ON A CANONICAL LIFT OF ARTIN'S REPRESENTATION TO LOOP BRAID GROUPS

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Abstract. Each pointed topological space has an associated π-*module*, obtained from action of its first homotopy group on its second homotopy group. For the 3-ball with a trivial link with n-components removed from its interior, its π-module $\mathcal{M}_n$ is of *free type*. In this paper we give an injection of the *(extended) loop braid group* into the group of automorphisms of $\mathcal{M}_n$. We give a topological interpretation of this injection, showing that it is both an extension of Artin's representation for braid groups and of Dahm's homomorphism for (extended) loop braid groups.

1. Introduction

A recent paper [BCK$^+$0] showed that *discrete higher gauge theory* (as discussed generally and informally by many authors, see e.g. [Mar91, §10.2], and compare with *continuous* higher gauge theory [Pfe03, BS07, BH11, FMP11, SW13] and several others) is well-defined. The technical engine of the construction notably reflects the *combinatorial homotopy* of Whitehead, Baues, Brown, *et al.* [Bau91, BHS11]. Relatively simple aspects of the construction such as loop-particle braiding [BFMM19] yield higher generalisations of classical results, for example in low-dimensional topology [Far09]. We discuss one such result, a lifting of Artin's representation of the *braid group* [Bir74] to the *(extended) loop braid group* [Lin08] (see [Dam17], for a survey).

1.1. **From mapping class groups to Artin-like representations.** Let X be an oriented topological manifold with boundary ∂X, and A a possibly empty subset in the interior of X. A *self-homeomorphism of the pair* (X, A) *relative to the boundary,* is a self-homeomorphism g of X that fixes ∂X pointwise, A setwise (i.e. $g(A) = A$), and preserves the orientation of X. Two such homeomorphisms f_0 and f_1 are (X, A)*-isotopic* if they can be included in a 1-parameter family $\{f_t\}_{t\in[0,1]}$ of self-homeomorphisms of (X, A) relative to the boundary, such that the map $X \times [0, 1] \to X$ sending (x, t) to $f_t(x)$ is continuous. The mapping class group $\mathrm{MCG}(X, A)$ is the group of (X, A)-isotopy classes of self-homeomorphisms of (X, A). We write $[g]$ for the class of g. Our convention for the product in $\mathrm{MCG}(X, A)$ is: $[g][g'] = [g \circ g']$.

A self-homeomorphism $g\colon (X, A) \to (X, A)$ relative to the boundary, takes an n-path $[0, 1]^n \xrightarrow{\gamma} X$ to an n-path $[0, 1]^n \xrightarrow{g\circ\gamma} X$. In particular g takes a loop based at a point $*$ in ∂X to another such loop. Furthermore a homeomorphism that fixes a set fixes its complement setwise. If γ avoids A then so does $g \circ \gamma$. The map g thus induces an automorphism of $\pi_1(X \setminus A, *)$ with $* \in \partial X$, that is: $[\gamma] \mapsto [g \circ \gamma]$.

Passing from g to $[g]$, this gives a well-defined group homomorphism

$$\tau\colon\ \mathrm{MCG}(X, A) \longrightarrow \mathrm{Aut}(\pi_1(X \setminus A, *)), \tag{1.1}$$

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where $[g \circ \gamma] = [g' \circ \gamma']$, if $g' \in [g]$ and $\gamma' \in [\gamma]$. If G is a group, our convention for the product in $\mathrm{Aut}(G)$, the automorphism group of G, is $fg = f \circ g$.

By considering based n-paths $([0,1]^n, \partial[0,1]^n) \to (X, *)$ we can in principle use other homotopy functors analogously in place of π_1, obtaining representations of mapping class groups this way. In this paper, we will use the homotopy functor which sends a pointed space $(Y, *)$ to the triple $\pi_{(1,2)}(Y, *)$ defined as $(\pi_1(Y, *), \pi_2(Y, *), \triangleright_{\pi_1})$, where $\triangleright_{\pi_1}$ is the usual action of π_1 on π_2. The underlying algebraic notion is that of a *π-module*, see for example [Bau91, Chapter 1, §1]. A π-module is a triple $\mathcal{G} = (G, A, \triangleright)$, where G is a group, A is an abelian group, and $\triangleright$ is a left-action by automorphisms of G on A; see Subsection 3.2. A morphism of π-modules is a pair of group morphisms that respect the actions.

Artin's representation of the braid group [Art47], and Dahm homomorphism for the extended loop braid group [Dah62, Gol81] can be seen as a special case of this construction. Let us recap here these two maps.

1.2. **Artin representation and Dahm homomorphism.** For $m \geq 1$, let D^m be the m-disc $[0,1]^m \subset \mathbb{R}^m$. Fix $d_n = \{p_1, ..., p_n\} \subset D^2 \setminus \partial D^2$ a set of n points in the interior of D^2, and $L_n = C_1 \cup ... \cup C_n \subset D^3 \setminus \partial D^3$ a set made out of the union of n disjoint, unknotted, oriented circles, that form a trivial link with n components in the interior of D^3.

The *braid group* B_n can be defined as the mapping class group of the pair (D^2, d_n) (for a survey, see for instance [BB05]). Analogously the *extended loop braid group* LB_n^{ext} is defined as the mapping class group $\mathrm{MCG}(D^3, L_n)$, meaning the group of self-homeomorphism of the pair (D^3, L_n), relative to the boundary of D^3, up to (D^3, L_n)-isotopy [Dam17]. Note that this definition appears also in [Gol81], in terms of *motion groups*. Homeomorphisms $(D^3, L_n) \to (D^3, L_n)$ do not necessarily preserve the orientation of L_n. If we consider the group $\mathrm{MCG}(D^3, \widehat{L_n})$ of isotopy classes of those homeomorphisms $(D^3, L_n) \to (D^3, L_n)$ that preserve also the orientation on L_n, the group obtained is the *loop braid group*, denoted by LB_n. The nomenclature "loop braid groups" is due to Lin [Lin08].

Let F_n be the free group of rank n generated by $\{x_1, \dots, x_n\}$. Then we have that $\pi_1(D^2 \setminus d_n, *)$ is isomorphic to F_n. Thus, in the case of the pair (D^2, d_n), Equation (1.1) becomes:

$$\theta \colon B_n \to \mathrm{Aut}(F_n).$$

Artin shows [Art47] that this is an injection.

We also have that $\pi_1(D^3 \setminus L_n, *) \cong F_n$, so the map $\theta \colon \mathrm{MCG}(D^3, L_n) \to \mathrm{Aut}(\pi_1(D^3 \setminus L_n, *))$ becomes:

$$\theta \colon LB_n^{\mathrm{ext}} \to \mathrm{Aut}(F_n).$$

This homomorphism is proven injective in [Dah62] and published in [Gol81, Theorem 5.2].

1.3. **Our main result.** The space $D^2 \setminus d_n$ is aspherical, and therefore its homotopy type can be recovered from the fundamental group. However $D^3 \setminus L_n$ is not aspherical, hence π_1 "forgets" more about $D^3 \setminus L_n$ than it does about $D^2 \setminus d_n$. In this paper, we work with $\pi_{(1,2)}(D^3 \setminus L_n, *)$, in the intent of retrieving some of the lost homotopical information when passing from $D^3 \setminus L_n$ to $\pi_1(D^3 \setminus L_n, *)$. How much homotopical information is retained is discussed in Subsection 4.3.

In particular, we will prove here that $\pi_{(1,2)}(D^3 \setminus L_n, *)$ is a π-module of *free type*[1]. In practice, with this we mean that $\pi_{(1,2)}(D^3 \setminus L_n, *) = \big(\pi_1(D^3 \setminus L_n, *), \pi_2(D^3 \setminus L_n, *), \triangleright_{\pi_1}\big)$, is isomorphic to $\mathcal{M}_n = (F_n, M_n, \triangleright)$, where M_n is the free $\mathbb{Z}[F_n]$-module generated by $\{K_1, \dots, K_n\}$. Given a π-module $\mathcal{G} = (G, A, \triangleright)$, a morphism $f = (f^1, f^2) \colon \mathcal{M}_n \to \mathcal{G}$ is therefore determined by the images $f^1(x_i)$ and $f^2(K_i)$; see Subsection 3.2.

In this paper we construct an inclusion $\Theta \colon LB_n^{\mathrm{ext}} \to \mathrm{Aut}(\mathcal{M}_n)$.

[1]The nomenclature "of free type" is borrowed from [BHS11, Definition 7.3.13].

1.4. **Structure of the paper.** In Section 2 we recall some notions that will be used throughout the paper. We also give a topological realisation of Artin's representation for the braid group, and a hands-on flavour of the lift we construct for the extended loop braid group. In Section 3 we present Dahm and Goldsmith's lift of Artin's representation for the extended loop braid group (Theorem 3.3). Then we introduce π-modules (Definition 3.5) and describe a lift of Artin's representation for extended loop braid groups in the π-module $\pi_{(1,2)}(D^3 \setminus L_n, *)$ of the 3-ball with a set of n (unlinked and unknotted) circles excised from its interior (Lemma 3.11 and Theorem 3.16). This is our first main result. In Section 4 we formalise the topological construction of the considered representation, in the second main result of this paper (Theorem 4.5).

2. On automorphisms of free groups and beyond

Here we first recall some constructions that are standard, but which will have useful lifts to higher dimensions later.

If $\mathcal{C}$ is a concrete category, then $\mathcal{U}\colon \mathcal{C} \to \mathcal{SET}$ is the forgetful functor. Let Grp be the category of groups, and, given a group G, define $\mathrm{Aut}(G) \subset \mathsf{Grp}(G,G)$ as the subset of invertible homomorphisms. The left adjoint $F^a\colon \mathcal{SET} \to \mathsf{Grp}$ takes a set to the free group on that set. We consider sets of form $x_{\underline{n}}$ defined to be $\{x_1, \ldots, x_n\}$, $n \in \mathbb{N}_0$, as a skeleton in the full subcategory $\mathcal{FINSET}$ of finite sets, and define

$$F_n = F^a(\{x_1, ..., x_n\}). \tag{2.1}$$

By the adjoint functor property ("freeness") elements of $\mathsf{Grp}(F_n, G)$ are uniquely specified by giving the image of each x_i. For example, for $\sigma \in Sym(n)$, the symmetric group on $\{1, \ldots, n\}$, define $t_\sigma \in \mathrm{Aut}(F_n)$ by:

$$t_\sigma : x_i \mapsto x_{\sigma(i)}. \tag{2.2}$$

A *non-automorphism* example in $\mathsf{Grp}(F_n, F_n)$ is given by $x_i \mapsto x_1$, for $i \in \{1, \ldots, n\}$. We use cycle notation for elements of $Sym(n)$, thus we have $t_{(12)}$ and so on; and define the *translation automorphism* by $t = t_{(1,2,\ldots,n)}$. We say an automorphism is *Coxeter geometric* (CG) if it acts trivially on all except at most two adjacent x_i, x_{i+1}, and on these it acts to produce elements of $F^a(\{x_i, x_{i+1}\})$. We say this action is local at $i, i+1$.

For example, the subset of automorphisms of form (2.2) is not CG in general. In particular the translation automorphism t is not CG. However the subset forms a subgroup and the subgroup is generated by the CG automorphism $t_{(12)}$ local at 1, 2 and $n-2$ translates thereof.

2.1. **On automorphisms of free groups realised topologically.** A topological realisation of F_n is given by $\pi_1(D^2 \setminus d_n, *)$. In order to explicitly write down an Artin representation $\theta\colon B_n \to \mathrm{Aut}(F_n)$, we specify d_n and a free basis for $\pi_1(D^2 \setminus d_n, *)$. We take the points $d_n = \{p_1, \ldots, p_n\}$ to be along a horizontal line. We consider paths $\widehat{x_i}$, $i = 1, \ldots, n$ passing clockwise around each p_i, as in Figure 1, such that their images intersect only at $* \in \partial D^2$. Let x_i be the homotopy class of $\widehat{x_i}$.

