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The computation of the greatest common divisor of three bivariate Bernstein polynomials defined in a rectangular domain

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Abstract

This paper considers the computation of the greatest common divisor (GCD) $d_{t_1,t_2}(x,y)$ of three bivariate Bernstein polynomials that are defined in a rectangular domain, where $t_1(t_2)$ is the degree of $d_{t_1,t_2}(x,y)$ when it is written as a polynomial in $x(y)$ whose coefficients are polynomials in $y(x)$. The Sylvester resultant matrix and its subresultant matrices are used for the computation of the degrees and coefficients of the GCD. It is shown that there are four forms of these matrices and that they differ in their computational properties. The most difficult part of the computation is the determination of t_1 and t_2, and two methods for this computation are described. One method is simple but inefficient, and the other method reduces the problem to the computation of the degree of the GCD of two univariate polynomials, which is more efficient. The basis functions of the polynomials include binomial terms, which span many orders of magnitude, even for polynomials of moderate degrees. It is shown that the adverse effects of this wide range of magnitudes and a significant reduction in the sensitivity of the degree of the GCD to noise are obtained when the polynomials are processed by three operations before computations are performed on them. Examples that demonstrate the theory are included in the paper.

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1. Introduction

This paper describes a robust numerical procedure for the computation of the greatest common divisor (GCD) of three bivariate Bernstein polynomials that are defined in a rectangular domain. This problem arises in computer-aided design (CAD) systems, for which the determination of the points of intersection of curves is important [6, 15, 16]. For the curves defined by the polynomials $(\hat{f}(x,y), \hat{g}(x,y), \hat{h}(x,y))$, this problem reduces to the determination of the solutions of the equations,

$$\hat{f}(x,y) = 0, \qquad \hat{g}(x,y) = 0 \qquad \text{and} \qquad \hat{h}(x,y) = 0,$$

from which it follows that the intersections are defined by the irreducible factors of the GCD of $(\hat{f}(x,y), \hat{g}(x,y), \hat{h}(x,y))$. The factorisation of multivariate polynomials and the computation of the GCD of two multivariate polynomials are considered in [7, 8, 10, 12, 14, 17]. The polynomials in these papers are expressed in the power basis, but the Bernstein basis is considered in this paper.

Corless et. al. [7] consider the computation of the GCD of two bivariate power basis polynomials $p(x,y)$ and $q(x,y)$. In the first stage, the independent variable x is set equal to a random number α and the GCD of the univariate polynomials $p(\alpha,y)$ and $q(\alpha,y)$ is computed. The independent variable y is then set equal to a sequence of random numbers β_i and the GCD of each set of polynomials $(p(x,\beta_i), q(x,\beta_i))$ is computed. The next stage requires a series of matrix computations that include a Vandermonde matrix, which may cause problems because it is ill-conditioned. Problems of instability may also occur due to a poor choice of points in an interpolation procedure. Noda and Sasaki [14] extend Euclid's algorithm for the computation of the GCD from univariate polynomials to multivariate polynomials, and the implementation is considered in detail in order to address problems of instability. The application of resultant matrices to computations on curves and surfaces is considered in [6, 15, 16] but the power basis is used, even though the Bernstein basis is used for the representation of curves and surfaces in CAD systems.

The computation of the GCD of two or more polynomials is ill-conditioned and thus the algorithms must guarantee that the computed GCD is numerically stable. Furthermore, GCD computations on Bernstein basis polynomials require more care than GCD computations on power basis polynomials because of the binomial terms in the Bernstein basis functions. It follows that even if the coefficients of the polynomials are of the same order of magnitude, the binomial terms may cause the entries in the matrices in the computations to span many orders of magnitude, which may yield numerical problems. These numerical and stability issues are minimised by processing the given polynomials by three operations before their GCD is computed. The examples in the paper show that the inclusion of these operations yields considerably improved results because the degrees in x and y of the GCD of the polynomials $(f(x,y), g(x,y), h(x,y))$ is much more clearly defined, and the error in the coefficients of the GCD is smaller, than their equivalents when the preprocessing operations are omitted.

All the terms in a bivariate Bernstein polynomial $f(x,y)$ in a triangular domain $\mathcal{T}$,

$$\mathcal{T}: \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq 1 - x - y \leq 1,$$

are of the same degree m,

$$f(x,y) = \sum_{i+j=0}^{m} a_{i,j} \binom{m}{i,j} (1-x-y)^{m-i-j} x^i y^j,$$

where

$$\binom{m}{i,j} = \binom{m}{j,i} = \frac{m!}{(m-i-j)!i!j!},$$

but a bivariate Bernstein polynomial $g(x,y)$ of degrees m and n in x and y, respectively, in a rectangular domain $\mathcal{R}$ is

$$g(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \hat{g}_{i,j} B_i^m(x) B_j^n(y), \qquad \mathcal{R}: \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1,$$

where

$$B_i^m(x) = \binom{m}{i}(1-x)^{m-i}x^i \qquad \text{and} \qquad B_j^n(y) = \binom{n}{j}(1-y)^{n-j}y^j.$$

The requirement to calculate two degrees, m and n, implies that GCD computations of polynomials defined in $\mathcal{R}$ are harder than GCD computations of polynomials defined in $\mathcal{T}$, for which only one degree need be calculated [5]. Other aspects of the GCD computation are similar for polynomials defined in $\mathcal{R}$ and $\mathcal{T}$, for example, the Sylvester matrix and its subresultant matrices for three polynomials defined in each domain can be written as the product of three matrices DTQ, where D and Q are diagonal matrices and T is a block matrix that includes Tœplitz matrices.

The form of a bivariate Bernstein polynomial defined in a rectangular domain is considered in Section 2 and it is shown in Section 3 that there are four forms of the Sylvester matrix and its subresultant matrices of three polynomials $(\hat{f}(x,y), \hat{g}(x,y), \hat{h}(x,y))$ defined in a rectangular domain. Significantly improved results are obtained when these polynomials are processed by three operations before computations are performed on their Sylvester matrices and subresultant matrices, and these operations are considered in Section 4. Two methods for the computation of the degree of the GCD of these three polynomials, and any two of these polynomials, are described in Section 5. One method is a simple extension of the method used for univariate polynomials but it is computationally expensive because the two-dimensional nature of the problem is retained and it therefore requires the computation of the singular value decomposition (SVD) of a large number of Sylvester matrices and subresultant matrices of bivariate polynomials. The second method is more efficient because it reduces to two problems, each of which requires the determination of the degree of the GCD of two univariate polynomials. The complexity of the algorithms discussed in this paper is considered in Section 6, and Section 7 contains examples that demonstrate the theory presented in the paper. The paper is summarised in Section 8.

The novel aspects of the paper are:

1. The four forms of the Sylvester matrix and its subresultant matrices of two or three polynomials defined in a rectangular domain are derived.
2. The preprocessing operations that are implemented on these polynomials, such that their GCD is numerically stable, are described.
3. An efficient algorithm for the computation of the degrees t_1 and t_2 of the GCD $d_{t_1,t_2}(x,y)$ of these polynomials, where $t_1(t_2)$ is the degree of $d_{t_1,t_2}(x,y)$ when it is written as a polynomial in $x(y)$ whose coefficients are polynomials in $y(x)$, is described.

2. Bernstein polynomials in a rectangular domain

A Bernstein polynomial $\hat{f}(x,y)$ defined in a rectangular domain is

$$\begin{aligned}\hat{f}(x,y) &= \sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2} \hat{a}_{i_1,i_2} B_{i_1}^{m_1}(x) B_{i_2}^{m_2}(y) \\ &= \sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2} \hat{a}_{i_1,i_2} \binom{m_1}{i_1}\binom{m_2}{i_2}(1-x)^{m_1-i_1}x^{i_1}(1-y)^{m_2-i_2}y^{i_2}, \quad (1)\end{aligned}$$

where $0 \leq x, y \leq 1$ and (m_1, m_2) is the degree of $\hat{f}(x,y)$, $\deg_x \hat{f}(x,y) = m_1$ and $\deg_y \hat{f}(x,y) = m_2$. The polynomial $\hat{f}(x,y)$ can be written as the product of a coefficient matrix and two vectors of basis functions,

$$\hat{f}(x,y) = \begin{bmatrix} B_0^{m_1}(x) & \cdots & B_{m_1}^{m_1}(x) \end{bmatrix} \begin{bmatrix} \hat{a}_{0,0} & \ldots & \hat{a}_{0,m_2} \\ \vdots & \vdots & \vdots \\ \hat{a}_{m_1,0} & \ldots & \hat{a}_{m_1,m_2} \end{bmatrix} \begin{bmatrix} B_0^{m_2}(y) \\ \vdots \\ B_{m_2}^{m_2}(y) \end{bmatrix}. \quad (2)$$

It can also be written as a polynomial in y whose coefficients are polynomials in x,

$$\hat{f}(x,y) = \hat{f}_0(x)B_0^{m_2}(y) + \hat{f}_1(x)B_1^{m_2}(y) + \cdots + \hat{f}_{m_2}(x)B_{m_2}^{m_2}(y),$$

where each of the univariate polynomials $\hat{f}_j(x), j = 0, \ldots, m_2$, is of degree m_1,

$$\hat{f}_j(x) = \sum_{i=0}^{m_1} \hat{a}_{i,j} B_i^{m_1}(x), \qquad j = 0, \ldots, m_2, \quad (3)$$

and the coefficients of $\hat{f}_j(x)$ form the $(j+1)$th column of the coefficient matrix in (2). The coefficients of $\hat{f}(x,y)$ are arranged to form the vector $\hat{\mathbf{f}}$,

$$\hat{\mathbf{f}} = \begin{bmatrix} \hat{\mathbf{f}}_0^T & \hat{\mathbf{f}}_1^T & \cdots & \hat{\mathbf{f}}_{m_2}^T \end{bmatrix}^T \in \mathbb{R}^{(m_1+1)(m_2+1)},$$

where $\hat{\mathbf{f}}_j \in \mathbb{R}^{m_1+1}, j = 0, \ldots, m_2$, contains the coefficients of the polynomial $\hat{f}_j(x)$,

$$\hat{\mathbf{f}}_j = \begin{bmatrix} \hat{a}_{0,j} & \hat{a}_{1,j} & \cdots & \hat{a}_{m_1,j} \end{bmatrix}^T,$$

and it therefore follows from (2) that $\hat{f}(x,y)$ can be written as

$$\hat{f}(x,y) = \begin{bmatrix} B_0^{m_1}(x) & \cdots & B_{m_1}^{m_1}(x) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{f}}_0 & \cdots & \hat{\mathbf{f}}_{m_2} \end{bmatrix} \begin{bmatrix} B_0^{m_2}(y) \\ \vdots \\ B_{m_2}^{m_2}(y) \end{bmatrix}.$$

The coefficients can also be ordered by considering $\hat{f}(x,y)$ a polynomial in x whose m_1+1 coefficients are polynomials in y.

2.1. Vector representation and multiplication

This section considers the multiplication of the polynomial $\hat{f}(x,y)$ that is defined in (1), and the polynomial

$$\begin{aligned} \hat{g}(x,y) &= \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \hat{b}_{i_1,i_2} B_{i_1}^{n_1}(x) B_{i_2}^{n_2}(y) \\ &= \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \hat{b}_{i_1,i_2} \binom{n_1}{i_1} \binom{n_2}{i_2} (1-x)^{n_1-i_1} x^{i_1} (1-y)^{n_2-i_2} y^{i_2}. \end{aligned} \quad (4)$$

The polynomial $\hat{g}(x,y)$ can be written as a polynomial in y whose n_2+1 coefficients are polynomials in x, each of degree n_1,

$$\hat{g}(x,y) = \hat{g}_0(x) B_0^{n_2}(y) + \hat{g}_1(x) B_1^{n_2}(y) + \cdots + \hat{g}_{n_2}(x) B_{n_2}^{n_2}(y),$$

where

$$\hat{g}_j(x) = \sum_{i=0}^{n_1} \hat{b}_{i,j} B_i^{n_1}(x), \qquad j = 0, \ldots, n_2.$$

The product of $\hat{f}(x,y)$ and $\hat{g}(x,y)$ is the polynomial $\hat{h}(x,y)$,

$$\hat{h}(x,y) = \hat{f}(x,y)\hat{g}(x,y) = \sum_{i_1=0}^{m_1+n_1} \sum_{i_2=0}^{m_2+n_2} \hat{c}_{i_1,i_2} B_{i_1}^{m_1+n_1}(x) B_{i_2}^{m_2+n_2}(y),$$

which can be considered a polynomial in y of degree m_2+n_2, where each polynomial in the set of coefficients $\{\hat{h}_i(x), i = 0, \ldots, m_2+n_2\}$ is a polynomial in x of degree m_1+n_1,

$$\hat{h}(x,y) = \hat{h}_0(x)B_0^{m_2+n_2}(y) + \hat{h}_1(x)B_1^{m_2+n_2}(y) + \cdots + \hat{h}_{m_2+n_2}(x)B_{m_2+n_2}^{m_2+n_2}(y).$$

