.3]) has weakly stable-type constructors, then left adjoint splitting $\mathcal{C}_!$ has the same constructors, but now strictly stable, thus solving the coherence problem. Unfortunately, condition (LF) is not satisfied in our constructive setting, essentially because cofibrant objects are generally not closed under dependent products.

We leave as an open problem the question of whether a comprehension category as in Theorem 8.1 can be split so as obtain a model of $ML(UA)$ and so, in particular, obtain a constructive version of the simplicial model of $ML(UA)$. It should be noted that this question is now completely independent from simplicial homotopy theory, as our results in this paper have obtained

all the necessary results to produce the structure in Theorem 8.1. We also leave as an open problem the question of whether our comprehension category supports other inductive types [9] and higher-inductive types in the sense of [43]. We expect these to work without the need to perform cofibrant replacements.

Finally, we mention as a potential area of further research the idea of designing a dependent-type theory with explicit substitutions [1] with rules matching the structure of the comprehension category of Theorem 8.1, extending the idea of substitution up to isomorphism of [15] to that of substitution up to weak equivalence. A conservativity result of such a system over $\text{ML}(\text{UA})$ would then be essentially equivalent to a solution to the coherence problem, cf. [16].

APPENDIX: THE COFIBRANT REPLACEMENT FUNCTOR

The goal of this appendix is to give an explicit description of the cofibrant replacement functor on simplicial sets (Proposition A.7) and to use it to discuss the size of this cofibrant replacement.

Definition A.1.

- (i) If $x \in X_n$ is an n -cell in a simplicial set X , the *degeneracy type* of x is the set of face maps $\delta : [i] \rightarrow [n]$ such that δ^*x is a degenerate cell.
- (ii) By a *degeneracy type* or *degeneracy n -type* we mean a subset of faces of $\Delta[n]$ that can appear as the degeneracy type of an n -cell of a simplicial set.

Note that given any n -cell $x \in X_n$, the factorisation of $x : \Delta[n] \rightarrow X$ as a degeneracy quotient followed by a degeneracy detecting morphism constructs a degeneracy quotient $y : \Delta[n] \rightarrow K$ such that y has the same degeneracy type as x . This induces a correspondence between degeneracy n -types and (isomorphism classes of) degeneracy quotients $\Delta[n] \rightarrow K$.

Lemma A.2. *A degeneracy n -type is the degeneracy type of a cell in a cofibrant object if and only if it is decidable as a subset of faces of $\Delta[n]$.*

Proof. Given an n -cell x in a cofibrant simplicial set X , given any face $\delta : [m] \rightarrow [n]$ it is decidable if δ^*x is degenerated or not, hence the degeneracy type of x is decidable. Conversely, given any degeneracy type p , it is the degeneracy type of a degeneracy quotient $\Delta[n] \rightarrow K$. Any cell of K is the image of a cell of $\Delta[n]$, and it is degenerated in K if and only if it is the image of a degenerated cell, or the image of a cell in p . Hence if p is decidable K is cofibrant. \square

We will now construct a ‘simplicial set of decidable degeneracy type D ’. Note that the construction would work exactly the same without the assumption ‘decidable’, with the exception that as CZF does not have power set, this might not be a simplicial set, but rather a simplicial class.

We start with a simplicial set D_0 whose n -cells are all the decidable subset of the faces of $\Delta[n]$, that is, decidable subsets of the set of finite subsets of $[n]$. As decidable subsets are just functions to $\{0, 1\}$, this is indeed a set. If $f : [m] \rightarrow [n]$ and $P \in D_0([n])$ one defines f^*P as the set of faces $[i] \rightarrow [m]$ such that the composite

$$[i] \rightarrow [m] \xrightarrow{f} [n]$$

is either in P or non-injective, which is indeed a decidable set of faces.

Lemma A.3. D_0 is a cofibrant simplicial set, and for any cofibrant simplicial set X , the map $X \rightarrow D_0$ sending each n cell to its degeneracy type is a simplicial and degeneracy detecting map.

