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Online Appendix

A. Proof of Proposition 1

Note that the process $e^{-rt}g(Y_t)$ is a supermartingale, because $r > \mu_2$. Therefore, the expectation

$$f(x,y) = \mathsf{E}\left[-c\int_0^{\tau_*^x} e^{-rt}\mathrm{d}t + e^{-r\tau_*^x} g(Y_{\tau_*}^y)\right],$$

is finite.

1. *(f is increasing in x)* Fix $\omega \in \Omega$ and $(x,y) \in \mathcal{E}$. Let $\varepsilon > 0$ be such that $x+\varepsilon \leq x^*$. Note that for every $t \geq 0$, it holds that

$$\tau_*^{x+\varepsilon} < \tau_*^x, \quad \mathsf{P}-\text{a.s.}$$

So,

$$\begin{aligned}
f(x+\varepsilon,y) &= \mathsf{E}\left[-c\int_0^{\tau_*^{x+\varepsilon}} e^{-rt}\mathrm{d}t\right] + \mathsf{E}\left[e^{-r\tau_*^{x+\varepsilon}} g(Y_{\tau_*^{x+\varepsilon}}^y)\right] \\
&> \mathsf{E}\left[-c\int_0^{\tau_*^x} e^{-rt}dt\right] + \mathsf{E}\left[e^{-r\tau_*^x} g(Y_{\tau_*}^y)\right] \\
&\geq \mathsf{E}\left[-c\int_0^{\tau_*^x} e^{-rt}\mathrm{d}t\right] + \mathsf{E}\left[e^{-r\tau_*^x} g(Y_{\tau_*}^y)\right] \\
&= f(x,y),
\end{aligned}$$

where the last inequality follows from the supermartingale property.

2. *(f is increasing in y)* Fix $\omega \in \Omega$ and $(x,y) \in \mathcal{E}$. Let $\varepsilon > 0$. For any $t \geq 0$ it holds that

$$\begin{aligned}
g\left(Y_t^{y+\varepsilon}\right) &= \frac{(y+\varepsilon)Y_t^1}{r-\mu_2} - \frac{\eta}{r} \\
&= \frac{\varepsilon Y_t^1}{r-\mu_2} + g\left(Y_t^y\right) \\
&> g\left(Y_t^y\right).
\end{aligned}$$

Hence, $f(x,y+\varepsilon) > f(x,y)$.

3. *(f is affine in y)* Fix $x \in \mathbb{R}$, $y', y'' \in \mathbb{R}_+$ and $\lambda \in (0,1)$. Define $y = \lambda y' + (1-\lambda)y''$. Note that for all $\omega \in \Omega$ and all $t \geq 0$ it holds that

$$g\left(Y_t^y(\omega)\right) = \frac{(\lambda y' + (1-\lambda)y'')Y_t^1(\omega)}{r-\mu_2} - \frac{\eta}{r} = \lambda g\left(Y_t^{y'}(\omega)\right) + (1-\lambda)g\left(Y_t^{y''}(\omega)\right).$$

Therefore, $f\left(x, \lambda y' + (1-\lambda)y''\right) = \lambda f(x,y') + (1-\lambda)f(x,y'')$.

4. *(there is a unique optimal investment trigger y^*)* Since the optimal stopping problem

$$v(y) = \sup_\tau \mathsf{E}\left[e^{-r\tau} f(0, Y_\tau^y)\right], \tag{A.1}$$

is a standard one, in the sense that the payoff function g is increasing and affine in y, the optimal stopping time will take the form of the first-hitting time (from below) of some trigger y^* that satisfies the free-boundary problem: find a function $\varphi \geq f(0, \cdot)$ and a trigger $y^* > 0$ such that

$$\begin{cases} \mathscr{L}\varphi - r\varphi = 0 & \text{on } (0, y^*) \\ \mathscr{L}\varphi - r\varphi \leq 0 & \text{on } (y^*, \infty) \\ \varphi(y^*) = f(0, y^*) \\ \varphi'(y^*) = f'_y(0, y^*) \end{cases},$$

where $\mathscr{L}$ is the characteristic operator, which, on C^2 is defined as

$$\mathscr{L}\varphi(y) = \frac{1}{2}\sigma_2 x^2 \varphi''(y) + \mu_2 y \varphi'(y), \quad \text{all } y > 0. \tag{A.2}$$

Note that v and any y^* that solves

$$\beta_1 f(0, y^*) = y f'_y(0, y^*), \tag{A.3}$$

solves this free-boundary problem.

To show that a solution to (A.2) solves the optimal stopping problem (A.1), let

$$\tau^y(y^*) \triangleq \inf \{ \, t \geq 0 \mid Y_t^y \geq y^* \, \} \, .$$

Let y^* and $\varphi \in C^2(\mathbb{R}_{++} \setminus \{y^*\})$ solve (A.2). It then follows from Dynkin's formula (see, e.g., Øksendal, 2000) that

$$\begin{aligned} \mathsf{E}[e^{-r\tau^y(y^*)} f(0, Y_{\tau^y(y^*)})] &= \mathsf{E}[e^{-r\tau^y(y^*)} \varphi(y^*)] \\ &= \varphi(y) + \mathsf{E}\left[\int_0^{\tau^y(y^*)} e^{-rt}(\mathscr{L}\varphi(Y_t) - r\varphi(Y_t))\mathrm{d}t\right] \\ &= \varphi(y). \end{aligned}$$

Now take any stopping time τ. Then it follows, again from Dynkin's formula, that

$$\begin{aligned} \mathsf{E}\left[e^{r\tau} f(0, Y_\tau^y)\right] &\leq \mathsf{E}\left[e^{r\tau} \varphi(0, Y_\tau^y)\right] \\ &= \varphi(y) + \mathsf{E}\left[\int_0^{\tau} e^{-rt}(\mathscr{L}\varphi(Y_t) - r\varphi(Y_t))\mathrm{d}t\right] \\ &\leq \varphi(y). \end{aligned}$$

Therefore, φ and y^* solve (A.2). . To show that (A.3) has a unique solution, define

$$\chi(y) \triangleq -\beta_1 f(0, y) + y f'_y(0, y),$$

and consider the equation $\chi(y) = 0$. Note that

$$\begin{aligned} \chi(0) &= -\beta_1 f(0, 0) + f'_y(0, 0) \\ &= \beta_1 \left[\frac{c}{r} + \frac{\kappa - c}{r} e^{-\alpha_1 x^*}\right] + \mathsf{E}\left[e^{-r\tau_*} \frac{Y_{\tau_*}^1}{r - \mu_1}\right] \\ &> 0. \end{aligned}$$

Since

$$f'_y(0,y) = \mathsf{E}\left[e^{-r\tau_*}\frac{Y^1_{\tau_*}}{r-\mu_1}\right] > 0,$$

it holds that $\chi(0) < 0$ for y large enough. Therefore, there is a unique $\tilde{y}^* > 0$ such that $\chi(\tilde{y}^*) = 0$.