The polynomial $\hat{h}(x,y)$ can also be expressed in the form

$$\begin{aligned} \hat{h}(x,y) \quad = \quad & \hat{f}_0(x)\hat{g}_0(x)B_0^{m_2}(y)B_0^{n_2}(y) + \\ & \Big(\hat{f}_0(x)\hat{g}_1(x)B_0^{m_2}(y)B_1^{n_2}(y) + \hat{f}_1(x)\hat{g}_0(x)B_1^{m_2}(y)B_0^{n_2}(y)\Big) + \\ & \Big(\hat{f}_0(x)\hat{g}_2(x)B_0^{m_2}(y)B_2^{n_2}(y) + \hat{f}_1(x)\hat{g}_1(x)B_1^{m_2}(y)B_1^{n_2}(y) + \\ & \hat{f}_2(x)\hat{g}_0(x)B_2^{m_2}(y)B_0^{n_2}(y)\Big) + \cdots + \hat{f}_{m_2}(x)\hat{g}_{n_2}(x)B_{m_2}^{m_2}(y)B_{n_2}^{n_2}(y), \end{aligned} \tag{5}$$

and the general expression for each term in this sum is

$$\hat{f}_s(x)\hat{g}_t(x)B_s^{m_2}(y)B_t^{n_2}(y) = \frac{\binom{m_2}{s}\binom{n_2}{t}}{\binom{m_2+n_2}{s+t}} B_{s+t}^{m_2+n_2}(y)\hat{f}_s(x)\hat{g}_t(x).$$

The expression (5) is therefore equal to

$$\begin{aligned} \hat{h}(x,y) \quad = \quad & B_0^{m_2+n_2}(y)\left(\hat{f}_0(x)\hat{g}_0(x)\frac{\binom{m_2}{0}\binom{n_2}{0}}{\binom{m_2+n_2}{0}}\right) + \\ & B_1^{m_2+n_2}(y)\left(\hat{f}_0(x)\hat{g}_1(x)\frac{\binom{m_2}{0}\binom{n_2}{1}}{\binom{m_2+n_2}{1}} + \hat{f}_1(x)\hat{g}_0(x)\frac{\binom{m_2}{1}\binom{n_2}{0}}{\binom{m_2+n_2}{1}}\right) + \\ & B_2^{m_2+n_2}(y)\left(\hat{f}_0(x)\hat{g}_2(x)\frac{\binom{m_2}{0}\binom{n_2}{2}}{\binom{m_2+n_2}{2}} + \hat{f}_1(x)\hat{g}_1(x)\frac{\binom{m_2}{1}\binom{n_2}{1}}{\binom{m_2+n_2}{2}} + \right. \\ & \left. \hat{f}_2(x)\hat{g}_0(x)\frac{\binom{m_2}{2}\binom{n_2}{0}}{\binom{m_2+n_2}{2}}\right) + \cdots + B_{m_2+n_2}^{m_2+n_2}(y)\left(\hat{f}_{m_2}(x)\hat{g}_{n_2}(x)\frac{\binom{m_2}{m_2}\binom{n_2}{n_2}}{\binom{m_2+n_2}{m_2+n_2}}\right), \end{aligned}$$

where each polynomial $p(x)$ in the term $B_i^{m_2+n_2}(y)p(x)$,$i = 0, \ldots, m_2+n_2$, is proportional to the product of two univariate Bernstein polynomials. This product can be written as a matrix-vector product, and thus the vector of the coefficients of $\hat{h}(x, y)$ can be written as a block matrix-vector product,

$$\hat{\mathbf{h}} = C_{n_1,n_2}(\hat{f}(x, y))\hat{\mathbf{g}} = \left(D^{-1}_{m_1+n_1,m_2+n_2} T_{n_1,n_2}(\hat{f}(x, y)) Q_{n_1,n_2}\right)\hat{\mathbf{g}}, \tag{6}$$

where $C_{n_1,n_2}(\hat{f}(x, y))$ is of order $(m_1 + n_1 + 1)(m_2 + n_2 + 1) \times (n_1 + 1)(n_2 + 1)$. The partitioned structure of $C_{n_1,n_2}(\hat{f}(x, y))$ is

$$\left[\begin{array}{cccc}
\frac{C_{n_1}(\hat{f}_0(x))\binom{m_2}{0}\binom{n_2}{0}}{\binom{m_2+n_2}{0}} & & & \\
\frac{C_{n_1}(\hat{f}_1(x))\binom{m_2}{1}\binom{n_2}{0}}{\binom{m_2+n_2}{1}} & \frac{C_{n_1}(\hat{f}_0(x))\binom{m_2}{0}\binom{n_2}{1}}{\binom{m_2+n_2}{1}} & & \\
\vdots & \frac{C_{n_1}(\hat{f}_1(x))\binom{m_2}{1}\binom{n_2}{1}}{\binom{m_2+n_2}{2}} & \ddots & \\
\vdots & \vdots & \ddots & \frac{C_{n_1}(\hat{f}_0(x))\binom{m_2}{0}\binom{n_2}{n_2}}{\binom{m_2+n_2}{n_2}} \\
\frac{C_{n_1}(\hat{f}_{m_2}(x))\binom{m_2}{m_2}\binom{n_2}{0}}{\binom{m_2+n_2}{m_2}} & \vdots & \ddots & \frac{C_{n_1}(\hat{f}_1(x))\binom{m_2}{1}\binom{n_2}{n_2}}{\binom{m_2+n_2}{n_2+1}} \\
 & \frac{C_{n_1}(\hat{f}_{m_2}(x))\binom{m_2}{m_2}\binom{n_2}{1}}{\binom{m_2+n_2}{m_2+1}} & \ddots & \vdots \\
 & & \ddots & \vdots \\
 & & & \frac{C_{n_1}(\hat{f}_{m_2}(x))\binom{m_2}{m_2}\binom{n_2}{n_2}}{\binom{m_2+n_2}{m_2+n_2}}
\end{array}\right],$$

where each matrix $C_{n_1}(\hat{f}_j(x))$ is the convolution matrix of a univariate Bernstein polynomial $\hat{f}_j(x)$ of degree m_1 that is multiplied by a univariate Bernstein polynomial of degree n_1. These matrices are therefore of order $(m_1 + n_1 + 1) \times (n_1 + 1)$ and they have the same structure as the univariate convolution matrix.

It follows from (6) that $C_{n_1,n_2}(\hat{f}(x, y))$ is equal to the product of three matrices, where $D^{-1}_{m_1+n_1,m_2+n_2}$ is diagonal and of order $(m_1+n_1+1)(m_2+n_2+1)$,

$$D^{-1}_{m_1+n_1,m_2+n_2} = \operatorname{diag}\left[\begin{array}{cccc} \frac{D^{-1}_{m_1+n_1}}{\binom{m_2+n_2}{0}} & \frac{D^{-1}_{m_1+n_1}}{\binom{m_2+n_2}{1}} & \cdots & \frac{D^{-1}_{m_1+n_1}}{\binom{m_2+n_2}{m_2+n_2}} \end{array}\right], \tag{7}$$

and the diagonal matrix

$$D^{-1}_{m_1+n_1} = \operatorname{diag}\left[\begin{array}{cccc} \frac{1}{\binom{m_1+n_1}{0}} & \frac{1}{\binom{m_1+n_1}{1}} & \cdots & \frac{1}{\binom{m_1+n_1}{m_1+n_1}} \end{array}\right],$$

arises from the multiplication of $\hat{f}_i(x)$, $i = 0, \ldots, m_2$, of degree m_1 by $\hat{g}_j(x)$, $j = 0, \ldots, n_2$, of degree n_1.

The block Tœplitz matrix $T_{n_1,n_2}(\hat{f}(x,y))$ of the bivariate polynomial $\hat{f}(x,y)$ is of order $(m_1+n_1+1)(m_2+n_2+1) \times (n_1+1)(n_2+2)$ and given by

$$\left[\begin{array}{cccc} T_{n_1}(\hat{f}_0(x))\binom{m_2}{0} & & & \\ T_{n_1}(\hat{f}_1(x))\binom{m_2}{1} & T_{n_1}(\hat{f}_0(x))\binom{m_2}{0} & & \\ \vdots & T_{n_1}(\hat{f}_1(x))\binom{m_2}{1} & \ddots & \\ \vdots & \vdots & \ddots & T_{n_1}(\hat{f}_0(x))\binom{m_2}{0} \\ T_{n_1}(\hat{f}_{m_2}(x))\binom{m_2}{m_2} & \vdots & \ddots & T_{n_1}(\hat{f}_1(x))\binom{m_2}{1} \\ & T_{n_1}(\hat{f}_{m_2}(x))\binom{m_2}{m_2} & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & T_{n_1}(\hat{f}_{m_2}(x))\binom{m_2}{m_2} \end{array}\right], \tag{8}$$

where $T_{n_1}(\hat{f}_j(x))$ is a Tœplitz matrix of order $(m_1+n_1+1) \times (n_1+1)$,

$$T_{n_1}(\hat{f}_j(x)) = \left[\begin{array}{cccc} \hat{a}_{0,j}\binom{m_1}{0} & & & \\ \hat{a}_{1,j}\binom{m_1}{1} & \hat{a}_{0,j}\binom{m_1}{0} & & \\ \vdots & \hat{a}_{1,j}\binom{m_1}{1} & \ddots & \\ \vdots & \vdots & \ddots & \hat{a}_{0,j}\binom{m_1}{0} \\ \hat{a}_{m_1,j}\binom{m_1}{m_1} & \vdots & \ddots & \hat{a}_{1,j}\binom{m_1}{1} \\ & \hat{a}_{m_1,j}\binom{m_1}{m_1} & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & \hat{a}_{m_1,j}\binom{m_1}{m_1} \end{array}\right],$$

for $j = 0, \ldots, m_2$, and $\hat{f}_j(x)$ is defined in (3). The diagonal matrix Q_{n_1,n_2} is of order $(n_1+1)(n_2+1)$,

$$Q_{n_1,n_2} = \operatorname{diag}\left[\begin{array}{cccc} Q_{n_1}\binom{n_2}{0} & Q_{n_1}\binom{n_2}{1} & \cdots & Q_{n_1}\binom{n_2}{n_2} \end{array}\right], \tag{9}$$

where Q_{n_1} is a diagonal matrix of order n_1+1,

$$Q_{n_1} = \operatorname{diag}\left[\begin{array}{cccc} \binom{n_1}{0} & \binom{n_1}{1} & \cdots & \binom{n_1}{n_1} \end{array}\right].$$

The vectors $\hat{\mathbf{g}}$ and $\hat{\mathbf{h}}$ in (6) are vectors of the coefficients of the polynomials $\hat{g}(x,y)$ and $\hat{h}(x,y)$,

$$\hat{\mathbf{g}} = \left[\begin{array}{cccc} \hat{\mathbf{g}}_0^T & \hat{\mathbf{g}}_1^T & \cdots & \hat{\mathbf{g}}_{n_2}^T \end{array}\right]^T \in \mathbb{R}^{(n_1+1)(n_2+1)},$$

where $\hat{\mathbf{g}}_j$ contains the coefficients of the polynomial $\hat{g}_j(x)$, which is of degree n_1,

$$\hat{\mathbf{g}}_j = \begin{bmatrix} \hat{b}_{0,j} & \hat{b}_{1,j} & \cdots & \hat{b}_{n_1,j} \end{bmatrix}^T \in \mathbb{R}^{n_1+1}, \qquad j = 0, \ldots, n_2,$$

and

$$\hat{\mathbf{h}} = \begin{bmatrix} \hat{\mathbf{h}}_0^T & \hat{\mathbf{h}}_1^T & \cdots & \hat{\mathbf{h}}_{m_2+n_2}^T \end{bmatrix}^T \in \mathbb{R}^{(m_1+n_1+1)(m_2+n_2+1)},$$

where $\hat{\mathbf{h}}_i$ contains the coefficients of the polynomial $\hat{h}_i(x)$, which is of degree $m_1 + n_1$,

$$\hat{\mathbf{h}}_i = \begin{bmatrix} \hat{c}_{0,i} & \hat{c}_{1,i} & \cdots & \hat{c}_{m_1+n_1,i} \end{bmatrix}^T \in \mathbb{R}^{m_1+n_1+1}, \qquad i = 0, \ldots, m_2 + n_2.$$