Proof. It is immediate to check that D_0 is a simplicial set. Moreover as it has decidable equality one can decide whether a cell x is degenerated by testing if it is equal to s^*d^*x for some non-trivial face d and degeneracy s , hence D_0 is cofibrant. Let $x \in X_n$ and $f : [m] \rightarrow [n]$, in order to check that the maps $X \rightarrow D_0$ sending each cell to its degeneracy type is simplicial, one needs to check that the degeneracy type of f^*x is indeed described from the degeneracy type of x using the formula for the functoriality of D_0 . For face maps $[i] \rightarrow [m]$ the cell i^*f^*x is degenerated as soon as $f \circ i$ is non-injective, and if $f \circ i$ is injective, then it depends on whether $f \circ i$ is in the degeneracy type of x or not.

If $f : [n] \rightarrow [m]$ is non-injective then for any $s \in (D_0)_m$ the identity map $[n] \rightarrow [n]$ is in f^*s . So a degenerate cell of D_0 always contains the maximal face. In particular the map $X \rightarrow D_0$ constructed above sends non-degenerate cell of X to non-degenerate cells of D_0 . \square

Lemma A.3 constructs a map $P : D_0 \rightarrow D_0$ sending any cell to its degeneracy type. As P preserves and detects degeneracy, it preserves the degeneracy type and hence $P \circ P = P$.

Definition A.4. The simplicial set D is the set of fix points of the idempotent P acting on D_0 .

Note that for any simplicial set X , the map $f : X \rightarrow D_0$ sending each cell to its degeneracy type preserves the degeneracy type, hence $Pf = f$. In particular a cell of D_0 is in D if and only if it is a degeneracy type. This leads to the following fact.

Lemma A.5. The n -cells of D are in bijection with decidable degeneracy n -types. Each cofibrant simplicial set X has a unique map detecting degeneracy to D , which is the map sending a cell to its degeneracy type.

Lemma A.6. Given a map $\partial\Delta[n] \rightarrow D$ there is a unique way to extend it into map $\Delta[n] \rightarrow D$ such that the non-degenerate n -cell of $\Delta[n]$ is sent to a non-degenerate cell.

Proof. If such an extension exists the n -cell x corresponding to $\Delta[n] \rightarrow D$ has to be as follows: x does not contain the face $[n] \rightarrow [n]$, and for all other faces $\delta : [i] \rightarrow [n]$ it is in x if and only if the composite $[i] \rightarrow \partial\Delta[n] \rightarrow D$ is a degenerate cell. So the uniqueness is clear. We only need to show that this set is indeed a degeneracy type. But if one forms $D \coprod_{\partial\Delta[n]} \Delta[n]$ then the new added n -cell has exactly this degeneracy type, so this concludes the proof. \square

Let $\varepsilon_X : \mathbb{L}X \rightarrow X$ be the cofibrant replacement of a simplicial set X constructed using Garner's version of the small object argument [21].

Proposition A.7. The map $\mathbb{L}X \rightarrow D \times X$ sending an n -cell to its degeneracy type $s \in D$ and its image in X induces a bijection between $\mathbb{L}X([n])$ and the set of pairs of a decidable degeneracy type s and a cell of x of degeneracy type larger than s .

Proof. Because of the stratified nature of the simplicial generating cofibration, $\mathbb{L}X$ can be written as the colimit of a sequence:

$$\mathbb{L}^{(0)}X \hookrightarrow \mathbb{L}^{(1)}X \hookrightarrow \dots \hookrightarrow \mathbb{L}^{(n)}X \hookrightarrow \dots,$$

where $\mathbb{L}^{(0)}X$ is just the set of 0-cell of X and the map $\mathbb{L}^{(n-1)}X \rightarrow \mathbb{L}^{(n)}X$ is constructed as the pushout of the coproduct of one copy of $\partial\Delta[n] \rightarrow \Delta[n]$ for each square of the form

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & \mathbb{L}^{(n-1)}X \\ \downarrow & & \downarrow \\ \Delta[n] & \longrightarrow & X. \end{array}$$

Indeed, the pushout constructing $\mathbb{L}^{(n)}X$ from $\mathbb{L}^{(n-1)}X$ does not change the k -skeleton for $k < n$ and the set of maps $\partial\Delta[k] \rightarrow \mathbb{L}^{(n-1)}X$ only depends on the k -skeleton of X , so one can always do all the necessary pushout by $\partial\Delta[k] \hookrightarrow \Delta[k]$ by for $k < n$ before those by $\partial\Delta[n] \hookrightarrow \Delta[n]$.