Note that $\pi_1(D^2 \setminus d_n, *)$ is free on $\{x_1, \ldots, x_n\}$. Also, $x_1 x_2 \ldots x_n$ is the homotopy class of a path that traces the boundary of D^2, clockwise, starting and finishing at $*$.

Note also that $(D^2 \setminus d_n, *)$ strongly deformation retracts into the pointed subspace made of the images of the paths, which is homeomorphic to $\bigvee_{i=1}^n (S^1, \star)$. Combining with Seifert – van Kampen theorem, this implies that $\pi_1(D^2 \setminus d_n, *)$ is freely generated by $\{x_1, \ldots, x_n\}$. The same type of argument will be used when we address the higher case in Section 4, and prove that $\pi_{(1,2)}(D^3 \setminus L_n, *)$ is of free type.

Using Equation (1.1), elements of $\mathrm{MCG}(D^2, d_n)$ induce elements of $\mathrm{Aut}(F_n)$. Consider for example the mapping class Σ_1 indicated by **a) - c)** of Figure 2. Note that this corresponds to

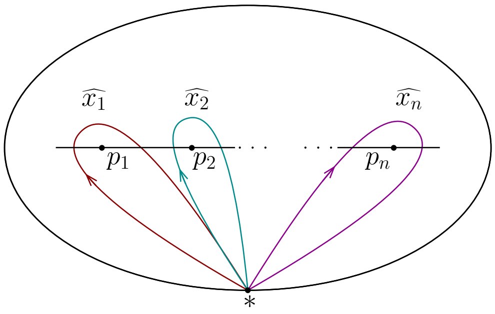

FIGURE 1. A free basis for $\pi_1(D^2 \setminus d_n, *)$.

the $1,2$ local automorphism $\mathcal{S}_1^1 \colon F_n \to F_n$ given by

$$x_1 \mapsto x_2, \qquad\qquad x_2 \mapsto x_2^{-1} x_1 x_2. \tag{2.3}$$

Define $\mathcal{S}_i^1 \colon F_n \to F_n$ as the translate of $\mathcal{S}_1^1$; an explicit formula is in Equation (3.2). From the topological realisation (or direct calculation) we have:

$$\mathcal{S}_i^1 \circ \mathcal{S}_{i+1}^1 \circ \mathcal{S}_i^1 = \mathcal{S}_{i+1}^1 \circ \mathcal{S}_i^1 \circ \mathcal{S}_{i+1}^1. \tag{2.4}$$

2.2. **Towards automorphisms in higher dimension.** Our task here is to describe automorphisms of a suitable lift of the free group to "higher dimension". Here dimension refers to

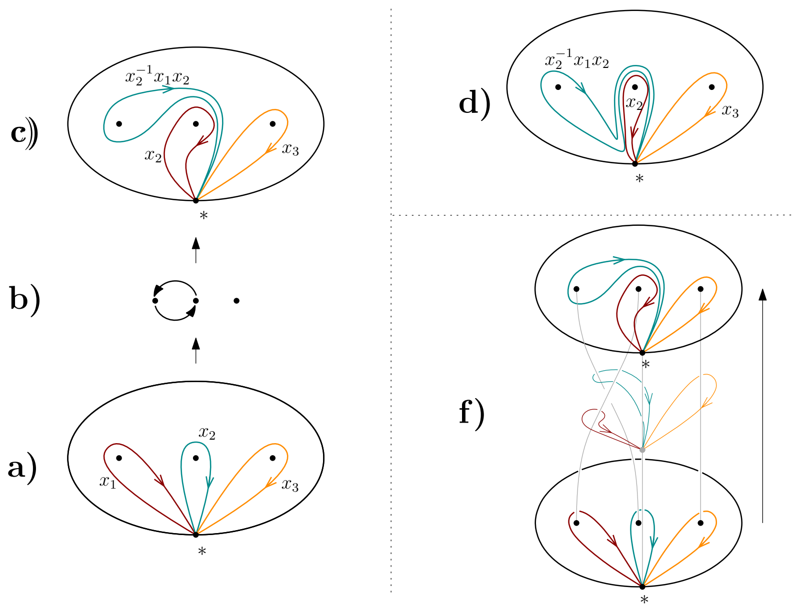

FIGURE 2. **a) - c)** An exchange of the marked points inducing a homeomorphism $(D^2, d_3) \to (D^2, d_3)$ and the corresponding automorphism of $\pi_1(D^2 \setminus d_3, *)$. **d)** The new paths are expressed in the original basis: thus x_1 has morphed to x_2 and x_2 has morphed to $x_2^{-1}x_1x_2$. **f)** The loops following the *timelines* of the braiding.

the topological interpretation of F_n as $\pi_1(D^2 \setminus d_n, *)$, whose building blocks are paths in the point-punctured disk $D^2 \setminus d_n$.

The lift involves the triple $\pi_{(1,2)}(D^3 \setminus L_n, *) = \big(\pi_1(D^3 \setminus L_n, *), \pi_2(D^3 \setminus L_n, *), \triangleright_{\pi_1}\big)$, where $\triangleright_{\pi_1}$ denotes the usual action of π_1 on π_2.

In the F_n case, freeness means that an automorphism is determined by the images of a set of free group generators. In the lift that we consider, the corresponding structure $\pi_{(1,2)}(D^3 \setminus L_n)$ is not a free group, but a π-module of free type. In particular, as we will prove in Subsection 3.2, $\pi_2(D^3 \setminus L_n, *)$ is a free $\mathbb{Z}$-module. In order to understand the basis we make some preparations.

Given a pointed topological space $(X, *)$, then the action of $\pi_1(X, *)$ on $\pi_2(X, *)$, means that $\pi_2(X, *)$, can naturally be equipped with the structure of $\mathbb{Z}[\pi_1(X, *)]$-module. We will give a visual idea of this structure for the case $X = D^3 \setminus L_n$.

We note that the *"balloons and hoops"* point of view we present in this subsection for the visual description of $\pi_{(1,2}(D^3 \setminus L_n, *)$ is essentially as in [BN15], where the term is coined. This balloons and hoops approach for understanding $\pi_{(1,2)}(D^3 \setminus L_n, *)$ was also used in [BFMM19, §4.5.1].

Note that there is not a canonical choice of the elements considered to be generating for example in $\pi_1(D^3 \setminus L_n, *)$. Varying the precise choice of L_n satisfying the "unlinked circles" characterisation affects what might be considered as a natural choice, and hence affects the construction — albeit only up to isomorphism. Later it will be convenient to work with L_n a row of circles confined to a plane in D^3. But in this section we will instead use for L_n a stack of circles confined to a single axis of rotational symmetry.

Note that by rotating $[0,1]^2 \times \{0\}$ about $\{0\} \times [0,1] \times \{0\}$ we obtain a topological D^3, and this rotation causes the points $p_1, \ldots, p_n$ to sweep out an $L_n = C_1 \cup \cdots \cup C_n$, one where the circles are "stacked" coaxially: see Figure 3, subfigures **a)**, **b)**. Note that we also assume that the circles $C_1, \ldots, C_n$ are stacked on top of each others, in decreasing height.

The group $\pi_1(D^3 \setminus L_n, *) \cong F_n$ is freely generated by (classes of) loops x_i around the circles as illustrated in Figure 3**(c)**. These classes have representatives that lie in the original copy of D^2. The image of the element called x_i in $\pi_1(D^2 \setminus d_n, *)$, as in Figure 3**(a)**, we again call x_i.

Now let g be a homeomorphism representing a certain class in $\mathrm{MCG}(D^2, d_n)$. This g restricts to $(0,1] \times [0,1]$: the omitted edge is the rotation axis, so rotating gives a self-homomorphism of $(0,1] \times [0,1] \times S^1$. Topologically, $(0,1] \times [0,1] \times S^1$ is D^3 with the axis removed. The latter homeomorphism thus extends to a self-homomorphism $r(g)$ of D^3 by inserting the constant function on the axis. This construction lifts to a well-defined group homomorphism

$$r\colon\ \mathrm{MCG}(D^2, d_n) \to \mathrm{MCG}(D^3, L_n).$$

In particular, consider the mapping class Σ_i in $\mathrm{MCG}(D^2, d_n)$ exchanging consecutive points p_i and p_{i+1}, as in Figure 2, which considers the case $n = 3$ and $i = 1$. Its image under r, which we also denote Σ_i, corresponds to an exchange of circles, as illustrated in Figure 4, for $n = 2$ and $i = 1$. To the mapping class $\Sigma_i \in \mathrm{MCG}(D^3, L_n)$ we will call *elementary braid permutation.*

One new aspect in the higher setting is that there are elements of $\mathrm{MCG}(D^3, L_n)$ that do not have representatives with the rotational symmetry, such as maps that exchange the circles by taking them off the rotation axis. Note that these induce elements of the automorphism group $\mathrm{Aut}(\pi_1(D^3 \setminus L_n))$ of type $t_{(i,i+1)}$. Also breaking the axial symmetry in this sense, are motions that flip a single circle C_i onto itself. One can see that this induces an automorphism local at i, given by $\mathcal{T}_i^1\colon x_i \mapsto x_i^{-1}$.

Another new aspect in the higher setting is that $\pi_2(D^3 \setminus L_n, *)$ is not trivial. An example of a non-trivial element of $\pi_2(D^3 \setminus L_n, *_i)$ is the homotopy class of a *wrapping square* $K_i'\colon (D^2, \partial D^2) \to (D^3 \setminus L_n, *_i)$, based at $*_i$ (a point in the rotating axis) that wraps a single circle C_i, including the disk of which C_i is boundary, exactly once, see Subfigures **d)** and **e)** in Figure 3 for the $i, n = 2$ case. Concretely K_i' is a positively oriented parametrisation of a *balloon* – i.e. a 2-sphere

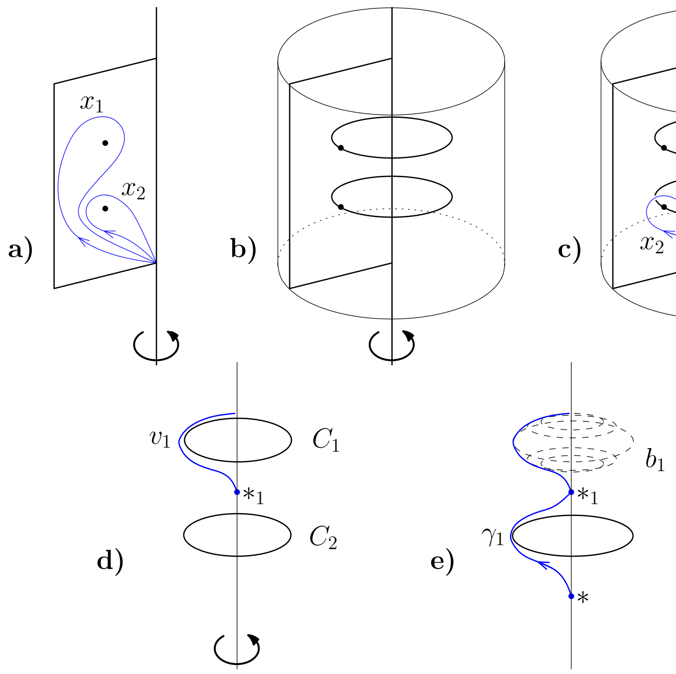

FIGURE 3. **a), b), c)** Rotating the two-punctured disk into $\mathbb{R}^3$. **d), e)** constructing a sphere (a "balloon") b_1, enclosing C_1, by rotating the path v_1. The element $K_1 \in \pi_2(D^3 \setminus L_n, *)$ is obtained from the homotopy class $K_1' \in \pi_2(D^3 \setminus L_n, *_1)$ of a positively oriented parametrisation $(D^2, \partial D^2) \to (b_1, *_1)$, of b_1, made into an element $K_1 \in \pi_2(D^3 \setminus L_n, *)$ by using the isomorphism $\pi_2(D^3 \setminus L_n, *_1) \to \pi_2(D^3 \setminus L_n, *)$ derived from the path γ_1.