Example 2.1. Consider the polynomial $\hat{f}(x, y)$ of degree $(m_1, m_2) = (2, 2)$,

$$\begin{aligned} \hat{f}(x,y) &= 7B_0^2(x)B_0^2(y) + 4B_1^2(x)B_0^2(y) + B_0^2(x)B_1^2(y) + \\ &\quad 6B_2^2(x)B_0^2(y) + 5B_1^2(x)B_1^2(y) + 2B_0^2(x)B_2^2(y) + \\ &\quad 3B_2^2(x)B_1^2(y) + 8B_1^2(x)B_2^2(y) + 9B_2^2(x)B_2^2(y) \\ &= \begin{bmatrix} B_0^2(x) & B_1^2(x) & B_2^2(x) \end{bmatrix} \begin{bmatrix} 7 & 1 & 2 \\ 4 & 5 & 8 \\ 6 & 3 & 9 \end{bmatrix} \begin{bmatrix} B_0^2(y) \\ B_1^2(y) \\ B_2^2(y) \end{bmatrix}, \end{aligned}$$

and the polynomial $\hat{g}(x, y)$ of degree $(n_1, n_2) = (1, 1)$,

$$\begin{aligned} \hat{g}(x,y) &= 2B_0^1(x)B_0^1(y) + 3B_1^1(x)B_0^1(y) + 4B_0^1(x)B_1^1(y) + 7B_1^1(x)B_1^1(y) \\ &= \begin{bmatrix} B_0^1(x) & B_1^1(x) \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} B_0^1(y) \\ B_1^1(y) \end{bmatrix}. \end{aligned}$$

The polynomial $\hat{f}(x, y)$ can be written as the sum of three polynomials, $\hat{f}_0(x)$, $\hat{f}_1(x)$ and $\hat{f}_2(x)$,

$$\hat{f}(x,y) = \hat{f}_0(x)B_0^2(y) + \hat{f}_1(x)B_1^2(y) + \hat{f}_2(x)B_2^2(y),$$

where

$$\hat{f}_0(x) = 7B_0^2(x) + 4B_1^2(x) + 6B_2^2(x),$$
$$\hat{f}_1(x) = B_0^2(x) + 5B_1^2(x) + 3B_2^2(x),$$
$$\hat{f}_2(x) = 2B_0^2(x) + 8B_1^2(x) + 9B_2^2(x).$$

The polynomial $\hat{g}(x,y)$ is the sum of the polynomials $\hat{g}_0(x)$ and $\hat{g}_1(x)$,

$$\hat{g}(x,y) = \hat{g}_0(x)B_0^1(y) + \hat{g}_1(x)B_1^1(y),$$

where

$$\hat{g}_0(x) = 2B_0^1(x) + 3B_1^1(x) \qquad \text{and} \qquad \hat{g}_1(x) = 4B_0^1(x) + 7B_1^1(x).$$

The product $\hat{h}(x,y)$, which is a polynomial of degree $(3,3)$, is the weighted sum of four polynomials $\hat{h}_0(x), \ldots, \hat{h}_3(x)$, where the weights are the univariate Bernstein basis functions in y,

$$\hat{h}(x,y) = \hat{h}_0(x)B_0^3(y) + \hat{h}_1(x)B_1^3(y) + \hat{h}_2(x)B_2^3(y) + \hat{h}_3(x)B_3^3(y).$$

The coefficients of $\hat{h}(x,y)$ are computed from (6) with $n_1 = n_2 = 1$,

$$\begin{bmatrix} \frac{C_1(\hat{f}_0(x))\binom{2}{0}\binom{1}{0}}{\binom{3}{0}} & \\ \frac{C_1(\hat{f}_1(x))\binom{2}{1}\binom{1}{0}}{\binom{3}{1}} & \frac{C_1(\hat{f}_0(x))\binom{2}{0}\binom{1}{1}}{\binom{3}{1}} \\ \frac{C_1(\hat{f}_2(x))\binom{2}{2}\binom{1}{0}}{\binom{3}{2}} & \frac{C_1(\hat{f}_1(x))\binom{2}{1}\binom{1}{1}}{\binom{3}{2}} \\ & \frac{C_1(\hat{f}_2(x))\binom{2}{2}\binom{1}{1}}{\binom{3}{3}} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{g}}_0 \\ \hat{\boldsymbol{g}}_1 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{h}}_0 \\ \hat{\boldsymbol{h}}_1 \\ \hat{\boldsymbol{h}}_2 \\ \hat{\boldsymbol{h}}_3 \end{bmatrix}.$$

Each of the block matrices in the coefficient matrix is of order 4×2, $\hat{\mathbf{g}} \in \mathbb{R}^4$ and $\hat{\mathbf{h}} \in \mathbb{R}^{16}$. $\square$

3. The Sylvester matrices of three polynomials that are defined in a rectangular domain

This section considers the forms of the Sylvester matrices and their subresultant matrices for the polynomials $(\hat{f}(x,y), \hat{g}(x,y), \hat{h}(x,y))$ that are defined in a rectangular domain,

$$\hat{f}(x,y) = \sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2} \hat{a}_{i_1,i_2} B_{i_1}^{m_1}(x) B_{i_2}^{m_2}(y), \tag{10}$$

$$\hat{g}(x,y) = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} \hat{b}_{i_1,i_2} B_{i_1}^{n_1}(x) B_{i_2}^{n_2}(y), \tag{11}$$

$$\hat{h}(x,y) = \sum_{i_1=0}^{p_1}\sum_{i_2=0}^{p_2} \hat{c}_{i_1,i_2} B_{i_1}^{p_1}(x) B_{i_2}^{p_2}(y). \tag{12}$$

If the polynomials have a GCD $\hat{d}_{t_1,t_2}(x,y)$ of degree (t_1,t_2),

$$\hat{d}_{t_1,t_2}(x,y) = \sum_{i_1=0}^{t_1}\sum_{i_2=0}^{t_2} \hat{e}_{i_1,i_2} B_{i_1}^{t_1}(x) B_{i_2}^{t_2}(y),$$

then there exist common divisors $\hat{d}_{k_1,k_2}(x,y)$ of degree (k_1,k_2), $k_1 = 0,\ldots,t_1$, and $k_2 = 0,\ldots,t_2$, such that

$$\hat{f}(x,y) = \hat{u}_{k_1,k_2}(x,y)\hat{d}_{k_1,k_2}(x,y), \tag{13}$$

$$\hat{g}(x,y) = \hat{v}_{k_1,k_2}(x,y)\hat{d}_{k_1,k_2}(x,y), \tag{14}$$

$$\hat{h}(x,y) = \hat{w}_{k_1,k_2}(x,y)\hat{d}_{k_1,k_2}(x,y), \tag{15}$$

where

$$\hat{u}_{k_1,k_2}(x,y) = \sum_{i_1=0}^{m_1-k_1}\sum_{i_2=0}^{m_2-k_2} \hat{u}_{i_1,i_2} B_{i_1}^{m_1-k_1}(x) B_{i_2}^{m_2-k_2}(y),$$

$$\hat{v}_{k_1,k_2}(x,y) = \sum_{i_1=0}^{n_1-k_1}\sum_{i_2=0}^{n_2-k_2} \hat{v}_{i_1,i_2} B_{i_1}^{n_1-k_1}(x) B_{i_2}^{n_2-k_2}(y),$$

$$\hat{w}_{k_1,k_2}(x,y) = \sum_{i_1=0}^{p_1-k_1}\sum_{i_2=0}^{p_2-k_2} \hat{w}_{i_1,i_2} B_{i_1}^{p_1-k_1}(x) B_{i_2}^{p_2-k_2}(y).$$

The elimination of $\hat{d}_{k_1,k_2}(x,y)$ between (13), (14) and (15) yields

$$\hat{f}(x,y)\hat{v}_{k_1,k_2}(x,y) - \hat{g}(x,y)\hat{u}_{k_1,k_2}(x,y) = 0, \tag{16}$$

$$\hat{f}(x,y)\hat{w}_{k_1,k_2}(x,y) - \hat{h}(x,y)\hat{u}_{k_1,k_2}(x,y) = 0, \tag{17}$$

$$\hat{h}(x,y)\hat{v}_{k_1,k_2}(x,y) - \hat{g}(x,y)\hat{w}_{k_1,k_2}(x,y) = 0, \tag{18}$$

and these three equations can be written in matrix form,

$$\tilde{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})\mathbf{x}_{k_1,k_2} = \mathbf{0}, \tag{19}$$

where $\hat{f} = \hat{f}(x,y), \hat{g} = \hat{g}(x,y), \hat{h} = \hat{h}(x,y)$ and $\tilde{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})$ is their (k_1,k_2)th 3×3 block subresultant matrix. Each non-zero entry in this matrix-vector product represents the product of two bivariate polynomials and it is therefore equal to the product of three matrices, as shown in (6). It follows that $\tilde{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})$ can be written as

$$\tilde{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h}) = \tilde{D}^{-1}_{k_1,k_2}\tilde{T}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})\tilde{Q}_{k_1,k_2}, \tag{20}$$

where $\tilde{D}^{-1}_{k_1,k_2}$ is equal to

$$\operatorname{diag}\left[\begin{array}{ccc} D^{-1}_{m_1+n_1-k_1,m_2+n_2-k_2} & D^{-1}_{m_1+p_1-k_1,m_2+p_2-k_2} & D^{-1}_{n_1+p_1-k_1,n_2+p_2-k_2}\end{array}\right],$$

and $D^{-1}_{m_1+n_1-k_1,m_2+n_2-k_2}$, which is of order $(m_1+n_1-k_1+1)(m_2+n_2-k_2+1)$, is defined in (7). The matrices $D^{-1}_{m_1+p_1-k_1,m_2+p_2-k_2}$ and $D^{-1}_{n_1+p_1-k_1,n_2+p_2-k_2}$ are defined similarly.

The block matrix $\tilde{T}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})$ is equal to

$$\left[\begin{array}{ccc} T_{n_1-k_1,n_2-k_2}(\hat{f}) & & T_{m_1-k_1,m_2-k_2}(\hat{g}) \\ & T_{p_1-k_1,p_2-k_2}(\hat{f}) & T_{m_1-k_1,m_2-k_2}(\hat{h}) \\ T_{n_1-k_1,n_2-k_2}(\hat{h}) & -T_{p_1-k_1,p_2-k_2}(\hat{g}) & \end{array}\right],$$

where each block has the form (8), and the diagonal matrix $\tilde{Q}_{k_1,k_2}$ is equal to

$$\tilde{Q}_{k_1,k_2} = \operatorname{diag}\left[\begin{array}{ccc} Q_{n_1-k_1,n_2-k_2} & Q_{p_1-k_1,p_2-k_2} & Q_{m_1-k_1,m_2-k_2}\end{array}\right],$$

where Q_{n_1,n_2} is defined in (9).

Equation (19) has a non-trivial solution for $k_1 = 0,\dots,t_1$, and $k_2 = 0,\dots,t_2$,

$$\mathbf{x}_{k_1,k_2} = \left[\begin{array}{ccc} \hat{\mathbf{v}}^T_{k_1,k_2} & \hat{\mathbf{w}}^T_{k_1,k_2} & -\hat{\mathbf{u}}^T_{k_1,k_2}\end{array}\right]^T, \tag{21}$$

where $\hat{\mathbf{u}}_{k_1,k_2}$, $\hat{\mathbf{v}}_{k_1,k_2}$ and $\hat{\mathbf{w}}_{k_1,k_2}$ are the vectors of the coefficients of the polynomials $\hat{u}_{k_1,k_2}(x,y)$, $\hat{v}_{k_1,k_2}(x,y)$ and $\hat{w}_{k_1,k_2}(x,y)$, respectively.