We will prove by induction on n that $\mathbb{L}^{(n)}X$ identifies with n -skeleton of the simplicial set mentioned in the proposition, that is the sub-simplicial set Y^n of $D \times X$ of pairs (s, x) such s is decidable, the degeneracy type of x is at least s and (s, x) is of dimension at most n , or a degeneracy of a cell of dimension at most n .

Note that as the degeneracy type of x is larger than s , s is the degeneracy type of the pair (s, x) , so the condition on (s, x) being a degeneracy is equivalent to the same condition on s .

In the case $n = 0$, there is only one degeneracy type in dimension 0, so both $\mathbb{L}^{(0)}X$ and Y^0 are the simplicial set of cells of X that are degeneracy of 0-cells.

Assume Y^{n-1} and $\mathbb{L}^{(n-1)}X$ are isomorphic as claimed. One only needs to check that the new non-degenerate n -cell that appears in Y^n and $\mathbb{L}^{(n)}X$ are in bijection compatible to their boundary and their image in X . In $\mathbb{L}^{(n)}X$ there is exactly one such cell for each square as above. In Y^n , any non-degenerate n -cells does produce such a square, and conversely given a square:

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & Y^{n-1} \\ \downarrow & & \downarrow \\ \Delta[n] & \longrightarrow & X \end{array}$$

Lemma A.6 above gives a unique map to extend $\partial\Delta[n] \rightarrow D$ to a non-degenerate n -cell of D , and the image of $\Delta[n]$ in X automatically has a larger degeneracy type that this extension so this gives a non-degenerate cell of Y^n generating this square. This is the unique way to get such a cell in Y^n to be non-degenerate: indeed a cell in Y^n is non-degenerate if and only if its image in D is non-degenerate. □

Corollary A.8. *The simplicial set D of decidable degeneracy type is the cofibrant replacement $\mathbb{L}1$ of 1 .*

Corollary A.9.

- (i) *Under the assumption that ‘subsets of small sets are small sets’, the map $\mathbb{L}X \rightarrow X$ is a small fibration for any simplicial set X .*

- (ii) In CZF, given a simplicial class X , the cofibrant replacement $\mathbb{L}X \rightarrow X$ exists as a simplicial class, is cofibrant, and the class map $\mathbb{L}X \rightarrow X$ is a trivial fibrations whose fibers are sets.

Proof. For part (i), the cells of $\mathbb{L}X$ over a given cell $x \in X([n])$ are the decidable degeneracy n -types s smaller than the degeneracy type of x . In particular this identifies with a subset of the set of all degeneracy n -types and hence form a small set by assumption.

For part (ii), the explicit description of $\mathbb{L}X$ given by Proposition A.7 immediately allows to make sense of it as a simplicial class. Indeed the class of (s, x) where $x \in X_n$, s is a decidable degeneracy type smaller than the degeneracy type of x is clearly a type. Such a pair (s, x) is degenerated if and only if s is, hence this is decidable, and using Lemma A.6 one can immediately see that the projection $\epsilon_X : \mathbb{L}X \rightarrow X$ is a trivial fibrations. Finally, the fiber over an n -cell $x \in X$, is isomorphic to a subsets of the set of decidable degeneracy n -types defined by Restricted Separation, hence it is a set. \square

ACKNOWLEDGEMENTS

Nicola Gambino gratefully acknowledges the support of EPSRC under grant EP/M01729X/1 and the AFOSR under agreement FA9550-17-1-0290 and FA9550-21-1-0007 and the hospitality of the School of Mathematics of the University of Manchester and the Centre for Advanced Study in Oslo, where the paper was written while on study leave from the University of Leeds. During the preparation of this work, Simon Henry was supported by the Czech Operational Programme “Research, Development and Education” Project ‘Postdoc@MUNI’ (No. CZ.02.2.69/0.0/0.0/16_027/0008360). We are also grateful to Steve Awodey, Marc Bezem, Paolo Capriotti, Thierry Coquand, André Joyal, Andy Pitts, Michael Rathjen, Christian Sattler, Thomas Streicher, Andrew Swan and Karol Szumiło for useful conversations.

