Mathematical Explanation Doesn’t Require Mathematical Truth

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# Abstract

The alleged existence of mathematical explanations of physical phenomena presents a *prima facie* difficulty for mathematical nominalism. If the best explanation of a physical phenomenon appeals to abstract mathematical objects, then nominalists owe an account of why they are able to accept such an explanation given that they do not accept the existence of abstracta. In some cases, where an explanation is couched in mathematical terms, it is open for a nominalist to argue that all the explanatory work resides in the explanation’s nominalistic content, and that it is this content that the nominalist believes when she accepts the mathematically-expressed explanation. However, Christopher Pincock suggests that there are some cases of mathematical explanations where the explanatory work does not reside in the nominalistic content of the explanations in question, but rather in more general structural features that the mathematics allows us to recognise. I agree with Pincock that in such cases mathematics does play a genuine explanatory role. However, I argue, we can understand how such structural explanations work from a nominalistic perspective, as the success of these explanations requires only that (a) the theorems of pure mathematics tell us what would have to be true in any system of objects satisfying the axioms of a given mathematical theory, and (b) the axioms of that theory are approximately true of the physical system to which the theory is applied.

According to recent explanatory versions of the Quine-Putnam indispensability argument for mathematical realism (QPIA), we have reason to believe in mathematical objects since mathematics plays an indispensable explanatory role in some of our best explanations of empirical phenomena. Some nominalists respond to this challenge by rejecting the indispensability of mathematics to these explanations. However, I agree with Christopher Pincock that in at least some cases (the honeycomb example being one), mathematics does do genuine explanatory work. I also agree with Pincock that mathematical explanations explain by picking out structural features of physical systems. Where I disagree, however, is the assumption that structural explanations that make use of mathematical theories to explain physical phenomena require those theories to be true (in the sense of consisting of bodies of truths about a domain of abstract objects). So, I will argue, mathematical explanation doesn’t require mathematical truth.

As Pincock explains, the original QPIA depends on a naturalist commitment to look to our best empirical scientific theories to determine our ontological commitments. In Quine’s view, our best scientific theory – that being, as Pincock puts it, “the regimented theory that maximizes the theoretical virtues” – quantifies over platonic mathematical entities, and is thus ontologically committed to such entities. So if naturalism requires us to believe our best scientific theory – at least “as a going concern” (Quine, 1975, p. 72) – then it looks as though naturalism requires us to believe in mathematical objects.

According to Pincock, there are two points of attack for nominalists in the naturalist tradition. “They may revise or deny Quine’s test for the ontological commitments of a scientific theory”, or “question the application of the Quinean theoretical virtues”, arguing that the best theory does not quantify over abstract objects. The latter approach is that pursued by Hartry Field (1980), who wishes to show that we can formulate nominalistically acceptable versions of our ordinary ‘platonistic’ (or ‘mathematically-stated’) scientific theories, and that these versions are preferable to platonistic interpretations. In particular, Field argues that his nominalistic version of Newtonian gravitational theory is preferable to the standard platonistic alternative in that it is able to provide *intrinsic explanations* of physical phenomena (that appeal only to causally relevant features), rather than the extrinsic explanations provided by their platonistic counterparts.

If in explaining the behavior of a physical system, one formulates one’s explanation in terms of relations between physical things and numbers, then the explanation is what I would call an extrinsic one. It is extrinsic because the role of the numbers is simply to serve as labels for some of the features of the physical system: there is no pretence that the properties of the numbers influence the physical system whose behavior is being explained. (Field 1985: 192-3)

Field’s contention is that explanations that are formulated in terms of the relation between physical and mathematical objects cannot be fundamental. The mathematical objects posited in these explanations are simply serving to enable us to represent, or index, relevant features of the physical system, and it is these features that are doing all the genuine explanatory work. Nominalistically-stated alternatives to ordinary platonistic scientific theories are preferable because their explanations, appealing only to physical features of physical systems, pick out the genuinely explanatorily relevant features.