The matrix $\tilde{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})$ can be defined in terms of its row partitions,

$$\tilde{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h}) = \left[\begin{array}{c} \mathcal{R}_{a,k_1,k_2} \\ \mathcal{R}_{b,k_1,k_2} \\ \mathcal{R}_{c,k_1,k_2}\end{array}\right], \tag{22}$$

where

$$\mathcal{R}_{a,k_1,k_2} = \left[\begin{array}{ccc} C_{n_1-k_1,n_2-k_2}(\hat{f}) & 0_a & C_{m_1-k_1,m_2-k_2}(\hat{g}) \end{array} \right], \tag{23}$$

$$\mathcal{R}_{b,k_1,k_2} = \left[\begin{array}{ccc} 0_b & C_{p_1-k_1,p_2-k_2}(\hat{f}) & C_{m_1-k_1,m_2-k_2}(\hat{h}) \end{array} \right], \tag{24}$$

$$\mathcal{R}_{c,k_1,k_2} = \left[\begin{array}{ccc} C_{n_1-k_1,n_2-k_2}(\hat{h}) & -C_{p_1-k_1,p_2-k_2}(\hat{g}) & 0_c \end{array} \right], \tag{25}$$

and 0_a, 0_b and 0_c are zero matrices of orders

$$\begin{array}{ll} 0_a: & (m_1+n_1-k_1+1)(m_2+n_2-k_2+1) \times (p_1-k_1+1)(p_2-k_2+1), \\ 0_b: & (m_1+p_1-k_1+1)(m_2+p_2-k_2+1) \times (n_1-k_1+1)(n_2-k_2+1), \\ 0_c: & (n_1+p_1-k_1+1)(n_2+p_2-k_2+1) \times (m_1-k_1+1)(m_2-k_2+1). \end{array}$$

The Sylvester subresultant matrix $\tilde{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})$ in (22) is one of four forms of this matrix for three polynomials [4, §4],[5, §3]. The other three forms follow because any two of the three equations (16), (17) and (18) allow the third equation to be derived. For example, (16) and (17) can be written in matrix form,

$$\hat{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})\mathbf{x}_{k_1,k_2} = \mathbf{0},$$

where $\mathbf{x}_{k_1,k_2}$ is defined in (21). The matrix $\hat{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})$ is the 2×3 block subresultant matrix and it is given by

$$\hat{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h}) = \hat{D}^{-1}_{k_1,k_2}\hat{T}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})\hat{Q}_{k_1,k_2} = \left[\begin{array}{c} \mathcal{R}_{a,k_1,k_2} \\ \mathcal{R}_{b,k_1,k_2} \end{array} \right], \tag{26}$$

where $\hat{D}^{-1}_{k_1,k_2}$,$\hat{T}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})$ and $\hat{Q}_{k_1,k_2}$ are similar to $\tilde{D}^{-1}_{k_1,k_2}$,$\tilde{T}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})$ and $\tilde{Q}_{k_1,k_2}$, and $\mathcal{R}_{a,k_1,k_2}$ and $\mathcal{R}_{b,k_1,k_2}$ are defined in (23) and (24), respectively. The other pairs of equations from (16), (17) and (18) yield Sylvester matrices and subresultant matrices that are defined in terms of $(\mathcal{R}_{a,k_1,k_2}, \mathcal{R}_{c,k_1,k_2})$ and $(\mathcal{R}_{b,k_1,k_2}, \mathcal{R}_{c,k_1,k_2})$, where $\mathcal{R}_{c,k_1,k_2}$ is defined in (25). It follows that there are four forms of the Sylvester matrix and its subresultant matrices, one 3×3 block matrix and three 2×3 block matrices.

Theorem 3.1 considers the computation of the degrees t_1 and t_2 of the GCD of $(\hat{f}(x,y),\ \hat{g}(x,y), \hat{h}(x,y))$ from $\tilde{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})$ and $\hat{S}_{k_1,k_2}(\hat{f}^*,\hat{g}^*,\hat{h}^*)$, where

$$k_1 = 1,\ldots,\min(m_1,n_1,p_1) \quad \text{and} \quad k_2 = 1,\ldots,\min(m_2,n_2,p_2), \tag{27}$$

and $\hat{S}_{k_1,k_2}(\hat{f}^*,\hat{g}^*,\hat{h}^*)$ denotes that the polynomials $(\hat{f}(x,y),\hat{g}(x,y),\hat{h}(x,y))$ can be considered in any order. Each ordering yields one of the three variants of the 2×3 block subresultant matrices discussed above. The proof of the theorem follows from the same result for univariate polynomials [1, 2, 4].

Theorem 3.1. The degrees t_1 and t_2 of the GCD of $(\hat{f}(x,y),\hat{g}(x,y),\hat{h}(x,y))$ are equal to, respectively, the largest indices k_1 and k_2 that are defined in (27), such that the subresultant matrices $\tilde{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})$ and $\hat{S}_{k_1,k_2}(\hat{f}^*,\hat{g}^*,\hat{h}^*)$ are rank deficient, where $t_1(t_2)$ is the degree of the GCD when it is written as a polynomial in $x(y)$ whose coefficients are polynomials in $y(x)$.

The four forms of the Sylvester matrix and its subresultant matrices are of the form $D^{-1}TQ$ where D^{-1} and Q are diagonal matrices and T is a matrix whose entries include binomial terms and the coefficients of the polynomials, as shown in (20). The matrices D^{-1} and Q are non-singular and thus

$$\text{rank } D^{-1}TQ = \text{rank } D^{-1}T = \text{rank } TQ = \text{rank } T,$$

from which it follows that the degree of the GCD of $(\hat{f}(x,y),\hat{g}(x,y),\hat{h}(x,y))$ can be calculated from the rank loss of each of the matrices $D^{-1}TQ, D^{-1}T, TQ$ and T.

Theorem 3.1 considers the GCD of three polynomials, but the coefficients of the polynomials are corrupted by noise in practical problems. It is therefore assumed that the given polynomials are coprime, but that they possess an approximate greatest common divisor (AGCD) that is near the GCD of the exact forms of the polynomials. An AGCD of three polynomials is defined in Definition 3.1.

Definition 3.1 (An AGCD). A polynomial $d(x,y)$ of degree (t_1,t_2) is an AGCD of $(\hat{f}(x,y),\hat{g}(x,y),\hat{h}(x,y))$ if it is the polynomial of maximum degree that is an exact divisor of $(\hat{f}(x,y)+\tilde{f}(x,y),\hat{g}(x,y)+\tilde{g}(x,y),\hat{h}(x,y)+\tilde{h}(x,y))$ for perturbations $\left\|\tilde{f}\right\| \leq \varepsilon_f$, $\|\tilde{g}\| \leq \varepsilon_g$ and $\left\|\tilde{h}\right\| \leq \varepsilon_h$, and $\left\|\tilde{f}\right\|^2 + \|\tilde{g}\|^2 + \left\|\tilde{h}\right\|^2$ is minimised over all polynomials of degree (t_1,t_2).

Example 3.1. Consider the Bernstein forms of the exact polynomials $\hat{f}(x,y)$, $\hat{g}(x,y)$ and $\hat{h}(x,y)$ of degrees $(m_1 = 17, m_2 = 13)$, $(n_1 = 20, n_2 = 19)$ and $(p_1 = 10, p_2 = 13)$ respectively, and whose factorised forms are

$$
\begin{aligned}
\hat{f}(x,y) &= (x-0.554687987932164654)^3(x+0.21657951321)\times \\
&\quad (x+y-0.46578784351654)^3(x+y+0.0124)^6\times \\
&\quad (x^2+y^2+0.5679814324687)^2, \\
\hat{g}(x,y) &= (x+0.21657951321)(x+y-0.46578784351654)^3\times \\
&\quad (x+y+0.4512)^6(x^2+y^2-0.00104751807)^3\times \\
&\quad (x^2+y^2+0.5679814324687)^2, \\
\hat{h}(x,y) &= (x+0.21657951321)(y-0.2465879841351465498)^4\times \\
&\quad (x+y-0.46578784351654)^3(12x^2+y^2-0.348798)\times \\
&\quad (x^2+y^2+0.5679814324687)^2.
\end{aligned}
$$

The polynomials $(\hat{f}(x,y),\hat{g}(x,y),\hat{h}(x,y))$ have a GCD $\hat{d}_{t_1,t_2}(x,y)$ of degree $(8,7)$,

$$
\begin{aligned}
\hat{d}(x,y) &= (x+0.21657951321)(x^2+y^2+0.5679814324687)^2\times \\
&\quad (x+y-0.46578784351654)^3.
\end{aligned}
$$

Noise was added to the coefficients of $(\hat{f}(x,y),\hat{g}(x,y),\hat{h}(x,y))$ such that the coefficients of the inexact polynomials $(f(x,y),g(x,y),h(x,y))$ are

$$
\begin{aligned}
a_{i_1,i_2} &= \hat{a}_{i_1,i_2} + \epsilon_{f,i_1,i_2}\hat{a}_{i_1,i_2}r_{f,i_1,i_2}, && i_1 = 0,\dots,m_1;\ i_2 = 0,\dots,m_2, \\
b_{j_1,j_2} &= \hat{b}_{j_1,j_2} + \epsilon_{g,j_1,j_2}\hat{b}_{j_1,j_2}r_{g,j_1,j_2}, && j_1 = 0,\dots,n_1;\ j_2 = 0,\dots,n_2, \\
c_{k_1,k_2} &= \hat{c}_{k_1,k_2} + \epsilon_{h,k_1,k_2}\hat{c}_{k_1,k_2}r_{h,k_1,k_2}, && k_1 = 0,\dots,p_1;\ k_2 = 0,\dots,p_2,
\end{aligned}
\tag{28}
$$

where $\{\epsilon_{f,i_1,i_2}\}$, $\{\epsilon_{g,j_1,j_2}\}$ and $\{\epsilon_{h,k_1,k_2}\}$ are uniformly distributed random variables in the interval $[10^{-12},10^{-10}]$, and $\{r_{f,i_1,i_2}\}$, $\{r_{g,j_1,j_2}\}$ and $\{r_{h,k_1,k_2}\}$ are uniformly distributed random variables in the interval $[-1, 1]$. The coefficients of $(f(x,y), g(x,y),\ h(x,y))$ are plotted in Figure 1, and it is seen that the coefficients of $f(x,y)$ and $g(x,y)$ span many more orders of magnitude than the coefficients of $h(x,y)$.

Heat maps of the binomial terms in the four variants of the 2×3 block matrix (26) for $k_1 = k_2 = 1$ are plotted in Figure 2 and it is seen that the binomial terms in the four non-zero block matrices in $\left\{ \hat{D}_{1,1}^{-1} \hat{T}_{1,1}(f,g,h) \hat{Q}_{1,1} \right\}$ span the smallest range. Figure 3 shows the minimum singular values of these variants and it is seen that $\{\hat{D}_{k_1,k_2}^{-1} \hat{T}_{k_1,k_2}(f,g,h) \hat{Q}_{k_1,k_2}\}$ is the optimal set of subresultant matrices for the computation of the degree of an AGCD because the separation between their numerically zero and non-zero minimum singular values is most clearly defined. This result is consistent with other results that require AGCD computations on Bernstein polynomials [3, 4, 5]. $\square$

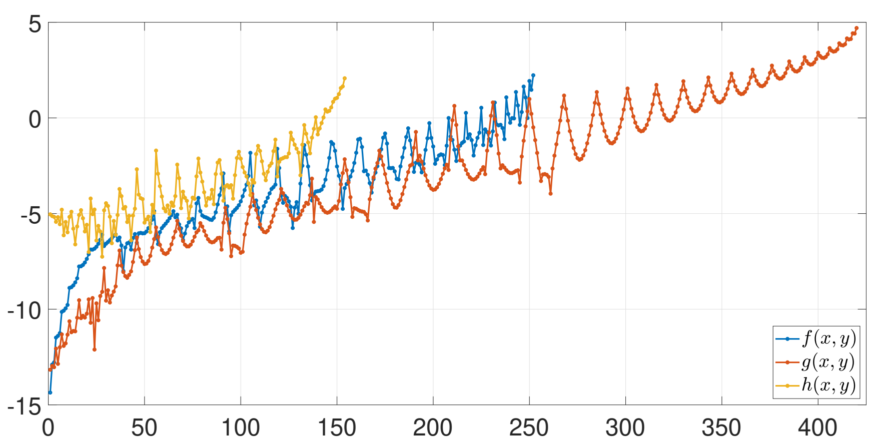
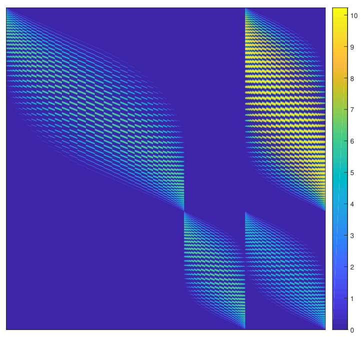
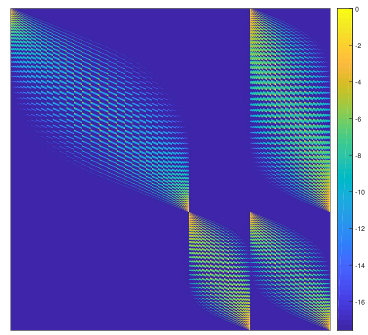


Figure 1: The coefficients of $(f(x,y), g(x,y), h(x,y))$, on a logarithmic scale, in Example 3.1.