■

B. Proof of Proposition 2

The proof of this proposition follows the following steps.

1. The optimal stopping problem (5) is rewritten in a more convenient form.
2. The corresponding free-boundary problem is stated.
3. The general solution to the HJB equation is found.
4. Fixing an arbitrary abandonment trigger, y_L, value-matching and smooth-pasting at y_L are solved to find an auxiliary value function.
5. This auxiliary value function is shown to be strictly convex with a unique point y_H at which smooth pasting with the NPV of operationalization is satisfied. If y_L is the point where abandonment and operationalization have the same NPV, it is shown that value-matching at y_H is not satisfied.
6. As y_L is decreased (and the auxiliary value function is changed such that value-matching and smooth-pasting at y_L remain satisfied) there will be a unique point y_H where smooth-pasting and value-matching are satisfied for the operationalization decision.
7. Those triggers give a continuation region $\mathcal{C} = (y_A, y_I)$ that solves the free-boundary problem.
8. It is shown that a solution to the free-boundary problem gives a solution to the optimal stopping problem.

1. The optimal stopping problem (5) can be written as

$$\begin{aligned} G(y) &= \sup_\tau \mathsf{E}\Big[-\kappa\int_0^\tau e^{-rt}\mathrm{d}t + e^{-r\tau}\max\Big\{\frac{Y^y_\tau}{r-\mu_2} - \frac{\eta}{r}, -K\Big\}\Big] \\ &= \sup_\tau \mathsf{E}\Big[-\kappa\int_0^\infty e^{-rt}\mathrm{d}t + e^{-r\tau}\Big(\kappa\int_0^\infty e^{-rt}\mathrm{d}t + \max\Big\{\frac{Y^y_\tau}{r-\mu_2} - \frac{\eta}{r}, -K\Big\}\Big)\Big] \\ &= -\frac{\kappa}{r} + \sup_\tau \mathsf{E}\left[e^{-r\tau}\max\left\{G_I(Y^y_\tau), G_A(Y^y_\tau)\right\}\right], \end{aligned} \tag{B.1}$$

where

$$G_I(y) \triangleq \frac{y}{r-\mu_2} - \frac{\eta}{r} + \frac{\kappa}{r}, \quad \text{and} \quad G_A(y) \triangleq -K + \frac{\kappa}{r}.$$

Note that G_I and G_A are increasing and non-increasing, respectively, that $G_A(0) > G_I(0)$ and $G_I(y) > G_A(y)$ for y large enough. They imply that there is a unique point $\bar{y} \in (0,1)$ such that $G_I(\bar{y}) = G_A(\bar{y})$.

Define the function $\hat{G} : \mathbb{R}_+ \to \mathbb{R}$ by

$$\hat{G}(y) = 1_{y \le \bar{y}} G_A(y) + 1_{y > \bar{y}} G_I(y).$$

Note that $\hat{G} = G_A \vee G_I$ and that $\hat{G}$ is C^2 on $\mathbb{R}_{++} \setminus \{\bar{y}\}$. Therefore, one can write

$$G(y) = -\frac{\kappa}{r} + \sup_\tau \mathsf{E}\left[e^{-r\tau}\hat{G}(Y_\tau^y)\right]. \tag{B.2}$$

2. From Peskir and Shiryaev (2006) it follows that we need to find (i) a function $G \in C^1$, with second derivatives locally bounded, which dominates $\hat{G}$ on $\mathbb{R}_+$, and (ii) a set $\mathcal{C} \subset \mathbb{R}_+$, that, together, solve the free boundary problem

$$\begin{cases} \mathscr{L}_Y G - rG = 0 \quad \text{on } \mathcal{C}, \text{ and} \quad \mathscr{L}_Y G - rG < 0 \quad \text{on } \mathbb{R}_{++} \setminus \mathcal{C} \\ G > \hat{G} \quad \text{on } \mathcal{C}, \text{ and} \quad G = \hat{G} \quad \text{on } \mathbb{R}_{++} \setminus \mathcal{C} \\ \frac{\partial G}{\partial y}_{|\partial\mathcal{C}} = \frac{\partial \hat{G}}{\partial y}_{|\partial\mathcal{C}}. \end{cases} \tag{B.3}$$

Here $\mathscr{L}_Y$ denotes the characteristic operator of the process Y, i.e. for any $\varphi \in C^2$,

$$\mathscr{L}_Y \varphi(y) = \frac{1}{2}\sigma_2^2 y^2 \varphi''(y) + \mu_2 y \varphi'(y).$$

Note that the condition

$$\mathscr{L}_Y G - rG < 0 \quad \text{on } \mathbb{R}_{++} \setminus \mathcal{C},$$

is always satisfied since $\hat{G}'' = 0$ on $\mathbb{R}_{++} \setminus \{\bar{y}\}$. Also note that $\hat{G} > 0$. [The proof that a solution to the free-boundary problem solves the optimal stopping problem is given under point 8 below.]

3. On $\mathbb{R}_{++}$, define the functions $\hat{\varphi} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $\check{\varphi} : \mathbb{R}_{++} \to \mathbb{R}_{++}$, by[12]

$$\hat{\varphi}(y) = y^{\beta_1}, \quad \text{and} \quad \check{\varphi}(y) = y^{\beta_2}, \tag{B.4}$$

where $\beta_1 > 1$ and $\beta_2 < 0$ are such that $\mathscr{Q}(\beta_1) = \mathscr{Q}(\beta_2) = 0$; cf. (1). The functions $\hat{\varphi}$ and $\check{\varphi}$ are the increasing and decreasing solutions to the differential equation $\mathscr{L}_Y \varphi - r\varphi = 0$, respectively. So, any solution to $\mathscr{L}_Y \varphi - r\varphi = 0$ is of the form

$$\varphi = \hat{A}\hat{\varphi} + \check{A}\check{\varphi},$$

where $\hat{A}$ and $\check{A}$ are arbitrary constants.