Many philosophers, even on the nominalist side, are sceptical about the prospects for completing Field’s nominalization project of finding nominalistically stated alternatives to our usual platonistic scientific theories, even if they would agree that, should such alternatives be found, the explanations they provide of physical phenomena would be preferable to explanations couched in mathematical terms. Pincock suggests that the alternative for nominalists who are sceptical of Field’s attempts to dispense with mathematics is to revise or deny Quine’s account of ontological commitment. In fact, there is some equivocation in the literature on what is meant by ‘ontological commitment’. We can talk about the ontological commitments of a theory, those being the objects that would have to exist in order for the theory to be true. This is the usage at work when Quine presents his criterion of ontological commitment: “A theory is committed to those and only those entities to which the bound variables of the theory must be capable of referring in order that the affirmations made in the theory be true.” (Quine 1948: 33) But we also sometimes talk about *our* ontological commitments in making use of a theory – the objects we would have to commit to believing in to make our use of the theory in question reasonable (see, for example, Mark Colyvan’s (2001, p. 11) presentation of the indispensability argument, whose first premise states that “*We ought to have ontological commitment to all and only those entities that are indispensable to our best scientific theories”*). With these two senses of ‘ontological commitment’ at work, the proposal to ‘revise or deny’ Quine’s account of ontological commitment, as Pincock puts it, can be taken in at least two ways.

Quine’s assumption is that *our* commitments in making use of a theory should line up with the theory’s ontological commitments, stressing, as Hilary Putnam (1979, p. 347) puts it, “the intellectual dishonesty of denying the existence of what one daily presupposes”. Some nominalists accept this alignment, but challenge Quine’s account of the ontological commitments of our discourse. For example, Jody Azzouni (2004) challenges Quine’s reading of the existential quantifier as ontologically committing, arguing instead that the commitments of our theories should be to objects corresponding to theoretical posits that are suitably *ontologically independent* of us. Others, though, myself included (Leng 2010), accept the ontologically committing reading of the existential quantifier *literally construed*, agreeing that the literal truth of theories that existentially quantify over numbers and electrons requires the existence of such objects in their domain of quantification. In such cases, the platonistic commitments of our scientific theories are resisted by denying that *we* are committed, by our successful use of such theories, to believing those theories to be literally true (and therefore to adopting their ontological commitments as our own). As I have defended the latter approach, I will focus on this here, though I suspect that the major differences between Azzouni’s views and mine lie primarily in emphasis rather than substance.

According to my fictionalist approach to mathematics and to the use of mathematics in empirical science, in formulating our empirical scientific theories we do so under the supposition that there are mathematical objects satisfying the assumptions of our mathematical theories, over and above any physical objects, and formulate our theories with that supposition in place, *without regard to whether that supposition is true*. In particular, to enable us to apply mathematics we make use of the theory of ZFU, that is, ZFC set theory with urelements, taking physical objects as urelements (non-sets that can themselves be members of sets). Once we have made the assumption that physical objects can be collected into sets, we can start applying mathematics to the physical world – making use of, for example, functions mapping sets of physical objects to associated real number quantities (their masses). For example, when we say that the mass of the Earth is 5.972 x 1024 kg, the literal content of this is to say that the earth is related by the ‘mass in kilos’ function to the real number 5.972 x 1024. However, the fundamental *nominalistic* features that make this claim an appropriate one to make against the backdrop of ZFU are facts about mass-relations between the Earth and a standard kilogram unit, which are themselves unmediated by real numbers. (The key feature in this case being the fact that, if you take 5.972 x 1024 objects standing in the ‘same mass as’ relation to a standard kilogram, then the resulting mereological sum itself stands in the ‘same mass as’ relation to the Earth.) It is inconceivable that we could actually represent this relation using only the quantifiers (though this is something that is expressible ‘in principle’), let alone make use of a claim so-expressed in doing any meaningful science, hence the immense value of ZFU and the mathematics of real numbers that allow us to express these underlying facts in terms of the relations to the real numbers that our mathematical theory supposes to exist. On all this, the platonist will agree, but will add that the confirmation that the resulting theory receives from its empirical successes confirms its set theoretic assumptions as true. As a fictionalist, on the other hand, I argue that the truth of the set theoretic assumptions – interpreted literally as claims about the existence of a realm of abstract objects – is irrelevant to the success of a theory of this kind. What the set theoretic assumptions enable us to do is to express efficiently claims about physical objects and their relations that we may not otherwise be able to express or process. In particular, the fictionalist claims, what is expressed is that the physical world is configured the way *it* would have to be in order for our platonistically expressed scientific theory to be true, while remaining agnostic about the existence of the additional mathematical objects posited by that theory.