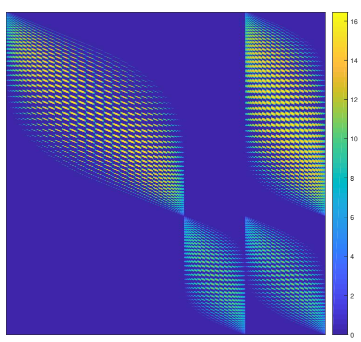
Example 3.1 shows that the ratio r of the entry of maximum magnitude to the entry of minimum magnitude assumes its minimum value for the set of subresultant matrices $\{\hat{D}_{k_1,k_2}^{-1} \hat{T}_{k_1,k_2}(\hat{f}, \hat{g}, \hat{h}) \hat{Q}_{k_1,k_2}\}$. The ratio r may be large, even for this optimal form of the Sylvester matrix and its subresultant matrices, and it is desirable to minimise r further because a large value may cause numerical problems. This minimisation is achieved by processing $(\hat{f}(x,y), \hat{g}(x,y), \hat{h}(x,y))$



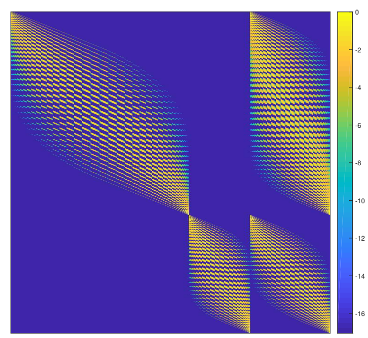
(a) $\hat{T}_{k_1,k_2}(f,g,h)$



(b) $\hat{D}^{-1}_{k_1,k_2}\hat{T}_{k_1,k_2}(f,g,h)$



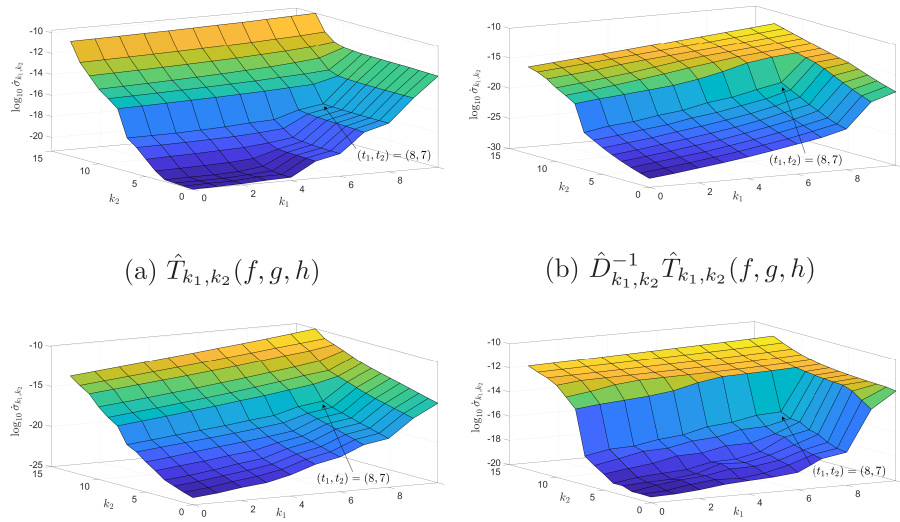
(c) $\hat{T}_{k_1,k_2}(f,g,h)\hat{Q}_{k_1,k_2}$



(d) $\hat{D}^{-1}_{k_1,k_2}\hat{T}_{k_1,k_2}(f,g,h)\hat{Q}_{k_1,k_2}$

Figure 2: Heat maps of the binomial terms, on a logarithmic scale, in the four variants of the 2×3 block subresultant matrix (26) for $k_1 = k_2 = 1$, where $f = f(x,y), g = g(x,y)$ and $h = h(x,y)$, in Example 3.1.

by three operations before computations are performed on their Sylvester matrices and subresultant matrices. These operations are considered in [3, 4, 5] for GCD computations on univariate and bivariate Bernstein polynomials, and it is shown that the inclusion of these operations yields considerably improved results. These operations for polynomials that are defined in a rectangular domain are considered in the next section.



(a) $\hat{T}_{k_1,k_2}(f, g, h)$ (b) $\hat{D}^{-1}_{k_1,k_2}\hat{T}_{k_1,k_2}(f, g, h)$

(c) $\hat{T}_{k_1,k_2}(f, g, h)\hat{Q}_{k_1,k_2}$ (d) $\hat{D}^{-1}_{k_1,k_2}\hat{T}_{k_1,k_2}(f, g, h)\hat{Q}_{k_1,k_2}$

Figure 3: The minimum singular values $\dot{\sigma}_{k_1,k_2}$ for the four variants of the 2×3 block subresultant matrix (26), where $f = f(x, y)$, $g = g(x, y)$ and $h = h(x, y)$, in Example 3.1.

4. Preprocessing operations

The preprocessing operations performed on bivariate polynomials are extensions of their forms for univariate polynomials. The first preprocessing operation requires the normalisation of the coefficients of each of the polynomials $(\hat{f}(x, y), \hat{g}(x, y), \hat{h}(x, y))$ by the geometric mean of its coefficients [5, §5]. This operation guarantees that each block matrix in the Sylvester matrices and their subresultant matrices is better balanced. The second preprocessing operation requires a change in the independent variables from (x, y) to (ω_1, ω_2),

$$\omega_1 = \theta_{1,k_1,k_2} x \qquad \text{and} \qquad \omega_2 = \theta_{2,k_1,k_2} y, \tag{29}$$

where θ_{1,k_1,k_2} and θ_{2,k_1,k_2} are constants and the orders k_1 and k_2 of the subresultant matrices are included in the notation because these constants must be determined for each subresultant matrix. The third preprocessing operation follows because the GCD of $(\hat{f}(x, y), \hat{g}(x, y), \hat{h}(x, y))$ is defined to within two arbitrary non-zero constants λ_{k_1,k_2} and ρ_{k_1,k_2},

$$\text{GCD}\,(\hat{f},\hat{g},\hat{h}) \sim \text{GCD}\,(\lambda_{k_1,k_2}\hat{f},\hat{g},\rho_{k_1,k_2}\hat{h}),$$

where $\sim$ denotes equivalence to within a non-zero constant. As in (29), the orders k_1 and k_2 of the subresultant matrices are included because the constants λ_{k_1,k_2} and ρ_{k_1,k_2} must be computed for each subresultant matrix. These constants, θ_{1,k_1,k_2} and θ_{2,k_1,k_2} are obtained from the solution of a linear programming (LP) problem [5, §5].

These operations yield the polynomials,

$$\begin{aligned}\lambda_{k_1,k_2}\tilde{f}_{k_1,k_2}(\omega_1,\omega_2) &= \lambda_{k_1,k_2}\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}\bar{a}_{i_1,i_2}\theta_{1,k_1,k_2}^{i_1}\theta_{2,k_1,k_2}^{i_2}\binom{m_1}{i_1}\binom{m_2}{i_2}\times\\ &\quad(1-\theta_{1,k_1,k_2}\omega_1)^{m_1-i_1}(1-\theta_{2,k_1,k_2}\omega_2)^{m_2-i_2}\omega_1^{i_1}\omega_2^{i_2},\end{aligned} \tag{30}$$

$$\begin{aligned}\tilde{g}_{k_1,k_2}(\omega_1,\omega_2) &= \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2}\bar{b}_{i_1,i_2}\theta_{1,k_1,k_2}^{i_1}\theta_{2,k_1,k_2}^{i_2}\binom{n_1}{i_1}\binom{n_2}{i_2}\times\\ &\quad(1-\theta_{1,k_1,k_2}\omega_1)^{n_1-i_1}(1-\theta_{2,k_1,k_2}\omega_2)^{n_2-i_2}\omega_1^{i_1}\omega_2^{i_2},\end{aligned} \tag{31}$$

$$\begin{aligned}\rho_{k_1,k_2}\tilde{h}_{k_1,k_2}(\omega_1,\omega_2) &= \rho_{k_1,k_2}\sum_{i_1=0}^{p_1}\sum_{i_2=0}^{p_2}\bar{c}_{i_1,i_2}\theta_{1,k_1,k_2}^{i_1}\theta_{2,k_1,k_2}^{i_2}\binom{p_1}{i_1}\binom{p_2}{i_2}\times\\ &\quad(1-\theta_{1,k_1,k_2}\omega_1)^{p_1-i_1}(1-\theta_{2,k_1,k_2}\omega_2)^{p_2-i_2}\omega_1^{i_1}\omega_2^{i_2},\end{aligned} \tag{32}$$

whose coefficients $\bar{a}_{i_1,i_2},\bar{b}_{i_1,i_2}$ and $\bar{c}_{i_1,i_2}$ are the normalised forms of the given coefficients $\hat{a}_{i_1,i_2},\hat{b}_{i_1,i_2}$ and $\hat{c}_{i_1,i_2}$, or their noisy forms, and they must be computed for each subresultant matrix. It is assumed in the sequel that the preprocessing operations have been implemented, such that all computations are performed on the polynomials (30), (31) and (32).

5. The computation of the degree of the GCD

Two methods for the computation of the degree of the GCD of the polynomials $(\hat{f}(x,y),\hat{g}(x,y),\hat{h}(x,y))$, and any two of these polynomials, are described in this section. The first method, called BiVariate Greatest Common Divisor (BVGCD), is an extension of the method for univariate polynomials and is computationally expensive. The second method, called BiVariate Degree Reducing Greatest Common Divisor (BVDRGCD), is much more efficient and uses degree elevated polynomials. These methods are described in Sections 5.1 and 5.2 respectively.

5.1. The BiVariate Greatest Common Divisor (BVGCD) method

The set of subresultant matrices of the bivariate polynomials $\hat{f}(x,y)$ and $\hat{g}(x,y)$ forms the two-dimensional array,

$$S_{k_1,k_2}(\hat{f},\hat{g}), \qquad k_1 = 1,\dots,\min(m_1,n_1);\ k_2 = 1,\dots,\min(m_2,n_2).$$

The rank of the (k_1,k_2)th subresultant matrix is defined by $\dot{\rho}_{k_1,k_2} = \log_{10}\dot{\sigma}_{k_1,k_2}$, where $\dot{\sigma}_{k_1,k_2}$ is its minimum (numerically) non-zero singular value. The degree of the GCD is therefore given by the maximum value of the difference $\delta\dot{\rho}_{k_1,k_2}$ in $\dot{\rho}_{k_1,k_2}$,

$$\delta\dot{\rho}_{k_1,k_2} = \dot{\rho}_{k_1+1,k_2+1} - \dot{\rho}_{k_1,k_2} = \log_{10}\frac{\dot{\sigma}_{k_1+1,k_2+1}}{\dot{\sigma}_{k_1,k_2}},$$

and thus the degree of the GCD is given by

$$(t_1,t_2) = \arg\max_{k_1,k_2}\{\delta\dot{\rho}_{k_1,k_2}\}.$$

This method for the computation of the degree of the GCD of two bivariate polynomials is computationally expensive because it is necessary to construct, preprocess and compute the singular values of $(\min(m_1,n_1)-1)\times(\min(m_2,n_2)-1)$ matrices. A more efficient method that reduces the bivariate GCD problem to two univariate GCD problems is considered in the next section.

5.2. The BiVariate Degree Reducing Greatest Common Divisor (BVDRGCD) method

The BVDRGCD method reduces the computation of the degrees t_1 and t_2 of the GCD of $\hat{f}(x,y)$ and $\hat{g}(x,y)$ from a two-dimensional problem to a one-dimensional problem that has two stages. The first stage determines a value

t^* such that either t_1 or t_2 can be deduced. Then, given either t_1 or t_2, the value of t_2 or t_1 is computed from the set of subresultant matrices $\{S_{t_1,k_2},\ k_2 = 1,\ldots,\min(m_2,n_2)\}$ or $\{S_{k_1,t_2},\ k_1 = 1,\ldots,\min(m_1,n_1)\}$. The first stage of the algorithm requires the degree elevation of $\hat{f}(x,y)$ and $\hat{g}(x,y)$ [9].

Consider the polynomials $\hat{f}(x,y)$ and $\hat{g}(x,y)$ of degrees (m_1,m_2) and (n_1,n_2), which are defined in (1) and (4) respectively, that have a GCD $\hat{d}_{t_1,t_2}(x,y)$ of degree (t_1,t_2). Let $\hat{f}(x,y)$ and $\hat{g}(x,y)$ be degree elevated by (p_1,p_2) and (q_1,q_2) respectively, such that their degree elevated forms are $\hat{f}^*(x,y)$ and $\hat{g}^*(x,y)$.[2] The polynomial $\hat{f}^*(x,y)$ can be written as

$$\hat{f}^*(x,y) = \hat{u}_{t_1,t_2,i_1,i_2}(x,y)\hat{d}_{t_1,t_2,p_1-i_1,p_2-i_2}(x,y),$$

where $i_1 = 0,\ldots,p_1$, $i_2 = 0,\ldots,p_2$, and the polynomial $\hat{u}_{t_1,t_2,i_1,i_2}(x,y)$ is the form of $\hat{u}_{t_1,t_2}(x,y)$ that has been degree elevated by (i_1,i_2), and the polynomial $\hat{d}_{t_1,t_2,p_1-i_1,p_2-i_2}(x,y)$ is the form of $\hat{d}_{t_1,t_2}(x,y)$ that has been degree elevated by (p_1-i_1,p_2-i_2),

$$\begin{aligned}\deg\ \hat{u}_{t_1,t_2,i_1,i_2}(x,y) &= (m_1-t_1+i_1, m_2-t_2+i_2)\,,\\ \deg\ \hat{d}_{t_1,t_2,p_1-i_1,p_2-i_2}(x,y) &= (t_1+p_1-i_1, t_2+p_2-i_2)\,.\end{aligned}$$