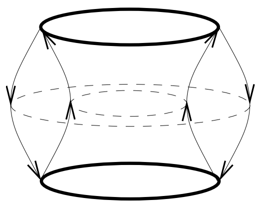

FIGURE 4. An "exchange" of coaxial circles, where the circle C_1 is originally on top. This is a smooth family of embeddings of $S^1 \sqcup S^1$ into D^3. It induces a diffeotopy $t \in [0,1] \mapsto \phi_t \in \text{Homeo}(D^3)$ of D^3, relative to the boundary, by the isotopy extension theorem (either before or after applying r). The end-value $\phi_1 \in \text{Homeo}(D^3)$ of the diffeotopy is a homeomorphism $(D^3, L_2) \to (D^3, L_2)$. The elementary braid permutation $\Sigma_1 \in \text{MCG}(D^3, L_2)$ is the mapping class of ϕ_1.

b_i containing C_i, and no other circle C_j, oriented by an exterior normal. If we connect $*_i$ to $*$ by a path γ_i that does not cross the disks spanned by the circles in L_n, and consider the usual isomorphism $\pi_2(D^3 \setminus L_n, *_i) \to \pi_2(D^3 \setminus L_n, *)$ derived from γ_i (see e.g. [Hat02, Page 343]), this gives a non-trivial element $K_i = [\gamma \triangleright K_i'] \in \pi_2(D^3 \setminus L_n, *)$.

We argue in Section 4, that $\pi_2(D^3 \setminus L_n, *)$ is freely generated, as an abelian group, by the elements $p \triangleright K_i$, where $i \in \{1, \ldots, n\}$ and $p \in \pi_1(D^3 \setminus L_n, *)$. Each $p \triangleright K_i$ can be visualised as a hoop p, connecting $*$ to $*$, which is then attached to K_i. In Figure 5 we show some examples of elements of $\pi_2(D^3 \setminus L_2, *)$.

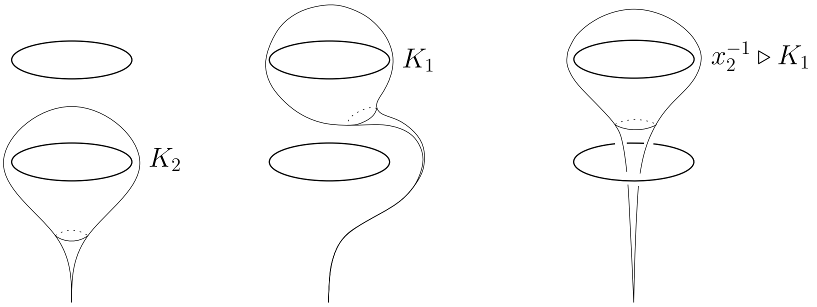

FIGURE 5. Elements in $\pi_2(D^3 \setminus L_2, *)$. We number the top circle 1. Note that the difference between the second and third figures is given by the path x_2^{-1} around circle C_2.

A crucial property of the classes $K_1, \ldots, K_n \in \pi_2(D^3 \setminus L_n, *)$ is that $K_1 + K_2 + \cdots + K_n$ can be represented by a parametrisation $(D^2, \partial D^2) \to (\partial D^3, *)$. With these points in mind, we lift consideration of the mapping class group action on the fundamental group to an action of the mapping class group $\mathrm{MCG}(D^3, L_n)$ on the triple:

$$\pi_{(1,2)}(D^3 \setminus L_n, *) = (\pi_1(D^3 \setminus L_n, *), \pi_2(D^3 \setminus L_n, *), \triangleright_\pi),$$

where $\triangleright_\pi$ denotes the usual action of $\pi_1(D^3 \setminus L_n, *)$ on $\pi_2(D^3 \setminus L_n, *)$. Since $\pi_2(D^3 \setminus L_n, *)$ is abelian, $\pi_{(1,2)}(D^3 \setminus L_n, *)$ is a π-module, a structure we will define in Section 3.2 .

3. The lifted Artin representation

We discussed in Subsection 1.2 that the homotopical information coming from $\pi_2(D^3 \setminus L_n, *)$ is not taken into account by Dahm's lift of Artin's representation. With this in mind, we proceed to define a *lifted Artin representation* for extended loop braid groups LB_n^{ext}. The codomain for our representation stores two levels of information, coming from the first and second homotopy groups of $D^3 \setminus L_n$, as well as the action of the first on the former. It will thus be a *π-module.* In this section, we recap some results about extended loop braid groups and π-modules, which will subsequently be used to define the lifted Artin representation.

3.1. **Loop braid groups and extended loop braid groups.** We recall [Art47] that the braid group $B_n = \mathrm{MCG}(D^2, d_n)$ is isomorphic to the group defined by generators $\{\sigma_1, \ldots, \sigma_{n-1}\}$, subject to relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{for } |i-j| > 1; \tag{3.1a}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \text{for } i = 1, \ldots, n-2. \tag{3.1b}$$

(Here and in subsequent presentations of groups, note that our convention is that multiplication is to be read from left to right.) The generator σ_i is the equivalence class of the mapping class Σ_i in $\mathrm{MCG}(D^2, d_n)$ exchanging the ith and $i+1$th point as in Figure 2.

Passing to the automorphisms of the free group F_n of rank n, let $\mathcal{S}_i^1\colon F_n \to F_n$ be defined on the generators $\{x_1, \ldots, x_n\}$ as:

$$\mathcal{S}_i^1\colon \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1}\, x_i\, x_{i+1}, \\ x_j \mapsto x_j, \text{ if } j < i \text{ or } j > i+1 \end{cases} . \tag{3.2}$$

Note from Figure 2 that Artin's representation $\theta\colon B_n \to \mathrm{Aut}(F_n)$ is such that, for $i = 1, \ldots, n-1$, $\Sigma_i \mapsto \mathcal{S}_i^1$, and this data determines θ.

Theorem 3.1 ([Art47], for a more recent proof see for instance [Han89, Theorem 5.1])**.** *The map $\theta\colon B_n \to \mathrm{Aut}(F_n)$ is injective, and an automorphism ϕ of F_n is in $\theta(B_n)$ if and only if the following two conditions are satisfied:*

(1) There exist $a_1, \ldots, a_n \in F_n$, and a permutation α of $\{1, \ldots, n\}$, such that $\phi(x_i) = a_i\, x_{\alpha(i)}\, a_i^{-1}$.
(2) $\phi(x_1\, x_2 \ldots x_n) = x_1\, x_2 \ldots x_n$.

Note that (2) holds since $x_1 x_2 \ldots x_n$ is the homotopy class of a path that traces the boundary of D^2, clockwise, starting and finishing at $*$. Hence $x_1 x_2 \ldots x_n$ is left untouched by all elements of $\mathrm{MCG}(D^2, d_n)$, since they are required to be the identity on ∂D^2.

Let us move one dimension up, in the realm of extended loop braid groups LB_n^{ext}, defined as the mapping class groups $\mathrm{MCG}(D^3, \widehat{L_n})$, and loop braid groups LB_n, defined as $\mathrm{MCG}(D^3, L_n)$; see Section 1.2. In this case we have the following result [BH13, BWC07, Dam17]:

Theorem 3.2. *The group LB_n is isomorphic to the abstract presented group defined by generators $\{\sigma_i, \rho_i \mid i = 1, \ldots, n-1\}$ subject to relations:*

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad \text{for } |i-j| > 1; \tag{3.3a}$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad \text{for } i = 1, \ldots, n-2; \tag{3.3b}$$
$$\rho_i\rho_j = \rho_j\rho_i, \qquad \text{for } |i-j| > 1; \tag{3.3c}$$
$$\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}, \qquad \text{for } i = 1, \ldots, n-2; \tag{3.3d}$$
$$\rho_i^2 = \mathrm{id}, \qquad \text{for } i = 1, \ldots, n-1; \tag{3.3e}$$
$$\rho_i\sigma_j = \sigma_j\rho_i, \qquad \text{for } |i-j| > 1; \tag{3.3f}$$
$$\rho_{i+1}\rho_i\sigma_{i+1} = \sigma_i\rho_{i+1}\rho_i \qquad \text{for } i = 1, \ldots, n-2; \tag{3.3g}$$
$$\sigma_{i+1}\sigma_i\rho_{i+1} = \rho_i\sigma_{i+1}\sigma_i, \qquad \text{for } i = 1, \ldots, n-2. \tag{3.3h}$$

Moreover the group LB_n^{ext} is isomorphic to the abstract presented group defined by generators $\{\sigma_i, \rho_i \mid i = 1, \ldots, n-1\} \cup \{\tau_j \mid j = 1, \ldots, n\}$ subject to relations (3.3a) *to* (3.3h) *above, together with:*

$$\tau_i\tau_j = \tau_j\tau_i, \qquad \text{for } i \neq j; \tag{3.4a}$$
$$\tau_i^2 = \mathrm{id}, \qquad \text{for } i = 1, \ldots, n; \tag{3.4b}$$
$$\sigma_i\tau_j = \tau_j\sigma_i, \qquad \text{for } j \neq i, i+1; \tag{3.4c}$$
$$\rho_i\tau_j = \tau_j\rho_i, \qquad \text{for } j \neq i, i+1; \tag{3.4d}$$
$$\tau_i\rho_i = \rho_i\tau_{i+1}, \qquad \text{for } i = 1, \ldots, n-1; \tag{3.4e}$$
$$\tau_i\sigma_i = \sigma_i\tau_{i+1}, \qquad \text{for } i = 1, \ldots, n-1; \tag{3.4f}$$
$$\tau_{i+1}\sigma_i = \rho_i\sigma_i^{-1}\rho_i\tau_i, \qquad \text{for } i = 1, \ldots, n-1. \tag{3.4g}$$

Let Σ_i denote the mapping class corresponding to the elementary braiding permutation in $\mathrm{MCG}(D^3, L_n)$, see Figure 6. In the coaxial configuration of L_n, $n = 2$, we may represent this as in Figure 4. Similarly, let ρ_i denote the mapping class of the *non-braiding permutation*; and τ_i be the mapping class corresponding to a 180 degrees flip of C_i with respect to the vertical axis, see Figure 6. The isomorphism between LB_n^{ext} and the abstract presented group sends $\Sigma_i \mapsto \sigma_i$, $\rho_i \mapsto \rho_i$, and $\tau_i \mapsto \tau_i$.

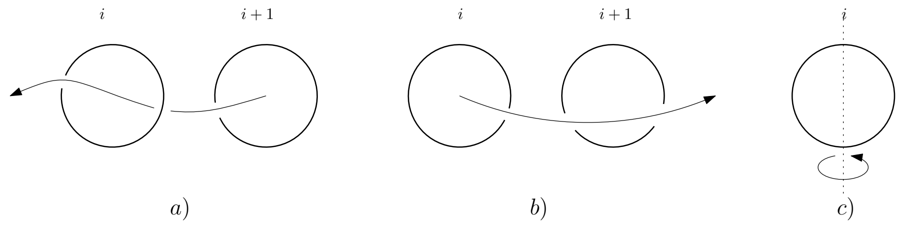

FIGURE 6. *a*) Pictorial idea of the mapping class Σ_i. *b*) The mapping class ρ_i. *c*) The mapping class τ_i.