To a great extent, the motivation behind this proposed understanding of our empirical scientific theories is the same as Field’s. In both cases, ordinary (platonistic) scientific theories are viewed as useful fictions, with mathematical assumptions (particularly the axioms of ZFU) providing a valuable expressive tool, enabling us to represent features of physical systems efficiently by appeal to the relations holding between appropriately related mathematical posits (such as associated real numbers). In both cases it is held that empirical scientists can and should continue to use platonistic scientific theories, without adopting the platonistic ontological commitments of those theories, but instead believing only the nominalistic content of the theories. The key difference is that, in Field’s case, scientific realism (understood as belief in the truth of our *best* scientific theories) is still vindicated, as although he proposes that we adopt an instrumentalist attitude to the platonistic commitments of the platonistic scientific theories that we actually use, this attitude is justified by appeal to alternative nominalistically stated alternatives to those theories, which we believe to be *better theories* than the ones we generally use, and whose truth would explain the instrumental success of their platonistic counterparts (as conservative extensions of these theories). The so-called ‘easy road’ nominalism of myself and others avoids the ‘hard road’ of finding nominalistically stated alternatives to our ordinary scientific theories (the labels ‘easy road’ and ‘hard road’ are due to Colyvan (2010)), but at the cost of giving up on ordinary scientific realism (understood as belief in the truth or approximate truth of our best scientific theories). Instead ‘easy roaders’ propose a weaker ‘nominalistic scientific realism’ (Balaguer 1998, p. 130), which consists in belief in the nominalistic content of our ordinary scientific theories, but not in their platonistic components, acknowledging that the best or even only way of expressing that nominalistic content may well have to appeal to mathematical posits.

Easy road nominalists, then, believe that our best scientific theories – the ones most brimming with the Quinean theoretical virtues – are likely to be platonistic. As such, one might expect these nominalists to be immune to challenges arising out of the alleged superior explanatory power of platonistic theories as compared with nominalistic alternatives. Since they have already conceded that our most virtuous theories – including those with greatest explanatory power – are likely to be couched in platonistic terms, one might expect them to be unmoved by appeals to the explanatory virtues of platonistic scientific theories. To some extent this is right: that a scientific explanation is formulated in mathematical terms is by itself no particular reason for concern. After all, easy roaders have conceded that mathematics may well be so useful as to be *indispensable* in expressing the nominalistic content of our theories – there may simply have no better means of expressing what we take to be the facts about the physical world except by saying that the physical world is the way it would have to be in order for our mathematically stated scientific theories to be true. The indispensability of mathematical posits in formulating our best explanations of empirical phenomena should not be particularly surprising to easy-roaders, who have already conceded the vast expressive power afforded by the framework of ZFU.

This, though, hides a central difficulty. The key reason why easy roaders claim that we are not ontologically committed to the mathematical objects posited by our best scientific theories is that these posits make their way into our scientific theories in order to enable us to express what are fundamentally *non-mathematical* facts about physical objects and their relations, and this is something that can be done regardless of whether there really are any mathematical objects satisfying the assumptions of our mathematical theories. The difficulty arises when one looks at purported mathematical explanations of physical phenomena. If easy roaders wish to claim that the sole contribution made by mathematics in our scientific theories is to express some nominalistic content, then they will have to claim that the sole contribution made by mathematics *in our scientific explanations* is likewise to express some nominalistic content. As such, although mathematical assumptions may be indispensable in formulating our explanations, it looks like any genuine explanatory work that is done by these explanations must be via their nominalistic content, not their platonistic assumptions. So just as Field is committed to claiming that mathematics only plays an ‘extrinsic’ explanatory role, so it seems must the easy road nominalists commit to holding that, although mathematics may be indispensable to some explanations of physical phenomena, all the genuine explanatory work of such explanations resides in the nominalistic content that the mathematics is being used to express.