The polynomial $\hat{f}^*(x,y)$ has a divisor of degree (t_1+p_1, t_2+p_2), which is the degree elevated form of $\hat{d}_{t_1,t_2}(x,y)$ defined by $i_1 = i_2 = 0$, that is, the degree of the degree elevated form of $\hat{d}_{t_1,t_2}(x,y)$ is its maximum possible value. Similarly, the polynomial $\hat{g}^*(x,y)$ can be written as

$$\hat{g}^*(x,y) = \hat{v}_{t_1,t_2,i_1,i_2}(x,y)\hat{d}_{t_1,t_2,q_1-i_1,q_2-i_2}(x,y),$$

where $i_1 = 0,\ldots,q_1$, $i_2 = 0,\ldots,q_2$,

$$\begin{aligned}\deg\ \hat{v}_{t_1,t_2,i_1,i_2}(x,y) &= (n_1-t_1+i_1, n_2-t_2+i_2)\,,\\ \deg\ \hat{d}_{t_1,t_2,q_1-i_1,q_2-i_2}(x,y) &= (t_1+q_1-i_1, t_2+q_2-i_2)\,,\end{aligned}$$

[2]The degree elevation (p_1,p_2) of $\hat{f}(x,y)$ must not be confused with the degree (p_1,p_2) of $\hat{h}(x,y)$ in (12).

and $\hat{g}^*(x,y)$ has a divisor of degree (t_1+q_1, t_2+q_2). The GCD of $\hat{f}(x,y)$ and $\hat{g}(x,y)$ is therefore given by $\hat{d}_{t_1,t_2,\min(p_1,q_1),\min(p_2,q_2)}(x,y)$, which is of degree $(t_1+\min(p_1,q_1), t_2+\min(p_2,q_2))$, and is a degree elevated form of $\hat{d}_{t_1,t_2}(x,y)$. The third and fourth subscripts, $(\min(p_1,q_1), \min(p_2,q_2))$, denote the number of degree elevations of $\hat{d}_{t_1,t_2}(x,y)$, and thus

$$S_{k_1,k_2}(\hat{f}^*, \hat{g}^*)\mathbf{x}_{k_1,k_2} = \mathbf{0},$$

has non-zero solutions for

$$k_1 = 1, \ldots, t_1 + \min(p_1,q_1), \qquad k_2 = 1, \ldots, t_2 + \min(p_2,q_2),$$

and only zero solutions for

$$k_1 = t_1 + \min(p_1,q_1) + 1, \ldots, \min(m^*, n^*),$$
$$k_2 = t_2 + \min(p_2,q_2) + 1, \ldots, \min(m^*, n^*),$$

where

$$m^* = \min\left(m_1+p_1, n_1+q_1\right) \qquad \text{and} \qquad n^* = \min\left(m_2+p_2, n_2+q_2\right).$$

If $S_{k,k}(\hat{f}^*, \hat{g}^*)$ is rank deficient, then the inequalities

$$k \leq t_1 + \min(p_1,q_1) \qquad \text{and} \qquad k \leq t_2 + \min(p_2,q_2),$$

are satisfied, and if $S_{k+1,k+1}(\hat{f}^*, \hat{g}^*)$ has full rank, then one or both of the inequalities

$$k+1 > t_1 + \min(p_1,q_1) \qquad \text{and} \qquad k+1 > t_2 + \min(p_2,q_2),$$

are satisfied.

Theorem 5.1. Let $(\hat{f}(x,y), \hat{g}(x,y), \hat{h}(x,y))$, which are defined in (10), (11) and (12) respectively, be degree elevated by (p_1,p_2), (q_1,q_2) and (r_1,r_2) respectively, thereby yielding the polynomials $(\hat{f}^*(x,y),\ \hat{g}^*(x,y), \hat{h}^*(x,y))$ respectively, such that $\hat{f}^*(x,y)$ is of degree (m^*,m^*), $\hat{g}^*(x,y)$ is of degree (n^*,n^*) and $\hat{h}^*(x,y)$ is of degree (p^*,p^*), where $m^* = \max(m_1,m_2)$, $n^* = \max(n_1,n_2)$ and $p^* =$

$\max(p_1, p_2)$. Let the integer t be such that $\dot{S}_{t,t}(\hat{f}^*, \hat{g}^*, \hat{h}^*)$ is rank deficient and $\dot{S}_{t+1,t+1}(\hat{f}^*, \hat{g}^*, \hat{h}^*)$ has full rank, where $\dot{S}_{k_1,k_2}(\hat{f}^*, \hat{g}^*, \hat{h}^*)$ is the Sylvester matrix or (k_1, k_2)th subresultant matrix of one of its 2×3 block matrix forms or its 3×3 block matrix form. Then either

$$t = t_1 + \min(p_1, q_1, r_1), \tag{33}$$

holds, or

$$t = t_2 + \min(p_2, q_2, r_2), \tag{34}$$

holds, or

$$t = t_1 + \min(p_1, q_1, r_1) \qquad \text{and} \qquad t = t_2 + \min(p_2, q_2, r_2). \tag{35}$$

hold.

Proof It follows from the rank deficiency of $\dot{S}_{t,t}(\hat{f}^*, \hat{g}^*, \hat{h}^*)$ that

$$t \leq t_1 + \min(p_1, q_1, r_1) \qquad \text{and} \qquad t \leq t_2 + \min(p_2, q_2, r_2), \tag{36}$$

and it follows from the full rank property of $\dot{S}_{t+1,t+1}(\hat{f}^*, \hat{g}^*, \hat{h}^*)$ that one of the situations S1, S2 and S3 holds:

S1: $t + 1 > t_1 + \min(p_1, q_1, r_1)$ and no constraint between t and t_2

S2: $t + 1 > t_2 + \min(p_2, q_2, r_2)$ and no constraint between t and t_1

S3: $t + 1 > t_1 + \min(p_1, q_1, r_1)$ and $t + 1 > t_2 + \min(p_2, q_2, r_2)$.

Equation (33) follows from the first inequality in (36) and the inequality in S1. Equation (34) follows similarly from the second inequality in (36) and the inequality in S2, and (35) follows from (36) and S3.

The three situations (33), (34) and (35) must be considered.

1. If (33) is satisfied, then t_2 is determined from the equations

$$\dot{S}_{t_1,k_2}(\hat{f}, \hat{g}, \hat{h})\mathbf{x}_{t_1,k_2} = \mathbf{0}, \qquad k_2 = 1, \ldots, \min(m_2, n_2, p_2).$$

These equations have non-trivial solutions for $k_2 = 1, \ldots, t_2$, and only trivial solutions for $k_2 = t_2 + 1, \ldots, \min(m_2, n_2, p_2)$. The degree t_2 is therefore determined from the change from rank deficiency to full rank of the matrices $\dot{S}_{t_1,k_2}(\hat{f}, \hat{g}, \hat{h})$, $k_2 = 1, \ldots, \min(m_2, n_2, p_2)$.

2. If (34) is satisfied, then following item 1 above, t_1 is determined from the change from rank deficiency to full rank of the matrices $\dot{S}_{k_1,t_2}(\hat{f},\hat{g},\hat{h})$, $k_1 = 1,\ldots,\min(m_1,n_1,p_1)$.
3. If (35) is satisfied, then

$$t_1 = t - \min(p_1,q_1,r_1) \qquad \text{and} \qquad t_2 = t - \min(p_2,q_2,r_2).$$

□

For a given value of t, it is necessary to determine the equation, either (33) or (34), that is satisfied. The subresultant matrices $\dot{S}_{t_1,k_2}(\hat{f},\hat{g},\hat{h})$ and $\dot{S}_{k_1,t_2}(\hat{f},\hat{g},\hat{h})$ are constructed and two candidate pairs (t_1,t_2) are computed. The degree of the GCD of $(\hat{f}(x,y),\hat{g}(x,y),\hat{h}(x,y))$ is the maximum of these candidate pairs.

Example 5.1. Let the degrees of the Bernstein polynomials $\hat{f}(x,y)$ and $\hat{g}(x,y)$ be, respectively, $(m_1,m_2) = (16,12)$ and $(n_1,n_2) = (14,10)$,

$$\hat{f}(x,y) = (x+y+0.0124)^5(x+0.56)^4(x^2+y^2+0.51)^2(x+y+1.12)^3,$$
$$\hat{g}(x,y) = (x+y+0.4512)^3(x+0.56)^4(x^2+y^2+0.51)^2(x+y+1.12)^3.$$

The factorised form of their GCD is

$$\hat{d}(x,y) = (x+0.56)^4(x^2+y^2+0.51)^2(x+y+1.12)^3.$$

Noise was added to the coefficients of $\hat{f}(x,y)$ and $\hat{g}(x,y)$ such that the coefficients of the inexact polynomials $f(x,y)$ and $g(x,y)$ are

$$a_{i_1,i_2} = \hat{a}_{i_1,i_2} + \epsilon_{f,i_1,i_2}\hat{a}_{i_1,i_2}r_{f,i_1,i_2}, \qquad i_1 = 0,\ldots,m_1;\ i_2 = 0,\ldots,m_2,$$
$$b_{j_1,j_2} = \hat{b}_{j_1,j_2} + \epsilon_{g,j_1,j_2}\hat{b}_{j_1,j_2}r_{g,j_1,j_2}, \qquad j_1 = 0,\ldots,n_1;\ j_2 = 0,\ldots,n_2,$$

where $\{\epsilon_{f,i_1,i_2}\} = \{\epsilon_{g,j_1,j_2}\} = 10^{-8}$, and $\{r_{f,i_1,i_2}\}$ and $\{r_{g,j_1,j_2}\}$ are uniformly distributed random variables in the interval $[-1,1]$.

The BVGCD method proceeds by preprocessing $f(x,y)$ and $g(x,y)$ for each subresultant matrix $S_{k_1,k_2}(\lambda_{k_1,k_2}\tilde{f}_{k_1,k_2},\tilde{g}_{k_1,k_2})$, $k_1 = 1,\ldots,14$, $k_2 = 1,\ldots,10$, where $\tilde{f}_{k_1,k_2} = \tilde{f}_{k_1,k_2}(\omega_1,\omega_2)$ and $\tilde{g}_{k_1,k_2} = \tilde{g}_{k_1,k_2}(\omega_1,\omega_2)$ are defined in (30) and (31) respectively, and computing the minimum singular value of each subresultant matrix. These singular values $\dot{\sigma}_{k_1,k_2}$ are plotted in Figure 4, and the degree of an AGCD is $(11,7)$ since the maximum entry in the set $\{\delta\dot{\rho}_{k_1,k_2}\}$ is $\delta\dot{\rho}_{11,7}$.

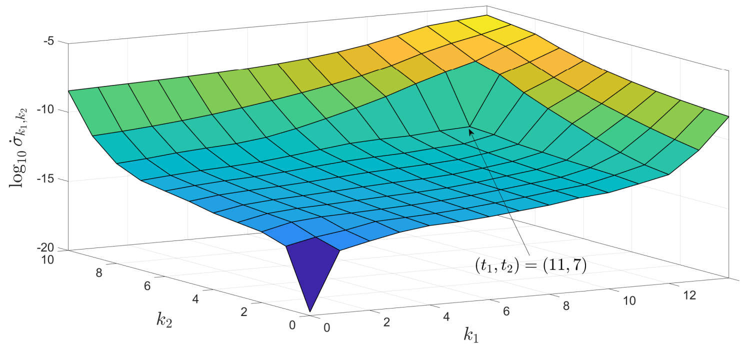


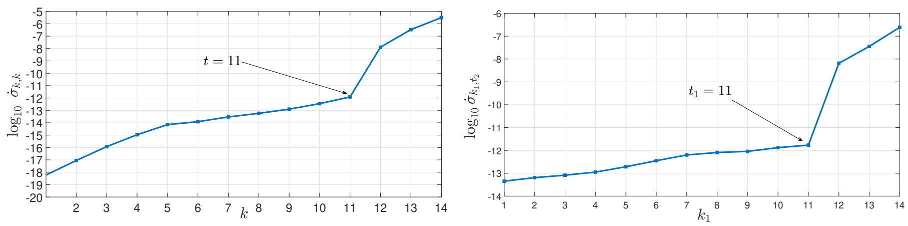
Figure 4: The minimum singular values $\dot{\sigma}_{k_1,k_2}$ of the preprocessed subresultant matrices $S_{k_1,k_2}(\lambda_{k_1,k_2}\tilde{f}_{k_1,k_2}, \tilde{g}_{k_1,k_2})$ in Example 5.1.