4. Fix $y_L \leq \bar{y}$ and define the mapping $y \mapsto \check{V}(y; y_L)$, by

$$\check{V}(y; y_L) = \hat{A}(y_L)\hat{\varphi}(y) + \check{A}(y_L)\check{\varphi}(y), \tag{B.5}$$

where the constants $\hat{A}(y_L)$ and $\check{A}(y_L)$ are given by

$$\hat{A}(y_L) = \frac{\check{\varphi}(y_L)G_A'(y_L) - \check{\varphi}'(y_L)G_A(y_L)}{\check{\varphi}(y_L)\hat{\varphi}'(y_L) - \check{\varphi}'(y_L)\hat{\varphi}(y_L)} = -\frac{\beta_2}{\beta_1 - \beta_2} y_L^{-\beta_1}\left(\frac{\kappa}{r} - K\right), \quad \text{and}$$
$$\check{A}(y_L) = \frac{\hat{\varphi}'(y_L)G_A(y_L) - \hat{\varphi}(y_L)G_A'(y_L)}{\check{\varphi}(y_L)\hat{\varphi}'(y_L) - \check{\varphi}'(y_L)\hat{\varphi}(y_L)} = \frac{\beta_1}{\beta_1 - \beta_2} y_L^{-\beta_2}\left(\frac{\kappa}{r} - K\right).$$

[12] The results in this part of the proof are standard and can be found in the literature, such as, for example, Borodin and Salminen (1996). They are collected here for ease of reference.

[These constants have been obtained from imposing value matching and smooth pasting at y_L.] From this it easily follows that

$$\begin{aligned}\frac{\partial \hat{A}(y_L)}{\partial y_L} &= \frac{\beta_1\beta_2}{\beta_1-\beta_2} y_L^{-\beta_1-1}\left(\frac{\kappa}{r}-K\right)<0, \quad \text{and}\\ \frac{\partial \check{A}(y_L)}{\partial y_L} &= -\frac{\beta_1\beta_2}{\beta_1-\beta_2} y_L^{-\beta_2-1}\left(\frac{\kappa}{r}-K\right)>0.\end{aligned}$$

5. Since $\hat{A}(y_L)>0$ and $\check{A}(y_L)>0$ for all $y_L\leq\bar{y}$, the mapping $y\mapsto\check{V}(y;\bar{y})$ is (strictly) convex. This follows from $\hat{\varphi}''>0$ and $\check{\varphi}''>0$. In addition, it satisfies $\check{V}(\cdot;\bar{y})\to\infty$ as $y\to\infty$ or $y\downarrow 0$. So, there is a unique point $y_H\in(\bar{y},\infty)$ such that $\check{V}'(y_H;\bar{y})=G_I'(y_H)$. Since $\check{V}'(\bar{y};\bar{y})=G_A'(\bar{y})=0$, at y_H it holds that $\check{V}(y_H;\bar{y})<G_I(y_H)$. Also, for y large enough, it holds that $\check{V}(y;\bar{y})>G_I(y)$.

6. Since $\hat{A}(y_L)$ decreases in y_L and $\check{A}(y_L)$ increases in y_L, the mapping $y\mapsto\check{V}(y;y_L)$ has the property that for every $y>y_L$ it holds that $\partial\check{V}(y;y_L)/\partial y_L<0$. So, the point $y_H\in(\bar{y},\infty)$ where $\check{V}'(y_H;y_L)=G_I'(y_H)$ is decreasing in y_L, as is the value $\check{V}(y_H;y_L)$. [This follows from G_I' being constant and $\check{V}'(y;y_L)$ being larger for every y when y_L is smaller. So, the value for which $\check{V}'(y_H;y_L)=G_I'(y_H)$ must be smaller for lower y_L.] Now decrease y_L from $\bar{y}$ to 0. There will be a unique y_A, with corresponding y_I at which $\check{V}(y_I;y_A)=G_I(y_I)$ and $\check{V}'(y_I;y_A)=G_I'(y_I)$. Figure 7 illustrates the argument.

7. The interval $\mathcal{C}=(y_A,y_I)$ and the proposed function G together solve the free-boundary problem (B.3). The fact that y_A and y_I are the unique triggers that make G a C^1 function on (y_A,y_I) follows by construction.

8. The proof that a solution to (B.3) is also a solution to the optimal stopping problem, i.e. that

$$\tilde{G}(y)\triangleq G(y)+\frac{\kappa}{r}=\sup_\tau \mathsf{E}\left[e^{-r\tau}\hat{G}(Y_\tau^y)\right]\triangleq\sup_\tau J^\tau(y),$$

is standard. Here I will sketch the main argument, for technical details see, e.g., Peskir and Shiryaev (2006). Obviously, it holds that $\tilde{G}(y)\leq\sup_\tau J^\tau(y)$. To prove the reverse inequality, take any stopping time τ. It now holds that

$$\begin{aligned}\tilde{G}(y)&=\mathsf{E}\left[e^{-r\tau}\tilde{G}(Y_\tau^y)\right]-\mathsf{E}\left[\int_0^\tau e^{-rt}(\mathscr{L}_Y-r)\tilde{G}(Y_t^y)dt\right]\\ &\geq\mathsf{E}\left[e^{-r\tau}\tilde{G}(Y_\tau^y)\right]\geq\mathsf{E}\left[e^{-r\tau}\hat{G}(Y_\tau^y)\right]=J^\tau(y),\end{aligned}$$

where the first equality follows from Dynkin's formula (Øksendal, 2000), the first inequality follows from $\mathscr{L}G^*-rG^*\leq 0$, and the second inequality follows from $\tilde{G}\geq\hat{G}$. Hence, $\tilde{G}(y)\geq\sup_\tau J^\tau(y)$.