Dahm [Dah62] generalised Artin's representation to general manifolds with a compact submanifold in its interior. Goldsmith published his result for the case of the pair $(\mathbb{R}^3, L_n)$, which gives the same group as the pair (D^3, L_n). Let us recall it:

Theorem 3.3 ([Gol81, Theorem 5.3]). *For $n \geq 1$, the map*

$$\theta \colon LB_n^{\mathrm{ext}} \longrightarrow \mathrm{Aut}(F_n).$$

is injective. Its image is the subgroup of $\mathrm{Aut}(F_n)$ *consisting of automorphisms of the form $x_i \mapsto a_i^{-1} x_{\alpha(i)}^{\pm 1} a_i$ where α is a permutation of $\{1, \ldots, n\}$ and $a_i \in F_n$. Moreover, this subgroup of* $\mathrm{Aut}(F_n)$ *is generated by the automorphisms $\{\mathcal{S}_i^1 \mid i = 1, \ldots, n-1\}$, $\{\mathcal{R}_i^1 \mid i = 1, \ldots, n-1\}$ and $\{\mathcal{T}_i^1 \mid i = 1, \ldots, n\}$, where $\mathcal{S}_i^1$ is as in Equation* (3.2), *and:*

$$\mathcal{R}_i^1 : \begin{cases} x_i \mapsto x_{i+1}; \\ x_{i+1} \mapsto x_i; \\ x_j \mapsto x_j, & \text{for } j \neq i, i+1. \end{cases} \tag{3.5}$$

$$\mathcal{T}_i^1 : \begin{cases} x_i \mapsto x_i^{-1}; \\ x_j \mapsto x_j, & \text{for } j \neq i. \end{cases} \tag{3.6}$$

This case of the map θ is known in the literature as *Dahm's homomorphism.* The generators of LB_n^{ext} of type σ_i, ρ_i and τ_i are respectively sent by Dahm's homomorphisms to automorphisms $\mathcal{S}_i^1$, $\mathcal{R}_i^1$ and $\mathcal{T}_i^1$.

Remark 3.4. A consequence of Theorem 3.3 is that LB_n injects into $\mathrm{Aut}(F_n)$ and its image is isomorphic to the group of automorphisms of the form $x_i \mapsto a_i^{-1} x_{\alpha(i)} a_i$, where α is a permutation and $a_i \in F_n$. This group is called *the group of basis conjugating automorphisms* of $\mathrm{Aut}(F_n)$, and is generated by the automorphisms $\{\mathcal{S}_i^1 \mid i = 1, \ldots, n-1\}$ and $\{\mathcal{R}_i^1 \mid i = 1, \ldots, n-1\}$.

From now on we will focus on the extended loop braid group, keeping in mind that through Remark 3.4 consequences of this work can be drawn on loop braid groups.

Dahm's homomorphism can be constructed with the same classical construction of Artin's representation in terms of mapping classes. Briefly, choose a base-point $*$ in the boundary of D^3.

Isotopies of maps $f\colon (D^3, L_n) \to (D^3, L_n)$ do not move the base-point. If $[f] \in LB_n^{\text{ext}}$ then $[f]$ yields a pointed-homotopy class $\Theta([f])\colon (D^3 \setminus L_n, *) \to (D^3 \setminus L_n, *)$. Let

$$\theta'([f])\colon \pi_1(D^3 \setminus L_n, *) \longrightarrow \pi_1(D^3 \setminus L_n, *) \tag{3.7}$$

be the induced map on fundamental groups. Algebraic topological considerations give that $\theta([f])$ coincides with $\theta'([f])$.

3.2. **The category of π-modules.**

Definition 3.5. A π-module is given by a triple $\mathcal{G} = (G, A, \triangleright)$, where G is a group, A is an abelian group, and $\triangleright$ is a left-action by automorphisms of G on A. Given two π-modules $\mathcal{G} = (G, A, \triangleright)$ and $\mathcal{G}' = (G', A', \triangleright')$, a morphism $f = (f^1, f^2)\colon \mathcal{G} \to \mathcal{G}'$ is a pair $(f^1\colon G \to G', f^2\colon A \to A')$ of homomorphisms that preserve actions: for each $g \in G$ and $a \in A$, we have $f^2(g \triangleright a) = f^1(g) \triangleright' f^2(a)$.

Our convention for the composition of π-module morphisms is $(f^1, f^2)(h^1, h^2) = (f^1 \circ h^1, f^2 \circ h^2)$. It is easy to see that the class of π-modules and their morphisms form a category. We write Agg for the category of π-modules, since π-modules are (abelian group)–group pairs. The category of π-modules was also used in [Bau91, Chapter I, §1 (1.7)]. Confer also for example [M.A74, §8], and [Sie93] where they are termed "second homotopy modules".

Consider the forgetful functor $\mathcal{U}\colon \mathsf{Agg} \to \mathcal{SET} \times \mathcal{SET}$ that takes $\mathcal{G} = (G, A, \triangleright)$ to the pair (G, A), made from the underlying sets of the groups G and A. Recall $F_n = F^a(\{1, 2, ..., n\})$ from (2.1): the left adjoint $\mathcal{F}^a$ takes $(\{1, 2, ..., n\}, \{1, 2, ..., n\})$ to a π-module that we construct next.

Lemma 3.6. *A morphism $f = (f^1, f^2)\colon \mathcal{G} \to \mathcal{G}'$ of π-modules is invertible if and only if both f^1 and f^2 are invertible homomorphisms.*

Proof. It follows from the definition of a morphism of π-modules. □

We denote the group of invertible morphisms $\mathcal{G} \to \mathcal{G}$ by $\mathrm{Aut}(\mathcal{G})$.

Definition 3.7. Let $\mathbb{Z}[G]$ denote the group ring of group G. Let M_n^G be the free $\mathbb{Z}[G]$-module on the symbols $\{K_1, \dots, K_n\}$, hence

$$M_n^G = \mathbb{Z}[G]\{K_1, \dots, K_n\} \cong \bigoplus_{i=1}^n \mathbb{Z}[G]K_i \cong \bigoplus_{g \in G} \bigoplus_{i=1}^n \mathbb{Z}(g, K_i)$$

equipped with the diagonal action of $\mathbb{Z}[G]$. We define $\mathcal{M}_n^G$ to be the π-module $(G, M_n^G, \triangleright)$. The action of G on M_n^G is induced by the $\mathbb{Z}[G]$-module structure. Define

$$M_n = M_n^{F_n}, \qquad \mathcal{M}_n^{(m)} = \mathcal{M}_n^{F_m}, \qquad \text{and therefore} \qquad \mathcal{M}_n = \mathcal{M}_n^{(n)}.$$

The following proposition describes how morphisms $\mathcal{M}_n \to \mathcal{G}$ can be uniquely specified by their value on the generators x_i and K_i, for $i = 1, \dots, n$. This result is also used, implicitly, in [BFMM19, §4.5.1].

Proposition 3.8. *Let $\mathcal{G} = (G, A, \blacktriangleright)$ be a π-module. There is a canonical one-to-one correspondence between morphisms $f = (f^1, f^2)\colon \mathcal{M}_n^{(m)} \to \mathcal{G}$ and pairs of tuples $(g_1, \dots, g_m) \in G^m$ and $(a_1, \dots, a_n) \in A^n$. The correspondence is such that: $g_i = f^1(x_i)$ and $a_i = f^2(K_i)$.*

Proof. Let us consider a pair of tuples $(g_1, \dots, g_m) \in G^m$ and $(a_1, \dots, a_n) \in A^n$. We associate to this pair a morphism $f = (f^1, f^2)$ as follows. We take f^1 to be the unique homomorphism $F_m \to G$ such that $f^1(x_i) = g_i$. As an abelian group, M_n^m is free on the set of pairs (g, K_i), where $g \in F_m$ and $i = 1, \dots, n$. We define $f^2\colon M_n^m \to A$ to be the unique homomorphism such that $f^2(g, K_i) = f^1(g) \blacktriangleright a_i$. Compatibility with group actions is preserved by construction. □

Remark 3.9. Because of Proposition 3.8, we say that $\mathcal{M}_n^{(m)}$, and in particular $\mathcal{M}_n$, is a π-module of *free type*. Indeed consider the forgetful functor $\mathcal{U}\colon \mathsf{Agg} \to \mathcal{SET} \times \mathcal{SET}$ that takes $\mathcal{G} = (G, A, \triangleright)$ to (G, A). This functor has a left adjoint $\mathcal{F}^a$, where given a pair (X, Y) of sets, $\mathcal{F}^a(X, Y) = (F(X), \mathbb{Z}[F(X)]Y, \triangleright)$, where $F(X)$ is the free group on X, and $\mathbb{Z}[F(X)]Y$ is the free $\mathbb{Z}[F(X)]$-module on Y, with the obvious action of $F(X)$. Hence:

$$\mathcal{M}_n^{(m)} \cong \mathcal{F}^a\big(\{1, \ldots, m\}, \{1, \ldots, n\}\big).$$

Remark 3.10. There exists a 'forgetful' functor $\mathsf{Agg} \to \mathsf{Grp}$, sending each π-module $\mathcal{G} = (G, A, \triangleright)$ to the semidirect product $G \ltimes_\triangleright A$. This functor is not full: given π-modules $\mathcal{G} = (G, A, \triangleright)$ and $\mathcal{G}' = (G', A', \triangleright)$, in general there exist group homomorphisms $G \ltimes_\triangleright A \to G' \ltimes_\triangleright A'$ that do not arise from π-module morphisms $\mathcal{G} \to \mathcal{G}'$. Moreover, this forgetful functor does not send π-modules of free type to free groups. Such freeness properties of π-modules, lost when we pass to semidirect products, are essential for our main theorem later on (see Theorem 3.16), when we construct a lifted Artin representation by giving its value on free generators.

As we will see in Section 4.3, maps between fundamental π-modules model pointed homotopy classes of maps between bouquets of 1- and 2-spheres. This connection to homotopy theory is lost once we pass to the underlying semidirect product of the fundamental π-modules.

We observed in Section 2.2 that $\pi_1(D^2 \setminus d_n, *)$ is a free group $F^a(\{x_1, ..., x_n\})$. As we will establish in Lemma 4.2 *et seq.*, we have $\pi_{(1,2)}(D^3 \setminus L_2, *) \cong \mathcal{F}^a\big(\{1, 2\}, \{1, 2\}\big)$. Then the automorphism of the π-module induced by a homeomorphism of (D^3, L_2), relative to the boundary, is prescribed by giving:

(i) the effect on the x_i generators of $\pi_1(D^3 \setminus L_2, *)$;
(ii) the effect on the K_i generators of $\pi_2(D^3 \setminus L_2, *)$ as a $\mathbb{Z}[\pi_1]$-module.

In particular we have the following.

Lemma 3.11. *Recall the elementary braid permutation $\Sigma_1 \in \mathrm{MCG}(D^3, L_2)$ illustrated in Figure 4 for $n = 2$. The image $\mathbf{\Sigma}$ of $\Sigma_1 \in \mathrm{MCG}(D^3, L_2)$ in $\mathrm{Aut}(\pi_{(1,2)}(D^3 \setminus L_2, *))$ is given by:*

$$\mathbf{\Sigma}(x_2) = x_2^{-1} x_1 x_2, \quad \mathbf{\Sigma}(x_1) = x_2,$$

$$\mathbf{\Sigma}(K_2) = x_2^{-1} \triangleright K_1 \tag{3.8}$$

The image of K_1 is determined by these since $\mathbf{\Sigma}$ fixes $K_1 + K_2$.

Remark 3.12. This result, and proof, is also used in [BFMM19, §4.5.1], where a dual form of the lifted Artin representation was treated, in the context of biracks derived from π-modules.

Proof. (Sketch) A proof can be deduced from the observation of Figures 5 and 6. It only remains to verify the last claim. Consider the representatives of $K_1 + K_2$ sketched in Figure 7. The last one is evidently not moved by Σ_1. We will go back to these ideas later in Section 4. □

Let us now treat this construction from a more algebraic viewpoint.

Definition 3.13. Define $\mathcal{M}$ as the monoidal category with object class $\mathbb{N}_0$ and $\mathcal{M}(n, n) = \mathsf{Agg}(\mathcal{M}_n, \mathcal{M}_n)$ with category composition as in Agg, and $\mathcal{M}(n, m)$ empty otherwise; and monoidal composition given by $a \otimes b = a + b$ on objects, and on morphisms as induced by the map $\{1, \ldots, n\} \sqcup \{1, \ldots m\} \mapsto \{1, \ldots, n, 1 + n, \ldots, m + n\}$ applied "diagonally" to the indices on pairs x_i, K_i. Note that the permutation (12) on $\{1, 2\}$ extends to make this a symmetric monoidal category.