Consider now the BVDRGCD method, the first stage of which requires that $f(x,y)$ and $g(x,y)$ be degree elevated to $(m^*, m^*) = (16, 16)$ and $(n^*, n^*) = (14, 14)$ respectively, that is, $f(x,y)$ and $g(x,y)$ are degree elevated by $(p_1, p_2) = (0, 4)$ and $(q_1, q_2) = (0, 4)$, respectively. The minimum singular values of the subresultant matrices $S_{k,k}(\lambda_{k,k}\tilde{f}^*_{k,k}, \tilde{g}^*_{k,k})$, $k = 1, \ldots, \min(m^*, n^*)$, were computed, where $\lambda_{k,k}\tilde{f}^*_{k,k}(\omega_1, \omega_2)$ and $\tilde{g}^*_{k,k}(\omega_1, \omega_2)$ are the degree elevated polynomials after preprocessing. These singular values are plotted in Figure 5(i) and it is seen that $t = 11$, and thus

$$\begin{aligned} &\text{either} \qquad t_1 = t - \min(p_1, q_1) = 11 - 0 = 11, \\ &\text{or} \qquad\quad\; t_2 = t - \min(p_2, q_2) = 11 - 4 = 7. \end{aligned}$$

Both t_1 and t_2 are correct, but assume that only t_2 is determined correctly. In this situation, the degree t_1 of an AGCD with respect to x is computed from the minimum singular values of the matrices $S_{k_1,t_2}(\lambda_{k_1,t_2}\tilde{f}_{k_1,t_2}, \tilde{g}_{k_1,t_2})$, $k_1 = 1, \ldots, 14$. They are plotted in Figure 5(ii) and it is seen that $t_1 = 11$.

The BVGCD method required the evaluation of 140 subresultant matrices, but the BVDRGCD method required the preprocessing and evaluation of only 28 subresultant matrices. □



(i) BVDRGCD Stage 1: The minimum singular values $\dot{\sigma}_{k,k}$ of $S_{k,k}(\lambda_{k,k}\tilde{f}^*_{k,k}, \tilde{g}^*_{k,k})$

(ii) BVDRGCD Stage 2a: The minimum singular values $\dot{\sigma}_{k_1,t_2}$ of $S_{k_1,t_2}(\lambda_{k_1,t_2}\tilde{f}_{k_1,t_2}, \tilde{g}_{k_1,t_2})$

Figure 5: The minimum singular values of the preprocessed subresultant matrices using the BVDRGCD method in Example 5.1.

5.3. The coefficients of an AGCD

The coefficients of an AGCD of $(\tilde{f}(\omega_1,\omega_2),\ \tilde{g}(\omega_1,\omega_2), \tilde{h}(\omega_1,\omega_2))$ are computed from one of the 2×3 block subresultant matrices, or the 3×3 block subresultant matrix $\tilde{S}_{t_1,t_2}(\lambda_{t_1,t_2}\tilde{f}_{t_1,t_2}, \tilde{g}_{t_1,t_2}, \rho_{t_1,t_2}\tilde{h}_{t_1,t_2})$, using the method described in [5] for bivariate Bernstein polynomials defined in a triangular domain. These coefficients are defined in the variables (ω_1,ω_2) and the transformation to the variables (x, y) follows from (29).

6. Complexity

The algorithms described in this paper involve computations on large matrices that have a block structure, where each block is of the form $D^{-1}TQ$. It is adequate to consider one of the four forms of the Sylvester matrix and its subresultant matrices because the determination of the complexity of the computations for the other three forms follows identically. Consider therefore the matrices $\tilde{S}_{k_1,k_2}(\hat{f},\hat{g},\hat{h})$, which are defined in (20) and of order $p\times q$ where

$$\begin{aligned} p \quad = \quad & (m_1+n_1-k_1+1)(m_2+n_2-k_2+1)+ \\ & (m_1+p_1-k_1+1)(m_2+p_2-k_2+1)+ \\ & (n_1+p_1-k_1+1)(n_2+p_2-k_2+1), \end{aligned}$$

and

$$
\begin{aligned}
q \quad = \quad & (m_1 - k_1 + 1)(m_2 - k_2 + 1) + (n_1 - k_1 + 1)(n_2 - k_2 + 1) + \\
& (p_1 - k_1 + 1)(p_2 - k_2 + 1),
\end{aligned}
$$

and thus the order of $\tilde{S}_{k_1,k_2}(\hat{f}, \hat{g}, \hat{h})$ is a quadratic function of the degrees $(m_1, m_2, n_1, n_2, p_1, p_2)$ and the subresultant indices (k_1, k_2).

The complexity of the computation of an AGCD is governed by the complexity of the preprocessing operations and the SVD of the matrices $\tilde{S}_{k_1,k_2}(\hat{f}, \hat{g}, \hat{h})$. The preprocessing operations require the solution of the LP problem,

$$
\min c^T x \qquad \text{subject to} \qquad Ax \geq b,
$$

where $A \in \mathbb{R}^{r \times s}$, r is the number of constraints,

$$
r = 2\left((m_1 + 1)(m_2 + 1) + (n_1 + 1)(n_2 + 1) + (p_1 + 1)(p_2 + 1)\right),
$$

and $s = 6$ is the number of variables. All the operations in the simplex method for the solution of the LP problem are simple matrix and vector calculations, the total cost of which is $O(rs)$ arithmetic operations if full pricing is used. The cost of refactorisation of the basis matrix in the simplex method is $O(r^3)$ operations and thus if $O(r)$ iterations are performed, the total cost of the simplex method is $O(r^4 + r^2 s) \approx O(r^4)$ arithmetic operations if $r \gg s$ [13, pp. 222, 270].

The degree and coefficients of an AGCD are calculated using the SVD of the matrices $\tilde{S}_{k_1,k_2}(\hat{f}, \hat{g}, \hat{h})$ after the preprocessing operations have been implemented. This decomposition requires $O(4p^2 q + 8pq^2 + 9q^3)$ arithmetic operations [11, p. 493], and the BVDRGCD method for the calculation of the degree of an AGCD requires that the SVD for two univariate AGCD computations be computed for each subresultant matrix, after $(\hat{f}(x,y), \hat{g}(x,y), \hat{h}(x,y))$ have been degree elevated. It follows that the total complexity of the SVD computations is significant, and the combination of this result and the complexity of the simplex method for the solution of the LP problem shows that the method described in this paper is computationally expensive.

7. Examples

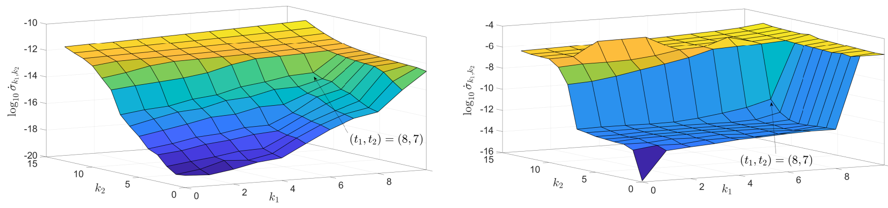
This section contains two examples that demonstrate an approximate factorisation of two or three Bernstein polynomials defined in a rectangular domain.

Example 7.1. Consider the polynomials in Example 3.1. Noise was added to their coefficients such that the inexact coefficients are given by (28), where $\{r_{f,i_1,i_2}\}$, $\{r_{g,j_1,j_2}\}$ and $\{r_{h,k_1,k_2}\}$ are uniformly distributed random variables in the interval $[-1,1]$, and $\{\epsilon_{f,i_1,i_2}\}$, $\{\epsilon_{g,j_1,j_2}\}$ and $\{\epsilon_{h,k_1,k_2}\}$ are uniformly distributed random variables in the interval $[10^{-10}, 10^{-8}]$.

Consider initially the BVGCD method for the determination of the degree of an AGCD of the perturbed polynomials. The minimum singular value $\dot{\sigma}_{k_1,k_2}$ of each matrix in the set $\{\tilde{S}_{k_1,k_2}(f,g,h)\}$ before the preprocessing operations are implemented is plotted in Figure 6(i) and the minimum singular value of each matrix in the set $\{\tilde{S}_{k_1,k_2}(\lambda_{k_1,k_2}\tilde{f}_{k_1,k_2}, \tilde{g}_{k_1,k_2}, \rho_{k_1,k_2}\tilde{h}_{k_1,k_2})\}$ after the preprocessing operations are implemented is plotted in Figure 6(ii). There does not exist a significant separation between the numerically zero and non-zero singular values of the unprocessed subresultant matrices (Figure 6(i)) and thus the degree of an AGCD cannot be determined reliably. There is, however, a distinct separation between the numerically zero and non-zero minimum singular values of the preprocessed subresultant matrices (Figure 6(ii)) and the degree of an AGCD is determined correctly and given by $(t_1, t_2) = (8, 7)$.

The BVDRGCD method was then applied, and $(f(x,y), g(x,y), h(x,y))$, which are of degrees $(17, 13)$, $(20, 19)$ and $(10, 13)$ respectively, were degree elevated such that $(f^*(x,y), g^*(x,y), h^*(x,y))$ are of degrees $m^* = 17$, $n^* = 20$ and $p^* = 13$. The number of degree elevations of $(f(x,y), g(x,y), h(x,y))$ are $(p_1, p_2) = (0, 4)$, $(q_1, q_2) = (0, 1)$ and $(r_1, r_2) = (3, 0)$. The minimum singular values of the set of subresultant matrices after the implementation of the preprocessing operations, $\{\tilde{S}_{k,k}(\lambda_{k,k}\tilde{f}^*_{k,k}, \tilde{g}^*_{k,k}, \rho_{k,k}\tilde{h}^*_{k,k})\}$, are plotted in Figure 7. It is seen that $t = 7$ and thus

$$\text{either} \quad t_1 = t - \min(p_1, q_1, r_1) = 7 \quad \text{or} \quad t_2 = t - \min(p_2, q_2, r_2) = 7.$$



(i) (ii)

Figure 6: The minimum singular values $\dot{\sigma}_{k_1,k_2}$ of the subresultant matrices (i) before and (ii) after the implementation of the preprocessing operations, in Example 7.1.

Assume that the candidate degree $t_1 = 7$ is correct, and thus the degree t_2 of an AGCD with respect to y is equal to the index of the last numerically rank deficient matrix in the set $\{\tilde{S}_{t_1,k_2}(\lambda_{t_1,k_2}\tilde{f}^*_{t_1,k_2}, \tilde{g}^*_{t_1,k_2}, \rho_{t_1,k_2}\tilde{h}^*_{t_1,k_2})\}$ of subresultant matrices. The minimum singular values of these subresultant matrices are plotted in Figure 8(i) and it is seen that the degree of an AGCD with respect to y is $t_2 = 7$.

Alternatively, assume that the candidate degree $t_2 = 7$ is correct, and thus the degree t_1 of an AGCD with respect to x is equal to the index of the last numerically rank deficient matrix in the set $\{\tilde{S}_{k_1,t_2}(\lambda_{k_1,t_2}\tilde{f}^*_{k_1,t_2}, \tilde{g}^*_{k_1,t_2}, \rho_{k_1,t_2}\tilde{h}^*_{k_1,t_2})\}$ of subresultant matrices. The minimum singular values of these matrices are plotted in Figure 8(ii), from which the degree of an AGCD with respect to x is $t_1 = 8$. The two candidate pairs are $(t_1, t_2) = (7, 7)$ and $(t_1, t_2) = (8, 7)$, and thus the degree of an AGCD is $(8, 7)$.

The BVGCD method required the evaluation of 150 subresultant matrices but the BVDRGCD method required the evaluation of only 36 subresultant matrices, which represents a significant reduction in computation time. □

Example 7.2. Consider the Bernstein forms of the exact polynomials $\hat{f}(x, y)$ and $\hat{g}(x, y)$ of degrees $(29, 15)$ and $(20, 17)$ respectively, and whose factorisations

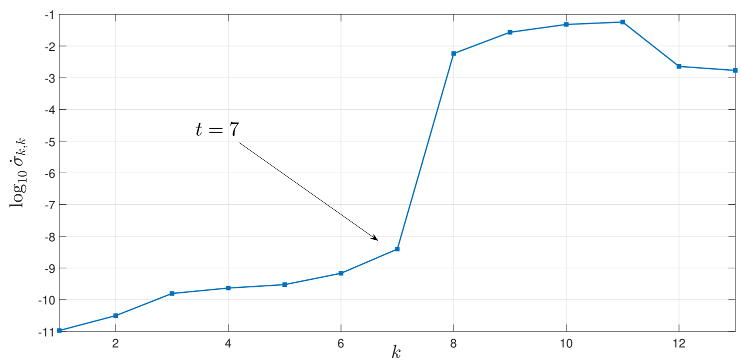
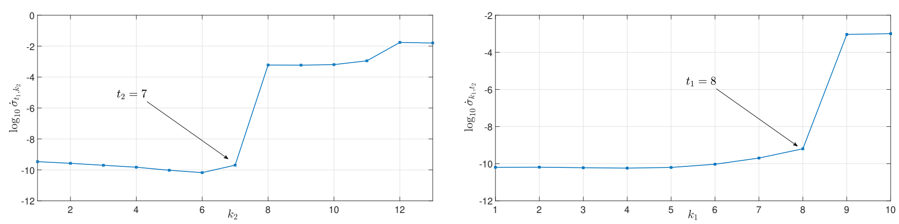


Figure 7: The minimum singular value $\dot{\sigma}_{k,k}$ of $\tilde{S}_{k,k}(\lambda_{k,k}\tilde{f}^*_{k,k}, \tilde{g}^*_{k,k}, \rho_{k,k}\tilde{h}^*_{k,k})$ in Example 7.1.