The difficulty here is that it looks as though there are some cases of mathematical explanations of physical phenomena where the mathematics itself is doing some genuine explanatory work, over and above any nominalistic content that the mathematics expresses. Pincock points to the Honeycomb Conjecture as an example of such a case (the example is due to Lyon and Colyvan (2008)). If we try to explain why bees build honeycombs the way they do, we can answer that this is an evolutionary adaptation that uses the minimum amount of wax. But if we push further and ask *why* this particular structure uses the minimum amount, then it looks like we have to answer that it’s because its 2-D cross sections are *hexagonal*, and hexagons are the maximally efficient choice of regular polygons to use in tilings, appealing to a theorem of geometry to do the explanatory work.[[1]](#footnote-1) If the only work being done by the mathematics in this explanation is to express some true nominalistic content, then it looks as though, if we were to describe the shape of the honeycomb in purely nominalistic terms, doing so should itself provide the resources for explaining why they are built this way rather than any other. But it is difficult to see how we could find any *better* understanding of what’s going on this way that does not appeal to the *hexagonal* shape of the honeycomb and the efficiency of this shape over others. The mathematical explanation in this case doesn’t seem to be playing proxy for some better intrinsic explanation that appeals only to fundamentally mathematics-free features of the physical situation. Rather, the mathematical features themselves seem to take central stage in providing us with an understanding of the phenomenon concerned.

I would like to agree with Pincock, then, that in this example the mathematical theorem itself, rather than any nominalistic content represented, is doing some genuine explanatory work. This is in contrast with many nominalists, who argue that in this and other cases (such as Alan Baker’s well-known (2005) example of periodical magicicada cicadas, insects whose prime-number length periods of 13 and 17 years are evolutionary optimal for avoiding periodical predators) any genuine explanatory work is achieved solely by the nominalistic content that is picked out by the mathematics used (see Daly and Langford 2009, Saatsi 2011, and myself in a previous incarnation (Leng, 2005), for accounts along these lines). The reason why I take it that Pincock is right here is that the explanation of the phenomenon in question appears to be via the structural features of the physical system, and that such *structural explanations* are most naturally presented in mathematical terms. However, it is precisely this structural feature of such explanations that neutralises them as a threat to nominalism. For, I will argue, mathematical explanations understood as structural explanations do not require the existence of any abstract mathematical objects or indeed structures, but only the (approximate) instantiation of axiomatically characterized mathematical theories.

What does it mean to say that mathematical explanations, such as the proposed explanation of the shape of the honeycomb by means of the Honeycomb Conjecture, are structural explanations? A structural explanation of a physical phenomenon explains that phenomenon as holding as a consequence of general (and typically mathematical) structural features instantiated in the physical circumstances to which it belongs (see, e.g., Bokulich 2008 (p. 149) for a characterization along these lines: “a structural explanation is one in which the explanandum is explained by showing how the (typically mathematical) structure of the theory itself limits what sorts of objects, properties, states, or behaviors are admissible within the framework of that theory, and then showing that the explanandum is in fact a consequence of that structure.”). But how do we identify which features are to count as general structural features? In mathematics, we can do so via structure-characterizing axioms. That is, we take it that the axioms of a given mathematical theory characterize relevant structures, the (2nd order) Peano axioms for example identifying what would have to be true of any given system of objects and their relations in order for it to count as an instance of a natural number structure, and Euclid’s axioms identifying what would have to be true of a given system of objects and their relations in order for it to count as an instance of a Euclidean geometry. If mathematical axioms are taken as structure-characterizing, then any consequences of those axioms can be taken to be consequences of the structure-characterizing features. To the extent that pure mathematicians are involved in drawing out the consequences of mathematical axioms, we can think of them as inquiring into structures and their features.

A natural interpretation of this picture, and one that I take it that Pincock assents to, is the platonic ‘*ante rem*’ structuralism of Stewart Shapiro (1997) and Michael Resnik (1997). According to both, mathematics is a body of truths *about axiomatically characterized mathematical structures*, and given that our axiomatic mathematical theories include axioms that are never instantiated in physical reality, in order to preserve the truth of mathematics these structures must exist *ante rem*, that is, as abstracta over and above any actual or potential physical instantiations. But an alternative picture (such as that of Geoffrey Hellman’s ‘*in re*’modal structuralism (1989)) takes mathematical axioms to be ‘structure-characterizing’ without requiring that there are any systems of objects instantiating our axioms. According to this picture, we take mathematical axioms as implicit definitions, stating what would have to be true of any system of objects in order for it to count as an instance of (say) a Euclidean geometry, without assuming that these axioms are in fact instantiated. When we derive consequences from such axioms, we can conclude that such consequences will also hold in any system of objects satisfying the axioms, *solely in virtue of the structural features* picked out by the axioms.