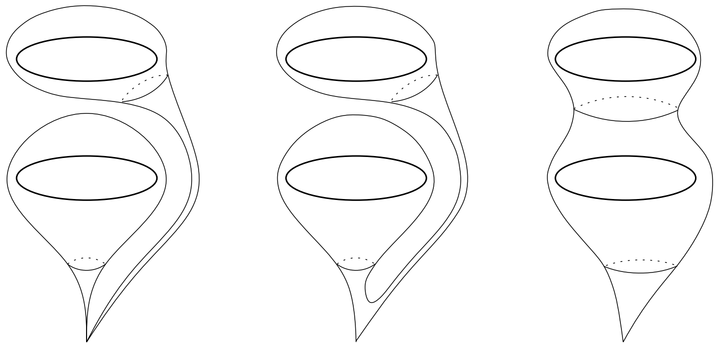

FIGURE 7. Representatives of the element $K_1 + K_2$.

Definition 3.14. Recall Proposition 3.8. Define $\mathcal{S}, \mathcal{R} \in \mathsf{Agg}(\mathcal{M}_2, \mathcal{M}_2)$ and $\mathcal{T} \in \mathsf{Agg}(\mathcal{M}_1, \mathcal{M}_1)$ to be the uniquely defined morphisms of π-modules such that:

$$\mathcal{S}^1(x_j) = \begin{cases} x_2, \text{ if } j = 1, \\ x_2^{-1} x_1 x_2, \text{ if } j = 2, \end{cases} \tag{3.9a}$$

$$\mathcal{S}^2(K_j) = \begin{cases} K_1 + K_2 - x_2^{-1} \triangleright K_1, \text{ if } j = 1, \\ x_2^{-1} \triangleright K_1, \text{ if } j = 2, \end{cases} \tag{3.9b}$$

$$\mathcal{R}^1(x_j) = \begin{cases} x_2, \text{ if } j = 1, \\ x_1, \text{ if } j = 2, \end{cases} \tag{3.10a}$$

$$\mathcal{R}^2(K_j) = \begin{cases} K_2, \text{ if } j = 1, \\ K_1, \text{ if } j = 2, \end{cases} \tag{3.10b}$$

$$\mathcal{T}^1(x_1) = x_1^{-1}, \qquad \mathcal{T}^2(K_j) = K_j. \tag{3.11a}$$

Define $\mathcal{S}_i, \mathcal{T}_i \in \mathsf{Agg}(\mathcal{M}_n, \mathcal{M}_n)$, for each $i = 1, \dots, n-1$, and $\mathcal{R}_i \in \mathsf{Agg}(\mathcal{M}_n, \mathcal{M}_n)$, for each $i = 1, \dots, n$, as:

$$\mathcal{S}_i = 1^{i-1} \otimes \mathcal{S} \otimes 1^{n-i-1}, \qquad \mathcal{R}_i = 1^{i-1} \otimes \mathcal{R} \otimes 1^{n-i-1}, \qquad \mathcal{T}_i = 1^{i-1} \otimes \mathcal{T} \otimes 1^{n-i}.$$

Explicitly:

$$\mathcal{S}_i^1(x_j) = \begin{cases} x_j, \text{ if } j < i \text{ or } j > i+1, \\ x_{i+1}, \text{ if } j = i, \\ x_{i+1}^{-1} x_i x_{i+1}, \text{ if } j = i+1, \end{cases} \tag{3.12a}$$

$$\mathcal{S}_i^2(K_j) = \begin{cases} K_j, \text{ if } j < i \text{ or } j > i+1, \\ K_i + K_{i+1} - x_{i+1}^{-1} \triangleright K_i, \text{ if } j = i, \\ x_{i+1}^{-1} \triangleright K_i, \text{ if } j = i+1; \end{cases} \tag{3.12b}$$

$$\mathcal{R}_i^1(x_j) = \begin{cases} x_j, \text{ if } j < i \text{ or } j > i+1, \\ x_{i+1}, \text{ if } j = i, \\ x_i, \text{ if } j = i+1, \end{cases} \tag{3.13a}$$

(3.13b)
$$\mathcal{R}_i^2(K_j) = \begin{cases} K_j, & \text{if } j < i \text{ or } j > i+1, \\ K_{i+1}, & \text{if } j = i, \\ K_i, & \text{if } j = i+1; \end{cases}$$

(3.14a)
$$\mathcal{T}_i^1(x_j) = \begin{cases} x_j, & \text{if } j \neq i, \\ x_i^{-1}, & \text{if } j = i, \end{cases} \qquad \text{and} \qquad \mathcal{T}_i^2(K_j) = K_j.$$

We remark that $\mathcal{T}_i^2 : M_n \to M_n$ is not the identity. For example $\mathcal{T}_i^2(x_i \triangleright K_j) = x_i^{-1} \triangleright K_j$, for each $j \in \{1, \dots, n\}$.

Lemma 3.15. *These homomorphisms of π-modules are invertible.*

Proof. The claim is true for $\mathcal{R}_i$ and the $\mathcal{T}_j$, because $\mathcal{R}_i^2 = \text{id}$ and $\mathcal{T}_j^2 = \text{id}$. Explicit calculation shows that the inverse of $\mathcal{S}_i$ is given by $\overline{\mathcal{S}_i} = (\overline{\mathcal{S}_i}^1, \overline{\mathcal{S}_i}^2)$ below:

(3.15a)
$$\overline{\mathcal{S}}_i^1(x_j) = \begin{cases} x_i x_{i+1} x_i^{-1}, & \text{if } j = i, \\ x_i, & \text{if } j = i+1, \\ x_j, & \text{if } j \neq i, i+1, \end{cases}$$

(3.15b)
$$\overline{\mathcal{S}}_i^2(K_j) = \begin{cases} x_i \triangleright K_{i+1}, & \text{if } j = i, \\ K_i + K_{i+1} - x_i \triangleright K_{i+1}, & \text{if } j = i+1, \\ K_j, & \text{if } j \neq i, i+1. \end{cases}$$

For the geometrical idea behind these calculations, see Lemma 3.11. □

The following is the first main result of this paper.

Theorem 3.16. *There exists a unique group homomorphism $\Theta\colon LB_n^{\text{ext}} \to \text{Aut}(\mathcal{M}_n)$, such that:*

$$\Sigma_i \mapsto \mathcal{S}_i, \quad \rho_i \mapsto \mathcal{R}_i, \quad \tau_j \mapsto \mathcal{T}_j,$$

where $i = 1, \dots, n-1$, $j = 1, \dots, n$, and Σ_i, ρ_i and τ_j are the generators of LB_n^{ext} as a mapping class group, as in Figure 6 . Furthermore, $\Theta\colon LB_n^{\text{ext}} \to \text{Aut}(\mathcal{M}_n)$ is injective.

To $\Theta\colon LB_n^{\text{ext}} \to \text{Aut}(\mathcal{M}_n)$, we call the *lifted Artin representation.*

Proof. These images are sufficient to determine a homomorphism by Theorem 3.2, and the comments just after. In Appendix A we check relations by explicit calculations.

Let $\mathcal{G} = (G, A, \triangleright)$ be a π-module. Then we have a morphism $F^1\colon \text{Aut}(\mathcal{G}) \to \text{Aut}(G)$, such that $F^1(f^1, f^2) = f^1$. The injectivity of Θ follows from the fact that $F^1 \circ \Theta$ coincides with Dahm's homomorphism $LB_n^{\text{ext}} \to \text{Aut}(F_n)$, see Theorem 3.3. □

A feature of Artin's representation, is that it provides a characterisation of *braid automorphisms* of F_n, as recalled in Theorem 3.1. Also in the case of extended loop braid groups, Goldsmith gives a characterisation elements of $\text{Aut}(F_n)$ that are images of elements of LB_n^{ext}, Theorem 3.3. It is thus natural to ask the following question.

Open Problem 3.17. Determine the image of $\Theta\colon LB_n^{\text{ext}} \to \text{Aut}(\mathcal{M}_n)$.

Remark 3.18. Note that for any $(f^1, f^2) \in \Theta(LB_n^{\text{ext}})$ it holds that:

(3.16)
$$f^2(K_1 + K_2 + \cdots + K_n) = K_1 + K_2 + \cdots + K_n.$$

This is an analogue for condition (2) in Theorem 3.1. As we will see in Remark 4.7, Equation (3.16) holds since $K_1 + K_2 + \cdots + K_n$ can be represented by a parametrisation $(D^2, \partial D^2) \to (\partial D^3, *)$, and hence is left untouched by all elements of $\text{MCG}(D^3, L_n)$, given that these mapping classes are the identity in ∂D^2.

Remark 3.19. A dual version of the lifted Artin representation, which considers finite dimensional representations of the loop braid group defined from biracks derived from π-modules, appeared in [BFMM19]. The underlying invariants of welded knots were initially defined in [KF08].

4. A topological interpretation of the lifted Artin representation

Let X be a path-connected space. A path $\gamma\colon [0,1] \to X$ canonically induces isomorphisms $\pi_2(X,\gamma(1)) \to \pi_2(X,\gamma(0))$, and also $\pi_1(X,\gamma(1)) \to \pi_1(X,\gamma(0))$; see e.g. [Hat02, Page 341]. Let $* \in X$ be a base-point. If we consider closed paths, starting and ending in $*$, this descends to an action of $\pi_1(X,*)$ on $\pi_2(X,*)$, by automorphisms; see e.g. [Hat02, page 342]. We denote by $\pi_{(1,2)}(X,*)$ the π-module obtained from $\pi_1(X,*)$ acting on $\pi_2(X,*)$ in this way. We call $\pi_{(1,2)}(X,*)$ the *fundamental π-module of* $(X,*)$ (confer with [Bau91, Chapter I, §1]). We let TOP_* be the category of pointed topological spaces, and pointed maps. Define also $\mathrm{TOP}_*/_{\cong}$ as the category with object the pointed spaces, and morphisms $(X,*) \to (Y,*)$ to be pointed maps $(X,*) \to (Y,*)$, considered up to pointed homotopy. The fundamental π-module extends to functors $\pi_{(1,2)}\colon \mathrm{TOP}_* \to \mathsf{Agg}$ and $\pi_{(1,2)}\colon \mathrm{TOP}_*/_{\cong} \to \mathsf{Agg}$. Given a pointed map $f\colon (X,*) \to (Y,*)$, the induced map of π-modules is denoted by $\pi_{(1,2)}(f)\colon \pi_{(1,2)}(X,*) \to \pi_{(1,2)}(Y,*)$. This is independent of the representative of the pointed homotopy class of f. Given a pointed homotopy class of pointed maps $[f]\colon (X,*) \to (Y,*)$, the induced morphism in Agg is also denoted by $\pi_{(1,2)}([f])\colon \pi_{(1,2)}(X,*) \to \pi_{(1,2)}(Y,*)$. Note that $\pi_{(1,2)}([f]) = \pi_{(1,2)}(f)$.

4.1. Some bouquets and their fundamental π-modules.

Definition 4.1. Fix, from now on, base points $\star \in S^1$ and $\star \in S^2$, hence $(S^1,\star)$ and $(S^2,\star)$ will be well-pointed. Let n be a positive integer. For $i = 1,\dots,n$, let S^1_i be a copy of the circle $S^1 \subset \mathbb{R}^2$, oriented counterclockwise, and S^2_i a copy of the 2-sphere $S^2 \subset \mathbb{R}^3$, oriented by the exterior normal. We define the following pointed spaces, with the CW-decompositions where $\star$ is the unique 0-cell, each S^1_i has a unique 1-cell, and each S^2_i has a unique 2-cell:

$$(\mathcal{H}_n,\star) = \bigvee_{i=1}^{n} \big((S^1_i,\star) \vee (S^2_i,\star)\big), \qquad (\mathcal{F}_n,\star) = \bigvee_{i=1}^{n} (S^1_i,\star), \qquad (\mathcal{J}_n,\star) = \bigvee_{i=1}^{n} (S^2_i,\star).$$

Let $A_i \in \pi_2(\mathcal{H}_n,\star)$ be given by the homotopy class of a positively oriented characteristic map $(D^2,\partial D^2) \to (S^2_i,\star) \subset (\mathcal{H}_n,\star)$, of the 2-cell S^1_i of $\mathcal{H}_n$, for $i = 1,\dots,n$. Let $x_i \in \pi_1(\mathcal{H}_n,\star)$ be given by the homotopy class of a positively oriented characteristic map $([0,1],\{0,1\}) \to (S^1_i,\star) \subset (\mathcal{H}_n,\star)$ of the 1-cell S^1_i of $\mathcal{H}_n$, for $i = 1,\dots,n$.