■

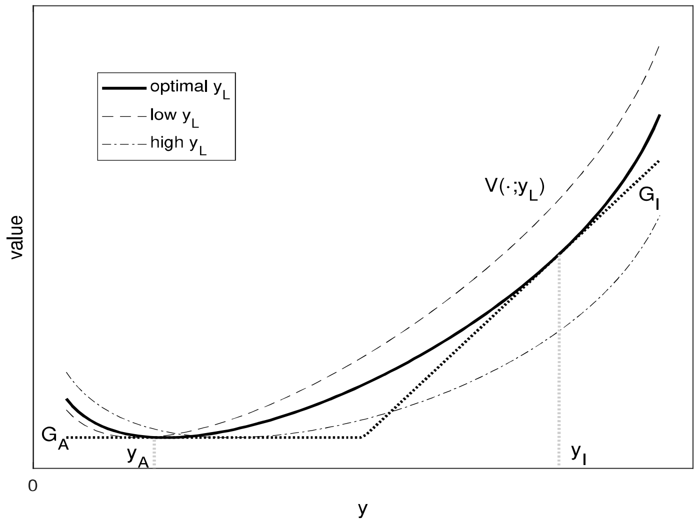

Figure 7: The function $y \mapsto \check{V}(y; y_L)$ for different values of y_L.

C. Proof of Proposition 3

Define

$$\hat{G}(x, y) = 1_{x<x^*}(-K) + 1_{x\geq x^*}G(y),$$

so that we can write

$$F(x, y) = -\frac{c}{r} + \sup_{\tau\leq\tau_*^x} \mathsf{E}\left[e^{-r\tau}\left(\frac{c}{r} + \hat{G}(X_\tau^x, Y_\tau^y)\right)\right].$$

1. *(Description of $x \mapsto b(x)$)* Let $(x, y) \in \mathcal{C}$, i.e. $F(x, y) > \hat{G}(x, y)$. There exists a stopping time $\tau \leq \tau_*^x$ such that

$$\begin{aligned} F(x, y) &\geq -\frac{c}{r} + \mathsf{E}\left[e^{-r\tau}\left(\frac{c}{r} + \hat{G}(X_\tau^x, Y_\tau^y)\right)\right] \\ &> \hat{G}(x, y) = -K. \end{aligned}$$

Let $\varepsilon > 0$. It then holds that

$$\begin{aligned} F(x, y + \varepsilon) &\geq -\frac{c}{r} + \mathsf{E}\left[e^{-r\tau}\left(\frac{c}{r} + \hat{G}(X_\tau^x, Y_\tau^{y+\varepsilon})\right)\right] \\ &= -\frac{c}{r} + \mathsf{E}\left[e^{-r\tau}\left(\frac{c}{r} + \hat{G}(X_\tau^x, Y_\tau^y) + 1_{\tau=\tau_*^x}\frac{\varepsilon Y_\tau^1}{r-\mu_2}\right)\right] \\ &> \hat{G}(x, y) + \mathsf{E}\left[1_{\tau=\tau_*^x}e^{-r\tau}\frac{\varepsilon Y_\tau^1}{r-\mu_2}\right] \\ &\geq \hat{G}(x, y) = -K = \hat{G}(x, y + \varepsilon). \end{aligned}$$

Therefore, $(x, y + \varepsilon) \in \mathcal{C}$. Since ε was chosen arbitrarily, we can write $b(x)$ as claimed.

2. *($y \mapsto F(x, y)$ is convex and increasing on $\mathcal{C}$)* First we prove convexity. Let $x \leq x^*$ and $y', y'' \in \mathbb{R}_+$. Fix $\lambda \in (0, 1)$ and define $y = \lambda y' + (1 - \lambda)y''$. Note that G is a convex function.

Therefore,

$$\begin{aligned}\hat{G}(x,y) &= 1_{x<x^*}(-K) + 1_{x=x^*}G(y)\\ &\leq 1_{x<x^*}(-K) + 1_{x=x^*}\left(\lambda G(y') + (1-\lambda)G(y'')\right)\\ &= \lambda\hat{G}(x,y') + (1-\lambda)\hat{G}(x,y''),\end{aligned}$$

which implies that $\hat{G}$ is convex in y. It then follows that

$$\begin{aligned}F(x,y) =& -\frac{c}{r} + \sup_{\tau\leq\tau_*^x}\mathsf{E}\left[e^{-r\tau}\left(\frac{c}{r} + \hat{G}(X_\tau^x, Y_\tau^y)\right)\right]\\ \leq& -\frac{c}{r} + \sup_{\tau\leq\tau_*^x}\mathsf{E}\left[e^{-r\tau}\left\{\lambda\left(\frac{c}{r} + \hat{G}(X_\tau^x, Y_\tau^{y'})\right) + (1-\lambda)\left[\lambda\left(\frac{c}{r} + \hat{G}(X_\tau^x, Y_\tau^{y''})\right)\right]\right\}\right]\\ \leq&\lambda\left\{-\frac{c}{r} + \sup_{\tau\leq\tau_*^x}\mathsf{E}\left[e^{-r\tau}\left(\frac{c}{r} + \hat{G}(X_\tau^x, Y_\tau^{y'})\right)\right]\right\}\\ &+ (1-\lambda)\left\{-\frac{c}{r} + \sup_{\tau\leq\tau_*^x}\mathsf{E}\left[e^{-r\tau}\left(\frac{c}{r} + \hat{G}(X_\tau^x, Y_\tau^{y''})\right)\right]\right\}\\ =&\lambda F(x,y') + (1-\lambda)F(x,y'').\end{aligned}$$

To prove that $y \mapsto F(x,y)$ is increasing on $\mathcal{C}$, consider the points $(x,y) \in \mathcal{C}$ and $(x, y+\varepsilon) \in \mathcal{C}$, for some $\varepsilon > 0$. Let τ^y be the optimal stopping time when the process (X,Y) starts at (x,y). There are two possible cases.