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. M. Abadi, L. Cardelli, P.-L. Curien, and J.-J. Lévy, *Explicit substitutions*, J. Funct. Programming **1** (1991), no. 4, 375–416.
2. P. Aczel, *The type theoretic interpretation of constructive set theory*, in A. Macintyre, L. Pacholski, and J. Paris (eds.), *Logic Colloquium '77*, pp. 55–66. Elsevier, North Holland, 1978.
3. P. Aczel and M. Rathjen, *Notes on constructive set theory*, 2010, <http://www1.maths.leeds.ac.uk/~rathjen/book.pdf>.
4. S. Awodey, *A cubical model of homotopy type theory*, Ann. Pure Appl. Logic **169** (2018), no. 12, 1270–1294.
5. S. Awodey and M. A. Warren, *Homotopy theoretic models of identity types*, Math. Proc. Cambridge Philos. Soc. **146** (2009), 45–55.
6. B. Barras, T. Coquand, and S. Huber, *A generalization of the Takeuti-Gandy interpretation*, Math. Structures Comput. Sci. **25** (2015), 1071–1099.
7. M. Bezem, T. Coquand, and S. Huber, *A model of type theory in cubical sets*, in R. Matthes and A. Schubert (eds.), 19th International Conference on Types for Proofs and Programs (TYPES 2013), vol. **26**, pp. 107–128. Schloss Dagstuhl, Leibniz-Zentrum für Informatik, 2014.

8. B. van den Berg and R. Garner, *Topological and simplicial models of identity types*, ACM Trans. Comput. Log. **13** (2012), no. 1, 1–44.
9. B. van den Berg and I. Moerdijk, *W-types in homotopy type theory*, Math. Structures Comput. Sci. **25** (2015), no. 5, 1100–1115.
10. M. Bezem, T. Coquand, and E. Parmann, *Non-constructivity in Kan simplicial sets*, in T. Altenkirch (ed.), 13th International Conference on Typed Lambda Calculi and Applications (TLCA 2015), vol. **38**, pp. 92–106. Schloss Dagstuhl, Leibniz-Zentrum für Informatik, 2015.
11. A. Carboni and P. Rosolini, *Locally cartesian closed exact completions*, J. Pure Appl. Algebra **154** (2000), no. 1–3, 103–116.
12. D.-C. Cisinski, *Univalent universes for elegant models of homotopy types*, arXiv:1406.0058, 2014.
13. P. Clairambault and P. Dybjer, *The biequivalence of locally cartesian closed categories and Martin-Löf type theories*, Math. Structures Comput. Sci. **24** (2014), no. 6, e240606.
14. C. Cohen, T. Coquand, S. Huber, and A. Mörtberg, *Cubical type theory: a constructive interpretation of the univalence axiom*, in T. Uustalu (ed.), 21st International Conference on Types for Proofs and Programs (TYPES 2015), vol. **69** of Leibniz International Proceedings in Informatics (LIPIcs), pp. 5:1–5:34, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
15. P.-L. Curien, *Substitution up to isomorphism*, Fund. Inform. **19** (1993), 51–85.
16. P.-L. Curien, R. Garner, and M. Hofmann, *Revisiting the categorical interpretation of type theory*, Theoret. Comput. Sci. **546** (2014), 99–119.
17. N. Gambino and R. Garner, *The identity type weak factorisation system*, Theoret. Comput. Sci. **409** (2008), 94–109.
18. N. Gambino and C. Sattler, *The Frobenius condition, right properness, and uniform fibrations*, J. Pure Appl. Algebra **221** (2017), no. 12, 3027–3068.
19. N. Gambino, C. Sattler, and K. Szumilo, *The constructive Kan-Quillen model structure: two new proofs*, Quart. J. Math., to appear. arXiv:1907.05394.
20. R. Garner, *On the strength of dependent products in the type theory of Martin-Löf*, Ann. Pure Appl. Logic **160** (2009), no. 1, 1–12.
21. R. Garner, *Understanding the small object argument*, Appl. Categ. Structures **17** (2009), no. 3, 247–285.
22. M. Grandis and W. Tholen, *Natural weak factorisation systems*, Arch. Math. **42** (2006), 397–408.
23. S. Henry, *Weak model categories in classical and constructive mathematics*, Theory Appl. Categ. **35** (2020), no. 24, 875–958.
24. S. Henry, *A constructive treatment of the Kan-Quillen model structure and of Kan's Ex^∞ functor*. arXiv:1905.06160, 2019.
25. M. Hofmann, *On the interpretation of type theory in locally cartesian closed categories*, in Computer Science Logic (CSL '94), vol. 933 of Lecture Notes in Computer Science, pp. 427–441. Springer, Berlin, Heidelberg, 1994.
26. M. Hofmann and T. Streicher, *Lifting Grothendieck universes*, Unpublished note, 1997. <http://www2.mathematik.tu-darmstadt.de/~streicher/NOTES/lift.pdf>.
27. B. Jacobs, *Categorical logic and type theory*, Elsevier, North Holland, 1999.
28. A. Joyal and M. Tierney, *Quasi-categories vs Segal spaces*, In Categories in algebra, geometry and mathematical physics, vol. 431 of Contemp. Math., pp. 277–326. American Mathematical Society, Providence, Rhode Island, 2007.
29. A. Joyal and M. Tierney, *Notes on simplicial homotopy theory. Lecture notes*, 2008, <http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern47.pdf>.
30. C. Kapulkin and P. LeFanu Lumsdaine, *The simplicial model of Univalent Foundations (after Voevodsky)*, J. Eur. Math. Soc., **23** (2021), no. 6, 2071–2126.
31. C. Kapulkin, P. LeFanu Lumsdaine, and V. Voevodsky, *Univalence in simplicial sets*, ArXiv:1203:2553, 2012.
32. P. LeFanu Lumsdaine and M. Warren, *The local universes model: an overlooked coherence construction for dependent type theories*, ACM Trans. Comput. Log. **16** (2014), no. 3, 23:1–23:31.
33. J. P. May, *Simplicial objects in algebraic topology*, Chicago University Press, Chicago, 1967.
34. B. Nordström, K. Petersson, and J. Smith, *Martin-Löf type theory*, in S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum (eds.), *Handbook of logic in computer science, Oxford logic guides*, chapter V, pp. 1–37. Oxford University Press, Oxford, 2001.