By the Seifert – van Kampen theorem, the group $\pi_1(\mathcal{H}_n,\star)$ is free on the $x_1,\dots,x_n \in \pi_1(\mathcal{H}_n,\star)$. Hence $\pi_1(\mathcal{H}_n,\star) \cong F_n$ in Section 2.1, canonically. This is because, since spaces are well-pointed, we have canonical isomorphisms of groups (see e.g. [Hat02, Theorem 1.20 and Example 1.21]):

$$\begin{aligned}\pi_1(\mathcal{H}_n,*) = \pi_1\Big(\bigvee_{i=1}^{n} \big((S^1_i,\star) \vee (S^2_i,\star)\big)\Big) &\cong \bigvee_{i=1}^{n} \pi_1\Big((S^1_i,\star) \vee (S^2_i,\star)\Big)\\ &\cong \bigvee_{i=1}^{n} \big(\pi_1(S^1_i,\star) \vee \pi_1(S^2_i,\star)\big) \cong \bigvee_{i=1}^{n} \pi_1(S^1_i,\star) \cong \bigvee_{i=1}^{n} \mathcal{F}^a(\{x_i\}) \cong F_n.\end{aligned}$$

In particular, the inclusion $(\mathcal{F}_n,\star) \to (\mathcal{H}_n,\star)$ canonically induces an isomorphism of groups $F_n \cong \pi_1(\mathcal{F}_n,\star) \xrightarrow{\cong} \pi_1(\mathcal{H}_n,\star) \cong F_n$.

We now determine $\pi_2(\mathcal{H}_n,\star)$ as a $\pi_1(\mathcal{H}_n,\star)$-module.

Lemma 4.2. *As an abelian group, $\pi_2(\mathcal{H}_n,*)$ is freely generated by the $g \triangleright A_i$, where $g \in \pi_1(\mathcal{H}_n,\star) \cong \pi_1(\mathcal{F}_n,\star) \cong F_n$, and $i = 1,\dots,n$. We have an isomorphism sending $g \triangleright A_i$ to (g,K_i):*

$$\pi_2(\mathcal{H}_n,*) \to M_n,$$

where we recall from Definition 3.7 that:

$$M_n = \mathbb{Z}[F_n]\{K_1, \ldots, K_n\} \cong \bigoplus_{i=1}^{n} \mathbb{Z}[F_n]K_i \cong \bigoplus_{g \in F_n} \bigoplus_{i=1}^{n} \mathbb{Z}(g, K_i).$$

In particular, we have a canonical isomorphism of π-modules:

$$\pi_{(1,2)}(\mathcal{H}_n, *) \cong \mathcal{M}_n.$$

Proof. We extend [Hat02, Example 4.27], which deals with case $n = 1$. The crucial ideas of the following argument are also in [BFMM19, §4.5.1].

Let $q\colon (\widehat{\mathcal{H}_n}, \hat{\star}) \to (\mathcal{H}_n, \star)$ be the universal cover of $\mathcal{H}_n$, where we fixed a $\hat{\star} \in q^{-1}(\star)$. Lifting elements of $\pi_2(\mathcal{H}_n, \star)$ to elements of $\pi_2(\widehat{\mathcal{H}_n}, \hat{\star})$ yields an isomorphism:

$$\Psi\colon \pi_2(\mathcal{H}_n, \star) \to \pi_2(\widehat{\mathcal{H}_n}, \hat{\star}),$$

see e.g. [Hat02, Proposition 4.1]. Let $\overline{q}\colon (\widehat{\mathcal{F}_n}, \hat{\star}) \to (\mathcal{F}_n, \star)$ be the universal cover of $\mathcal{F}_n$, where we fixed $\hat{\star} \in \overline{q}^{-1}(\star)$. The crucial observation, as in [Hat02, Example 4.27], is that that $\widehat{\mathcal{H}_n}$ is obtained from $\widehat{\mathcal{F}_n}$ by attaching a copy of $(\mathcal{J}_n, \star) = \bigvee_{i=1}^{n}(S_i^2, \star)$, along $\star$, to each element of $\overline{q}^{-1}(\star)$.

Covering space theory gives a one-to-one correspondence $\pi_1(\mathcal{H}_n, \star) \to q^{-1}(\star) = \overline{q}^{-1}(\star)$. We choose the convention sending $g \in \pi_1(\mathcal{H}_n, \star)$ to $\hat{\star} \triangleleft g$, where on the right-hand-side we considered the monodromy right-action of $\pi_1(\mathcal{H}_n, \star) \cong \pi_1(\mathcal{F}_n, \star)$ on $q^{-1}(\star) = \overline{q}^{-1}(\star)$.

For each $g \in \pi_1(\mathcal{H}_n, \star)$, consider a copy of $(\mathcal{J}_{n,g}, \star_g)$ of $(\mathcal{J}_n, \star)$. We have a pushout diagram:

$$\begin{array}{ccc}
\bigsqcup_{g \in \pi_1(\mathcal{H}_n, \star)} \{\hat{\star} \triangleleft g\} & \longrightarrow & \bigsqcup_{g \in \pi_1(\mathcal{H}_n, \star)} \mathcal{J}_{n,g} \\
\downarrow & & \downarrow \\
\widehat{\mathcal{F}_n} & \xrightarrow{\iota_n} & \widehat{\mathcal{H}_n}
\end{array}$$

in the category of topological spaces. Here the top horizontal arrow is induced by $\hat{\star} \triangleleft g \mapsto \star_g$. In particular, the top horizontal arrow is a cofibration. Consequently $\iota_n\colon \widehat{\mathcal{F}_n} \to \widehat{\mathcal{H}_n}$ is a cofibration; see e.g. [May99, Page 44]. Since $\widehat{\mathcal{F}_n}$ is contractible, we hence have a homotopy equivalence $\widehat{\mathcal{H}_n} \to \widehat{\mathcal{H}_n}/\widehat{\mathcal{F}_n}$, obtained by collapsing $\widehat{\mathcal{F}_n}$ to a point: call it $\star$; see e.g. [Hat02, Proposition 0.17]. Now note that $(\widehat{\mathcal{H}_n}/\widehat{\mathcal{F}_n}, \star)$ is canonically homeomorphic to $\bigvee_{g \in \pi_1(\mathcal{H}_n, \star)} \bigvee_{i=1}^{n}(S_{g,i}^2, \star)$, where $S_{g,i}^2 = S^2$. Therefore we have a canonical isomorphism $\pi_2(\widehat{\mathcal{H}_n}, \star) \cong \pi_2\big(\bigvee_{g \in \pi_1(\mathcal{H}_n, \star)} \bigvee_{i=1}^{n}(S_{g,i}^2, \star)\big)$.

By [Hat02, Example 4.26], $\pi_2\big(\bigvee_{g \in \pi_1(\mathcal{H}_n, \star)} \bigvee_{i=1}^{n}(S_{g,i}^2, \star)\big)$ is the free abelian group on the homotopy classes $A_{g,i}$ yielded by the inclusions $(S^2, \star) \xrightarrow{\cong} (S_{g,i}^2, \star) \to \bigvee_{g \in \pi_1(\mathcal{H}_n, \star)} \bigvee_{i=1}^{n}(S_{g,i}^2, \star)$. Therefore we have:

$$\pi_2(\widehat{\mathcal{H}}_n, \hat{\star}) \cong \bigoplus_{g \in \pi_1(\mathcal{H}_n, \star)} \bigoplus_{i=1}^{n} \mathbb{Z}(g, K_i)$$

canonically, where $\mathbb{Z}(g, K_i) \cong \mathbb{Z}$, and its positive generator is identified with

$$A_{g,i} \in \pi_2\Big(\bigvee_{g \in \pi_1(\mathcal{H}_n, \star)} \bigvee_{i=1}^{n}(S_{g,i}^2, \star)\Big) \cong \pi_2(\widehat{\mathcal{H}_n}, \star).$$

With our conventions, the positive generator of $\mathbb{Z}(g, K_i)$ corresponds to $\Psi(g \triangleright A_i) \in \pi_2(\widehat{\mathcal{H}_n}, \star)$, for each $g \in \pi_1(\mathcal{H}_n, \star)$ and each $i \in \{1, \ldots, n\}$. Given that $\Psi\colon \pi_2(\mathcal{H}_n, \star) \to \pi_2(\widehat{\mathcal{H}_n}, \hat{\star})$ is an isomorphism, and that $\pi_1(\mathcal{H}_n, \star) \cong F_n$, this completes the proof. $\square$

Remark 4.3. Confer with [BH82, Chapter 5] and also [Hue12, Section 3]. Let $\Pi_2(X)$ denote the fundamental crossed module of a reduced 2-dimensional CW-complex X, with its skeletal filtration $(X, X^1, \{\star\})$, where the base-point $\star$ coincides with the unique 0-cell. Explicitly, $\Pi_2(X) = (\partial\colon \pi_2(X, X^1, \star) \to \pi_1(X^1, \star), \triangleright)$, where ∂ is the map appearing in the penultimate stage of the homotopy long exact sequence of the pointed pair $(X, X^1, \star)$, and $\triangleright$ is the usual action of π_1 on relative π_2. Another proof of Lemma 4.2 can be obtained in the following way:

First of all note that $\Pi_2(\mathcal{H}_n) \cong \bigvee_{i=1}^n (\Pi_2(S_i^1) \vee \Pi_2(S_i^2))$, since the fundamental crossed module functor sends wedge products of well-pointed pairs of spaces to wedge products of crossed modules: see for example [Bro99, Theorem 4.1]. By checking that the associated universal properties hold, one can conclude that the boundary map ∂ in the crossed module $\Pi_2(\mathcal{H}_n)$ is trivial, and also:

$$\pi_1(\mathcal{H}_n, \star) \cong \pi_1(\mathcal{H}_n^1, \star) = \pi_1(\mathcal{F}_n, \star) \cong F_n$$

and

$$\pi_2(\mathcal{H}_n, \mathcal{H}_n^1, \star) = \pi_2(\mathcal{H}_n, \mathcal{F}_n, \star) \cong M_n.$$

Now since $\mathcal{F}_n$ is aspherical, it then follows, by applying the homotopy long exact sequence of $(\mathcal{H}_n, \mathcal{F}_n, \star)$, that $\pi_2(\mathcal{H}_n, \star) \cong \ker(\partial) = \pi_2(\mathcal{H}_n, \mathcal{H}_n^1, \star) \cong M_n$. By construction this homomorphism preserves the action of $\pi_1(\mathcal{F}_n, \star) \cong F_n$.