1. $\tau^y = \tau_*^x$. In this case τ^y is also the optimal stopping time for the starting point $(x, y+\varepsilon)$. On the event $\{Y_{\tau^y}^y > \check{y}\}$ (which has positive probability) it holds that

$$\begin{aligned}&\left(-c\int_0^{\tau^y} e^{-rt}\mathrm{d}t + e^{-r\tau^y}G(Y_{\tau^y}^{y+\varepsilon})\right) - \left(-c\int_0^{\tau^y} e^{-rt}\mathrm{d}t + e^{-r\tau^y}G(Y_{\tau^y}^{y})\right)\\ =&\left(-c\int_0^{\tau^y} e^{-rt}\mathrm{d}t + e^{-r\tau^y}\left(\frac{(y+\varepsilon)Y_{\tau^y}^1}{r-\mu_2} - \frac{\eta}{r}\right)\right) - \left(-c\int_0^{\tau^y} e^{-rt}\mathrm{d}t + e^{-r\tau^y}\left(\frac{yY_{\tau^y}^1}{r-\mu_2} - \frac{\eta}{r}\right)\right)\\ =& e^{-rt}\frac{\varepsilon Y_{\tau^y}^1}{r-\mu_2} > 0.\end{aligned}$$

2. $\tau^y < \tau_*^x$. Let $\tau^{y+\varepsilon}$ be the optimal stopping time starting at $(x, y+\varepsilon)$. Since $\tau^{y+\varepsilon} \geq \tau^y$, P-a.s., it then holds that

$$\begin{aligned}&\left(-c\int_0^{\tau^{y+\varepsilon}} e^{-rt}\mathrm{d}t + e^{-r\tau^{y+\varepsilon}}G(Y_{\tau^{y+\epsilon}})\right) - \left(-c\int_0^{\tau^y} e^{-rt}\mathrm{d}t + e^{-r\tau^y}G(Y_{\tau^y})\right)\\ =&\left(-c\int_0^{\tau^y} e^{-rt}\mathrm{d}t + e^{-r\tau^y}F(X_{\tau^y}Y_{\tau^y})\right) - \left(-c\int_0^{\tau^y} e^{-rt}\mathrm{d}t + e^{-r\tau^y}(-K)\right)\\ =&e^{-r\tau^y}\left[F(X_{\tau^y}, Y_{\tau^y}) - K\right] \geq 0.\end{aligned}$$

Noting that $\mathsf{P}(\tau^y = \tau_*^x) > 0$, because, otherwise, $(x,y) \notin \mathcal{C}$, leads to the conclusion that $F(x, y+\varepsilon) - F(x,y) > 0$.

3. *($x \mapsto F(x,y)$ is non-decreasing on $\mathcal{C}$)* We first show that $x \mapsto F(x,y)$ is non-decreasing on $\mathcal{C}$. Take $(x,y) \in \mathcal{C}$, $\varepsilon > 0$ such that $x + \varepsilon < x^*$ and a stopping time $\tau^1 \leq \tau_*^x$, P-a.s. Define $\tau^2 = \tau^1 \wedge \tau_*^{x+\varepsilon}$. Consider the following mutually exclusive and collectively exhaustive events.

1. $\{\ X^x_{\tau^1}(\omega) < x^*\ \} \cap \{\ X^{x+\epsilon}_{\tau^2}(\omega) < x^*\ \}$. On this event $\tau^1 = \tau^2$ and, thus,

$$\begin{aligned}
&\left(-c\int_0^{\tau^2} e^{-rt}\mathrm{d}t + e^{-r\tau^2}G(X^{x+\varepsilon}_{\tau^2})\right) - \left(-c\int_0^{\tau^1} e^{-rt}\mathrm{d}t + e^{-r\tau^1}G(X^{x+\varepsilon}_{\tau^1})\right)\\
=&\left(-c\int_0^{\tau^1} e^{-rt}\mathrm{d}t + e^{-r\tau^1}(-K)\right) - \left(-c\int_0^{\tau^1} e^{-rt}\mathrm{d}t + e^{-r\tau^1}(-K)\right)\\
=&0.
\end{aligned}$$

2. $\{\ X^x_{\tau^1}(\omega) < x^*\ \} \cap \{\ X^{x+\epsilon}_{\tau^2}(\omega) = x^*\ \}$. On this event it holds that $\tau^2 < \tau^1$ and we get

$$\begin{aligned}
&\left(-c\int_0^{\tau^2} e^{-rt}\mathrm{d}t + e^{-r\tau^2}G(X^{x+\varepsilon}_{\tau^2})\right) - \left(-c\int_0^{\tau^1} e^{-rt}\mathrm{d}t + e^{-r\tau^1}G(X^{x+\varepsilon}_{\tau^1})\right)\\
=&\left(-c\int_0^{\tau_*^{x+\varepsilon}} e^{-rt}\mathrm{d}t + e^{-r\tau_*^{x+\varepsilon}}G(Y^y_{\tau_*^{x+\varepsilon}})\right) - \left(-c\int_0^{\tau^1} e^{-rt}\mathrm{d}t + e^{-r\tau^1}(-K)\right)\\
\geq&\left(-c\int_0^{\tau_*^{x+\varepsilon}} e^{-rt}\mathrm{d}t + e^{-r\tau_*^{x+\varepsilon}}(-K)\right) - \left(-c\int_0^{\tau^1} e^{-rt}\mathrm{d}t + e^{-r\tau^1}(-K)\right)\\
>&\left(-c\int_0^{\tau^1} e^{-rt}\mathrm{d}t + e^{-r\tau^1}(-K)\right) - \left(-c\int_0^{\tau^1} e^{-rt}\mathrm{d}t + e^{-r\tau^1}(-K)\right)\\
=&0.
\end{aligned}$$

3. $\{\ X^x_{\tau^1}(\omega) = X^{x+\epsilon}_{\tau^2}(\omega) = x^*\ \}$. On this event it holds that $\tau^1 > \tau^2$. Consider the process H, defined by

$$H_t \triangleq -\frac{c}{r} + e^{-rt}\left(\frac{c}{r} + G(Y_t)\right).$$

Note that $\mathsf{E}[H_t|\mathscr{F}_s] \leq X_s$ for all $t > s$, i.e., that H is a supermartingale. This follows from

$$\begin{aligned}
\mathsf{E}[H_t|\mathscr{F}_s] &= -\left(1-e^{-rt}\right)\frac{c}{r} + e^{-rs}\mathsf{E}\Big[e^{-r(t-s)}G(Y_t)\Big|\mathscr{F}_s\Big]\\
&\leq -\left(1-e^{-rt}\right)\frac{c}{r} + e^{-rs}G(Y_s)\\
&\leq -\left(1-e^{-rs}\right)\frac{c}{r} + e^{-rs}G(Y_s)\\
&= H_s,
\end{aligned}$$

where the first inequality follows from the fact that $e^{-r\cdot}G(Y_\cdot)$ is a supermartingale, it being the solution to an optimal stopping problem. From this it follows that

$$\left(-c\int_0^{\tau^2} e^{-rt}\mathrm{d}t + e^{-r\tau^2}G(X^{x+\varepsilon}_{\tau^2})\right) - \left(-c\int_0^{\tau^1} e^{-rt}\mathrm{d}t + e^{-r\tau^1}G(X^{x+\varepsilon}_{\tau^1})\right) \geq 0.$$

Since τ^1 was taken to be an arbitrary stopping time, it, therefore, follows that $F(x+\varepsilon, y) \geq F(x, y)$.