(i) BVDRGCD Stage 2a: The minimum singular values $\dot{\sigma}_{t_1,k_2}$ of $\tilde{S}_{t_1,k_2}(\lambda_{t_1,k_2}\tilde{f}^*_{t_1,k_2}, \tilde{g}^*_{t_1,k_2}, \rho_{t_1,k_2}\tilde{h}^*_{t_1,k_2})$

(ii) BVDRGCD Stage 2b: The minimum singular values $\dot{\sigma}_{k_1,t_2}$ of $\tilde{S}_{k_1,t_2}(\lambda_{k_1,t_2}\tilde{f}^*_{k_1,t_2}, \tilde{g}^*_{k_1,t_2}, \rho_{k_1,t_2}\tilde{h}^*_{k_1,t_2})$

Figure 8: BVDRGCD Stage 2: The minimum singular values in the second stage of the BVDRGCD method in Example 7.1.

are

$$
\begin{aligned}
\hat{f}(x,y) &= (x - 0.8365498798)^3(x - 0.145487821)^{10} \times \\
&\quad (x - 0.126479841321)^5(x + y - 0.16546978321)^2 \times \\
&\quad (x + y + 0.5679814354)^3(x + y^2 - 0.2564878)^4 \times \\
&\quad (x^2 + y^2 - 0.46549871232156), \\
\hat{g}(x,y) &= (x - 0.8365498798)^3(x - 0.126479841321)^5 \times \\
&\quad (y - 0.45489789123123)^5(x + y - 0.35648979126321)^3 \times \\
&\quad (x + y - 0.16546978321)^2(x + y + 0.5679814354)^3 \times \\
&\quad (x^2 + y^2 - 0.46549871232156)(x^2 + y^2 - 0.45489789123123).
\end{aligned}
$$

The polynomials $\hat{f}(x,y)$ and $\hat{g}(x,y)$ have a GCD $\hat{d}_{t_1,t_2}(x,y)$ of degree $(t_1,t_2) = (15,7)$,

$$\begin{aligned}\hat{d}_{t_1,t_2}(x,y) &= (x-0.8365498798)^3(x-0.126479841321)^5 \times \\ &\quad (x+y-0.16546978321)^2(x+y+0.5679814354)^3 \times \\ &\quad (x^2+y^2-0.46549871232156).\end{aligned}$$

Noise was added to the coefficients of $\hat{f}(x,y)$ and $\hat{g}(x,y)$, such that the coefficients of the inexact polynomials $f(x,y)$ and $g(x,y)$ are

$$a_{i_1,i_2} = \hat{a}_{i_1,i_2} + \epsilon_{f,i_1,i_2}\hat{a}_{i_1,i_2}r_{f,i_1,i_2} \quad \text{and} \quad b_{j_1,j_2} = \hat{b}_{j_1,j_2} + \epsilon_{g,j_1,j_2}\hat{b}_{j_1,j_2}r_{g,j_1,j_2},$$

where $\{\epsilon_{f,i_1,i_2}\}$ and $\{\epsilon_{g,j_1,j_2}\}$ are uniformly distributed random variables in the interval $[10^{-11}, 10^{-10}]$, and $\{r_{f,i_1,i_2}\}$ and $\{r_{g,j_1,j_2}\}$ are uniformly distributed random variables in the interval $[-1,1]$.

The preprocessing operations were applied to the inexact polynomials $f(x,y)$ and $g(x,y)$, thus yielding the polynomials $\lambda_{1,1}\tilde{f}(\omega_1,\omega_2)$ and $\tilde{g}(\omega_1,\omega_2)$. The coefficients of the polynomials $f(x,y)$ and $g(x,y)$ spanned approximately 18 orders of magnitude, but the coefficients of the preprocessed polynomials $\lambda_{1,1}\tilde{f}(\omega_1,\omega_2)$ and $\tilde{g}(\omega_1,\omega_2)$ spanned approximately 6 orders of magnitude.

The degree of an AGCD of $f(x,y)$ and $g(x,y)$ was computed using the methods BVGCD and BVDRGCD. Consider initially the BVGCD method, for which the minimum singular values $\dot{\sigma}_{k_1,k_2}$ of the subresultant matrices before and after the application of the preprocessing operations are plotted in Figure 9. It is clear that the degree of an AGCD cannot be determined from the minimum singular values of the subresultant matrices that have not been preprocessed, but it can be determined from these singular values of the subresultant matrices that have been preprocessed. The maximum change is between $\dot{\rho}_{15,7} = \log_{10}\dot{\sigma}_{15,7}$ and $\dot{\rho}_{16,8} = \log_{10}\dot{\sigma}_{16,8}$, and thus the degree of an AGCD is $(t_1,t_2) = (15,7)$.

The BVDRGCD method was also used to compute the degree of an AGCD. The polynomials $f(x,y)$ and $g(x,y)$ were degree elevated by $(p_1,p_2) = (0,14)$ and $(q_1,q_2) = (0,3)$ respectively, such that the degree elevated forms $f^*(x,y)$ and $g^*(x,y)$ are of degrees $m^* = 29$ and $n^* = 20$, respectively. The minimum singular values of the set of subresultant matrices $\{S_{k,k}(\lambda_{k,k}\tilde{f}^*_{k,k}, \tilde{g}^*_{k,k})\}$ after the application of the preprocessing operations are plotted in Figure 10, and it is seen that the degree of an AGCD is $t = 10$. It follows that

$$\text{either} \qquad t_1 = t - \min(p_1, q_1) = 10 \qquad \text{or} \qquad t_2 = t - \min(p_2, q_2) = 7.$$

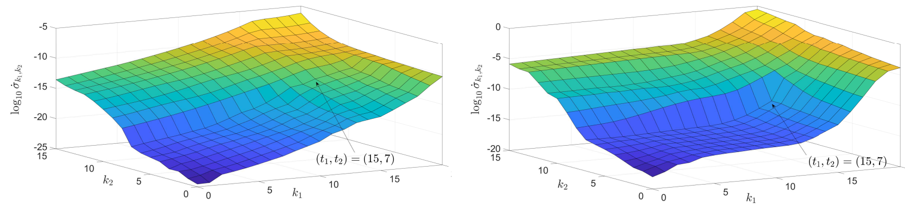
Assume that the degree $t_1 = 10$ is correct, in which case t_2 is equal to the index of the last numerically rank deficient subresultant matrix in the set $\{S_{t_1,k_2}(\lambda_{t_1,k_2}\tilde{f}^*_{t_1,k_2}, \tilde{g}^*_{t_1,k_2})\}$. The minimum singular values of these subresultant matrices are plotted in Figure 11(i) and it is seen that the degree of an AGCD with respect to y is $t_2 = 7$. Alternatively, if the degree $t_2 = 7$ is correct, then t_1 is equal to the index of the last numerically rank deficient subresultant matrix in the set $\{S_{k_1,t_2}(\lambda_{k_1,t_2}\tilde{f}^*_{k_1,t_2}, \tilde{g}^*_{k_1,t_2})\}$. The minimum singular values of these subresultant matrices are plotted in Figure 11(ii) and it is seen that the degree of an AGCD with respect to x is $t_1 = 15$. The degree of an AGCD is therefore equal to either $(10, 7)$ or $(15, 7)$, and thus $(t_1, t_2) = (15, 7)$.

The coefficients of the coprime polynomials $u_{t_1,t_2}(x, y)$ and $v_{t_1,t_2}(x, y)$, and AGCD $d_{t_1,t_2}(x, y)$, were computed using the method described in Section 5.3 and the errors are shown in Table 1. The coprime polynomials and AGCD could not be computed if the preprocessing operations were not implemented because the degree of an AGCD was not defined. □

Error $u_{t_1,t_2}(x, y)$	1.455132e-06
Error $v_{t_1,t_2}(x, y)$	5.197778e-06
Error $d_{t_1,t_2}(x, y)$	2.548405e-06

Table 1: Errors in the approximate coprime polynomials $u_{t_1,t_2}(x, y)$ and $v_{t_1,t_2}(x, y)$, and AGCD $d_{t_1,t_2}(x, y)$, after the preprocessing operations are implemented in Example 7.2.

The examples show that the Sylvester matrix and its subresultant matrices can be used to calculate an AGCD of bivariate Bernstein polynomials, which arises in intersection problems in CAD systems. Specifically, consider cubic curves and bicubic patches, which are frequently used in CAD systems. The



(i) The minimum singular values $\dot{\sigma}_{k_1,k_2}$ of $S_{k_1,k_2}(f, g)$ before preprocessing

(ii) The minimum singular values $\dot{\sigma}_{k_1,k_2}$ of $S_{k_1,k_2}(\lambda_{k_1,k_2}\tilde{f}_{k_1,k_2}, \tilde{g}_{k_1,k_2})$ after preprocessing

Figure 9: The minimum singular values $\dot{\sigma}_{k_1,k_2}$ of the subresultant matrices (i) before and (ii) after the implementation of the preprocessing operations, in Example 7.2.

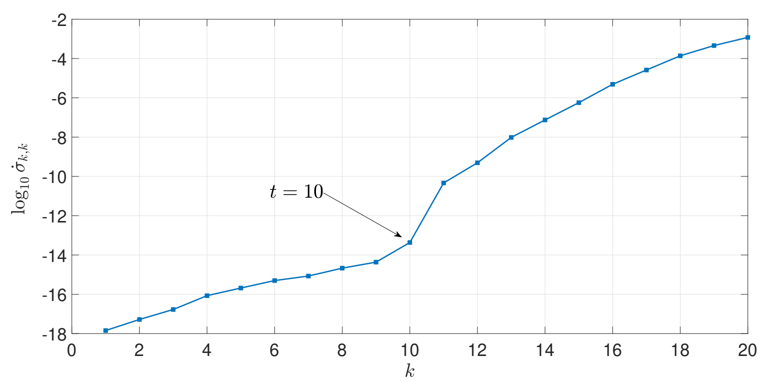
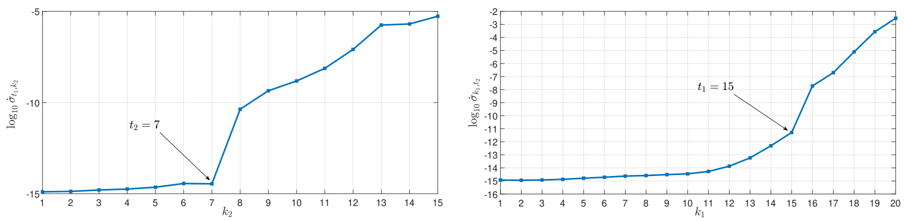


Figure 10: BVDRGCD Stage 1: The minimum singular values $\dot{\sigma}_{k,k}$ of the subresultant matrices $S_{k,k}(\lambda_{k,k}\tilde{f}^*_{k,k}, \tilde{g}^*_{k,k})$ in Example 7.2.

implicit form of a parametric cubic curve is a polynomial of degree three and thus the intersection points of two planar parametric cubic curves are equal to the roots of a polynomial of degree nine. A general bicubic patch is an algebraic surface of degree 18, the implicit equation of which has 1330 coefficients, and thus a ray intersects a bicubic patch at a maximum of 18 points. Also, a bicubic patch intersects a quadric surface in a curve of degree 36, and it is noted that an intersection curve of degree t may be formed by a collection of several distinct curves whose degrees sum to t. The degrees of the curves in the examples in Sections 3, 5 and 7 are approximately equal to the degrees of these curves of



(i) BVDRGCD Stage 2a: The minimum singular values $\dot{\sigma}_{t_1,k_2}$ of the matrices $S_{t_1,k_2}(\lambda_{t_1,k_2}\tilde{f}^*_{t_1,k_2}, \tilde{g}^*_{t_1,k_2})$

(ii) BVDRGCD Stage 2b: The minimum singular values $\dot{\sigma}_{k_1,t_2}$ of the matrices $S_{k_1,t_2}(\lambda_{k_1,t_2}\tilde{f}^*_{k_1,t_2}, \tilde{g}^*_{k_1,t_2})$

Figure 11: The minimum singular values of the subresultant matrices in the second stage of the BVDRGCD method in Example 7.2.

intersection that arise in CAD systems, and thus the examples are representative of these curves of intersection.

8. Summary

This paper has considered an approximate factorisation of two or three polynomials defined in a rectangular domain. The Sylvester matrix and its subresultant matrices of the polynomials were formed and it was shown that the polynomials must be processed before computations are performed on these matrices in order to mitigate the adverse effects of the large range of the magnitudes of their entries. It was shown that there are four forms of the Sylvester matrix and its subresultant matrices. One of these forms is a 3×3 block matrix and the three other forms are 2×3 block matrices. Two methods for the computation of the degree of an AGCD of the polynomials were presented and it was shown that one method is efficient because it reduces the problem from one AGCD computation on bivariate polynomials to two AGCD computations on two univariate polynomials.

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