4.2. **Fundamental π-module for** $(D^3 \setminus L_n, *)$**.** Recall that $L_n = \cup_{i=1}^n C_i$ is an unlinked union of unlinked circles in the interior of D^3. It is now convenient to consider that all circles are confined to the vertical plane $\{y = 1/2\}$. We take each C_i to be a circle centred at $(i/(n+1), 1/2, 1/2)$, with radius $1/2n$, and oriented clockwise, for the point of view of an observer sitting in the $y = 0$ plane. Furthermore, we consider a base-point located in the $\{z = 0\}$ plane. Analogously to the 2-dimensional case in Subsection 2.1, we have:

Proposition 4.4. *Let $*$ be a base point for $D^3 \setminus L_n$ contained in $\partial D^3 \cap \{z = 0\}$. The spaces $(D^3 \setminus L_n, *)$ and $(\mathcal{H}_n, \star)$ are pointed homotopic. Moreover, we have a deformation retraction from $(D^3 \setminus L_n, *)$ onto a certain homeomorphic image of $(\mathcal{H}_n, \star)$. Here $(\mathcal{H}_n, \star)$ embeds in $(D^3 \setminus L_n, *)$ in a way where each oriented circle S_i^1 encircles the oriented circle $C_i \subset D^3$, forming a Hopf link, with positive linking number, and each oriented S_i^2 embeds as a positively oriented "balloon" (i.e. a 2-sphere, oriented by the exterior normal $\vec{n}_i$) containing C_i, and no other circle, see Figure 8.*

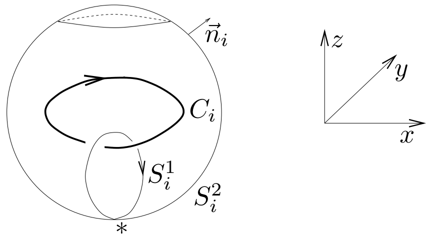

FIGURE 8. Embedding of $(S_i^1, \star) \vee (S_i^2, \star)$ inside $(D^3 \setminus L_n, *)$.

From the previous lemma, it follows that the inclusion $(\mathcal{H}_n, \star)$ inside $\pi_2(D^3 \setminus L_n, *)$ induces an isomorphism of π-modules $\pi_{(1,2)}(\mathcal{H}_n, \star) \cong \pi_{(1,2)}(D^3 \setminus L_n, *)$. Hence $\pi_{(1,2)}(D^3 \setminus L_n, *) \cong \mathcal{M}_n$.

Let f be a self-homeomorphism of the pair (D^3, L_n), relative to the boundary. Isotopies of such homeomorphisms are also considered to be relative to the boundary. Hence each element $[f] \in \mathrm{MCG}(D^3, L_n) = LB_n^{\mathrm{ext}}$ yields, by restricting to $D^3 \setminus L_n$, a pointed homotopy class of pointed maps $\Upsilon([f])\colon (D^3 \setminus L_n, *) \to (D^3 \setminus L_n, *)$; here recall that we imposed $* \in \partial D^3$. The induced map of fundamental π-modules:

$$\pi_{(1,2)}\big(\Upsilon([f])\big)\colon \pi_{(1,2)}(D^3 \setminus L_n, *) \cong \mathcal{M}_n \longrightarrow \pi_{(1,2)}(D^3 \setminus L_n, *) \cong \mathcal{M}_n$$

is denoted $\Theta'([f])\colon \mathcal{M}_n \longrightarrow \mathcal{M}_n$. The functoriality of $\pi_{(1,2)}\colon \mathrm{TOP}_*/_{\cong} \to \mathsf{Agg}$ gives that $\Theta'([f][g]) = \Theta'([f\circ g]) = \Theta'([f])\circ\Theta'([g])$, for each $[f],[g] \in \mathrm{MCG}(D^3, L_n)$. We thus have a group homomorphism $\Theta'\colon LB_n^{\mathrm{ext}} = \mathrm{MCG}(D^3, L_n) \to \mathrm{Aut}(\mathcal{M}_n)$. The following theorem gives the topological interpretation of the lifted Artin representation.

Theorem 4.5. *The homomorphism $\Theta'\colon LB_n^{\mathrm{ext}} \to \mathrm{Aut}(\mathcal{M}_n)$ coincides with the homomorphism $\Theta\colon LB_n^{\mathrm{ext}} \to \mathrm{Aut}(\mathcal{M}_n)$ in Theorem 3.16. In particular Θ' is injective.*

The main ideas of the following proof are also in [BFMM19, §4.5.1].

Proof. **(Sketch)** We recall what each generator g of the extended loop braid group in Theorem 3.2 incarnates geometrically. Let $\mathrm{Homeo}(D^3)$ be the group of homeomorphisms $D^3 \to D^3$, relative to the boundary. Consider an ambient isotopy $t \in [0,1] \mapsto \phi_t^g \in \mathrm{Homeo}(D^3)$ of D^3, for each generator g of LB_n^{ext}, as in Figure 9. Concretely $t \mapsto \phi_t^g$ is obtained by applying the isotopy extension theorem, as in [Hir76, Chapter 8. 1.3 Theorem], to the smooth isotopies outlined in Figure 9. Each ambient isotopy $t \mapsto \phi_t^g$ is relative to the boundary of D^3, and satisfies $\phi_0^g = \mathrm{id}_{D^3}$. Moreover, note that ϕ_1^g is a homeomorphism $(D^3, L_n) \to (D^3, L_n)$. The elements of $\mathrm{MCG}(D^3, L_n)$ corresponding to each of the generators g of LB_n^{ext} in Theorem 3.2 are obtained by evaluating the ambient isotopies ϕ_t^g at $t = 1$. Our motions are the ones in [Dam17, §3], with the interval $[0,1]$ taken with reversed extremes due to different conventions for the product in $\mathrm{MCG}(D^3, L_n)$.

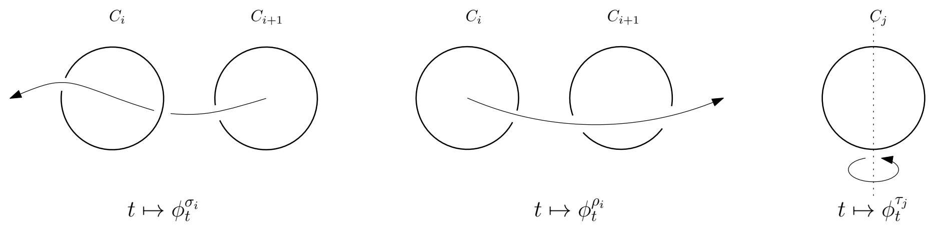

FIGURE 9. Our conventions for the elements Σ_i, ρ_i and τ_j in $\mathrm{MCG}(D^3, L_n)$, obtained by evaluating (respectively) the shown ambient isotopies at $t = 1$.

Recall $\pi_{(1,2)}(D^3, L_n) \cong \mathcal{M}_n = (F_n, M_n, \triangleright)$, where M_n is the free $\mathbb{Z}[F_n]$-module generated by $\{K_1, \ldots, K_n\}$; see Definition 3.7. Our conventions for the generators $x_j \in \pi_1(D^3 \setminus L_n, *)$ and $K_j \in \pi_2(D^3 \setminus L_n, *)$, where $j = 1, \ldots, n$, are depicted on the left-hand side of Figure 10. In particular, K_j is obtained from a positively oriented parametrisation $(D^2, \partial D^2) \to (D^3 \setminus L_n, *_j)$ of a "balloon" (i.e. a 2-sphere, oriented by an exterior normal), based in $*_j$, as shown, containing the circle C_j only; acted on by the path γ_j, a straight line connecting $*$ and $*_j$, in order to obtain an element of $\pi_2(D^3 \setminus L_n, *)$. This point of view is as in [BN15], [BFMM19, §4.5.1] and [Far09, §2.1.3]. Also, x_j is given by the loop in Figure 10, which is then conjugated by the path γ_j, in order to yield an element of $\pi_1(D^3 \setminus L_n, *)$. By applying the deformation retration in Proposition 4.4, we hence obtain the generators of $\pi_{(1,2)}(\mathcal{H}_n, \star)$ as in Lemma 4.2.

Let us see how the generators of $\mathrm{MCG}(D^3, L_n)$ in Theorem 3.2 act on the x_is and the K_js. We use the same notation to denote $g \in LB_n^{\mathrm{ext}}$ and the map $\Theta'(g)\colon \mathcal{M}_n \to \mathcal{M}_n$.

Whenever $|i - j| > 1$, nothing happens when we apply Σ_i and ρ_i to x_j and to K_j. Also, ρ_i is such that:

$$\rho_i\colon \qquad x_i \mapsto x_{i+1}, \qquad x_{i+1} \mapsto x_i, \qquad K_i \mapsto K_{i+1}, \qquad K_{i+1} \mapsto K_i.$$

Hence ρ_i coincides with $\mathcal{R}_i$ in (3.10a) and (3.10b).

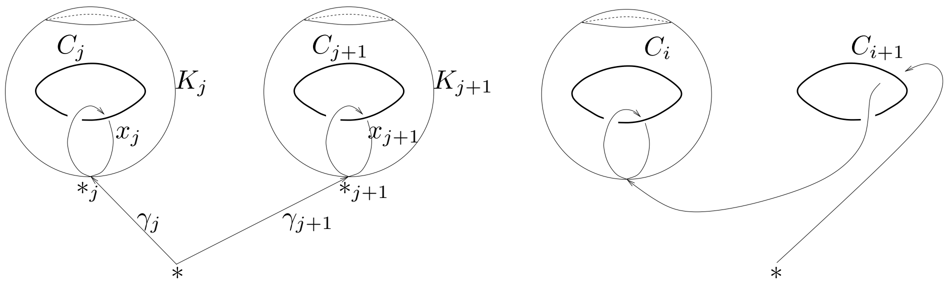

FIGURE 10. On the left of the figure: our conventions for the generators $x_j \in \pi_1(D^3 \setminus L_n, *)$ and $K_j \in \pi_2(D^3 \setminus L_n, *)$, in the vicinity of the circle C_j. On the right of the figure: the result of acting with Σ_i on x_{i+1} and on K_{i+1}.

If we apply Σ_i to x_i we get x_{i+1}. The right-hand-side of Figure 10 indicates what happens to x_{i+1} and K_{i+1} when we apply Σ_i. Hence:

$$\Sigma_i\colon \qquad x_{i+1} \mapsto x_{i+1}^{-1}\, x_i\, x_{i+1}, \qquad K_{i+1} \mapsto x_{i+1}^{-1} \triangleright K_i.$$

Let us now determine $\Sigma_i(K_{i+1})$. Note that when the circle C_{i+1} goes inside the circle C_i it "drags" the balloon representing the class $K_i \in \pi_2(D^3 \setminus L_n, *)$. However $\Sigma_i(K_i+K_{i+1}) = K_i+K_{i+1}$. This is because $K_i + K_{i+1} \in \pi_2(D^3 \setminus L_n, *)$ can be seen as being represented by a bigger balloon $K_{i,i+1}$ containing $C_i \cup C_{i+1}$, see Figure 11. Moreover, the ambient isotopy $t \in [0,1] \mapsto \phi_t^{\sigma_i}$ from which we define $\Sigma_i = \phi_1^{\sigma_i}$ can be chosen to happen inside the bigger balloon $K_{i,i+1}$, hence not moving $K_{i,i+1}$. In more precise terms, we choose an ambient isotopy $t \in [0,1] \mapsto \phi_t^{\sigma_i}$ supported in a compact set contained in the region bounded by $K_{i,i+1}$.

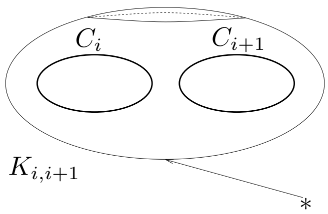

FIGURE 11. The balloon $K_{i,i+1}$ representing $K_i + K_{i+1} \in \pi_2(D^3 \setminus L_n, *)$.

We obtain $\Sigma_i(K_{i,i+1}) = K_{i,i+1}$. Therefore:

$$\Sigma_i(K_i) + \Sigma_i(K_{i+1}) = \Sigma_i(K_i + K_{i+1}) = K_i + K_{i+1},$$

and

$$\Sigma_i(K_i) = K_i + K_{i+1} - \Sigma_i(K_{i+1}) = K_i + K_{i+1} - x_{i+1}^{-1} \triangleright K_{i+1}.$$

Hence Σ_i coincides with $\mathcal{S}_i$ in (3.9a) and (3.9b). That τ_i's coincide with $\mathcal{T}_i$'s in (3.11a) can be seen in a similar way. Note that $\tau_i(K_i) = K_i$, since the ambient isotopy yielding K_i can be chosen to happen inside the balloon representing K_i. □

Remark 4.6. The restriction of the representation $\Theta\colon LB_n^{\mathrm{ext}} \to \mathrm{Aut}(\mathcal{M}_n)$ to the loop braid group LB_n, as well as its topological interpretation, are also discussed in [BFMM19, §4.5.1].