4. *($x \mapsto b(x)$ is non-increasing and continuous)* Let $(x, y) \in \mathcal{C}$ with $x < x^*$. Then there exists a stopping time $\tau \leq \tau_*^x$ such that

$$\begin{aligned}
F(x,y) &\geq -\frac{c}{r} + \mathsf{E}\left[e^{-r\tau}\left(\frac{c}{r} + \hat{G}(X^x_\tau, Y^y_\tau)\right)\right]\\
&> \hat{G}(x,y) = -K.
\end{aligned}$$

Let $\varepsilon > 0$ be such that $x + \varepsilon \leq x^*$, and define $\hat{\tau} \triangleq \tau \wedge \tau_*^{x+\varepsilon}$. Note that $\hat{\tau} \leq \tau$, P-a.s.. Since,

$$\begin{aligned} F(x+\varepsilon, y) &\geq -\frac{c}{r} + \mathsf{E}\left[e^{-r\hat{\tau}}\left(\frac{c}{r} + \hat{G}(X_{\hat{\tau}}^{x+\varepsilon}, Y_{\hat{\tau}}^{y})\right)\right] \\ &\geq -\frac{c}{r} + \mathsf{E}\left[e^{-r\tau}\left(\frac{c}{r} + \hat{G}(X_{\tau}^{x}, Y_{\tau}^{y})\right)\right] \\ &> \hat{G}(x,y) = -K = \hat{G}(x+\varepsilon, y), \end{aligned}$$

it holds that $(x+\varepsilon, y) \in \mathcal{C}$ and, thus, that $b(x+\varepsilon) \leq b(x)$.

To prove continuity, suppose, ad absurdum, that at some $\hat{x} < x^*$ there is a jump in $b(x)$. This jump can only be downwards, as b is a non-increasing function. Consider $(\hat{x}, b(\hat{x}))$. It holds that $F(\hat{x}, b(\hat{x})) = -K$. Now consider the right-hand limit of $\hat{x}$ on the boundary i.e. $F(\hat{x}+, b(\hat{x}+))$. Since both points are on the boundary, it holds that $F(\hat{x}, b(\hat{x})) = F(\hat{x}+, b(\hat{x}+)) = -K$. However, because of the downwards jump, it must be the case that $b(\hat{x}) > b(\hat{x}+)$. Now consider the point $F(\hat{x}+, b(\hat{x}))$. Since $F_y(x,y) > 0$, we know that $F(\hat{x}+, b(\hat{x})) > F(\hat{x}+, b(\hat{x}+))$. So, it, therefore, also holds that $F(\hat{x}+, b(\hat{x})) > F(\hat{x}, b(\hat{x}))$. This, however, contradicts results from the general theory of optimal stopping, e.g., Krylov (1980, Theorem 3.1.5), that show that the value function F is continuous. ■

D. Proof of Proposition 4

1. As in the proof of Proposition 1, there is a trigger y^* such that the optimal stopping time is the first exit time of the interval $(0, y^*)$. As before, the trigger y^* should satisfy the first-order condition

$$\chi(y) \triangleq y F_y'(0,y) - \beta_1 F(0,y).$$

It holds that

$$\chi(0) = -\beta_1 F(0,0) = \beta_1 K > 0.$$

Note that $F(0,\cdot) > f(0,\cdot)$ and that, for y large enough, $F(0,y)$ and $f(0,y)$ can be arbitrarily close. [Intuitively, when y is very large the exit option is so unlikely to ever be exercised that it is worthless.] Since $F(0,\cdot)$ is a convex function and $f(0,\cdot)$ is an affine function it must, therefore, hold that $f_y'(0,\cdot) > F_y'(\cdot)$. For the optimal investment trigger of the problem without exit option, $\tilde{y}^*$ it, therefore, holds that

$$\beta_1 F(0,\tilde{y}^*) \geq \beta_1 f(0,\tilde{y}^*) = \tilde{y}^* f_y'(0,\tilde{y}^*) > \tilde{y}^* F_y'(0,\tilde{y}^*),$$

so that $\chi(\tilde{y}^*) < 0$. Therefore, there exists $y^* \in (0,\tilde{y}^*)$ such that $\chi(y^*) = 0$.

2. Take the continuation region to be $\mathcal{C} = (0, y^*)$ and consider the value function

$$V(y) = \begin{cases} \varphi(y) & \text{if } y < y^* \\ F(0,y) & \text{if } y \geq y^* \end{cases},$$

where

$$\varphi(y) \triangleq \left(\frac{y}{y^*}\right)^{\beta_1} F(0, y^*).$$

On $\mathcal{C}$ it holds, by construction, that $\mathscr{L}_Y V - rV = 0$. It remains to check that $\mathscr{L}_Y V - rV \leq 0$ on $\mathcal{C}^c$. To establish this we use the facts that (i) $x \mapsto F(x, y)$ is a convex mapping, (ii) $\mathscr{L}_{X,Y} F = rF$ on $(0, x^*] \times \mathbb{R}_+$, where $\mathscr{L}_{X,Y}$ is the characteristic operator of the process (X, Y), i.e., for $\psi \in C^2$,

$$\mathscr{L}_{X,Y}\psi(x,y) \triangleq \frac{1}{2}\sigma_1^2\psi''_{xx}(x,y) + \frac{1}{2}\sigma_2^2 y^2 \psi''_{yy}(x,y) + \mu_1\psi'_x(x,y) + \mu_2 y \psi'_y(x,y),$$

and (iii) $F'_x \geq 0$. It then follows that, on $[y^*, \infty)$,

$$\begin{aligned}
\mathscr{L}_Y V(y) - rV(y) &= \mathscr{L}_Y F(0,y) - rF(0,y) \\
&= \frac{1}{2}\sigma_2^2 y^2 F''_{yy}(0,y) + \mu_2 y F'_y(x,y) - rF(0,y) \\
&= -\frac{1}{2}\sigma_1^2 F''_{xx}(0,y) - \mu_1 F'_x(0,y) \\
&\leq -\frac{1}{2}\sigma_1^2 F''_{xx}(0,y) \leq 0.
\end{aligned}$$