Hellman’s modal structuralism provides an interpretation of the claims of pure mathematics as bodies of truths. In particular, in Hellman’s picture, a mathematical proposition P expressed within the context of an axiomatic mathematical theory T should be interpreted not at face value, but rather as (roughly) the claim that ‘The axioms of T are logically possible, and have P as a logical consequence’ (which will be true in those cases where P is a theorem of a consistent axiomatic theory). Like the *ante rem* structuralist, Hellman is motivated in his structuralist account to provide an interpretation of mathematics according to which standard mathematics comes out as true. As a mathematical fictionalist, I am less concerned with preserving the truth of standard mathematics, so long as we can explain the sense in which the ordinary mathematical claims of ordinary mathematicians can be seen as *correct* if not literally true. So I reject Hellman’s modal-structural reinterpretations of mathematical claims in favour of a face value reading. Nevertheless, in terms of our account of mathematics as practised, Hellman and I are on the same page: in both cases we take it that in pure mathematics (or at least, in mature, axiomatic mathematics – see Leng (2010a) for an account of pre-axiomatic theorizing) mathematicians are involved in drawing out the consequences of mathematical axioms. Thus, for my fictionalist account of pure mathematics, when a mathematical theorem is justified as being mathematically correct or ‘true in the fiction’, all that is meant is that it is a consequence of the fiction-characterizing axioms.

What does this mean for the prospects for a fictionalist understanding mathematical explanations as structural explanations? Pincock complains that my account posits a ‘mysterious link’ between a fictional claim and a real world claim. Why, he wonders, should the fictional truth of the mathematical theorem tell us anything about the real world honeycomb? My answer to this is that the mathematical theorem is a consequence of structure-characterizing axioms (in this case, the axioms for Euclidean geometry). Those axioms are themselves *approximately* satisfied by physical points and lines on the scale that the conjecture applies. So taking a cross section of the bees’ honeycomb as a plane, with nodes in the honeycomb as points and walls as straight lines, the Euclidean axioms are (approximately) true when interpreted as about this physical system. As such, Euclidean theorems are likewise (approximately) true of this physical system, including the theorem concerning the efficiency of hexagonal tilings as opposed to other regular shapes. This explanation of the bees’ choice of the honeycomb shape as the most efficient use of wax thus requires no ‘mysterious link’ between a fictional claim and the real world, but only the straightforward interpretation of hitherto uninterpreted mathematical axioms as (approximate) truths about a physical system.

While I agree, then, with Pincock that mathematics sometimes plays an explanatory role over and above any specific nominalistic content that the mathematics is being used to represent, I do not think that *mathematical objects*, either abstract particulars (such as numbers) or abstract universals (such as *ante rem* structures) play any role in these explanations. Rather, mathematical theories provide for *structural explanations,* where a structural explanation of a given phenomenon is given whenever (a) mathematical axioms can be interpreted as (approximately) true of the relevant physical system, and (b) the occurrence of the phenomenon can be derived as a consequence of those axioms under that interpretation. Here mathematics plays an important *modal* role in convincing us that the phenomenon in question *had to occur* given the structural features of the physical system (as characterized by the mathematical axioms that are instantiated), a feature that would be hard to replicate in a mathematics-free description of the physical system that did not indicate general structural features best described by mathematical axioms. But for mathematical theories to play this role we need only for their axioms to be consistent and for their theorems to be logical consequences of those axioms, not for the axioms to be true of any structure over and above a given physical instantiation.

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1. While most philosophers of mathematics discussing the case have accepted that the Honeycomb Conjecture is at least relevant in explaining the shape of actual honeycombs, the example isn’t universally accepted, with Tim Räz arguing that the three dimensional structure of the bee’s honeycomb complicates the issue in a way that throws doubt on the relevance of the 2-dimensional mathematical result. I will ignore these reservations here, in part because I take it that the conjecture is still relevant to the efficient construction of the two-dimensional cross-sections of the honeycomb, but also because, even if this explanation turns out not to be the correct one, its form as a structural explanation of structural features of the physical world is one that I believe many good explanations will take. [↑](#footnote-ref-1)