Remark 4.7. In Remark 3.18, we promised an interpretation of the property $\Theta(g)^2(K_1+\cdots+K_n) = K_1+\cdots+K_n$, for each $g \in \mathrm{MCG}(D^3, L_n)$. Observe that, given our conventions, $M = K_1+\cdots+K_n$ is homotopic to the element of $\pi_2(D^3 \setminus L_n, *)$ yielded by the inclusion of $(\partial D^3, *)$ inside $(D^3 \setminus L_n, *)$. Since all elements of $\mathrm{MCG}(D^3, L_n)$ restrict to the identity over ∂D^3, it follows that $\Theta(g)^2(M) = M$, for all $g \in \mathrm{MCG}(D^3, L_n) = LB_n^{\mathrm{ext}}$.

4.3. **Remarks on connections to Baues' combinatorial homotopy.** Given a pointed space $(X,*)$, we denote by $\mathcal{E}(X,*)$ the group of pointed homotopy equivalences $(X,*) \to (X,*)$, up to pointed homotopy. Note that $\mathcal{E}(D^3 \setminus L_n,*) \cong \mathcal{E}(\mathcal{H}_n,\star)$. If $* \in \partial D^3$, we have a group homomorphism $\Upsilon\colon LB_n^{\mathrm{ext}} \to \mathcal{E}(D^3 \setminus L_n,*)$, obtained by restricting $f\colon (D^3,L_n) \to (D^3,L_n)$ to $D^3 \setminus L_n$. The fundamental π-module functor $\mathrm{TOP}_*/_{\cong} \to \mathsf{Agg}$ gives another group homomorphism $\mathcal{E}(D^3 \setminus L_n,*) \to \mathrm{Aut}(\pi_{(1,2)}(D^3 \setminus L_n,*)) \cong \mathrm{Aut}(\mathcal{M}_n)$. Composing with Υ, this yields a group homomorphism $\Theta'\colon LB_n \to \mathrm{Aut}(\mathcal{M}_n)$, which by Theorem 4.5, coincides with $\Theta\colon LB_n \to \mathrm{Aut}(\mathcal{M}_n)$.

Note that taking into account our results, it follows from elementary algebraic topological techniques, or as a particular case of [Bau91, Theorem III(7.1)] or [Bau08, Corollary VI(3.5)], that $\mathcal{E}(D^3 \setminus L_n,*) \cong \mathcal{E}(\mathcal{H}_n,\star) \cong \mathrm{Aut}(\mathcal{M}_n)$. This follows since the construction in *loc cit* implies that pointed homotopy classes of pointed maps between bouquets of 1- and 2-spheres are in one-to-one correspondence with maps between their fundamental crossed modules, which is this case reduce to π-modules (meaning that they have a trivial boundary map ∂). This result is a generalisation of the well known fact that $\mathcal{E}(D^2 \setminus d_n,*) \cong \mathcal{E}(\mathcal{F}_n,*) \cong \mathrm{Aut}(F_n)$. Hence in this $(D^3 \setminus L_n,*)$ case, π-modules retain all of the homotopy theoretical information necessary to combinatorially model the group $\mathcal{E}(D^3 \setminus L_n,*)$. Therefore, the lifted Artin representation $\Theta\colon \mathrm{MCG}(D^3,L_n) \to \mathrm{Aut}(\mathcal{M}_n)$ models the pointed homotopy type of the restriction of a mapping class in $\mathrm{MCG}(D^3,L_n)$ to $(D^3 \setminus L_n,*)$.

Appendix A. Calculations

In this Appendix we explicitly verify that relations from Theorem 3.16 hold. It will be enough to consider the case $n=3$. In order to reduce indices, with respect to Theorem 3.16 we replace $\{x_1,x_2,x_3\}$ with $\{x,y,z\}$ and $\{K_1,K_2,K_3\}$ with $\{K,L,M\}$. We will only show computation for generators $\{K,L,M\}$, since on $\{x,y,z\}$ our representation coincides with Dahm's homomorphism. Note that the way the generators x,y,z change by applying Artin representation is fundamental for the lifted Artin representation to work at the level of $\pi_2(D^3 \setminus L_n,*)$. For instance note that in the first calculation below, $y^{-1} \triangleright K$ is sent by $\mathcal{S}_2$ to $z^{-1} \triangleright K$, since y is sent to z.

- $\mathcal{S}_1 \circ \mathcal{S}_2 \circ \mathcal{S}_1 = \mathcal{S}_2 \circ \mathcal{S}_1 \circ \mathcal{S}_2$.
  Applying $\mathcal{S}_1 \circ \mathcal{S}_2 \circ \mathcal{S}_1$ one obtains:

$$\begin{cases} K \longmapsto & K + L - y^{-1} \triangleright K \longmapsto K + L + M - z^{-1} \triangleright L - z^{-1} \triangleright K \longmapsto \\ & K + L - y^{-1} \triangleright K + y^{-1} \triangleright K + M - z^{-1} \triangleright y^{-1} \triangleright K - z^{-1} \triangleright (K + L - y^{-1} \triangleright K) \\ & = K + L + M - z^{-1} \triangleright (K + L), \\ L \longmapsto & y^{-1} \triangleright K \longmapsto z^{-1} \triangleright K \longmapsto z^{-1} \triangleright (K + L - y^{-1} \triangleright K), \\ M \longmapsto & M \longmapsto z^{-1} \triangleright L \longmapsto z^{-1} \triangleright y^{-1} \triangleright K \end{cases}$$

while applying $\mathcal{S}_2 \circ \mathcal{S}_1 \circ \mathcal{S}_2$ one obtains:

$$\begin{cases} K \longmapsto & K \longmapsto K + L - y^{-1} \triangleright K \longmapsto K + L + M - z^{-1} \triangleright L - z^{-1} \triangleright K, \\ L \longmapsto & L + M - z^{-1} \triangleright L \longmapsto y^{-1} \triangleright K + M - z^{-1} \triangleright y^{-1} \triangleright K \\ & \longmapsto z^{-1} \triangleright K + z^{-1} \triangleright L - (z^{-1}y^{-1}z) \triangleright z^{-1} \triangleright K \\ M \longmapsto & z^{-1} \triangleright L \mapsto z^{-1} \triangleright y^{-1} \triangleright K \mapsto (z^{-1}y^{-1}z) \triangleright z^{-1} \triangleright K. \end{cases}$$

- $\mathcal{R}_2 \circ \mathcal{R}_1 \circ \mathcal{S}_2 = \mathcal{S}_1 \circ \mathcal{R}_2 \circ \mathcal{R}_1$.
  Applying $\mathcal{R}_2 \circ \mathcal{R}_1 \circ \mathcal{S}_2$ one obtains:

$$\begin{cases} K \longmapsto & K \longmapsto L \longmapsto M \\ L \longmapsto & L + M - z^{-1} \triangleright L \longmapsto K + M - z^{-1} \triangleright K \longmapsto K + L - y^{-1} \triangleright K \\ M \longmapsto & z^{-1} \triangleright L \mapsto z^{-1} \triangleright K \mapsto y^{-1} \triangleright K \end{cases}$$

while applying $\mathcal{S}_1 \circ \mathcal{R}_2 \circ \mathcal{R}_1$ one obtains:

$$\begin{cases} K \longmapsto & L \longmapsto M \longmapsto M \\ L \longmapsto & K \longmapsto K \longmapsto K + L - y^{-1} \triangleright K \\ M \longmapsto & M \longmapsto L \longmapsto y^{-1} \triangleright K. \end{cases}$$

- $\mathcal{S}_2 \circ \mathcal{S}_1 \circ \mathcal{R}_2 = \mathcal{R}_1 \circ \mathcal{S}_2 \circ \mathcal{S}_1$.
  Applying $\mathcal{S}_2 \circ \mathcal{S}_1 \circ \mathcal{R}_2$ one obtains:

$$\begin{cases} K \longmapsto & K \longmapsto K + L - y^{-1} \triangleright K \longmapsto K + L + M - z^{-1} \triangleright L - z^{-1} \triangleright K \\ L \longmapsto & M \longmapsto M \longmapsto z^{-1} \triangleright L \\ M \longmapsto & L \longmapsto y^{-1} \triangleright K \longmapsto z^{-1} \triangleright K \end{cases}$$

while applying $\mathcal{R}_1 \circ \mathcal{S}_2 \circ \mathcal{S}_1$ one obtains:

$$\begin{cases} K \longmapsto & K + L - y^{-1} \triangleright K \longmapsto K + L + M - z^{-1} \triangleright L - z^{-1} \triangleright K \\ & \longmapsto L + K + M - z^{-1} \triangleright K - z^{-1} \triangleright L \\ L \longmapsto & y^{-1} \triangleright K \longmapsto z^{-1} \triangleright K \longmapsto z^{-1} \triangleright L \\ M \longmapsto & M \longmapsto z^{-1} \triangleright L \longmapsto z^{-1} \triangleright K. \end{cases}$$

- $\mathcal{T}_1 \circ \mathcal{R}_1 = \mathcal{R}_1 \circ \mathcal{T}_2$.
  Applying $\mathcal{T}_1 \circ \mathcal{R}_1$ one obtains:

$$\begin{cases} K \longmapsto & L \longmapsto L \\ L \longmapsto & K \longmapsto K \\ M \longmapsto & M \longmapsto M \end{cases}$$

while applying $\mathcal{R}_1 \circ \mathcal{T}_2$ one obtains:

$$\begin{cases} K \longmapsto & K \longmapsto L \\ L \longmapsto & L \longmapsto K \\ M \longmapsto & M \longmapsto M. \end{cases}$$

- $\mathcal{T}_1 \circ \mathcal{S}_1 = \mathcal{S}_1 \circ \mathcal{T}_2$.
  Applying $\mathcal{T}_1 \circ \mathcal{S}_1$ one obtains:

$$\begin{cases} K \longmapsto & K + L - y^{-1} \triangleright K \longmapsto K + L + y^{-1} \triangleright K \\ L \longmapsto & y^{-1} \triangleright K \longmapsto y^{-1} \triangleright K \\ M \longmapsto & M \longmapsto M \end{cases}$$

while applying $\mathcal{S}_1 \circ \mathcal{T}_2$ one obtains:

$$\begin{cases} K \longmapsto & K + L - y^{-1} \triangleright K \longmapsto K + L + y^{-1} \triangleright K \\ L \longmapsto & y^{-1} \triangleright K \longmapsto y^{-1} \triangleright K \\ M \longmapsto & M \longmapsto M. \end{cases}$$

- $\mathcal{T}_2 \circ \mathcal{S}_1 = \mathcal{R}_1 \circ \mathcal{S}_1^{-1} \circ \mathcal{R}_1 \circ \mathcal{T}_1$.
  Applying $\mathcal{T}_2 \circ \mathcal{S}_1$ one obtains:

$$\begin{cases} K \longmapsto & K + L - y^{-1} \triangleright K \longmapsto K + L - y \triangleright K \\ L \longmapsto & y^{-1} \triangleright K \mapsto y \triangleright K \\ M \longmapsto & M \longmapsto M \end{cases}$$

while applying $\mathcal{R}_1 \circ \mathcal{S}_1^{-1} \circ \mathcal{R}_1 \circ \mathcal{T}_1$ one obtains:

$$\begin{cases} K \longmapsto & K \longmapsto L \longmapsto K + L - x \triangleright L \longmapsto L + K - y \triangleright K \\ L \longmapsto & L \longmapsto K \longmapsto x \triangleright L \longmapsto y \triangleright K \\ M \longmapsto & M \longmapsto M \longmapsto M \longmapsto M. \end{cases}$$

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