3. The proof that V and $\mathcal{C}$ solve the optimal stopping problem (7) follows along the same lines as in point 8 of the proof of Proposition 2. ∎

E. Proof of Proposition 5

1. Suppose that C and φ solve (13). Let

$$\tau_C^{x,y} \triangleq \inf\{t \geq 0 : (X_t^x, Y_t^y) \notin C\},$$

be the first exit time of C. Since $\mu_1 > 0$ it holds that $\tau_C^{x,y} < \infty$, a.s., so that for all $(x, y) \in \mathcal{E}$, Dynkin's formula gives that

$$\begin{aligned}
\mathsf{E}\left[e^{-r\tau_C^{x,y}}\varphi(X^x_{\tau_C^{x,y}}, Y^y_{\tau_C^{x,y}})\right] &= \varphi(x,y)\mathsf{E}\left[\int_0^{\tau_C^{x,y}} e^{-rt}\left(\mathscr{L}\varphi(X_t^x, Y_t^y) - r\varphi(X_t^x, Y_t^y)\right)\mathrm{d}t\right] \\
&= \varphi(x,y)\mathsf{E}\left[\int_0^{\tau_C^{x,y}} e^{-rt} c\,\mathrm{d}t\right] \\
\iff \varphi(x,y) &= \mathsf{E}\left[-c\int_0^{\tau_C^{x,y}} e^{-rt}\mathrm{d}t + e^{-r\tau_C^{x,y}}\left(1_{\tau_C^{x,y} < \tau_*^x}(-K) + 1_{\tau_C^{x,y} \geq \tau_*^x} G(Y^y_{\tau_*^x})\right)\right].
\end{aligned}$$

2. Let τ be any stopping time. For every $x < x^*$ it then holds that $\hat{\tau} \triangleq \tau \wedge \tau_*^x < \infty$ a.s., so that Dynkin's formula gives

$$\begin{aligned}
&\mathsf{E}\Big[-c\int_0^{\hat\tau} e^{-rt}\mathrm{d}t + e^{-r\hat\tau}\left(1_{\hat\tau < \tau_*^x}(-K) + 1_{\hat\tau = \tau_*^x} G(Y^y_{\hat\tau})\right)\Big] \\
&\quad \leq \mathsf{E}\Big[-\int_0^{\hat\tau} e^{-rt}\left(\mathscr{L}\varphi(X_t^x, Y_t^y) - r\varphi(X_t^x, Y_t^y)\right)\mathrm{d}t + e^{-r\hat\tau}\left(1_{\hat\tau < \tau_*^x}(-K) + 1_{\hat\tau = \tau_*^x} G(Y^y_{\hat\tau})\right)\Big] \\
&\quad = \varphi(x,y) - \mathsf{E}\Big[e^{-r\hat\tau}\varphi(X^x_{\hat\tau}, Y^y_{\hat\tau})\Big] + \mathsf{E}\Big[e^{-r\hat\tau}\left(1_{\hat\tau < \tau_*^x}(-K) + 1_{\hat\tau = \tau_*^x} G(Y^y_{\hat\tau})\right)\Big] \\
&\quad = \varphi(x,y).
\end{aligned}$$

Therefore, $F = \varphi$ and the first exit time of C is the optimal stopping time. ∎

F. Additional Figures

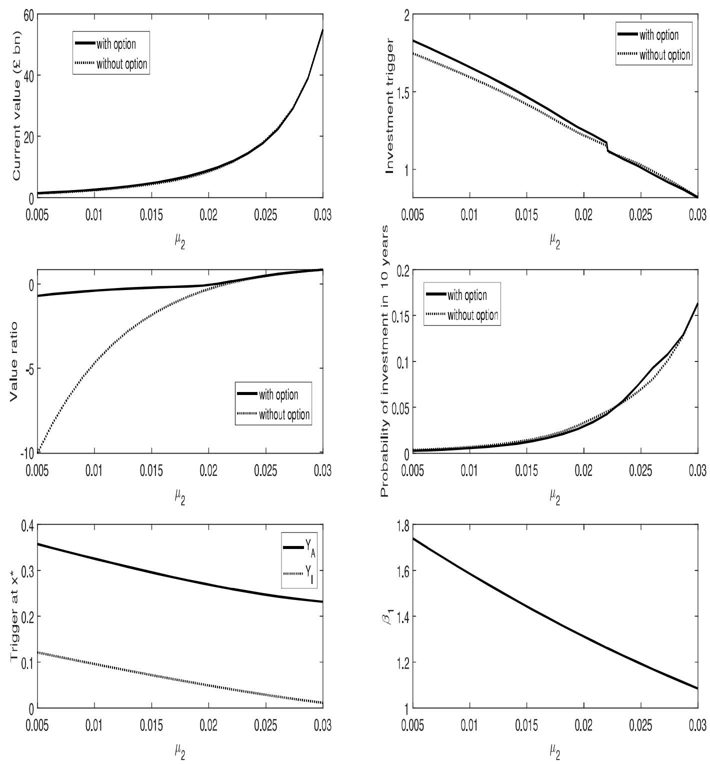

Figure 8: Value at $t = 0$ (top-left panel), optimal investment trigger (top-right panel), Value ration (middle-left panel), probability of investment in 10 years (middle-right panel), investment triggers upon completion in the model with abandonment option (bottom-left panel), and β_1 (bottom-right panel) as a function of μ_2. All other parameter values are as in the base-case scenario.

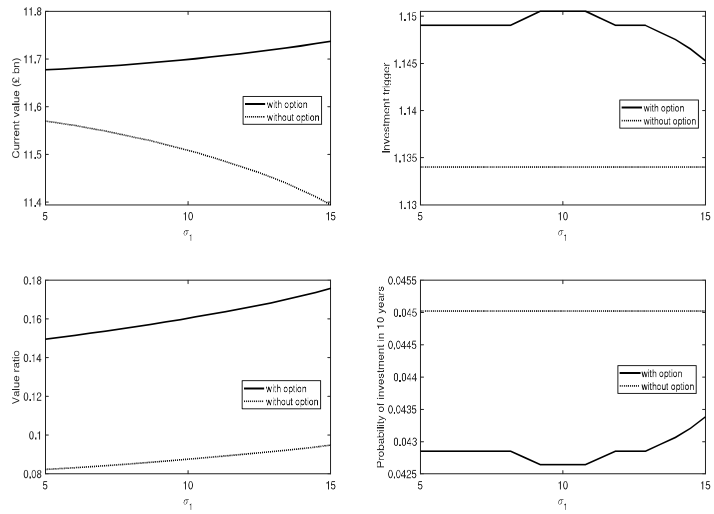

Figure 9: Value at $t = 0$ (top-left panel), optimal investment trigger (top-right panel), Value ration (bottom-left panel), and the probability of investment in 10 years (bottom-right panel) as a function of σ_1. All other parameter values are as in the base-case scenario.