35. I. Orton and A. M. Pitts, *Axioms for modelling cubical type theory in a topos*, Log. Methods Comput. Sci. **14** (2018), no. (4:23), 1–33. Special issue for CSL 2016.
36. C. Sattler, *The equivalence extension property and model categories*, arXiv:1704.06911, 2017.
37. C. Sattler, *Failure of the algebraic universe of types with prism-based Kan composition in simplicial sets*. Unpublished note, 2018.
38. M. Shulman, *The univalence axiom for elegant Reedy presheaves*, Homology, Homotopy Appl. **17** (2015), no. 2, 81–106.
39. M. Shulman, *Univalence for inverse diagrams and homotopy canonicity*, Math. Structures Comput. Sci. **25** (2015), no. 5, 1203–1277.
40. M. Shulman, *All $(\infty, 1)$ -toposes have strict univalent universes*, arXiv:1904.07004, 2019.
41. R. Stenzel, *On univalence, Rezk completeness and presentable quasi-categories*, PhD thesis, University of Leeds, 2019. <http://etheses.whiterose.ac.uk/24342>.
42. T. Streicher, *A model of type theory in simplicial sets: a brief introduction to Voevodsky's homotopy type theory*, J. Appl. Log. **12** (2014), no. 1, 45–49.
43. The Univalent Foundations Program, *Homotopy type theory: univalent foundations of mathematics*, 2013, <https://homotopytypetheory.org/book>, Institute for Advanced Study.
44. V. Voevodsky, *Notes on type systems*. Unpublished note, 2009. http://www.math.ias.edu/~vladimir/Site3/Univalent_Foundations_files/expressions_current.pdf.
45. V. Voevodsky, *An experimental library of formalized mathematics based on the univalent foundations*, Math. Structures Comput. Sci. **25** (2015), no. 5, 1278–1294.