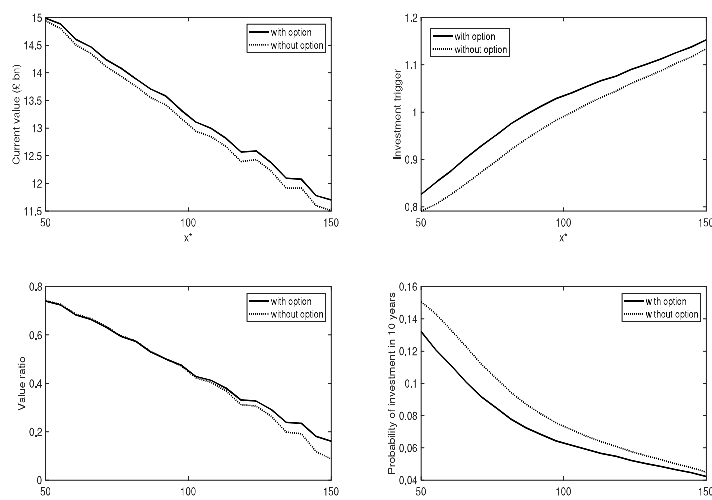

Figure 10: Value at $t = 0$ (top-left panel), optimal investment trigger (top-right panel), Value ration (bottom-left panel), and the probability of investment in 10 years (bottom-right panel) as a function of x^*. All other parameter values are as in the base-case scenario.

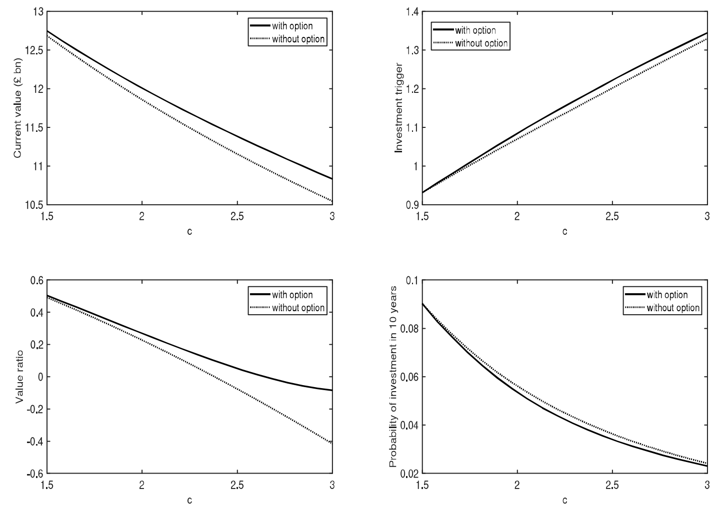

Figure 11: Value at $t = 0$ (top-left panel), optimal investment trigger (top-right panel), Value ration (bottom-left panel), and the probability of investment in 10 years (bottom-right panel) as a function of c. All other parameter values are as in the base-case scenario.

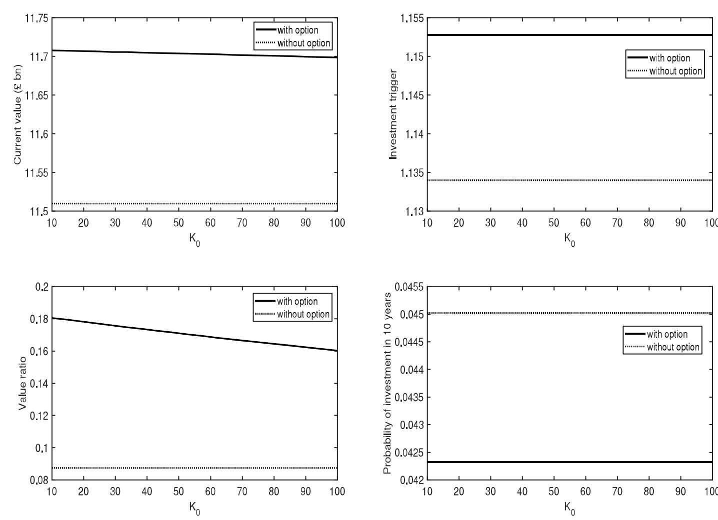

Figure 12: Value at $t = 0$ (top-left panel), optimal investment trigger (top-right panel), Value ration (bottom-left panel), and the probability of investment in 10 years (bottom-right panel) as a function of K_0. All other parameter values are as in the base-case scenario.

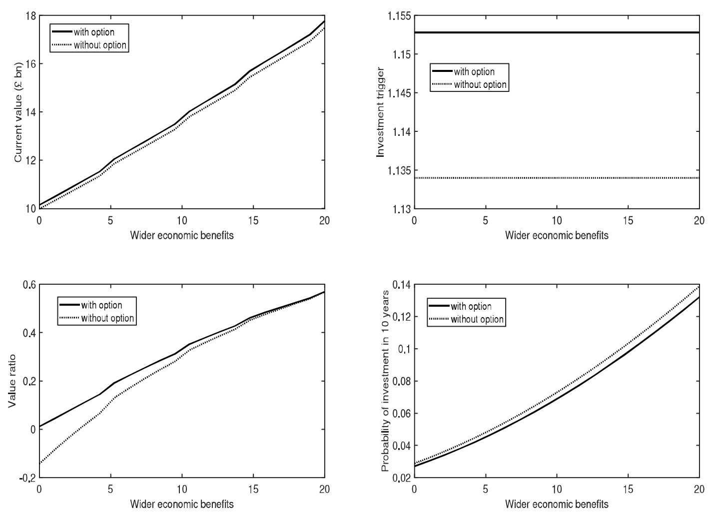

Figure 13: Comparative statics for wider economic benefits: the project's current value (top-left panel), the optimal investment trigger (top-right panel), the value ratio (bottom-left panel), and the probability of investment within 10 years (bottom-right panel). All other parameter values are as in the base-case scenario.